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## SOME PROPERTIES OF UPPER BASIC SUBGROUPS

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In what follows the word "group" will mean "primary Abelian group" and unless otherwise specified the notation and terminology will be that of L. FUCHS in [1]. One exception to this will be that a direct sum of the groups  $A$  and  $B$  will be denoted  $A \oplus B$  and  $A + B$  will indicate a sum which is not necessarily direct.

The following definition and theorems will be used:

**DEFINITION A.** (IRWIN [2]). Let  $G$  be a group. If  $H$  is a subgroup of  $G$  maximal with respect to disjointness from the elements of infinite height of  $G$  then  $H$  will be called a high subgroup of  $G$ .

**THEOREM B.** (IRWIN [2]). Let  $G$  be a group. If  $H$  is a high subgroup of  $G$  then  $H$  is pure in  $G$  and  $H$  contains a basic subgroup of  $G$ .

**THEOREM C.** (MITCHELL and MITCHELL [5]). Let  $G$  be an infinite reduced  $p$ -group and  $B$  a basic subgroup of  $G$  such that  $G/B = \sum_{\alpha \in I} (G_\alpha/B)$  where  $G_\alpha/B \cong Z(P^\alpha)$  for all  $\alpha \in I$ . Then  $G = H \oplus K$  and  $B = H \oplus L$  where  $L$  is a basic subgroup of  $K$  such that  $r(K/L) = r(G/B) = |I|$  and  $|K| = \text{maximum } \{x_0 | I|\}$ .

**THEOREM D.** (MITCHELL and MITCHELL [5]). Let  $G$  be an infinite Abelian  $p$ -group such that  $G^1 = 0$  or  $|G^1| \geq x_0$ . Then  $G = H \oplus K$  where  $H$  is a direct sum of cyclic groups and every basic subgroup of  $K$  is both an upper basic subgroup of  $K$  and a lower basic subgroup of  $K$ .

This paper deals with the following problems concerning properties of upper basic subgroups of infinite Abelian  $p$ -groups. It is evident that some of these problems follow from others but as it often happens the proofs precede in the reverse direction.

**PROBLEM I.** If  $G$  is a reduced  $p$ -group such that  $G = H \oplus K$  where  $K$  is a direct sum of cyclic groups and  $B$  is an upper basic subgroup of  $H$ , then is  $B \oplus K$  an upper basic subgroup of  $G$ ?

**PROBLEM II.** If  $G$  is a reduced  $p$ -group such that  $G = H \oplus K$ , and  $A$  and  $B$  are upper basic subgroups of  $H$  and  $K$  respectively, then is  $A \oplus B$  an upper basic subgroup of  $G$ ?

**PROBLEM III.** If  $G$  is a reduced  $p$ -group, and  $H$  is a high subgroup of  $G$ , and  $B$  is an upper basic subgroup of  $H$ , then is  $B$  an upper basic subgroup of  $G$ ?

**PROBLEM IV.** If  $G$  is a reduced  $p$ -group and  $B$  is a basic subgroup of  $G$ , then is  $B$  always contained in an upper basic subgroup of  $G$ ?

Problems I, II, and III are solved for all reduced Abelian  $p$ -groups and a partial solution to Problems IV is given leading to a class of groups which includes several previously defined classes. We begin with the following lemmas.

**LEMMA 1.** If  $G$  is a  $p$ -group without elements of infinite height such that every basic subgroup of  $G$  is both an upper and lower basic subgroup of  $G$ , and such that final rank  $(G) = |G|$ . Then  $G$  cannot be decomposed as  $G = H \oplus F$  where  $F$  is a direct sum of cyclic groups, and  $|H| < |G|$ .

**PROOF.** Suppose such a decomposition of  $G$  does exist, and let  $B$  be a basic subgroup of  $H$ . Now  $B \oplus F$  is a basic subgroup of  $G$  and rank  $(G/(B \oplus F)) = \text{rank}(H/B) + \text{rank}(F/F) = \text{rank}(H/B) \leq |H| < |G|$ . Since final rank  $(G) = |G|$  there exists a basic subgroup  $A$  of  $G$  by Theorem 31.4, page 105, in [1] such that rank  $(G/A) = |G|$ . But these two facts contradict the hypothesis that every basic subgroup of  $G$  is both an upper and lower basic subgroup of  $G$ .

**LEMMA 2.** Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus F$  where  $F$  is a direct sum of cyclic groups, and suppose that every basic subgroup of  $H$  is both an upper and a lower basic subgroup of  $H$ . If final rank  $(H) = |H|$ ,  $|F| < |H|$ , and  $B$  is a basic subgroup of  $H$ , then  $B \oplus F$  is an upper basic subgroup of  $G$ .

**PROOF.** Suppose that  $B \oplus F$  is not an upper basic subgroup of  $G$ , and let  $A$  be an upper basic subgroup of  $G$ . By Theorem C we know that  $G = L \oplus A'$  and  $A = A' \oplus A''$  where  $|L| = \text{maximum}(\aleph_0, \text{rank}(G/A))$ . If  $|L| \leq \aleph_0$  then by Theorem 33.4, page 113, in [1] we have that  $G$  is a direct sum of cyclic groups, and therefore  $H$  is a direct sum of cyclic groups. Thus  $H$  must be bounded since each of its basic subgroups is both an upper and lower basic subgroup. Therefore  $B = H$  and so  $B \oplus F = H \oplus F$  is an upper basic subgroup of  $G$ . We can now assume that  $\aleph_0 < |L| = \text{rank}(G/A) < \text{rank}(G/(B \oplus F)) \leq |H|$ . Now write  $G = L \oplus S' \oplus S''$  where  $A' = S' \oplus S''$ , and where  $L \oplus S'$  contains  $F$  and  $|L \oplus S'| < |H|$ . But we know  $H \cong G/F \cong [(L \oplus S')/F] \oplus S''$ , which contradicts Lemma 1 when applied to  $H$ , therefore  $B \oplus F$  must be an upper basic subgroup of  $G$ .

**LEMMA 3.** Let  $G$  be a  $p$ -group without elements of infinite height with  $G = H \oplus F$  where  $F$  is a direct sum of cyclic groups, final rank  $(H) = |H|$ , and every basic subgroup of  $H$  is both an upper and lower basic subgroup of  $H$ . If  $B$  is a basic subgroup of  $H$ , then  $B \oplus F$  is an upper basic subgroup of  $G$ .

**PROOF.** Suppose that  $B \oplus F$  is not an upper basic subgroup of  $G$ , and let  $A$  be an upper basic subgroup of  $G$ . As in Lemma 2 we can assume that rank  $(G/A) > \aleph_0$ . By Theorem C we can write  $G = L \oplus A'$  and  $A = A' \oplus A''$  where  $|L| = \text{rank}(G/A) < \text{rank}(G/(B \oplus F)) \leq |H|$ . Now we can write  $G = H \oplus F' \oplus F''$  where  $F = F' \oplus F''$  and  $H \oplus F'$  contains  $L$  and  $|F' + L| < |H|$ . Consider the group  $H \oplus F'$ . By Lemma 2 we know that  $B \oplus F'$  is

an upper basic subgroup of  $H \oplus F'$ . But  $H \oplus F'$  contains  $L$  which is a summand of  $G$  so that we can write  $H \oplus F' = L \oplus [(H \oplus F') \cap A']$ . Let  $S = (H \oplus F') \cap A'$ . Now observe that  $\text{rank } ((H \oplus F')/(A'' \oplus S)) = \text{rank } (L/A'') \leq |L| < |H|$ . Since final rank  $(H) = |H|$ , and every basic subgroup of  $H$  is both an upper and lower basic subgroup of  $H$  we know that  $\text{rank } (H/B) = |H|$  which contradicts  $B \oplus F'$  being an upper basic subgroup of  $H \oplus F'$ . Thus  $B \oplus F$  must have been an upper basic subgroup of  $G$ .

**LEMMA 4.** Let  $G$  be a reduced  $p$ -group such that every basic subgroup of  $G$  is an upper and lower basic subgroup of  $G$ . Suppose  $G = H \oplus K$  where  $K$  is a direct sum of cyclic groups. Then  $H$  has the property that each of its basic subgroups is both an upper and lower basic subgroup of  $H$ .

**PROOF.** Suppose there exists two basic subgroups  $A$  and  $B$  of  $H$  such that the rank  $(H/A) \neq \text{rank } (H/B)$ . Then we know that  $A \oplus K$  and  $B \oplus K$  are basic subgroups of  $G$  such that  $\text{rank } (G/(A \oplus K)) = \text{rank } (H/A) \neq \text{rank } (H/B) = \text{rank } (G/(B \oplus K))$ , and this contradicts the hypothesis on  $G$ .

**THEOREM 5.** Let  $G$  be a  $p$ -group without elements of infinite height such that  $G = H \oplus F$  where  $F$  is a direct sum of cyclic groups. If  $B$  is an upper basic subgroup of  $H$ , then  $B \oplus F$  is an upper basic subgroup of  $G$ .

**PROOF.** If  $H$  is a finite group then  $B \oplus F = G$  since  $B$  is basic in  $H$  thus  $B \oplus F$  is upper basic. Since  $B$  is an upper basic subgroup of  $H$  we can write, by Theorem D,  $H = H' \oplus B''$  and  $B = B' \oplus B''$  where  $|H'| = \text{maximum}(\aleph_0, \text{rank } (H/B))$ , and where every basic subgroup of  $H'$  is both an upper and lower basic subgroup of  $H'$ . As in Lemma 2 we can assume that  $\text{rank } (H/B) > \aleph_0$ . Now if final rank  $(H') < |H'|$  then we can write  $H' = H'' \oplus H'''$  and  $B' = A \oplus H'''$  where final rank  $(H'') = |H''|$ . By Lemma 4 every basic subgroup of  $H''$  is both an upper and lower basic subgroup of  $H''$ . Thus  $G = H'' \oplus H''' \oplus B'' \oplus F$  where  $H''' \oplus B'' \oplus F$  is a direct sum of cyclic groups, and, hence, by Lemma 3 we have that  $A \oplus H''' \oplus B'' \oplus F$  is an upper basic subgroup of  $G$ , but  $A \oplus H''' \oplus B'' \oplus F = B \oplus F$ .

**COROLLARY 6.** Let  $G$  be a  $p$ -group without elements of infinite height such that  $G = H \oplus K = S \oplus T$  where final rank  $(H) = |H|$  and final rank  $(S) = |S|$ . Suppose that  $K$  and  $T$  are direct sums of cyclic groups, and that every basic subgroup of  $H$  and  $S$  is an upper and lower basic subgroup. Then final rank  $(H) = \text{final rank } (S)$ .

**PROOF.** Let  $A$  and  $B$  be basic subgroup of  $H$  and  $S$  respectively. Now we have final rank  $(H) = \text{rank } (H/A) = \text{rank } (G/(A \oplus K)) = \text{rank } (G/(B \oplus T)) = \text{final rank } (S)$  since  $A \oplus K$  and  $B \oplus T$  are upper basic subgroups of  $G$  by Theorem 5.

Theorem 5 has solved Problem I for  $p$ -groups without elements of infinite height. We will now use these results to attack Problem II.

**LEMMA 7.** Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus K$  where final rank  $(G) = |G|$ , final rank  $(H) = |H|$ , final rank  $(K) = |K|$ , and  $H$  and  $K$  have the property that every basic subgroup is an upper and lower basic subgroup. If  $G = L \oplus F$  where  $|L| < |G|$ , and  $F$  is a direct sum of cyclic groups, then  $|G| = |H| = |K|$ .

**PROOF.** Suppose that  $|H| < |G|$ , then  $|K| = |G|$ . Now write  $G$  as  $G = L \oplus F' \oplus F''$  where  $L \oplus F'$  contains  $H$ , and  $|L \oplus F'| < |G|$ . This is possible since  $|L| < |G|$ , and  $|H| < |G|$ . Now notice that  $K \cong G/H \cong \cong [(L \oplus F')/H] \oplus F''$ , and  $|(L \oplus F')/H| \leq |L \oplus F'| < |G| = |K|$ . But this contradicts Lemma 1 when applied to  $K$ , therefore we must have  $|G| = |H| = |K|$ .

**LEMMA 8.** Let  $G$  be a  $p$ -group without elements of infinite height and suppose that  $G = H \oplus K$  where the final rank  $(H) = |H|$ , and final rank  $(K) = |K|$ , final rank  $(G) = |G|$ , and every basic subgroup of  $H$  or  $K$  is both an upper and lower basic subgroup. Let  $A$  and  $B$  be basic subgroups of  $H$  and  $K$  respectively. If either  $|H| < |G|$  or  $|K| < |G|$ , then  $A \oplus B$  is an upper basic subgroup of  $G$ .

**PROOF.** Assume that  $A \oplus B$  is not an upper basic subgroup of  $G$ . Let  $S$  be an upper basic subgroup of  $G$ . Now by Theorem C we know that  $G = L \oplus F$  where  $F$  is a direct sum of cyclic groups, and  $|L| = \text{maximum}(\aleph_0, \text{rank}(G/S))$ . If  $|L| \leq \aleph_0$ , then  $G$  is a direct sum of cyclic groups by Theorem 33.4, page 113, in [1]. But this means that  $H$  and  $K$  are bounded since the bounded direct sum of cyclic groups are the only direct sums of cyclic groups which have the property that every basic subgroup is both an upper and lower basic subgroup. Thus  $G$  is a bounded  $p$ -group and hence has only one basic subgroup which contradicts the assumption that  $A \oplus B$  is not an upper basic subgroup. Therefore  $\aleph_0 < \text{rank}(G/S)$ , and  $|L| = \text{rank}(G/S)$ .

Since  $A \oplus B$  is not an upper basic subgroup we know that  $\text{rank}(G/S) < |G|$ , but this contradicts Lemma 7. Thus  $A \oplus B$  must be an upper basic subgroup of  $G$ .

**LEMMA 9.** Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus K$  where final rank  $(H) = |H|$ ,  $|K| < |H|$ , and every basic subgroup of  $H$  is both an upper and lower basic subgroup of  $H$ . If  $B$  is an upper basic subgroup of  $K$ , and  $A$  is a basic subgroup of  $H$ , then  $A \oplus B$  is an upper basic subgroup of  $G$ .

**PROOF.** If  $G$  is finite the proof is trivial. By Theorem D, we can write  $K = L \oplus B'$  and  $B = B' \oplus B''$  where every basic subgroup of  $L$  is both an upper and lower basic subgroup of  $L$ . We can also assume that final rank  $(L) = |L|$ . Consider the group  $H \oplus L$ . This group satisfies the hypotheses of Lemma 8, and so  $A \oplus B''$  is an upper basic subgroup of  $H \oplus L$ . Now by Theorem 5 we have  $A \oplus B'' \oplus B' = A \oplus B$  is an upper basic subgroup of  $G$ .

**LEMMA 10.** Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus K$  where final rank  $(H) = |H|$ , final rank  $(K) = |K|$ , final rank  $(G) = |G|$ , and every basic subgroup of  $H$  and  $K$  is both an upper and lower basic subgroup. If  $A$  and  $B$  are basic subgroups of  $H$  and  $K$  respectively, then  $A \oplus B$  is an upper basic subgroup of  $G$ .

**PROOF.** By Lemma 8 we can assume that  $|H| = |K| = |G|$ . Suppose that  $A \oplus B$  is not an upper basic subgroup of  $G$ , and let  $S$  be an upper basic subgroup of  $G$ . Now by Theorem D we have  $G = L \oplus S'$  and  $S = S' \oplus S''$  where  $L$  has the property that every basic subgroup of  $L$  is both an upper and a lower

basic subgroup of  $L$ , and  $|L| = \text{maximum}(\aleph_0, \text{rank}(G/S))$ . As in the proof of Lemma 8 we can assume that  $\aleph_0 < |L| = \text{rank}(G/S) < |G|$ . We may also assume that final rank  $(L) = |L|$ . Consider the group  $H+L$ . Since  $H+L$  contains the groups  $H$  and  $L$ , both of which are summands of  $G$ , we know we can write  $H+L = H \oplus [(H+L) \cap K]$ , and  $H+L = L \oplus [(H+L) \cap S']$ . Let  $K' = (H+L) \cap K$ , and let  $T$  be an upper basic subgroup of  $K'$ . The following equation  $|L| \stackrel{i}{=} \text{rank}(L/S'') \stackrel{ii}{=} \text{rank}((L+H)/(S'' \oplus [(H+L) \cap S'])) \stackrel{iii}{=} \text{rank}((H+L)/(A \oplus T)) \stackrel{iv}{=} \text{rank}(H/A) + \text{rank}(K'/T)$  holds since:

- (i) Follows since final rank  $(L) = |L|$ .
- (ii) Follows from an isomorphism theorem.
- (iii) By Theorem 5 we have that  $S'' \oplus [(H+L) \cap S']$  is an upper basic subgroup of  $H+L$ . Since  $K' \cong (H+L)/H \cong L/(L \cap H)$ , we know that  $|K'| \leq |L| < |H|$ , and hence by Lemma 9 we have  $A \oplus T$  is an upper basic subgroup of  $H+L$ . Thus the equality follows since both basic subgroups are upper basic subgroups.
- (iv) Follows from an isomorphism theorem.

Now  $|L| = \text{rank}(H/A) + \text{rank}(K'/T) = |H| + \text{rank}(K'/T)$ , since final rank  $(H) = |H|$ , and every basic subgroup of  $H$  is both an upper and a lower basic subgroup of  $H$ . Thus we have that  $|H| \leq |L|$ , but this is a contradiction since  $|L| < |G| = |H|$ . Therefore  $A \oplus B$  must have been an upper basic subgroup of  $G$ .

**THEOREM 11.** *Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus K$ , and let  $A$  and  $B$  be upper basic subgroups of  $H$  and  $K$  respectively. Then  $A \oplus B$  is an upper basic subgroup of  $G$ .*

**PROOF.** If either  $H$  or  $K$  is finite then by Theorem 5,  $A \oplus B$  is upper basic in  $G$ . By Theorem D we have  $H = H' \oplus A'$  and  $A = A' \oplus A''$  where every basic subgroup of  $H'$  is both an upper and a lower basic subgroup of  $H'$ . Similarly we can write  $K = K' \oplus B'$  and  $B = B'' \oplus B'$  where every basic subgroup of  $K'$  is both an upper and a lower basic subgroup of  $K'$ . Now we can write  $H'' = H'' \oplus A'''$  and  $A''' = A''' \oplus N$  where final rank  $(H'') = |H''|$ . Similarly we can write  $K'' = K'' \oplus B'''$  and  $B''' = B''' \oplus M$  where final rank  $(K'') = |K''|$ . Notice that by Lemma 4 we also know that  $H''$  and  $K''$  have the property that every basic subgroup is both an upper and a lower basic subgroup. Thus  $G = (H'' \oplus K'') \oplus (A' \oplus A''' \oplus B' \oplus B''')$  and by applying Theorem 5 and Lemma 10 the proof is completed.

**COROLLARY 12.** *Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that  $G = H \oplus K$  where every basic subgroup of  $H$  or  $K$  is both an upper and a lower basic subgroup, and suppose that final rank  $(G) = |G|$ . Then every basic subgroup of  $G$  is an upper and lower basic subgroup of  $G$ .*

**PROOF.** Let  $A$  and  $B$  be upper basic subgroups of  $H$  and  $K$  respectively. By Theorem 11 we have that  $A \oplus B$  is an upper basic subgroup of  $G$ . We will now show that  $A \oplus B$  is a lower basic subgroup of  $G$ . Notice that final rank  $(G) = \text{final rank}(H) + \text{final rank}(K) = \text{rank}(H/A) + \text{rank}(K/B) = \text{rank}(G/(A \oplus B))$ .

Theorem 11 has answered Problem II affirmatively for  $p$ -groups without elements of infinite height. Theorem 13 answers Problem III and with its aid we will extend the results of Theorems 5 and 11 to arbitrary reduced  $p$ -groups.

**THEOREM 13.** *Let  $G$  be a reduced  $p$ -group, and let  $H$  be a high subgroup of  $G$ . If  $B$  is an upper basic subgroup of  $H$ , then  $B$  is an upper basic subgroup of  $G$ .*

**PROOF.** Suppose that  $B$  is not upper basic in  $G$ , and let  $A$  be an upper basic subgroup of  $G$  and  $K$  a high subgroup of  $G$  containing  $A$  and recall that  $r(G/A) < |G|$  or  $B$  would have been an upper basic subgroup. Now consider the following cases.

Case (i). Suppose that  $\text{rank}(K/A) \leq \aleph_0$ , then  $G$  is a  $\Sigma$ -group and by Theorem 5, page 1380, in [2] we know that  $\text{rank}(G/H) = \text{rank}(G/K)$ . Since  $B$  is an upper basic subgroup of  $H$  and  $G$  is a  $\Sigma$ -group, then  $B = H$ , and thus  $B$  is an upper basic subgroup of  $G$ .

Case (ii). Suppose that  $\text{rank}(K/A) > \aleph_0$ , then  $\text{rank}(G/A) > \aleph_0$ . By Theorem C we have  $G = L \oplus A'$  and  $A = A' \oplus A''$  where  $|L| = \text{rank}(G/A)$ . Now  $K$  contains  $A'$  and hence  $K = A' \oplus R$  where  $R = L \cap K$ . Let  $\eta : G \rightarrow G/G^1$  be the natural quotient map. Under the map  $\eta$  we have  $H \cong \eta(H)$ ,  $K \cong \eta(K)$ , and  $A \cong \eta(A)$  since all subgroups involved are disjoint from  $G^1$ . Also  $[\eta(H)]_p = [\eta(K)]_p$  by Theorem 6, page 1382, in [2]. Since  $K = A' \oplus R$  we have that  $\eta(K) = \eta(A') \oplus \eta(R)$ . Now by Theorem 12, page 25, in [4] we know that  $\eta(A')_p = \bigcup_{i=1}^{\infty} S_i$  where  $S_i$  is a subgroup of elements of

bounded height in  $\eta(G)$  and consequently in  $\eta(H)$  and  $\eta(K)$  since both are pure in  $\eta(G)$ . By Theorem 29.5, page 99, in [1] there exists a basic subgroup  $M$  of  $\eta(H)$  such that  $M \supset \eta(A')_p$ . Let  $N = \eta^{-1}(M) \cap H$ , since  $\eta$  is an isomorphism between  $H$  and  $\eta(H)$  we know  $N$  is a basic subgroup of  $H$ . If  $\text{rank}(H/N) \leq \aleph_0$  an argument as in Case (i) would complete the proof. Thus assume  $\text{rank}(H/N) > \aleph_0$  and consider  $\text{rank}(H/N) = \text{rank}(\eta(H)/M) = |(\eta(H)/M)_p| = |\eta(H)_p/M_p| = |\eta(K)_p/M_p| = |(\eta(A')_p \oplus \eta(R)_p)/M_p|$ , but  $M_p$  contains  $\eta(A')_p$ . Hence  $\text{rank}(H/N) \leq |R_p| \leq |L_p| = \text{rank}(G/A)$ . We will now show that  $\text{rank}(H/N) < \text{rank}(H/B)$ . First notice that  $G/N \cong H/N \oplus G/H$ , and hence  $|G/N| = |H/N| + |G/H| \leq \text{rank}(G/A) + |G/H| = \text{rank}(G/A) + |G/K|$ , and since  $|G/K| \leq \text{rank}(G/A)$ , we have that  $\text{rank}(G/N) \leq \text{rank}(G/A) + \text{rank}(G/A) = \text{rank}(G/A)$ . Therefore  $N$  is an upper basic subgroup of  $G$ . We assumed  $B$  is not an upper basic subgroup of  $G$ , and so the rank  $(G/B) > \text{rank}(G/N)$ . Notice that  $\text{rank}(G/B) = \text{rank}(G/H) + \text{rank}(H/B)$ , and  $\text{rank}(G/N) = \text{rank}(G/H) + \text{rank}(H/N)$ , so that  $\text{rank}(H/B) > \text{rank}(H/N)$  which contradicts  $B$  being an upper basic subgroup of  $H$ . Therefore  $B$  is an upper basic subgroup of  $G$ .

**COROLLARY 14.** *Let  $G$  be a  $p$ -group, and let  $H$  and  $K$  be high subgroups of  $G$ , and let  $A$  and  $B$  be upper basic subgroups of  $H$  and  $K$  respectively. Then  $\text{rank}(H/A) = \text{rank}(K/B)$ .*

**PROOF.** Follows easily from the proof of the last theorem.

Now we turn our attention to extending the results of Theorems 5 and 11. In this view we first prove:

**THEOREM 15.** Let  $G$  be a reduced  $p$ -group such that  $G = H \oplus K$ . Let  $A$  and  $B$  be upper basic subgroups of  $H$  and  $K$  respectively. Then  $A \oplus B$  is an upper basic subgroup of  $G$ .

**PROOF.** Let  $M$  and  $N$  be high subgroups of  $H$  and  $K$  respectively, which contain  $A$  and  $B$  respectively. Now suppose that  $S$  and  $T$  are upper basic subgroups of  $M$  and  $N$  respectively. By Theorem 11 we know that  $S \oplus T$  is an upper basic subgroup of  $M \oplus N$ , and hence by Theorem 13,  $S \oplus T$  is an upper basic subgroup of  $G$ . Now  $\text{rank}(G/(S \oplus T)) = \text{rank}(H/S) + \text{rank}(K/T)$ , and since  $S$  and  $T$  are upper basic subgroups of  $M$  and  $N$  respectively, we have by Theorem 13 that they are upper basics of  $H$  and  $K$  respectively. Thus we know that  $\text{rank}(H/S) = \text{rank}(H/A)$ , and  $\text{rank}(K/T) = \text{rank}(K/B)$ . Therefore  $\text{rank}(G/(S \oplus T)) = \text{rank}(H/S) + \text{rank}(K/T) = \text{rank}(H/A) + \text{rank}(K/B) = \text{rank}(G/(A \oplus B))$ , and hence  $A \oplus B$  is an upper basic subgroup of  $G$ .

**THEOREM 16.** Let  $G$  be a reduced  $p$ -group such that  $G = H \oplus K$  where  $K$  is a direct sum of cyclic groups. Let  $B$  be an upper basic subgroup of  $H$ . Then  $B \oplus K$  is an upper basic subgroup of  $G$ .

**PROOF.** The proof follows easily from Theorem 15.

I now turn my attention to the partial results obtained for Problem IV. We first prove the following lemma.

**LEMMA 17.** Let  $F = H \oplus K$  be a direct sum of cyclic  $p$ -groups, and suppose  $|K| < |F|$  and that  $\aleph_0 < |K|$  is not a limit cardinal. Let  $M$  be a pure subgroup of  $K$ , and let  $B \oplus M$  be a basic subgroup of  $F$  such that the  $\text{rank}(F/(B \oplus M)) > |K|$ . Then there exists a basic subgroup  $A \oplus M$  such that  $A \oplus M$  contains  $B \oplus M$ , and the  $\text{rank}(F/(A \oplus M)) \leq |K|$ .

**PROOF.** Consider  $F/B = D/B \oplus R/B$  where  $R/B$  is reduced and can be chosen to contain  $(B \oplus M)/B$ , and where  $D/B$  is divisible. Now consider the group  $D + M$ . First notice that  $D \cap M = 0$  since  $D \cap B = B$  and  $B \cap M = 0$ . Thus  $D + M = D \oplus M$ , and as a subgroup of a direct sum of cyclic groups is itself a direct sum of cyclics. To show that  $D \oplus M$  is a basic subgroup we need only prove that  $D \oplus M$  is pure, but  $(D \oplus M)/(B \oplus M) \cong D/B$  which is divisible and hence  $D \oplus M$  is pure. Therefore  $D \oplus M$  is a basic subgroup of  $F$  which contains  $B \oplus M$ , and notice that  $\text{rank}(F/(D \oplus M)) \leq |F/D| = |R/B|$ , but by Theorem 30.1, page 102, in [1] we know that  $|R/B| \leq |(B \oplus M)/B|^{\aleph_0} = |M|^{\aleph_0} \leq |K|^{\aleph_0} = |K|$  since  $|K|$  is not a limit cardinal. This completes the proof.

**THEOREM 18.** Let  $G$  be a  $p$ -group without elements of infinite height, and let  $B$  be a basic subgroup of  $G$ . Let  $A$  be an upper basic subgroup of  $G$ , and suppose that  $\text{rank}(G/A)$  is not a limit cardinal larger than  $\aleph_0$ . Then  $B$  is contained in an upper basic subgroup of  $G$ .

**PROOF.** If  $\text{rank}(G/A) \leq \aleph_0$ , then  $G$  is a direct sum of cyclic groups and hence  $B$  is contained in an upper basic subgroup of  $G$ , namely,  $G$  itself. Thus we may assume that  $\aleph_0 < |L| = \text{rank}(G/A)$ . By Theorem C we can write  $G = L \oplus A'$  where  $A = A' \oplus A''$  and  $|L| = \text{rank}(G/A)$ . Let  $L$  be the homomorphic image of the free  $p$ -group  $K$  with pure kernel  $M$  and where we can

assume  $|K| = |L|$ . Now  $(A' \oplus K)/M \cong L \oplus A'$ , and suppose  $(B' \oplus M)/M \cong B$ . If  $\text{rank}(G/B) = \text{rank}(G/A)$  we know that  $B$  is already an upper basic subgroup of  $G$  and we are done, so that we can assume that  $\text{rank}(G/B) > \text{rank}(G/A) = |L|$ . Thus  $\text{rank}((K \oplus A')/(B' \oplus M)) > |L| = |K|$ , and by Lemma 17 there exists a basic subgroup  $B'' \oplus M$  containing  $B' \oplus M$  and such that  $\text{rank}((K \oplus A')/(B'' \oplus M)) = |K|$ . Let  $S \cong (B'' \oplus M)/M$ . We know that  $S$  is a basic subgroup of  $G$  which contains  $B$  and  $S$  is upper basic subgroup of  $G$  since  $\text{rank}(G/S) = \text{rank}((K \oplus A')/(B'' \oplus M)) = |L| = \text{rank}(G/A)$ .

**THEOREM 19.** *Let  $G$  be a reduced  $p$ -group and let  $B$  be a basic subgroup of  $G$ . If there exists a high subgroup  $H$  of  $G$  which contains  $B$ , and an upper basic subgroup  $A$  of  $H$  containing  $B$ , then  $B$  is contained in an upper basic subgroup of  $G$ .*

**PROOF.** If  $A$  is an upper basic subgroup of  $H$ , then  $A$  is an upper basic subgroup of  $G$  by Theorem 13.

Although I have not been able to construct a proof, I still conjecture that every basic subgroup of a  $p$ -group is contained in an upper basic subgroup. In either case it would be interesting to characterize the class of  $p$ -groups for which this statement is true. To this end I will now list several classes of  $p$ -groups which do have this property.

Let  $C$  be the class of  $p$ -groups which have the property that every basic subgroup is contained in an upper basic subgroup.

**THEOREM 20.** *The Class  $C$  contains all  $p$ -groups which are divisible direct sum with a bounded group.*

**PROOF.** This follows immediately from the fact that such groups have only one basic subgroup, and it is by necessity an upper basic subgroup.

**THEOREM 21.** *The Class  $C$  contains all  $\Sigma$ - $p$ -groups.*

**PROOF.** Let  $G$  be a  $\Sigma$ - $p$ -group, and  $B$  a basic subgroup of  $G$ . Now  $B$  can be put in a high subgroup of  $H$  of  $G$ , and since  $G$  is a  $\Sigma$ -group we know that  $H$  is an upper basic subgroup of  $G$ .

**THEOREM 22.** *Let  $G$  be a  $p$ -group without elements of infinite height. Suppose that the final rank of  $G$  is equal to its cardinality, and that  $B$  is a basic subgroup of  $G$ . If  $\text{rank}(G/B)$  is equal to the cardinality of  $G$ , then  $G$  is in the Class  $C$ .*

**PROOF.** This follows from Theorem 18.

**COROLLARY 23.** *The Class  $C$  contains all closed  $p$ -groups.*

The following unsolved problems arise from the preceding work.

- (i) If  $G$  is a reduced Abelian  $p$ -group such that  $G = \sum_{x \in I} G_x$  and  $B_x$  is upper basic in  $G_x$  then is it true that  $\sum_{x \in I} B_x$  is upper basic in  $G$ .
- (ii) Does the Class  $C$  defined above indeed contain all reduced Abelian  $p$ -groups?

- (iii) If the answer to (ii) is negative then does  $C$  contain
- a) fully starred groups,
  - b)  $p$ -groups with property that every infinite subgroups can be put into a summand of the same cardinality.
  - c) The Class of  $Q$  groups, (see [3]).

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## A NOTE ON THE VANISHING OF POWER SUMS

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In his book "Eine neue Methode in der Analysis und deren Anwendungen", [9], P. TURÁN deals extensively with minimax problems for generalised power sums  $\sum_{j=1}^n b_j z_j^\nu$ , where the maximum, with respect to  $\nu$ , is taken over  $n$  consecutive non-negative integers. One of the theorems (VII, p. 38), now generally known as Turán's first main theorem runs as follows:

Given  $n$  complex numbers  $b_1, b_2, \dots, b_n$  and  $n$  complex numbers  $z_1, z_2, \dots, z_n$  with  $\min_{j=1, \dots, n} |z_j| = 1$ . Then one has for every integer  $m$ ,  $m \geq -1$ ,<sup>2</sup>

$$(1) \quad \max_{\nu=m+1, \dots, m+n} \left| \sum_{j=1}^n b_j z_j^\nu \right| \geq \left( \frac{n}{2e(m+n)} \right)^n \left| \sum_{j=1}^n b_j \right|.$$

Since the edition of this book in 1953, further applications in various parts of analysis and number theory appear regularly in the literature. Other authors discuss the possibilities of sharpening Turán's results or consider related problems with the  $\nu$  varying over intervals of length  $> n$ . See e.g. [1], [2], [3], [7].

Turán found that possible application of the theory on the problem of twin primes leads to a study of analogous questions when  $\nu$  runs through intervals of length  $\leq n-1$ . For shorter intervals nothing more can be said in general than the trivial

$$(2) \quad \max_{\nu=m+1, \dots, m+n-1} \left| \sum_{j=1}^n b_j z_j^\nu \right| \geq 0.$$

If one knew however the systems with

$$(3) \quad \max_{\nu=m+1, \dots, m+n-1} \left| \sum_{j=1}^n b_j z_j^\nu \right| = 0$$

<sup>1</sup> This paper was prepared when the author was staying at Budapest with a grant from the Netherlands Organisation for the Advancement of Pure Research (Z.W.O).

<sup>2</sup> The non-interesting case  $n = 1, m = -1$  is not covered by the formula.

one could hope to get, under a side condition expressing in one way or another a restriction to systems "not to close" to those satisfying (3), non-trivial estimations.

VERA T. SÓS and P. TURÁN, [8], started the study of systems satisfying

$$(4) \quad \max_{r=m+1, \dots, m+n-1} \left| \sum_{j=1}^n z_j^r \right| = 0$$

and gave a complete characterisation of the cases  $m = 0$ ,  $m = 1$  and  $m = 2$ . See also [10]. The only systems satisfying (4) with  $m = 0$  are given by the zeros of an equation

$$(5) \quad z^n + a = 0, \text{ } a \text{ complex.}$$

Two non-trivial solutions of (4) are called independent if they cannot be carried over into each other by a multiplication with a complex number, followed by a relabelling. Let  $B(m, n)$  denote the number of non-trivial independent solutions of (4). UCHIYAMA, [11], proved the remarkable formula

$$B(m, n) = B(n, m).$$

One has e.g.  $B(0, n) = 1$ ,  $B(1, n) = 1$ ,  $B(2, n) = \left[ \frac{n}{2} \right] + 1$ ,  $B(3, n) = \left[ \frac{n^2 + 3n}{6} \right] + 1$ .

Now TURÁN conjectured that any system of  $n$  complex numbers ( $n > 2$ ) satisfying (4) for infinitely many values of  $m$  is given by the  $n$  solutions of equation (5). ERDŐS conjectured that the same holds for any system satisfying (4) for two instead of infinitely many values of  $m$ .

This however turned out not to be true. Mr. D. CANTOR and Mr. R. TIJDEMAN proved independently that if for a system of  $n$  complex numbers, (4) holds for two different values of  $m$ , it holds for infinitely many values of  $m$ . Moreover one has that the numbers are  $A^{\text{th}}$  roots of unity. Both gave examples with  $A \geq 2n$ .<sup>3</sup>

In this connection the following result of ERDŐS, the proof of which depends upon (1), is of interest.

Given  $n$  complex numbers  $b_1, b_2, \dots, b_n$ ,  $\operatorname{Re} b_j > 0$ ,  $j = 1, 2, \dots, n$ , and  $n$  complex numbers  $z_1, z_2, \dots, z_n$  with  $|z_1| = \dots = |z_l| > |z_{l+1}| \geq \dots \geq |z_n|$ ,  $1 \leq l \leq n$ .

Let there exist a sequence  $\{m_v\}_{v=1}^{\infty}$  of indices and a positive integer  $k$  such that  
 $f(m_v + 1) = f(m_v + 2) = \dots = f(m_v + k) = 0$ ,  $v = 1, 2, \dots$ ,  
where

$$f(v) = \sum_{j=1}^n b_j z_j^v, \quad v = 1, 2, \dots$$

Then

$$l \geq k + 1.$$

The purpose of this note is to show that if the sequence  $\{f(v)\}$  of generalised power sums contains infinitely many gaps of  $\left[ \frac{n}{2} \right]$  consecutive zeros, the numbers  $z_1, z_2, \dots, z_n$  must be roots of unity. More precisely one has the following

<sup>3</sup> One either has  $A = n$  or  $A \geq 2n$ .

**THEOREM.** Let  $b_1, b_2, \dots, b_n$  be  $n$  non-zero complex numbers,  $z_1, z_2, \dots, z_n$   $n$  different non-zero complex numbers,  $z_1 = 1$ .

Put  $f(v) = \sum_{j=1}^n b_j z_j^v$ ,  $v = 1, 2, \dots$ ;  $k = \left\lfloor \frac{n}{2} \right\rfloor$  and suppose there exists a sequence  $\{m_v\}_{v=1}^\infty$  of indices such that

$$f(m_v + 1) = f(m_v + 2) = \dots = f(m_v + k) = 0, v = 1, 2, \dots$$

Then the numbers  $z_1, z_2, \dots, z_n$  are roots of unity.

**COROLLARY.** If the sequence  $\{f(v)\}_{v=1}^\infty$  contains infinitely many times  $\left\lfloor \frac{n}{2} \right\rfloor$  consecutive zeros and one gap of more than  $\left\lfloor \frac{n}{2} \right\rfloor$  consecutive zeros, then it contains infinitely many of these longer gaps.

The number  $\left\lfloor \frac{n}{2} \right\rfloor$  in this theorem is best in the sense that there do exist sets of  $n$  points, not all lying on the same circle, with an infinity of gaps of  $\left\lfloor \frac{n}{2} \right\rfloor - 1$  consecutive zeros in their power sum sequence. Indeed, for even  $n$ , the set

$$z_l = \exp(4\pi il/n), z_{nl+1} = 2 \exp(4\pi il/n), l = 0, 1, \dots, \frac{n}{2} - 1$$

provides us with an example. For odd  $n$  a similar example can be constructed. The proof of the theorem depends on a result of MAHLER. See [5] and [6] and also LECH, [4]. Shortly Mahler's theorem says that if infinitely many Taylor coefficients of a rational function vanish, then these zero coefficients occur periodically. The proof uses  $p$ -adic methods and as far as I know every attempt to give it without such methods failed until now.

Now since the function  $F(z) = \sum_{v=1}^\infty f(v) z^v$  is a rational function of  $z$  there exist two positive integers  $m$  and  $A$  such that

$$(6) \quad f(m+lA+1) = f(m+lA+2) = \dots = f(m+lA+k) = 0, l = 0, 1, 2, \dots$$

Putting  $b_j z_j^{m+1} = c_j, z_j^A = \zeta_j, j = 1, 2, \dots, n$ , the equalities  $f(m+lA+1) = 0$   $l = 0, 1, 2, \dots$  take the form

$$(7) \quad \begin{cases} c_1 + c_2 + \dots + c_n = 0 \\ c_1 \zeta_1 + c_2 \zeta_2 + \dots + c_n \zeta_n = 0 \\ c_1 \zeta_1^2 + c_2 \zeta_2^2 + \dots + c_n \zeta_n^2 = 0 \\ \dots \dots \dots \end{cases}$$

Suppose that there are among the numbers  $\zeta_1, \zeta_2, \dots, \zeta_n$  just  $t$  different ones. After an appropriate relabelling we then may write

$$\zeta_1 = \zeta_1 = \dots = \zeta_{n_1}$$

$$\zeta_2 = \zeta_{n_1+1} = \dots = \zeta_{n_1+n_2}$$

.....

$$\zeta_t = \zeta_{n_1+\dots+n_{t-1}+1} = \dots = \zeta_{n_1+\dots+n_t}$$

with  $n_1 \leq n_2 \leq \dots \leq n_t$ . Since the numbers  $\xi_1, \xi_2, \dots, \xi_t$  are pairwise unequal it follows from (7) and Vandermonde that

$$c_1 + c_2 + \dots + c_{n_1} = 0$$

i.e.

$$b_1 z_1^{m+1} + b_2 z_2^{m+1} + \dots + b_{n_1} z_{n_1}^{m+1} = 0.$$

We have not yet made full use of (6). Starting with  $f(m+lA+2) = 0$ ,  $l = 0, 1, 2, \dots$  we prove in the same way

$$b_1 z_1^{m+2} + b_2 z_2^{m+2} + \dots + b_{n_1} z_{n_1}^{m+2} = 0.$$

Indeed, we find that the sequence  $\{f^*(v)\}_{v=1}^{\infty}$ ,  $f^*(v) = \sum_{j=1}^{n_1} b_j z_j^v$ , of power sums of the  $n_1$  numbers  $z_1, z_2, \dots, z_{n_1}$  with coefficients  $b_1, b_2, \dots, b_{n_1}$  contains a gap of  $k$  consecutive zeros.

Let  $t > 1$ ; then  $n_1 \leq \left[ \frac{n}{2} \right] = k$ . This implies, again using Vandermonde, that  $b_1 = b_2 = \dots = b_{n_1} = 0$ , since the numbers  $z_1, z_2, \dots, z_{n_1}$  are pairwise unequal and non-zero. But this is contrary to our hypothesis. We conclude then that  $t = 1$  i.e.

$$1 = z_1^A = z_2^A = \dots = z_n^A.$$

This proves the theorem.

Finally I want to seize this opportunity to express my gratitude to Prof. TURÁN who introduced me into his theory and whose critical remarks changed the form of this paper for the good.

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# SOME INFINITE SYSTEMS OF LINEAR EQUATIONS IN STATISTICS

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## 1. §. Introduction

Miss SHEELA BILDIKAR doing research under the supervision of Professor G. P. PATIL on discreet distribution theory at McGill University, Montreal, Canada, has been investigating the problem of *identifying a distribution* which when mixed with a negative binomial distribution gives rise to another negative binomial distribution.<sup>1</sup> She has reduced the problem to that of finding a suitable solution of the infinite system of linear equations  $\mathbf{Ax} = \mathbf{b}$ , given by

$$(1.1) \sum_{j=0}^{\infty} \binom{i+j+k-1}{i} p^i (1-p)^{k+j} x_j = \binom{i+k-1}{i} \left(\frac{p}{q}\right)^i \left(\frac{q-p}{q}\right)^k, \text{ for } i=0, 1, 2, \dots,$$

where  $k$  is a positive number, and  $0 < p < q < 1$ .

It is required that the solution  $x_j$  should represent the frequency function  $f_j(x)$  of a random variable, i.e. that  $x_j \geq 0$ ,  $\sum x_j = 1$ , and it should be proved that such a solution is unique.

Miss BILDIKAR has also considered another system of linear equations, the matrix of which apart from the first row and the first column (of index 0) is of the same type as before, namely:

$$(1.2) \begin{cases} \sum_{j=1}^{\infty} (1-\Theta)^j x_j = \frac{\log \varphi}{\log (\varphi - \Theta)}, \text{ and} \\ \frac{\Theta^i x_0}{-i \log (1-\Theta)} + \sum_{j=1}^{\infty} \binom{i+j-1}{i} \Theta^i (1-\Theta)^j x_j = \frac{1}{-i \log (\varphi - \Theta)} \left(\frac{\Theta}{\varphi}\right)^i, \end{cases}$$

for  $i = 1, 2, \dots$ ,

where  $0 < \Theta < \varphi < 1$ .

<sup>1</sup> A short summary of our results will appear in the paper „Identifiability of countable mixtures of discrete probability distributions using methods of infinite matrices” by G. P. PATIL and SHEELA BILDIKAR in the proceedings of Cambridge Philosophical Society.

She has observed that in both problems the matrix  $\mathbf{A}$  and its transpose  $\mathbf{A}'$  satisfy Pólya's conditions [see 1 pp. 31–35 or 3]<sup>2</sup>, so that infinitely many solutions of (1.1) and (1.2) exist such that each series  $\sum a_{ij}x_j$  converges absolutely. Moreover  $\mathbf{A}$  has infinitely many right-hand and left-hand reciprocals obtainable by Pólya's method [see 1], but she could not obtain the required solution, nor could she find a unique two-sided reciprocal  $\mathbf{A}^{-1}$  which would lead to a solution  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$ .

Observing that for  $k = 1$  the matrix in (1.1) is the Laurent summability matrix [see 4 or 2:  $S_n$ ], and that it can be expressed as the product  $\mathbf{ET}$  of the lower triangular Euler matrix  $\mathbf{E}$  and the upper triangular Taylor matrix  $\mathbf{T}$  [2 and 4], we obtain in Section 3 for integral  $k$  a solution of the form  $\mathbf{x} = \mathbf{T}^{-1}(\mathbf{E}^{-1}\mathbf{b})$  which has the required property, and which then is proved to be valid and unique for any positive  $k$ . An interesting feature of the solution is that the matrix  $\mathbf{T}^{-1}\mathbf{E}^{-1}$  does not exist, so that our method does not lead to a two-sided reciprocal of  $\mathbf{A}$ , and that on the other hand  $\mathbf{A}$  has infinitely many two-sided reciprocals which can be constructed by Pólya's method.

In Section 4 we find the required solution for the system (1.2) by solving all equations but the first for the unknowns  $x_1, x_2, \dots$  in terms of  $x_0$ , and then determining  $x_0$ .

Section 2 is about infinite matrices stating known results, and establishing general results which are used in the later sections, and which may be of interest by themselves.

## 2. §. Products and reciprocals of infinite matrices

Throughout this section we shall consider infinite matrices of real elements  $\mathbf{A} = [a_{ij}]$ ,  $i, j = 0, 1, 2, \dots$ , and column-vectors  $\mathbf{x} = \{x_i\}$ . An infinite matrix is called *row-finite* if each row contains only a finite number of non-zero elements, and *column-finite* if its transpose is row-finite. A row-finite matrix is called *normal* if it is lower triangular with non-zero diagonal elements, i.e. if  $a_{ii} \neq 0$  and  $a_{ij} = 0$  for  $j > i$ . A matrix is called *upper-normal* if its transpose is normal.

The following two results are well known:

- (2.1) When  $\mathbf{A}$  and  $\mathbf{B}$  are row-finite or when  $\mathbf{B}$  and  $\mathbf{C}$  are column-finite the product  $\mathbf{ABC}$  exists and is associative.
  - (2.2) Normal (upper normal) matrices form a group under matrix multiplication.
- The next two results are almost obvious:
- (2.3) If  $\mathbf{A}$  is row-finite and  $\mathbf{Bx}$  exists, then  $(\mathbf{AB})\mathbf{x}$  exists, and  $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx})$ .

For if  $\sum_j b_{rj}x_j$  is convergent, then so is

$$\sum_{r=0}^{r_i} a_{ir} \sum_{j=0}^{\infty} b_{rj} x_j = \sum_{j=0}^{\infty} \left( \sum_{r=0}^{r_i} a_{ir} b_{rj} \right) x_j.$$

- (2.4) If  $\mathbf{N}$  is normal then  $\mathbf{Ax} = \mathbf{y}$  and  $(\mathbf{NA})\mathbf{x} = \mathbf{Ny}$  imply each other.

This follows from the previous results since  $\mathbf{Ax} = \mathbf{y} \Rightarrow (\mathbf{NA})\mathbf{x} = \mathbf{Ny}$ , and  $(\mathbf{NA})\mathbf{x} = \mathbf{Ny} \Rightarrow \mathbf{N}^{-1}((\mathbf{NA})\mathbf{x}) = \mathbf{N}^{-1}(\mathbf{Ny}) \Rightarrow \mathbf{Ax} = \mathbf{y}$ .

<sup>2</sup> Numbers in square brackets indicate references at the end of this paper.

The remaining two results are believed to be new.

- (2.5) *An infinite matrix  $\mathbf{A}$  can be expressed as the product  $\mathbf{NU}$  of a normal matrix  $\mathbf{N}$  and an upper normal matrix  $\mathbf{U}$  if and only if each leading  $i \times i$  submatrix  $\mathbf{A}_i$  of  $\mathbf{A}$  is non-singular.*

**PROOF.** The necessity is obvious from considering the elements in the first  $i$  rows and  $i$  columns of the product  $\mathbf{NU}$ .

To prove sufficiency we first observe that  $\det(\mathbf{A}_0) = a_{00} \neq 0$ , hence post-multiplying by an upper normal column combinator  $\mathbf{C}_0$  we can reduce all other elements in that row to zero. This operation leaves  $\det(\mathbf{A}_1)$  unchanged, hence the diagonal element in the next row of  $\mathbf{AC}_0$  is not zero, and we can operate similarly, reducing to zero all elements to the right of the diagonal. Proceeding in the same way we obtain a normal matrix  $\mathbf{N}$  as an infinite right product  $\mathbf{AC}_0\mathbf{C}_1\mathbf{C}_2\dots$ . We next show that this product exists and that it is the product of  $\mathbf{A}$  and of an upper normal matrix  $\mathbf{C} = \mathbf{C}_0\mathbf{C}_1\mathbf{C}_2\dots$ . The matrices  $\mathbf{C}_0, \mathbf{C}_1, \dots$  are all upper normal; post-multiplication by  $\mathbf{C}_{i+1}, \mathbf{C}_{i+2}, \dots$  leave the first  $i$  columns of both  $\mathbf{C}_0\mathbf{C}_1\dots\mathbf{C}_i$  and  $\mathbf{AC}_0\mathbf{C}_1\dots\mathbf{C}_i$  unchanged. Hence the infinite right products

$$\mathbf{C} = \lim_{i \rightarrow \infty} \mathbf{C}_0\mathbf{C}_1\dots\mathbf{C}_i, \quad \mathbf{N} = \lim_{i \rightarrow \infty} (\mathbf{AC}_0\mathbf{C}_1\dots\mathbf{C}_i) = \lim_{i \rightarrow \infty} \mathbf{A}(\mathbf{C}_0\mathbf{C}_1\dots\mathbf{C}_i)$$

all exist so that  $\mathbf{AC} = \mathbf{N}$ . But  $\mathbf{C}$  is upper normal, it has an upper normal two-sided reciprocal  $\mathbf{U}$ , hence  $\mathbf{A} = \mathbf{A}(\mathbf{CU}) = (\mathbf{AC})\mathbf{U} = \mathbf{NU}$ .

- (2.6) *If a matrix  $\mathbf{A}$  and its transpose  $\mathbf{A}'$  are both Pólya matrices, then  $\mathbf{A}$ , and hence  $\mathbf{A}'$ , has infinitely many two-sided reciprocals.*

**PROOF.** Since the Pólya property is retained when a finite number of rows and columns are removed, the systems

$$(i) \quad \sum_{j=m}^n a_{ij}x_j = s_i \quad \text{and} \quad (ii) \quad \sum_{i=n}^m y_ia_{ij} = t_j$$

have infinitely many solutions for any  $m$  and  $n$ , and for any vectors  $\mathbf{s}$  and  $\mathbf{t}$ . Hence we can define in succession the first row of the reciprocal, then the first column, then the next row, then the next column, and so on, by solving alternately a system of type (i) and type (ii), taking in succession  $m = 0, n = 0, m = 1, n = 1$ , and so on, and taking for  $\mathbf{s}$  successive columns and for  $\mathbf{t}$  successive rows of the unit matrix from which are subtracted the finite sums  $\sum_{j < m} a_{ij}x_j$  and  $\sum_{i < n} y_ia_{ij}$  containing those elements  $x_j$  and  $y_i$  which have already been determined.

### 3. §. Solution of the system (1.1)

Assuming that  $\mathbf{Ax}$  exists and that  $k$  is a positive integer we first multiply the equation  $\mathbf{Ax} = \mathbf{b}$  on the left by the *normal* matrix  $\mathbf{N}$ , given by

$$(3.1) \quad n_{ij} = \binom{i+k-1}{j+k-1} (-1)^{i-j} p^{-j} (1-p)^{-i-k}.$$

Hence, by (2.4)  $\mathbf{Ax} = \mathbf{b}$  and  $(\mathbf{NA})\mathbf{x} = \mathbf{Nb}$  are equivalent. Now

$$\sum_{r=0}^i n_{ir} a_{rj} = (1-p)^{j-i} \sum_{r=0}^i (-1)^{i-r} \binom{i+k-1}{r+k-1} \binom{r+j+k-1}{r},$$

and the last sum is the coefficient of  $z^i$  in the power series of  $(1-z)^{i+k-1}/(1-z)^{j+k} = 1/(1-z)^{j-i+1}$ . Thus  $\mathbf{NA} = \mathbf{U}$  is an upper normal matrix, where

$$(3.2) \quad u_{ij} = \binom{j}{i} (1-p)^{j-i},$$

and we have obtained the equivalent equation

$$(3.3) \quad \mathbf{Ux} = \mathbf{c}, \text{ where } \mathbf{c} = \mathbf{Nb}.$$

We note that  $\mathbf{U}$  is a Pólya matrix, since  $u_{0j} > 0$  and for each  $i$   $u_{ij}/u_{i+1,j} \rightarrow 0$  as  $j \rightarrow \infty$ , so that (3.3) has infinitely many solutions. Since  $\Sigma a_{ij}x_j$  should converge, every solution satisfies the condition  $x_j(1-p)^j \rightarrow 0$ , showing that  $x_j$  need not be bounded. But we can show that:

(3.4) *For any vector  $\mathbf{c}$ ,  $\mathbf{Ux} = \mathbf{c}$  cannot have more than one bounded solution.*

For if  $x_j$  is bounded, then the series  $f(z) = \Sigma x_j z^j$  converges on the disc  $|z| < 1$ , and  $f(z)$  can be expanded at  $z = 1-p$  into a Taylor series convergent on the disc  $|z - (1-p)| < p$  which is contained in the unit disc. In the new expansion the coefficient of  $(z - (1-p))^i$  is

$$\frac{f^{(i)}(1-p)}{i!} = \sum_{j=i}^i \binom{j}{i} (1-p)^{j-i} x_j.$$

Thus when  $x_j$  is bounded and satisfies (3.3), then that coefficient is  $c_i$ , and hence on the smaller disc

$$(3.5) \quad f(z) = \Sigma x_j z^j = \Sigma c_i (z - (1-p))^i,$$

so that  $x_j$  is uniquely defined.

COROLLARY. (3.4) remains true if "  $x_j$  is bounded" is replaced by " $\limsup |x_j|^{1/j} \leq 1$ ", and also when  $u_{ij} = \binom{j}{i} p^i (1-p)^{j-i}$ .

From (3.5) we obtain

$$(3.6) \quad x_i = \sum_{j=i}^i \binom{j}{i} (p-1)^{j-i} c_j,$$

showing that when a bounded solution exists it is given by  $\mathbf{x} = \mathbf{U}^{-1}\mathbf{c}$ , where  $\mathbf{U}^{-1}$  is the two-sided upper normal reciprocal of  $\mathbf{U}$ .

To simplify the final calculations we first solve the system (1.1) when  $k = 1$ . Then  $b_j = (1-p/q)(p/q)^j$ , hence

$$c_i = \sum_{j=0}^i n_{ij} b_j = \frac{q-p}{q(1-p)^{i+1}} \sum_{j=0}^i \binom{i}{j} \frac{(-1)^{i-j}}{q^j} = \frac{q-p}{q(1-p)} \left\{ \frac{1-q}{q(1-p)} \right\}^i,$$

and therefore by (3.6)

$$x_i = \frac{q-p}{q(1-p)} \left\{ \frac{1-q}{q(1-p)} \right\}^i \sum_{j=i}^{\infty} \binom{j}{i} \left( \frac{q-1}{q} \right)^{j-i} = \frac{q-p}{1-p} \left( \frac{1-q}{1-p} \right)^i.$$

These values of  $x_i$  satisfy the conditions  $x_i > 0$ ,  $\Sigma x_i = 1$ , and

$$\sum_{j=0}^{\infty} a_{ij} x_j = (q-p)p^i \sum_{j=0}^{\infty} \binom{i+j}{i} (1-q)^j = \frac{q-p}{q} \left( \frac{p}{q} \right)^i,$$

showing that we have obtained for  $k = 1$  the required solution, which is unique by (3.4).

Next we observe that for  $|z| < 1$  we can differentiate  $(k-1)$ -times term by term the series  $\sum \binom{i+j}{i} z^j = (1-z)^{-i-1}$ , and then dividing the result by  $(k-1)!$  we obtain

$$\begin{aligned} \sum_{j=k-1}^{\infty} \binom{i+j}{i} \binom{j}{k-1} z^{j-k+1} &= \sum_{j=0}^{\infty} \binom{i+j+k-1}{i} \binom{j+k-1}{k-1} z^j = \\ &= \binom{i+k-1}{i} \frac{1}{(1-z)^{i+k}}. \end{aligned}$$

Hence if we take  $x_j = \lambda \binom{i+k-1}{k-1} \left( \frac{1-q}{1-p} \right)^j$ , substitution into (1.1) gives a constant multiple of  $b_i$ , and we find that

$$(3.7) \quad x_j = \left( \frac{q-p}{1-p} \right)^k \binom{j+k-1}{k-1} \left( \frac{1-q}{1-p} \right)^j$$

is a solution of the system for  $k = 1, 2, \dots$  such that  $x_i > 0$ ,  $\Sigma x_i = 1$ , hence (3.7) is the required unique solution.

We remark that from  $\mathbf{N}\mathbf{A} = \mathbf{U}$  follows that  $\mathbf{U}^{-1}(\mathbf{N}\mathbf{A}) = \mathbf{I}$ , but  $\mathbf{U}^{-1}\mathbf{N}$  does not exist, since the series  $\sum u_{ir} n_{rj}$  do not converge. Hence our method is not based on the use of a two-sided reciprocal of  $\mathbf{A}$ . By (2.6)  $\mathbf{A}$  has infinitely many two-sided reciprocals, but none of those would lead to the above solution.

Finally we remark that the solution given in (3.7) satisfies (1.1) if  $k$  is any real number. When  $k > 0$ ,  $x_i$  is positive, otherwise the terms alternate in sign, and  $\Sigma x_i = 1$  for any real  $k$ . But when  $k$  is not a positive integer, the matrix in (3.1) is not a normal matrix, hence the equivalence of  $\mathbf{Ax} = \mathbf{b}$  with  $(\mathbf{NA})\mathbf{x} = \mathbf{Ux} = \mathbf{Nb}$  is not evident.

But we can still prove that the solution (3.7) is unique. We first observe that if  $\mathbf{Au} = \mathbf{b}$  and  $\mathbf{Av} = \mathbf{b}$  then  $\mathbf{A}(u-v) = \mathbf{0}$ , and therefore if the solution would not be unique, then  $\mathbf{A}\alpha = \mathbf{0}$  would have a non-zero solution, i.e. the system of equations

$$(3.8) \quad \sum_{j=0}^{\infty} \binom{i+j+k-1}{i} p^i (1-p)^{k+j} \alpha_j = 0$$

would be satisfied for  $i = 0, 1, 2, \dots$  with  $\alpha_j$  bounded.

If we now define for  $i = 1, 2, \dots$  and for  $0 \leq t < 1$  the functions  $f_i(t)$  of the real variable  $t$  by

$$\sum_{j=0}^{\infty} \binom{i+j+k-1}{i} \alpha_j t^{i+j+k} = f_i(t),$$

then each series can be differentiated term by term for  $0 < t < 1$ , and

$$(3.9) \quad f_i(t) = \frac{i+1}{t^2} f_{i+1}(t).$$

By (3.8)  $f_i(1-p) = 0$ , and hence by (3.9) all derivatives of  $f_i(t)$  exist and vanish at  $t = 1-p$ . Now, if we define  $g(t)$  by

$$g(t) = \sum_{j=0}^{\infty} (k+j) \alpha_j t^j,$$

then

$$g(t) = t^{-k} f_1(t),$$

and hence all derivatives of  $g(t)$  vanish at  $t = 1-p$ . But  $g(z)$  is regular for  $|z| < 1$ , hence the derivatives of the function of the complex variable  $z$  vanish at  $z = 1-p$ . Thus  $g(z)$  is zero for  $|z-(1-p)| < p$ , and therefore  $g(z)$  vanishes for  $|z| < 1$ , so that  $\alpha_j = 0$  for  $j = 0, 1, 2, \dots$ . This proves the uniqueness for  $k > 0$ .

#### 4. §. Solution of the system (1.2)

Here we re-write the system in the form:

$$(4.1) \quad \sum_{j \geq 1} \binom{i+j-1}{i} \Theta^i (1-\Theta)^j x_j = b_i - x_0 c_i, \quad \text{for } i = 1, 2, \dots,$$

$$(4.2) \quad \sum_{j \geq 1} (1-\Theta)^j x_j = \frac{\gamma}{\beta},$$

$$\text{where } 0 < \Theta < \varphi < 1, \quad b_i = \frac{1}{i\beta} \left\{ \frac{\Theta}{\varphi} \right\}^i, \quad c_i = \frac{\Theta^i}{i\alpha},$$

$$\alpha = -\log(1-\Theta), \quad \beta = -\log(\varphi-\Theta), \quad \gamma = -\log \varphi.$$

We multiply on the left by the normal matrix  $N$  the matrix of the equations (4.1), where

$$n_{ij} = (-1)^{i-j} \binom{i-1}{j-1} \Theta^{-j} (1-\Theta)^{-i}, \quad i, j = 1, 2, \dots,$$

and obtain for the element in the  $i$ -th row  $j$ -th column of the product

$$\begin{aligned} (1-\Theta)^{j-i} \sum_{r=1}^i (-1)^{i-r} \binom{i-1}{r-1} \binom{r+j-1}{r} &= \\ &= (1-\Theta)^{j-i} \sum_{r=0}^{i-1} (-1)^{i-1-r} \binom{i-1}{r} \binom{r+j}{j-1}. \end{aligned}$$

The last sum is the coefficient of  $z^i$  in the power series for

$$(1-z)^{i-1}/(1-z)^j = 1/(1-z)^{j-i+1}, \text{ hence}$$

$$\sum_{r=0}^i n_{ir} a_{rj} = \binom{j}{i} (1-\theta)^{j-i},$$

so that  $\mathbf{N}\mathbf{A} = \mathbf{U}$  is an upper normal matrix, essentially the same as  $\mathbf{U}$  in section 3, except that the first row and first column are removed. Hence using an argument similar to that in (3.4) with  $x_0 = 0$ , we can prove that  $\mathbf{Ux} = \mathbf{z}$  has at most one bounded solution.

The equations equivalent to (4.1) are in matrix form

$$(4.3) \quad \mathbf{Ux} = \mathbf{Nb} - x_0 \mathbf{Nc}.$$

Calculating  $\mathbf{Nb}$  we obtain

$$\begin{aligned} \sum_{j=1}^i n_{ij} b_j &= \frac{(1-\theta)^{-i}}{\beta} \sum_{j=1}^i (-1)^{i-j} \binom{i-1}{j-1} \frac{1}{j\varphi^j} \\ &= \frac{(1-\theta)^{-i}}{i\beta} \sum_{j=1}^i (-1)^{i-j} \binom{i}{j} \frac{1}{\varphi^j} = \frac{(1-\theta)^{-i}}{i\beta} \left\{ \left( \frac{1}{\varphi} - 1 \right)^i - (-1)^i \right\}, \end{aligned}$$

and since  $c_j = \varphi^j b_j \beta / \alpha$ , we have at once

$$\sum_{j=1}^i n_{ij} c_j = \frac{(1-\theta)^{-i}(-1)^i}{i\alpha}.$$

Hence  $\mathbf{Ux} = \mathbf{d}$ , where

$$(4.4) \quad d_i = \frac{(1-\theta)^{-i}}{i\beta} \left\{ \left( \frac{1-\varphi}{\varphi} \right)^i + \left( \frac{\beta x_0 - \alpha}{\alpha} \right) (-1)^i \right\}.$$

The reciprocal of  $\mathbf{U}$  is given again by the formula

$$u_{ij}^{-1} = \binom{j}{i} (\theta - 1)^{j-i},$$

and we see that  $u_{ij}^{-1} d_j \rightarrow 0$  as  $j \rightarrow \infty$  if and only if  $\varphi > \frac{1}{2}$  and

$$(4.5) \quad x_0 = \alpha/\beta.$$

Hence to obtain a solution of the form  $\mathbf{x} = \mathbf{U}^{-1}\mathbf{d}$  we assume that those conditions are satisfied and obtain

$$\begin{aligned} (4.6) \quad x_i &= \sum_{j \neq i} u_{ij}^{-1} d_j = \sum_{j \neq i} (1-\theta)^{-i} \binom{j}{i} \frac{(-1)^{j-i}}{j\beta} \left( \frac{1-\varphi}{\varphi} \right)^j \\ &= \frac{1}{i\beta} \left( \frac{1-\varphi}{\varphi(1-\theta)} \right)^i \sum_{j \neq i} \binom{j-1}{i-1} \left( \frac{1-\varphi}{\varphi} \right)^{j-i} = \frac{1}{i\beta} \left( \frac{1-\varphi}{1-\theta} \right)^i. \end{aligned}$$

Substituting the values  $x_i$  into the left-hand side of (4.1)

$$\begin{aligned}\sum_{j=1}^{\infty} a_{ij} x_j &= \frac{\Theta^i}{\beta} \sum_{j=1}^{\infty} \binom{i+j-1}{i} \frac{(1-\varphi)^j}{j} = \frac{\Theta^i}{i\beta} \sum_{j=1}^{\infty} \binom{i+j-1}{j} (1-\varphi)^j = \\ &= \frac{\Theta^i}{i\beta} \left( \frac{1}{\varphi^i} - 1 \right) = b_i - \frac{\alpha}{\beta} c_i,\end{aligned}$$

showing that the values (4.5) and (4.6) satisfy (4.1). Also

$$\sum_{j=1}^{\infty} (1-\Theta)^j x_j = \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{(1-\varphi)^j}{j} = \frac{-\log \varphi}{\beta} = \frac{\gamma}{\beta},$$

hence (4.2) is satisfied. We observe that all the equations are satisfied independently of the condition  $\varphi > \frac{1}{2}$  which had to be imposed to insure that  $\mathbf{U}^{-1}\mathbf{d}$  should exist.

Finally we see that  $x_i > 0$ , and that

$$\sum_{j=0}^{\infty} x_j = \frac{\alpha}{\beta} + \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{1-\varphi}{1-\Theta} \right)^j = \frac{\alpha}{\beta} - \frac{1}{\beta} \log \left( \frac{\varphi-\Theta}{1-\Theta} \right) = 1,$$

so that

$$x_0 = \frac{\log(1-\Theta)}{\log(\varphi-\Theta)}, \quad x_j = \frac{1}{-j \log(\varphi-\Theta)} \left( \frac{1-\varphi}{1-\Theta} \right)^j$$

is the required unique solution.

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# ENDOMORPHISM RADICALS WHICH CHARACTERIZE SOME DIVISIBLE GROUPS

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## 0. Introduction<sup>1</sup>

Let  $p$  be a prime, let  $P_p$  be the ring of  $p$ -adic integers, and represent  $P_p$  faithfully as the endomorphism ring  $E(C(p^\infty))$  of the quasicyclic group  $C(p^\infty)$ . It is well known [3] that  $J(E(C(p^\infty)))$ , the Jacobson radical of  $P_p$ , is just  $(p)$ , the principal ideal in  $P_p$  generated by the  $p$ -adic integer  $p$ . There is a standard result [3] on the radical of the  $n$ -by- $n$  matric ring over  $P_p$  which allows us to compute  $J(E(G))$  where  $G$  is the direct sum of  $n$  copies of  $C(p^\infty)$ . In fact, this particular endomorphism radical comes out to be  $(pI_n)$  where  $I_n$  is the unity matrix for the matric ring in question. Now let  $m$  be any infinite cardinal number; let  $M_{rf}(P_p, m)$  be the ring of  $m$ -by- $m$  row-finite matrices over  $P_p$ ; and let  $M_{rc}(P_p, m)$  be the ring of  $m$ -by- $m$  row-convergent matrices over  $P_p$ , this matric ring to be interpreted as the endomorphism ring of the direct sum of  $m$  copies of  $C(p^\infty)$ . See [2] for definitions. Let  $I$  denote the unity matrix for  $M_{rc}$  or for  $M_{rf}$ .

We shall show that  $J(M_{rc}(P_p, m)) = M_{rc}(J(P_p), m) = (pI)$  but that  $J(M_{rf}(P_p, m))$  is only some proper portion of the ideal  $(pI)$  in  $M_{rf}$ . We shall, in fact, exhibit a large subideal in  $J(M_{rf})$ ; but the apparent lack of an effective diagonalization process for  $M_{rf}$  seems to indicate that the structure problem for  $J(M_{rf})$  is still open. Not only does this difference in behavior arise from the fact that  $M_{rf}$  is not an ideal in  $M_{rc}$ , but the disparity is reinforced by the presence of row-finite matrices over  $P_p$  with inverses which are row-convergent but not row-finite. For instance, let  $A$  be the countably infinite matrix over  $P_p$  with entries  $a_{ij}$  where, for all pertinent values of  $i$ ,  $a_{i,i+1} = -p$  and  $a_{ii} = 1$ , while  $a_{ij} = 0$  if  $j \neq i, i+1$ . Although  $A$  is row-finite, it has an inverse  $B = (b_{ij})$  given by  $b_{ii} = 1$ ,  $b_{ij} = p^{j-i}$  if  $i < j$ , and  $b_{ij} = 0$  if  $i > j$ , a row-convergent, but not row-finite, matrix. See [5] for the genesis of this example.

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Since integers induce endomorphisms on abelian groups, one can speak of an integer as lying, or as not lying, in a subring of  $E(G)$ ; for example,  $0 \in J(E(G))$ , but  $1 \notin J(E(G))$ . It is well known [1] [4] [7] that the structure of an abelian group can, in some instances, be related to properties of its endomorphism-ring radical. The non-trivial divisible abelian groups with non-zero integers in their endomorphism radicals will be shown to be the torsion divisible groups with only a finite number of primary components.

### 1. The row-convergent case

As in PATTERTON [5], Theorem 1,  $J \leq M$  where  $J = J(M_{rc}(P_p, m))$  and  $M = M_{rc}(J(P_p), m)$ . Let  $G$  be the discrete direct sum of  $m$  copies of  $C(p^\infty)$ , and let  $A$  be a member of  $M$ . Suppose that all but possibly one of the components of some  $x \in G$  are zero. Although we make no countability assumptions about  $m$ , we can simplify notation by writing  $x = (x_1, 0, 0, \dots)$ . Suppose that, for such an  $x$ , we have  $x(I - A) = (0, 0, \dots)$ . Since  $A = (a_{ij})$ , a comparison of components gives  $x_1(1 - a_{11}) = 0$ . But  $a_{11} \in (p)$  implies that  $1 - a_{11}$  is invertible in  $P_p$ , whence  $x_1 = 0$ .

Assume inductively that, if any non-zero  $x' \in G$  has  $n$  or fewer non-zero components, then  $x'(I - A) \neq 0$ , no matter which  $A \in M$  is chosen. Now suppose that  $x \in G$  has  $n+1$  non-zero components and that  $x(I - A) = 0$  for some  $A \in M$ . Again with no assumptions on the countability of non-finite  $m$ , we can write  $x = (x_1, \dots, x_{n+1}, 0, 0, \dots)$  where each  $x_i \neq 0$ ,  $1 \leq i \leq n+1$ . From  $(0, 0, \dots) = x(I - A)$  we derive  $m$  identities, among which are the following  $n+1$ :

$$(A) \quad \sum_{j=1}^{n+1} x_j a_{ji} = x_i, \quad 1 \leq i \leq n+1.$$

From the last of these,  $x_{n+1} = \left| \sum_{j=1}^n x_j a_{j,n+1} \right| (1 - a_{n+1,n+1})^{-1}$ , using the fact that  $a_{n+1,n+1} \in J(P_p)$ . Substituting in (A) and writing  $a'_{ji} = a_{ji} + (1 - a_{n+1,n+1})^{-1} a_{j,n+1} a_{n+1,i}$ , we have

$$(B) \quad x_i = \sum_{j=1}^n x_j a'_{ji}, \quad 1 \leq i \leq n.$$

Let  $A'$  be the  $m$ -by- $m$  matrix with entries  $a'_{ji}$  whenever  $1 \leq i, j \leq n$  and with zero entries elsewhere. By construction,  $A' \in M$ , and the equations (B) show that  $x' = (x_1, \dots, x_n, 0, 0, \dots)$  is nullified by  $I - A'$ , contradicting the induction hypothesis. We have thus established that each  $I - A$  for  $A \in M$  represents a monendomorphism on  $G$ .

Let  $C(p^\infty)$  have generators  $\{c(m)\}$  where  $pc(1) = 0$  and  $pc(m+1) = c(m)$ ,  $m = 1, 2, \dots$ . For  $A \in M$ ,  $(c(1), 0, 0, \dots)(I - A) = (c(1), 0, 0, \dots)$ . Suppose that  $x \in G$  has each of its non-zero components  $x_i$  in the form  $k_i c(1)$  where  $k_i$  is an integer obeying  $0 < k_i < p$ . By what we have just done, there exists

$y \in G$  such that  $y(I - A) = x$ . Now  $(c(m+1), 0, 0, \dots)(I - A) = (t_1c(m+1), t_2c(m), t_3c(m), \dots, t_nc(m), 0, 0, \dots)$  where the  $t_i$  are appropriate integers and where, in particular,  $p$  does not divide  $t_1$ . We can then find an integer  $t$  such that  $tt_1 \equiv 1 \pmod{p^{m+1}}$ , whence  $t[(c(m+1), 0, 0, \dots) - z](I - A)$  will simplify to  $(c(m+1), 0, 0, \dots)$  if, by the induction hypothesis, we choose  $z \in G$  in such a way that  $z(I - A) = (0, t_2c(m), t_3c(m), \dots, t_nc(m), 0, 0, \dots)$ . Thus, each  $I - A$  represents an ependomorphism on  $G$ . We now have  $I - A$  regular and  $A$  quasiregular. But  $A$  is the most general right multiple of  $pI$  in  $M_{rc}(P_p, m)$  since, if one factors out  $p$  from each entry of  $A$ , the resulting matrix  $B$  has, for each positive integer  $k$ , almost all its entries in each row divisible by  $p^k$ , placing  $B$  in  $M_{rc}(P_p, m)$ . At once,  $pI$  is a radical element as is each  $A$  of the ideal  $(pI)$ . But these  $A$  are precisely the elements of  $M$ . We have proved the following:

**THEOREM 1.** *If  $m$  is an infinite cardinal and if  $I_m$  is the  $m$ -by- $m$  identity matrix, then  $J(M_{rc}(P_p, m)) = M_{rc}(J(P_p), m) = (pI_m)$ .*

Theorem 1 shows that the result of PATTERSON [6] [5] on row-finite matrices does not extend to row-convergent matrices. For,  $J(P_p) = (p)$  does not satisfy the right-vanishing condition of LEVITZKI; yet the radical of the row-convergent  $m$ -by- $m$  matric ring is the row-convergent matric ring over the radical elements.

**THEOREM 2.** *Let  $G$  be a non-trivial divisible abelian group. Then  $J(E(G))$  possesses an integer  $n \geq 2$  if and only if  $G$  is a direct sum  $\sum_{i=1}^m \sum_{m_i} C(p_i^\infty)$  where the  $m$  distinct primes  $p_i$  are chosen from among the  $m'$  distinct prime divisors of  $n$ , where the cardinals  $m_i$  are arbitrary non-zero, and where each inner summation is carried over  $m_i$  copies of  $C(p_i^\infty)$ .*

**PROOF.** Suppose that  $G$  has the structure of a divisible torsion group with only a finite number  $m$  of non-trivial primary components. Let the  $p_i$ -primary component  $G_i$  of  $G$  be the direct sum of  $m_i$  copies of  $C(p_i^\infty)$ . If  $m_i$  is finite, one can use the standard radical-matrix theorem [3] in conjunction with the fact that  $p_i \in J(P_{p_i})$  to show that each integer in the ideal on  $p_i$  lies in  $J(E(G_i))$ . If the given  $m_i$  happens to be infinite, then Theorem 1 allows us to reach the same conclusion. Since the  $p_i$  are distinct,  $E(G)$  is the ring direct sum of the  $E(G_i)$ , and  $J(E(G))$  is the ring direct sum of the  $J(E(G_i))$ . Thus,  $n = p_1p_2 \dots p_m \in J(E(G))$ , as we wished to show.

Conversely, suppose that  $G$  is divisible with  $n \in J(E(G))$ . If  $A$  is any direct summand of  $G$ , an easy argument shows that  $n \in J(E(A))$ . Should  $A = R$ , the additive group of rationals, or should  $A = C(p^\infty)$  for some prime  $p$  which does not divide  $n$ , then  $n$  is an automorphism on  $A$  with inverse automorphism, say  $v$ . Since  $n \in J(E(A))$  the identity automorphism  $nv$  on  $A$  would have to be quasiregular, an impossibility. Thus  $G$  has the desired structure.

We should note that the divisible groups with the minimum condition [2] are among the groups having the structure described in the theorem.

**COROLLARY 1.** *The torsion divisible groups with only a finite number of primary components are precisely the divisible groups which have non-zero integers in their endomorphism-ring radicals.*

**COROLLARY 2.** Let  $G$  be a non-trivial divisible group. If  $J(E(G))$  possesses an integer  $n \geq 2$ , then it possesses a least square-free integer  $n' \geq 2$ .

**PROOF.** By the theorem,  $G$  has a finite number  $m$  of distinct  $p_i$ -primary components. The square-free integer  $n'$  is just  $p_1 p_2 \dots p_m$ .

**COROLLARY 3.** Let  $G$  be a non-trivial abelian group on which 2 is an epimorphism. Then  $2 \in J(E(G))$  if and only if  $G$  is a direct sum of copies of  $C(2^n)$ .

**PROOF.** If  $2 \in J(E(G))$ , and if  $m$  is any integer, then  $1 - 2m$  is an automorphism on  $G$ , so that  $G$  is divisible. The theorem now applies.

## 2. The row-finite case

The PATERSON result [6] [5] on the radicals of row-finite matrix rings shows that  $M' = M_{rf}(J(P_p), m)$  must contain at least one member of the complement of  $J' = J(M_{rf}(P_p, m))$ . Since  $M' = (pI)$ ,  $pI$  itself cannot be in  $J'$ . We can show somewhat more in this direction. Let  $\Lambda$  be an index set of infinite cardinal  $m$ , and well order  $\Lambda$ . For each  $\alpha \in \Lambda$ , let  $v_\alpha$  be a non-zero member of  $J(P_p)$ . Let  $V = (v_{\alpha\beta})$  be the matrix with entries  $v_{\alpha\alpha} = v_\alpha$  for each  $\alpha \in \Lambda$  and with zeros elsewhere, a member of  $M'$ . Let the matrix  $C = (c_{\alpha\beta})$  have the entries  $c_{\alpha, \alpha+1} = 1$  whenever  $\alpha \in \Lambda$  obeys  $\alpha < \omega$ , and let  $C$  have zero entries in all other positions. Then  $VC = (w_{\alpha\beta})$  has the entries  $w_{\alpha, \alpha+1} = v_\alpha$  for  $\alpha < \omega$ , zeros elsewhere. If  $V$  were to be a radical element, then  $VC$  would have to be quasiregular, with some quasi-inverse  $D = (d_{\alpha\beta}) \in M_{rf}(P_p, m)$ . Denoting the initial members of well-ordered  $\Lambda$  by  $1, 2, \dots$ , we see that the first row of  $D + VC$  turns out to be  $(d_{11}, d_{12} + v_1, d_{13}, d_{14}, \dots)$ ; of  $DVC$ ,  $(0, d_{11}v_1, d_{12}v_2, d_{13}v_3, \dots)$ ; of  $VCD$ ,  $(v_1d_{21}, v_1d_{22}, v_1d_{23}, \dots)$ . But  $DVC = D + VC = VCD$ , and a comparison of components gives  $d_{11} = 0$ ,  $d_{12} = -v_1$ ,  $d_{13} = d_{12}v_2 = -v_1v_2$ , and, inductively,  $d_{1, n+1} = -v_1v_2 \dots v_n \neq 0$ . The quasi-inverse of  $VC$  cannot, therefore, lie in  $M_{rf}(P_p, m)$  even though Theorem 1 does place it in  $M_{rc}(P_p, m)$ . It is not difficult to see that, if  $X = (x_{\alpha\beta}) \in M'$  where  $x_{\alpha\beta} = 0$  if  $\alpha \neq \beta$ , and where  $x_{\alpha\alpha} \neq 0$  for more than a finite number of  $\alpha \in \Lambda$ , then  $X \notin J'$ , if one proceeds roughly as above.

On the other hand, for  $\gamma \in \Lambda$ , let  $K_\gamma = (k_{\alpha\beta}) \in M'$  have zero columns except, perhaps, at index  $\gamma$ . Let  $U_\gamma = (u_{\alpha\beta}) \in M'$  be the matrix which has zero columns except, perhaps, at index  $\gamma$ , where, for each  $\beta \in \Lambda$ ,  $u_{\beta\gamma} = -k_{\beta\gamma}(1 - k_{1\gamma})^{-1}$ . Then  $K_\gamma$  has quasi-inverse  $U_\gamma$ . Since the left multiples of matrices like  $K_\gamma$  are matrices like  $K_\gamma$ , all quasiregular, each  $K_\gamma \in J'$ ; and all matrices  $K \in M'$  having at most a finite number of non-zero columns lie in  $J'$ . The set  $N = N(J(P_p), m)$  of all such  $K$  is an ideal of  $M'$  where  $N \leq J' \leq M'$ . [6], [5].

Let  $S = S(J(P_p), m)$  be the discrete ring direct sum of  $m$  copies of  $J(P_p)$ ;  $S$  can be represented faithfully on the main diagonal as a subset of  $N$ : for map  $t = \{t_\alpha\} \in S$  onto  $T = (t_{\alpha\beta}) \in M'$  where  $t_{\alpha\beta} = 0$  if  $\alpha \neq \beta$ , and where  $t_{\alpha\alpha} = t_\alpha$  for each  $\alpha \in \Lambda$ . Since almost all  $t_\alpha$  are 0, it is clear that  $T \in N$ . Likewise, let  $S' = S'(J(P_p), m)$  be the complete ring direct sum of  $m$  copies of  $J(P_p)$ , and represent  $S'$  faithfully on the main diagonal as a subset of  $M'$ . Some easily established results can be summarized as follows:

THEOREM 3.  $S'/S$  can be injected both into  $M'/J'$  and into  $M'/N$  in such a way that a 40-vertex, 16-line, three-dimensional diagram, involving the rings 0,  $S$ ,  $S'$ ,  $S'/S$ ,  $J'$ ,  $N$ ,  $M'$ ,  $J'/N$ ,  $M'/N$ , and  $M'/J'$ , is commutative and exact.

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# HOMOMORPHISMEN ENDLICHER ORDNUNG

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## Einleitung

Die Aufgabe, die Gruppe  $\text{Hom}(A, W)$  aller Homomorphismen zweier beliebiger abelscher Gruppen  $A$  und  $W$  durch Invarianten eindeutig zu beschreiben, ist sicher unlösbar, denn das würde die Kenntnis der Struktur jeder beliebigen abelschen Gruppe  $X \cong \text{Hom}(Z, X)$  voraussetzen [ $Z$  ist die unendlich zyklische Gruppe]. Man wird sich also durchaus mit einer Beschreibung zufrieden geben, in der noch die Gruppen  $A$  und  $W$  und ihre Untergruppen und Faktorgruppen, soweit sie wohl definiert sind, vorkommen.

Außerdem ist zu vermuten (und zu hoffen), daß zu einer Charakterisierung von  $\text{Hom}(A, W)$  nicht alle Invarianten von  $A$  und  $W$  gebraucht werden. Dennoch ist die Situation so gut wie aussichtslos.

Erstaunlicherweise ist dagegen für beliebige abelsche Gruppen  $A$  und  $W$  eine Beschreibung der Torsionsuntergruppe  $\text{THom}(A, W)$  aller Homomorphismen endlicher Ordnung möglich. Es zeigt sich nämlich, daß nur sehr wenige Invarianten der Gruppe  $A$  gebraucht werden, nämlich für jede Primzahl  $p$  die Invarianten der Faktorgruppen

$$(*) \quad A/(TA + pA) \text{ und } A/p^i A \text{ für } i = 1, 2, \dots,$$

welche ja als beschränkte  $p$ -Gruppen leicht zu beschreiben sind. Außerdem gehen in die Beschreibung noch die  $p$ -Komponenten  $W_p$  von  $W$  und ihre Untergruppen  $W[p^i]$  aller Elemente von  $p^i$  teilender Ordnung für  $i = 1, 2, \dots$  ein: Es ist  $\text{THom}(A, W)$  der Torsionsuntergruppe einer cartesischen Summe von  $W_p$  und  $W[p^i]$  isomorph, wobei über  $p$  und  $i$  und gewisse Mächtigkeiten  $r(p, i)$ , die man aus  $(*)$  erhält, summiert wird.

Das bedeutet, daß bei der  $\text{THom}$ -Bildung viel von der Struktur von  $A$  verloren geht. Für die  $p$ -Komponenten  $\text{Hom}(A, W)_p$  von  $\text{THom}(A, W)$  können wir das noch deutlicher aussprechen: es gibt Untergruppen  $B$  von  $A$ , sogenannte  $p$ -Basisuntergruppen, die eine direkte Summe zyklischer Gruppen sind, so daß  $\text{Hom}(A, W)_p$  und  $\text{Hom}(B, W)_p$  in natürlicher Weise isomorph sind. Für Gruppen  $A_1$  und  $A_2$  mit isomorphen  $p$ -Basisuntergruppen  $B_1 \cong B_2$  gilt also die Iso-

morphe  $\text{Hom}(A_1, W)_p \cong \text{Hom}(A_2, W)_p$ . Bezuglich der Frage, welche Gruppen  $A$  isomorphe  $p$ -Basisuntergruppen  $B$  haben, wird auf BOYER [1] verwiesen.

Da  $\text{Hom}(A, W)_p$  durch eine direkte Summe zyklischer Untergruppen von  $A$  und durch  $W_p$  charakterisiert wird, so erhebt sich die naheliegende Frage, ob  $\text{Hom}(A, W)_p$  selbst durch eine direkte Summe zyklischer Untergruppen eindeutig bestimmt und damit eine abgeschlossene  $p$ -Gruppe im Sinne von FUCHS [2; pp. 114–117] ist. Die Antwort kann man an Satz 2.3 (c) ablesen: dann und nur dann ist  $\text{Hom}(A, W)_p$  abgeschlossen, wenn  $W_p$  abgeschlossen oder  $A/TA = p(A/TA)$  ist. Insbesondere ist  $\text{Hom}(A_p, W)_p$  stets eine abgeschlossene  $p$ -Gruppe, ein sicher schon bekanntes Resultat.

Schließlich zerfällt  $\text{THom}(A, W)$  über der Untergruppe  $\text{THom}(A/TA, W)$  mit einem zu  $\text{THom}(TA, W)$  isomorphen direkten Komplement.

### 1. Induzierte und Erweiterbare Homomorphismen

Ist  $B$  eine Untergruppe der Gruppe  $A$ , so induziert jeder Homomorphismus  $\sigma$  von  $A$  in  $W$  einen Homomorphismus  $\sigma'$  von  $B$  in  $W$ . Die Abbildung von  $\sigma$  auf  $\sigma'$ , d.h. die Restriktion von  $A$  auf  $B$ , ist ein natürlicher Homomorphismus von  $\text{Hom}(A, W)$  in  $\text{Hom}(B, W)$ . Wir bezeichnen das Bild von  $\text{Hom}(A, W)$  unter dieser Abbildung mit  $\text{Hom}(A/B, W)$ , d.h. die Gesamtheit der Homomorphismen von  $B$  in  $W$ , die zu Homomorphismen von  $A$  in  $W$  erweitert werden können. Der Kern dieser natürlichen Abbildung ist die Gesamtheit der Homomorphismen von  $A$  in  $W$ , die  $B$  auf 0 abbilden; wir bezeichnen diese Untergruppe von  $\text{Hom}(A, W)$  mit  $\text{Hom}(A/B, W)$ :

$$(1.1) \quad \text{Hom}(A/W)/\text{Hom}(A/B, W) \cong \text{Hom}(A|B, W) \subseteq \text{Hom}(B, W).$$

Analog gilt diese Beziehung (1.1) auch für die Torsionsuntergruppe  $\text{THom}(A, W)$ , d.h. die Gesamtheit der Homomorphismen endlicher Ordnung; bzw. für die  $p$ -Komponente  $\text{Hom}(A, W)_p$ , d.h. die Gesamtheit der Homomorphismen von Primzahlpotenzordnung.

Ist  $p$  eine Primzahl, so heißt  $G$  eine  $p$ -teilbare Gruppe, wenn  $G = pG$  ist. Die Untergruppe  $U$  von  $G$  heißt  $p$ -rein in  $G$ , wenn  $p^iU = p^iG \cap U$  für  $i = 1, 2, \dots$  gilt. Eine Gruppe heißt teilbar, wenn sie  $p$ -teilbar für jede Primzahl  $p$  ist; und eine Untergruppe heißt rein, wenn sie  $p$ -rein für jede Primzahl  $p$  ist.

**LEMMA 1.2.** Aus  $B \subseteq U \subseteq A$  und der  $p$ -Teilbarkeit von  $U/B$  folgt

$$\text{Hom}(A/B, W)_p = \text{Hom}(A/U, W)_p.$$

**LEMMA 1.2\*.** Aus  $B \subseteq U \subseteq A$  und der Teilbarkeit von  $U/B$  folgt

$$\text{THom}(A/B, W) = \text{THom}(A/U, W).$$

**BEWEIS.** Bei beiden Lemmata ist die Inklusion  $\supseteq$  trivial. Ist umgekehrt  $\sigma$  ein Homomorphismus, der  $B$  auf 0 abbildet, und  $q$  seine Ordnung, so ergibt sich

$$U\sigma = (U/B)\sigma = q(U/B)\sigma = (U/B)q\sigma = 0,$$

denn für das erste Lemma ist  $q$  eine  $p$ -Potenz und  $U/B$  ist  $p$ -teilbar und für das zweite Lemma ist  $q$  eine ganze Zahl und  $U/B$  ist teilbar. Also gilt jeweils auch die Inklusion  $\subseteq$ .

**LEMMA 1.3.** Ist  $B$  eine  $p$ -reine Untergruppe von  $A$ , so folgt

$$\text{Hom}(A|B, W)_p = \text{Hom}(B, W)_p.$$

**LEMMA 1.3\*.** Ist  $B$  eine reine Untergruppe von  $A$ , so folgt

$$\text{THom}(A|B, W) = \text{THom}(B, W).$$

**BEWEIS.** Für beide Lemmata folgt die Inklusion  $\subseteq$  aus (1.1). Ist umgekehrt  $\sigma$  ein Homomorphismus von  $B$  in  $W$ , ist  $K$  der Kern von  $\sigma$  und  $q$  die Ordnung von  $\sigma$ , so sind wegen

$$q(B/K) \cong q(B\sigma) = Bq\sigma = 0$$

die Elemente in der Faktorgruppe  $B/K$  von  $q$  teilender Ordnung. Für das erste Lemma ist  $q$  eine  $p$ -Potenz und folglich  $B/K$  sogar eine  $p$ -Gruppe. Ist  $n$  eine ganze Zahl, so folgt aus  $nB = nA \cap B$ :

$$\begin{aligned} n(B/K) &= [nB + K]/K = [(nA \cap B) + K]/K = [(nA + K) \cap B]/K = \\ &= (nA + K)/K \cap B/K = n(A/K) \cap B/K. \end{aligned}$$

Im ersten Lemma ist  $B$  eine  $p$ -reine Untergruppe, woraus sich also zunächst die  $p$ -Reinheit von  $B/K$  in  $A/K$  ergibt; dann folgt aber sogar die Reinheit, da  $B/K$  eine  $p$ -Gruppe ist. Im zweiten Lemma ist  $B$  rein in  $A$  und also ergibt sich auch hier die Reinheit von  $B/K$  in  $A/K$ .

Als reine und beschränkte Untergruppe ist  $B/K$  ein direkter Summand von  $A/K$ ; FUCHS [2; Th. 24.5]. Ist etwa  $A/K = B/K \oplus C/K$  mit einer  $B \cap C = K$  erfüllenden Untergruppe  $C$  von  $A$ , so existiert ein und nur ein Homomorphismus  $\tau$  von  $A$  in  $W$  mit

$$b\tau = b\sigma \text{ für } b \in B \text{ und } C\tau = 0.$$

Es ist klar, daß  $\tau$  und  $\sigma$  die gleiche Ordnung haben und daß  $\tau$  gerade  $\sigma$  in  $B$  induziert.

**KOROLLAR 1.4.** Ist  $B$  eine  $p$ -reine Untergruppe von  $A$  mit  $p$ -teilbarer Faktorgruppe  $A/B$ , so besteht die natürliche Isomorphie

$$\text{Hom}(A, W)_p \cong \text{Hom}(A|B, W)_p = \text{Hom}(B, W)_p.$$

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$$\text{THom}(A, W) \cong \text{THom}(A|B, W) = \text{THom}(B, W).$$

## 2. $p$ -Basisuntergruppen

In der Anwendung von Korollar 1.4 und 1.4\* wird man danach trachten, daß  $B$  zusätzlich eine einfache Struktur hat, etwa eine direkte Summe zyklischer Gruppen ist. Das wird für Korollar 1.4\* nur in besonders günstigen Fällen möglich sein. Für Korollar 1.4 ist die Situation wesentlich vorteilhafter: jede Gruppe  $A$  enthält eine Untergruppe  $B$  mit

- (i)  $B$  ist eine direkte Summe zyklischer Gruppen, deren Ordnungen unendlich und/oder  $p$ -Potenzen sind;
- (ii)  $B$  ist  $p$ -rein in  $A$ ;
- (iii)  $A/B$  ist  $p$ -teilbar.

Diese von FUCHS [3] eingeführten Untergruppen heißen  $p$ -Basisuntergruppen. Der Bequemlichkeit halber wiederholen wir hier ihre Konstruktion. FUCHS [3] nennt eine Menge von Elementen  $a_i \neq 0$  aus  $A$  eine  $p$ -unabhängige Menge, wenn für jede endliche Linearkombination  $\sum \alpha_i a_i \in p^n A$  entweder  $\alpha_i a_i = 0$  oder  $p^n \mid \alpha_i$  folgt. Dies ist gleichwertig damit, daß die von den Elementen  $a_i$  erzeugte Untergruppe  $B$  die Bedingungen (i) und (ii) erfüllt. Nach dem Zorn'schen Lemma existieren maximale  $p$ -unabhängige Mengen, und eine  $p$ -unabhängige Menge ist genau dann maximal, wenn die von ihr erzeugte Untergruppe  $B$  zusätzlich noch (iii) erfüllt.

Also existieren immer  $p$ -Basisuntergruppen  $B$  in  $A$  und diese sind (zur selben Primzahl  $p$ ) untereinander isomorph. Wir setzen

(2.1)  $B = B_0 \oplus B_p$  dabei ist

$$B_0 \cong \sum_{r_0}^0 Z \quad \text{eine freie Gruppe vom Range } r_0,$$

$$B_p = \sum_{i=1}^{\infty} B_i \quad \text{die } p\text{-Komponente von } B \text{ und}$$

$$B_i \cong \sum_{r_i}^0 Z(p^i) \quad \text{für } i = 1, 2, \dots \text{ eine direkte Summe zyklischer Gruppen der Ordnung } p^i; \text{ der Rang von } B_i \text{ sei } r_i.$$

Der freie Anteil  $B_0$  von  $B$  repräsentiert in  $A/B_p$  bzw.  $A/A_p$  bzw.  $A/TA$  jeweils eine  $p$ -Basisuntergruppe; analog repräsentiert die  $p$ -Komponente  $B_p$  von  $B$  jeweils eine  $p$ -Basisuntergruppe von  $A_p$  bzw.  $TA$  bzw.  $A/B_0$ ; es ist  $B_p$  sogar eine Basisuntergruppe von  $A_p$  im üblichen Sinne.

Die Ränge  $r_i$  von  $B_i$  für  $i = 0, 1, 2, \dots$  sind Invarianten der Gruppe  $A$  und werden folgendermaßen berechnet:

(2.2)  $r_0$  ist gleich dem Rang der elementar-abelschen  $p$ -Gruppen

$$B_0/pB_0 \cong A/(TA + pA) \cong [A/TA]/p[A/TA];$$

$r_i$  für  $1 \leq i \leq n-1$  und  $n$  beliebig ist gleich der Anzahl der zyklischen direkten Summanden der Ordnung  $p^i$  in einer der Gruppen

$$B/p^n B \cong A/p^n A \text{ bzw. } A_p/p^n A_p \text{ bzw. } TA/p^n TA.$$

Nach diesem Zitat aus FUCHS [3] sind wir in der Lage die Torsionsuntergruppe  $T\text{Hom}(A, W)$  zu beschreiben.

SATZ 2.3. Für jede Primzahl  $p$  und jede  $p$ -Basisuntergruppe  $B$  von  $A$  gilt:

- (a)  $\text{Hom}(A, W)_p \cong \text{Hom}(A|B, W)_p = \text{Hom}(B, W)_p$   
mit einem natürlichen Isomorphismus.
- (b)  $\text{Hom}(A, W)_p = \text{Hom}(A/B_p, W)_p \oplus \text{Hom}(A/B_0, W)_p$   
 $\text{Hom}(A/B_p, W)_p = \text{Hom}(A/A_p, W)_p = \text{Hom}(A/TA, W)_p \cong \text{Hom}(B_0, W)_p$   
 $\text{Hom}(A/B_0, W)_p \cong \text{Hom}(B_p, W)_p \cong \text{Hom}(A_p, W)_p \cong \text{Hom}(TA, W)_p.$

$$(c) \quad \text{Hom}(A, W)_p \cong \left( \sum_{r_0}^* W_p \oplus \sum_{i=1}^* \sum_{r_i}^* W[p^i] \right)_p,$$

wobei der erste direkte Summand zu  $\text{Hom}(A/B_p, W)_p$  und der zweite zu  $\text{Hom}(A/B_0, W)_p$  isomorph ist.

$$(d) \quad \text{THom}(A, W) \cong \text{THom}(A/TA, W) \oplus \text{THom}(TA, W)$$

in dem Sinne, daß  $\text{THom}(A, W)$  über der Untergruppe  $\text{THom}(A/TA, W)$  zerfällt mit einem zu

$\text{THom}(A, W)/\text{THom}(A/TA, W) \cong \text{THom}(A|TA, W) = \text{THom}(TA, W)$  isomorphen direkten Komplement.

Beweis. Aus der Definition der  $p$ -Basisuntergruppe und Korollar 1.4 ergibt sich (a).

Aus (a) und der Zerlegung  $B = B_0 \oplus B_p$  folgt die erste Zeile von (b). Aus der  $p$ -Teilbarkeit von  $A_p/B_p$  und  $TA/A_p$  und Lemma 1.2, bzw. aus der Tatsache, daß  $B_0$  eine  $p$ -Basisuntergruppe von  $A/TA$  repräsentiert, und Korollar 1.4 ergibt sich die zweite Zeile von (b). Da  $B_p$  eine  $p$ -Basisuntergruppe von  $A_p$  bzw.  $TA$  bzw.  $A/B_0$  ist, so liefert Korollar 1.4 die dritte Zeile von (b).

Um (c) zu zeigen, benutzt man (a), die Zerlegung  $B = \sum_{i=0}^* B_i$  und Fuchs [2; Th. 54.3].

Schließlich hat man noch

$$\text{THom}(A, W) = \sum_p^0 \text{Hom}(A, W)_p.$$

Aus (b) ergibt sich

$$\text{Hom}(A, W)_p \cong \text{Hom}(A/TA, W)_p \oplus \text{Hom}(A_p, W)_p,$$

so daß

$$\text{THom}(A, W) \cong \sum_p^0 \text{Hom}(A/TA, W)_p \oplus \sum_p^0 \text{Hom}(A_p, W)_p$$

und schließlich

$$\text{THom}(A, W) \cong \text{THom}(A/TA, W) \oplus \text{THom}(TA, W).$$

Aus der Reinheit von  $TA$  in  $A$  und (1.1) sowie Lemma 1.3\* folgt dann (d).

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# AXIALLY SYMMETRIC PACKING OF EQUAL CIRCLES ON A SPHERE

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## 1. Introduction

Many interesting problems can be found in the study of the most efficient packing of circles and spheres. Excellent summaries of the known results are given by FEJES TÓTH [1] and COXETER [2]. It is the purpose of the present paper to summarize the known results for one class of these problems and to augment these results with a few corrections and additions. This general problem is the determination of the largest angular diameter  $a(n)$  of  $n$  equal circular rings (or spherical caps) which can be packed on the surface of a sphere without overlapping.

## 2. Known solutions

The most important contributions to the known solutions are the papers by SCHÜTTE and VAN DER WAERDEN [3, 4]. They derived and proved the best solutions for  $n = 2, 3, 4, 5, 6, 7, 8$  and 9. Conjectures for other values, namely, for  $n = 10, 11, 12, 13, 14, 15, 16, 20, 24, 32, 42$  and 122 were made. FEJES TÓTH [5] proved the conjecture for  $n = 12$ . DANZER's proofs for  $n = 10$  and  $n = 11$  have not yet been published. ROBINSON [6] proved the conjecture for  $n = 24$ . JUCOVIČ [7] added conjectured solutions for  $n = 17, 25$  and 33. The author [10, 11] submitted improved arrangements for  $n = 33$ , and a conjecture for  $n = 18$ . STROHMAIER [12] added conjectured solutions for  $n = 18, 21, 22, 26, 30, 31$  and 52.

## 3. Improvements for $n = 32$

It was conjectured that the best arrangement for 32 circles resulted from placing the centers of the circles on the vertices and face centers of a regular icosahedron (or a regular dodecahedron). This gives

$$a(32) = \text{arc tan} (3 - \sqrt{5}) = \text{arc tan } 0.763932023 = 37^\circ 22' 39'',$$

which is the angular distance between a vertex and a face center. LEECH [8, pp. 89–90] and FEJES TÓTH [9, p. 236] have already given reasons to believe that this value can be improved by a modified arrangement, but no new values for  $a(32)$  had been computed.

Instead of using icosahedral symmetry for the arrangement of the circles, let us use the axial symmetry indicated by  $32\{1, 5, 5, 5, 5, 5, 5, 5, 1\}$ . This indicates that circles are centered at the opposite poles of a sphere while six rings of five equally spaced circles are placed at six intermediate latitudes. This arrangement is shown in the plan and elevation views of Figure 1. Note that the circles of each ring lie equally between the longitudes of the circles of the adjacent rings, except that at the equator, the adjacent rings are skewed to reduce the latitude between them. Computation by approximation and successive interpolation yields  $a(32) = 37^\circ 25' 51''$ , which is more than three minutes of arc larger than for the icosahedral arrangement.

This value can be verified as follows:

Let  $PA = 2z$ . Then,

$$\tan z = \cos 36^\circ \tan 37^\circ 25' 51'' = 0,809017 \cdot 0,76541080 = 0,61923035$$

$$z = 31^\circ 46' 01\frac{1}{2}'' , 2z = 63^\circ 32' 03''$$

$$\sin w = \sin 36^\circ \sin 2z = 0,58778525 \cdot 0,89520028 = 0,52618552$$

$$w = 31^\circ 44' 53''$$

$$\cos u = \cos a / \cos w = 0,79408765 / 0,85037008 = 0,93381419$$

$$u = 20^\circ 57' 45''$$

$$\tan v = \cos 36^\circ \tan 2z = 0,809017 \cdot 2,00868849 = 1,62506314$$

$$v = 58^\circ 23' 36\frac{1}{2}''$$

$$\begin{aligned} \sin b &= \frac{1}{2}(1/\cos a/2) = 0,5279139, \sin e = \sin w / \sin a = \\ &= 0,52618552 / 0,60780326 = \\ &= 0,86571684 \end{aligned}$$

$$b = 31^\circ 51' 52\frac{1}{2}'' , 2b = 63^\circ 43' 45''$$

$$e = 59^\circ 57' 53''$$

$$2b + e = 123^\circ 41' 38''$$

$$f = 180^\circ - (2b + e) = 56^\circ 18' 22'', PB = u + v$$

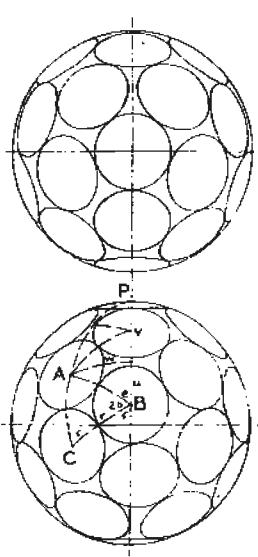
$$\begin{aligned} \tan r &= \cot(u + v) / \cos f = \cot 79^\circ 21' 21\frac{1}{2}'' / \cos 56^\circ 18' 22'' = \\ &= 0,18794037 / 0,55475569 = 0,33878043 \end{aligned}$$

$$r = 18^\circ 42' 55\frac{1}{2}''$$

$$BC = 2r = 37^\circ 25' 51'' = a.$$

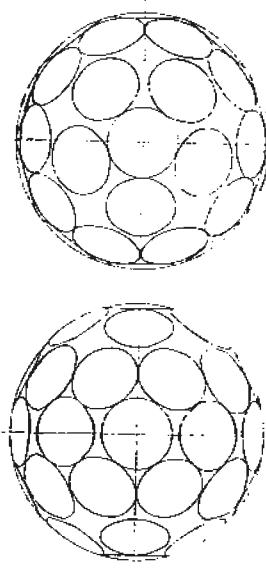
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Fig. 1



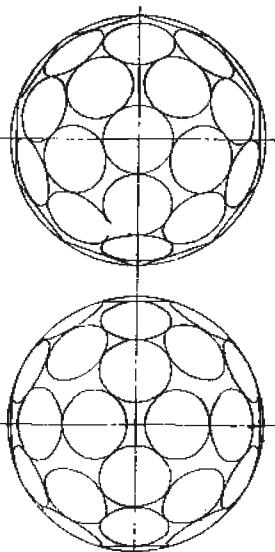
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Fig. 2



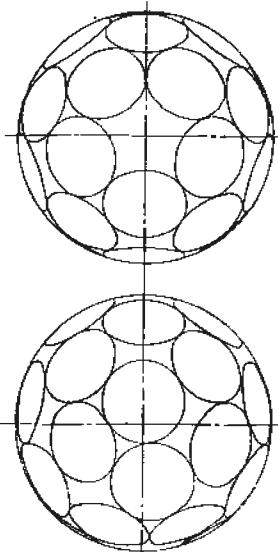
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Fig. 3



50

Fig. 4



#### 4. Values for $n = 42$

The icosahedral arrangement for 42 circles is obtained by placing the centers of the circles at the vertices of a regular icosahedron and slightly to one side (or the other) of the midpoints of the edges. The best value, so determined, yields

$$a(42, \text{icosahedral}) = 31^\circ 46'.$$

However, the value for the axially symmetric arrangement  $42\{1,5,10,10,10,5,1\}$ , shown in Figure 2, as described by VAN DER WAERDEN, is  $a(42) = 32^\circ 00'$ . A further improvement is obtained by the arrangement  $42\{1,5,5,5,(10),5,5,5,1\}$ , shown in Figure 3, for which  $a(42) = 32^\circ 08'$ . As in the case of  $n = 32$ , the icosahedral arrangement for  $n = 42$  yields a smaller value than for the axially symmetric arrangements.

#### 5. Values for $n = 30$ and $n = 31$

One might consider the regular rhombic triacontahedron as likely to yield a satisfactory arrangement of 30 circles on the sphere. A circle can be inscribed in each face. These circles touch other circles in the adjacent faces. The value of  $a(30, \text{rhombic})$  is  $36^\circ$ , which is the supplement of the dihedral angle.

However, the axial arrangement  $30\{5,5,5,5,5,5\}$  gives a larger value, namely,  $a(30, \text{axial}) = 38^\circ 09' 40''$ . As in the case of  $n = 32$ , the positions of the centers of the circles on the latitudes adjacent to the equator are skewed with respect to each other. This is seen in the lower view of Figure 4. It is not evident in the top view, nor in the views shown by STROHMAJER. This accounts for the increase of 96 seconds of arc over the angular diameter given by STROHMAJER.

A similar improvement can be made for  $n = 31$ . One ring near the equator is skewed with respect to the other ring near the equator giving the value  $a(31) = 37^\circ 42' 36''$ , whereas STROHMAJER's arrangement gave only  $37^\circ 40'$ .

#### 6. Axial arrangements

There are no general rules for determining the most efficient way of packing  $n$  given equal circles on a sphere. One usually expects the highly symmetric arrangements to give extremal values. But this is not always the case, as shown in the foregoing. For at least the smaller values of  $n$ , the axially symmetric arrangements seem to give the best packings.

One of the reasons for this departure is the fact that the axial arrangements allow more contacts between the circles. In these arrangements, there are usually four contacts of a circle with its neighbors, namely, two in the higher latitude and two in the lower latitude. The more symmetric arrangements usually provide only three contacts.

Various axial arrangements were considered. They include the following nine indicated types:

$$\begin{aligned} &\{3,3,\dots,3,3\}, \quad \{1,3,3,\dots,3,3,1\}, \quad \{1,3,3,\dots,3,3\}, \\ &\{4,4,\dots,4,4\}, \quad \{1,4,4,\dots,4,4,1\}, \quad \{1,4,4,\dots,4,4\}, \\ &\{5,5,\dots,5,5\}, \quad \{1,5,5,\dots,5,5,1\}, \quad \{1,5,5,\dots,5,5\}. \end{aligned}$$

## 7. Tabulation

Table 1 summarizes the results of the computations on various arrangements. For some values of  $n$ , (like  $n = 17$ ), several forms apply, and they give distinct packings. For some others, like  $n = 12$ , different forms give identical packings.

As the rings are placed about a pole, each ring is placed as close to the pole as possible. In some cases, this requires that the circles in the added ring be placed unsymmetrically in the gaps provided by the previous ring. See, for example,  $n = 32$ . Also, the circles in the ring are not always equally spaced. In the symbol for the arrangement, a parenthesis about a number indicates that the circles in this ring are not equally spaced. However, the circles are arranged into groups which preserve the form of the axial symmetry that is being used. Furthermore, as in  $n = 33$ , different arrangements arise by the exercise of the choice of shifting circles to one side or the other in successive rings.

The largest known values of  $a(n)$  which have been computed are shown in Table 2. These include previously published results of the author and others [10, 11]. Values enclosed in parentheses have not yet been proved to be the best possible. This table includes the results of STROHMAIER [12] for  $n = 18$ , 21, 22 and 52. However, other arrangements and better values are given for  $n = 26$ , 30 and 31.

The quantity  $D_n$ , which represents the density of packing on the sphere, is given by  $D_n = n[1 - \cos 0.5 a(n)]/2$ . If the  $n$  circles have unit diameter, the sphere has the radius  $R_n = 1/\sqrt{2 - 2 \cos a(n)}$ .

A collection of drawings for the arrangements in Table 2, from  $n = 13$  to  $n = 52$ , is shown in the plates.

### Note

Since submission of this paper, several new results have been obtained by the author. One result is an improved arrangement for 33 circles which yields an angular diameter of  $36^{\circ}15'32''$  for each circle. Another result is an arrangement of 19 circles which yields an angular diameter of  $47^{\circ}25'22''$  for each circle. These results are described in notes to be published in *Elemente der Mathematik*. Also, improved results for 21 and 22 circles have been obtained, and an arrangement of 28 circles has been determined.

Meanwhile, Dr. RAPHAEL M. ROBINSON has obtained an improved result for  $n = 22$ , and new conjectures for  $n = 44, 48, 60, 80, 110$  and 120. Some of these are described in an abstract in the *Notices of the American Mathematical Society*, vol. 13, No. 6, October 1966. The complete results will be published later.

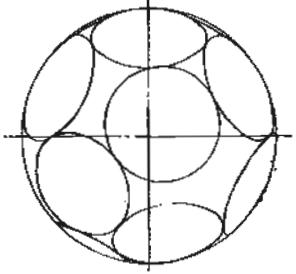
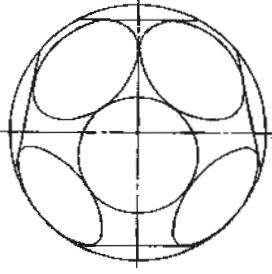
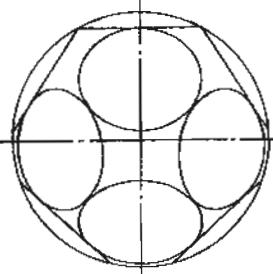
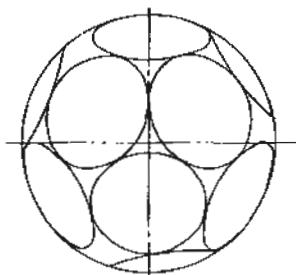
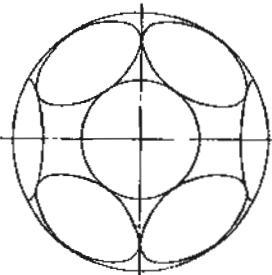
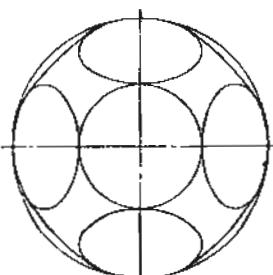
Table 1.

<i>n</i>	Angular diameter of <i>n</i> circles packed on sphere			
	Type of axial symmetry			
	2-fold	3-fold	4-fold	5-fold
9		70°32'		
10				
11				
12		63°26'	60°00'	63°26'
13			57°08'	
14	55°40'		* 54°44'	
15		53°26'		52°30'
16			52°14'	51°47'
17	51°02'		50°45'	51°02'
18		47°26'	49°33'	
19				
20		47°29'		45°31'
21				44°57'
22			43°43'	44°24'
23				
24			43°41'	
25				41°24'
26		40°44'		41°01'30"
27				40°41'
28				
29				
30				38°09'40"
31				37°42'36"
32				37°25'51"
33		35°25'		
34				
35				33°56'
36				33°40'30"
37				33°30'
38				
39				
40				32°25'
41				32°19'
42				32°08'
52				28°46'

\* Not stable

Table 2.

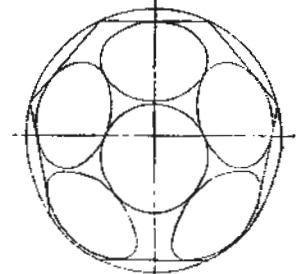
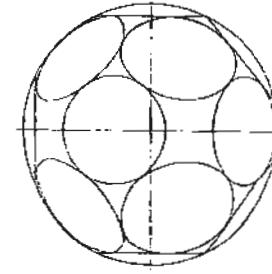
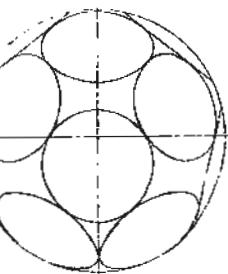
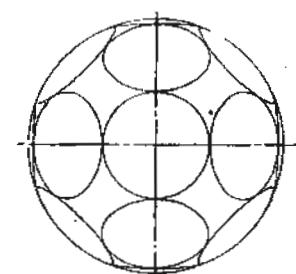
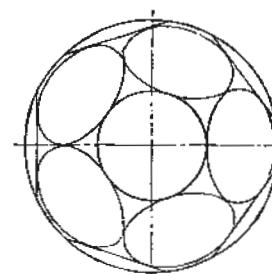
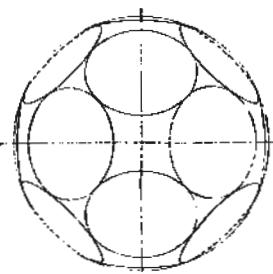
Axial symmetry packing of $n$ equal circles on the sphere					
$n$	arrangement	diameter	$R_n$	$D_n$	source
2		180°	0.5	1	Schütte & van der Waerden
3		120°	0.577	0.75	Schütte & van der Waerden
4	1,3 or 2,2 Reg.	109°28'16"	0.612	0.845	Schütte & van der Waerden
5	Degenerate	90°	0.707	0.732	Schütte & van der Waerden
6	1,4,1 or 3,3 Reg.	90°	0.707	0.878	Schütte & van der Waerden
7	1,3,3	77°51'58"	0.795	0.777	Schütte & van der Waerden
8	4,4	74°51'31"	0.822	0.823	Schütte & van der Waerden
9	3,3,3	70°31'44"	0.866	0.825	Schütte & van der Waerden
10	2,4,4	66°19'	0.916	0.812	Danzer
11	Degenerate	63°26'06"	0.951	0.802	Danzer
12	1,5,5,1 or 3,3,3,3 Reg.	63°26'06"	0.951	0.896	Fejes Tóth
13	1,4,4,4	(57°08')	(1.045)	(0.791)	Schütte & van der Waerden
14	1,(4),2,2,(4),1	(55°40')	(1.070)	(0.810)	Schütte & van der Waerden
15	3,3,3,3,3	(53°39')	(1.097)	(0.808)	Schütte & van der Waerden
16	4,4,4,4	(52°14')	(1.135)	(0.816)	Schütte & van der Waerden
17	1,5,5,5,1	(51°02')	(1.161)	(0.829)	Jucović
18	1,4,4,4,4,1	(49°33')	(1.193)	(0.828)	Strohmajer, Goldberg
19					
20	1,3,3,(6),3,3,1	(47°26')	(1.242)	(0.845)	van der Waerden
21	1,5,5,5,5	(44°57')	(1.308)	(0.798)	Strohmajer
22	1,5,5,5,5,1	(44°24')	(1.335)	(0.815)	Strohmajer
23					
24	4,4,4,4,4,4	43°41'	1.343	0.861	Robinson
25	5,5,5,5,5	(41°24')	(1.414)	(0.807)	Jucović
26	1,5,5,5,5,5	(41°01'30")	(1.428)	(0.823)	Goldberg
27	1,5,5,5,5,5,1	(40°41')	(1.437)	(0.842)	Goldberg
28					
29					
30	5,5,5,5,5,5	(38°09'40")	(1.530)	(0.824)	Goldberg
31	1,5,5,5,5,5,5	(37°42'36")	(1.548)	(0.830)	Goldberg
32	1,5,5,5,5,5,5,1	(37°25'51")	(1.558)	(0.846)	Goldberg
33	3,3,(6),(9),(6),3,3	(35°25')	(1.644)	(0.782)	Goldberg
34					
35	5,5,5,(10),5,5	(33°56')	(1.713)	(0.763)	Goldberg
36	5,5,5,(10),5,5,1	(33°40'30")	(1.726)	(0.771)	Goldberg
36	5,5,5,(10),5,5,1	(33°40'30")	(1.726)	(0.771)	Goldberg
37	1,5,5,5,(10),5,5,1	(33°30')	(1.735)	(0.785)	Goldberg
38					
39					
40	5,5,5,(10),5,5,5	(32°25')	(1.792)	(0.795)	Goldberg
41	1,5,5,5,(10),5,5,5	(32°19')	(1.795)	(0.810)	Goldberg
42	1,5,5,5,(10),5,5,5,1	(32°08')	(1.805)	(0.820)	Goldberg
52	1,5,10,10,10,10,5,1	(28°46')	(2.013)	(0.815)	Strohmajer
122		(19°21')	(2.977)	(0.868)	van der Waerden
~	Infinite plane			0.9069	Thue



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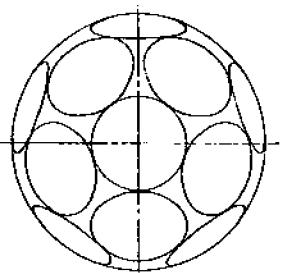
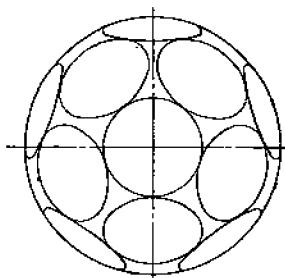
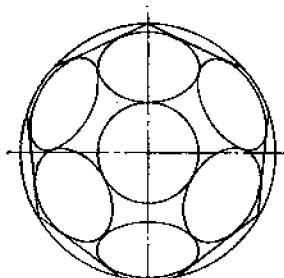
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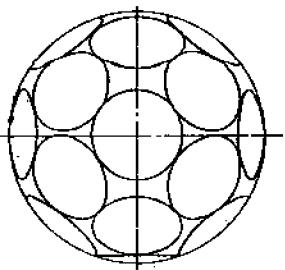
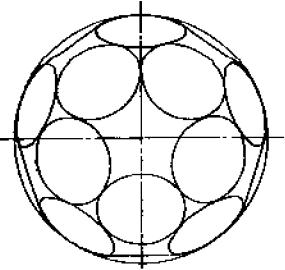
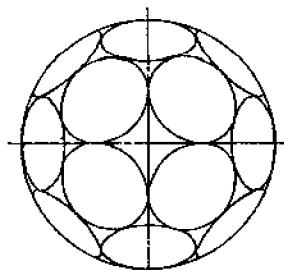
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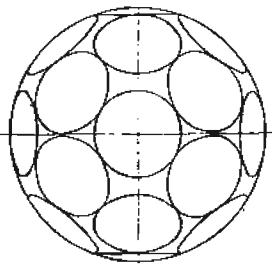
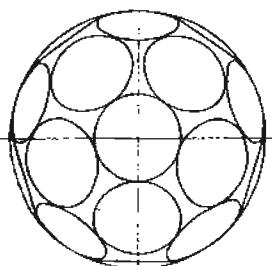
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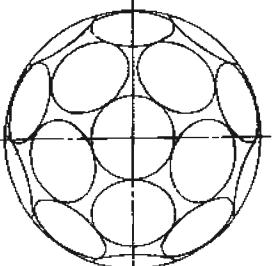
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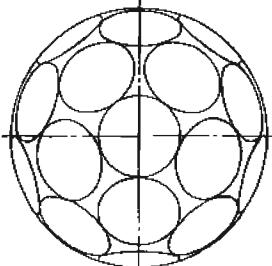
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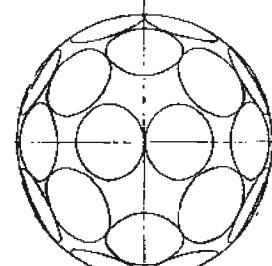
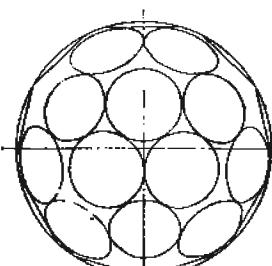
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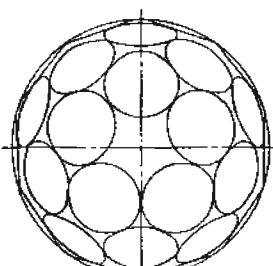
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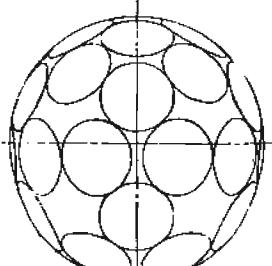
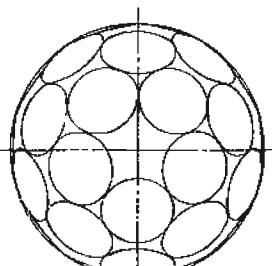
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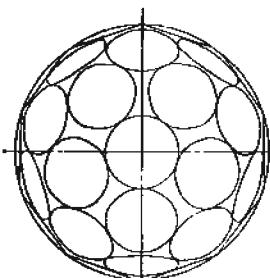
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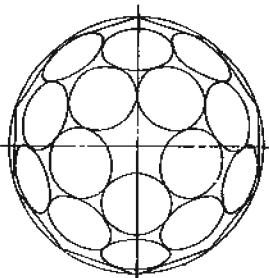
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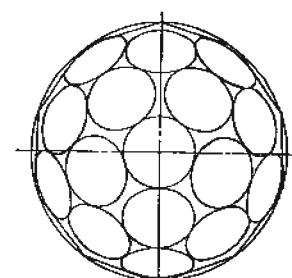
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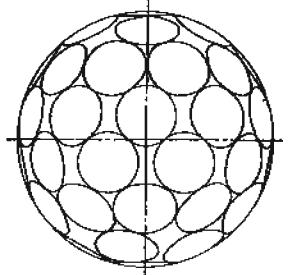
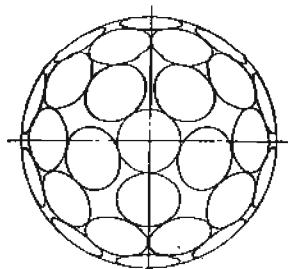
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## ON THE FUNCTIONAL EQUATIONS

$$f_1(x_1 + \dots + x_n)^2 = \left[ \sum_{(i_1, \dots, i_n)} f_1(x_{i_1}) \dots f_n(x_{i_n}) \right]^2$$

By

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(Received June 28, 1966)

In this paper we shall answer the question: When all the solutions of the equation

$$(1) \quad f_1(x_1 + \dots + x_n)^2 = \left[ \sum_{(i_1, \dots, i_n) \in P_k} f_1(x_{i_1}) \dots f_n(x_{i_n}) \right]^2$$

(where  $P_k$  is a set of  $k$  permutations of the numbers  $1, \dots, n$ ) can be obtained from the solutions of the equations

$$(1') \quad f_1(x_1 + \dots + x_n) = \sum_{(i_1, \dots, i_n) \in P_k} f_1(x_{i_1}) \dots f_n(x_{i_n}),$$

and

$$(1'') \quad f_1(x_1 + \dots + x_n) = - \sum_{(i_1, \dots, i_n) \in P_k} f_1(x_{i_1}) \dots f_n(x_{i_n}).$$

Such a problem was set first by J. ACZÉL, K. FLADT, M. HOSZSÚ ([1]) for the equation

$$(2) \quad f_1(x_1 + x_2)^2 = [f_1(x_1)f_2(x_2) + f_1(x_2)f_2(x_1)]^2$$

and for the equations

$$(2') \quad f_1(x_1 + x_2) = f_1(x_1)f_2(x_2) + f_1(x_2)f_2(x_1)$$

and

$$(2'') \quad f_1(x_1 + x_2) = -f_1(x_1)f_2(x_2) - f_1(x_2)f_2(x_1).$$

M. HOSZSÚ ([2]) proved that all the solutions of the equation

$$(3) \quad f(x_1 + x_2)^2 = [f(x_1) + f(x_2)]^2$$

can be obtained from those of Cauchy's equation

$$(3') \quad f(x_1 + x_2) = f(x_1) + f(x_2)$$

under the assumption that  $x$  and  $f(x)$  are real.

E. VINCZE ([4]) proved the same theorem for  $x$  from a commutative semi-group and for  $f(x)$  complex.

The equivalence of equation (2) and equations (2'), (2'') was proved by H. ŚWIATAK ([3]) under the assumption that the functions  $f_1(x)$  and  $f_2(x)$  are real and continuous. In some cases (e.g. for the solutions with  $f_1(0) \neq 0$ ) this assumption can be weakened but it is not always so. The functions

$$f_1(x) = \begin{cases} -xe^{ax} & \text{for } x < 0 \\ xe^{ax} & \text{for } x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} -e^{ax} & \text{for } x < 0 \\ e^{ax} & \text{for } x \geq 0 \end{cases}$$

satisfy (2) but they satisfy neither (2') nor (2''), though  $f_1(x)$  is continuous everywhere and  $f_2(x)$  has only one point of discontinuity.

We shall investigate (1), (1'), (1'') under the assumption that

$$f_i : X \rightarrow R \quad (i = 1, \dots, n),$$

where  $X$  is a connected, commutative topological group (with the unity  $e$ , and with the group operation  $+$ ) such that the equivalence classes of the relation  $\sim$  defined in  $X \times \dots \times X$  by

$$(x_1, \dots, x_n) \sim (x_1^*, \dots, x_n^*) \Leftrightarrow x_1 + \dots + x_n = x_1^* + \dots + x_n^*$$

are connected sets, and  $R = (-\infty, \infty)$ .

**LEMMA 1.** Every continuous solution of equation (1) satisfies the equation

$$(1^*) \quad f_i(x_1 + \dots + x_n) = \epsilon(x_1 + \dots + x_n) \sum_{(i_1, \dots, i_n) \in P_k} f_1(x_{i_1}) \dots f_n(x_{i_n}).$$

where  $|\epsilon(x)| = 1$ .

**PROOF.** Equation (1) can be written as

$$f_1(x_1 + \dots + x_n) = \epsilon(x_1, \dots, x_n) \sum_{(i_1, \dots, i_n) \in P_k} f_1(x_{i_1}) \dots f_n(x_{i_n}),$$

where  $|\epsilon(x_1, \dots, x_n)| = 1$ .

Let us fix an arbitrary point  $(x_1^*, \dots, x_n^*) \in X \times \dots \times X$  such that  $f_1(x_1^* + \dots + x_n^*) \neq 0$ , and let us write

$$A = \{(x_1, \dots, x_n) : x_1 + \dots + x_n = x_1^* + \dots + x_n^*, \epsilon(x_1, \dots, x_n) = \epsilon(x_1^*, \dots, x_n^*)\},$$

$$B = \{(x_1, \dots, x_n) : x_1 + \dots + x_n = x_1^* + \dots + x_n^*, \epsilon(x_1, \dots, x_n) = -\epsilon(x_1^*, \dots, x_n^*)\}.$$

Since the functions  $f_1(x), \dots, f_n(x)$  are continuous, we have

$$(4) \quad A = \bar{A}, \quad B = \bar{B}$$

( $\bar{A}$  is the closure of the set  $A$ ).

By the definition of  $A$  and  $B$

$$(5) \quad A \cup B = \{(x_1, \dots, x_n) : x_1 + \dots + x_n = x_1^* + \dots + x_n^*\}.$$

Moreover,

$$(6) \quad A \cap B = \emptyset$$

since in the contrary case we should have  $f_1(x_1^* + \dots + x_n^*) = 0$  contrary to the definition of the point  $(x_1^*, \dots, x_n^*)$ .

Notice that  $A \cup B$  is the equivalence class of the relation  $\sim$ , and we have assumed that it is connected. Therefore it follows from (4), (5), (6) that

$$A = \emptyset \quad \text{or} \quad B = \emptyset.$$

Since  $(x_1^*, \dots, x_n^*) \in A$ , we have  $A \neq \emptyset$ , and  $B = \emptyset$  i.e.  $\epsilon(x_1, \dots, x_n) = \epsilon(x_1^*, \dots, x_n^*)$  at all the points  $(x_1, \dots, x_n)$  such that  $x_1 + \dots + x_n = x_1^* + \dots + x_n^*$ . The point  $(x_1^*, \dots, x_n^*)$  is an arbitrary point at which  $f_1(x_1^* + \dots + x_n^*) \neq 0$ . Thus we have  $\epsilon(x_1, \dots, x_n) = \epsilon(x_1 + \dots + x_n)$  at all the points  $(x_1, \dots, x_n)$  such that  $f_1(x_1 + \dots + x_n) \neq 0$ . If  $f_1(x_1 + \dots + x_n) = 0$ , we can write  $\epsilon(x_1, \dots, x_n) = \epsilon(x_1 + \dots + x_n)$ , too. Therefore equation (1) can be written as (1\*). Q.E.D.

LEMMA 11. All the solutions of equation (1\*) can be obtained from those of equations (1'), (1'').

PROOF. If either  $f_1(x) \equiv 0$  or  $f_1(x) \neq 0$  everywhere, the proof is obvious.

Suppose that  $f_1(x) \neq 0$  but it vanishes at some points and consider two cases:

$$1^\circ \quad f_1(e) = 0,$$

$$2^\circ \quad f_1(e) \neq 0 \text{ and } f_1(a) = 0 \text{ for some } a \neq e.$$

In case 1° we put  $x_1 = x, x_2 = \dots = x_n = e$  into (1\*) and we obtain

$$f_1(x) = \epsilon(x) k_{11} F_{1e} f_1(x),$$

where  $k_{11}$  is the number of components of the right-hand side of (1\*) with  $f_1(x_i)$  and  $F_{1e} = f_2(e) \dots f_n(e)$ . Hence it follows that  $\epsilon(x) = \text{const}$  at all the points where  $f_1(x) \neq 0$ . At the points where  $f_1(x) = 0$  the value of  $\epsilon(x)$  may be arbitrary and therefore all the solutions of equation (1\*) with  $f_1(e) = 0$  can be obtained from those of equations (1'), (1'').

In case 2° it must be  $k_{11} \neq 0$  and  $f_i(a) \neq 0$  for  $i = 2, \dots, n$ . In fact; in the contrary case it follows from (1\*) (after substitution of  $x_1 = x - (n-1)a, x_2 = \dots = x_n = a$ ) that  $f_1(x) = 0$ . Substituting in (1\*)  $x_1 = -(n-2)a, x_2 = \dots = x_n = a$  and taking into account  $f_1(a) = 0$ , we obtain

$$\epsilon(a) k_{11} F_{1a} f_1(-(n-2)a) = 0$$

where  $F_{1a} = f_2(a) \dots f_n(a)$ .

Since  $F_{1a} \neq 0$ , it is  $f_1(-(n-2)a) = 0$ . Now putting in (1\*)  $x_1 = x, x_2 = -(n-2)a, x_3 = \dots = x_n = a$  we obtain

$$f_1(x) = \epsilon(x) C_{na} f_1(x),$$

where  $C_{na}$  is a constant. Hence it follows that  $\epsilon(x) = \text{const}$  at all the points where  $f_i(x) \neq 0$ . Since  $\epsilon(x)$  may be arbitrary at the points where  $f_i(x) = 0$ , all the solutions of equation (1\*) with  $n > 2$  can be obtained from those of equations (1'), (1''). For  $n = 2$  the proof is such as in [3].

Thus we proved that in both cases all the solutions of equation (1\*) can be obtained from those of equations (1'), (1''). Q.E.D.

By Lemma I and Lemma II we obtain the following

**THEOREM.** *All the continuous solutions of equation (1) can be obtained from those of equations (1'), (1'').*

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## ON $R_0$ -TOPOLOGICAL SPACES

By

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(Received September 30, 1966)

In the well-known survey of topological spaces, P. S. ALEKSANDROV [1] remarks "in many problems of topology it has proved natural to consider  $T_0$ -spaces, and it seems that there is no need to consider any broader class of spaces". However, recent developments in general topology show that the  $R_0$ -axiom plays a fundamental role in several situations and in some respects it is more natural than the  $T_0$ -axiom. In this note I wish to point out briefly the importance of the  $R_0$ -axiom.

**DEFINITION:** A topological space  $(X, \tau)$  is  $R_0$  iff for each  $G \in \tau$ ,  $x \in G$  implies  $\bar{x} \subset G$ . (Here  $\bar{x}$  denotes the  $\tau$ -closure of  $\{x\}$ .)

N. A. SHANIN [8] first defined the above axiom.

K. MORITA [4] gave a characterization in terms of coverings of the space. The above terminology is due to A. S. DAVIS [2] who rediscovered the axiom and gave several interesting characterizations. He also showed that  $T_1 = R_0 + T_0$  and that  $T_0$  is independent of  $R_0$ .

The  $R_0$ -axiom has a certain symmetry (in fact M. W. LODATO [3] calls such a topological space *symmetric*) which is due to the fact that in such a space  $x \in \bar{y}$  iff  $y \in \bar{x}$ . This symmetry of the  $R_0$ -axiom is reflected in the existence of a symmetric "indexed system of neighborhoods" (A. S. DAVIS [2]) and in the local symmetry condition satisfied by the Pervin quasi-uniformity of  $X$  (S. A. NAIMPALLY [6]). The intersection of all the entourages of a compatible quasi-uniformity of  $X$  is symmetric when  $X$  is  $R_0$  but on the other hand it is anti-symmetric when  $X$  is  $T_0$  (M.-G. MURDESHWAR and S. A. NAIMPALLY [5]).

Unexpectedly the  $R_0$ -axiom has been found to be essential in different problems. M. W. LODATO [3] found that every  $R_0$ -space has a compatible generalized proximity which he considered. In D. E. SANDERSON's work [7] on certain classes of non-continuous functions, the  $R_0$ -axiom enters several results in a natural way.

I will conclude this note by showing how the  $R_0$ -axiom enters a new comparison between various topologies on the set  $X$ . The idea of one cover being a refinement of another has played an important role in topology especially in paracompactness and metrization. Therefore, it is natural to attempt to define a preorder  $<$  among the topologies on  $X$  as follows:  $\tau_1 < \tau_2$  iff every  $\tau_1$ -cover of  $X$  has a  $\tau_2$ -refinement (refinement of a cover is always a cover). It is easy to show that  $<$  is reflexive and transitive but that it is not anti-symmetric in general. If  $X = \{a, b, c\}$ ,  $\tau_1 = \emptyset, X, \{a\}, \{a, b\}$ ,  $\tau_2 = \emptyset, X, \{b\}, \{b, c\}$  then  $\tau_1 < \tau_2$ ,  $\tau_2 < \tau_1$  but  $\tau_1, \tau_2$  are not comparable in the usual sense. We note that  $\tau_1$  and  $\tau_2$  are both  $T_0$ . The situation, however, is quite different if the topologies satisfy the  $R_0$ -axiom, as the following result shows.

**THEOREM.** *If  $\tau_1 < \tau_2$  and  $\tau_1$  is  $R_0$  then  $\tau_1 \subseteq \tau_2$ .*  
*(Obviously if  $\tau_1 \subseteq \tau_2$  then  $\tau_1 < \tau_2$ .)*

**PROOF.** Let  $G \in \tau_1$  and  $p \in G$ . Then  $\bar{p} \subset G$  and so  $\{G, \bar{p}^c\}$  is a  $\tau_1$ -cover of  $X$ . By hypothesis there exists a  $\tau_2$ -cover of  $X$  which is a refinement of the above cover. Thus there exists an  $H \in \tau_2$  such that  $p \in H \subset G$  and  $G \in \tau_2$ .

**COROLLARY.** *If  $\tau_1$  is  $R_0$  then  $\tau_1 < \tau_2$  iff  $\tau_1 \subseteq \tau_2$ .*

This note was written with support from N.R.C. (Canada) and the Summer Research Institute of the Canadian Mathematical Congress.

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## NOTE ON CAYLEY'S GROUPTHEORETICAL THEOREM

By

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(Received November 30, 1966)

Let  $G$  be any finite group and let  $P = \{P_g; g \in G\}$ , where

$$P_g = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ gg_1 & gg_2 & \cdots & gg_n \end{pmatrix}.$$

$P$  is a subgroup of the symmetric group  $S_n$ . Cayley's theorem asserts that  $G$  and  $P$  are isomorphic under the correspondence  $g \rightarrow P_g$ . We give in this paper a refinement of Cayley's theorem and from this we derive simple proofs of two interesting theorems.

Throughout the present paper, we denote with the symbol  $o(G)$  the order of the group  $G$ , with the symbol  $o(g)$  the order of the element  $g \in G$ , with the symbol  $\{g\}$  the cyclic subgroup of the group  $G$  generated by the element  $g \in G$  and with the symbol  $[G : \{g\}]$  the index of the subgroup  $\{g\}$  in  $G$ .

**LEMMA 1.** *In the representation of the permutation  $P_g$  as a product of disjoint cycles, the number of these cycles equal  $[G : \{g\}]$ .*

**PROOF.** The elements of the cycles are identical with the elements of the classes in the decomposition

$$G = \{g\}g_{k_1} + \{g\}g_{k_2} + \dots + \{g\}g_{k_i} \quad (g_{k_1} = e)^1$$

of the group, because in the permutation  $P_g$  we have:

$$g_{k_f} \rightarrow gg_{k_f} \rightarrow g^2g_{k_f} \rightarrow \dots \rightarrow g^{o(g)}g_{k_f} = g_{k_f} \quad (f = 1, 2, \dots, i).$$

**LEMMA 2.** *Let  $s = c_1c_2 \dots c_i$  be the representation of any finite permutation  $s$  as a product of cycles and let  $o(c_f) = n_f$  ( $f = 1, 2, \dots, i$ ). Then the sign of the permutation  $s$  is given by*

$$\operatorname{sgn} s = (-1)^{n_1 + \dots + n_i - i}.$$

**PROOF.** Let  $c_f = (l_1, l_2, \dots, l_{n_f})$  ( $f = 1, 2, \dots, i$ ). The representation of the  $c_f$  as product of transpositions  $c_f = (l_1, l_2)(l_1, l_3) \dots (l_1, l_{n_f})$  implies  $\operatorname{sgn} c_f = (-1)^{n_f - 1}$ . The equality  $\operatorname{sgn} s = \operatorname{sgn} c_1 \cdot \operatorname{sgn} c_2 \dots \operatorname{sgn} c_i$  completes the proof.

From the Lemmas 1 and 2 it follows the

<sup>1</sup>  $e$  denotes the unite element of  $G$ .

LEMMA 3.  $\operatorname{sgn} P_g = (-1)^{o(G) - [G : \{g\}]}$ .

THEOREM. Let  $G$  be any finite group and  $o(G) = 2^\alpha q$ , with  $\alpha \geq 0$  and with  $q (\geq 1)$  odd. If  $o(G)$  is odd ( $\alpha = 0$ ), or if  $o(G)$  is even ( $\alpha \neq 0$ ), but  $G$  contains no element of order  $2^\alpha$ , then the group  $P$ , which is isomorphic in Cayley's representation to the group  $G$ , is a subgroup of the alternating group  $A_n$ . If  $o(G)$  is even ( $\alpha \neq 0$ ) and  $G$  contains at least one element of order  $2^\alpha$ , then the group  $P$  is a subgroup of the symmetric group  $S_n$ , which is not contained in  $A_n$ .

PROOF. If  $o(G)$  is odd, then  $[G : \{g\}]$  is odd for every element  $g \in G$ . It follows from the Lemma 3, that  $\operatorname{sgn} P_g = 1$  and consequently  $P \subseteq A_n$ .

Let  $o(G)$  be even and for every element  $g \in G$  let be  $o(g) \neq 2^\alpha$ . Then  $[G : \{g\}]$  is even for every element  $g \in G$ . It follows from the Lemma 3, that  $\operatorname{sgn} P_g = 1$  and consequently  $P \subseteq A_n$ .

Let  $o(G)$  be even and let  $g$  be an element of the group  $G$  having the property  $o(g) = 2^\alpha$ . Then  $[G : \{g\}]$  is odd. It follows from the Lemma 3, that  $\operatorname{sgn} P_g = -1$  and thus  $P \not\subseteq A_n$ .

COROLLARY 1. Let  $G$  be a group of even order:  $o(G) = 2^\alpha \cdot q$  ( $\alpha \neq 0$ ). If  $G$  contains an element of order  $2^\alpha$ , then  $G$  contains a subgroup of index  $2$ .<sup>2</sup>

PROOF. It follows from the Theorem, that in this case we have  $P \not\subseteq A_n$  and consequently among the permutations of the group  $P$  there are odd ones. The number of even and odd permutations is equal in  $P$ . Thus the set of all even permutations of the group  $P$  forms a subgroup of index 2.  $P \cong G$  isomorphism completes the proof.

COROLLARY 2. Let  $G$  be any finite group and  $o(G) = 2^\alpha \cdot q$ , with  $\alpha \geq 0$  and with  $q (\geq 1)$  odd. If  $G$  contains at least one element of order  $2^\alpha$ , then  $G$  is solvable.<sup>3</sup>

PROOF. We make use of induction with respect to  $\alpha$ . If we have  $\alpha = 0$ , then  $o(G)$  is odd, thus  $G$  is solvable [3]. Let  $\alpha$  be a positive number. In accordance with Corollary 1, there is a subgroup  $N$  in the group  $G$ , with  $[G : N] = 2$ .  $N$  is a maximal normal subgroup of  $G$  and  $o(N) = 2^{\alpha-1}q$ . If  $g \in G$  is an element of order  $2^\alpha$ , then  $o(g^2) = 2^{\alpha-1}$ . But  $\operatorname{sgn} P_{g^2} = 1$  and in the isomorphism  $G \cong P$  the subgroup  $N$  corresponds to such a subgroup of  $P$ , which is formed from all its even permutations. Thus  $g^2 \in N$ . The premise of the induction ensures the solvability of  $N$ . Thus  $G$  is solvable.

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<sup>2</sup> A proof of this theorem is given by J. DÉNES [2].

<sup>3</sup> Our proof uses the validity of the theorem in the particular case  $\alpha = 0$ : any finite group of odd order is solvable. This assertion forms the content of a famous problem due to W. BURNSIDE [1] and resolved by W. FETT and I. G. THOMPSON [3].

## CLOSE PACKING OF SEGMENTS

By

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We start with the problem of finding the densest arrangement of unit segments in the plane under the condition that the segments are not allowed to get closer to one another than a prescribed distance  $2r$ . In other words, we want to find the densest packing of parallel domains of radius  $r$  of unit segments.

It is known [1] that the density of any packing of congruent centro-symmetric convex discs cannot exceed the density of the densest lattice-packing of the discs. It easily follows that the arrangement of segments we are looking for looks like that shown in Fig. 1. The aim of the present paper is to show that the density of this arrangement cannot be surpassed by incongruent segments of unit average length. Fig. 2 illustrates a densest packing of incongruent segments of unit average length having the same value of  $r$  and the same density as the arrangement exhibited in Fig. 1.

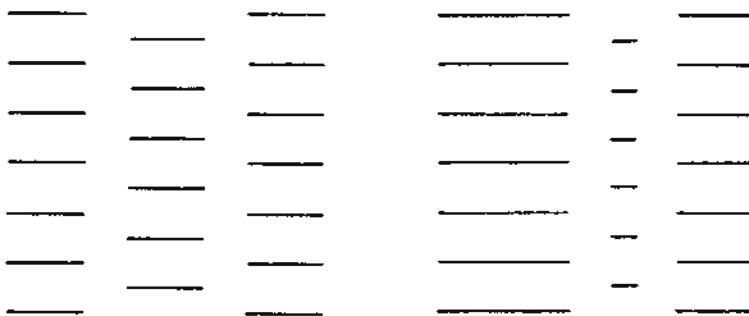


Fig. 1

Fig. 2

Our result is contained in the following

**THEOREM.** If a convex polygon  $p$  with at most six sides contains  $n$  segments of average length  $l$  such that any two of them have a distance not less than  $2r$ , then

$$(1) \quad n \leq \frac{a + br + cr^2}{2lr + \sqrt{12}r^2},$$

where  $a$  is the area of  $p$ ,  $b$  its perimeter and  $c$  the area of its indicatrix.

The indicatrix [2] of a polygon  $p$  is defined as a polygon circumscribed about the unit circle whose sides have the same outer normal directions as those of  $p$ . Thus the numerator on the right side equals the area of the polygon  $p$ , which arises from  $p$  by translating each of its sides outwards through the distance  $r$ . The denominator may be interpreted in a similar way. We consider a segment of length  $l$  as the limiting figure of an equiangular hexagon. The denominator is equal to the area of the hexagon arising from our degenerate hexagon by translating each of its sides outwards through the distance  $r$  (Fig. 3).

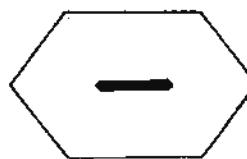


Fig. 3

We consider a set of segments scattered over the plane in such a way that the distance between any two of them is  $\geq 2r$ . Let  $q$  be a square of area  $a(q)$ ,  $n(q)$  the number of the segments contained in  $q$ ,  $l(q)$  the average length of these segments and  $q \rightarrow \infty$  a limiting process referring to a sequence of unlimitedly increasing squares. We suppose that the limiting values  $N = \lim_{q \rightarrow \infty} n(q)/a(q)$  and  $l = \lim_{q \rightarrow \infty} l(q)$  exist and that they do not depend on the choice of the set of squares.  $N$  and  $l$  may be interpreted as the number density and the average length of the segments.

In virtue of inequality (1), we have

$$\frac{n(q)}{a(q)} \leq \frac{1 + \frac{4r}{\sqrt{a(q)}} + \frac{4r^2}{a(q)}}{2l(q)r + \sqrt{12}r^2},$$

whence

$$N \leq \frac{1}{2lr + \sqrt{12}r^2},$$

in accordance with our statement in the introduction.

The proof of (1) rests on known arguments [3] using the following

**LEMMA.** Let  $p$  be the parallel domain of radius  $r$  of a segment of length  $l$ . If  $p$  is contained in a convex  $k$ -gon of area  $t$ , then

$$(2) \quad t \geq 2lr + r^2k \tan \frac{\pi}{k}.$$

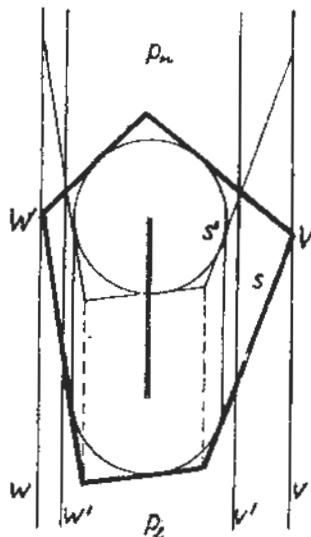


Fig. 4

To see this, we imagine the segment to be in a vertical position and consider the vertical supporting lines  $v$  and  $w$  of the  $k$ -gon, which we denote by  $s$  (Fig. 4). Let  $V$  and  $W$  be two vertices of  $s$  lying on  $v$  and  $w$ , respectively.  $V$  and  $W$  decompose the boundary of  $s$  into an „upper” and a „lower” polygonal line  $V \dots W$  and  $W \dots V$ . We consider  $s$  as being the intersection of the upper and lower convex pointsets  $p_u$  and  $p_l$  bounded by  $w$ ,  $W \dots V$ ,  $v$  and  $v$ ,  $V \dots W$ ,  $w$ , respectively. We translate  $p_u$  upwards through the distance  $l$ , obtaining a new  $k$ -gon  $s'$  of area  $t'$ , having the vertical supporting lines  $v'$  and  $w'$ . We have

$$t = t' + ld + l'',$$

where  $d$  is the distance between  $v'$  and  $w'$  and  $l''$  is the area of the part of  $p$  outside the strip bounded by  $v'$  and  $w'$ . But since  $s'$  contains a circle of radius  $r$ , we have  $d \geq 2r$  and  $l'' \geq r^2k \tan \frac{\pi}{k}$ . This completes the proof of the lemma.

In (2) equality holds if and only if  $k$  is even and  $s$  arises from a regular  $k$ -gon of inradius  $r$  by a telescopic elongation in the direction of a side through the distance  $l$ .

Now we turn to the proof of the theorem.

We suppose that the segments become inflated, each turning into its parallel domain at a steadily increasing distance  $\varrho$ . At first the growth will proceed unimpeded. But at  $\varrho = r$  some of the parallel domains will abut. We

consider the common supporting lines of the adjacent domains and suppose that the further growth is limited by these lines, preventing the domains to overlap. Continuing this process, each segment will turn into a convex polygon or an infinite polygonal region. Let  $h_1, \dots, h_n$  be the intersections of these polygons or polygonal regions with  $p_r$ . Since  $h_i$  contains the parallel domain at distance  $r$  of the corresponding segment, we have

$$t_i \geq 2rl_i + r^2k_i \tan \frac{\pi}{k_i},$$

where  $t_i$  is the area of  $h_i$ ,  $k_i$  its number of sides and  $l_i$  the length of the segment. But as a simple consequence of Euler's formula, we have

$$k_1 + \dots + k_n \leq 6n.$$

Furthermore, since  $k \tan \frac{\pi}{k}$  is a convex decreasing function of  $k \geq 3$ , we have

$$a + br + cr^2 \geq \sum_{i=1}^n t_i \geq 2rln + nr^2 6 \tan \frac{\pi}{6},$$

as required.

#### References

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- [2] Ib. p. 158.
- [3] Ib. pp. 163–167.

# ÜBER DIE SCHNITTPUNKTE VON GERADEN UND ZYKLEN IN DER HYPERBOLISCHEN GEOMETRIE

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Es ist bekannt<sup>1</sup>, daß beiden nachstehenden Sätze auf Grund der Axiome<sup>2</sup> der Verknüpfung, der Anordnung, der Kongruenz und der hyperbolischen Parallelaxiome beweisbar sind.

SATZ A<sub>1</sub>. Wenn ein Punkt einer Geraden im Inneren eines Kreises liegt, so hat die Gerade einen Punkt mit dem Kreise gemein.

SATZ B<sub>11</sub>. Wenn ein Kreis einen Punkt im Inneren und einen Punkt im Äußeren eines anderen Kreises hat, so haben die beiden Kreise einen Punkt gemein.

In dieser Arbeit zeigen wir, daß die Existenz der Schnittpunkte einer Geraden und eines Zyklus<sup>3</sup> bzw. zweier Zyklen aus den Axiomen I–IV folgt.

<sup>1</sup> S.z.B. J. STROMMER [4].

<sup>2</sup> D. HILBERT [2]; Axiome I–IV.

<sup>3</sup> Wir verstehen unter einem *Zykel* die Gesamtheit der Punkte, die durch Spiegelung eines Punktes *P* an den Geraden eines Büschels entstehen. Haben die Geraden des Büschels einen (eigentlichen) Punkt gemein, so ist der Zykel ein gewöhnlicher *Kreis*. Sind die Geraden des Büschels parallel zu einander, so ist der Zykel ein *Grenzkreis*. Haben die Geraden des Büschels ein gemeinsames Lot, so ist er eine *Abstandslinie*, deren Grundlinie dieses Lot ist. — Ein Punkt heißt *innerer Punkt* des Zyklus, wenn der Zykel ein Kreis ist und der Punkt im Inneren dieses Kreises liegt, oder wenn der Zykel ein Grenzkreis ist und der Punkt auf dem aus einem Punkte desselben parallel zu seinen Achsen gezogenen Halbstrahl liegt, oder wenn der Zykel eine Abstandslinie ist und der Punkt auf dem aus einem Punkte desselben senkrecht zu ihrer Grundlinie gezogenen Halbstrahl liegt; alle übrigen Punkte, die nicht zum Zykel gehören, liegen außerhalb des Zyklus.

Zum Beweis verwenden wir die Abbildung der Halbebene auf sich durch komplementäre Ordinaten (die L-Transformation). Wir betrachten den oberhalb einer Geraden, der x-Achse, gelegenen Teil der Ebene, fällen von irgend einem Punkte  $P$  aus das Lot  $PX = y$  und ordnen wir dem Punkte  $P$  den Punkt  $P'$  auf der Halbgerade  $\overset{\rightarrow}{XP}$  zu, dessen Abstand  $P'X = y'$  von  $x$  durch

$$\Pi(y') + \Pi(y) = \frac{\pi}{2}$$

bestimmt ist, wobei  $\Pi(y)$  den zum Lote  $y$  gehörenden Parallelwinkel bezeichnet. Diese Abbildung heißt die L-Transformation;  $y$  und  $y'$  sind komplementäre Strecken. Aus der Definition der L-Transformation folgt, daß die Abbildung ein-eindeutig und involutorisch ist. Ferner werden die, in einem Punkte der x-Achse errichteten senkrechten Halbgeraden auf sich selbst abgebildet und die Bilder der Abstandslinien, deren gemeinsame Grundlinie die x-Achse ist, sind wiederum Abstandslinien, deren Grundlinie ebenfalls die x-Achse ist. H. LIEBMANN hat auf Grund der Axiome I – IV die folgenden Sätze bewiesen<sup>4</sup>.

**SATZ 1.** *Der Geraden, die eine Ende  $E$  der x-Achse mit dem Ende der in irgend einem Punkte  $O$  der x-Achse errichteten senkrechten Halbgeraden verbindet, entspricht ein Grenzkreis, der  $O$  enthält und dessen Achse  $E$  zum Ende hat; und umgekehrt.*

**SATZ 2.** *Der Abstandslinie, die von der Geraden, die die x-Achse in einem beliebigen Punkte  $O$  senkrecht schneidet, den konstanten Abstand  $c$  hat, entspricht der Halbstrahl durch  $O$ , der mit der x-Achse den Winkel  $\gamma = \Pi(c)$  einschließt; und umgekehrt.*

**SATZ 3.** *Hat eine Gerade mit der x-Achse ein Lot gemein, so entspricht ihr der Halbkreis, dessen Mittelpunkt der Schnittpunkt  $O$  des gemeinsamen Lotes und der x-Achse ist und dessen Halbmesser und das gemeinsame Lot komplementäre Strecken sind (d.h. ihre Parallelwinkel ergänzen sich zu  $\frac{\pi}{2}$ ); und umgekehrt.*

**SATZ 4.** *Die Abbildung ist winkeltreu<sup>5</sup>.*

Wir werden zuerst Sätze  $A_2$ ,  $A_3$  über die Existenz der Schnittpunkte einer Geraden und eines Zykels (1), und dann mit Hilfe der Sätze  $A_2$ ,  $A_3$  und I – 4 einige Eigenschaften der L-Transformation beweisen (2). Hierauf zeigen wir die Existenz der Schnittpunkte zweier Zyklen (3).

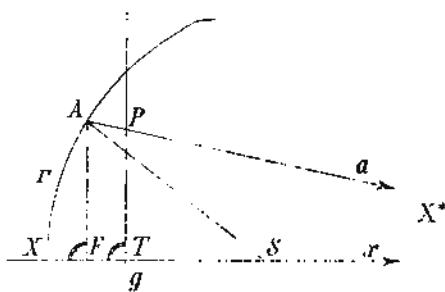
**1. SATZ  $A_2$ .** *Wenn ein Punkt einer Geraden im Inneren eines Grenzkreises liegt, so hat die Gerade einen Punkt mit dem Grenzkreis gemein.*

<sup>4</sup> H. LIEBMANN [3], S. 38 – 40.

<sup>5</sup> Die Tangente in einem Punkt  $P$  eines Zykels ist senkrecht auf die Gerade durch  $P$ , die Element des zu dem Zyklus zugehörigen Büschels ist. Leicht beweisbar ist, daß die von  $P$  verschiedenen Punkte der Tangente außerhalb des Zykels sind. Unter dem Winkel einer Geraden und eines Zykels versteht man den Winkel der Geraden und der zu dem Schnittpunkt zugehörigen Tangente. Wir verstehen unter dem Winkel zweier Zyklen den Winkel der zu dem Schnittpunkt zugehörigen Tangenten.

**BEWEIS.** Wenn die Gerade und die Achse des Grenzkreises parallel sind, so hat die Gerade einen und nur einen Punkt mit dem Grenzkreis gemein<sup>6</sup>.

Wenn die Gerade  $g$  mit der Achse des Grenzkreises nicht parallel ist, bezeichnen wir den im Inneren des Grenzkreises  $\Gamma$  liegenden Punkt der Geraden  $g$  mit  $P$ , die Achse des Grenzkreises  $\Gamma$  durch  $P$  mit  $a$  (wenn  $a \perp g$  ist), den Schnittpunkt der Grenzkreise  $\Gamma$  und der Geraden  $a$  mit  $A$ , die auf  $g$  senkrechte und mit  $a$  parallele Gerade mit  $x^*$ , den gemeinsamen Punkt des Grenzkreises  $\Gamma$  und der Geraden  $x$  mit  $X$  und endlich den gemeinsamen Punkt der Geraden  $g$  und  $x$  mit  $T$  (Fig. I).



Fällen wir das Lot durch  $A$  auf  $x$  und bezeichnen den Fußpunkt mit  $F$ , die Halbgeraden durch  $X$ ,  $F$  und  $T$ , die parallel und gleichgerichtet mit der Achse des Grenzkreises  $\Gamma$  sind, mit  $XX^*$ ,  $FX^*$  und  $TX^*$ . Es ist leicht beweisbar, daß die Halbgerade  $FX^*$  die Halbgerade  $TX^*$  enthält, und darum den Punkt  $T$  auch; nämlich wenn die Geraden  $AF$  und  $g$  einen gemeinsamen Punkt hätten, gäbe es zwei Geraden durch denselben Punkt, die senkrecht auf dieselbe Geraden stehen. Der Punkt  $F$  ist zwischen  $X$  und  $T$ . Der Punkt  $X$  liegt nicht auf der Halbgeraden  $FX^*$ , nämlich für den beliebigen Punkt  $S$  von  $FX^*$   $\angle X^*SA > R > \angle SAP$  gilt, wo  $R$  ein rechter Winkel ist.

In der L-Transformation in Bezug auf  $x$  entspricht die Gerade  $g$  sich selbst und dem Grenzkreise  $\Gamma$  eine mit der Halbgeraden  $XX^*$  und  $X^*Y$  parallele Gerade  $\Gamma'$ ; wo  $X^*Y$  aus den Punkt  $X$  ausgehende, auf die Gerade  $x$  senkrechte Halbgerade ist. Betrachten wir die mit der Halbgerade  $X^*Y$  parallele Halbgerade  $TY^*$ . Es ist klar, daß  $\angle XTY^* < R$  ist, folglich die Gerade  $g$  im Inneren des Winkels  $Y^*TX^*$  liegt, und deshalb haben die Geraden  $\Gamma'$  und  $g$  einen gemeinsamen Punkt  $G$ .

Wegen des ein-eindeutigen Charakters der L-Transformation haben der Grenzkreis  $\Gamma$  und die Gerade  $g$  einen gemeinsamen Punkt  $G$ , der in der Transformation dem Punkt  $G'$  entspricht. Wenn  $g \perp a$  ist, liegt der Beweis auf der Hand.

<sup>6</sup> S.z.B. H. LIEBMANN [3], S. 35.

<sup>7</sup> Über die Existenz der Gerade  $x$  s. D. HILBERT [2], S. 140 – 144.

SATZ A<sub>3</sub>. Wenn eine Gerade einen Punkt im Inneren einer Abstandslinie hat, und dieser Punkt und die Abstandslinie auf derselben Seite der Grundlinie von der Abstandslinie liegen, so hat die Gerade einen Punkt mit der Abstandslinie gemein.

Beweis. Wir bezeichnen die Abstandslinie mit  $\Delta$ , die Grundlinie und den konstanten Abstand der Abstandslinie mit  $l$  bzw.  $d$ , endlich die gegebene Gerade mit  $g$ .

Wenn die Gerade  $g$  auf die Gerade  $l$  senkrecht ist, folgt aus der Definition der Abstandslinie, daß die Gerade einen Punkt mit der Abstandslinie gemein hat.

Wenn die Gerade  $g$  die Gerade  $l$  nicht rechtwinklig schneidet, oder  $g$  und  $l$  parallel sind, so gibt es immer solche Geraden  $x$ , die parallel mit  $g$  und senkrecht auf  $l$  sind<sup>8</sup>.

In der L-Transformation in Bezug auf  $x$  entspricht der Abstandslinie  $\Delta$  eine Halbgerade  $\Delta'$ , die mit der Geraden  $x$  den Winkel  $\delta = \Pi(d)$  bestimmt und der Geraden  $g$  entspricht ein Grenzkreis  $g'$ , dessen Achse die Gerade  $x$  ist. Wenn die Geraden  $g$  und  $l$  parallel sind, hat der Grenzkreis  $g'$  einen gemeinsamen Punkt  $L$  mit der Geraden  $x$  und  $l$ . Weil der Winkel  $\delta = \Pi(d) < \frac{\pi}{2}$  ist, hat die

Gerade  $\Delta'$  noch einen Punkt  $D'$  mit dem Grenzkreis  $g'$  gemein, der mit  $L$  korrespondierender Punkt ist.

Wenn die Gerade  $g$  die Gerade  $l$  nicht rechtwinklig schneidet, geht der Grenzkreis  $g'$  durch den Punkt  $S$ , der der Schnittpunkt der Geraden  $x$  und  $s$  ist, wo  $s$  auf  $x$  senkrecht steht und mit  $g$  parallel ist. Die mit  $g$  parallele und aus dem Punkt  $S$  ausgehende Halbgerade von  $x$  besitzt einen gemeinsamen Punkt  $L$  mit der Geraden  $l$ ; sonst hätten die Geraden  $s$  und  $l$  nach dem Axiom vom Pasch einen gemeinsamen Punkt, es gäbe also zwei Geraden aus diesem Punkt, die auf  $x$  senkrecht stehen. Die Gerade  $\Delta'$  und der Grenzkreis  $g'$  haben wegen des Satzes A<sub>2</sub> einen gemeinsamen Punkt  $D'$ . Weil die Abbildung ein-eindeutig ist, existiert der gemeinsame Punkt  $D$  von der Geraden  $g$  und der Abstandslinie  $\Delta$ .

Wenn die Geraden  $g$  und  $l$  ein gemeinsames Lot haben, so betrachten wir die L-Transformation bezüglich dieses gemeinsamen Lotes  $x$ . In dieser Abbildung entspricht die Gerade  $g$  sich selbst und der Abstandslinie  $\Delta$  die Gerade  $\Delta'$ , die durch einen Punkt  $L$  durchgeht und mit der Geraden  $x$  den Winkel  $\delta = \Pi(d)$  bestimmt, wo  $L$  der Schnittpunkt von der Geraden  $x$  und  $l$  ist. Die Gerade  $g$  hat einen Punkt  $P$  im Inneren der Abstandslinie, dessen Abstand von  $l$  kleiner als  $d$  ist, deswegen  $LM < d$  ist, wo  $M$  der Schnittpunkt von  $x$  und  $g$  ist. Die Gerade  $g$  hat einen Punkt mit der Geraden  $\Delta'$  gemein, weil  $\Pi(LM) > \Pi(d)$  ist. Wegen der Ein-eindeutigkeit der L-Transformation haben die Abstandslinie  $\Delta$  und die Gerade  $g$  auch einen gemeinsamen Punkt.

\* S. Fußnote 7.

BEMERKUNG. Auf Grund des Satzes A<sub>3</sub> können wir den folgenden Satz leicht beweisen:

*Wenn eine Gerade einen Punkt im Inneren und einen Punkt im Äußeren einer Abstandslinie hat, so haben die Gerade und die Abstandslinie einen Punkt gemein.*

**2. HILFSATZ I.** *Wenn die Endpunkte einer Strecke auf einem Zykel oder im Inneren des Zyklus liegen, so liegen die Punkte der Strecke im Inneren des Zyklus.*

BEWEIS. Bezeichnen wir die Endpunkte der Strecke mit A und B und einen beliebigen Punkt der Strecke AB mit S.

Wenn A und B auf dem Zykel liegen, so nehmen wir die Achsen des Zyklus durch A, B und S. Wenn der Zykel ein gewöhnlicher Kreis ist, so schneiden sich die Achsen in dem Mittelpunkt O des Kreises. Einer von den Winkel OSA und OSB ist größer (oder gleich) als  $\frac{\pi}{2}$ ; es sei z. B.  $\angle OSA \geq \frac{\pi}{2}$ , so ist  $OA > OS$  in dem Dreieck OSA, und deswegen der Punkt S im Inneren des Kreises ist.

Wenn der Zykel ein Grenzkreis ist, so bezeichnen wir die parallelen Achsen durch A, B und S mit AX, BX und SX. Einer von den Winkeln XSA und XSB ist größer (oder gleich) als  $\frac{\pi}{2}$ ; es sei z. B.  $\angle XSA \geq \frac{\pi}{2}$ . Bezeichnen wir den gemeinsamen Punkt des Grenzkreises und der Geraden SX mit R, so ist  $\angle XAR = \angle XRA < \frac{\pi}{2}$ , folglich der Punkt S auf der Halbgeraden RX liegt.

Wenn der Zykel eine Abstandslinie ist, so schneiden die Achsen durch A, B und S die Grundlinie in den Punkten T<sub>1</sub>, T<sub>2</sub> und T<sub>3</sub>; und AT<sub>1</sub> = BT<sub>2</sub>. Einer von den Winkeln T<sub>3</sub>SA und T<sub>3</sub>SB ist größer (oder gleich) als  $\frac{\pi}{2}$ ; es sei z. B.  $\angle T_3SA \geq \frac{\pi}{2}$ . Bezeichnen wir den gemeinsamen Punkt der Geraden ST<sub>3</sub> und der Abstandslinie mit R, so ist  $\angle T_1AR = \angle T_3RA < \frac{\pi}{2}$ .

Es seien A und A\* symmetrisch in der Spiegelung an der Geraden RT<sub>3</sub>, und P der gemeinsame Punkt der Geraden AA\* und RT<sub>3</sub>, so ist PT<sub>3</sub> < AT<sub>1</sub>, folglich ST<sub>3</sub> < AT<sub>1</sub>, und S auf der Strecke RT<sub>3</sub> liegt.

Falls der Punkt A auf dem Zykel, der Punkt B im Inneren des Zyklus liegt, und wenn der Zykel ein gewöhnlicher Kreis mit dem Mittelpunkt O ist, so ist in dem Dreieck OBA OA > OB, folglich ist  $\angle ABO > \angle OAB$ . Der Punkt S liegt auf der Strecke AB, so ist in dem Dreieck OSA  $\angle ASO > \angle ABO$ , und  $OA > OS$ .

Wenn der Zykel ein Grenzkreis ist, so bezeichnen wir die Achsen durch A, B, S mit AX, BX und SX. Der Punkt B ist ein innerer Punkt und  $\angle XBA > \angle XAB$ . Weil die Halbgeraden BX und SX parallel sind, deswegen ist  $\angle XBA + \angle XSB < \pi$ , ferner  $\angle XSA + \angle XSB = \pi$  und  $\angle XSA > \angle XAB$ , also  $\angle XSA > \angle SAX$ , das bedeutet aber, daß der Punkt S ein innerer Punkt ist.

Wenn der Zykel eine Abstandslinie ist, und der innere Punkt  $B$  und der Punkt  $A$  auf verschiedenen Seiten der Grundlinie liegen, so ist der Beweis trivial. Wenn die Punkte  $A$  und  $B$  auf derselben Seite der Grundlinie liegen, so ist der Punkt  $S$  auch in dieser Halbebene. Wir bezeichnen die Fußpunkte der aus den Punkten  $A$ ,  $B$  und  $S$  auf der Grundlinie gefällten Lote mit  $T_1$ ,  $T_2$  und  $T_3$ , so ist  $\angle T_2BA > \angle T_1AB$ . In dem Viereck  $SBT_2T_3$  ist  $\angle T_2BA + \angle T_3SB < \pi$ , ferner  $\angle T_3SA + \angle T_3SB = \pi$ , also  $\angle T_2BA < \angle T_3SA$ , und so  $\angle T_1AS < \angle T_3SA$ , also  $S$  ein innerer Punkt ist.

Wenn die Punkte  $A$  und  $B$  innere Punkte eines Zykels sind – und wenn dieser Zykel eine Abstandslinie ist, einer vor den Punkten  $A$  und  $B$  auf einer Seite ihrer Grundlinie mit ihr liegt – so haben der Zykel und die Gerade  $AB$  einen gemeinsamen Punkt  $Q$ . Es ist leicht zu beweisen, daß der Punkt  $Q$  außerhalb der Strecke  $AB$  liegt, also eine von der Strecken  $QA$  und  $QB$  die Strecke  $AB$  enthält, und die Punkte der Strecke  $AB$  im Inneren des Zykels liegen. Wenn die Punkte  $A$ ,  $B$  und die Punkte der Abstandslinie auf verschiedenen Seiten der Grundlinie sind, so ist der Beweis trivial.

**HILFSATZ II.** *Wenn ein Punkt  $A$  im Inneren und ein anderer Punkt  $B$  im Äußeren eines Zykels liegen, so haben der Zykel und die Strecke  $AB$  einen gemeinsamen Punkt.*

**BEWEIS.** Wenn die Gerade  $AB$  eine Achse des Zykels ist, so folgt der Satz aus den Erwähnten.

Wenn die Gerade  $AB$  nicht eine Achse des Zykels ist – und wenn dieser Zykel eine Abstandslinie ist, die Gerade  $AB$  und die Grundlinie der Abstandslinie einen gemeinsamen Lot haben – so gibt es immer eine Achse des Zykels, die auf  $AB$  senkrecht steht<sup>9</sup>; wir bezeichnen seinen Fußpunkt mit  $F$ . In Bezug auf diese Achse sind die Gerade  $AB$  und der Zykel symmetrisch, also die mit  $A$  und  $B$  symmetrische Punkte  $\bar{A}$  und  $\bar{B}$  auch innerer bzw. äußerer Punkte sind. Der Punkt  $F$  liegt zwischen  $A$  und  $\bar{A}$ , also  $F$  ein innerer Punkt ist, also der Punkt  $A$  oder  $\bar{A}$  wegen des Hilfsatzes I. zwischen  $B$  und  $F$  liegt. Auf Grund der Sätze  $A_1 - A_3$  haben die Gerade  $AB$  und der Zykel einen Punkt  $G$  gemein und wegen der Symmetrie auch einen anderen Punkt  $\bar{G}$  gemein. Der Punkt  $G$  liegt nicht zwischen  $A$  und  $\bar{A}$ , und  $B$  bzw.  $\bar{B}$  liegen nicht zwischen  $G$  und  $A$  bzw.  $\bar{G}$  und  $\bar{A}$ , folglich  $G$  bzw.  $\bar{G}$  sind an der Strecken  $AB$  bzw.  $\bar{AB}$ .

Betrachten wir die anderen Fälle. Wenn die Gerade  $AB$  die Grundlinie schneidet oder mit der Grundlinie parallel ist, so liegt der gemeinsame Punkt wegen des Hilfsatzes I. auf der Halbgeraden  $BA$ . Wenn die Halbgerade  $BA$  mit der Grundlinie parallel ist, so liegt der gemeinsame Punkt auf der Strecke  $AB$ ; die übrigen Punkte der Halbgeraden liegen näher zu der Grundlinie, als der Punkt  $A$ . Wenn die Gerade die Grundlinie in einem Punkt  $H$  schneidet, so sind die sämtlichen Punkte der Strecke  $AH$  im Inneren der Abstandslinie. Die anderen Punkte der Halbgeraden  $AH$  sind auch innere Punkte, nämlich diese Punkte und die Punkte der Abstandslinie in Bezug auf die Grundlinie auf verschiedenen Seiten liegen. Der gemeinsame Punkt ist zwischen  $A$  und  $B$ . Damit haben wir den Hilfsatz II vollständig bewiesen.

<sup>9</sup> S. D. HILBERT [2], S. 140–144.

Mit Verwendung der Sätze  $A_1 - A_3$  und Hilfsätze I – II werden wir eine weitere Eigenschaft der L-Transformation auf Grund der HILBERT-schen Axiomengruppen I – IV der hyperbolischen Geometrie rein geometrisch beweisen<sup>10</sup>.

Es soll vor allem das Folgende bemerkt werden: Betrachten wir einen Halbkreis über dem Durchmesser  $AB$ , der im Inneren einen Punkt  $P$  eines durch  $A$  und  $B$  durchgehenden Zyklus hat. Nehmen wir die Achsen des Zyklus, die zwischen den von  $A$  und  $B$  ausgehenden Achsen  $a$  und  $b$  liegen. Wir beweisen, daß der auf einer solchen Achse  $s$  liegender Punkt des Zyklus im Inneren des Halbkreises ist.

Die Punkte  $A$ ,  $B$  und  $P$  bestimmen einen Zyklus. Wenn dieser Zyklus ein gewöhnlicher Kreis ist, sein Mittelpunkt und der Halbkreis auf verschiedenen Seiten der Geraden  $AB$  liegen. Ist dieser Zyklus ein Grenzkreis, so enthält die Mittelsenkrechte der Strecke  $AB$  eine Halbgerade, so daß  $P$  und diese Halbgerade auf verschiedenen Seiten der Geraden  $AB$  liegen, und die Achsen des Zyklus mit dieser Halbgeraden parallel sind. Wenn der Zyklus eine Abstandslinie ist, die Gründlinie dieser Abstandslinie liegt in Bezug auf der Geraden  $AB$  mit dem Halbkreis auf verschiedener Halbebene. Diese drei Tatsachen sind leicht beweisbar.

Betrachten wir die Achse  $s$ , und setzen wir voraus, daß die Mittelsenkrechte  $m$  der Strecke  $AB$  und  $s$  verschiedene Geraden sind, ferner  $a$  und  $s$  in Bezug auf  $m$  in verschiedenen Halbebene liegen. Es gibt immer eine Gerade  $t$ , so daß in der Spiegelung an der Geraden  $t$   $a$  und  $s$  symmetrisch sind. Die Gerade  $t$  ist ein Element vom Büschel des Zyklus. Bezeichnen wir die gemeinsamen Punkte von  $AB$  und  $m$ ,  $s$  bzw.  $t$  mit  $M$ ,  $S$  bzw.  $T$ . Der Punkt  $T$  liegt zwischen den Punkten  $M$  und  $A$ . Setzen wir nämlich voraus, daß der Punkt  $T$  zwischen  $M$  und  $S$  liegt und zugleich die Achse  $t$  zwischen den Achsen  $m$  und  $s$  ist. Wenn wir bei den Spiegelungen an den Geraden  $t$  bzw.  $m$ , die mit  $s$  symmetrische Geraden mit  $\bar{s}$  bzw.  $s'$  bezeichnen, so liegt die Achse  $\bar{s}$  zwischen den Achsen  $t$  und  $s'$ <sup>11</sup>. Weil die Achse  $s'$  zwischen  $a$  und  $m$  liegt, die Achsen  $a$  und  $s$  sind nicht symmetrisch in der Spiegelung an der Geraden  $t$ . Der gemeinsame Punkt  $T$  von  $t$  und  $AB$  liegt zwischen  $A$  und  $M$ , deshalb mit  $A$  auf der Geraden  $t$  symmetrischer Punkt  $\bar{A}$  im Inneren des Halbkreises liegt, nämlich  $AT = \bar{A}T < AM$  und  $\bar{A}T > \bar{A}M$  ist, also  $\bar{A}M < AM$  gilt.

<sup>10</sup> S. den analytischen bzw. raumgeometrischen Beweis F. HAUSDORFF, Analytische Beiträge zur nicht-euklidischen Geometrie, Ber. Verh. Sächs. Ges. Wiss. Leipzig, Math.-Phys. Kl. II, 51 (1899), 161–214. bzw. H. LIEBMANN, Synthetische Ableitung der Kreisverwandtschaften in der Lobatschefskischen Geometrie, Ber. Verh. Sächs. Ges. Wiss. Leipzig, Math.-Phys. Kl. II, 54 (1902), 244–260.

<sup>11</sup> Die Achse  $t$  liegt in einem Büschel zwischen der zwei Achsen  $m$  und  $s$ . Wir zeigen, daß  $\bar{s}$  zwischen  $t$  und  $s'$  ist. Wenn die Elemente des Büschels einen gemeinsamen Punkt oder ein gemeinsames Lot haben, liegt der Beweis auf der Hand. Wenn die Elemente des Büschels mit einander parallel sind, bezeichnen wir einen beliebigen Punkt von  $s'$  mit  $E$ , die Fußpunkte von der Senkrechten aus  $E$  auf  $t$  bzw.  $m$  mit  $G$  bzw.  $J$  ferner die gemeinsame Punkte von  $EJ$  und  $s$ ,  $EG$  und  $m$  bzw.  $EG$  und  $s$  mit  $K$ ,  $F$  und  $H$ . Die Punkte  $E$  und  $K$  sind in Bezug auf  $m$  symmetrisch. Weil  $EF > EJ = JK$  und  $EK > EF + FH$  ist, gilt  $EF > FH$ ; folglich der Punkt  $\bar{H}$  – der symmetrisch mit  $H$  in der Spiegelung auf  $t$  ist – zwischen  $E$  und  $G$  liegt.

SATZ 5. Einem Zykel, der die  $x$ -Achse in den Punkten  $A$  und  $B$  nicht senkrecht schneidet, entspricht eine Abstandslinie, deren Grundlinie die Bildgerade des Halbkreises über dem Durchmesser  $AB$  ist und deren Abstand von der Grundlinie gleich mit dem Parallelwinkel ist, das zu dem Schnittwinkel des Zykels mit der  $x$ -Achse gehört. Wenn der oberhalb der  $x$ -Achse liegende Teil des gegebenen Zykels im Inneren des Kreises über  $AB$  ist, so trennt die Grundlinie dieser Abstandslinie dieselbe und die  $x$ -Achse voneinander; im anderen Fall liegt sie zwischen ihrer Grundlinie und der  $x$ -Achse. — Umgekehrt entspricht einer Abstandslinie, deren Grundlinie mit der  $x$ -Achse ein gemeinsames Lot hat, ein Zykel, der die  $x$ -Achse in zwei Punkten mit einem Winkel schneidet, der gleich dem Parallelwinkel ihres konstanten Abstandes von der Grundlinie ist.

BEWEIS. Es sei  $P$  ein beliebiger Punkt des gegebenen Zykels (Fig. 2). Wir können einen und nur einen Zykel durch  $A$  und  $P$  bzw.  $B$  und  $P$  schreiben, der die  $x$ -Achse unter rechten Winkel schneidet. Die Geraden, die diesen beiden

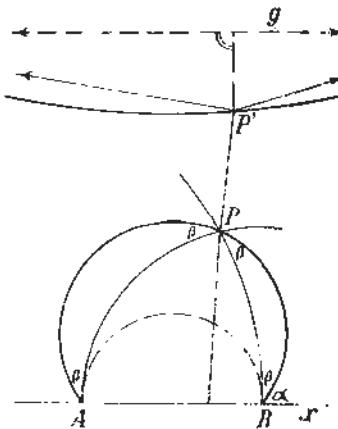


Fig. 2

Zyklen entsprechen (Sätze 1, 2 und 3) schneiden sich im Punkte  $P'$ , der als Bildpunkt dem Punkte  $P$  entspricht und sind parallel zu der Geraden  $g$ , die dem Halbkreis über  $AB$  entspricht (Satz 3). Aus der Tatsache, daß zwei Zyklen, die zwei Punkte gemein haben, sich in diesen Punkten unter gleichen Winkeln schneiden, folgt nun, daß die beiden Zyklen, die die  $x$ -Achse in dem Punkte  $A$  bzw.  $B$  rechtwinklig schneiden, sich unter dem Winkel  $\pi - 2\beta$  schneiden, wobei  $\beta$  der Schnittwinkel des Kreises über dem Halbmesser  $AB$  mit dem gegebenen Zykel ist. Bezeichnen wir den Schnittwinkel des gegebenen Zykels mit der  $x$ -

Achse mit  $\alpha$ , so ist  $\alpha + \beta = \frac{\pi}{2}$ , also  $\pi - 2\beta = 2\alpha$ . Da die Abbildung winkeltreu

ist, so ist auch der Winkel beiden Bildgeraden gleich  $2\alpha$  und somit sind die aus den Punkten, die den Punkten des gegebenen Zykels als Bildpunkten entsprechen, auf die Gerade  $g$  gefällten Lote untereinander gleich und liegen die Bildpunkte selbst jenseits oder diesseits der Geraden  $g$ , je nachdem die betreffenden Punkte des Zykels innerhalb oder außerhalb des Kreises über  $AB$  sind.

Diese letztere Tatsache läßt sich leicht auf Grund der Eigenschaft der komplementären Ordinaten einsehen, da  $r'_1 < r'_2$  gilt, falls  $r'_1 > r_2$  ist. Damit haben wir bewiesen, daß die Punkte, die den Punkten des gegebenen Zyklus entsprechen, auf der Abstandslinie liegen, die zu der Geraden  $g$  in dem Abstand  $\Delta(\alpha) = a$  gezogen werden kann, wobei  $\Delta(\alpha)$  das zu dem Winkel  $\alpha$  gehörige Parallelot bedeutet.

Es soll nun gezeigt werden, daß irgend ein Punkt dieser Abstandslinie das Bild eines Punktes des gegebenen Zyklus ist. Zu dem Zwecke setzen wir voraus, daß  $Q'$  ein Punkt der Abstandslinie ist, der einem Punkte  $Q$  entspricht, der nicht zu dem gegebenen Zyklus gehört.

Wenn der oberhalb der  $x$ -Achse liegende Teil des gegebenen Zyklus ins Innere des Kreises über  $AB$  fällt, so liegt der Fußpunkt des von irgend einem Punkte der Abstandslinie auf die  $x$ -Achse gefällten Lotes zwischen  $A$  und  $B$  und hiermit hat das Lot mit dem Zyklus einen von  $Q$  verschiedenen Punkt  $T$  gemein, zu dem ebenfalls  $Q'$  als Bildpunkt gehört, was nicht möglich ist, weil die **L**-Transformation ein-eindeutig ist.

Ist nun der gegebene Zyklus ein gewöhnlicher Kreis, dessen oberhalb der  $x$ -Achse liegender Teil außerhalb des Kreises über  $AB$  ist, und wäre  $Q$  ein innerer Punkt des gegebenen Kreises, so können wir in ähnlicher Weise, wie vorher, auf Widerspruch gelangen. — Wenn aber  $Q$  außerhalb des gegebenen Kreises liegt (Fig. 3), so haben der Kreis und der Zyklus durch  $A$  und  $Q$ , der die  $x$ -Achse

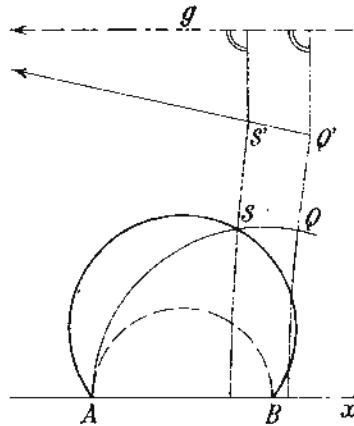


Fig. 3

rechteckig schneidet, eine gemeinsame Achse<sup>12</sup>. Der Kreis und der Zyklus haben den Punkt  $S$  gemein, der in der Spiegelung an der gemeinsamen Achse dem Punkt  $A$  entspricht. Der Bildpunkt  $S'$  liegt an einer Geraden, die mit der Grundlinie der Abstandslinie parallel ist, dann wäre aber  $S'$  und  $Q'$  gleich weit von der Grundlinie, was unmöglich ist. Die obenerwähnte Abstandslinie und die  $x$ -Achse haben offenbar keinen gemeinsamen Punkt. — Der Beweis der Umkehrung obiges Teilsatzes ist infolge des involutorischen Charakters der **L**-Transformation trivial.

<sup>12</sup> S. D. HILBERT [2], S. 140 – 144.

Ist weiter der gegebene Zykel ein Grenzkreis, dessen oberhalb der  $x$ -Achse liegender Teil außerhalb des Kreises über  $AB$  liegt (Fig. 4), so ist die  $x$ -Achse die Tangente in der Mitte der Strecke  $AB$  der Abstandslinie, die diejenigen

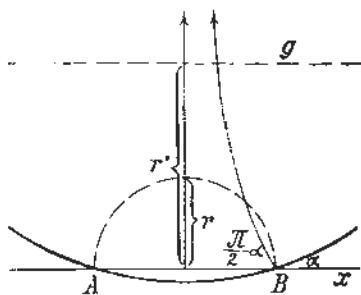


Fig. 4

Punkten enthält, die den Punkten dieses Grenzkreises entsprechen. Ist  $2r = AB$  und  $r'$  die Länge des gemeinsamen Lotes der  $x$ -Achse und der Geraden  $g$ , so gilt

$$\Pi(r) + \Pi(r') = \frac{\pi}{2}.$$

Der Grenzkreis muß die  $x$ -Achse unter dem Winkel  $\Pi(r')$  schneiden, weil die Achse des Grenzkreises durch  $A$  die  $x$ -Achse unter dem Winkel  $\Pi(r)$  schneidet. Der konstante Abstand dieser Abstandslinie von ihrer Grundlinie  $g$ , die Länge des gemeinsamen Lotes der  $x$ -Achse und der Geraden  $g$  sind also gleich miteinander.

Aus der Voraussetzung nunmehr, daß es einen Punkt  $Q'$  der Abstandslinie gibt, der nicht das Bild eines Punktes des gegebenen Grenzkreises ist, können wir ebenso wie vorher auf Widerspruch gelangen. — Die Umkehrung dieser Tatsache können wir auch wegen des involutorischen Charakters der Abbildung leicht beweisen.

Ist endlich der gegebene Zykel (der die  $x$ -Achse in den Punkten  $A$  und  $B$  nicht senkrecht schneidet) eine Abstandslinie, deren oberhalb der  $x$ -Achse liegender Teil außerhalb des Kreises über  $AB$  liegt, so — wegen der zwei vorliegenden Fälle — muß die Abstandslinie, die diejenigen Punkte enthält, die den Punkten der gegebenen Abstandslinie entsprechen, die  $x$ -Achse in zwei Punkten  $C$  und  $D$  schneiden. Dann muß aber wegen des involutorischen Charakters der Abbildung die Abstandslinie — die durch  $C$  und  $D$  geht — das Bild der gegebenen Abstandslinie durch  $A$  und  $B$  sein. — Die Umkehrung ist auch hier trivial.

Damit haben wir den Satz 5 vollständig bewiesen.

**3. SATZ B<sub>12</sub>.** Wenn ein Kreis einen Punkt im Inneren und einen Punkt im Äußeren eines Grenzkreises hat, so haben der Kreis und der Grenzkreis einen Punkt gemein.

**SATZ B<sub>21</sub>.** Wenn ein Grenzkreis einen Punkt im Inneren und einen Punkt im Äußeren eines Kreises hat, so haben der Grenzkreis und der Kreis einen Punkt gemein.

**Beweis.** Wir werden die Sätze B<sub>12</sub> und B<sub>21</sub> gleichzeitig beweisen. Betrachten wir die Achse des Grenzkreises  $\Gamma$ , die durch den Mittelpunkt  $O$  des Kreises  $\Omega$  übergeht. Diese Gerade  $x$  schneidet  $\Omega$  in den Punkten  $K_1$  und  $K_2$ , und  $\Gamma$  in dem Punkt  $G$ . Wir beweisen, daß der Punkt  $G$  — im Falle des Satzes B<sub>12</sub> und auch des Satzes B<sub>21</sub> — zwischen den Punkten  $K_1$ ,  $K_2$  liegt.

Im Falle vom Satz B<sub>12</sub> gibt es einen Punkt  $A$  des Kreises  $\Omega$  im Inneren und einen Punkt  $B$  von  $\Omega$  im Äußeren des Grenzkreises ist. Zu dem Zwecke setzen wir voraus, daß die Punkte  $K_1$  und  $K_2$  im Äußeren des Grenzkreises sind. Da der Punkt  $A$  ein innerer Punkt des Grenzkreises ist, ist deshalb des Fußpunkt  $S$  von der durch  $A$  auf  $x$  senkrechten Geraden auch ein innerer Punkt, also  $OS > OK_1 = OK_2$  gilt. Das Dreieck  $OAS$  ist ein rechtwinkliges Dreieck, folglich ist  $OA > OS$ . Der Punkt  $A$  ist ein Randpunkt von  $\Omega$  deswegen sind die zwei Ungleichungen mit einander im Widerspruch.

Setzen wir voraus, daß die beiden Punkte  $K_1$  und  $K_2$  im Inneren des Grenzkreises sind; jetzt werden wir auch zu einem Widerspruch gelangen. Betrachten wir die Gerade, die durch den im Äußeren des Grenzkreises liegenden Punkt  $B$  und durch den Punkt  $O$  geht. Nach dem Satz A<sub>2</sub> haben diese Gerade und der Grenzkreis  $\Gamma$  einen Punkt  $T$  gemein. In dem Dreieck  $OTG$  ist  $\angle TGO > \angle GTO$ , folglich  $OT > GO$ . Da  $OB > OT$  und  $GO > OK_1 = OK_2$  sind, also wäre wegen der vorherigen Ungleichung  $OB > OK_1$ , was unmöglich ist.

Im Falle vom Satz B<sub>21</sub> ist der Punkt  $A$  des Grenzkreises  $\Gamma$  im Inneren und der Punkt  $B$  von  $\Gamma$  im Äußeren des Kreises  $\Omega$ . Zuerst werden wir beweisen, daß beide Punkte  $K_1$  und  $K_2$  nicht im Äußeren des Grenzkreises liegen. Zu diesem Zwecke setzen wir voraus, daß die Punkte  $K_1$ ,  $K_2$  im Äußeren des Grenzkreises sind. Für den Punkt  $A$  ist  $OA < OK_1 = OK_2$ . Betrachten wir das Dreieck  $OAG$ ; es ist leicht beweisbar, daß in diesem Dreieck  $\angle OGA$  ein stumpfer Winkel ist, deshalb  $OA > OG$  gilt. Aus der indirekten Bedingung folgt, daß  $OG > OK_1 = OK_2$  ist, also  $OA > OK_1 = OK_2$ , was unmöglich ist. Wenn die Punkte  $K_1$ ,  $K_2$  im Inneren des Grenzkreises sind, so betrachten wir das Dreieck  $OAG$ , in dem  $\angleAGO > \angleGAO$  ist, also  $AO > OG$  gilt. Da  $OG > OK_1 = OK_2$  ist, besteht  $OA > OK_1 = OK_2$ . Das ist aber unmöglich.

Betrachten wir die L-Transformation, deren Achse die Gerade  $x$  ist. Wir bezeichnen die Geraden in der Halbebene der Transformation mit  $k_1$ ,  $k_2$  und  $g$ , die durch  $K_1$ ,  $K_2$  und  $G$  durchgehen und auf  $x$  senkrecht stehen. Es sei der Punkt mit  $K_2$  bezeichnet, der im Inneren des Grenzkreises liegt. In der L-Transformation ist das Bild von  $\Omega$  eine Gerade  $\Omega'$ , die parallel mit der Halbgeraden  $k_1$ ,  $k_2$  ist, und das Bild von  $\Gamma$  eine Gerade  $\Gamma'$  ist, die mit der Halbgeraden  $g$  und  $x$  parallel ist (Fig. 5). Die Halbgeraden  $\Gamma'$  und  $k_2$  haben einen gemeinsamen Punkt, weil die mit  $g$  parallele Halbgerade  $v$  durch  $K_2$  mit der Halbgeraden  $\Gamma'$  auch parallel ist; und die Halbgeraden  $v$  und  $K_2G$  einen spitzen Winkel bestimmen, liegt die Halbgerade  $k_2$  im Inneren des Winkels, den die aus dem Punkt  $K_2$  ausgehende und mit  $\Gamma'$  parallele Halbgeraden bestimmen. So folgt, daß die Halbgerade  $k_2$  die Gerade  $\Gamma'$  in einem Punkt  $V$  schneidet. Jetzt müssen wir noch beweisen, daß die Geraden  $\Omega'$  und  $g$  einen gemeinsamen Punkt haben.

Betrachten wir die Halbgeraden  $u_1$  und  $u_2$ , die aus dem Punkt  $G$  ausgehen und mit  $\Omega'$  parallel sind. Diese Halbgeraden sind parallel mit  $k_1$  und  $k_2$ , folglich bestimmen die Halbgeraden  $u_1$  und  $GK_1$  bzw.  $u_2$  und  $GK_2$  einen spitzen Winkel.

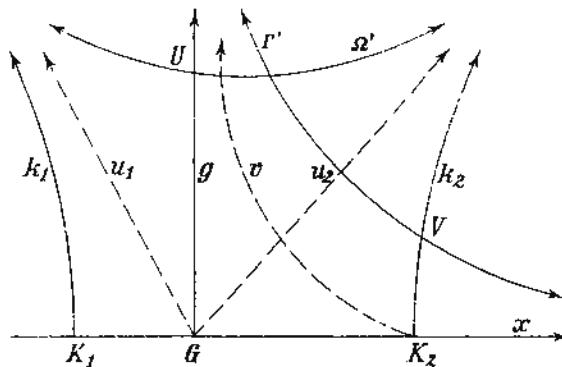


Fig. 5

Aus dieser Tatsache folgt, daß die Halbgerade  $g$  im Inneren des Winkels liegt, der durch die Halbgeraden  $u_1$  und  $u_2$  bestimmt ist, also schneidet  $g$  die Gerade  $\Omega'$  in einem Punkt  $U$ . Es ist bekannt, aus der Arbeit von J. STROMMER<sup>13</sup>, daß die aus  $U$  ausgehende mit  $k_2$  parallele Halbgerade  $\Omega'$  und die aus  $V$  ausgehende mit  $g$  parallele Halbgerade  $\Gamma'$  einen gemeinsamen Punkt haben (Fig. 6). Wegen des ein-eindeutigen Charakters der L-Transformation haben die Zyklen  $\Gamma$  und  $\Omega$  einen Punkt gemein.

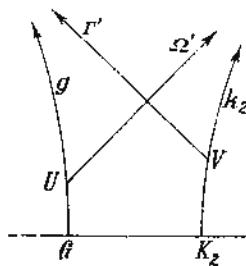


Fig. 6

SATZ B<sub>13</sub>. Wenn ein Kreis einen Punkt im Inneren und einen Punkt im Äußeren einer Abstandslinie hat, so haben der Kreis und die Abstandslinie einen Punkt gemein.

SATZ B<sub>31</sub>. Wenn eine Abstandslinie einen Punkt im Inneren und einen Punkt im Äußeren eines Kreises hat, so haben die Abstandslinie und der Kreis einen Punkt gemein.

<sup>13</sup> S. J. STROMMER [4], S. 194.

**BEWEIS.** Wir werden die Sätze  $B_{13}$  und  $B_{31}$  gleichzeitig beweisen. Betrachten wir die Gerade  $x$ , die durch den Mittelpunkt  $O$  des Kreises  $\Omega$  durchgeht und auf der Grundlinie  $l$  der Abstandslinie  $\Delta$  senkrecht steht. Die Gerade  $x$  schneidet den Kreis in den Punkten  $K_1$  und  $K_2$ , und die Abstandslinie in  $D$ . Wir beweisen, daß in beiden Fällen der Punkt  $D$  zwischen den Punkten  $K_1, K_2$  liegt. Im Falle vom Satz  $B_{13}$  ist der Punkt  $A$  des Kreises  $\Omega$  im Inneren und der Punkt  $B$  von  $\Omega$  im Äußeren der Abstandslinie  $\Delta$ . Wir zeigen, daß einer von den Punkten  $K_1, K_2$  im Inneren und der andere Punkt im Äußeren der Abstandslinie ist. Das zu beweisen setzen wir voraus, daß die Punkte  $K_1, K_2$  im Äußeren der Abstandslinie liegen. Weil der Punkt  $A$  im Inneren der Abstandslinie liegt, folglich ist der Fußpunkt  $S$  von der Geraden, die durch  $A$  geht und auf  $x$  senkrecht steht, auch ein innerer Punkt, also gilt:  $OS > OK_1 = OK_2$ . Das Dreieck  $OAS$  ist ein rechtwinkliges Dreieck, deshalb ist  $OA > OS$ . Weil der Punkt  $A$  ein Randpunkt des Kreises ist, stehen die Gleichheit  $OA = OK_1$  und die vorigen Ungleichungen im Widerspruch. Wenn die Punkte  $K_1$  und  $K_2$  im Inneren der Abstandslinie sind, gelangen wir auch zu einem Widerspruch. Nehmen wir nämlich die Verbindungsgerade der Punkte  $O$  und  $B$ . Nach dem Satz  $A_3$  ist leicht zu beweisen, daß die Gerade  $OB$  und die Abstandslinie  $\Delta$  einen Punkt  $T$  gemein haben. In dem Dreieck  $OTD$  ist  $\angle TDO > \angle DTO$ , also  $OT > DO$ . Weil  $OB > OT$  gilt und  $DO > OK_1 = OK_2$  besteht, folglich ist  $OB > OK_1$ , was unmöglich ist.

Im Falle vom Satz  $B_{31}$  ist der Punkt  $A$  der Abstandslinie  $\Delta$  im Inneren und der Punkt  $B$  von  $\Delta$  im Äußeren des Kreises  $\Omega$ . Wir beweisen, daß beide Punkte  $K_1$  und  $K_2$  nicht im Äußeren der Abstandslinie liegen. Zu dem Zwecke setzen wir voraus, daß die Punkte  $K_1, K_2$  im Äußeren liegen. Für den Punkt  $A$  besteht  $OA < OK_1 = OK_2$ . Betrachten wir das Dreieck  $ODA$ ; es ist leicht beweisbar, daß in diesem Dreieck der Winkel  $ODA$  stumpfer Winkel ist, so ist  $OA > OD$ . Aus der indirekten Bedingung folgt, daß  $OD > OK_1 = OK_2$  ist, also  $OA > OK_1 = OK_2$  gilt, was unmöglich ist. Setzen wir voraus, daß die Punkte  $K_1, K_2$  im Inneren der Abstandslinie sind, so folgt, daß in dem Dreieck  $ODA$   $\angle ADO > \angle DAO$  ist, folglich besteht  $AO > OD$ . Weil  $OD > OK_1 = OK_2$  ist, deswegen gilt  $OA > OK_1 = OK_2$ , was unmöglich ist.

Betrachten wir die L-Transformation, deren Achse die Gerade  $x$  ist. Bezeichnen wir die Halbgeraden durch die Punkte  $K_1, K_2$  und  $D$ , die auf  $x$  senkrecht stehen und in der Halbebene der L-Transformation liegen mit  $k_1, k_2$  und  $d$ . In der L-Transformation ist das Bild von  $\Omega$  eine Gerade  $\Omega'$ , die parallel mit der Halbgeraden  $k_1, k_2$  ist, und das Bild von  $\Delta$  ist eine Gerade  $\Delta'$ , die parallel mit der Halbgeraden  $d$  ist, und durch den Schnittpunkt  $L$  durchgeht, wo  $L$  der Schnittpunkt von  $x$  und  $l$  ist und die Gerade  $l$  die Grundlinie der Abstandslinie ist. Sei der Punkt  $K_1$  im Äußeren und der Punkt  $K_2$  im Inneren der Abstandslinie. Der Punkt  $L$  liegt entweder zwischen  $D$  und  $K_2$ , oder  $K_2$  liegt zwischen  $D$  und  $L$ . (Den Fall, wo  $K_2$  und  $L$  zusammenfallen, werden wir später betrachten.) Wenn der Punkt  $K_2$  zwischen  $D$  und  $L$  liegt, so ist  $K_2L < DL$  und folglich schneidet die Gerade  $\Delta'$  die Halbgerade  $k_2$  in einem Punkt  $V$ . Die Gerade  $\Omega'$  schneidet die Halbgerade  $d$  in beiden Fällen in einem Punkt  $U$ , weil die Halbgerade  $d$  im Inneren des Winkels liegt, der durch die mit  $k_1$  und  $k_2$  parallelen und aus dem Punkt  $D$  ausgehenden Geraden  $u_1$  und  $u_2$  bestimmt ist. Also die Gerade  $\Omega'$  schneidet die auf  $x$  senkrecht stehende Halbgerade  $d$  in einem Punkt  $U$  und ist parallel mit der auf  $x$  senkrecht stehenden Halbgeraden  $k_2$ ,

ferner die Halbgerade  $\Delta'$  schneidet die Halbgerade  $k_2$  in einem Punkt  $V$  und ist parallel mit  $d$ . Es ist möglich einzusehen — wie vorher — daß der gemeinsame Punkt von der Geraden  $\Omega'$  und  $\Delta'$  existiert.

Wenn der Punkt  $L$  zwischen  $D$  und  $K_2$  liegt bzw. wenn die Punkte  $K_2$  und  $L$  zusammenfallen, betrachten wir die Verbindungsgerade von  $L$  und  $U$ , und den Winkel  $ULD$  (Fig. 7). Der Winkel  $ULD < \pi(DL)$  ist, wo  $\pi(DL)$  der Win-

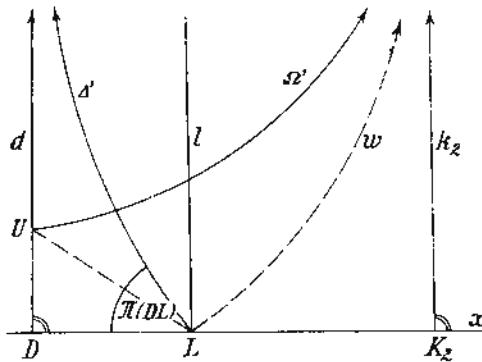


Fig. 7

kel der Halbgeraden  $\Delta'$  und  $LD$  ist. Die mit  $k_2$  parallele Halbgerade  $w$  durch  $L$  (im Falle wo  $L = K_2$  ist, sei  $w = l$ ) und die Halbgerade  $LK_2$  bestimmen einen Winkel, der kleiner (oder gleich) als  $\frac{\pi}{2}$  ist, folglich liegt die Halbgerade  $\Delta'$  im Inneren des Winkels, der durch die Halbgeraden  $LU$  und  $w$  bestimmt ist und deswegen schneidet die Halbgerade  $\Delta'$  die Halbgerade  $\Omega'$ . Wegen des eindeutigen Charakters der L-Transformation haben die Abstandslinie  $\Delta$  und der Kreis  $\Omega$  einen Punkt gemein, damit haben wir die Sätze  $B_{13}$  und  $B_{31}$  vollständig bewiesen.

**SATZ  $B_{23}$ .** Wenn ein Grenzkreis einen Punkt im Inneren und einen Punkt im Äußeren einer Abstandslinie hat, so haben der Grenzkreis und die Abstandslinie einen Punkt gemein.

**SATZ  $B_{32}$ .** Wenn eine Abstandslinie einen Punkt im Inneren und einen Punkt im Äußeren eines Grenzkreises hat, so haben die Abstandslinie und der Grenzkreis einen Punkt gemein.

**BEWEIS.** Wir werden die Sätze  $B_{23}$  und  $B_{32}$  gleichzeitig beweisen. Wenn die Grundlinie der Abstandslinie  $\Delta$  und eine Achse des Grenzkreises  $\Gamma$  zusammenfallen, so ist es genug nur den Satz  $B_{23}$  zu beweisen, nämlich den Beweis des Satzes  $B_{32}$  kann man auf den Satz  $B_{23}$  zurückführen. Bezeichnen wir den Punkt

des Grenzkreises  $\Gamma$  mit  $A$  bzw.  $B$ , der im Inneren bzw. Äußeren der Abstandslinie liegt. Betrachten wir die L-Transformation, deren  $x$ -Achse die Grundlinie der Abstandslinie ist. Das Bild der Abstandslinie  $\Delta$  ist eine andere Abstandslinie  $\Delta'$ , und das Bild des Grenzkreises  $\Gamma$  ist eine Gerade  $\Gamma'$  die parallel mit der Halbgeraden  $g$  und  $\overset{\rightarrow}{x}$  ist, wo  $g$  die auf  $x$  senkrechte und aus dem gemeinsamen Punkt  $G$  von  $\Gamma$  und  $x$  ausgehende Halbgerade, und  $\overset{\rightarrow}{x}$  die Halbgerade von  $x$  ist, die eine Achse des Grenzkreises ist. Es ist leicht beweisbar, daß der Punkt  $B'$  im Inneren der Abstandslinie  $\Delta'$ , in der Halbebene der L-Transformation liegt. Aus dem Satz A<sub>3</sub> folgt, daß die Gerade  $\Gamma'$  und die Abstandslinie  $\Delta'$ , ferner – wegen des ein-eindeutigen Charakters der L-Transformation – der Grenzkreis  $\Gamma$  und die Abstandslinie  $\Delta$  einen gemeinsamen Punkt haben.

Im Falle des Satzes B<sub>32</sub> kann man einsehen, daß der Grenzkreis einen Punkt im Inneren und einen Punkt im Äußeren der Abstandslinie hat, nämlich der gemeinsame Punkt  $A_1$  der Grundlinie von der Abstandslinie und des Grenzkreises  $\Gamma$  ein innerer Punkt ist. Man bekommt einen äußeren Punkt  $B_1$ , wenn man den gemeinsamen Punkt der auf die Grundlinie der Abstandslinie senkrecht stehenden Geraden, der durch  $A$  geht, und des Grenzkreises  $\Gamma$  nimmt. Der Punkt  $A$  ist nämlich im Inneren des Grenzkreises, und folglich ist der Punkt  $A$  zwischen  $B_1$  und  $L$ , wo  $L$  der gemeinsame Punkt der Grundlinie der Abstandslinie  $\Delta$  und der durch  $A$  auf die Grundlinie senkrecht stehenden Geraden ist. Der Punkt  $B_1$  ist ein äußerer Punkt.

Wenn die Grundlinie der Abstandslinie  $\Delta$  und die Achsen des Grenzkreises  $\Gamma$  mit einander nicht parallel sind, so gibt es immer eine Gerade  $x$ , die eine Achse von  $\Gamma$  ist und auf die Grundlinie  $l$  der Abstandslinie  $\Delta$  senkrecht steht. Bezeichnen wir den gemeinsamen Punkt von  $x$  und  $\Gamma$  bzw.  $x$  und  $\Delta$  mit  $G$  bzw.  $D$ , ferner den Schnittpunkt von  $l$  und  $x$  mit  $L$ . Im Falle der Sätze B<sub>23</sub> und B<sub>32</sub> werden wir beweisen, daß der Punkt  $D$  im Inneren des Grenzkreises liegt.

Wenn die Achsen mit der Halbgeraden  $DL$  parallel sind, so ist  $G$  kein innerer Punkt der Abstandslinie  $\Delta$ , weil in diesem Falle sämtliche Punkte des Grenzkreises im Inneren der Abstandslinie wären. Wenn die Punkte  $G$  und  $D$  zusammenfallen, so liegen die Punkte des Grenzkreises  $\Gamma^*$  im Inneren der Abstandslinie  $\Delta$ . (Wir bezeichnen mit  $\Gamma^*$  den mit dem Grenzkreis  $\Gamma$  koaxialen Grenzkreis durch den Punkt  $D$ , bzw. den Grenzkreis  $\Gamma$ , wenn  $\Gamma = \Gamma^*$  ist.) Setzen wir voraus, daß der Grenzkreis  $\Gamma^*$  einen Punkt  $G_1$  hat, der im Äußeren der Abstandslinie liegt.

Bezeichnen wir den Fußpunkt der aus diesem Punkt auf die Grundlinie  $l$  senkrecht stehenden Geraden mit  $L_1$ , dann gehört ein Punkt  $D_1$  von  $\Delta$  zu der Strecke  $G_1L_1$ . Betrachten wir die Halbgerade  $G_1X^*$ , die durch  $G_1$  mit der Halbgeraden  $DL$  parallel ist. Wegen des Zusammenhangs der äußeren und inneren Winkel des Dreieckes  $DD_1G_1$  ist  $\angle L_1D_1D > \angle L_1G_1D$ , ferner  $\angle L_1G_1D > \angle X^*G_1D = \angle LDG_1 > \angle D_1DL$ . Also  $\angle L_1D_1D > \angle D_1DL$  ist, folglich führt die indirekte Annahme zu einem Widerspruch. Nach dem Hilfsatz I. sind die inneren Punkte des Grenzkreises  $\Gamma^*$  im Inneren der Abstandslinie. Wenn der Punkt  $G$  auf der Halbgeraden  $DL$  liegt, so sind die Punkte des Grenzkreises  $\Gamma$  im Inneren des Grenzkreises  $\Gamma^*$ , und folglich im Inneren der Abstandslinie.

Wenn die Achsen des Grenzkreises mit der Halbgeraden  $LD$  parallel sind, so ist der Punkt  $G$  im Inneren der Abstandslinie. Setzen wir voraus, daß der Punkt  $G$  im Äußeren der Abstandslinie liegt. In diesem Falle für einen beliebigen

Punkt  $G_2$  von  $\Gamma$  die Ungleichung  $\frac{\pi}{2} < \angle G_2 GL < \angle G_2 DL$  gilt. Bezeichnen wir

den Fußpunkt der auf  $l$  senkrechten Geraden durch  $G_2$  mit  $L_2$ , so ist  $\angle L_2 G_2 D < \angle G_2 DL$ , also der Punkt  $G_2$  liegt im Äußeren der Abstandslinie, was unmöglich ist.

Bezeichnen wir die Halbgeraden durch  $D$  und  $G$ , die auf einer Achse des Grenzkreises  $\Gamma$  liegen, mit  $DX^*$  und  $GX^*$ , ferner den Punkt auf  $\Gamma$  bzw.  $A$ , der im Inneren von  $A$  bzw.  $\Gamma$  liegt, mit  $A$ .

Im Falle des Satzes  $B_{23}$ , wenn der Punkt  $G$  im Äußeren der Abstandslinie ist, folgt aus dem Dreieck  $ADG$ , daß  $\angle ADX^* > \angle AGX^*$  ist, also der Punkt  $D$  im Inneren des Grenzkreises liegt. Wenn der Punkt  $G$  im Inneren der Abstandslinie ist, so ist  $\angle AGX^* < \angle ADX^*$ , also der Punkt  $D$  im Inneren des Grenzkreises  $\Gamma$  liegt.

Im Falle des Satzes  $B_{32}$ , wenn der Punkt  $G$  im Inneren der Abstandslinie liegt,  $\angle AGX^*$  ein spitzer Winkel und  $\angle ADX^*$  ein stumpfer Winkel ist, also  $\angle AGX^* < \angle ADX^*$  gilt, folglich ist  $D$  ein innerer Punkt von  $\Gamma$ . Wenn der Punkt  $G$  außerhalb der Abstandslinie liegt, so besteht  $\angle AGX^* < \angle ADL$ , also der Punkt  $D$  im Inneren des Grenzkreises  $\Gamma$  ist.

Wir wählen die Gerade durch  $D, G$  als die  $x$ -Achse der  $L$ -Transformation, so ist das Bild des Grenzkreises  $\Gamma$  eine Gerade  $\Gamma'$ , die mit der aus dem Punkt  $G$  ausgehenden und auf  $x$  senkrechten Halbgeraden und der auf  $x$  liegenden Achse des Grenzkreises parallel ist. Das Bild der Abstandslinie ist eine Halbgerade durch  $L$ , die mit der auf  $x$  senkrecht stehenden und aus dem Punkt  $D$  ausgehenden Halbgeraden parallel ist.

Wenn der Punkt  $D$  zwischen  $G$  und  $L$  liegt, so betrachten wir die Halbgerade  $z$  durch  $L$ , die mit der Halbgeraden  $g$  parallel ist. Da  $GL > DL$  ist, folglich  $H(GL) < H(DL)$  besteht, deswegen die Gerade  $A'$  im Inneren des durch die Halbgeraden  $z$  und  $LX^*$  bestimmten Winkels liegt, also die Halbgerade  $A'$  und die Gerade  $\Gamma'$  schneiden sich (Fig. 8).

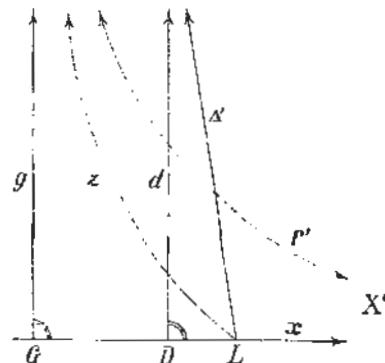


Fig. 8

Wenn der Punkt  $D$  außerhalb der Strecke  $GL$  liegt, so ist es leicht beweisbar, daß die Gerade  $\Delta'$  im Inneren des Winkels liegt, den die Halbgerade  $LX^*$  und die mit  $g$  parallele und aus dem Punkt  $L$  ausgehende Halbgerade bestimmen; also die Halbgerade  $\Delta'$  und die Gerade  $\Gamma'$  schneiden sich (Fig. 9).

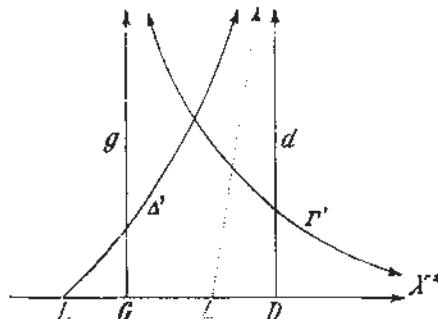


Fig. 9

Der Grenzkreis  $\Gamma$  und die Abstandslinie  $\Delta$  haben — wegen des ein-eindeutigen Charakters der  $L$ -Transformation — einen Punkt gemein, wenn die Geraden  $\Gamma'$  und  $\Delta'$  sich schneiden.

**SATZ B<sub>22</sub>.** Wenn ein Grenzkreis einen Punkt im Inneren und einen Punkt im Äußeren eines anderen Grenzkreises hat, so haben die beiden Grenzkreise einen Punkt gemein.

**BEWEIS.** Bezeichnen wir die Grenzkreise mit  $\Gamma_1$  und  $\Gamma_2$ , und den im Inneren des Grenzkreises  $\Gamma_2$  liegenden Punkt des Grenzkreises  $\Gamma_1$  mit  $A$ , den außerhalb des Grenzkreises  $\Gamma_2$  liegenden Punkt von  $\Gamma_1$  mit  $B$ . Die Punkte der Grenzkreise  $\Gamma_1$  und  $\Gamma_2$  durch die zwei verschiedenen Strahlbüschel bestimmt sind. In diesem Falle gibt es immer eine Gerade  $x$ , die eine Achse von  $\Gamma_1$  und  $\Gamma_2$  enthält, und diese Achsen gegensinnige Halbgeraden der Geraden  $x$  sind. Die Punkte  $G_1$  und  $G_2$  von  $\Gamma_1$  bzw.  $\Gamma_2$  liegen auf der Geraden  $x$ . Wir werden beweisen, daß der Punkt  $G_1$  auf der aus dem Punkt  $G_2$  ausgehenden Achse  $G_2X_2$  des Grenzkreises  $\Gamma_2$  liegt. Zu dem Zwecke setzen wir voraus, daß der Punkt  $G_1$  außerhalb der Halbgeraden  $G_2X_2$  liegt. In diesem Falle ist  $\angle AG_2G_1$  ein stumpfer Winkel in dem Dreieck  $G_1G_2A$ , also  $\angle AG_1G_2$  ist ein spitzer Winkel. Es folgt, daß  $\angle AG_1X_1$  ein stumpfer Winkel ist, wo die Halbgerade  $G_1X_1$  die aus dem Punkt  $G_1$  ausgehende Achse des Grenzkreises  $\Gamma_1$  ist. Der Winkel  $AG_1X_1$  muß ein spitzer Winkel sein, also die Ungleichung  $\angle AG_1X_1 > \frac{\pi}{2}$  unmöglich ist. Damit haben

wir bewiesen, daß die Halbgerade  $G_2X_2$  den Punkt  $G_1$  enthält und folglich der Punkt  $G_2$  auf der Halbgeraden  $G_1X_1$  liegt.

Betrachten wir die  $L$ -Transformation, deren Achse die Gerade  $x$  ist. Das Bild des Grenzkreises  $\Gamma_1$  ist eine Gerade  $\Gamma'_1$ , die mit der Halbgeraden  $G_1X_1$  und mit der auf  $x$  senkrecht stehenden Halbgeraden  $g_1$  durch  $G_1$  parallel ist. Ebenso entspricht der Grenzkreis  $\Gamma_2$  eine Gerade  $\Gamma'_2$ , die mit der Geraden  $G_2X_2$  und mit der auf  $x$  senkrechten Halbgeraden  $g_2$  durch  $G_2$  parallel ist.

Es ist leicht beweisbar, daß  $\Gamma'_1$  und  $g_2$  bzw.  $\Gamma'_2$  und  $g_1$  je einen gemeinsamen Punkt  $V$  bzw.  $U$  haben. Auf Grund dieser Tatsache folgt — wie vorher — daß die Geraden  $\Gamma'_1$  und  $\Gamma'_2$  einen Punkt gemein haben, folglich haben die Grenzkreise  $\Gamma_1$  und  $\Gamma_2$  auch einen gemeinsamen Punkt wegen des ein-eindeutigen Charakters der **L**-Transformation.

**SATZ B<sub>33</sub>.** *Wenn eine Abstandslinie einen Punkt im Inneren und einen Punkt im Äußeren einer anderen Abstandslinie hat, so haben die beiden Abstandslinien einen Punkt gemein.*

**Beweis.** Bezeichnen wir die Abstandslinien mit  $A_1$  bzw.  $A_2$ , ferner den Punkt der Abstandslinie  $A_1$ , der im Inneren der Abstandslinie  $A_2$  liegt, mit  $A$ , und den Punkt, der außerhalb der Abstandslinie  $A_2$  ist, mit  $B$ . Es ist offenbar, daß die Bedingungen vom Satz B<sub>33</sub> nicht erfüllt sind, wenn die durch die Abstandslinien und ihre Grundlinien bestimmten Halbebenen keine gemeinsame Punkte haben.

Wenn die Grundlinien mit einander parallel sind, ferner die Abstandslinie  $A_1$  und die Grundlinie  $l_2$  der Abstandslinie  $A_2$  auf derselben Seite der Grundlinie  $l_1$  liegen und umgekehrt, so gibt es immer eine Gerade  $x$ , die auf der Geraden  $l_1$  senkrecht steht und mit der Geraden  $l_2$  ein gemeinsames Lot hat; so daß die Länge des gemeinsamen Lotes größer, als der Abstand des Punktes  $B$  von  $l_2$  ist<sup>14</sup>. Wählen wir eine solche Gerade als die x-Achse der **L**-Transformation. Das Bild von  $A_1$  ist eine Gerade  $A'_1$ , durch den Schnittpunkt  $L_1$  des Kreises  $A'_2$  und der Gerade  $x$ , wo  $A'_2$  das Bild der Abstandslinie  $A_2$  bedeutet. Es folgt aus den Hilfsätzen I—II und den Sätzen 1—5, daß der Bildpunkt  $A'$  von  $A$  im Inneren des Kreises  $A'_2$  liegt und der Punkt  $B'$  außerhalb des Kreises  $A'_2$  ist, also in der Halbebene der **L**-Transformation haben die Strecke  $A'B'$  von  $A'_1$  und der Kreis  $A'_2$  einen Punkt gemein.

Wenn die Grundlinien  $l_1$  und  $l_2$  mit einander parallel sind, und im Bezug auf der Grundlinie  $l_1$  die Abstandslinie  $A_1$  und die Gerade  $l_2$  in gegenseitiger Halbebenen liegen, so ist es beweisbar, daß die Abstandslinie  $A_2$  einen Punkt im Inneren und einen Punkt im Äußeren der Abstandslinie  $A_1$  hat. Lassen wir ein Lot aus dem Punkt  $A$  auf der Grundlinie  $l_2$  fallen, und bezeichnen wir den Fußpunkt mit  $F$ . Die Gerade  $AF$  schneidet die Abstandslinie  $A_2$  in dem Punkt  $X$ , und die Grundlinie  $l_1$  in dem Punkt  $Z$ . Der Punkt  $Z$  liegt zwischen  $A$  und  $F$ , und der Punkt  $X$  wegen des Hilfsatzes I. liegt außerhalb der Abstandslinie  $A_1$ . Die Grundlinie  $l_1$  schneidet die Abstandslinie  $A_2$  in dem Punkt  $Y$ . Dieser Punkt liegt im Inneren der Abstandslinie  $A_1$ . (Diese Bemerkung ist notwendig, weil in diesem Falte die Rollen der Abstandslinien  $A_1$  und  $A_2$  nicht symmetrisch sind.)

Wenn die Grundlinien  $l_1$  und  $l_2$  sich schneiden, oder mit einander parallel sind und im Bezug auf der Grundlinie  $l_2$  die Abstandslinie  $A_2$  und die Gerade  $l_1$  in gegenseitiger Halbebene liegen, so gibt es immer eine Gerade  $x$ , die senkrecht auf  $l_1$  steht und mit der Geraden  $l_2$  ein gemeinsames Lot hat so, daß die Gerade  $x$  und die Abstandslinie  $A_2$  im Bezug auf  $l_2$  in gegenseitiger Halbebene liegen<sup>15</sup>.

<sup>14</sup> S. die Fußnote 7.

<sup>15</sup> S. die Fußnote 7.

Betrachten wir die **L**-Transformation, deren Achse die Gerade  $x$  ist. Das Bild von  $\Delta_2$  ist ein Zykel, der die  $x$ -Achse in zwei Punkten schneidet, und das Bild von  $\Delta_1$  ist eine Gerade. Nach dem Hilfsatz I. ist leicht beweisbar, daß der Bildpunkt  $A'$  von  $A$  im Äußeren und der Bildpunkt  $B'$  von  $B$  im Inneren des Zyklus  $\Delta'_2$  liegen. Wegen des Hilfsatzes II. hat die Strecke  $A'B'$  von  $\Delta'_1$  mit dem Zykel  $\Delta'_2$  einen Punkt gemein in der Halbebene der **L**-Transformation. Weil die **L**-Transformation ein-eindeutig ist, haben die Abstandslinie  $\Delta_1$  und  $\Delta_2$  einen Punkt gemein in allen betrachteten Fällen.

Wenn die Grundlinie  $l_1$  und  $l_2$  der Abstandslinie  $\Delta_1$  bzw.  $\Delta_2$  ein gemeinsames Lot haben, bezeichnen wir die Schnittpunkte des gemeinsamen Lotes  $x$  und der Abstandslinien  $\Delta_1$ ,  $\Delta_2$  bzw. der Geraden  $l_1$ ,  $l_2$  mit  $D_1$ ,  $D_2$  bzw.  $L_1$ ,  $L_2$ . Wenn die Abstandslinie  $\Delta_1$  und die Grundlinie  $l_2$  im Bezug auf  $l_1$  und die Abstandslinie  $\Delta_2$  und die Grundlinie  $l_1$  im Bezug auf  $l_2$  in derselben Halbebene liegen, so — abgesehen von den Fällen, die infolge unserer Annahmen trivialenweise ausgeschlossen sind — besteht  $L_1L_2 < L_1D_1 + L_2D_2$ . Wenn also  $L_1L_2 > L_1D_1$  und  $L_1L_2 > L_2D_2$  sind, ist die Anordnung von den Punkten  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$ :  $(L_1D_2D_1L_2)$ . Wenn eine von den Strecken  $L_1D_1$  und  $L_2D_2$ , z. B.  $L_1D_1 < L_1L_2$  ist, und die andere Strecke  $L_2D_2 > L_1L_2$ , ist die Anordnung der Punkte  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$ :  $(D_2L_1D_1L_2)$ ; zuletzt wenn  $L_1D_1 > L_1L_2$  und  $L_2D_2 > L_1L_2$  sind, ist die Anordnung der Punkte  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$ :  $(D_2L_1L_2D_1)$ . Bezeichnen wir die auf  $x$  senkrecht stehenden Halbgeraden durch  $d_1$  und  $d_2$  in der Halbebene der **L**-Transformation mit  $d_1$  und  $d_2$ . Das Bild der Abstandslinie  $\Delta_1$  ist eine mit der Halbgerade  $d_1$  parallele Halbgerade  $\Delta'_1$  durch  $L_1$ , und das Bild von  $\Delta_2$  ist eine mit der Halbgerade  $d_2$  parallele Halbgerade  $\Delta'_2$  durch  $L_2$ . Wenn die Anordnung der Punkte  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$  ( $L_1D_2D_1L_2$ ) ist, so schneidet die Halbgerade  $\Delta'_1$  die Halbgerade  $d_2$  in einem Punkt  $V$  und die Halbgerade  $\Delta'_2$  die Halbgerade  $d_1$  in einem Punkt  $U$ . Die Halbgeraden  $\Delta'_1$  und  $\Delta'_2$  haben einen Punkt gemein, was man — wie vorher — beweisen kann.

Wenn die Anordnung von  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$  ( $D_2L_1D_1L_2$ ) oder ( $D_2L_1L_2D_1$ ) ist, so liegt die Halbgerade  $\Delta'_1$  im Inneren des durch die Halbgerade  $L_1L_2$  und die mit der  $d_2$  parallele Halbgerade durch  $L_1$  bestimmten Winkels, also  $\Delta'_1$  und  $\Delta'_2$  schneiden sich.

Wenn die Abstandslinie  $\Delta_1$  und  $\Delta_2$  in derselben Halbebene im Bezug auf einer der beiden Grundlinien — z. B. auf  $l_1$  — liegen, aber die Abstandslinie  $\Delta_2$  und die Grundlinie  $l_1$  in gegenseitigen Halbebenen im Bezug auf der Grundlinie  $l_2$  sind, so ist die Anordnung von  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$ :  $(D_1D_2L_2L_1)$ .

Es ist unmöglich, daß die Punkte  $D_1$  und  $D_2$  zusammenfallen, weil die zur  $l_1$  zugehörige Abstandslinie  $\Delta_1^*$  durch  $D_1 = D_2$  in diesem Falle im Inneren der Abstandslinie  $\Delta_2$  liegt. Betrachten wir einen beliebigen Punkt  $T$  von  $\Delta_1^*$ , der nicht in derselben Halbebene wie die Gerade  $l_1$  im Bezug auf  $l_2$  liegt. Bezeichnen wir den Fußpunkt von dem Lot aus  $T$  auf  $l_1$  mit  $U$  und den Schnittpunkt der Geraden  $l_2$  und dieses Lotes mit  $V$ . Es folgt, daß  $TU = D_1L_1 = TV + VU$  ist, wo  $VU > L_1L_2$  ist. Wir bezeichnen den Fußpunkt von dem Lot aus  $T$  auf  $l_2$  mit  $W$ , so ist  $TW < TV$  und folglich  $TW < D_1L_2$ . Die Anordnung von  $L_1$ ,  $L_2$ ,  $D_1$ ,  $D_2$  kann nicht ( $D_2D_1L_2L_1$ ) sein, nämlich die Abstandslinie  $\Delta_1$  im Inneren der Abstandslinie  $\Delta_1^{**}$  liegt, wo  $\Delta_1^{**}$  die Abstandslinie durch  $D_1$  mit Grundlinie  $l_2$  ist. Es folgt aus dem Hilfsatz I., daß die Abstandslinie  $\Delta_1$  im Inneren der Abstandslinie  $\Delta_2$  liegt.

Betrachten wir die **L**-Transformation, deren Achse die Gerade  $x$  ist. Das Bild der Abstandslinie  $\Delta_1$  ist eine Halbgerade  $\Delta'_1$ , die die Halbgerade  $d_2$  in einem Punkt  $L_2^*$  schneidet. Die entsprechende Halbgerade  $\Delta'_2$  der Abstandslinie  $\Delta_2$  und die Halbgerade  $\Delta'_1$  haben einen gemeinsamen Punkt; diese Tatsache folgt, wenn wir das Axiom von Pasch für das Dreieck  $D_2L_1L_2^*$  und die Gerade  $\Delta'_2$  verwenden.

Da die **L**-Transformation ein-eindeutig ist, haben die Abstandslinien  $\Delta_1$  und  $\Delta_2$  einen Punkt gemein.

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## A REMARK ON ADDITIVE ARITHMETICAL FUNCTIONS

By

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To Professor F. KÁRTESZI on his 60 th birthday

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A function of a positive integer  $f(n)$  is said to be restrictedly additive (or simply additive) if  $(n_1, n_2) = 1$  implies  $f(n_1 n_2) = f(n_1) + f(n_2)$ . If this equation is satisfied for any pair of integers  $n_1$  and  $n_2$ , then we say that  $f(n)$  is completely additive.

ERDŐS [1] has proved the following two assertions.

(A) If  $f(n)$  is restrictedly additive and monotonic, then it is a constant multiple of  $\log n$ .

(B) If  $f(n)$  is restrictedly additive and if  $f(n+1) - f(n) = o(1)$ , then it is a constant multiple of  $\log n$ .

New proofs of these assertions have been given by MOSER and LAMBEK [2], SCHOENBERG [3], A. RÉNYI [4], BESICOVITCH [5], VERA T. SÓS [6] and CSÁSZÁR [7]. P. TURÁN [8] obtained an important application of these assertions to the characterization of Dirichlet's  $L$ -functions.

Using the ideas of BESICOVITCH [5] to the proof of (B) we prove the following.

**THEOREM.** Let  $\epsilon(n)$  be a sequence tending to zero as  $n \rightarrow \infty$ . If  $f(n)$  is restrictedly additive and

$$f(n+1) - f(n) \geq -\epsilon(n),$$

then it is a constant multiple of  $\log n$ .

It is evident that from our Theorem follow (A) and (B).

**PROOF.** Without loss of generality we can assume that  $\epsilon(n)$  is a non-negative sequence tending to zero monotonically.

Firstly we prove, that  $f(n)$  is completely additive. Let  $p$  be an arbitrary prime or a power of a prime and  $\epsilon > 0$ . There exists an integer  $l$  such that

$$(2) \quad f(n+1) - f(n) \geq -\epsilon \text{ for } n \geq p^{l-1}.$$

We take  $k$  large and use the inequality (2) a few times. So we have

$$\begin{aligned} f(p^k) &\leq f(p^k + p) + p\varepsilon = f(p) + f(p^{k-1} + 1) + p\varepsilon \leq \\ &\leq f(p) + f(p^{k-1} + p) + p\varepsilon + (p-1)\varepsilon \leq \dots \leq (k-l)f(p) + f(p^l + 1) + (k-l)p\varepsilon. \end{aligned}$$

Similarly

$$f(p^k) \geq f(p^k - p) - p\varepsilon \geq \dots \geq (k-l)f(p) + f(p^l - 1) - (k-l)p\varepsilon.$$

From these inequalities follows, that

$$\lim_{k \rightarrow \infty} \frac{f(p^k)}{k} = f(p).$$

Using this relation for  $p = q^v$  and  $p = q$ , where  $q$  is an arbitrary prime, we obtain that

$$f(q^v) = \lim_{k \rightarrow \infty} \frac{f(q^{vk})}{k} = v \lim_{vk \rightarrow \infty} \frac{f(q^{vk})}{vk} = vf(q),$$

that is,  $f(n)$  is completely additive.

Let now  $p$  be a prime. We take  $n$  large and write it in the form

$$n = a_0 p^v + \dots + a_{v-1} p + a_v$$

with  $v > l$ , and  $1 \leq a_0 < p$ ,  $0 \leq a_i < p$  for  $i = 1, 2, \dots, v$ .

Using the inequality (2) we have

$$\begin{aligned} f(n) &\geq f(a_0 p^v + \dots + a_{v-1} p) - \varepsilon p = f(p) + f(a_0 p^{v-1} + \dots + a_{v-1}) - \varepsilon p \geq \\ &\geq \dots \geq (v-l+1)f(p) + f(a_0 p^{l-1} + \dots + a_{l-1}) - \varepsilon(v-l+1)p. \end{aligned}$$

Writing

$$\max_{m < p^l} |f(m)| = M,$$

we have

$$f(n) \geq (v-l+1)f(p) - M - \varepsilon(v-l+1)p.$$

Observing that

$$p^v \leq n < p^{v+1},$$

we obtained that

$$\lim_{n \rightarrow \infty} \frac{\nu}{\log n} = \frac{1}{\log p}.$$

Hence

$$(3) \quad \lim_{n \rightarrow \infty} \frac{f(n)}{\log n} \geq \frac{f(p)}{\log p}.$$

Suppose that there exist prime numbers  $p, q$  such that for example  $\frac{f(q)}{\log q} < \frac{f(p)}{\log p}$ . Then choosing  $n = q^n$  we obtain from [3] that

$$\frac{f(q)}{\log q} = \lim_{n \rightarrow \infty} \frac{f(q^n)}{\log q^n} \geq \frac{f(p)}{\log p},$$

which is a contradiction.

So  $\frac{f(p)}{\log p}$  is a constant for every prime and the theorem is proved.

REMARK. As Prof. P. TURÁN communicated to me after completing my manuscript our theorem was stated without proof by P. ERDŐS in [9].

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## NOTE ON THE REPRESENTATION OF PARTIALLY ORDERED GROUPS

By

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In his paper [3] CH. HOLLAND proved that every lattice-ordered group is  $\sigma$ -isomorphic to a subgroup of the group of all monotone permutations of some fully ordered set. Later, I gave a necessary and sufficient condition for groups to be  $\sigma$ -isomorphic to a subgroup of the group of monotone permutations of some fully ordered set [1]. In paper [1] I did not prove the fulfilment of this condition for lattice-ordered groups.

The aim of this paper is to establish another condition. Both the condition and the proof are simpler than the previous one. We can show easily that this condition holds for lattice-ordered groups.

Let  $S$  be a fully ordered set and let  $\leqq$  denote the order-relation on  $S$ . We shall call a permutation  $\alpha$  of  $S$  *monotone* if  $a \leqq b$  ( $a, b \in S$ ) implies  $\alpha^a \leqq \alpha^b$ . The monotone permutations of  $S$  form a subgroup  $\bar{S}$  of the group of all permutations of  $S$ . The relation:

" $\alpha \leqq \beta$  if and only if, for each  $a \in S$ ,  $\alpha^a \leqq \beta^a$  holds"

is obviously an order on  $\bar{S}$ . It is easy to see that  $\bar{S}$  is a lattice-ordered group with respect to this order relation. We shall call a group  $\sigma$ -isomorphic to a subgroup of some  $\bar{S}$  *monotone*.

**THEOREM 1.** *A partially ordered group  $G$  is monotone if and only if  $G$  contains a family of subsemigroups  $T_\lambda$  such that<sup>1,2</sup>:*

- (i)  $\bigcap_{\lambda} T_\lambda = P$ , the positive cone of  $G$ ,
- (ii)  $T_\lambda \cup T_\lambda^{-1} = G$ , for each  $\lambda$ .

<sup>1</sup> The subsemigroups  $T_\lambda$  are convex, because of  $P \subseteq T_\lambda$ .

<sup>2</sup> If, in addition, these are normal subsemigroups, we get just the condition for  $G$  to be a vector group (See [2] p. 42). Hence, all vector groups are monotone.

**PROOF.** Let first  $G$  be monotone, i.e.  $G \subseteq \bar{S}$  for some fully ordered set  $S$ . Further, for each  $a \in S$  we define  $T_a \subseteq G$  as follows:  $\alpha \in T_a$  if and only if  $a^\alpha \geq a$ . Let  $\alpha \in P$ ; then  $a^\alpha \geq a^\sigma \geq a$  because of  $\alpha \geq \sigma$  i.e.  $T_\alpha \supseteq P$ . If, for each  $a \in S$ ,  $\alpha \in T_a$ , i.e.  $a^\alpha \geq a$  holds for every  $a \in S$  then  $\alpha \geq \varepsilon$ , proving (i). Now,  $\alpha \notin T_a$  means  $a^\alpha < a$  whence  $a = (a^\alpha)^{\alpha^{-1}} < a^{\alpha^{-1}}$  follows. Hence  $\alpha^{-1} \in T_a$  and (ii) is fulfilled. It remains to prove the  $T_a$ 's are subsemigroups. Let  $\alpha, \beta \in T_a$ ; thus  $a^\alpha \geq a$  and  $a^\beta \geq a$ . We get  $a^{\alpha\beta} \geq a^\beta \geq a$ , i.e.  $\alpha\beta \in T_a$ .

Conversely, let us suppose that there exist subsemigroups  $T_\lambda$  in  $G$  satisfying the stated properties. Without loss of generality we may regard the set of indices  $\lambda$  to be ordered. Clearly,  $G_\lambda = T_\lambda \cap T_{\lambda^{-1}}$  is a subgroup of  $G$ . Let the set  $S$  consist of the right cosets  $G_\lambda\alpha$  ( $\alpha \in G$  and for all  $\lambda$ ) and put

$$(iii) \quad G_\lambda\alpha \leq G_\mu\beta \text{ if either } \lambda < \mu \text{ or } \lambda = \mu \text{ and } \beta\alpha^{-1} \in T_\lambda.$$

We have first to prove that this relation is independent of the representatives of the cosets. This is, for  $\lambda \neq \mu$ , obvious. If  $\alpha' = \sigma\alpha$  and  $\beta' = \tau\beta$  with  $\sigma, \tau \in G_\lambda$  then  $\sigma^{-1} \in G_\lambda$  and so  $\tau, \sigma^{-1} \in T_\lambda$  follows. Since  $G_\lambda\alpha \leq G_\lambda\beta$  means  $\beta\alpha^{-1} \in T_\lambda$ , we get  $\beta'(\alpha')^{-1} = \tau\beta(\sigma\alpha)^{-1} = \tau(\beta\alpha^{-1})\sigma^{-1} \in T_\lambda$ , i.e.  $G_\lambda\alpha' \leq G_\lambda\beta'$ .

We prove that the relation defined under (iii) is a full order. Reflexivity, antisymmetry and transitivity are obvious. It is also obvious that for  $\lambda \neq \mu$  either  $G_\lambda\alpha \leq G_\mu\beta$  or  $G_\mu\beta \leq G_\lambda\alpha$  holds. If  $G_\lambda\alpha \leq G_\lambda\beta$  does not hold then  $\beta\alpha^{-1} \in T_\lambda$  and thus, by (ii),  $\alpha\beta^{-1} = (\beta\alpha^{-1})^{-1} \in T_\lambda$ ; hence  $G_\lambda\beta \leq G_\lambda\alpha$ .

Now, let us correspond to each  $\sigma \in G$  the mapping  $\bar{\sigma}$  of  $S$  into itself, defined by

$$(iv) \quad (G_\lambda\alpha)^{\bar{\sigma}} = G_{\lambda}\alpha\sigma.$$

$G$  being a group  $\bar{\sigma}$ , it is clearly, a permutation of  $S$ , moreover, it is a monotone permutation. In the case  $\lambda < \mu$  we get from  $G_\lambda\alpha \leq G_\mu\beta$  that  $(G_\lambda\alpha)^{\bar{\sigma}} = G_\lambda\alpha\sigma \leq G_\mu\beta\sigma = (G_\mu\beta)^{\bar{\sigma}}$  and from  $G_\lambda\alpha \leq G_\lambda\beta$  it follows  $(\beta\sigma)(\alpha\sigma)^{-1} = \beta\alpha^{-1} \in T_\lambda$ , that means  $(G_\lambda\alpha)^{\bar{\sigma}} \leq (G_\lambda\beta)^{\bar{\sigma}}$ .

Next we prove that the elements of the form  $\bar{\sigma}$  form a subgroup  $\bar{G}$  of  $\bar{S}$   $\sigma$ -isomorphic to  $G$ .

1°  $\bar{\sigma}\bar{\tau} = \bar{\sigma}\bar{\tau}$  because of  $(G_\lambda\alpha)(\sigma\tau) = ((G_\lambda\alpha)\sigma)\tau$ .

2°  $\sigma \geq \varepsilon$  implies  $\bar{\sigma} \geq \bar{\varepsilon}$  ( $\bar{\varepsilon}$  is the identity of  $\bar{S}$ ). Indeed, from  $\sigma \in P$  it follows  $(\alpha\sigma)\alpha^{-1} = \alpha\sigma\alpha^{-1} \in P \subseteq T_\lambda$  for each  $T_\lambda$ , thus  $(G_\lambda\alpha)^{\bar{\sigma}} = G_\lambda\alpha \leq G_\lambda\alpha\sigma = (G_\lambda\alpha)^{\bar{\sigma}}$ .

3° If  $\bar{\sigma} \geq \bar{\varepsilon}$  then  $\sigma \geq \varepsilon$ . Indeed, from  $\bar{\sigma} \geq \bar{\varepsilon}$  we obtain  $G_\lambda\alpha \leq G_\lambda\alpha\sigma$  for each  $\alpha \in G$ . In the special case  $\alpha = \varepsilon$  we get  $G_\lambda \leq G_\lambda\sigma$ , i.e.  $\sigma$  is contained in every  $T_\lambda$ , thus  $\sigma \in P$  by (i).

4° If  $\bar{\sigma} = \bar{\tau}$  then  $\sigma = \tau$ . This is an obvious consequence of 3°.

1° and 4° prove that the mapping  $\sigma \leftrightarrow \bar{\sigma}$  is a group-isomorphism between  $G$  and  $\bar{G}$  while 2° and 3° guarantee that this mapping is an  $\sigma$ -isomorphism. This completes the proof of Theorem 1.

In the rest of this paper we shall prove

**THEOREM 2.** *Every lattice-ordered group is monotone.*

In the proof of Theorem 2 we shall need a lemma.

Let  $G$  be a lattice-ordered group with the positive cone  $P$ , and let  $C$  be an  $l$ -subsemigroup (i.e. sublattice-subsemigroup) of  $G$  containing  $P$ . Let further  $(C, y)$  denote the  $l$ -subsemigroup generated by  $C$  and  $y$  and  $C(y)$  the set of all  $t \in G$  for which there exist a natural integer  $n$  and an element  $u$  of  $C$  such that  $(yu)^n \leq t$ .

LEMMA. For  $y < e$  we have  $(C, y) = C(y)$ .

PROOF.  $(C, y)$ ; just as  $C$ , is convex because it contains  $P$ , while  $C(y)$  is convex, by definition. For every  $u \in C$  and natural integer  $n$ ,  $(yu)^n \in (C, y)$  and, by the convexity of  $(C, y)$ ,  $C(y) \subseteq (C, y)$ . In order to prove  $(C, y) \subseteq C(y)$ , we show that  $y \in C(y)$ ,  $C \subseteq C(y)$  and  $C(y)$  is an  $l$ -subsemigroup of  $G$ . Using  $e \in C$ ,  $ye \leq y$  proves  $y \in C(y)$  while from  $u \in C$  it follows, by  $y < e$ , that  $yu \leq u$  i.e.  $u \in C(y)$ . Now, let  $t_1, t_2 \in C(y)$ . It is obvious, by the convexity of  $C(y)$ , that  $t_1 \cup t_2 \in C(y)$ . By our assumption, there are elements  $u_1, u_2$  in  $C$  and natural integers  $n_1, n_2$  such that  $(yu_i)^n \leq t_i$ , for  $i = 1, 2$ . Let  $u = e \cap u_1 \cap u_2$  and  $n = \max(n_1, n_2)$ . Since  $C$  is a sublattice containing  $e$ , therefore  $u \in C$  also holds. From  $u \leq e$  we get  $(yu)^n \leq t_1, t_2$ . This means  $(yu)^n \leq t_1 \cap t_2$  and  $(yu)^{2n} \leq t_1 \cdot t_2$  which completes the proof of the Lemma.

PROOF OF THEOREM 2. Let  $a \in G$  and  $a \notin P$ . Then  $P$  is an  $l$ -subsemigroup containing  $P$  but excluding  $a$ . By Zorn's Lemma there exists an  $l$ -subsemigroup  $T_a \supseteq P$  which is maximal with respect to the property of not containing  $a$ . If we let  $a$  run over all  $a \notin P$ , then for these subsemigroups  $T_a$  (i) obviously holds. To prove (ii), let us suppose that both  $x \in T_a$  and  $x^{-1} \notin T_a$  are valid. Hence, by  $P \subseteq T_a$ , neither  $e \cap x$  nor  $e \cap x^{-1}$  belongs to  $T_a$ . Therefore, by the maximality of  $T_a$ , both  $a \in (T_a, e \cap x)$  and  $a \in (T_a, e \cap x^{-1})$  hold. In view of the Lemma, there are elements  $u_1, u_2$  in  $T_a$  and natural integers  $n_1, n_2$  such that  $((e \cap x)u_1)^{n_1} \leq a$  and  $((e \cap x^{-1})u_2)^{n_2} \leq a$ . By our assumption on  $T_a$ ,  $u = e \cap u_1 \cap u_2 \in T_a$  and, clearly,  $u \leq e$ . Then, we get  $((e \cap x)u)^n \leq a$  and  $((e \cap x^{-1})u)^n \leq a$ , where  $n = \max(n_1, n_2)$ . Now  $z = [(e \cap x)u] \cup [(e \cap x^{-1})u]]^{2n-1}$  is a union of products in which either  $(e \cap x)u$  or  $(e \cap x^{-1})u$  occurs at least  $n$  times. However, this means that each of these products is either  $\leq ((e \cap x)u)^n$  or  $\leq ((e \cap x^{-1})u)^n$  because the other factors are less than or equal to  $e$ . Thus  $z \leq a$ . Further, from  $(e \cap x)U(e \cap x^{-1}) = e$  we get  $[(e \cap x)u] \cup [(e \cap x^{-1})u] = [(e \cap x) \cup (e \cap x^{-1})]u = u$  i.e.  $u^{2n-1} = z \leq a$ . Hence  $a \in T_a$ , a contradiction. Therefore, either  $x \in T_a$  or  $x^{-1} \in T_a$ , and (ii) is proved<sup>3</sup>.

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<sup>3</sup> Mr. A. BIGARD pointed out to me that the isomorphism, given in [3], is a lattice-isomorphism too. One can prove, that the isomorphism, defined in Theorem 1., is a lattice-isomorphism too, for lattice-ordered groups, because the subsemigroups, in Theorem 2., are  $l$ -subsemigroups.



# ON THE SUM OF DIGITS OF PRIME NUMBERS

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To Professor F. KÁRTESZ on his 60 th birthday

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## 1. Introduction

Let  $k > 1$  be a fixed positive integer. Then we can use any integer  $n$  to be uniquely represented as follows

$$(1) \quad n = a_1 k^{n_1} + a_2 k^{n_2} + \dots + a_t k^{n_t},$$

where  $n_1 > n_2 > \dots > n_t \geq 0$  are integers; so are  $a_1, a_2, \dots, a_t$  each of which no larger than  $k-1$ . We set  $\alpha(n) = \sum_{i=1}^t a_i$  and  $A(x) = \sum_{n \leq x} \alpha(n)$ .

In the case of  $k=2$  R. BELLMAN and H. SHAPIRO proved the following relation [1]:

$$(2) \quad A(x) = \frac{x \log x}{2 \log 2} + O(x \log \log x).$$

S. C. TANG has extended their result to the general case [2], i.e. he proved that

$$(3) \quad A(x) = \frac{k-1}{2} \frac{x \log x}{\log k} + O(x).$$

It is natural to ask that what is the asymptotical behavior of

$$(4) \quad B(x) = \sum_{p \leq x} \alpha(p),$$

where in the sum  $p$  runs through the prime numbers.

In this paper we shall prove that assuming the validity of the density hypothesis for the Riemann's zeta function we have

$$(5) \quad B(x) = \frac{k-1}{2} \frac{x}{\log k} + O\left(\frac{x}{(\log \log x)^{1/3}}\right), \quad \text{as } x \rightarrow \infty.$$

In a forthcoming paper we shall investigate with MR. MOGYORÓDI the limit distribution of the values of  $\alpha(p)$ . It seems difficult to prove the existence of the asymptotic of  $\alpha(p)$  without any conjectures.

## 2. Lemmas and a correction to my paper [3]

Let

$$A(n) = \begin{cases} \log p, & \text{when } n = p^r, p \text{ prime, } r = 1, 2, \dots, \\ 0 & \text{otherwise,} \end{cases}$$

$\xi(s)$  — the zeta function of Riemann,

$$(5) \quad \psi(x) = \sum_{n \leq x} A(n), \quad \pi(x) = \sum_{p \leq x} 1,$$

where  $p$  runs through the primes,

$$(6) \quad \Delta_K(x) = \psi(x) - \psi(x-K-1) - (K+1),$$

$$(7) \quad \varrho_K(x) = \pi(x+K) - \pi(x) - \frac{K}{\log x},$$

where  $K$  is a positive number.

Let further  $N(\sigma_0, T)$  denote the number of the roots of  $\xi(s)$  in the rectangle  $\sigma_0 \leq \sigma \leq 1, |t| \leq T$  ( $s = \sigma + it$ ).

In [3] I asserted the following:

Let  $h(x)$  a monoton non-decreasing function of  $x$ , which tends to infinity as  $x \rightarrow +\infty$  and which satisfies the relation

$$(8) \quad 1 \leq h(x) = O(x^{1/4}),$$

and let

$$(9) \quad K = \Phi(x) = (\log x)^{7.5} h(x).$$

Let further  $m(x)$  be the Lebesgue's measure of those  $y$ -s in the interval  $1 \leq y \leq x$ , for which the interval  $(y, y+K)$  does not contain primes. If

$$(10) \quad N(\sigma, T) = O(T^{2(1-\sigma)} \log^2 T) \quad \text{for } \frac{1}{2} \leq \sigma \leq 1, \quad 1 \leq T < \infty,$$

then

$$m(x) = O\left(\frac{x}{h(x)}\right).$$

Indeed I proved this assertion assuming the inequality

$$(8') \quad 1 \leq h(x) = O(\log x)$$

instead of (8), only.

More accurately I proved that

$$(11) \quad \sum_{n \leq x} A_K^2(n) = O\left(\frac{K^2 x}{h(x)}\right),$$

if (8') is satisfied ( $K$  as in (9)). (See [3] p. 64.)

From this it follows easily that

$$(12) \quad \sum_{n \leq x} |\varrho_K(n)|^2 = O\left(\frac{K^2 x}{h(x) \log^2 x}\right),$$

which we shall formulate as

**LEMMA 1.** Let  $h(x)$  be a function of  $x$  tending monotonically to infinity as  $x \rightarrow +\infty$ , and satisfying  $h(x) = O(\log x)$ . Let  $K = (\log x)^{7.5} h(x)$ . Then assuming (10) we have (12).

**LEMMA 2** [4]. For  $x \geq 1, y \geq 1$  we have

$$\pi(x+y) - \pi(x) < c \frac{y}{\log y},$$

where  $c$  is an absolute constant.

### 3. Formulation and proof of the Theorem

**THEOREM.** Assuming that the density hypothesis is true in the form

$$(13) \quad N(\sigma, T) = O(T^{2(1-\sigma)} \log^2 T) \quad \text{for} \quad \frac{1}{2} \leq \sigma \leq 1,$$

we have

$$(14) \quad B(x) = \frac{k-1}{2} \frac{x}{\log k} + O\left(\frac{x}{(\log \log x)^{1/3}}\right).$$

**PROOF.** In the sequel  $c_1, c_2, \dots$  denote constants depending on  $k$  only. From (1) it follows evidently, that for any  $n \leq y$  the inequality

$$(15) \quad \alpha(n) \leq c_1 \log y$$

is satisfied.

Let  $l$  be a natural number satisfying the inequalities

$$(16) \quad K \leq k^l \leq kK,$$

where  $K$  is as in (9).

Let further  $A_j$  denote the set of integers in the interval

$$[k^l j, k^l(j+1)), \quad \text{for } j = 0, 1, \dots, j_0,$$

where

$$(17) \quad j_0 = \left\lceil \frac{x}{k^l} \right\rceil.$$

It is trivial that these sets are disjoint and their union contains any natural number smaller than  $x$ .

Further, if  $n \in A_j$ , then

$$\alpha(j) \leq \alpha(n) \leq \alpha(j) + (k-1)l.$$

So

$$B(x) = \sum_{p \leq x} \alpha(p) = \sum_{j=0}^{j_0} \sum_{p \in A_j} \alpha(p) - \sum_{\substack{p > x \\ p \in A_{j_0}}} \alpha(p).$$

Using the inequalities (15) and (16) we have

$$\sum_{\substack{p > x \\ p \in A_{j_0}}} \alpha(p) < c_1 \log 2x \cdot k^l < c_2 K \log x,$$

and so

$$B(x) = \sum_{j=0}^{j_0} \alpha(j) (\pi(k^l(j+1)) - \pi(k^l j)) + O(l \pi(2x)) + O(K \log x).$$

From (8') and (9) it follows that  $K = O((\log x)^{8/5})$ , and hence

$$l = O(\log \log x).$$

So the second and third term on the right hand side are

$$O\left(\frac{x}{\log x} \log \log x\right),$$

where the  $O$  depends on  $k$  only.

We write the first sum in the form

$$\sum_1 + \sum_2 = \sum_{j=0}^{j_0} \alpha(j) \frac{k^l}{\log x} + \sum_{j=0}^{j_0} \alpha(j) \varrho_{k^l}(k^l j),$$

and we have

$$B(x) = \sum_1 + \sum_2 + O\left(\frac{x \log \log x}{\log x}\right).$$

From the result of TANG [2] (see (2)) we obtain

$$\sum_1 = \frac{k^l}{\log x} A(j_0) = \frac{k-1}{2} \frac{x}{\log k} + O\left(\frac{x \log \log x}{\log x}\right).$$

For the proof we need to prove the inequality

$$\sum_2 = O\left(\frac{x}{(\log \log x)^{1/3}}\right)$$

only.

Let now  $A$  be a natural number  $< k^l$ . For the integers  $u$  in  $1 \leq u \leq A$  we have

$$|\varrho_{k^l}(n+u) - \varrho_{k^l}(n)| \leq \pi(n+k^l+A) - \pi(n+k^l) + \pi(n+A) - \pi(n).$$

Further using Lemma 2, the right hand side is

$$\leq c_3 \frac{A}{\log A}, \quad \text{if } n = x.$$

Hence it follows that

$$|\varrho_{k^l}(k^l j)| \leq \frac{1}{A+1} \sum_{u=0}^A |\varrho_{k^l}(k^l j + u)| + c_3 \frac{A}{\log A},$$

and so

$$\sum_3 = \sum_{j=0}^{j_0} |\varrho_{k^l}(k^l j)| \leq \frac{1}{A+1} \sum_{j=0}^{j_0} \sum_{u=0}^A |\varrho_{k^l}(k^l j + u)| + O\left(j_0 \frac{A}{\log A}\right)$$

holds. Since  $A < k^l$ , so a natural number  $n$  is represented in the form  $n = k^l j + u$  ( $j = 0, \dots, j_0$ ;  $u = 0, \dots, A$ ) at most once, and the number of represented numbers equals to  $(A+1)(j_0+1)$ . Using the Hölder-inequality and Lemma 1 we have that the double sum is smaller than

$$\begin{aligned} \frac{1}{A+1} \left\{ \sum_{n=k^l j+u} 1 \right\}^{1/2} \left\{ \sum_{n=2x} |\varrho_{k^l}(n)|^2 \right\}^{1/2} &= O\left(\frac{1}{A} \left(\frac{x A}{k^l}\right)^{1/2} \left(\frac{K^2 x}{h(x) \log^2 x}\right)^{1/2}\right) = \\ &= O\left(\frac{x}{\log x} \left(\frac{K}{A h(x)}\right)^{1/2}\right). \end{aligned}$$

Further

$$O\left(j_0 \frac{A}{\log A}\right) = O\left(\frac{x}{K} \frac{A}{\log A}\right).$$

Using the inequality

$$\sum_2 = O(\log x \cdot \sum_3),$$

we have

$$\sum_2 = O\left(x \left(\frac{K}{A h(x)}\right)^{1/2}\right) + O\left(\frac{Ax}{K} \frac{\log x}{\log A}\right).$$

Let now choose  $h(x) = \log x$ ,  $A = \frac{K(\log \log x)^{2/3}}{\log x}$ , hence it follows that

$$\sum_2 = O\left(\frac{x}{(\log \log x)^{1/3}}\right),$$

and the Theorem is proved.

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# STRUCTURAL PROPERTIES OF CONTINUOUS FUNCTIONS CONNECTED WITH THE ORDER OF RATIONAL APPROXIMATION, I

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## § 1. Introduction

An important topic of the constructive theory of functions is the investigation of structural properties of continuous functions. Namely, what can be stated on a continuous function if we know the order of some kind of approximation? The well-known theorems of S. N. BERNSTEIN solved the problem in connection with the polynomial approximation. As far as I know the first results in the direction of the rational approximation are due to A. A. GONČAR. He extended the problem to functions defined on an arbitrary finite perfect set, and proved structural properties on it — apart from a set of measure 0. But these theorems are not too applicable for practical purposes. Such practical purposes were expounded by P. SzÜSZ and P. TURÁN [1]. They raised the problem of characterisation of functions *without an exceptional set*, and proposed to investigate the role of the minimal distance between the considered interval and the poles of the approximating rational function. It was a deep insight which proves to be very useful in the following.

Let

$$z_k = \alpha_k \pm \beta_k i \quad (\beta_k \geq 0; k = 1, 2, \dots, m; m \leq n)$$

be the poles of the rational function  $R_n(x)$  (the suffix in  $R_n(x)$  always denotes the degree of  $R_n(x)$ ), and

$$E_1 = [-1, +1], \quad E_2 = \text{int } E_1 = (-1, +1), \quad E_3 = (-\infty, +\infty).$$

Define

$$(1.1) \quad \delta(R_n) = \min (1; \min_k \min_{x \in E_1} |x - z_k|) = \min (1; \min_{|z_k| \leq 1} \beta_k; \min_{|z_k| > 1} \sqrt{(|\alpha_k| - 1)^2 + \beta_k^2})$$

and

$$(1.2) \quad A(R_n) = \min_k \min_{x \in E_3} |x - z_k| = \min_k \beta_k.$$

Notice that if  $R_n(x)$  is especially a polynomial then

$$(1.3) \quad \delta(R_n) = 1.$$

Now let  $\{R_n(x)\}_{n=0}^{\infty}$  be a sequence of approximating rational functions to a continuous function  $f(x)$ ; i.e.

$$(1.4) \quad R_n(f; E_i) \stackrel{\text{def}}{=} \max_{x \in E_i} |f(x) - R_n(x)| \rightarrow 0 \quad (n \rightarrow \infty, i = 1 \text{ or } 3).$$

Without loss of generality we may assume

$$(1.5) \quad R_{n+1}(f; E_i) \leq R_n(f; E_i) \quad (n = 0, 1, \dots; i = 1 \text{ or } 3).$$

## § 2. Theorems and remarks

In the following let  $p \geq 1$  be real and

$$q = \begin{cases} 0 & \text{if } p \text{ integer} \\ [p] & \text{otherwise.} \end{cases}$$

We state the following theorems, using the previous notations.

**THEOREM 1.** If

$$(2.1) \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) \quad \text{finite}$$

and with arbitrary but fixed  $p > 1$  and  $\varepsilon > 0$  the condition

$$(2.2) \quad R_n(f; E_3) = O\left(\frac{\min(\Delta(R_n); \min_{0 \leq i \leq q} \Delta(R_{[np]+i}))}{n^{1+\varepsilon}}\right)$$

holds then  $f(x)$  is differentiable in  $E_3$ .

**THEOREM 2.** If with arbitrary but fixed  $p > 1$  and  $\varepsilon > 0$  the condition

$$(2.3) \quad R_n(f; E_1) = O\left(\frac{\min(\delta(R_n); \min_{0 \leq i \leq q} \delta(R_{[np]+i}))}{n^{1+\varepsilon}}\right)$$

holds then  $f(x)$  is differentiable in  $E_2$ .

**THEOREM 3.** The conditions of the previous theorems essentially cannot be weakened. Namely, these conditions with  $p = 1$  and  $\varepsilon = 0$  even in the stronger form

$$(2.4) \quad R_n(f; E_3) = o\left(\frac{\Delta(R_n)}{n}\right) \text{ and } R_n(f; E_1) = o\left(\frac{\delta(R_n)}{n}\right)$$

do not ensure the differentiability of  $f(x)$  in  $E_3$  and  $E_2$ , respectively.

**THEOREM 4.** If (2.1) holds,  $0 < \alpha \leq 1$ , and with arbitrary but fixed  $p > 1$ ,  $\varepsilon > 0$

$$(2.5) \quad R_n(f; E_3) = O\left(\frac{[\min(\Delta(R_n); \min_{0 \leq i \leq q} \Delta(R_{[np]+i}))]^{\alpha}}{n^{\alpha+\varepsilon}}\right)$$

then  $f(x) \in \text{Lip } \alpha$  in  $E_3$ .

**THEOREM 5.** If  $0 < \alpha \leq 1$  and with arbitrary but fixed  $p > 1$  and  $\varepsilon > 0$  the condition

$$(2.6) \quad R_n(f; E_1) = O\left(\frac{[\min(\delta(R_n); \min_{0 \leq i \leq q} \delta(R_{[np]+i}))]^{\alpha}}{n^{\alpha+\varepsilon}}\right)$$

holds then  $f(x) \in \text{Lip } \alpha$  in every closed subinterval of  $E_1$ .

**REMARKS.** 1. If

$$\Delta(R_{n+1}) \leq \Delta(R_n) \quad \text{or} \quad \delta(R_{n+1}) \leq \delta(R_n) \quad (n = 0, 1, \dots)$$

(and this is practically the most important case) then the conditions (2.2), (2.3), (2.5), (2.6) can be replaced by

$$(2.2)^* \quad R_n(f; E_3) = O\left(\frac{\Delta(R_{[np]+1})}{n^{1+\varepsilon}}\right)$$

$$(2.3)^* \quad R_n(f; E_1) = O\left(\frac{\delta(R_{[np]+1})}{n^{1+\varepsilon}}\right)$$

$$(2.5)^* \quad R_n(f; E_3) = O\left(\frac{\Delta(R_{[np]+1})^{\alpha}}{n^{\alpha+\varepsilon}}\right)$$

$$(2.6)^* \quad R_n(f; E_1) = O\left(\frac{\delta(R_{[np]+1})^{\alpha}}{n^{\alpha+\varepsilon}}\right).$$

2. If  $R_n(x)$  are especially polynomials then by (1.1) we have e.g. from (2.3)

$$R_n(f; E_1) = O\left(\frac{1}{n^{1+\varepsilon}}\right)$$

which essentially gives the corresponding BERNSTEIN's theorem, apart from the fact that Theorem 2 ensures the differentiability of  $f(x)$  only in  $E_2$ . At present I am unable to prove that the condition (2.3) ensures the differentiability of  $f(x)$  in  $E_1$ .

3. There remain some other open questions. Whether our theorems remain valid if we assume  $p = 1$  and  $\varepsilon > 0$  or  $p > 1$  and  $\varepsilon = 0$ ? In connection with Theorem 4 and 5 the case  $p = 1$ ,  $\varepsilon = 0$  is also unsettled.

We shall come back to further questions in a next paper.

### § 3. Two lemmas

In the proof of our theorems the most important is the estimation of the derivative of rational functions *without* exceptional set.

**LEMMA 1.** Let  $n$  be even,  $R_n(x)$  be a rational function of degree  $n$ ,

$$(3.1) \quad \sup_{x \in E_3} |R_n(x)| \leq M < +\infty.$$

Then

$$(3.2) \quad |R'_n(x)| \leq \frac{2nM}{A(R_n)} \quad (x \in E_3).$$

**PROOF.** We quote the following theorem of V. N. RUSAK [2]: If the function

$$r_n(x) = \frac{P_n(x)}{\sqrt{h_{2n}(x)}}$$

— where  $P_n(x)$  is a polynomial of degree  $n$  at most,  $h_{2n}(x) = \prod_{k=1}^n [(\alpha_k - x)^2 + \beta_k^2]$  ( $\beta_k > 0$ ) — satisfies the condition

$$(3.3) \quad \sup_{x \in E_3} |r_n(x)| \leq 1$$

then

$$(3.4) \quad |r'_n(x)| \leq \sum_{k=1}^n \frac{\beta_k}{(\alpha_k - x)^2 + \beta_k^2} \quad (x \in E_3).$$

It is worthy of note that this theorem is the most important in our proofs. We make use of it with

$$r_n(x) = \frac{1}{M} R_n(x) = \frac{1}{M} \cdot \frac{P_n(x)}{Q_n(x)} \quad \text{where} \quad h_{2n}(x) = Q_n(x)^2 = \prod_{k=1}^{\frac{n}{2}} [(\alpha_k - x)^2 + \beta_k^2]^2.$$

Then condition (3.3) holds because of (3.1). Thus we have by (3.4) and (1.2)

$$\frac{1}{M} |R'_n(x)| \leq 2 \sum_{k=1}^{\frac{n}{2}} \frac{\beta_k}{(\alpha_k - x)^2 + \beta_k^2} \leq \frac{2n}{A(R_n)} \quad (x \in E_3)$$

i.e. (3.2) holds.

**LEMMA 2.** Let  $R_n(x)$  be a rational function of degree  $n$  at most and

$$(3.5) \quad \max_{x \in E_1} |R_n(x)| \leq M < +\infty.$$

Then

$$|R'_n(x)| \leq \frac{12\sqrt{2}}{(1+x)\sqrt{1-x^2}} \cdot \frac{nM}{\delta(R_n)} \quad (x \in E_2)$$

**PROOF.** Let

$$r_{2n}(x) = R_n \left( \frac{1-x^2}{1+x^2} \right) \quad (x \in E_3)$$

Clearly  $r_{2n}(x)$  is a rational function of degree  $2n$  at most. By (3.5)

$$\sup_{x \in E_3} |r_{2n}(x)| \leq M$$

further

$$R'_n(x) = -\frac{1}{(1+x)\sqrt{1-x^2}} r'_{2n}\left(\sqrt{\frac{1-x}{1+x}}\right) \quad (x \in E_2).$$

Thus by Lemma 1

$$|R'_n(x)| \leq \frac{1}{(1+x)\sqrt{1-x^2}} \cdot \frac{2 \cdot 2n \cdot M}{A(r_{2n})} \quad (x \in E_2).$$

Correspondingly, it remains to prove the inequality

$$A(r_{2n}) \geq \frac{\delta(R_n)}{3\sqrt{2}}.$$

An easy calculation shows that the absolute value of the imaginary part of a pole of  $r_{2n}(x)$  is

$$c = \sqrt{\frac{\sqrt{(1-a^2-b^2)^2+4b^2} + a^2+b^2-1}{2[(1+a)^2+b^2]}}$$

if  $a+bi$  is a pole of  $R_n(x)$ . We have to prove

$$c \geq \frac{\delta(R_n)}{3\sqrt{2}}.$$

Evidently we may assume  $a \geq 0, b \geq 0$ .

*Case 1:*  $b \geq a+1$ . Then by (1.1)

$$(3.6) \quad c = \sqrt{\frac{a^2+b^2+2b-1}{2(a^2+b^2+2a+1)}} = \sqrt{\frac{1}{2} + \frac{b-a-1}{a^2+b^2+2a+1}} \geq \frac{1}{\sqrt{2}} \geq \frac{\delta(R_n)}{\sqrt{2}}.$$

*Case 2:*  $b < a+1$ .

*Case 2.1:*  $a \leq 1$ .

*Case 2.1.1:*  $a^2+b^2 \leq 1$ . Then

$$\begin{aligned} c &= \frac{2b}{\sqrt{2(a^2+b^2+2a+1)[\sqrt{(1-a^2-b^2)^2+4b^2} + (1-a^2-b^2)]}} \geq \frac{2b}{\sqrt{2 \cdot 4 \cdot 2}} = \\ &= \frac{b}{2} \geq \frac{\delta(R_n)}{2}. \end{aligned}$$

*Case 2.1.2:  $a^2 + b^2 > 1$ .*

*Case 2.1.2.1:  $b \geq 1$ . From (3.6), being  $a > b - 1 \geq 0$*

$$c \geq \sqrt{\frac{2b^2}{2(4+4)}} = \frac{b}{2\sqrt{2}} \geq \frac{\delta(R_n)}{2\sqrt{2}}.$$

*Case 2.1.2.2:  $b < 1$ . From (3.6)*

$$c \geq \sqrt{\frac{2b}{2 \cdot 5}} = \frac{\sqrt{b}}{\sqrt{5}} > \frac{b}{\sqrt{5}} \geq \frac{\delta(R_n)}{\sqrt{5}}.$$

*Case 2.2:  $a > 1$ .*

*Case 2.2.1:  $(a-1)^2 + b^2 \leq 1$ . Then  $a \leq 2$  and  $(a+1)^2 + b^2 \leq 9$ .  
Thus by (3.6) and (1.1)*

$$c \geq \sqrt{\frac{a^2 + b^2 - 1}{2 \cdot 9}} \geq \sqrt{\frac{(a-1)^2 + b^2}{18}} \geq \frac{\delta(R_n)}{3\sqrt{2}}.$$

*Case 2.2.2:  $(a-1)^2 + b^2 > 1$ . Then by (3.6)*

$$c \geq \sqrt{\frac{1}{2} - \frac{a+1-b}{(a+1)^2 + b^2}} \geq \sqrt{\frac{1}{2} - \frac{a+1}{4a+1}} \geq \frac{1}{\sqrt{10}} \geq \frac{\delta(R_n)}{\sqrt{10}},$$

qu. e.d.

#### § 4. Proof of the theorems

The proof of Theorem 1 and Theorem 2 runs parallelly and uses Lemma 1 and Lemma 2, respectively. We prove only Theorem 2.

PROOF OF THEOREM 2. Let

$$(4.1) \quad \bar{R}_0(x) = R_{[p]}(x), \quad \bar{R}_k(x) = R_{[p^{k+1}]}(x) - R_{[p^k]}(x) \quad (k = 1, 2, \dots).$$

Then by (1.4)

$$(4.2) \quad f(x) = \sum_{k=0}^{\infty} \bar{R}_k(x) \quad (x \in E_1).$$

We obtain from (1.4) and (2.3)

$$(4.3) \quad |\bar{R}_k(x)| \leq |f(x) - R_{[p^{k+1}]}(x)| + |f(x) - R_{[p^k]}(x)| \leq 2R_{[p^k]}(f; E_1) \leq O\left(\frac{\min(\delta(R_{[p^k]}); \min_{0 \leq i \leq q} \delta(R_{[p^k]p+i}))}{[p^k]^{1+\epsilon}}\right) \leq O\left(\frac{\min(\delta(R_{[p^k]}); \delta(R_{[p^{k+1}]}))}{p^{k(1+\epsilon)}}\right) \quad (x \in E_1)$$

being

$$[p^{k+1}] = [[p^k]p] + i$$

with a suitable  $0 \leq i \leq q$ . Clearly  $\bar{R}_k(x)$  is a rational function of degree  $[p^{k+1}] + [p^k] \leq 2[p^{k+1}]$  at most, and

$$(4.4) \quad \delta(\bar{R}_k) = \min (\delta(R_{[p^k]}), \delta(R_{[p^{k+1}]})).$$

Thus by Lemma 2

$$\begin{aligned} |\bar{R}'_k(x)| &\leq \frac{12\sqrt{2}}{(1+x)\sqrt{1-x^2}} \cdot \frac{2[p^{k+1}]}{\delta(\bar{R}_k)} \cdot O\left(\frac{\delta(\bar{R}_k)}{p^{k(1+i)}}\right) = \\ &= \frac{1}{(1+x)\sqrt{1-x^2}} O\left(\frac{1}{(p^e)^k}\right) \quad (x \in E_2). \end{aligned}$$

Hence

$$\sum_{k=0}^{\infty} |\bar{R}'_k(x)| = \frac{O(1)}{(1+x)\sqrt{1-x^2}} \left( \sum_{k=0}^{\infty} \frac{1}{(p^e)^k} \right) < \infty \quad \text{if } x \in E_2,$$

correspondingly  $f'(x)$  exists and

$$f'(x) = \sum_{k=0}^n \bar{R}'_k(x) \quad (x \in E_2).$$

**PROOF OF THEOREM 3.** Let

$$R_{2n+1}(x) = R_{2n}(x) = \sum_{k=1}^n \frac{xk \log k}{4^k x^2 + k^2 \log^2 k}.$$

Evidently

$$\left| \frac{xk \log k}{4^k x^2 + k^2 \log^2 k} \right| \leq \frac{1}{2^{k+1}} \quad (x \in E_3; k=1, 2, \dots)$$

thus  $f(x) = \lim_{n \rightarrow \infty} R_n(x)$  exists and is continuous in  $E_3$ . We have

$$R_n(f; E_1) \leq R_n(f; E_3) \leq \sum_{k=\lceil \frac{n}{2} \rceil + 1}^{\infty} \left| \frac{xk \log k}{4^k x^2 + k^2 \log^2 k} \right| \leq \sum_{k=\lceil \frac{n}{2} \rceil + 1}^{\infty} \frac{1}{2^{k+1}} \leq \frac{1}{2^{\frac{n}{2}}}$$

and

$$\delta(R_n) = \mathcal{A}(R_n) \geq \frac{n \log \frac{n}{2}}{2 \cdot 2^{\frac{n}{2}}}.$$

Thus (2.4) holds. On the other hand if  $0 < |x| \leq \frac{n \log n}{2^n}$  then

$$\frac{f(x) - f(0)}{x - 0} = \sum_{k=1}^n \frac{k \log k}{4^k x^2 + k^2 \log^2 k} \geq \sum_{k=1}^n \frac{k \log k}{4^k x^2 + k^2 \log^2 k} \geq \frac{1}{2} \sum_{k=1}^n \frac{1}{k \log k}$$

i.e.

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = +\infty,$$

qu.e.d.

PROOF OF THEOREM 5. Using the notation (4.1) we have (4.2). Similarly as in (4.3), we get by (1.4), (2.4) and (4.4)

$$(4.5) \quad |\bar{R}_k(x)| \leq 2R_{[p^k]}(f; E_1) \leq O\left(\frac{[\min(\delta(R_{[p^k]}); \min_{0 \leq i \leq q} \delta(R_{[[p^k]p^i+1]}))]^\alpha}{[p^k]^{\alpha+\epsilon}}\right) \leq \\ \leq O\left(\frac{[\min(\delta(R_{[p^k]}); \delta(R_{[p^{k+1}]})]]^\alpha}{p^{k(\alpha+\epsilon)}}\right) = O\left(\frac{\delta(\bar{R}_k)^\alpha}{p^{k(\alpha+\epsilon)}}\right) \quad (x \in E_1).$$

Let  $0 < t < 1$  arbitrary but fixed. Then by Lemma 2

$$(4.6) \quad |\bar{R}'_k(x)| \leq \frac{12\sqrt{2}}{t\sqrt{t^2}} \cdot \frac{2[p^{k+1}]}{\delta(\bar{R}_k)} \cdot O\left(\frac{\delta(\bar{R}_k)^\alpha}{p^{k(\alpha+\epsilon)}}\right) = \frac{1}{t^2} O\left(\frac{p^{k(1-\alpha-\epsilon)}}{\delta(\bar{R}_k)^{1-\alpha}}\right). \\ (x \in [-1+t, 1-t]).$$

Now let  $h \geq 0$  be arbitrary and  $x, y \in [-1+t, 1-t]$ ,  $|x-y| \leq h$ . We get from (4.2)

$$|f(x)-f(y)| \leq \sum'_k |\bar{R}_k(x) - \bar{R}_k(y)| + \sum''_k |\bar{R}_k(x) - \bar{R}_k(y)|$$

where  $\sum'_k$  is extended over that  $k$ 's for which

$$p^k h \leq \delta(\bar{R}_k)$$

and  $\sum''_k$  contains the remaining  $k$ 's. We obtain by (4.6)

$$\sum'_k |\bar{R}_k(x) - \bar{R}_k(y)| = |x-y| \cdot \sum'_k |\bar{R}'_k(\xi_k)| \leq \frac{h}{t^2} O\left(\sum_{k=0}^{\infty} \frac{p^{k(1-\alpha-\epsilon)}}{p^{k(1-\alpha)} h^{1-\alpha}}\right) = \\ = \frac{h^\alpha}{t^2} O\left(\sum_{k=0}^{\infty} \frac{1}{(p^\epsilon)^k}\right) \quad (\xi_k \in [x, y])$$

and by (4.5)

$$\sum''_k |\bar{R}_k(x) - \bar{R}_k(y)| \leq 2 \sum''_k \max_{x \in E_1} |\bar{R}_k(x)| = O\left(\sum_{k=0}^{\infty} \frac{\delta(\bar{R}_k)^\alpha}{p^{k(\alpha+\epsilon)}}\right) = \\ = O\left(\sum_{k=0}^{\infty} \frac{p^{ka} h^\alpha}{p^{k(\alpha+\epsilon)}}\right) = h^\alpha O\left(\sum_{k=0}^{\infty} \frac{1}{(p^\epsilon)^k}\right).$$

But  $\sum \frac{1}{(p^\epsilon)^k}$  converges and so

$$|f(x)-f(y)| = \frac{O(1)}{t^2} h^\alpha \quad (x, y \in [-1+t, 1-t]),$$

qu.e.d. We omit the proof of Theorem 4 which hardly differs from that of Theorem 5.

#### References

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# ON AN EXTENSION OF THE PRIME-NUMBER THEOREM

By

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To Prof. F. KÁRTESZI on the occasion of his 60 th birthday.

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It is a well-known fact, that the prime-number theorem is equivalent to the statement

$$\sum_{n \leq x} \lambda(n) = o(x),$$

where  $\lambda(n)$  stands for Liouville's function. The method introduced by A. SELBERG in the theory of numbers enables one to prove the following more general.

**THEOREM.** Let  $f(n)$  be an arbitrary absolutely multiplicative function i.e.

$$f(ab) = f(a)f(b)$$

for natural  $a$  and  $b$ . Suppose that  $f(n)$  takes the three values  $\pm 1, 0$  only. Suppose further that

$$\sum_{n \leq x} \frac{f(n)}{n} = O_R(1)$$

holds. Then

$$F(x) \stackrel{\text{def}}{=} \sum_{n \leq x} f(n) = o(x).$$

The technique used by the proof is essentially the same what E. WRIGHT developed in his paper [1]. Maybe, the way of POSTNIKOV and ROMANOV [2] leads to the mentioned result too. For a generalisation of this theorem see [3].

The proof consists of two different parts. In the first of them using Selbergs' inequality we deduce an integral-inequality for real valued multiplicative functions, namely

$$(1) \quad |F(x)| \log^2 x \leq 2 \int_1^x \left| F\left(\frac{x}{t}\right) \right| \log t dt + O(x \log x),$$

which in formal respect corresponds to the inequality for  $R(x)$  in [1]. In the second one we deduce from (1) the statement of the theorem.

Before concluding this preliminary part we should like to mention that in the notation

$$(2) \quad V(\eta) = e^{-\eta} F(e^\eta)$$

the condition made for the values of  $f(n)$  ensures that

$$\int |V(\eta)| d\eta$$

is little on little intervals. Namely as easy to see that this is the case on intervals, where  $V(\eta)$  preserves his sign. On the other hand the function (2) can change his sign on intervals only which contain at least one root of the equation

$$V(\eta) = 0.$$

This latter fact, ensured by the prescribed values of  $f(n)$  turns out to be the crucial point in the proof of the theorem.

The author is indebted to Prof. P. TURÁN for his encouragement and for several remarks concerning the paper.

**PROOF OF THE INEQUALITY (1).** We need the following lemmas.

**LEMMA 1.** Suppose that  $c_1, c_2, \dots$  is a sequence of numbers, that

$$c(t) = \sum_{n \leq t} c_n$$

and that  $f(t)$  is any function of  $t$ . Then

$$\sum_{n \leq x} c_n f(n) = \sum_{n \leq x-1} c(n) \{f(n) - f(n+1)\} + c(x) f([x]).$$

If in addition,  $c_j = 0$  for  $j < n_1$  and  $f(t)$  has a continuous derivative for  $t = n_1$  then

$$\sum_{n \leq x} c_n f(n) = c(x) f(x) - \int_{n_1}^x c(t) f'(t) dt.$$

For the proof of this Lemma see [4], theorem 421, p. 346.

**LEMMA 2. (Selberg's formula)**

$$(3) \quad \sum_{n \leq x} \Lambda(n) \log n + \sum_{mn \leq x} \Lambda(m) \Lambda(n) = 2x \log x + O(x),$$

where  $\Lambda(n)$  stands for von Mangoldt's function. For the proof see [5].

**LEMMA 3.** Let  $f(n)$  be any multiplicative function satisfying the condition

$$|f(n)| \leq 1$$

for all natural values of  $n$ . Then

$$(4) \quad F(x) \log x - \sum_{n \leq x} f(n) \Lambda(n) F\left(\frac{x}{n}\right) = O(x).$$

PROOF OF LEMMA 3. From the trivial formula

$$\sum_{n \leq x} f(n) \log \frac{x}{n} = O(x)$$

one can deduce that

$$F(x) \log x - \sum_{n \leq x} f(n) \log n = F(x) \log x - \sum_{p \leq x} f(p) \log p F\left(\frac{x}{p}\right) + O(x) = O(x).$$

Now considering that

$$A(p) = \log p$$

and

$$\sum_{\substack{n \leq x \\ n \neq p}} f(n) A(n) F\left(\frac{x}{n}\right) = O(x),$$

we get at once the statement of the Lemma.

DEDUCTION OF (1) FROM LEMMA 3. From (4) replacing  $n$  by  $m$  and  $x$  by  $\frac{x}{n}$  one can obtain

$$F\left(\frac{x}{n}\right) \log \frac{x}{n} - \sum_{m \leq \frac{x}{n}} f(m) A(m) F\left(\frac{x}{mn}\right) = O\left(\frac{x}{n}\right).$$

So we have

$$\begin{aligned} & \left\{ F(x) \log x - \sum_{n \leq x} f(n) A(n) \log \frac{x}{n} \right\} \log x + \sum_{n \leq x} f(n) A(n) \left\{ F\left(\frac{x}{n}\right) \log \frac{x}{n} - \right. \\ & \quad \left. - \sum_{m \leq \frac{x}{n}} f(m) A(m) F\left(\frac{x}{mn}\right) \right\} = O(x \log x). \end{aligned}$$

Whence after rearranging the terms .

$$F(x) \log^2 x = \sum_{n \leq x} f(n) A(n) \log n F\left(\frac{x}{n}\right) + \sum_{mn \leq x} f(m) f(n) A(m) A(n) F\left(\frac{x}{mn}\right) + O(x \log x)$$

or in another form

$$(5) \quad |F(x)| \log^2 x \leq \sum_{n \leq x} a_n \left| F\left(\frac{x}{n}\right) \right| + O(x \log x),$$

where

$$a_n = A(n) \log n + \sum_{hk=n} A(h) A(k).$$

Applying Lemma 2 one has

$$\sum_{n \leq x} a_n = 2x \log x + O(x).$$

Next we show that

$$(6) \quad \sum_{n \leq x} a_n \left| F\left(\frac{x}{n}\right) \right| = 2 \int_1^x \left| F\left(\frac{x}{t}\right) \right| \log t dt + O(x \log x),$$

and this will finish the proof of (1). At the deduction of (6) we had to suppose that the inequality

$$-1 \leq f(n) \leq 1$$

holds for every value of  $n$ .

We start with observing that if  $t > t' \geq 0$

$$\begin{aligned} |F(t)| - |F(t')| &\leq |F(t) - F(t')| = \\ &= |(F(t) + G(t)) - (F(t') + G(t')) - G(t) + G(t')| \\ &\leq H(t) - H(t'), \end{aligned}$$

where

$$H(t) \stackrel{\text{def}}{=} F(t) + 2G(t),$$

$$G(t) \stackrel{\text{def}}{=} \sum_{n \leq t} 1.$$

We mention that  $H(t)$  is a steadily increasing function of the natural parameter  $t$  and  $H(t) = O(t)$  holds.

Using Lemma 1, we obtain that

$$(7) \quad \sum_{n \leq x-1} n \left\{ H\left(\frac{x}{n}\right) - H\left(\frac{x}{n+1}\right) \right\} = \sum_{n \leq x} H\left(\frac{x}{n}\right) - [x]H\left(\frac{x}{[x]}\right) = \\ = O\left(x \sum_{n \leq x} \frac{1}{n}\right) + O(x) = O(x \log x).$$

Now we prove (6) in two stages. Let

$$C_1 = 0, \quad c_n = a_n - 2 \int_{n-1}^n \log t dt, \quad f(n) = \left| F\left(\frac{x}{n}\right) \right|.$$

Applying Lemma 1 with this choice, we have

$$C(x) = \sum_{n \leq x} a_n - 2 \int_1^{[x]} \log t dt = O(x),$$

and

$$(8) \quad \begin{aligned} \sum_{n \leq x} a_n \left| F\left(\frac{x}{n}\right) \right| - 2 \sum_{2 \leq n \leq x} \left| F\left(\frac{x}{n}\right) \right| \int_{n-1}^n \log t dt = \\ = \sum_{n \leq x-1} C(n) \left\{ \left| F\left(\frac{x}{n}\right) \right| - \left| F\left(\frac{x}{n+1}\right) \right| \right\} + C(x) \left| P\left(\frac{x}{[x]}\right) \right| = \\ = O\left( \sum_{n \leq x-1} n \left\{ H\left(\frac{x}{n}\right) - H\left(\frac{x}{n+1}\right) \right\} \right) + O(x) = O(x) \end{aligned}$$

by (7).

Next

$$\begin{aligned} & \left| \left| F\left(\frac{x}{n}\right) \right| \int_{n-1}^n \log t dt - \int_{n-1}^n \left| F\left(\frac{x}{t}\right) \right| \log t dt \right| \leq \\ & \leq \int_{n-1}^n \left| \left| F\left(\frac{x}{n}\right) \right| \left| F\left(\frac{x}{t}\right) \right| \right| \log t dt \leq \int_n^{n-1} \left\{ H\left(\frac{x}{t}\right) - H\left(\frac{x}{n}\right) \right\} \log t dt \\ & \leq (n-1) \left\{ H\left(\frac{x}{n-1}\right) - H\left(\frac{x}{n}\right) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} (9) \quad & \sum_{2 \leq n \leq x} \left| F\left(\frac{x}{n}\right) \right| \int_{n-1}^x \left| F\left(\frac{x}{t}\right) \right| \log t dt = \\ & = O\left( \sum_{n \leq x-1} n \left\{ H\left(\frac{x}{n-1}\right) - H\left(\frac{x}{n}\right) \right\} \right) + O(x \log x) = O(x \log x). \end{aligned}$$

(8) and (9) give together (6) and so the proof of the inequality (1) is finished. Introducing the function  $V(\xi)$

$$V(\xi) \stackrel{\text{def}}{=} e^{-\xi} F(e^\xi)$$

we can write the inequality (1) in an another form. Writing  $x = e^\xi$ ,  $t = xe^{-\eta}$  we have

$$\int_1^x \left| F\left(\frac{x}{t}\right) \right| \log t dt = x \int_0^\xi |V(\eta)| (\xi - \eta) d\eta = x \int_0^\xi |V(\eta)| \int_\eta^\xi d\xi d\eta = x \int_0^\xi \int_0^\xi |V(\eta)| d\eta d\xi$$

on changing the order of integration. So inequality (1) becomes

$$(10) \quad |V(\xi)| \xi^2 \leq 2 \int_0^\xi \int_0^\xi |V(\eta)| d\eta d\xi + O(\xi).$$

Taking

$$\alpha = \overline{\lim}_{\xi \rightarrow \infty} |V(\xi)|, \quad \beta = \overline{\lim}_{\xi \rightarrow \infty} \frac{1}{\xi} \int_0^\xi |V(\eta)| d\eta$$

both of these numbers are bounded, and from (10) one can deduce easily the inequality

$$(11) \quad \alpha \leq \beta.$$

So it suffices to prove that (11) can hold in the case  $\alpha = 0$  only. So we suppose in the sequel that  $\alpha > 0$  and deduce a contradiction from this assumption.

The proof requires two further Lemmas.

**LEMMA 4.** Let  $f(n)$  be an arbitrary real valued absolutely multiplicative function, satisfying

$$\sum_{n \leq x} \frac{f(n)}{n} = O_R(1) \quad \text{and} \quad |f(n)| \leq 1.$$

Then in the notations used before, we have for every pair  $\xi_2 \geq \xi_1 > 0$  of positive numbers

$$\left| \int_{\xi_1}^{\xi_2} V(\eta) d\eta \right| < A,$$

where  $A$  denotes an absolute constant.

**PROOF OF LEMMA 4.** We begin by proving that

$$\sum_{n \leq x} \frac{f(n)}{n} = O(1)$$

holds. This can be shown as follows.

$$\sum_{d|n} f(d) = \prod_{p|n} \left( \sum_{r=0}^m (f(p))^r \right) \cong 0$$

by the supposed absolutely multiplicativity of  $f(n)$  and by  $|f(n)| \leq 1$ . So one have

$$0 \leq \sum_{n \leq x} \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d}}} 1 = \sum_{d \leq x} f(d) \left[ \frac{x}{d} \right].$$

On the other-hand

$$\sum_{n \leq x} f(n) \left\{ \frac{x}{n} - \left[ \frac{x}{n} \right] \right\} = O(x),$$

so that

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{x} \sum_{n \leq x} f(n) \left[ \frac{x}{n} \right] + O(1),$$

from which

$$\sum_{n \leq x} \frac{f(n)}{n} = O_L(1)$$

follows. So the remark made is proved. If we put now  $x = e^\xi$ ,  $t = e^n$  we have

$$\int_1^\xi V(\eta) d\eta = \int_1^x \frac{F(t)}{t^2} dt = \sum_{n \leq x} \frac{f(n)}{n} - \frac{F(x)}{x}$$

which follows at once applying Lemma I with

$$c_n = f(n),$$

$$f(t) = \frac{1}{t}.$$

So we obtain, that

$$\left| \int_0^{\xi} V(\eta) d\eta \right| = O(1).$$

Hence we have

$$\begin{aligned} \left| \int_{\xi_1}^{\xi_2} V(\eta) d\eta \right| &= \left| \int_0^{\xi_2} V(\eta) d\eta - \int_0^{\xi_1} V(\eta) d\eta \right| \leq \\ &\leq \left| \int_0^{\xi_2} V(\eta) d\eta \right| + \left| \int_0^{\xi_1} V(\eta) d\eta \right| = O(1) + O(1) = O(1) \end{aligned}$$

and this gives the statement of the Lemma.

LEMMA 5. If  $\eta_0 > 0$ , and  $V(\eta_0) = u$  then

$$\int_0^{\alpha} |V(\eta_0 + \tau)| d\tau = \frac{1}{2} \alpha^2 + O\left(\frac{1}{\eta_0}\right)$$

where the meaning of  $\alpha$  is the same as in (11).

PROOF OF LEMMA 5. Let

$$T(x) \stackrel{\text{def}}{=} \sum_{n \leq x} 1 = [x].$$

Then

$$T(x) \log x + \sum_{n \leq x} A(n) T\left(\frac{x}{n}\right) = 2x \log x + O(x).$$

Combining this with (4) we have

$$(12) \quad \{T(x) + F(x)\} \log x + \sum_{n \leq x} A(n) Q_n\left(\frac{x}{n}\right) = 2x \log x + O(x),$$

where  $Q_n(x)$  stands for

$$Q_n(x) \stackrel{\text{def}}{=} \sum_{m \leq x} (1 - f(n)/f(m)).$$

The functions  $T(x) + F(x)$  and  $Q_n(x)$  are non-negative and non-decreasing, so for any pair  $x_0, x \geq x_0$  of positive numbers we have

$$0 \leq \{T(x) + F(x)\} \log x - \{T(x_0) + F(x_0)\} \log x_0 \leq 2x \log x - 2x_0 \log x_0 + O(x).$$

From this we can deduce that

$$(13) \quad |F(x) \log x - F(x_0) \log x_0| \leq x \log x - x_0 \log x_0 + O(x)$$

using the definition of  $T(x)$ . If we put now  $x = e^{\eta_0 + \tau}$ ,  $x_0 = e^{\eta_0}$  so we have  $F(x_0) = 0$  and (13) becomes

$$|V(\eta_0 + \tau)| \leq \left| 1 - \frac{\eta_0}{\eta_0 + \tau} e^{-\tau} \right| + O\left(\frac{1}{\eta_0}\right) = 1 - e^{-\tau} + O\left(\frac{1}{\eta_0}\right) = \tau + O\left(\frac{1}{\eta_0}\right).$$

Hence

$$\int_0^\zeta |V(\eta_0 + \tau)| d\tau \leq \int_0^\zeta \tau d\tau + O\left(\frac{1}{\eta_0}\right) = \frac{1}{2} \alpha^2 + O\left(\frac{1}{\eta_0}\right)$$

which gives the statement of the Lemma.

Now we are in position to finish the proof of the Theorem. We write

$$\delta = \frac{2A + 3\alpha^2}{2\alpha} > \alpha$$

take  $\zeta$  be any positive number and consider the behaviour of  $V(\eta)$  in the interval  $\zeta \leq \eta \leq \zeta + \delta - \alpha$ .  $V(\eta)$  can change sign in an interval of the mentioned kind only if there is an  $\eta_0 \in [\zeta, \zeta + \delta - \alpha]$  for which  $V(\eta_0) = 0$ . This is a consequence of the fact that  $f(n)$  takes the three values  $\pm 1, 0$  only. Hence in our interval either  $V(\eta_0) = 0$  for some  $\eta_0$ , or  $V(\eta)$  preserves his sign.

So in the first case using Lemma 5

$$\begin{aligned} \int_{\zeta}^{\zeta + \delta} |V(\eta)| d\eta &= \int_{\zeta}^{\eta_0} + \int_{\eta_0}^{\eta_0 + \alpha} + \int_{\eta_0 + \alpha}^{\zeta + \delta} |V(\eta)| d\eta \leq \\ &\leq \alpha(\eta_0 - \zeta) + \frac{1}{2} \alpha^2 + \alpha(\zeta + \delta - \eta_0 - \alpha) + o(1) \\ &= \alpha \left( \delta - \frac{1}{2} \alpha \right) + o(1) = \alpha' \delta + o(1) \end{aligned}$$

for large  $\zeta$ , where

$$\alpha' = \alpha \left( 1 - \frac{1}{2} \frac{\alpha}{\delta} \right) < \alpha.$$

In the second one we have

$$\int_{\zeta}^{\zeta + \delta - \alpha} |V(\eta)| d\eta = \left| \int_{\zeta}^{\zeta + \delta - \alpha} V(\eta) d\eta \right| < A$$

by Lemma 4. So in this case

$$\int_{\zeta}^{\zeta+\delta} |V(\eta)| d\eta = \int_{\zeta}^{\zeta+\delta-\alpha} + \int_{\zeta+\delta-\alpha}^{\zeta+\delta} |V(\eta)| d\eta < A + \alpha^2 + o(1) = \alpha''\delta + o(1),$$

where

$$\alpha'' = \frac{A + \alpha^2}{\delta} := \alpha \left[ \frac{2A + 2\alpha^2}{2A + 3\alpha^2} \right] = \alpha \left( 1 - \frac{1}{2} \frac{\alpha}{\delta} \right) = \alpha'.$$

Hence we have always

$$\int_{\zeta}^{\zeta+\delta} |V(\eta)| d\eta \leq \alpha'\delta + o(1),$$

where  $o(1) \rightarrow 0$  as  $\zeta \rightarrow \infty$ .

If  $M = \left[ \frac{\xi}{\delta} \right]$ , then

$$\begin{aligned} \int_0^{\xi} |V(\eta)| d\eta &= \sum_{m=0}^{M-1} \int_{m\delta}^{(m+1)\delta} |V(\eta)| d\eta + \int_{M\delta}^{\xi} |V(\eta)| d\eta \leq \\ &\leq \alpha'M\delta + o(M) + O(1) = \alpha'\xi + o(\xi). \end{aligned}$$

Hence

$$\beta = \lim_{\xi \rightarrow \infty} \frac{1}{\xi} \int_0^{\xi} |V(\eta)| d\eta = \alpha' < \alpha,$$

and this contradicts to the inequality (11). This contradiction completes the proof of our Theorem.

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# ÜBER W-FOURIERREIHEN MIT NICHTNEGATIVEN PARTIALSUMMEN

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Es sei  $\{\varphi_n(t)\}$  ( $n = 0, 1, 2, \dots$ ) ein im Intervall  $[0, 1]$  definiertes Funktionensystem mit

$$(1) \quad |\varphi_n(t)| = 1 \quad (t \in [0, 1], n = 0, 1, 2, \dots),$$

ferner sei  $\{\psi_n(t)\}$  ( $n = 0, 1, 2, \dots$ ) das von  $\{\varphi_n(t)\}$  erzeugte W-System, d.h.  $\psi_0(t) \equiv 1$  und für  $n \geq 1$  mit der dyadischen Entwicklung

$$(2) \quad n = 2^{r_1} + 2^{r_2} + \dots + 2^{r_s} \quad (r_1 > r_2 > \dots > r_s \geq 0)$$

ist

$$(3) \quad \psi_n(t) = \varphi_{r_1}(t)\varphi_{r_2}(t) \dots \varphi_{r_s}(t).$$

Wir nehmen an, daß das System  $\{\varphi_n(t)\}$  stark-multiplikativ orthogonal ist, d.h. daß das System  $\{\psi_n(t)\}$  orthogonal ist.<sup>1</sup>

Wir bezeichnen mit

$$(4) \quad S_n(t; f) = \sum_{r=0}^{n-1} \psi_r(t) \left( \int_0^1 f(u) \psi_r(u) du \right)$$

die  $n$ -te Partialsumme der nach dem orthogonalen System  $\{\psi_n(t)\}$  fortschreitenden Fourier-Entwicklung von  $f(t)$ .

In dieser Arbeit werden wir den folgenden Satz beweisen.

SATZ. Es existiert eine Funktion  $f(t) \in L[0, 1]$ , für welche

$$(5) \quad f(t) \notin L^2[0, 1],$$

$$(6) \quad S_n(t; f) \geq 0 \quad (n = 1, 2, 3, \dots; t \in [0, 1])$$

gilt.

P. TURÁN [2] hat einen analogen Satz für das trigonometrische System bewiesen.

<sup>1</sup> Siehe ALEXITS [1], S. 165.

### § 1. Bezeichnungen

Zur Konstruktion der Funktion  $f(t)$  führen wir die Funktionen

$$(1.1) \quad D_n(t) = \sum_{r=0}^{n-1} \varphi_r(t) \quad (n = 1, 2, 3, \dots)$$

und die folgenden Mengen ein:

$$(1.2) \quad \begin{aligned} E_v^{(0)} &= \{t : t \in [0, 1], \varphi_v(t) = 1\}, \\ E_v^{(1)} &= \{t : t \in [0, 1], \varphi_v(t) = -1\}; \end{aligned} \quad (v = 0, 1, 2, \dots);$$

$$(1.3) \quad H_0 = [0, 1], \quad H_k = \bigcap_{v=0}^{k-1} E_v^{(0)} \quad (k = 1, 2, 3, \dots);$$

$$(1.4) \quad F_{-1} = \bigcap_{v=0}^{\infty} E_v^{(0)}, \quad F_k = H_k \cap E_k^{(1)} \quad (k = 0, 1, 2, \dots).$$

Es ist leicht zu zeigen, daß für  $n = 2^k + n'$  ( $0 < n' \leq 2^k$ )

$$(1.5) \quad D_n(t) = D_{2^k}(t) + \varphi_k(t)D_{n'}(t)$$

gilt, woraus sich für  $n' = 2^k$  die Gleichung

$$(1.6) \quad D_{2^{k+1}}(t) = \prod_{v=0}^k (1 + \varphi_v(t)) = \begin{cases} 2^{k+1}(t \in H_{k+1}) \\ 0 (t \in [0, 1] - H_{k+1}) \end{cases} \quad (k = 0, 1, 2, \dots)$$

ergibt, und nach (1.1)

$$D_1(t) \equiv 1 \quad (t \in [0, 1] = H_0).$$

Aus der Definitionen (1.2), (1.3), (1.4) und aus (1) folgt, daß

- a)  $E_v^{(0)} \cap E_v^{(1)} = \emptyset, \quad E_v^{(0)} \cup E_v^{(1)} = [0, 1] \quad (v = 0, 1, 2, \dots);$
- b)  $H_k \subset H_{k-1} \quad (k = 1, 2, \dots);$
- c)  $H_k = H_k \cap (E_k^{(0)} \cup E_k^{(1)}) = H_{k+1} \cup F_k, \quad F_k = H_k - H_{k+1} \quad (k = 0, 1, 2, \dots);$
- d)  $[0, 1] = H_0 = H_k \cup \left( \bigcup_{v=0}^{k-1} (H_v - H_{v+1}) \right) = H_k \cup \left( \bigcup_{v=0}^{k-1} F_v \right) \quad (k = 1, 2, \dots);$
- e)  $F_k \cap F_{k'} = \emptyset \quad (k \neq k'), \quad \bigcup_{v=-1}^{\infty} F_v = [0, 1], \quad H_k \cap F_v = \emptyset \quad (v < k)$

gilt.

Zum Beweis unseres Satzes benutzen wir den folgenden

**Satz.** Ist

$$n = 2^{k_1} + 2^{k_2} + \dots + 2^{k_s} \quad (k_1 > k_2 > \dots > k_s \geq 0)$$

die dyadische Darstellung von  $n \geq 1$ , dann gilt:

$$D_n(t) = \psi_n(t) \sum_{i=0}^s \varphi_{k_i}(t) D^{-k_i}(t) \quad (t \in [0, 1]).$$

(Siehe: [3], Hilfssatz I.)

Hieraus folgt (in der Bezeichnung  $k_1 = k$ ), daß

$$(1.8) \quad |D_n(t)| \leq \sum_{i=1}^s D_{2^k i}(t) \leq \sum_{i=1}^k D_{2^k i}(t) \stackrel{\text{def}}{=} B_k(t) \quad (t \in [0, 1]),$$

woraus sich nach (1.6) und (1.7) c), d) die Gleichung

$$(1.9) \quad B_k(t) = \begin{cases} \sum_{j=0}^v 2^j = 2^{v+1} - 1 & (t \in F_v, v = 0, 1, 2, \dots, k-1), \\ \sum_{j=0}^k 2^j = 2^{k+1} - 1 & (t \in H_k) \end{cases}$$

ergibt.

Führen wir noch die Funktionen

$$(1.10) \quad f_k(t) = (1 - \varphi_k(t)) D_{2^k}(t) \quad (k = 0, 1, 2, \dots)$$

ein. Dann gilt nach (1.2), (1.3) und (1.4):

$$(1.11) \quad f_k(t) = \begin{cases} 2^{k+1} & (t \in H_k \cap E_k^{(1)} = F_k), \\ 0 & (t \in [0, 1] - F_k). \end{cases}$$

Aus der Gleichung

$$\int_0^1 \psi_r(t) dt = \begin{cases} 1 & (r \neq 0), \\ 0 & (r = 0) \end{cases}$$

und aus (2), (3) und (1.10) ergibt sich

$$\int_0^1 f_k(t) dt = 1,$$

woraus nach (1.11)

$$(1.12) \quad |F_k| = 2^{-(k+1)} \quad (k = 0, 1, 2, \dots)$$

folgt, wobei  $|F_k|$  das Lebesguesche Maß von  $F_k$  bedeutet.

## § 2. Beweis des Satzes

Es sei

$$(2.1) \quad f(t) = \sum_{r=0}^{\infty} \frac{f_r(t)}{2^{r/2}} \quad (t \in [0, 1]).$$

wo  $f_v(t)$  die in (1.10) definierte Funktionen sind. Aus (1.7) e) (1.11) und (1.12) ergibt sich, daß

$$\int_0^1 |f(t)|^p dt = \sum_{v=0}^{\infty} \int_{F_v} \frac{|f_v(t)|^p}{2^{v/2}} dt = \sum_{v=0}^{\infty} \frac{2^{(v+1)p}}{2^{v+1} \cdot 2^{p \cdot v/2}} = \sum_{v=0}^{\infty} \frac{2^{p-1}}{2^{(1-p/2)v}} = \begin{cases} < \infty & (p < 2), \\ = \infty & (p = 2) \end{cases}$$

besteht. Die Funktion  $f(t)$  erfüllt also die Forderungen  $f(t) \in L[0, 1]$ ,  $f(t) \notin L^2[0, 1]$ .

Zum Beweis von (6) schreiben wir  $n = 2^k + n'$ , wo  $0 < n' \leq 2^k$  ist. Da für alle  $n \geq 1$

$$\begin{aligned} S_n(t; f) &= \sum_{v=0}^{\infty} \frac{S_n(t; f_v)}{2^{v/2}} = \sum_{v=0}^{k-1} \frac{f_v(t)}{2^{v/2}} + \frac{D_{2^k}(t) - \varphi_k(t)D_{n'}(t)}{2^{k/2}} + \sum_{v=k+1}^{\infty} \frac{D_{2^k}(t) + \varphi_k(t)D_{n'}(t)}{2^{v/2}} = \\ &= \sum_{v=0}^{k-1} \frac{f_v(t)}{2^{v/2}} + \frac{D_{2^k}(t) - \varphi_k(t)D_{n'}(t)}{2^{k/2}} + \frac{D_{2^k}(t) + \varphi_k(t)D_{n'}(t)}{2^{k/2}} (\sqrt{2} + 1) = \\ &= A_k(t) + \frac{D_{2^k}(t)}{2^{k/2}} (\sqrt{2} + 2) + \frac{\varphi_k(t)D_{n'}(t)}{2^{k/2}} \sqrt{2} \end{aligned}$$

gilt, wo

$$\begin{aligned} A_k(t) &= \sum_{v=0}^{k-1} \frac{f_v(t)}{2^{v/2}} = \begin{cases} 2 \cdot 2^{v/2} & (t \in F_v, v = 0, 1, 2, \dots, k-1), \\ 0 & (t \in H_k = [0, 1] - \bigcup_{v=0}^{k-1} F_v), \end{cases} \\ D_{2^k}(t) &= \begin{cases} 2^k & (t \in H_k), \\ 0 & (t \in [0, 1] - H_k) \end{cases} \end{aligned}$$

bedeutet, und nach (1.8)

$$|\varphi_k(t)D_{n'}(t)| \leq B_k(t) = \begin{cases} 2^{v+1} - 1 & (t \in F_v, v = 0, 1, 2, \dots, k-1), \\ 2^{k+1} - 1 & (t \in H_k) \end{cases}$$

gilt, so ergibt sich

$$S_n(t; f) \geq \frac{\sqrt{2} + 2}{2^{k/2}} 2^k - \frac{\sqrt{2}}{2^{k/2}} (2^{k+1} - 1) = \frac{\sqrt{2}}{2^{k/2}} + 2^{k/2} (2 - \sqrt{2}) > 0 \quad (t \in H_k)$$

und

$$\begin{aligned} S_n(t; f) &\geq 2 \cdot 2^{k/2} - \sqrt{2} \frac{2^{k+1} - 1}{2^{k/2}} \geq 2^{k/2} (2 - 2^{v/2 + 3/2 - k/2}) \geq \\ &\geq 2^{k/2} \left( 2 - 2^{-\frac{k-1+3-k}{2}} \right) = 0 \quad (v = 0, 1, 2, \dots, k-1), \end{aligned}$$

woraus nach (1.7) d) die Behauptung (6) folgt.

Damit haben wir unseren Satz bewiesen.

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# ОБ ОДНОМ НОВОМ ИНТЕРПОЛЯЦИОННОМ ПРОЦЕССЕ

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## § 1. Содержание работы

Одна из задач настоящей работы — построить такие простые тригонометрические многочлены  $p_n(t)$ , которые совпадают в равноотстоящих узлах

$$(1.1) \quad t_k = \frac{2\pi k}{2n+1}.$$

с  $2\pi$ -периодической непрерывной функцией  $f(t)$  и для которых выполняется неравенство Д. Джексона

$$(1.2) \quad |f(t) - p_n(t)| = O\left[\omega\left(\frac{1}{n}\right)\right] \quad (n = 1, 2, 3, \dots),$$

где  $\omega(\delta)$  — модуль непрерывности функции  $f(t)$ . В § 2. мы докажем, что этому неравенству удовлетворяют следующие, совпадающие в узлах  $t_k$  с функцией  $f(t)$ , тригонометрические многочлены  $4n$ -ого порядка:

$$(1.3) \quad p_n(t) = \sum_{k=-n}^n f(t_k) u_k(t) \quad (n = 0, 1, 2, \dots),$$

где

$$(1.4) \quad u_k(t) = 4l_k^2(t) - 3l_k^4(t)$$

и  $l_k(t)$  фундаментальные многочлены Лагранжева тригонометрического интерполирования

$$(1.5) \quad l_k(t) = \frac{\sin \frac{2n+1}{2} (t-t_k)}{(2n+1) \sin \frac{1}{2} (t-t_k)}.$$

Заметим, что в работе [1] Д. Л. БЕРМАН доказал, что для алгебраических интерполяционных многочленов С. Н. Бернштейна выполняется неравенство Д. Джексона. В работе [2] Г. ФРАЙД построил алгебраические интерполяционные многочлены другого вида, для которых также выполняется неравенство Д. Джексона. Исследования Г. Фрайда были продолжены в работах [3]–[5]. Тригонометрические интерполяционные многочлены (1.3) похожи на алгебраические интерполяционные многочлены Г. Фрайда.

Сопоставим каждой непрерывной на отрезке  $[-1, +1]$  функции  $g(x)$  следующие алгебраические многочлены  $4n$ -той степени:<sup>1</sup>

$$(1.6) \quad q_n(x) = \sum_{k=-n}^n g(\cos t_k) u_k(\arccos x) \quad (n = 0, 1, 2, \dots),$$

которые совпадают в узлах  $\cos t_k$  с функцией  $g(x)$ . В § 3. и § 4. мы докажем, что на отрезке  $[-1, +1]$  они удовлетворяют неравенству А. Ф. Тимана

$$(1.7) \quad |g(x) - q_n(x)| = O\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \omega\left(\frac{|x|}{n^2}\right)\right] \quad (n = 1, 2, 3, \dots),$$

где  $\omega(s)$  — модуль непрерывности функции  $g(x)$ .

Заметим, что интерполяционные многочлены, удовлетворяющие неравенству А. Ф. Тимана, впервые изучались в работе Г. ФРАЙДА и П. ВЕРТЭШИ [6].

В последних двух параграфах мы сделаем некоторые замечания относительно фундаментальных многочленов  $u_k(t)$  и относительно возможностей и затруднений, связанных с обобщением результатов работы на другие узлы интерполирования и другие фундаментальные многочлены, содержащие выражение  $I_k(t)$  в степенях, выше четвертой.

<sup>1</sup> Из (1.5) следует известное равенство

$$(1.8) \quad I_k(t) = \frac{1}{2n+1} \left[ 1 + 2 \sum_{j=1}^n \cos j(t-t_k) \right] \quad (k=0, \pm 1, \dots, \pm n).$$

Поэтому

$$(1.9) \quad u_k(t) = \sum_{j=0}^{4n} c_j \cos jt_k \quad (k=0, \pm 1, \dots, \pm n),$$

где  $c_j$  некоторые числа, и

$$(1.10) \quad u_k(t) + u_{-k}(t) = 2 \sum_{j=0}^{4n} c_j \cos jt_k \cos jt \quad (k=1, 2, \dots, n).$$

Отсюда заключаем, что, если  $t = \arccos x$ , то эти функции и функция  $u_0(t)$  суть алгебраические многочлены  $4n$ -той степени от переменной  $x$  и поэтому функции  $q_n(x)$  — также многочлены  $4n$ -той степени.

## § 2. Новое доказательство неравенства Д. Джексона (1.2)

В работе [7] А. Х. ТУРЕЦКИЙ опубликовал (без доказательства) тождество

$$(2.1) \quad \sum_{k=-n}^n l_k^3(t) = \frac{1}{(2n+1)^2} [3n^2 + 3n + 1 + n(n+1) \cos(2n+1)t],$$

$$(2.2) \quad \sum_{k=-n}^n l_k^4(t) = \frac{1}{(2n+1)^2} \left[ \frac{1}{3} (8n^2 + 8n + 3) + \frac{4}{3} n(n+1) \cos(2n+1)t \right] \\ (n = 0, 1, 2, \dots).$$

(Мы докажем их в § 6.). Используя их, получаем:

$$(2.3) \quad \sum_{k=-n}^n u_k(t) = 1 \quad (n = 0, 1, 2, \dots).$$

(Мы докажем это равенство в § 5., не используя тождеств А. Х. Турацкого.)<sup>2</sup>

С помощью этого тождества разность  $p_n(t) - f(t)$  может быть представлена в удобном для ее оценки виде:

$$(2.4) \quad f(t) - p_n(t) = \sum_{k=-n}^n [f(t) - f(t_k)] u_k(t) \quad (n = 0, 1, 2, \dots).$$

Обозначим через  $t_j$  ближайший к точке  $t$  узел, т.е. пусть

$$(2.5) \quad |t - t_j| \leq \frac{\pi}{2n+1}.$$

Ввиду того, что функция  $f(t)$   $2\pi$ -периодична и в силу (1.5)  $f(t_k)$  и  $l_k(t)$  не изменятся, если мы увеличим или уменьшим  $k$  на число, кратное  $2n+1$ . Поэтому

$$(2.6) \quad f(t) - p_n(t) = \sum_{k=j-n}^{j+n} [f(t) - f(t_k)] u_k(t) \quad (n = 0, 1, 2, \dots).$$

<sup>2</sup> Аналогичные тождества выполняются для фундаментальных многочленов Лагранжа и тригонометрического интерполирования  $l_k(t)$  и для многочленов  $l_k^2(t)$ . Однако кроме тождества (2.3) мы в дальнейшем используем также тот факт, что в выражении  $u_k(t)$  величина  $l_k(t)$  фигурирует в третьей и четвертой степени. Простейшие интерполяционные многочлены

$$\sum_{k=-n}^n f(t_k) l_k(t)$$

и интерполяционные многочлены Д. Джексона

$$\sum_{k=-n}^n f(t_k) l_k^2(t)$$

могут не удовлетворять неравенству (1.2).

Оценим фигурирующие здесь величины. Так как в силу (1.8)

$$(2.7) \quad |l_k(t)| \leq 1,$$

то ввиду (1.4)

$$(2.8) \quad |u_k(t)| \leq 7|l_k(t)|^3.$$

Если  $k \neq j$ , то из (1.5) получаем:

$$(2.9) \quad |l_k(t)| \leq \frac{1}{(2n+1) \sin \frac{1}{2}|t-t_k|} \leq \frac{\pi}{(2n+1)|t-t_k|}.$$

Кроме того при  $j-n \leq k \leq j+n$ ,  $n \geq 1$

$$(2.10) \quad |f(t)-f(t_k)| \leq \omega(|t-t_k|) \leq (n|t-t_k|+1)\omega\left(\frac{1}{n}\right).$$

Таким образом при  $n \geq 1$

$$(2.11) \quad |f(t)-p_n(t)| \leq 7\omega\left(\frac{1}{n}\right) \left[ n|t-t_j| + 1 + \frac{\pi^3}{(2n+1)^3} \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} \left( \frac{n}{|t-t_k|^2} + \frac{1}{|t-t_k|^3} \right) \right]$$

Так как имеет место (2.5) и

$$(2.12) \quad |t-t_k| \geq \frac{2|k-j|-1}{2n+1} \pi,$$

то при  $n \geq 1$

$$(2.13) \quad |f(t)-p_n(t)| \leq 7\omega\left(\frac{1}{n}\right) \left\{ \frac{\pi}{2} + 1 + 2 \sum_{i=1}^{\infty} \left[ \frac{\pi}{2(2i-1)^2} + \frac{1}{(2i-1)^3} \right] \right\}$$

и поэтому

$$(2.14) \quad |f(t)-p_n(t)| = O\left[\omega\left(\frac{1}{n}\right)\right] \quad (n = 1, 2, 3, \dots),$$

что и требовалось доказать.

### § 3. Новое доказательство неравенства А. Ф. Тимана (1.7)

Чтобы оценить разность  $g(x)-q_n(x)$ , мы, полагая  $t = \arccos x$ , представим ее в виде

$$(3.1) \quad g(x)-q_n(x) = \sum_{k=j-n}^{j+n} [g(\cos t) - g(\cos t_k)] u_k(t) \quad (n = 0, 1, 2, \dots),$$

аналогичном равенству (2.4). Чтобы упростить выкладки, перепишем это равенство так:

$$(3.2) \quad g - q_n = \sum_{k=j-n}^{j+n} (g - g_k)(4l_k^3 - 3l_k^4).$$

Разобьем эту сумму на 4 части:

$$(3.3) \quad g - q_n = (g - g_j)(4l_j^3 - 3l_j^4) + 4 \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} (g - g_k) l_k^4 + 3 \sum_{k=j+1}^{j+n} (g - g_k) l_k^3 + \\ + 3 \sum_{k=j-n}^{j-1} (g - g_k) l_k^3.$$

Мы покажем, что все они имеют вид

$$(3.4) \quad O\left[\omega\left(\frac{|\sin t|}{n}\right) + \omega\left(\frac{|\cos t|}{n^2}\right)\right] = O\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \omega\left(\frac{|x|}{n^2}\right)\right]$$

и поэтому такова же и их сумма  $g - q_n$ , что нам и следует доказать.

Прежде всего заметим, что ввиду (2.7)

$$|g - g_j| \cdot |4l_j^3 - 3l_j^4| \leq 7|g - g_j|.$$

В следующем параграфе мы докажем, что

$$(3.5) \quad |g(\cos t) - g(\cos t_k)| \leq \left(2n \sin \frac{1}{2}|t - t_k| + 1\right) \omega\left(\frac{|\sin t|}{n}\right) + \\ + \left[2\left(n \sin \frac{1}{2}|t - t_k|\right)^2 + 1\right] \omega\left(\frac{|\cos t|}{n^2}\right) \quad (j-n \leq k \leq j+n).$$

Поэтому и ввиду (2.5)

$$(3.6) \quad |g - g_j| \cdot |4l_j^3 - 3l_j^4| \leq 7 \left[ \left(\frac{\pi}{2} + 1\right) \omega\left(\frac{|\sin t|}{n}\right) + \left(\frac{\pi^2}{8} + 1\right) \omega\left(\frac{|\cos t|}{n^2}\right) \right],$$

т.е. первый член суммы (3.3) действительно может быть представлен в виде (3.4).

Ввиду (3.5) и (2.9), вводя обозначение

$$(3.7) \quad s_k = (2n+1) \sin \frac{1}{2}(t - t_k),$$

получаем:

$$(3.8) \quad \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} |g - g_k| l_k^4 \leq \omega\left(\frac{|\sin t|}{n}\right) \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} \left(\frac{1}{|s_k^3|} + \frac{1}{s_k^4}\right) + \\ + \omega\left(\frac{|\cos t|}{n^2}\right) \sum_{\substack{k=j-n \\ k \neq j}}^{j+n} \left(\frac{1}{2s_k^2} + \frac{1}{s_k^4}\right).$$

Но ввиду (2.12)

$$(3.9) \quad |s_k| \geq \frac{2\pi+1}{\pi} |t-t_k| \geq 2|k-j|-1 \quad (k = j \pm 1, j \pm 2, \dots, j \pm n)$$

и поэтому

$$\sum_{\substack{k=j-n \\ k \neq j}}^{j+n} |g-g_k| |l_k|^4 \leq 2 \left\{ \omega \left( \frac{|\sin t|}{n} \right) \sum_{i=1}^{\infty} \left[ \frac{1}{(2i-1)^3} + \frac{1}{(2i-1)^4} \right] + \right. \\ \left. + \omega \left( \frac{|\cos t|}{n^2} \right) \sum_{i=1}^{\infty} \left[ \frac{1}{2(2i-1)^2} + \frac{1}{2(i-1)^4} \right] \right\},$$

т.е. вторая часть суммы (3.3) также имеет вид (3.4).

Третья и четвертая часть суммы (3.3) оценивается одним и тем же способом, поэтому мы будем исследовать лишь третью часть. Члены этой суммы оцениваются попарно (в этом заключается одна из основных идей доказательства). Если  $n$  нечетно, то последний член суммы

$$\sum_{k=j+1}^{j+n} (g-g_k) l_k^3$$

оценивается отдельно тем же способом, что и члены второй суммы. В дальнейшем мы будем предполагать, что  $n$  четно. Пусть  $k = j+1, j+3, \dots, j+n-1$ . Тогда

$$(3.10) \quad |(g-g_k)l_k^3 + (g-g_{k+1})l_{k+1}^3| \leq |g-g_k| |l_k^3 + l_{k+1}^3| + |g_k - g_{k+1}| \cdot |l_{k+1}|^3.$$

В следующем параграфе мы докажем, что

$$(3.11) \quad |l_k^3(t) + l_{k+1}^3(t)| \leq \frac{3\pi}{\left[ (2n+1) \sin \frac{1}{2} (t-t_k) \right]^4} \quad (j < k < j+n),$$

$$(3.12) \quad |g(\cos t_k) - g(\cos t_{k+1})| \leq 4\omega \left( \frac{|\sin t|}{n} \right) + \\ + \left( 2\pi n \sin \frac{1}{2} |t_{k+1}-t| + 1 \right) \omega \left( \frac{|\cos t|}{n^2} \right) \quad (j < k < j+n).$$

Поэтому и ввиду (3.5), (3.7) и (2.9) при  $j < k < j+n$

$$(3.13) \quad |(g-g_k)l_k^3 + (g-g_{k+1})l_{k+1}^3| = \omega \left( \frac{|\sin t|}{n} \right) \left( \frac{3\pi}{|s_k|^3} + \frac{3\pi}{s_k^4} + \frac{4}{|s_{k+1}|^3} \right) + \\ + \omega \left( \frac{|\cos t|}{n^2} \right) \left( \frac{3\pi}{2s_k^2} + \frac{3\pi}{s_k^4} + \frac{\pi}{s_{k+1}^2} + \frac{1}{|s_{k+1}|^3} \right).$$

Отсюда и из (3.9) видно, что

$$(3.14) \quad \left| \sum_{k=j+1}^{j+n} (g - g_k) I_k^3 \right| \leq \omega \left( \frac{|\sin t|}{n} \right) \left\{ 3\pi \left[ \sum_{i=0}^{\infty} \frac{1}{(4i+1)^3} + \frac{1}{(4i+1)^4} \right] + \right.$$

$$+ 4 \sum_{i=1}^{\infty} \frac{1}{(4i-1)^3} \left. \right\} + \omega \left( \frac{|\cos t|}{n^2} \right) \left\{ 3\pi \sum_{i=0}^{\infty} \left[ \frac{1}{2(4i+1)^2} + \frac{1}{(4i+1)^4} \right] + \right.$$

$$+ \left. \sum_{i=1}^{\infty} \left[ \frac{\pi}{(4i-1)^2} + \frac{1}{(4i-1)^3} \right] \right\},$$

т.е. и третья часть суммы (3.3) имеет вид (3.4).

#### § 4. Доказательство вспомогательных неравенств (3.5), (3.11) и (3.12)

Чтобы доказать (3.5), заметим, что

$$(4.1) \quad \cos t_k - \cos t = 2 \sin \frac{1}{2} (t - t_k) \sin \frac{1}{2} (t + t_k) =$$

$$= 2 \sin \frac{1}{2} (t - t_k) \left[ \sin t \cos \frac{1}{2} (t - t_k) - \cos t \sin \frac{1}{2} (t - t_k) \right],$$

поэтому

$$(4.2) \quad |\cos t - \cos t_k| \leq 2 \sin \frac{1}{2} |t - t_k| \left( |\sin t| + \sin \frac{1}{2} |t - t_k| \cdot |\cos t| \right)$$

и

$$(4.3) \quad |g(\cos t) - g(\cos t_k)| \leq \omega(|\cos t - \cos t_k|) \leq$$

$$\leq \left( 2n \sin \frac{1}{2} |t - t_k| + 1 \right) \omega \left( \frac{|\sin t|}{n} \right) + \left[ 2 \left( n \sin \frac{1}{2} |t - t_k| \right)^2 + 1 \right] \omega \left( \frac{|\cos t|}{n^2} \right),$$

т.е. (3.5) действительно имеет место.

Докажем (3.12). Так как при  $j < k < j + n$

$$(4.4) \quad \cos t_k - \cos t_{k+1} = 2 \sin \frac{1}{2} (t_{k+1} - t_k) \sin \frac{1}{2} (t_k + t_{k+1}) \leq$$

$$\leq \frac{2\pi}{2n+1} \sin t_{k+1} = \frac{2\pi}{2n+1} [\sin t \cos (t_{k+1} - t) + \cos t \sin (t_{k+1} - t)] \leq$$

$$\leq \frac{2\pi}{2n+1} |\sin t| + \frac{4\pi}{2n+1} \sin \frac{1}{2} |t_{k+1} - t| \cdot |\cos t|,$$

то

$$(4.5) \quad |g(\cos t_k) - g(\cos t_{k+1})| \leq \omega(|\cos t_k - \cos t_{k+1}|) \leq \\ \leq 4\omega\left(\frac{|\sin t|}{n}\right) + \left(2\pi n \sin \frac{1}{2} |t_{k+1} - t| + 1\right) \omega\left(\frac{|\cos t|}{n^2}\right),$$

т.е. (3.12) тоже имеет место.

Докажем неравенство (3.11). Так как

$$(4.6) \quad l_k^3 + l_{k+1}^3 = (l_k + l_{k+1})(l_k^2 - l_k l_{k+1} + l_{k+1}^2),$$

и ввиду (1.5) и (3.7)

$$(4.7) \quad |l_k + l_{k+1}| \leq \left| \frac{1}{s_k} - \frac{1}{s_{k+1}} \right| = \left| \frac{s_{k+1} - s_k}{s_k \cdot s_{k+1}} \right| \leq \frac{\pi}{|s_k| \cdot |s_{k+1}|},$$

то при  $j < k < j + n$

$$(4.8) \quad |l_k^3 + l_{k+1}^3| \leq \frac{\pi}{|s_k| \cdot |s_{k+1}|} \left( \frac{1}{s_k^2} + \frac{1}{|s_k| \cdot |s_{k+1}|} + \frac{1}{s_{k+1}^2} \right) \leq \frac{3\pi}{s_k^4},$$

что и требовалось доказать.

## § 5. Замечания о фундаментальных многочленах

1. Заметим, что кроме получаемого из (1.4) и (1.5) очевидного условия

$$(5.1) \quad u_k(t_j) = \begin{cases} 1, & \text{если } k = j \\ 0, & \text{если } k \neq j \end{cases} \quad (k, j = 0, \pm 1, \pm 2, \dots, \pm n),$$

функции

$$(5.2) \quad u_k(t) = 4l_k^3(t) - 3l_k^4(t)$$

удовлетворяют еще условиям

$$(5.3) \quad u'_k(t_j) = u''_k(t_j) = 0 \quad (k, j = 0, \pm 1, \dots, \pm n).$$

Эти равенства получаются дифференцированием формулы (5.2):

$$(5.4) \quad u'_k = 12(l_k^2 l'_k - l_k^3 l''_k),$$

$$(5.5) \quad u''_k = 12(2l_k l'^2_k + l_k^2 l''_k - 3l_k^2 l'^2_k - l_k^3 l'''_k),$$

ссылкой на очевидное следствие формулы (1.5):

$$(5.6) \quad l_k(t_j) = \begin{cases} 1, & \text{если } k = j, \\ 0, & \text{если } k \neq j, \end{cases}$$

и следствие формулы (1.8):

$$(5.7) \quad l'_k(t_k) = 0.$$

2. Чтобы доказать тождество (2.3), не используя тождеств А. Х. Турсецкого (2.1) и (2.2), заметим, что ввиду (5.5)

$$(5.8) \quad u'''_k = 12(2l_k'^3 + 6l_k'l_k'' + l_k^2l_k''' - 6l_kl_k'^3 - 9l_k^2l_k'' - l_k^3l_k'''),$$

и что ввиду (1.8)

$$(5.9) \quad l'''_k(t_k) = 0.$$

Поэтому и ввиду (5.6) и (5.7)

$$(5.10) \quad u'''_k(t_j) = \begin{cases} 0, & \text{если } j = k, \\ 24l'_k(t_j)^3, & \text{если } j \neq k. \end{cases}$$

Из (1.5) при  $j \neq k$  получаем:

$$(5.11) \quad l'_k(t_j) = \frac{(-1)^{|j-k|}}{2 \sin \frac{1}{2}(t_j - t_k)},$$

и поэтому

$$(5.12) \quad u'''_{j+i}(t_j) = -u'''_{j-i}(t_j) \quad (i = 1, 2, \dots, n).$$

Следовательно при  $j = 0, \pm 1, \dots, \pm n$

$$(5.13) \quad \sum_{k=-n}^n u'''_k(t_j) = \sum_{k=j-n}^{j+n} u'''_k(t_j) = 0.$$

Ввиду (5.1) и (5.3) при  $j = 0, \pm 1, \dots, \pm n$

$$(5.14) \quad \sum_{k=-n}^n u_k(t_j) = 1,$$

$$(5.15) \quad \sum_{k=-n}^n u'_k(t_j) = 0,$$

$$(5.16) \quad \sum_{k=-n}^n u''_k(t_j) = 0.$$

Мы видим, что в  $2n+1$  точках  $t_j$  тригонометрические многочлены 4п-ого порядка

$$(5.17) \quad \sum_{k=-n}^n u_k(t)$$

и 1 и их первые три производные совпадают. Поэтому имеет место тождество (2.3), что мы и хотели доказать.

3. Пусть  $t = \arccos x$ ,

$$(5.18) \quad v_0(x) = u_0(t), \quad v_k(x) = u_k(t) + u_{-k}(t) \quad (k = 1, 2, \dots, n).$$

Ввиду (5.1) и (5.3)

$$(5.19) \quad v_k(\cos t_j) = \begin{cases} 1, & \text{если } j=k \\ 0, & \text{если } j \neq k \end{cases} \quad (j, k = 0, 1, \dots, n),$$

$$(5.20) \quad v'_k(\cos t_j) = v''_k(\cos t_j) = 0 \quad (j = 1, 2, \dots, n; \quad k = 0, 1, \dots, n).$$

## § 6. Замечания о возможностях обобщения фигурирующих в работе многочленов

1. Укажем метод, которым легко можно вычислить суммы

$$(6.1) \quad S_m = \sum_{k=-n}^n l_k^m(t)$$

не только при  $m = 1, 2, 3, 4$ , но и при  $m = 5, 6, 7, \dots$ . Пользуясь им, можно строить интерполяционные многочлены, содержащие величины  $l_k(t)$  в степени выше четвертой и удовлетворяющие неравенству Д. Джексона.

Пусть

$$(6.2) \quad z = \exp it.$$

Ввиду (1.8)

$$(6.3) \quad (2n+1)l_0(t) = \sum_{j=-n}^n z^j = z^{-n}(1-z^{2n+1})(1-z)^{-1}.$$

Поэтому

$$(6.4) \quad [(2n+1)l_0(t)]^m = \sum_{j=-mn}^{mn} c_{j,m} z^j,$$

где  $c_{j,m}$  некоторые числа, причем

$$(6.5) \quad c_{-j,m} = c_{j,m} \quad (j = 1, 2, \dots, mn).$$

Их можно вычислить пользуясь тождеством

$$(6.6) \quad [(2n+1)l_0(t)]^m = z^{-mn}(1-z^{2n+1})^m \sum_{j=0}^{\infty} \frac{(j+1)(j+2)\dots(j+m-1)}{(m-1)!} z^j.$$

Нам нужно вычислить числа  $c_{j,m}$  лишь при  $j$ , кратном  $2n+1$ . Например

$$(6.7) \quad c_{0,3} = \frac{1}{2} [(3n+1)(3n+2)-3n(n+1)] = 3n^2+3n+1,$$

$$(6.8) \quad c_{2n+1,3} = \frac{1}{2} n(n+1),$$

$$(6.9) \quad c_{0,4} = \frac{1}{6} [(4n+1)(4n+2)(4n+3) - 4 \cdot 2n(2n+1)(2n+2)] = \\ = \frac{1}{3} (2n+1)(8n^2 + 8n + 3),$$

$$(6.10) \quad c_{2n+1,4} = \frac{1}{6} 2n(2n+1)(2n+2) = \frac{2}{3} (2n+1)n(n+1).$$

Ввиду (1.5)

$$(6.11) \quad (2n+1)^m \sum_{k=-n}^n l_k^m(t) = (2n+1)^m \sum_{k=-n}^n l_0^m(t-t_k) = \\ = \sum_{k=-n}^n \sum_{j=-mn}^{mn} c_{j,m} \exp ij(t-t_k) = \sum_{j=-mn}^{mn} c_{j,m} z^j \sum_{k=-n}^n \exp(-ijt_k).$$

Очевидно

$$(6.12) \quad \sum_{k=-n}^n \exp(-ijt_k) = \\ = \sum_{k=-n}^n \exp\left(-\frac{2\pi ijk}{2n+1}\right) = \begin{cases} 2n+1, & \text{если } j \text{ делится на } 2n+1, \\ \exp\left(-\frac{2\pi ijn}{2n+1}\right) \frac{1-\exp 2\pi ij}{1-\exp \frac{2\pi ij}{2n+1}}, & \text{в противном случае.} \end{cases}$$

Поэтому

$$(6.13) \quad (2n+1)^{m-1} \sum_{k=-n}^n l_k^m(t) = \\ = \sum_{|j|=0}^{\left[\frac{mn}{2n+1}\right]} c_{j(2n+1),m} z^{j(2n+1)} = c_0 + 2 \sum_{j=1}^{\left[\frac{mn}{2n+1}\right]} c_{j(2n+1),m} \cos j(2n+1)t.$$

В частности при  $m = 3$  и  $4$  из (6.7)–(6.10) и (6.13) получаем тождества А. Х. Турацкого (2.1) и (2.2).

2. Пусть

$$(6.14) \quad t_k = \frac{\pi k}{n} \quad [k=0, \pm 1, \dots, \pm(n-1), n]$$

или

$$(6.15) \quad t_k = \frac{2k-1}{2n} \pi \quad [k=0, \pm 1, \dots, \pm(n-1), n],$$

и

$$(6.16)$$

$$(2n+1)l_k(t) = \sin n(t-t_k) \operatorname{ctg} \frac{1}{2}(t-t_k) = \frac{1}{2} + \sum_{k=1}^{n-1} \cos(t-t_k) + \frac{1}{2} \cos n(t-t_n).$$

С помощью этих фундаментальных многочленов можно строить алгебраические интерполяционные многочлены, соответствующие узлам Чебышева  $\cos \frac{2k-1}{2n}\pi$  и узлам  $\cos \frac{k\pi}{n}$ , симметричным относительно точки  $x=0$ , в то время как узлы  $\cos \frac{2k\pi}{2n+1}$  несимметричны. Однако в этом случае тождество, аналогичное тождеству А. Х. Турацкого (2.2), кроме членов вида  $c_0 + c_{2n}\cos 2nx$  содержит еще  $c_{4n}\cos 4nx$  и поэтому для этих случаев нельзя построить выражения, аналогичные выражениям (1.3) и (1.6), в которых фигурирует лишь третья и четвертая степень фундаментальных многочленов  $l_k(t)$  и которые поэтому удовлетворяют неравенству Д. Джексона (1.2) и неравенству А. Ф. Тимана (1.7).

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## ON THE THEORY OF POWER SERIES

By

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To Prof. F. KÁRTESZI on the occasion of 60 th birthday.

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Let  $f(z)$  be regular in  $|z| < 1$  and let it have there the representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

In the present paper we are going to investigate the class  $Q$  of the power series (1) with  $f(z) \neq 0$  in  $|z| < 1$ . Let  $f(z) \in Q$ . Through the whole paper we will make use of the notation

$$(2) \quad \sqrt{f(z)} = \sum_{n=0}^{\infty} b_n z^n, \quad b_0 = \sqrt{a_0}$$

where we take the branch of  $\sqrt{f(z)}$  with the normalisation

$$-\frac{\pi}{2} \leq \arg b_0 < \frac{\pi}{2}.$$

Let us denote as usual by  $H_\alpha$  for any  $\alpha > 0$  the class of the power series (1) satisfying

$$\sup_{0 < r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^\alpha d\theta < +\infty.$$

I. A well-known theorem of F. RIESZ [1] pp. 542–543 asserts that in order that both of the conjugate trigonometric series

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

(3) and

$$\sum_{n=1}^{\infty} (-\beta_n \cos nx + \alpha_n \sin nx)$$

shall be a Fourier series, the fulfilment of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_1$$

is necessary and sufficient, where  $a_n = \alpha_n - i\beta_n$ .

For the class  $Q$  the statement of this theorem is equivalent to  $\sqrt{f(z)} \in H_2$ , which condition in the notation of (2) can be formulated as

$$(4) \quad \sum_{n=1}^{\infty} |b_n|^2 < +\infty.$$

See for example [1] p. 541.

Let now  $\{\lambda_n\}$  be a given decreasing, convex sequence of positive numbers with  $\lambda_n \rightarrow 0$  and satisfying

$$(5) \quad \sum_{n=1}^{\infty} \frac{\lambda_n}{n} = +\infty.$$

Then by the quoted theorem of F. RIESZ

$$f(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \lambda_n z^n \in H_1$$

holds. Namely the two conjugate series

$$\frac{\lambda_0}{2} + \sum_{n=1}^{\infty} \lambda_n \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n \sin nx$$

are so determined, that the first of them turns out to be the Fourier series of an  $F(x) \in L[-\pi, \pi]$  with  $F(x) \geq 0$ , while the second one does not represent a Fourier series by (5). See [1] p. 100 and p. 652.

Noticing that by the Poisson representation

$$\operatorname{Re} f(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) P(r; x-t) dt, \quad r = |z|$$

holds,  $\operatorname{Re} f(z) > 0$  follows in  $|z| < 1$ , using the non-negativity of  $P(r; u)$ . So  $f(z) \in Q$  follows and in the notation of (2) we have

$$(6) \quad \sum_{n=0}^{\infty} b_n^2 = +\infty$$

taking into account that in this case the  $b_n$ -s are real numbers.

By (6) one can obtain using a theorem of Paley [1] p. 277 on trigonometric series with non-negative coefficients, that

$$b_0^2 + \sum_{n=1}^{\infty} b_n^2 \cos nx$$

is the Fourier series of an unbounded function. In other words taking

$$(7) \quad h(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} b_n^2 z^n$$

and using the  $O$  notation

$$\operatorname{Re} h(z) \neq O(1)$$

holds.

In the present paper our aim is to prove, that however holds the following

**THEOREM.** Let  $f(z) \in Q$  satisfying  $\operatorname{Re} f(z) > 0$  in  $|z| < 1$ . Let the meaning of the  $b_n$ -s be given by (2). Let

$$h(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then for  $|z| < 1$  we have

$$Ih(z) = O(1).$$

To the proof of the Theorem we need the following result due to M. RIESZ [1] p. 580.

II. Suppose that  $F(x) \in L[-\pi, \pi]$  and let

$$(8) \quad F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

Then the function  $\tilde{F}(x)$  defined by the conjugate series of (8), satisfies the relation

$$\int_{-\pi}^{\pi} |\tilde{F}(x)|^\lambda dx < B_\lambda \int_{-\pi}^{\pi} |F(x)|^\lambda dx$$

for every  $\lambda$  with  $0 < \lambda < 1$ , where  $B_\lambda$  depends only on  $\lambda$ .

So especially  $F(x) \in L^\lambda[-\pi, \pi]$  holds for the mentioned values of  $\lambda$ .

By the assumption of our Theorem we have for

$$\int_{-\pi}^{\pi} |\operatorname{Re} f(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} \operatorname{Re} f(re^{i\theta}) d\theta = \operatorname{Re} f(0).$$

So using II. one obtains

$$\int_{-\pi}^{\pi} |If(re^{i\theta})| d\theta < B_\lambda \int_{-\pi}^{\pi} |\operatorname{Re} f(re^{i\theta})| d\theta < B_\lambda \operatorname{Re} f(0).$$

for every  $\lambda$  with  $0 < \lambda < 1$ . Hence  $f(z) \in H_\lambda$  holds for the mentioned  $\lambda$ -s. From this  $\sqrt{f(z)} \in H_{2\lambda}$  follows for  $0 < \lambda < 1$ .

In the notation  $b_n = \gamma_n - i\delta_n$  we have by the quoted theorem of F. RIESZ that

$$\operatorname{Re} b_0 + \sum_{n=1}^{\infty} (\gamma_n \cos nx + \delta_n \sin nx)$$

and

$$\sum_{n=1}^{\infty} (-\delta_n \cos nx + \gamma_n \sin nx)$$

are both Fourier-series. Let us denote by  $U(x)$  and  $V(x)$  their sums. By the Poisson representation already mentioned

$$\operatorname{Re} \sqrt{f(z)} = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t) P(r; x-t) dt, \quad z = re^{ix}$$

$$IV \sqrt{f(z)} = \frac{1}{\pi} \int_{-\pi}^{\pi} V(t) P(r; x-t) dt.$$

By a known result of Fatou

$$\operatorname{Re} \sqrt[f]{z} \rightarrow U(x) \quad \text{and} \quad I \sqrt[f]{z} \rightarrow V(x)$$

holds for a.e.  $x$  in  $[-\pi, \pi]$ . By the assumption  $\operatorname{Re} f(z) \geq 0$  for  $|z| < 1$  it follows that

$$(9) \quad (\operatorname{Re} \sqrt[f]{z})^2 - (I \sqrt[f]{z})^2 \geq 0$$

i.e.

$$(10) \quad |I \sqrt[f]{z}| \leq |\operatorname{Re} \sqrt[f]{z}|.$$

From (9) now follows using  $f(z) \neq 0$  in  $|z| < 1$ , that  $\operatorname{Re} \sqrt[f]{z}$  preserves his sign in  $|z| < 1$  and by the normalisation

$$-\frac{\pi}{2} \leq \arg \sqrt[f]{0} < \frac{\pi}{2}$$

it proves that this sign is positive. So by the quoted result of Fatou

$$(11) \quad |V(x)| \leq U(x)$$

holds for a.e.  $x \in [-\pi, \pi]$ .

From  $\sqrt[f]{z} \in H_{2\lambda}$  ( $0 < \lambda < 1$ ) it follows that  $U(x), V(x) \in L^{2\lambda}[-\pi, \pi]$  holds for every  $\lambda$  with  $0 < \lambda < 1$ . Now we appeal to a theorem of HAUSDORFF—YOUNG [1] p. 211.

The quoted theorem ensures that

$$\sum_{K=0}^{\infty} |b_n|^K < +\infty$$

for every  $K > 2$ .

On the other hand the series

$$(12) \quad \sum_{n=1}^{\infty} (-2\gamma_n \delta_n \cos nx + (\gamma_n^2 - \delta_n^2) \sin nx)$$

has the property that the series

$$\sum_{n=1}^{\infty} \{|2\gamma_n \delta_n|^{\alpha} + |\gamma_n^2 - \delta_n^2|^{\alpha}\}$$

converges for  $\alpha > 1$ . Namely

$$\sum_{n=1}^{\infty} \{|2\gamma_n \delta_n|^{\alpha} + |\gamma_n^2 - \delta_n^2|^{\alpha}\} \leq 2 \sum_{n=1}^{\infty} (\sqrt{(2\gamma_n \delta_n)^2 + (\gamma_n^2 - \delta_n^2)^2})^{\alpha} = 2 \sum_{n=1}^{\infty} |b_n|^{2\alpha} < +\infty.$$

So we obtain that (12) is the Fourier-series of an  $G(x)$  with  $G(x) \in L^q[-\pi, \pi]$  for every  $q > 0$ . Here we used the converse statement of the Hausdorff—Young theorem mentioned before. So  $G(x) \in L^2[-\pi, \pi]$  holds especially.

On the other hand

$$(13) \quad G(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} U(t)V(x-t) dt.$$

Using (13) one can obtain that the functions

$$H(x) \stackrel{\text{def}}{=} 2 \operatorname{Re} f(0) + 2G(x),$$

and

$$K(x) \stackrel{\text{def}}{=} 2 \operatorname{Re} f(0) - 2G(x)$$

have the representations

$$(14) \quad H(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} (U(t) + V(t)) (U(x-t) + V(x-t)) dt,$$

$$K(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} (U(t) - V(t)) (U(x-t) - V(x-t)) dt.$$

Noting that by (11) the expressions under the sign of integration are non-negative in (14) for a.e.  $x \in [-\pi, \pi]$

$$(15) \quad H(x) \geq 0, K(x) \geq 0 \text{ for a.e. } x \in [-\pi, \pi]$$

follows. Now by  $G(x) \in L^2[-\pi, \pi]$ ,  $H(x), K(x) \in L^2[-\pi, \pi]$  holds.

So we can apply the theorem of L. CARLESON [2]. This ensures that the series

$$(16) \quad \text{and} \quad 2 \operatorname{Re} f(0) + 2 \left\{ \sum_{n=1}^{\infty} (-2\gamma_n \delta_n \cos nx + (\gamma_n^2 - \delta_n^2) \sin nx) \right\}$$

$$2 \operatorname{Re} f(0) - 2 \left\{ \sum_{n=1}^{\infty} (-2\gamma_n \delta_n \cos nx + (\gamma_n^2 - \delta_n^2) \sin nx) \right\}$$

converge to  $H(x)$  and  $K(x)$  respectively for a.e.  $x \in [-\pi, \pi]$ .

So we have

$$(17) \quad H(x) = 2 \operatorname{Re} f(0) + 2 \left\{ \sum_{n=1}^{\infty} (-2\gamma_n \delta_n \cos nx + (\gamma_n^2 - \delta_n^2) \sin nx) \right\} \geq 0$$

$$K(x) = 2 \operatorname{Re} f(0) - 2 \left\{ \sum_{n=1}^{\infty} (-2\gamma_n \delta_n \cos nx + (\gamma_n^2 - \delta_n^2) \sin nx) \right\} \geq 0.$$

From (17) one can deduce by investigating the functions

$$\frac{1}{2}(H(x)+H(-x)) \quad \text{and} \quad \frac{1}{2}(K(x)+K(-x))$$

that

$$(18) \quad \left| \sum_{n=1}^{\infty} 2\gamma_n \delta_n \cos nx \right| \leq \operatorname{Re} f(0)$$

holds for a.e.  $x \in [-\pi, \pi]$ . So by (17) and (18) one can deduce that

$$(19) \quad \left| \sum_{n=1}^{\infty} (\gamma_n^2 - \delta_n^2) \sin nx \right| \leq K$$

fulfils for a.e.  $x \in [-\pi, \pi]$  with a suitable value of  $K$ . So finally we have that

$$(20) \quad |G(x)| \leq \operatorname{Re} f(0)$$

holds for a.e.  $x \in [-\pi, \pi]$ . Making use of the representation

$$(21) \quad ih(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(t) P(r; x-t) dt$$

the statement of the Theorem now follows by (20).

Our Theorem has the following

**COROLLARY.** Suppose that the  $b_n$ -s in the Theorem are real. This is automatically fulfilled if  $f(z)$  has real coefficients. Then in the notation

$$h_n(x) \stackrel{\text{def}}{=} \sum_{r=1}^n b_r^2 \sin rx \quad (n = 1, 2, \dots).$$

$h_n(x) = O(1)$  holds uniformly in  $n$  and  $x$ .

This follows immediately from the Theorem noticing that in this case the  $G(x)$  of the Theorem has the representation

$$G(x) = \sum_{r=1}^{\infty} b_r^2 \sin rx$$

and using Paley's theorem, which asserts that if a Fourier sine series of a bounded function has positive coefficients, then its partial sums are uniformly bounded. [1] p. 277.

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# ОДНО УСЛОВИЕ РАСХОДИМОСТИ ТРИГОНОМЕТРИЧЕСКОГО ИНТЕРПОЛИРОВАНИЯ

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1. В работе [1] Г. ФАБЕР доказал, что, какими бы не были узлы интерполирования, существуют непрерывные функции, для которых Лагранжево интерполирование не сходится равномерно. В статьях [2]—[5] было установлено, каким может быть порядок модуля непрерывности таких функций, если известен лишь порядок чисел Лебега узлов интерполирования, или имеется еще какая-нибудь информация об этих узлах. Аналогичные вопросы для тригонометрического интерполирования изучались в работах [2] и [6]—[8]. В настоящей статье мы докажем еще две теоремы такого рода и покажем, что некоторые из прелыдущих теорем являются их следствиями.

2. Чтобы иметь возможность более точно сформулировать результаты работы, введем следующие обозначения.

Если  $f(x)$  непрерывная  $2\pi$ -периодическая функция, то

$$||f(x)|| = \max |f(x)|$$

ее норма, а

$$\omega(f, t) = \max_{\substack{0 \leq h \leq t \\ -\infty < x < +\infty}} |f(x+h) - f(x)|$$

ее модуль непрерывности.  $\omega(t)$  — модуль непрерывности некоторой фиксированной  $2\pi$ -периодической непрерывной функции.  $C(\omega)$  — множество  $2\pi$ -периодических функций, удовлетворяющих условию

$$\omega(f, t) = O[\omega(t)].$$

$x_{kn}$  ( $k = 0, 1, \dots, 2n$ ,  $n = 0, 1, 2, \dots$ ) — узлы тригонометрического интерполирования, удовлетворяющие условию

$$0 \leq x_{0n} < x_{1n} < \dots < x_{2n,n} < 2\pi . (n = 0, 1, 2, \dots).$$

$d_n$  — наименьшее расстояние между соседними узлами, т.е.

$$(2.1) \quad d_n = \min_{0 \leq k \leq 2n} (x_{k+1, n} - x_{kn}) \quad (n = 0, 1, 2, \dots),$$

где

$$x_{2n+1, n} = x_{0n} + 2\pi.$$

$\lambda_n$  — числа Лебега, соответствующие узлам  $x_{kn}$ , т.е.

$$(2.2) \quad \lambda_n = \max \sum_{k=0}^{2n} |l_{kn}(x)| \quad (n = 0, 1, 2, \dots),$$

где  $l_{kn}(x)$  фундаментальные многочлены тригонометрического интерполяирования:

$$l_{kn}(x) = \prod_{i=0}^{2n} \frac{\sin \frac{1}{2} (x - x_{in})}{\sin \frac{1}{2} (x_{kn} - x_{in})} \quad (k = 0, 1, \dots, 2n; \quad n = 0, 1, 2, \dots).$$

$p_n(x)$  — тригонометрические интерполяционные многочлены, соответствующие узлам  $x_{kn}$  и функции  $f(x)$ :

$$p_n(x) = \sum_{k=0}^{2n} f(x_{kn}) l_{kn}(x) \quad (n = 0, 1, 2, \dots).$$

3. Теперь мы уже можем более точно сформулировать результаты работы:

**ТЕОРЕМА 1: Если**

$$(3.1) \quad \liminf_{t \rightarrow +0} \frac{t}{\omega(t)} = 0,$$

и

$$(3.2) \quad \limsup_{n \rightarrow \infty} \omega(d_n) \lambda_n > 0,$$

то для некоторой функции  $f(x) \in C(\omega)$

$$(3.3) \quad \limsup_{n \rightarrow \infty} ||f(x) - p_n(x)|| > 0$$

(т.е. тригонометрический интерполяционный процесс не сходится равномерно).

**ТЕОРЕМА 2. Если**

$$(3.4) \quad \limsup_{n \rightarrow \infty} \omega(d_n) \lambda_n = \infty,$$

то для некоторой функции  $f(x) \in C(\omega)$

$$(3.5) \quad \limsup_{n \rightarrow \infty} ||p_n(x)|| = \infty$$

(т.е. последовательность  $p_n(x)$  неограничена).

4. Приведем некоторые следствия теорем 1 и 2.

В [6] было доказано, что всегда

$$d_n < \frac{1}{n\lambda_n} \quad (n=1, 2, 3, \dots).$$

Отсюда и из теоремы 1 получаем такое следствие: если

$$\limsup_{n \rightarrow \infty} \omega \left( \frac{1}{n\lambda_n} \right) \lambda_n > 0,$$

то для некоторой функции  $f(x)$  из  $C(\omega)$  имеет место (3.3). Этот результат уже был опубликован в [6].

Из теоремы 2 получаем: если

$$\limsup_{n \rightarrow \infty} \omega \left( \frac{1}{n\lambda_n} \right) \lambda_n = \infty,$$

то для некоторой функции  $f(x)$  из  $C(\omega)$  имеет место (3.5). Этот результат уже был опубликован в [8].

В этих следствиях мы не налагали никаких ограничений на величину  $d_n$ . Потребуем теперь, чтобы ее порядок был наибольшим:

$$(4.1) \quad d_n \geq \frac{c}{n} \quad (n=1, 2, 3, \dots),$$

где  $c$  некоторое положительное число. Так как

$$\omega(d_n) \geq \omega \left( \frac{c}{n} \right) \geq \frac{1}{1 + \frac{1}{c}} \omega \left( \frac{1}{c} \cdot \frac{c}{n} \right) = \frac{c}{1+c} \omega \left( \frac{1}{n} \right),$$

то из теоремы 1 получаем:

**СЛЕДСТВИЕ 1.** Если выполняются условия (3.1), (4.1) и

$$\limsup_{n \rightarrow \infty} \omega \left( \frac{1}{n} \right) \lambda_n > 0,$$

то существует функция  $f(x)$  из  $C(\omega)$ , для которой имеет место неравенство (3.3).

Частный случай этого следствия был опубликован в [7].

Из теоремы 2 получаем:

**СЛЕДСТВИЕ 2.** Если выполняется условие (4.1) и

$$\limsup_{n \rightarrow \infty} \omega \left( \frac{1}{n} \right) \lambda_n = \infty,$$

то для некоторой функции  $f(x)$  из  $C(\omega)$  имеет место (3.5).

Частный случай этого следствия был опубликован в [8].

5. Приступая к доказательству теоремы 1, построим некоторые вспомогательные функции и приведем некоторые их свойства.

Обозначим через  $z_n$  одну из точек отрезка  $[0, 2\pi]$ , в которой функция Лебега принимает свое наибольшее значение:

$$(5.1) \quad \sum_{k=0}^{2n} |l_{kn}(z_n)| = \lambda_n \quad (n = 0, 1, 2, \dots).$$

Пусть  $2\pi$ -периодическая непрерывная функция  $g_n(x)$  ( $n = 0, 1, 2, \dots$ ) линейна между узлами  $x_{kn}$  и  $x_{k+1,n}$  ( $k = 0, 1, \dots, 2n$ ) и

$$(5.2) \quad g_n(x_{kn}) = \operatorname{sign} l_{kn}(z_n) \quad (k = 0, 1, \dots, 2n+1),$$

где

$$l_{2n+1,n}(x) = l_{0n}(x).$$

Очевидно

$$(5.3) \quad |g_n(x)| \leq 1 \quad (n = 0, 1, 2, \dots).$$

Принимая во внимание (2.1), получаем:

$$(5.4) \quad |g_n(x+h) - g_n(x)| \leq \frac{2h}{d_n} \quad (h \geq 0, n = 0, 1, 2, \dots).$$

Поэтому

$$(5.5) \quad g_n(x) \in C(\omega) \quad (n = 0, 1, 2, \dots).$$

Обозначим через  $q_{nm}(x)$  тригонометрический интерполяционный многочлен порядка  $n$  функции  $g_m(x)$ . Тогда ввиду (5.2) и (5.1)

$$(5.6) \quad q_{nn}(z_n) = \sum_{k=0}^{2n} g_n(x_{kn}) l_{kn}(z_n) = \sum_{k=0}^{2n} |l_{kn}(z_n)| = \lambda_n \quad (n = 0, 1, 2, \dots).$$

Принимая во внимание (5.3) и (2.2), получаем:

$$(5.7) \quad |q_{nm}(x)| = \left| \sum_{k=0}^{2n} g_m(x_{kn}) l_{kn}(x) \right| \leq \lambda_n \quad (m, n = 0, 1, 2, \dots).$$

Если при некотором  $m$

$$\limsup_{n \rightarrow \infty} ||g_m(x) - q_{nm}(x)|| > 0,$$

то ввиду (5.5) теорема 1 доказана. Поэтому можно считать, что

$$(5.8) \quad \lim_{n \rightarrow \infty} ||g_m(x) - q_{nm}(x)|| = 0 \quad (m = 0, 1, 2, \dots).$$

6. Продолжая доказательство теоремы 1 построим фигурирующую в ней функцию.

Так как имеет место (3.2), (3.1), (5.8) и положительный модуль непрерывности  $\omega(f)$  удовлетворяет условию

$$(6.1) \quad \lim_{t \rightarrow 0} \omega(t) = 0,$$

то положительное число  $c$  и возрастающую последовательность натуральных чисел  $n_i$  можно выбрать так, чтобы выполнялись следующие условия:

$$(6.2) \quad \omega(d_{n_i})\lambda_{n_i} \geq c \quad (i = 1, 2, 3, \dots),$$

$$(6.3) \quad \frac{\omega(d_{n_{i+1}})}{d_{n_{i+1}}} \geq 2 \frac{\omega(d_{n_i})}{d_{n_i}} \quad (i = 1, 2, 3, \dots),$$

$$(6.4) \quad |g_{n_i}(x) - q_{n_k n_i}(x)| \leq \frac{1}{2} \quad (i = 1, 2, 3, \dots; k > i),$$

$$(6.5) \quad \omega(d_{n_i}) \leq \frac{c}{3}, \quad \omega(d_{n_{i+1}}) \leq \frac{1}{3} \omega(d_{n_i}) \quad (i = 1, 2, 3, \dots),$$

$$(6.6) \quad \omega(d_{n_{i+1}})\lambda_{n_i} \leq \frac{c}{3} \quad (i = 1, 2, 3, \dots).$$

Пусть

$$(6.7) \quad f(x) = \sum_{i=1}^{\infty} \omega(d_{n_i}) g_{n_i}(x).$$

Ввиду (6.5) и (5.3) ряд справа сходится. Так как функции  $g_n(x)$   $2\pi$ -периодичны, то и  $f(x)$  такова.

7. Продолжая доказательство теоремы 1, покажем, что

$$(7.1) \quad |f(x+h) - f(x)| \leq 11\omega(h),$$

и поэтому  $f \in C(\omega)$ .

Ввиду (6.7)

$$(7.2) \quad |f(x+h) - f(x)| \leq \sum_{i=1}^{\infty} \omega(d_{n_i}) |g_{n_i}(x+h) - g_{n_i}(x)|.$$

Обозначим через  $j$  наименьший индекс, при котором

$$(7.3) \quad d_{n_j} \leq h.$$

(Ввиду (6.5) подпоследовательность  $d_{n_i}$  монотонно убывает.)  
Если  $i < j$ , то ввиду (5.4), (6.3) и неравенства

$$(7.4) \quad \frac{\omega(H)}{H} \leq 2 \frac{\omega(h)}{h} \quad (0 < h < H)$$

получаем:

$$(7.5) \quad \omega(d_{n_i}) |g_{n_i}(x+h) - g_{n_i}(x)| \leq \omega(d_{n_i}) \frac{2h}{d_{n_i}} \leq 2^{i-j+2} \frac{\omega(d_{n_{j-1}})}{d_{n_{j-1}}} h \leq 2^{i-j+3} \omega(h).$$

Если  $i \geq j$ , то используя (5.3), (6.5), (7.3) и монотонность модуля непрерывности  $\omega(t)$ , получаем:

$$(7.6) \quad \omega(d_{n_i}) |g_{n_i}(x+h) - g_{n_i}(x)| \leq 2\omega(d_{n_i}) \leq 2 \cdot 3^{j-i} \omega(d_{n_j}) \leq 2 \cdot 3^{j-i} \omega(h).$$

Из (7.2), (7.5) и (7.6) получаем (7.1):

$$|f(x+h)-f(x)| \leq \left( \sum_{i=1}^{j-1} 2^{i-j+3} + \sum_{i=j}^{\infty} 2 \cdot 3^{j-i} \right) \omega(h) < 11\omega(h).$$

**8.** Чтобы завершить доказательство теоремы 1, покажем, что

$$(8.1) \quad f(z_{n_k}) - p_{n_k}(z_{n_k}) \equiv \frac{c}{4} (3^{1-k} - 1) \quad (k = 1, 2, 3, \dots)$$

и поэтому имеет место неравенство (3.3).

Ввиду (6.7) при  $k = 1, 2, 3, \dots$

$$(8.2) \quad f(z_{n_k}) - p_{n_k}(z_{n_k}) = \sum_{i=1}^{\infty} \omega(d_{n_i}) [g_{n_i}(z_{n_k}) - q_{n_k n_i}(z_{n_k})].$$

Если  $i < k$ , то, используя (6.4) и (6.5), получаем:

$$(8.3) \quad \omega(d_{n_i}) |g_{n_i}(z_{n_k}) - q_{n_k n_i}(z_{n_k})| \leq \frac{c}{2 \cdot 3^i}.$$

Так как имеет место (5.3) и (6.5), то при  $i \geq k$

$$(8.4) \quad \omega(d_{n_i}) |g_{n_i}(z_{n_k})| \leq \omega(d_{n_i}) \leq \frac{c}{3^i}.$$

Ввиду (5.6) и (6.2)

$$(8.5) \quad \omega(d_{n_k}) q_{n_k n_k}(z_{n_k}) = \omega(d_{n_k}) \lambda_{n_k} \leq c.$$

Наконец, если  $i > k$ , то ввиду (5.7), (6.5) и (6.6)

$$(8.6) \quad \omega(d_{n_i}) |q_{n_k n_i}(z_{n_k})| \leq \omega(d_{n_i}) \lambda_{n_k} \leq 3^{k-i+1} \omega(d_{n_{k+1}}) \lambda_{n_k} \leq c 3^{k-i}.$$

Резюмируя (8.2)–(8.6), приходим к (8.1):

$$f(z_{n_k}) - p_{n_k}(z_{n_k}) \leq c \left( \frac{1}{2} \sum_{i=1}^{k-1} 3^{-i} + \sum_{i=k}^{\infty} 3^{-i} - 1 + \sum_{i=k+1}^{\infty} 3^{k-i} \right) < \frac{c}{4} (3^{1-k} - 1).$$

**9.** Приступаем к доказательству теоремы 2.

Если при некотором фиксированном  $m$

$$\limsup_{n \rightarrow \infty} ||q_{nm}(x)|| = \infty,$$

то теорема 2 доказана. Поэтому можно считать, что

$$(9.1) \quad ||q_{nm}(x)|| \leq c_m \quad (m, n = 0, 1, 2, \dots),$$

где  $c_m$  некоторая числовая последовательность.

Определим возрастающую последовательность натуральных чисел  $n_i$  так, чтобы выполнялись условия

$$(9.2) \quad \omega(d_{n_k})\lambda_{n_k} \geq k2^k \quad (k=1, 2, 3, \dots),$$

$$(9.3) \quad \omega(d_{n_k})\lambda_{n_k} \geq 3 \cdot 2^k \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})c_{n_i} \quad (k=2, 3, 4, \dots),$$

$$(9.4) \quad \omega(d_{n_i}) \leq \frac{1}{2} \omega(d_{n_{i-1}}) \quad (i=2, 3, 4, \dots).$$

Это возможно ввиду (3.4) и (6.1).

Пусть

$$(9.5) \quad f(x) = \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})g_{n_i}(x).$$

Ряд справа сходится ввиду (9.4) и (5.3), его сумма  $2\pi$ -периодична, так как таковы функции  $g_n(x)$ , определенные в 5.

**10.** Продолжая доказательство теоремы 2, покажем, что

$$(10.1) \quad |f(x+h) - f(x)| \leq 4\omega(h)$$

и поэтому  $f(x) \in C(\omega)$ .

Из (9.5) получаем:

$$(10.2) \quad |f(x+h) - f(x)| \leq \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})|g_{n_i}(x+h) - g_{n_i}(x)|.$$

Если  $i < j$  ( $j$  определено в 7), то ввиду (5.4) и (7.4)

$$(10.3) \quad 2^{-i}\omega(d_{n_i})|g_{n_i}(x+h) - g_{n_i}(x)| \leq 2^{1-i}\frac{\omega(d_{n_i})}{d_{n_i}}h \leq 2^{1-i}\omega(h).$$

Если  $i \geq j$ , то из (5.3) и (7.3), принимая во внимание монотонность модуля непрерывности  $\omega(l)$ , получаем.

$$(10.4) \quad 2^{-i}\omega(d_{n_i})|g_{n_i}(x+h) - g_{n_i}(x)| \leq 2^{1-i}\omega(d_{n_i}) \leq 2^{1-i}\omega(h).$$

Резюмируя (10.2)–(10.4), приходим к (10.1).

**11.** Чтобы завершить доказательство теоремы 2, покажем, что

$$(11.1) \quad p_{n_k}(z_{n_k}) \equiv \frac{k}{3} \quad (k=1, 2, 3, \dots)$$

и поэтому имеет место (3.5).

Ввиду (9.5)

$$(11.2) \quad p_{n_k}(z_{n_k}) = \sum_{i=1}^{\infty} 2^{-i}\omega(d_{n_i})q_{n_k n_i}(z_{n_k}) \quad (k=1, 2, 3, \dots).$$

Принимая во внимание (5.6), получаем:

$$(11.3) \quad 2^{-k}\omega(d_{n_k})q_{n_k n_k}(z_{n_k}) = 2^{-k}\omega(d_{n_k})\lambda_{n_k}.$$

Из (9.1) и (9.3) следует:

$$(11.4) \quad \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})|q_{n_k n_i}(z_{n_k})| \leq \sum_{i=1}^{k-1} 2^{-i}\omega(d_{n_i})c_{n_i} \leq \frac{1}{3} 2^{-k}\omega(d_{n_k})\lambda_{n_k}.$$

Наконец, ввиду (5.7) и (9.4)

$$(11.5) \quad \sum_{i=k+1}^{\infty} 2^{-i}\omega(d_{n_i})|q_{n_k n_i}(z_{n_k})| \leq \lambda_{n_k} \sum_{i=k+1}^{\infty} 2^{-i}\omega(d_{n_i}) \leq \frac{1}{3} \lambda_{n_k} 2^{-k}\omega(d_{n_k}).$$

Резюмируя (11.2)–(11.5) и (9.2), получаем (11.1):

$$p_{n_k}(z_{n_k}) \geq \frac{1}{3} 2^{-k}\omega(d_{n_k})\lambda_{n_k} \geq \frac{k}{3} \quad (k = 1, 2, 3, \dots).$$

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## ON DIOPHANTINE APPROXIMATION

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In the present paper we make use of the following notations.

Let  $\xi$  be a given real number. Denote by  $\{\xi\}$  and  $|\xi|$  the fractional part of  $\xi$ , and the distance of  $\xi$  from the nearest integer respectively. The numbers  $\xi$  and  $\eta$  will be called equivalent, if

$$\eta = \frac{r\xi+s}{t\xi+u}, \quad \left| \frac{r}{t} \frac{s}{u} \right| = \pm 1$$

holds with suitable integers  $r, s, t$  and  $u$ . The notation  $q$  we reserve for natural numbers. By  $\alpha$  we will denote an irrational number.

Let as usual

$$\{\alpha\} = [a_1, a_2, a_3, \dots]$$

and

$$\frac{p_n}{q_n} = [a_1, a_2, \dots, a_n].$$

We put further

$$\alpha_n = [a_n, a_{n+1}, a_{n+2}, \dots], \quad \alpha_n^* = [a_n, a_{n-1}, \dots, a_1].$$

Let  $\beta$  be any irrational number with  $0 < \beta < 1$ . Let it has the representation

$$\beta = [b_1, b_2, b_3, \dots]$$

and denote its  $n$ 'th convergent

$$\frac{P_n}{Q_n} = [b_1, b_2, \dots, b_n].$$

Let further the meaning of  $\beta_n$  and  $\beta_n^*$  be given by

$$\beta_n = [b_n, b_{n+1}, b_{n+2}, \dots], \quad \beta_n^* = [b_n, b_{n-1}, \dots, b_1].$$

We denote in the sequel by  $\gamma$ ,  $\delta$ ,  $\vartheta$  irrational numbers, by  $\lambda$  and  $\sigma$  natural numbers.

1. A. V. PRASAD [3] proved, that for all natural  $q$  there exists a  $C_m$ , depending only on  $m$  so, that for any irrational  $\alpha$

$$(1) \quad ||q||q\alpha|| \leq C_m^{-1}$$

has at least  $m$  solutions in  $q$ . He determined the best possible value of  $C_m$  for every  $m$ . For the number  $\alpha = \frac{\sqrt{5}-1}{2}$  the sign of equality can not be omitted.

The result of Prasad was improved by T. N. SINHA [4] who showed, that if we omit the numbers  $\alpha$  equivalent to  $\frac{\sqrt{5}-1}{2}$ , the value of  $C_m$  in (1) can be increased. He determined the exact value of  $C_m^*$  in the case mentioned.

Let  $\delta$  be a quadratic irrationality. Put

$$\nu = \nu(\delta) = \lim_{q \rightarrow \infty} q||q\delta|| \quad (q = 1, 2, 3, \dots).$$

We prove in the sequel the following statement. If the inequality

$$q||q\alpha|| < \nu$$

has an infinity of solutions for every  $\alpha$  equivalent to  $\delta$  (in the sequel  $\alpha \sim \delta$ ), then for every  $\delta$  and for every natural  $m$  there exists a  $C_m$  depending on  $\delta$  only so that

$$q||q\alpha|| \leq C_m^{-1}$$

has at least  $m$  solutions for every  $\alpha \sim \delta$ . Here the exact value of  $C_m$  can be given explicitly.

2. Let  $A$ ,  $B$  and  $C$  rational numbers, put

$$f(x, y) = Ax^2 + Bxy + Cy^2$$

where  $f(x, y)$  stands for an indefinite quadratic form, for which

$$f(\vartheta, 1) = 0$$

has irrational roots, and denote by  $\nu$  and  $\mu$

$$\nu = \lim_{q \rightarrow \infty} q||q\vartheta||$$

$$\mu = \min |f(x, y)| \quad x, y \text{ integers}, \quad (x, y) \neq (0, 0)$$

respectively. It is known, that if there exist integers  $x_1, y_1, x_2, y_2$  with

$$(1) \quad f(x_1, y_1) = \mu, \quad f(x_2, y_2) = -\mu,$$

then for all  $\alpha \sim \delta$

$$(2) \quad q||q\alpha|| < \nu$$

has infinitely many solutions. We shall prove that if (2) has infinitely many solutions for every  $\alpha \sim \vartheta$ , then the existence of the integers  $x_1, y_1, x_2, y_2$  with (1) is guaranteed.

3. Let

$$(1) \quad K(x_1, x_2, \dots, x_n) = \begin{vmatrix} x_1 & -1 \\ 1 & x_2 & -1 \\ & 1 & x_3 & -1 \\ \cdots & & \cdots & \cdots \\ & & 1 & x_{n-1} & -1 \\ & & & 1 & x_n \end{vmatrix}$$

where the elements different from the described are all equal to zero.

As easy to see

$$(2) \quad P_n = K(b_2, b_3, \dots, b_n)$$

$$(3) \quad Q_n = K(b_1, b_2, \dots, b_n).$$

Further it is known that

$$(4) \quad \frac{Q_{n-1}}{Q_n} = [b_n, b_{n-1}, \dots, b_1]$$

$$(5) \quad Q_n ||Q_n \beta|| = (b_n + \beta_{n+1} + \beta_{n-1}^*)^{-1}$$

$$(6) \quad Q_n P_{n-1} - Q_{n-1} P_n = (-1)^n.$$

It is known further that from

$$(7) \quad Q ||Q \beta|| < 0,5$$

$Q = Q_n$  follows, with a suitable choice of the natural number  $n$ . We put further

$$(8) \quad \begin{cases} P_{\sigma, \lambda} = K(b_{\lambda+2}, b_{\lambda+3}, \dots, b_{\lambda+\sigma}) & \lambda = 0, 1, 2, \dots; \sigma = 2, 3, 4, \dots \\ Q_{\sigma, \lambda} = K(b_{\lambda+1}, b_{\lambda+2}, \dots, b_{\lambda+\sigma}) & \lambda = 0, 1, 2, \dots, \sigma = 1, 2, 3, \dots \\ P_{-1, \lambda} = 1 P_{0, \lambda} = 0, \quad Q_{-1, \lambda} = 0, \quad Q_{0, \lambda} = 1 & \lambda = 0, 1, 2, \dots \end{cases}$$

We state the following result, which can be found in a little modified form in [2] as

LEMMA 1.

$$(9) \quad P_{n, \lambda} = b_{n+\lambda} P_{n-1, \lambda} + P_{n-2, \lambda} \quad (n = 1, 2, 3, \dots)$$

$$(10) \quad Q_{n, \lambda} = b_{n+\lambda} Q_{n-1, \lambda} + Q_{n-2, \lambda} \quad (n = 1, 2, 3, \dots)$$

$$(11) \quad P_{\sigma+\lambda-1} = P_{\lambda-1} P_{\sigma-1, \lambda} + P_{\lambda} Q_{\sigma-1, \lambda}$$

$$(12) \quad Q_{\sigma+\lambda-1} = Q_{\lambda-1} P_{\sigma-1, \lambda} + Q_{\lambda} Q_{\sigma-1, \lambda}$$

$$(13) \quad P_{\sigma+\lambda-1} P_{\lambda-1} - P_{\sigma+\lambda-1} Q_{\lambda-1} = (-1) \varphi_{\sigma-1, \lambda}.$$

For the sake of completeness we give a short outline of the proof.

**PROOF OF LEMMA 1.** (9) and (10) can be obtained by standard calculations. To prove (11) and (12) we may suppose that  $\lambda > 1$ ,  $\sigma > 1$ . (11) and (12) can be obtained from determinants

$$\begin{aligned} P_{\sigma+\lambda-1} &= K(b_2, b_3, \dots, b_{\sigma+\lambda-1}) \\ Q_{\sigma+\lambda-1} &= K(b_1, b_2, \dots, b_{\sigma+\lambda-1}) \end{aligned}$$

respectively, using the Laplace development with respect to the first  $\lambda-1$  and  $\lambda$  rows respectively.

By (11), (12) and (6) we have

$$\begin{aligned} Q_{\sigma+\lambda-1} P_{\lambda-1} - Q_{\sigma+\lambda-1} Q_{\lambda-1} &= Q_{\lambda-1} P_{\sigma-1, \lambda} P_{\lambda-1} + Q_{\lambda} Q_{\sigma-1, \lambda} P_{\lambda-1} - \\ &- P_{\lambda-1} P_{\sigma-1, \lambda} Q_{\lambda-1} - P_{\lambda} Q_{\sigma-1, \lambda} Q_{\lambda-1} = (-1)^{\lambda} Q_{\sigma-1, \lambda} \end{aligned}$$

so (13) is proved.

Let  $\{b_n\}$  be a sequence of natural numbers satisfying

$$(14) \quad b_{k+1} = b_k \quad (k = 1, 2, 3, \dots)$$

for a suitable choice of the natural number  $l$ . In the sequel we always suppose that for the sequence  $\{b_n\}$  (14) fulfills.

**LEMMA 2.** In the notation introduced in (8) we have

$$(15) \quad Q_{l-1, \lambda} = P_{\lambda-1} Q_{\sigma-1, \lambda} + Q_{\lambda-1} Q_{\sigma, \lambda}$$

$$(16) \quad P_{l-1, \lambda} + Q_{l, \lambda} = P_{l-1} + Q_{l-1},$$

where  $\lambda + \sigma = l$ .

**PROOF.** To get (15) in the case  $\lambda > 1$ , we had to develop the determinant  $Q_{l-1, \lambda}$  using Laplace's method with respect to the last  $\lambda-1$  columns. To prove (16) we start with the determinant  $Q_{l, \lambda}$ . Using the Laplace-development with respect to the last  $\lambda$  columns one can deduce

$$(17) \quad Q_{l, \lambda} = P_{\lambda} Q_{\sigma-1, \lambda} + Q_{\lambda} Q_{\sigma, \lambda}.$$

On the other hand the Laplace-development of  $P_{l-1, \lambda}$  with respect its last  $\lambda-1$  columns gives

$$(18) \quad P_{l-1, \lambda} = P_{\lambda-1} Q_{\sigma-2, \lambda+1} + Q_{\lambda-1} Q_{\sigma-1, \lambda+1}.$$

Similarly the Laplace-development of the determinants  $Q_l$  and  $P_{l-1}$  with respect they first  $\lambda$  resp.  $\lambda-1$  columns respectively give

$$(19) \quad Q_l = Q_{\lambda} Q_{\sigma, \lambda} + Q_{\lambda-1, \lambda+1}$$

$$(20) \quad P_{l-1} = P_{\lambda} Q_{\sigma-1, \lambda} + P_{\lambda-1} Q_{\sigma-2, \lambda+1}.$$

By (17), (18), (19) and (20) we have (16). So the proof of Lemma 2 is completed.  
It is known, that if

$$(21) \quad f_{\lambda}(x, y) = Q_{l-1, \lambda} x^2 + (Q_{l, \lambda} - P_{l-1, \lambda}) xy - P_{l, \lambda} y^2$$

then

$$f_{\lambda}(\beta_{kl+\lambda+1}, 1) = 0.$$

In other words

$$(22) \quad \beta_{kl+\lambda+1} = \frac{\sqrt{D} - Q_{l,\lambda} + P_{l-1,\lambda}}{2Q_{l-1,\lambda}}$$

where  $D$  is given by

$$(23) \quad D = (Q_{l,\lambda} - P_{l-1,\lambda})^2 - 4Q_{l-1,\lambda}P_{l,\lambda} > 0$$

and the value of  $D$  is independent of  $\lambda$ .

We state now

LEMMA 3. If for some natural numbers  $l$  and  $\lambda$

$$(24) \quad Q_{l-1,\lambda} = Q_{l-1}$$

holds, then

$$(25) \quad b_{l+\lambda} + \beta_{kl+\lambda+1} + [b_{kl+\lambda-1}, b_{kl+\lambda-2}, \dots, b_{\lambda+1}] = b_l + \beta_{k+1}\beta_{k-1}^*.$$

PROOF. Suppose  $l > 1$ . First using (24) we deduce

$$(26) \quad Q_{kl-1,\lambda} = Q_{kl-1} \quad (k = 2, 3, 4, \dots).$$

We use induction on  $k$ . For  $k = 1$  the statement reduces to the assumption of the lemma. Suppose its validity for  $k-1$ . Taking the Laplace development of  $Q_{kl-1}$  with respect to its last  $l-1$  columns we get

$$(27) \quad Q_{kl-1} = Q_{(k-1)l}Q_{l-1} + Q_{l-1}P_{l-1}.$$

By (13) substituting  $\sigma = (k-2)l$ ,  $\lambda = l$  one has

$$(28) \quad Q_{(k-1)l-1}P_{l-1} = P_{(k-1)l-1}Q_{l-1} + (-1)^l Q_{(k-1)l-1,l} = \\ = P_{(k-1)l-1}Q_{l-1} + (-1)^l Q_{(k-1)l-1}.$$

Using (27) and (28)

$$(29) \quad Q_{kl-1} = Q_{l-1}(Q_{(k-1)l} + P_{(k-1)l-1}) + (-1)^l Q_{(k-1)l-1}$$

follows.

Applying (29) with  $\beta_{\lambda+1}$  instead of  $\beta$  we get

$$(30) \quad Q_{(k-1)l-1,\lambda} = Q_{l-1,\lambda}(Q_{(k-1)l,\lambda} + P_{(k-1)l-1,\lambda}) + (-1)^l Q_{(k-1)l-1}.$$

By (24), (29), (30) and (16) using the hypothesis on  $k-1$  the assertion (26) follows.

By (26) it is enough to prove (25) in the case  $k = 1$  only. Using (22), (4) and (10)

$$\begin{aligned} b_{l+\lambda} + \beta_{l+\lambda-1} + [b_{l+\lambda-1}, b_{l+\lambda-2}, \dots, b_{\lambda+1}] = \\ = \frac{\sqrt{D} - Q_{l,\lambda} + P_{l-1,\lambda} + 2(b_{l+\lambda}Q_{l-1,\lambda}Q_{l-2,\lambda})}{2Q_{l-1,\lambda}} = \frac{\sqrt{D}P_{l-1,\lambda} + Q_{l,\lambda}}{2Q_{l-1,\lambda}}. \end{aligned}$$

Our deduction remains true in the case  $\lambda = 0$  too. So

$$b_l + \beta_{l+1} + \beta_{l+1}^* = \frac{\sqrt{D} + P_{l-1} + Q_l}{2Q_{l-1}}$$

and the statement follows by (16) and (24).

We quote the following

**LEMMA 4.** Let  $A$ ,  $B$  and  $C$  be integers. Put

$$f(x, y) = Ax^2 + Bxy + Cy^2.$$

Suppose that  $f(x, y)$  is an indefinite form. Let  $f(\gamma_1, 1) = 0$ ,  $f(\gamma_2, 1) = 0$ , suppose that  $\gamma_1$  and  $\gamma_2$  are irrational numbers satisfying

$$\begin{aligned} 0 &< \gamma_1 < 1 \\ \gamma_2 &< 1. \end{aligned}$$

Then the continued fractions

$$\begin{aligned} \gamma_1 &= [c_1, c_2, \dots, \overline{c_n}] \\ -\frac{1}{\gamma_2} &= [\overline{c_n, c_{n-1}, \dots, c_1}] \end{aligned}$$

are pure periodical. On the other-hand if  $\gamma_1$  satisfies  $0 < \gamma_1 < 1$  and the continued fraction of  $\gamma_1$  is pure periodical, then

$$\gamma_2 < -1.$$

For the proof see [2] pp. 81–83.

**LEMMA 5.** If

$$\beta = [b_1, b_2, \dots, b_l].$$

then

$$(31) \quad \lim_{n \rightarrow \infty} Q_{nl} ||Q_{nl}\beta|| = \frac{Q_{l-1}}{\sqrt{D}},$$

where

$$D = (Q_l - P_{l-1})^2 - 4Q_{l-1}P_l.$$

**PROOF.**

$$Q_{nl} ||Q_{nl}\beta|| = (b_l + \beta + \beta_{nl}^*{}^{-1})^{-1} = (\beta + \beta_{nl}^*{}^{-1})^{-1}.$$

Using Lemma 4

$$\lim_{n \rightarrow \infty} \beta_{nl}^*{}^{-1} = \frac{\sqrt{D} + Q_l - P_{l-1}}{2Q_{l-1}}$$

follows. Taking into account that

$$\beta = \frac{\sqrt{D} - Q_l + P_{l-1}}{2Q_{l-1}}$$

we obtain the statement.

**4. THEOREM 1.** Let  $\delta$  be a quadratical irrationality having the continued fraction representation

$$\delta = [d_1, d_2, \dots, d_l, \overline{c_1, c_2, \dots, c_l}],$$

where  $l$  denotes the length of the primitive period. Put

$$\gamma = [\overline{c_1, c_2, \dots, c_l}],$$

$$\frac{A_k}{B_k} = [c_1, c_2, \dots, c_n]$$

and suppose that the sequence  $\{c_n\}$  has the property

$$c_{k+l} = c_k \quad (k = 1, 2, 3, \dots)$$

for some natural  $l$ .

Chose  $h$  with the property that

$$\lim_{n \rightarrow \infty} B_{nl+h} ||B_{nl+h}\gamma|| = \mu_h$$

should have its possible minimal value. Let

$$\beta = [\overline{b_1, b_2, \dots, b_l}] = \overline{\{c_{h+1}, c_{h+2}, \dots, c_{h+l}\}}.$$

We assert, that

**ASSUMPTION:**

- A) If  $l$  is odd and  $k = 2l$ .
- B) If  $l$  is even and  $k = l$ , and there exists an odd  $\lambda$  with

$$Q_{l-1, \lambda} = Q_{l-1}.$$

**PROPOSITION:** then for all  $\alpha \sim \delta$  and for any natural number  $m$  the inequality

$$(1) \qquad q ||q\alpha|| \leq C_m^{-1}$$

has at least  $m$  solutions, where  $C_m$  stands for

$$(2) \qquad c_m = b_l + \beta_{km+1} + \beta_{km-1}^*$$

and the value of  $C_m$  is the best possible one.

**ASSUMPTION:**

- C) If  $l$  is even and from

$$Q_{l-1, \lambda} = Q_{l-1}$$

it follows, that  $\lambda$  is even,

**PROPOSITION:** then there exists an  $\alpha \sim \delta$  so that the inequality

$$(3) \qquad q ||q\alpha|| < v$$

has at most a finite number of solutions, where

$$v = \lim_{q \rightarrow \infty} q ||q\delta||.$$

**REMARK.** From Lemma 2 it follows, that  $C_m$  is uniquely determined.

**PROOF OF THEOREM 1.** It will be convenient to prove the assertions A, B and C separately.

**PROOF OF A).** If  $\alpha = \beta$ , then by (3.5), (1) yields for  $q = q_m$ , ( $n = k, 2k, \dots, mk$ ) and it is clear that the value of  $C_m$  can not be improved. If  $\alpha \neq \beta$  then

$$\alpha = [a_1, a_2, \dots, a_s, \overline{b_1, b_2, \dots, b_l}], \quad (a_{s+1} = b_1, a_{s+2} = b_2, \dots).$$

In the case  $a_s = b_l, a_{s-1} = b_{l-1}, \dots, a_1 = b_{l-s+1}$  ( $s < l$ ) we put

$$n = \begin{cases} s, s+k, \dots, s+(m-1)k & \text{if } s \text{ is even,} \\ s+l, s+3l, \dots, s+(2m-1)l & \text{if } s \text{ is odd.} \end{cases}$$

In all other cases the existence of an  $r$  with  $0 < r \leq s$ , satisfying  $a_s = b_l, a_{s-1} = b_{l-1}, \dots, a_{r+1} = b_{l-s+r+1}, a_r \neq b_{l-s+r}$  is guaranteed.

Let now  $n$  be given by

$$n = \begin{cases} s, s+k, \dots, s+(m-1)k & \text{if } s \text{ is even and } \begin{cases} a_r > b_{l-s+r}, r \text{ even} \\ a_r < b_{l-s+r}, r \text{ odd} \end{cases} \\ s+l, s+3l, \dots, s+(2m-1)l & \text{if } s \text{ is odd and } \begin{cases} a_r > b_{l-s+r}, r \text{ odd} \\ a_r < b_{l-s+r}, r \text{ even.} \end{cases} \end{cases}$$

In these cases

$$a_n + \alpha_{n+1} + \alpha_{n-1}^* \geq b_l + \beta_{mk+1} + \beta_{mk-1}^*,$$

so by (3.5)

$$q_n ||q_n \alpha|| \leq C_m^{-1}.$$

**PROOF OF B).** If  $\alpha = \beta$ , then by (3.5)(1) holds with  $q = q_n$  ( $n = \lambda, \lambda+k, \dots, \lambda+(m-1)k$ ) and as easy to see the value of  $C_m$  can not be improved.

If  $\alpha \neq \beta$  then

$$\alpha = [a_1, a_2, \dots, a_s, \overline{b_1 b_2, \dots, b_l}] \quad (a_{s+1} = b_1, a_{s+2} = b_2, \dots).$$

In the case  $a_s = b_l, a_{s-1} = b_{l-1}, \dots, a_1 = b_{l-s+1}$ , choose  $n$  with

$$n = \begin{cases} s, s+k, \dots, s+(m-1)k & \text{if } s \text{ is even} \\ s+\lambda, s+\lambda+k, \dots, s+\lambda+(m-1)k & \text{if } s \text{ is odd.} \end{cases}$$

In the remaining cases there is an  $r$  with  $0 < r \leq s$ , satisfying  $a_s = b_l, a_{s-1} = b_{l-1}, \dots, a_{r+1} = b_{l-s+r+1}, a_r \neq b_{l-s+r}$ . Choose  $n$  now by

$$n = \begin{cases} s, s+k, \dots, s+(m-1)k & \text{if } s \text{ is even and } \begin{cases} a_r > b_{l-s+r}, r \text{ is odd} \\ a_r < b_{l-s+r}, r \text{ is even} \end{cases} \\ s+\lambda, s+\lambda+k, \dots, s+\lambda+(m-1)k & \text{if } s \text{ is odd and } \begin{cases} a_r > b_{l-s+r}, r \text{ is even} \\ a_r < b_{l-s+r}, r \text{ is odd.} \end{cases} \end{cases}$$

So we have

$$a_n + \alpha_{n+1} + \alpha_{n-1}^* \geq b_l + \beta_{mk+1} + \beta_{mk-1}^*$$

and

$$\begin{aligned} a_n + \alpha_{k+1} + \alpha_{n-1}^* &\geq b_{\lambda+l} + \beta_{mk+\lambda+1} + [b_{mk+\lambda-1}, b_{mk+\lambda-2}, b_{\lambda+1}] = \\ &= b_l + \beta_{mk+1} + \beta_{mk-1}^* \end{aligned}$$

corresponding the cases mentioned respectively.

By (3.5) now follows that

$$q_n ||q_n \alpha|| \leq C_m^{-1}.$$

So the proof of B) is completed.

PROOF OF C). Let  $\alpha$  has the representation

$$\alpha = [b, b_2, b_3, \dots, \overline{b_l, b_1, b_2, \dots, b_l}], b > b_1.$$

Denote by  $0 = \lambda_1 < \lambda_2 < \dots < \lambda_r < l$  the non-negative integers  $u$  with  $u < l$  and

$$Q_{l-1, u} = Q_{l-1}.$$

If  $n \equiv \lambda_i \pmod{l}$ , for some  $i$  with  $1 \leq i \leq r$ , then

$$\begin{aligned} q_n ||q_n \alpha|| &= (b_{\lambda_i} + \alpha_{n+1} + \alpha_{n-1}^*)^{-1} > \\ &> (b_{m_i} + \beta_{ml+\lambda_i+1} + [b_{ml+\lambda_i-1}, b_{ml+\lambda_i-2}, \dots, b_{\lambda_i+1}])^{-1} \end{aligned}$$

holds for every  $m$  ( $m = 1, 2, \dots$ ). From this

$$q_n ||q_n \alpha|| \geq \gamma$$

follows.

In the remaining case

$$n \equiv k \pmod{l}, \quad (0 < k < l)$$

where  $k$  is different from the numbers  $\lambda_i$  ( $1 \leq i \leq r$ ). Then by the choice of  $\beta$ , and using Lemma 5 it follows that

$$\lim_{m \rightarrow \infty} q_{ml+k} ||q_{ml+k} \alpha|| = \frac{Q_{l-1, k}}{\sqrt{D}} > \frac{Q_{l-1}}{\sqrt{D}} = \nu.$$

Taking into account that from

$$q ||q \alpha|| > \frac{1}{2}$$

$q = q_n$  follows, we get that (3) has at most a finite number of solutions.

So the proof of C) is completed.

5. We are now in position to prove the following

THEOREM 2. Let  $A, B$ , and  $C$  be integers,

$$F(x, y) = Ax^2 + Bxy + Cy^2,$$

where

$$D = D(F) = B^2 - 4AC > 0.$$

Let  $\vartheta$  be an irrational number with  $F(\vartheta, 1) = 0$ . Put

$$(1) \quad \nu = \nu(\vartheta) = \lim q ||q \vartheta||,$$

$$(2) \quad \mu = \min_{(x,y) \neq (0,0)} |F(x, y)|.$$

where  $(x, y)$  runs on the lattice-points.

If for any  $\alpha \sim \vartheta$

$$q||q\alpha|| < r$$

has infinitely many solutions in  $q$ , then there exist lattice-points  $R_1 = R_1(x_1, y_1)$ ,  $R_2 = R_2(x_2, y_2)$  with

$$F(x_1, y_1) = \mu, \quad F(x_2, y_2) = -\mu.$$

PROOF OF THEOREM 2. Let  $\delta = \{\vartheta\}$ . Suppose that  $\delta$  has the representation

$$\delta = [d_1, d_2, \dots, \overline{d_l, c_1, c_2, \dots, c_l}].$$

Let the choice of  $\beta$  be the same as in Theorem 1. Put

$$f(x, y) = Q_{l-1}x^2 + (Q_l - P_{l-1})xy - P_l y^2,$$

$$\mu' = \min |f(x, y)|,$$

where the minimum is taken on the lattice points different from  $(0, 0)$  as before.

Now by (3.31), (1) using the choice of  $\beta$  and appealing to Lemma 5 we have  $\mu' = Q_{l-1}$ . So

$$\mu' = f(1, 0)$$

follows.

From Theorem 1 we can deduce that if  $l$  is even, then there exists an odd  $\lambda$  with  $Q_{l-1, \lambda} = Q_{l-1}$ . So we have

$$f(P_{l-1}, Q_{l-1}) = P_{l-1}(Q_{l-1}P_{l-1} - P_{l-1}Q_{l-1}) + Q_{l-1}(Q_l P_{l-1} - P_l Q_{l-1}).$$

By (3.13) and (3.15)

$$f(P_{l-1}, Q_{l-1}) = (-1)^l P_{l-1} Q_{l-1, \lambda} + (-1)^l Q_{l-1} Q_{l-1, \lambda} = (-1)^l Q_{l-1, \lambda} = -Q_{l-1} = -\mu'.$$

So the existence of  $R_3 = R_3(x_3, y_3)$ ,  $R_4 = R_4(x_4, y_4)$  with

$$f(x_3, y_3) = \mu', \quad f(x_4, y_4) = -\mu'$$

is guaranteed.

The same result for  $f(x, y)$  can be deduced in the case if  $l$  is odd. Then simply

$$f(P_{l-1}, Q_{l-1}) = Q_{l-1}(Q_l P_{l-1} - P_l Q_{l-1}) = (-1)^l Q_{l-1} = -\mu'.$$

Observing that by  $\beta \sim \vartheta$  one can find suitable integers  $a, b, c, d$  with

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$$

and satisfying

$$f(x, y) = rF(ax+by, cx+dy),$$

with a rational  $r$ . Now taking

$$\begin{aligned} x_1 &= ax_3 + by_3 & x_2 &= ax_4 + by_4 \\ y_1 &= cx_3 + dy_3 & y_2 &= cx_4 + dy_4 \end{aligned}$$

the statement of Theorem 2 follows.

6. Let  $a$  be a fixed natural number. Put  $A_n B_n^{-1}$  for the  $n$ 'th convergent to

$$\vartheta = \frac{\sqrt{a^2+4} - a}{2} = [a, a, a, \dots].$$

Put further

$$c_m = \frac{\sqrt{a^2+4} + a}{2} + \frac{A_{2m-1}}{B_{2m-1}} \quad (m=1, 2, 3, \dots).$$

In these notations hold the following

**THEOREM 3.** If  $a_n \geq a$  has infinitely many solutions in  $n$ , where  $a_n$  denotes the  $n$ th partial quotients of  $\alpha$ , then

$$(1) \quad q ||q\alpha|| \leq c_m^{-1}$$

has at least  $m$  solutions. The value of  $c_m$  can not be improved.

**PROOF OF THEOREM 3.** In the case  $\alpha \sim \vartheta$  the statement is a consequence of the prop. A) in Theorem 1, and it is clear that in the case  $\alpha \sim \vartheta$  the value of  $c_m$  can not be improved.

In the remaining case we ded with two subcases separately. In the first one  $a_n > a$  holds for infinitely many  $n$ . Then using a know theorem of A. HURWITZ [1], chapter I., we have that

$$q_n ||q_n \alpha|| < \frac{1}{\sqrt{a^2+2a+5}}$$

has an infinite number of solutions. Noticing that

$$\sqrt{a^2+2a+5} < \frac{\sqrt{a^2+4} + a}{2} + \frac{1}{a} \geq \frac{\sqrt{a^2+4} + a}{2} + \frac{A_{2m-1}}{B_{2m-1}} = c_m$$

holds, we get the statement of the Theorem 3.

We have now to deal with the case only, when  $a_n \leq a$  holds for all values of  $n$  large enough.

If for an infinite number of  $n$ ,  $a_n = a$ ,  $a_{n-1} < a$ ,  $a_{n-1} < a$  ( $a > 1$ ) then

$$\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots] > [a, a, a, \dots] = \vartheta = \frac{\sqrt{a^2+4} - a}{2},$$

$$\alpha_{n-1}^* = [a_{n-1}, a_{n-2}, \dots, a_1] \geq \frac{1}{a} = \frac{A_1}{B_1} \geq \frac{A_{2m-1}}{B_{2m-1}}$$

and the statement follows by (3.5). So we may suppose that for all  $n$  with  $a_n = a$  one of the equations  $a_{n-1} = a$ ,  $a_{n+1} = a$  holds if the index  $n$  is sufficiently large.

If  $a_{n-1} = a_n = a$ ,  $a_{n+1} = a-2$  ( $a > 2$ ) holds for infinitely many  $n$ , then

$$\alpha_{n+1} \geq \frac{1}{a-1} \quad \alpha_{n-1}^* \geq \frac{1}{a+1}.$$

It is easy to see, that

$$\frac{1}{a-1} + \frac{1}{a+1} \geq \frac{\sqrt{a^2+4}-a}{2} + \frac{1}{a}$$

so (1) is fulfilled with  $q = q_n$ . The case  $a_{n-1} \leq a-2$ ,  $a_n = a_{n+1} = a$  can be treated similarly.

If  $a > 1$  and  $a_{n-1} = a_n = a$ ,  $a_{n+1} = a-1$ ,  $a_{n+2} \geq a-1$  is fulfilled for an infinite number of  $n$ , so we have

$$\alpha_{n+1} = [a-1, \dots] > [a-1, 1, a, a] > [a-1, 1, a+1] = \frac{a-2}{a^2+2a-1}$$

$$\alpha_{n+1}^* = [a, \dots, a_1] \geq [a, a-1] = \frac{a-2}{a^2-a+1}.$$

Since

$$[1, 1, 2, 2] + [2, 1] > \sqrt{2} - 1 + 0,5$$

so we may suppose that  $a > 2$ . Now

$$\frac{a+2}{a^2+2a-1} + \frac{a-1}{a^2-a+1} \geq \frac{\sqrt{a^2+4}-a}{2} + \frac{1}{a}$$

is equivalent to

$$4a^8 - 4a^7 - 16a^6 + 16a^5 - 28a^4 + 44a^3 - 24a^2 + 4 \geq 0.$$

Using

$$4a^8 - 4a^7 - 16a^6 \geq 0 \quad (a \geq 3),$$

$$16a^5 - 28a^4 \geq 0,$$

$$44a^3 - 24a^2 + 4 \geq 0,$$

so (1) is fulfilled with  $q = q_n$ . The case  $a_{n-1} = a-1$ ,  $a_n = a_{n+1} = a$ ,  $a_{n+2} \geq a-1$  is similar. So the proof of Theorem 3 is completed.

We conclude by remarking, that Theorem 3 in the cases  $a = 1$  and  $a = 2$  represents the results do to A. V. PRASAD [3] and T. N. SINHA [4] respectively.

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# ON THE ROLE OF THE STRUCTURAL PROPERTIES OF FUNCTIONS IN THE THEORY OF THEIR WALSH—FOURIER EXPANSIONS

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To F. KÁRTESZI on his 60th birthday.  
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1. In the theory of orthogonal series theorems of the following type are of importance: If for a given orthonormal system  $\{\varphi_n(x)\}_{n=0}^{\infty}$  ( $x \in \langle a, b \rangle$ ) the coefficients of the series

$$(1) \quad \sum c_n \varphi_n(x)$$

satisfy a condition of the form

$$(2) \quad \sum c_n^2 \lambda(n) < \infty$$

— where  $\lambda(n)$  ( $> 0$ ) is a convenient sequence tending to infinity — then the series (1) converges, or it is summable in a suitable sense on a set  $E(mE > 0)$ . (In the following a condition of type (2) will be called a coefficient-condition.)

One of the most frequently used coefficient-conditions is the well known condition of RADEMACHER [1] and MENCHOV [2], which states, that the validity of

$$(3) \quad \sum c_n^2 (\log n)^2 < \infty$$

implies the convergence of the series (1) almost everywhere (a.e.) for any orthonormal system  $\{\varphi_n(x)\}$ .

Similarly every series (1) is summable ( $C, \alpha > 0$ ) a.e., provided that the coefficient-condition

$$(4) \quad \sum c_n^2 (\log \log n)^2 < \infty$$

holds (MENCHOV [3] and KACZMARZ [4]), and so on.

For special orthonormal systems the sufficient condition (3) can be replaced by weaker ones. So for example in the case of the trigonometric series, the series

$$(5) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges a.e. if for its coefficients the condition

$$(6) \quad \sum_{n=0}^{\infty} (a_n^2 + b_n^2) \log n < \infty$$

is fulfilled (KOLMOGOROV—SELIVERTSOV [5] and PLESSNER [6]).

Since the Lebesque functions [7] of the Walsh orthonormal system [8] have the order of magnitude  $O(\log n)$  (N. J. FINE [9]), therefore applying the generalisation of the above mentioned theorem of KOLMOGOROV and SELIVERTOV [10] we get, that a Walsh-series

$$(7) \quad \sum c_n w_n(x)$$

a.e. converges in the interval  $< 0, 1 >$  if for its coefficients the condition

$$(8) \quad \sum c_n^2 \log n < \infty$$

holds.

Quite similar results are known for a wide class of orthonormal polinomial systems (B. Sz.-Nagy [11]) and polinomial-like orthogonal systems [12].

All the conditions in the examples mentioned above – in accordance with the general condition (2) – imply the validity of  $\sum c_n^2 < \infty$  and so the series in question are orthogonal development of functions  $f(x) \in L^2$  by the theorem of RIESZ – FISCHER [13] [14], but besides this fact the coefficient-condition do not give any immediate information about the structur of the functions represented by the orthonormal-series.

Therefore arises the following question: What kind of structural properties can guarantee for a given function  $f(x) \in L^2$ , that its Fourier-coefficients

$$(9) \quad c_n = \int_a^b f(t) \varphi_n(t) dt$$

– with respect to a given orthonormal system  $\{\varphi_n\}$  – allow a prescribed coefficientcondition of type (2)?

A further question is the following: In which cases is it possible to give such a structural condition to the function  $f(x)$ , which is just equivalent to a given coefficient-condition (2) for its Fourier coefficients (9)? Long ago such kind of theorems are known in the case of special orthogonal systems and of special conditions (2) (ALEXITS [15], STETCHKIN [16], KOLMOGOROV [17], ULANOV [18]).

In 1957 ALEXITS and KRÁLIK [19] showed, that the greatest part of these above mentioned theorems can be proved by a general method for a wide class of coefficient-conditions (2), supposing that the groundsystem  $\{\varphi_n(x)\}$  has a property  $A$ , which is defined by:

**DEFINITION [1, 1].** An orthonormal system  $\{\varphi_n(x)\}$  ( $x \in < a, b >$ ) has the *property A*, if for the Fourier coefficients (9) of any function  $f(x) \in L^2 < a, b >$  the following three assertions are true:

$$(10) \quad a) \quad \int_a^b \left| f(x) - \sum_{k=0}^{n-1} c_k \varphi_k(x) \right|^2 dx = \sum_{k=n}^{\infty} c_k^2 = R_n$$

(i.e.  $\{\varphi_n(x)\}$  is a complete system in  $L^2 < a, b >$ .)

$$(11) \quad b) \quad R_n = O \left( \omega_2 \left( \frac{1}{n}, f \right) \right)^2,$$

$$c) \quad \left( \omega_2 \left( \frac{1}{n}, f \right) \right)^2 = O \left( \frac{1}{n^2} \sum_{k=1}^n k R_k \right),$$

where  $\omega_2(\delta, f)$  means the  $L^2$ -continuity modulus (or quadratic-integral modulus) of the function  $f(x)$ , which is defined by the equation

$$(11a) \quad \omega_2(\delta, f) = \sup_{|h| \leq \delta} \left\{ \int_a^b |f(x+h) - f(x)|^2 dx \right\}^{\frac{1}{2}}.$$

The above mentioned result of ALEXITS and KRÁLIK is the

**THEOREM [1, 1].** *If a given orthonormal system  $\{\varphi_n(x)\}$  ( $x \in [a, b]$ ) has the property A, then the necessary and sufficient condition, that the Fourier coefficients (9) of a function  $f(x) \in L^2[a, b]$  satisfy a condition of type (2) – with a non-decreasing from below concave function  $\lambda(x) > 0$  ( $x \in [1, \infty)$ ) – is, that for the  $L^2$ -continuity modulus of  $f(x)$  the relation*

$$(12) \quad \omega_2(\delta, f) = O\left(\frac{1}{\sqrt{\Phi(\delta^{-1})}}\right)$$

holds by such an increasing function  $\Phi(x) > 0$  ( $x \in [1, \infty)$ ) which allows the estimate

$$(13) \quad \int_1^\infty \frac{\lambda'(x)}{\Phi(x)} dx < \infty.$$

Since for the trigonometrical system the property A is valid [15], and since this system is similar from many point of views to the Walsh-system, therefore arises the following question: has the Walsh-system the property A?

In the third chapter of this note we shall prove (Theorem [3, 1]) that the Walsh-system has an analogous property to A, if we replace the notion of  $\omega_2(\delta, f)$  by  $\omega_2(\delta, f)$ , which is defined also in the third chapter.

Finally in 4 we shall give some corollaries of theorem [3, 1].

2. in this point we recall some notions and theorems to be used in the sequel. The most part of these results are due to N. J. FINE [9], and therefore the proofs will be omitted.

**DEFINITION [2, 1].** Let us denote by  $X$  the set of all real numbers of the interval  $0 \leq x \leq 1$ . We give the elements of  $X$  in a diadic representation

$$(14) \quad x = 0.x_1, x_2, x_3, \dots, x_n, \dots \quad (x_n = 0, 1),$$

where – to avoid the ambiguity – the diadic rationals  $x = p2^{-q}$  ( $q = 1, 2, \dots$ ,  $p = 0, 1, 2, \dots, 2^q$ ) shall have their finite expansion (i.e.  $x_n = 0$ , if  $n \geq q+1$ ).

**DEFINITION [2, 2].** Let  $\{r_n(x)\}_{n=0}^\infty$  be the Rademacher system given for any  $n$  and any  $x \in X$  in (14) by the equations

$$(15) \quad r_n(x) = \begin{cases} +1, & \text{if } x_{n+1} = 0 \\ -1, & \text{if } x_{n+1} = 1. \end{cases}$$

**DEFINITION [2, 3].** The orthonormal functions of the Walsh' system are defined (PALEY [20]) on  $X$  by

$$(16) \quad w_0(x) = 1$$

$$w_n(x) = r_{n_1}(x)r_{n_2}(x)\dots r_{n_k}(x) \quad (n \geq 1)$$

$$\text{for } n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_k}$$

where the integers  $n_i$  are uniquely determined by  $n_{i+1} < n_i$ .

**DEFINITION [2, 4].** Let  $G$  be the group of all such sequences  $\bar{x} = \{x_n\}_{n=1}^{\infty}$  which contain only the elements  $x_n = 1$  or  $x_n = 0$ , and the group-operation  $\oplus$  will be defined by the following formulas:

$$(17) \quad \begin{aligned} \text{if } \bar{x} = \{x_n\} \in G \text{ and } \bar{y} = \{y_n\} \in G \\ \text{then } \bar{x} \stackrel{\text{def}}{\oplus} \bar{y} = \{x_n + y_n \pmod{2}\}_{n=1}^{\infty}, \end{aligned}$$

where the sign  $+$  in the right bracket means the ordinary addition modulo 2.

The  $G$  is evidently an additive Abelian group, and each element  $\bar{x} \in G$  is (of) second order i.e.  $\bar{x} \oplus \bar{x} = \bar{0} = \{0, 0, 0, \dots\}$ .

**DEFINITION [2, 5].** Ordering to each element  $\bar{x} = \{x_n\} \in G$  the real number

$$(18) \quad \lambda(\bar{x}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n} = x,$$

we get such a function  $\lambda(\bar{x})$ , which maps the elements of  $G$  into the interval  $X$  defined in [2, 1] (in sign  $\lambda(\bar{x})$ ) ( $G \rightarrow X$ ).

**DEFINITION [2,6].** Let be ordered to each value  $x (= 0.x_1, x_2, \dots, x_n, \dots) \in X$  such an element  $\bar{x} = \mu(x) \in G$ , for which the sequence-elements of  $\bar{x}$  agree term by term with the digits of  $x \in X$  i.e. we define the function  $\mu(x)$  ( $X \rightarrow G$ ) by

$$(19) \quad \mu(x) = \mu(0.x_1, x_2, \dots, x_n, \dots) = \{x_1, x_2, \dots, x_n, \dots\} = \bar{x} \in G.$$

**THEOREM [2,1].** For all  $x \in X$  the equation

$$(20) \quad \lambda(\mu(x)) = x$$

is true, and conversely, if  $\lambda(\bar{x})$  is not a dyadic rational, then

$$(21) \quad \mu(\lambda(\bar{x})) = \bar{x} \quad \left| \lambda(\bar{x}) \neq \frac{p}{2^q} \right|$$

also holds.

Since the functions  $\lambda(\bar{x})$  ( $G \rightarrow X$ ) and  $\mu(x)$  ( $X \rightarrow G$ ) are defined for each element of  $G$  resp.  $X$ , we can introduce a new operation on  $X$  by the

**DEFINITION [2,7].** To each pair of elements  $x, y \in X$  let be ordered the element  $z = x \dot{+} y \in X$  by the following manner

$$(22) \quad z = x \dot{+} y = \lambda(\mu(x) \oplus \mu(y)).$$

This operation plays a fundamental part in the investigations of N. J. FINE [9] and similarly does in this work.

Since in group  $G$

$$(23) \quad \varrho(\bar{x}, \bar{y}) = \lambda(\bar{x} \oplus \bar{y})$$

is a metric, and  $\lambda(\bar{x}) (G \rightarrow X)$  is a contraction with respect to  $\varrho$ , i.e.

$$(24) \quad |\lambda(\bar{x}) - \lambda(\bar{y})| \leq \varrho(\bar{x}, \bar{y})$$

holds, therefore it is easy to prove the important

**THEOREM [2,2].** *For every pair of numbers  $x, h \in X$  the inequality*

$$(25) \quad |(x \dotplus h) - x| \leq h$$

*is valid, i.e.  $x - h \leq x \dotplus h \leq x + h$  (where + and - mean the ordinary addition and subtraction).*

From this result we get at once

**THEOREM [2,3].** *If  $\omega(\delta, f)$  denotes the ordinary continuity modulus of the continuous function  $f(x) (x \in X)$  i.e. if*

$$(26) \quad \omega(\delta, f) = \sup_{|h| \leq \delta} |f(x+h) - f(x)|,$$

*then for every  $h (0 \leq h \leq \delta < 1) \in X$*

$$|f(x \dotplus h) - f(x)| \leq \omega(\delta, f)$$

*is true.*

Introducing then the modulus of continuity  $\hat{\omega}(\delta, f)$  with respect to the operation  $\dot{+}$  by

$$(27) \quad \hat{\omega}(\delta, f) \stackrel{\text{def}}{=} \sup_{h \leq \delta} |f(x \dotplus h) - f(x)|,$$

we get, that for every continuous function  $f(x) (x \in X)$

$$(28) \quad \hat{\omega}(\delta, f) \leq \omega(\delta, f).$$

From the definitions [2,2], [2,3], [2,4] and [2,5] follows that the system of Walsh-functions can be regarded as the multiplicative Abelian character-group of the group  $G$ . In consequence of it we get from the theorem [2,1] and from the definition [2,7] the

**THEOREM [2,4].** *If  $x \in X$  is fixed, then for every  $y \in X$  – apart from the elements of a denumerable subset  $X_{\alpha_0}(x) \subset X$  – the equation*

$$(29) \quad w_n(x \dotplus y) = w_n(x)w_n(y)$$

*is valid for every Walsh-function  $w_n(x)$ .*

An other very important property of the operation  $\dot{+}$  is, that it generates – as the ordinary addition – a measurepreserving transformation with respect to the Lebesgue-measure. A consequence of it is the

**THEOREM [2,5].** *If  $f(x)$  is a Lebesgue integrable function on  $X$ , then for every fixed  $y (\in X)$   $f(x \dotplus y)$  is also integrable and the equation*

$$(30) \quad \int_0^1 f(x \dotplus y) dx = \int_0^1 f(x) dx$$

*is true.*

Finally from theorems [2,4] and [2,5] we get a very important theorem with respect to the Walsh-functions:

**THEOREM [2,6].** If  $\{c_n\}_{n=0}^{\infty}$  is the sequence of the Walsh–Fourier coefficients of the function  $f(x) \in L^2[0, 1]$  i.e. if

$$(31) \quad c_n = \int_0^1 f(t)w_n(t) dt \quad (n = 0, 1, 2, \dots),$$

then for every fixed  $h \in X$  the Walsh–Fourier coefficients

$$\gamma_n = \int_0^1 f(x+h)w_n(x) dx$$

of the function  $f(x+h)$  allow the representation

$$(32) \quad \gamma_n = c_n w_n(h).$$

3. Using theorem [2,6] we can give an answer to the question mentioned in 1.

For our purpose we introduce the following

**DEFINITION [3,1].** Let  $\dot{\omega}_2(\delta, f)$  be the  $L^2$ -continuity modulus of a given function  $f(x) \in L^2[0, 1]$  with respect to the operation  $\dot{+}$  (defined in (22)) i.e.:

$$(33) \quad \dot{\omega}_2(\delta, f) = \sup_{h \leq \delta} \left\{ \int_0^1 |f(x+h) - f(x)|^2 dx \right\}^{1/2}$$

**THEOREM [3,1].** If for a given function  $f(x) \in L^2[0, 1]$  we consider the series  $\sum_{n=0}^{\infty} c_n^2$  (the square sum of the Walsh–Fourier-coefficients of  $f(x)$ ) then the restsums  $R_n = \sum_{k=n}^{\infty} c_k^2$  of it can be estimated for every  $n$  by the following inequalities:

$$(34) \quad \frac{1}{4} \left[ \dot{\omega}_2 \left( \frac{1}{n}, f \right) \right]^2 \leq R_n \leq \frac{1}{2} \left[ \dot{\omega}_2 \left( \frac{4}{n}, f \right) \right]^2.$$

**PROOF.** Let be  $\sum c_n w_n(x)$  the Walsh–Fourier expansion of a given  $f(x) \in L^2[0, 1]$  i.e.:

$$(35) \quad f(x) \sim \sum_{n=0}^{\infty} c_n w_n(x) = \sum_{n=0}^{\infty} \left( \int_0^1 f(t)w_n(t) dt \right) w_n(x),$$

then – according to the theorem [2,6] – we get for an arbitrary  $h \in [0, 1]$  the following expansion

$$(36) \quad f(x+h) \sim \sum_{n=0}^{\infty} (c_n w_n(h)) w_n(x),$$

and therefore

$$(37) \quad f(x+h) - f(x) \sim \sum_{n=0}^{\infty} c_n [w_n(h) - 1] w_n(x).$$

Since  $f(x+h) \in L^2 <0, 1>$  then, taking into account the completeness of the Walsh-system, we get for every  $h \in <0, 1>$  from (33) and (37) the inequality

$$(38) \quad \sum_{n=0}^{\infty} c_n^2 [1 - w_n(h)]^2 = \int_0^1 |f(x+h) - f(x)|^2 dx \leq [\dot{\omega}_2(h, f)]^2.$$

Being  $|w_n(h)| = 1$  for every  $n$  and  $h$ , the estimate

$$(39) \quad \sum_{n=0}^{\infty} c_n^2 [1 - w_n(h)]^2 \leq 4 \sum_{n=0}^{\infty} c_n^2 < \infty$$

is true ( $f(x) \in L^2 <0, 1>$ ), and so the series on the left of (38) uniformly converges in the interval  $0 \leq h \leq 1$ . Integrating term by term we obtain from (38) the relation

$$(40) \quad \sum_{n=0}^{\infty} \frac{1}{\delta} \int_0^{\delta} c_n^2 [1 - w_n(h)]^2 dh \leq \frac{1}{\delta} \int_0^{\delta} [\dot{\omega}_2(h, f)]^2 dh$$

for every  $0 < \delta < 1$ .

Since  $\dot{\omega}_2(h, f)$  is a non-decreasing function of  $h$  it follows from (40) that

$$(41) \quad \sum_{n=0}^{\infty} \frac{1}{\delta} \int_0^{\delta} c_n^2 [1 - w_n(h)]^2 dh \leq [\dot{\omega}_2(\delta, f)]^2.$$

Finally, considering that for any  $h$  the equality  $w_n(h) = \pm 1$  holds — and therefore  $[1 - w_n(h)]^2 = 2[1 - w_n(h)]$  — we obtain from the last inequality, that the following estimate

$$(42) \quad 2 \sum_{n=0}^{\infty} \frac{1}{\delta} c_n^2 \int_0^{\delta} [1 - w_n(h)] dh \leq [\dot{\omega}_2(\delta, f)]^2$$

is valid for any  $\delta \in (0, 1)$ .

Let now  $\delta$  be given by  $\delta = 2^{-m}$ , where  $m$  denotes an arbitrary but fixed natural number.

It follows from the definition [2,3], that for the indices  $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_k} < 2^{m+1}$  ( $n_i > n_{i+1}$ ) the Walsh functions  $w_n(h)$  are identically equal to 1 on the interval  $<0, 2^{-m}>$ , and so, for such a  $\delta$ , (42) gives, that

$$(43) \quad 2 \sum_{n=2^{m+1}}^{\infty} 2^m c_n^2 \int_0^{2^{-m}} [1 - w_n(h)] dh \leq [\dot{\omega}_2(2^{-m}, f)]^2.$$

Again on the ground of the definition [2,3] we obtain, that for the indices  $n \geq 2^{m+1}$  the Walsh-function  $w_n(h)$  take in the interval  $<0, 2^{-m}>$  the values +1 and -1 on subsets  $E_+$  resp.  $E_-$  having both of them the measure  $2^{-(m+1)}$ .

Consequently (43) can be written in the following form:

$$(44) \quad 2 \sum_{n=2^{m+1}}^{\infty} 2^m c_n^2 \frac{1}{2^{m+1}} 2 \leq [\dot{\omega}_2(2^{-m}, f)]^2,$$

and so we get for any  $m \geq 1$  the estimate

$$(45) \quad R_{2^{m+1}} = \sum_{n=2^{m+1}}^{\infty} c_n^2 \leq \frac{1}{2} [\dot{\omega}_2(2^{-m}, f)]^2,$$

At last, let  $n (> 1)$  be an arbitrary but fixed natural number and let  $m (\geq 0)$  be chosen by the inequality

$$(46) \quad 2^{m+1} \leq n < 2^{m+2}.$$

According to (45), for such an index  $n$  the inequality

$$(47) \quad \sum_{k=n}^{\infty} c_k^2 \leq \sum_{k=2^{m+1}}^{\infty} c_k^2 \leq \frac{1}{2} [\dot{\omega}_2(2^{-m}, f)]^2$$

is valid.

Since from (46) one has

$$n = 2^{m+1} + \bar{n} \quad (\text{where } \bar{n} < 2^{m+1}),$$

so the relation

$$(48) \quad \frac{4}{n} = \frac{4}{2^{m+1} + \bar{n}} = \frac{1}{2^{m-1} + \frac{\bar{n}}{4}} > \frac{1}{2 \cdot 2^{m-1}} = \frac{1}{2^m}$$

holds. Considering that  $\dot{\omega}_2(\delta, f)$  is a non-decreasing function of  $\delta$ , it follows from (47) and (48), that for any  $n (> 1)$

$$(49) \quad R_n = \sum_{k=n}^{\infty} c_k^2 \leq \frac{1}{2} \left[ \dot{\omega}_2 \left( \frac{4}{n}, f \right) \right]^2.$$

and so the right hand side of (34) is proved.

To prove that the left hand side of (34) holds too, we start again with (38). Using (38)

$$(50) \quad \int_0^1 |f(x+h) - f(x)|^2 dx = \sum_{n=0}^{\infty} c_n^2 [1 - w_n(h)]^2.$$

For any natural value of the index  $n$  let  $m$  be defined by

$$2^m \leq n < 2^{m+1},$$

so one have

$$(51) \quad \frac{1}{n} \leq \frac{1}{2^m}.$$

From (16) we obtain that for every index  $k < 2^{m+1}$  the functions  $w_k(h)$  are identically equal to 1 on the interval  $< 0, 2^{-m} >$  and then, from (50) and (51) we get for any  $h < \frac{1}{n}$  the inequality

$$(52) \quad \int_0^1 |f(x+h) - f(x)|^2 dx = \sum_{k=2^{m+1}}^{\infty} c_k^2 [1 - w_k(h)]^2 \leq \sum_{k=n}^{\infty} c_k^2 [1 - w_k(h)]^2 \leq \\ \leq 4 \sum_{k=n}^{\infty} c_k^2,$$

from which — using the definition [3,1] —, the inequality

$$(53) \quad \left[ \dot{\omega}_2 \left( \frac{1}{n}, f \right) \right]^2 = \sup_{h \leq \delta} \int_0^1 |f(x+h) - f(x)|^2 dx \leq 4 \sum_{k=n}^{\infty} c_k^2$$

follows, and so the Theorem [3,1] is completely proved.

4. The inequality (34) of the Theorem [3,1] shows that the condition b) of the definition [1,1] is fulfilled by the Walsh system with the modulus  $\dot{\omega}_2$  instead of  $\omega_2$  and in the place of c) we have the estimate

$$(54) \quad \left[ \dot{\omega}_2 \left( \frac{1}{n}, f \right) \right]^2 = O(R_n).$$

After these we are in position to prove a theorem corresponding to Theorem [1,1]. The proof make use of some ideas applied by ALEXITS and KRÁLIK to the proof of the Theorem [1,1].

**THEOREM [4,1].** *In order that the Walsh-Fourier coefficients  $\{c_n\}$  of a function*

$$(55) \quad f(x) (\in L^2 < 0, 1 >) \sim \sum_{n=0}^{\infty} c_n w_n(x)$$

*shall satisfy a condition*

$$(56) \quad \sum_{n=1}^{\infty} c_n^2 \lambda(n) < \infty,$$

*— with a non decreasing function  $\lambda(x) > 0$  ( $x \in < 1, \infty >$ ), concave from below —, the existence of an increasing function  $\Phi(x) > 0$  ( $x \in < 1, \infty >$ ) with the following two conditions*

$$(57) \quad \int_1^{\infty} \frac{\lambda'(x)}{\Phi(x)} dx < \infty$$

$$(58) \quad \dot{\omega}_2(\delta, f) \leq \dot{\omega}_2(4\delta, f) = O \left( \frac{1}{\sqrt{\Phi(\delta^{-1})}} \right)$$

*is necessary and sufficient.*

**PROOF.** Since  $\sum c_n^2 < \infty$  we get from the monotonicity of  $\lambda(x)$  that

$$(59) \quad \sum_{n=1}^{\infty} c_n^2 \lambda(n) = \sum_{n=1}^{\infty} c_n^2 \left[ \int_1^n \lambda'(x) dx - \lambda(1) \right] = O(1) + \sum_{n=1}^{\infty} c_n^2 \int_1^n \lambda'(x) dx$$

holds, and so —  $\lambda'(x)$  being monotonically decreasing, — we obtain the relations

$$(60) \quad \sum_{n=1}^{\infty} c_n^2 \sum_{k=2}^n \lambda'(k) \leq \sum_{n=1}^{\infty} c_n^2 \int_1^n \lambda'(x) dx \leq \sum_{n=1}^{\infty} c_n^2 \sum_{k=1}^n \lambda'(k).$$

(60) shows, that the series (56) is equiconvergent with the series

$$(61) \quad \sum_{n=1}^{\infty} c_n^2 \sum_{k=1}^n \lambda'(k) = \sum_{n=1}^{\infty} \lambda'(n) \sum_{k=n}^{\infty} c_k^2 = \sum_{n=1}^{\infty} \lambda'(n) R_n.$$

Considering this last result, it follows: If for a function  $f(x) \in L^2(-1, 1)$  there exists a function  $\Phi(x)$  with the properties (58) and (59), then by (34)

$$(62) \quad \sum_{n=1}^{\infty} \lambda'(n) R_n = O(1) \sum_{n=1}^{\infty} \lambda'(n) \left[ \dot{\omega}_2 \left( \frac{4}{n}, f \right) \right]^2 = O(1) \sum_{n=1}^{\infty} \lambda'(n) \frac{1}{\Phi(n)} = \\ = O(1) \int_1^{\infty} \frac{\lambda'(x)}{\Phi(x)} dx < \infty,$$

and therefore (56) is fulfilled for the Walsh—Fourier coefficients of  $f(x)$ .

Conversely, if the condition (56) is valid for the Walsh—Fourier coefficients of  $f(x)$ , then being  $\sum \lambda'(n) R_n$  convergent too, we obtain from (34), that the relation

$$(63) \quad \sum_{n=1}^{\infty} \lambda'(n) \left[ \dot{\omega}_2 \left( \frac{1}{n}, f \right) \right]^2 < \infty$$

holds, and therefore there exists such a decreasing function  $\psi(x)$  ( $x \in (-1, \infty)$ ) for which the relations

$$(64) \quad \left[ \dot{\omega}_2 \left( \frac{1}{x}, f \right) \right]^2 \leq \psi(x), \\ \sum_{n=1}^{\infty} \lambda'(n) \psi(n) < \infty$$

simultaneously are fulfilled. Introducing then the non-decreasing function  $\Phi(x) = \frac{1}{\psi(x)}$  we get from the last inequalities, that for this function  $\Phi(x)$  the estimates

$$(65) \quad \dot{\omega}_2(\delta, f) = O \left( \frac{1}{\sqrt{\Phi(\delta^{-1})}} \right) \\ \int_1^{\infty} \frac{\lambda'(x)}{\Phi(x)} dx < \infty,$$

are valid, and so the Theorem [4,1] is completely proved.

If we apply our last theorem to the important class of continuous functions  $f(x)$  ( $x \in (-\infty, 1)$ ) — taking into account, that for the ordinary modulus of continuity  $\omega(\delta, f)$  the relations (l.c. (28))

$$(66) \quad \dot{\omega}_2(\delta, f) \leq \dot{\omega}(\delta, f) \leq \omega(\delta, f)$$

hold — we obtain the

**THEOREM [4,2].** *For the Walsh—Fourier coefficients  $\{c_n\}$  of a function  $f(x) \in C(-\infty, 1)$  the conditions of the type (56) are always fulfilled, if there exists an increasing function  $\Phi(x) > 0$  ( $x \in (-\infty, \infty)$ ) with the following two properties:*

$$(67) \quad \begin{aligned} \text{a)} \quad & \omega(\delta, f) = O\left(\frac{1}{\sqrt{\Phi(\delta^{-1})}}\right) \\ \text{b)} \quad & \int_1^\infty \frac{\lambda'(x)}{\Phi(x)} dx < \infty. \end{aligned}$$

Applying the generalisation of the Kolmogorov—Seliverstov-theorem mentioned in 1. we can assert the following corollary of Theorem [4,2] (l.c. (7), (8)).

**THEOREM [4,3].** *If the ordinary continuity modulus of  $f(x) \in C(-\infty, 1)$  ( $f(0) = f(1)$ ) allows an estimate of the form*

$$(68) \quad \omega(\delta, f) = O(|\log \delta|^{-(\frac{1}{2} + \epsilon)}),$$

*then the Walsh—Fourier expansion of  $f(x)$  converges a.e. on  $(-\infty, 1)$ .*

Similarly, applying a theorem of ORLICZ [21], one can prove (by Theorem [4,2]) the

**THEOREM [4,4].** *If for a given function  $f(x) \in C(-\infty, 1)$  ( $f(0) = f(1)$ ) there exists such an increasing function  $\Phi(x) > 0$  ( $x \in (-\infty, \infty)$ ), for which the relations*

$$(69) \quad \begin{aligned} \text{a)} \quad & \omega(\delta, f) = O\left(\frac{1}{\sqrt{\Phi(\delta^{-1})}}\right) \\ \text{b)} \quad & \int_2^\infty \frac{(\log \log x)^{1+\epsilon} \log x}{x \Phi(x)} dx < \infty \end{aligned}$$

*are valid, then the Walsh—Fourier expansions of  $f(x)$  converges a.e. on  $(-\infty, 1)$  with every arrangement of its terms.*

Among others we mention finally, that the Theorem [4,2] and its corollaries can be formulated in a localised form [19], because for the Walsh system the localisation—theorem of RIEMANN is true [9].

For example, if the local modulus of continuity of  $f(x)$

$$(70) \quad \omega(\delta, f; x_0 - \eta, x_0 + \eta) \stackrel{\text{def}}{=} \sup_{\substack{|x_1 - x_2| \leq \delta < \eta \\ (x_1, x_2 \in (x_0 - \eta, x_0 + \eta))}} |f(x_1) - f(x_2)|$$

has the order of magnitude

$$\omega(\delta, f; x_0 - \eta, x_0 + \eta) = O(|\log \delta|^{-(\frac{1}{2} + \epsilon)}),$$

then the Walsh—Fourier expansion of  $f(x)$  converges a.e. on the interval  $(x_0 - \eta, x_0 + \eta)$ .

This last result improves the local-condition of Fine [9] according to it, the condition

$$\omega(\delta, f; x_0 - \eta, x_0 + \eta) = o(|\log \delta|^{-1})$$

is sufficient for the convergence on the interval  $< x_0 - \eta, x_0 + \eta >$ .

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## ON RAMSEY-TYPE PROBLEMS

By

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In the present paper we deal with graphs having a finite number of vertices, single edges and no loops. The number of vertices of the graph  $G$  will be denoted by  $\pi(G)$ , the edge between the vertices  $A$  and  $B$  by  $AB$ . To indicate that a vertex or an edge belongs to the graph, we use the symbol  $\in$ . A graph is called complete, if any two of its vertices are connected by an edge. Complete graphs with  $k$  vertices are called complete  $k$ -tuples. The graph having as vertices the points  $U_1, U_2, \dots, U_{k+1}$  and edges  $U_1U_2, U_2U_3, \dots, U_kU_{k+1}$  is a path of length  $k$ ,  $U_1, U_{k+1}$  are its endpoints. Adding the edge  $U_{k+1}U_1$  we get a circuit of length  $k+1$ .  $\bar{G}$  will denote the complementary graph or complement of  $G$  (i.e. two vertices in  $\bar{G}$  are adjacent if and only if they are not adjacent in  $G$ ). A graph is connected if for each pair of its vertices there exists a path in  $G$  having these vertices as endpoints.

According to the well known theorem of RAMSEY [1] there exists for every system of natural numbers  $(k_1, k_2, \dots, k_r)$  a natural number  $N(k_1, k_2, \dots, k_r)$  with the property that for  $n \geq N(k_1, \dots, k_r)$  dividing the edges of a complete graph of  $n$  vertices into  $r$  distinct classes (colouring every edge with one of  $r$  different colours) at least for one  $i$  ( $1 \leq i \leq r$ ) the  $i$ -th class contains a complete  $k_i$ -tuple (there exists a one-coloured complete  $k_i$ -tuple.) The least value of  $N(k_1, \dots, k_r)$  is unknown for the general case. (For special cases see [2].)

Colouring the edges of complete graphs with  $r$  different colours, we may investigate problems about the existence of other special types of one-coloured subgraphs instead of one-coloured  $k$ -tuples — as in Ramsey-theorem. In this paper we shall consider two different types of graphs, namely:

- a) paths of given length and
- b) connected graphs.

Let  $g(k, l)$  denote the least integer for which in case  $\pi(G) \geq g(k, l)$  either  $G$  contains a path of length  $k$ , or  $\bar{G}$  one of length  $l$ .

Our main purpose is to prove the following

**THEOREM 1.** For  $k \geq l$  we have

$$(1) \quad g(k, l) = k + \left\lceil \frac{l+1}{2} \right\rceil.$$

Considering the other special case of this type of problems, let  $f_r(n)$  denote the greatest integer with the property, that colouring the edges of a complete  $n$ -tuple  $g$  with  $r$  colours arbitrarily, there exists always a one-coloured connected subgraph with at least  $f_r(n)$  vertices.

It is easy to see the following remark of P. ERDŐS: if a graph is not connected then its complement is connected, i.e.  $f_2(n) = n$ . We shall prove

**THEOREM 2.**

$$(2) \quad f_3(n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Now we turn to the proof of Theorem 1. First we prove  $g(k, l) \leq k + \left\lceil \frac{l+1}{2} \right\rceil$  by induction on  $k$ . For  $k=1$  the Theorem evidently holds and let us suppose that for all  $k$ -s less than this the statement is true. Let us consider a graph  $G$  with  $k + \left\lceil \frac{l+1}{2} \right\rceil$  vertices. If  $l < k$ , then for any subgraph of  $G$  with  $k-1 + \left\lceil \frac{l+1}{2} \right\rceil$  points holds that either itself contains a path of length  $k-1$ , or its complement a path of length  $l$ . For  $l=k$  we consider a subgraph with  $k-1 + \left\lceil \frac{l}{2} \right\rceil$  points. This or its complement contains a path of length  $k-1$ . Thus in every case can be supposed, that the length of the longest path of  $G$  is  $k-1$ . Let  $U_1, U_2, \dots, U_k$  be the consecutive vertices of such a path and  $U = \{U_1, \dots, U_k\}$ . We denote the remaining vertices by  $V_1, \dots, V_{\left\lceil \frac{l+1}{2} \right\rceil}$  and the set of them by  $V = \{V_1, \dots, V_{\left\lceil \frac{l+1}{2} \right\rceil}\}$ .

It clearly holds that

- (i) for all  $V_i \in V$  either  $V_i U_j \in \bar{G}$  or  $V_i U_{j+1} \in \bar{G}$
- (ii) for all  $V_i \in V$   $V_i U_1 \in \bar{G}$  and  $V_i U_k \in \bar{G}$
- (iii) for  $V_{i1}, V_{i2}, V_{i3} \in V$  and  $U_j, U_{j+1} \in U$

at least one of the latest points is connected in  $\bar{G}$  with at least two of  $V_{i1}, V_{i2}, V_{i3}$ .

Consider a maximal path of  $\bar{G}$  not containing  $U_1, U_k$  with the property that any edge of it connects a point of  $U$  with a point of  $V$ , and its endpoints are in  $V$ ; let us denote the endpoints by  $A$  and  $B$ , and the path by  $S$ . If  $S$  contains all points of  $V$ , then by adding the edges  $U_1 A, B U_k$  we have a path of length  $2 \left\lceil \frac{l+1}{2} \right\rceil \geq l$  in  $\bar{G}$ . So we may suppose that the set of points  $V$  not contained by  $S$  is not empty. Let this set be called  $W$ . Consider a maximal path  $q$  of  $\bar{G}$  not containing  $U_1, U_k$  and having no common points with  $S$ , such that any edge of it connects a point of  $U$  with a point of  $W$  and the endpoints of it,

called by  $C$  and  $D$ , are in  $W$ . We show that all points of  $V$  are contained either in  $S$  or in  $q$ . Suppose that  $X \in V$  but  $X \notin S, X \notin q$ . It is clear, that the number of vertices of  $S$  and  $q$  in  $U$  is at most  $\left[ \frac{l+1}{2} \right] - 3 < \left[ \frac{k-3}{2} \right] = \left[ \frac{k-2-1}{2} \right]$  since  $l \leq k$ . So there exist two points  $U_i, U_{i+1} \in \{U_2, \dots, U_{k-1}\}$  which do not belong either to  $S$  or to  $q$ . Applying (iii) for  $A, C, X \in V$  and  $U_i, U_{i+1} \in U$  we have a contradiction to the maximal properties of  $S$  and  $q$ .

So the sum of the length of  $S$  and  $q$  is  $2\left[ \frac{l+1}{2} \right] - 4$ . We add them the edges

$U_1A, BU_k, U_kC, DU_1$  and so we have a circuit of length  $2\left[ \frac{l+1}{2} \right]$  in  $\bar{G}$ . For odd  $l$  this contains a desired path with length  $l$ . For even  $l$  an easy reasoning shows that there are  $U_i, U_{i+1} \in U$  which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length  $l$  in  $\bar{G}$ . That completes the proof.

Now we give examples for graphs  $G$  with  $k + \left[ \frac{l+1}{2} \right] - 1$  points that have no path of length  $k$ , and for them at the same time  $\bar{G}$  have no path of length  $l$ .

a) Let  $G$  consist of the disjoint graphs  $H_1, H_2$  with  $k$  and  $\left[ \frac{l+1}{2} \right] - 1$  points respectively, where the graph  $H_1$  is complete.

b) For even  $l$  we can leave one of the edges of  $H_1$ . These graphs possess obviously the desired property.<sup>1</sup>

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph  $G$  into three classes, i.e. let the edges of  $G$  be coloured with red, yellow and blue colours. So we get the graphs  $G_r, G_y$  and  $G_b$  formed by the red, yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of  $G_r$ . Let us take a maximal red-connected subgraph  $R$ . It may be supposed that  $R$  is not empty and  $\pi(R) < \pi(G) = n$ . Let  $B$  be a point of  $G$  such that  $B \notin R$ . Since  $R$  is a maximal connected subgraph of  $G$ ,  $BR_i$  is not red for  $R_i \in R$ . So one may suppose that there are at least  $\frac{1}{2}\pi(R)$  points of  $R$  which are connected with  $B$  by blue edges.

Let  $V$  denote the set of these points of  $R$  and  $W$  be the maximal blue-connected subgraph that contains  $B$ . If  $Y$  is a point such that  $Y \notin R$  and  $Y \notin W$  then  $YV_i$  is yellow for  $V_i \in V$ . Let  $Q$  denote the maximal yellow-connected subgraph that contains  $Y$ . If there is no such  $Y$ ,  $Q$  denotes the empty set.  $R, W, Q$  contain together all points of  $G$ . Namely any points  $S \notin R$  is connected with a

<sup>1</sup> The weaker result  $g(k, l) \leq k+l$  can be easily proved. Let us consider any vertex  $P$  and a pair of paths of  $G$  and  $\bar{G}$  without common vertices except  $P$ . It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all  $P$  and all pairs.) From that the statement follows.

$V_i \in V$  either by a blue or a yellow edge, i.e. either  $S \in W$  or  $S \in Q$ . In both cases

$$\pi(W) + \pi(Q) \geq (\pi(G) - \pi(R)) + 2\pi(V) \geq n - \pi(R) + 2 \frac{\pi(R)}{2} = n,$$

and then

$$\max(\pi(W), \pi(Q)) \geq \frac{n}{2}$$

which completes the proof of  $f_3(n) \geq \left[ \frac{n+1}{2} \right]$ .

For the proof of  $f_3(n) \leq \left[ \frac{n+1}{2} \right]$  we prove the more general

LEMMA. For odd  $r$ ,  $n = (r+1)v$  ( $v = 1, 2, \dots$ )

$$(3) \quad f_r(n) \leq \frac{2}{r+1} n.$$

Here we use the following theorem:

The edges of a complete graph  $G_0$  with  $2k$  vertices can be coloured with  $2k-1$  colours so that the edges having common vertices have different colours ([3]). Let  $H$  be a graph with  $r+1$  vertices, and consider a colouring mentioned above for  $2k = r+1$ . Let us replace any vertex of  $H$  by an arbitrarily coloured complete  $v$ -tuple. Let the edges which connect vertices from two different  $v$ -tuples have the same colour as the edge connecting the corresponding vertices in  $H$ . This graph clearly satisfies the requirements and this proves (3).

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# A CENTRAL LIMIT THEOREM FOR THE SUMS OF A RANDOM NUMBER OF RANDOM VARIABLES

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To Professor F. KÁRTESZI on his 60th birthday.

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I. Let  $\{\Omega, \mathcal{A}, P\}$  be a probability space, i.e. let  $\mathcal{A}$  be a  $\sigma$ -algebra of some subsets of  $\Omega$  and let  $P$  be a measure defined on the measurable space  $\{\Omega, \mathcal{A}\}$  satisfying the condition  $P(\Omega) = 1$ . The elements of  $\mathcal{A}$  will be called events. Random variables are measurable functions defined on  $\Omega$ .  $\bar{A}$  stands for the event  $\Omega - A$ .

Let  $\xi_1, \xi_2, \dots$  be a sequence of random variables. In this paper we suppose that  $\xi_1, \xi_2, \dots$  are  $m$ -dependent and the sums  $S_n = \xi_1 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) form a martingale.

**DEFINITION 1.** *The sequence  $\xi_1, \xi_2, \dots$  is called a sequence of  $m$ -dependent random variables, if for all positive integers  $r, s, n$  such that  $1 \leq r < s \leq n$  the random variables  $(\xi_1, \dots, \xi_r)$  and  $(\xi_s, \dots, \xi_n)$  are independent provided that  $s-r > m$  ( $m$  is a nonnegative integer).*

**DEFINITION 2.** *The sequence  $S_1, S_2, \dots$  is a martingale if for every  $n$  we have with probability 1*

$$M(S_{n+1} | S_1, S_2, \dots, S_n) = S_n,$$

where the symbol  $M(\cdot | \cdot)$  denotes the conditional mathematical expectation.

A trivial example satisfying to Definitions 1. and 2. is the sequence of independent random variables with mean value 0. Another example is the following: let  $r_1(x), r_2(x), \dots$  be the sequence of the Rademacher functions defined on the interval  $[0, 1]$ . Let  $\Omega = [0, 1]$  and let  $A$  be the  $\sigma$ -algebra of the Borel measurable subsets of  $\Omega$ , and  $P$  the Lebesgue-measure on  $\Omega$ . Then the Rademacher functions are independent with respect to  $P$ . Define the random variable  $\xi_n$  as follows:

$$\xi_n = \prod_{k=n}^{n+m} r_k(x),$$

where  $m$  is a fixed integer. ( $n = 1, 2, \dots$ ). Then evidently the sequence  $\{\xi_n\}$  is a sequence of  $m$ -dependent random variables. It is also trivial that  $M(\xi_n) = 0$ . We verify now that  $M(\xi_{n+1} | \xi_1, \xi_2, \dots, \xi_n) = 0$  with probability 1. In fact, for every possible choice of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , where  $\varepsilon_i = \pm 1$ , ( $i = 1, 2, \dots, n$ ), we have

$$\begin{aligned} M(\xi_{n+1} | \xi_1 = \varepsilon_1, \dots, \xi_n = \varepsilon_n) &= \frac{1}{P(\xi_1 = \varepsilon_1, \dots, \xi_n = \varepsilon_n)} \int_{\{\xi_1 = \varepsilon_1, \dots, \xi_n = \varepsilon_n\}} \xi_{n+1} dx = \\ &= \frac{\prod_{i=1}^n \varepsilon_i}{P(\xi_1 = \varepsilon_1, \dots, \xi_n = \varepsilon_n)} \int_0^1 \xi_{n+1} \cdot \prod_{i=1}^n \frac{\xi_i + \varepsilon_i}{2} dx. \end{aligned}$$

A general term in the integrand is

$$\frac{\xi_{n+1} \xi_{i_1} \xi_{i_2} \dots \xi_{i_k} \xi_{j_1} \xi_{j_2} \dots \xi_{j_{n-k}}}{2^n},$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  and  $1 \leq j_1 < j_2 < \dots < j_{n-k} \leq n$  and  $i_l \neq j_p$  ( $l = 1, \dots, k$ ;  $p = 1, \dots, n-k$ ). In this product the Rademacher function  $r_{n+m+1}(x)$  plays role only in  $\xi_{n+1}$ . So it is independent of the other random variables in the product. Thus the integral of the general term is zero and we have

$$M(\xi_{n+1} | \xi_1 = \varepsilon_1, \dots, \xi_n = \varepsilon_n) = 0.$$

From this it follows that the sequence  $S_n = \xi_1 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) forms a martingale. We can take, of course, in this example any sequence  $\{\eta_n\}$  of independent random variables for which  $M(\eta_n) = 0$ , instead of  $r_n(x)$ .

Let

$$B_n^2 = M(S_n^2), \quad F_n(x) = P(S_n < B_n x), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

DIANANDA proved ([2]) that in the case of  $m$ -dependent random variables

$$\lim_{n \rightarrow +\infty} F_n(x) = \Phi(x),$$

if

1.  $M(\xi_j^2) \leq C$ ,  $M(\xi_j) = 0$ ,  $j = 1, 2, \dots$  where  $C$  is a positive constant,
2.  $\lim_{n \rightarrow +\infty} B_n^2/n$  exists and is positive,
3.  $\frac{1}{n} \sum_{j=1}^n \int_{|x| > \epsilon \sqrt{n}} x^2 dV_j(x) \rightarrow 0$  ( $n \rightarrow +\infty$ ),

where  $\epsilon > 0$  is an arbitrary fixed number and  $V_j(x)$  denotes the distribution function of  $\xi_j$ .

The aim of the present paper is to prove among others the following

**THEOREM 2.** Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables for which  $\{S_n\}$  ( $n = 1, 2, \dots$ ) is a martingale. Let us suppose that the conditions 1., 2. and 3. are fulfilled. Let further  $\{v_n\}$  be a sequence of positive integer-valued random variables for which  $v_n/n$  converges in probability to a positive random variable  $v$ . ( $P(v > 0) = 1$ .) Then

$$\lim_{n \rightarrow +\infty} P(S_{v_n} < B_{v_n}x) = \Phi(x).$$

Such assertions seem to be applicable to Monte-Carlo methods, to large-sample theory of sequential estimations, etc.

2. In what follows we need some further definitions and lemmas.

**DEFINITION 3.** The sequence  $\eta_1, \eta_2, \dots$  of random variables is called strongly mixing with limiting distribution  $F(x)$  if for any event  $B$ , with  $P(B) > 0$ , we have

$$\lim_{n \rightarrow +\infty} P(\eta_n < x|B) = F(x)$$

in every continuity point  $x$  of  $F(x)$ .

**DEFINITION 4.** The sequence  $\eta_1, \eta_2, \dots$  of random variables is called stable, if for any event  $B$ , with  $P(B) > 0$ , we have

$$\lim_{n \rightarrow +\infty} P(\eta_n < x|B) = F_B(x)$$

where  $F_B(x)$  is a distribution function and the limiting relation holds for every continuity point  $x$  of  $F_B(x)$ .

We remark that every strongly mixing sequence of random variables is also stable.

It can be easily seen, that all the discontinuity points of  $F_B(x)$  are also discontinuity points of  $F_B(x)$ . Thus the set of the discontinuity points of  $F_B(x)$  for all  $B$  is denumerable. Hence one can prove that the expression

$$Q_x(B) = P(B)F_B(x)$$

for every fixed  $x$  is a measure in  $B \in A$ , which is absolutely continuous with respect to  $P$ . Thus by the Radon–Nikodym theorem

$$Q_x(B) = \int_B \alpha_x(\omega) dP(\omega) \quad (\omega \in \Omega)$$

where  $\alpha_x(\omega)$  is the Radon–Nikodym derivative of  $Q$  with respect to  $P$  and hence it is determined uniquely modulo  $P$ . For every fixed  $x$   $\alpha_x(\omega)$  will be called the local density of the stable sequence  $\eta_1, \eta_2, \dots$

A necessary and sufficient condition concerning the stability of a sequence  $\eta_1, \eta_2, \dots$  is the following:

$$\lim_{n \rightarrow +\infty} P(\eta_n < x|B_k) = F_{B_k}(x), \quad (k = 0, 1, 2, \dots)$$

where  $B_k = \{\eta_k < x\}$  if  $k$  is any fixed positive integer, further  $B_0 = \Omega$ .

If we have

$$\lim_{n \rightarrow +\infty} P(\eta_n < x | B_k) = F(x), \quad (k = 0, 1, 2, \dots)$$

where  $B_k = \{\eta_k < x\}$ , ( $k = 1, 2, \dots$ ) and  $B_0 = \Omega$ , further  $F(x)$  is a distribution function not depending on  $k$ , then the sequence  $\eta_1, \eta_2, \dots$  is strongly mixing with limiting distribution  $F(x)$ .

We mention also that if a sequence  $\eta_1, \eta_2, \dots$  of random variables defined on the probability space  $\{\Omega, A, P\}$  is strongly mixing (resp. stable) with limiting distribution  $F(x)$  (with local density  $\alpha_x(\omega)$ ) and if  $Q$  is a probability measure which is absolutely continuous with respect to  $P$ , then it is also strongly mixing (resp. stable) on the probability space  $\{\Omega, A, Q\}$  with the same limiting distribution  $F(x)$  (with the same local density  $\alpha_x(\omega)$ ).

(For these definitions and results see [3]).

**LEMMA 1.** *If the conditions 1., 2. and 3. are satisfied then the sequence  $S_n/B_n$  is strongly mixing with limiting distribution  $\Phi(x)$ .*

**PROOF.** Put  $n > k+m+1$  and let us fix  $k$ .

$$P(S_n < B_n x | S_k < B_k x) = P(S_n - S_{k+m+1} + S_{k+m+1} < B_n x | S_k < B_k x).$$

Since  $S_{k+m+1}/B_n$  converges in probability to zero as  $n \rightarrow +\infty$  and  $S_n - S_{k+m+1}$  is independent of  $S_k$ , by a theorem due to H. CRAMÉR [7] we see that

$$\lim_{n \rightarrow +\infty} P(S_n < B_n x | S_k < B_k x) = \Phi(x).$$

**LEMMA 2.** *Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables and let  $k_n < m_n$  be two sequences of positive integers tending to infinity. Let further  $Q$  be any probability measure, which is absolutely continuous with respect to  $P$  and the Radon–Nikodym derivative of which is square integrable. If  $A_n$  is any event defined by the random variables  $\xi_{k_n}, \xi_{k_n+1}, \dots, \xi_{m_n}$ , then*

$$\limsup_{n \rightarrow \infty} P(A_n) = \limsup_{n \rightarrow \infty} Q(A_n).$$

**PROOF.** Let  $I_n$  be the indicator of the event  $A_n$ , i.e. let  $I_n = 1$ , if  $\omega \in A_n$ , and  $I_n = 0$  if  $\omega \notin A_n$ . Put  $f_n = I_n - P(A_n)$ , ( $n = 1, 2, \dots$ ). We have  $M(f_n) = 0$ ,  $M(f_n^2) = P(A_n)(1 - P(A_n))$ .

Further

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n f_k dP(\omega) = \lim_{n \rightarrow +\infty} (P(A_n A_k) - P(A_n)P(A_k)) = 0,$$

since, if  $n$  is large enough,  $A_n$  is independent of  $A_k$ . Let us consider the space  $H$  of all square integrable random variables. We define the inner product of  $\xi \in H$  and  $\eta \in H$  by

$$(\xi, \eta) = \int_{\Omega} \xi \eta dP.$$

Then  $H$  is a Hilbert-space and  $f_n \in H$ . It can be proved that if  $\|f_n\| < K$  ( $n = 1, 2, \dots$ ), where  $K$  is independent of  $n$ , further  $\lim_{n \rightarrow +\infty} (f_n, f_k) = 0$  for any fixed  $k$ , then for any square integrable  $\eta$  we have

$$\lim_{n \rightarrow +\infty} (f_n, \eta) = 0 \quad (\text{see e.g. [4]})$$

Let us put now  $\eta = dQ/dP$  (i.e. the Radon–Nikodym derivative of  $Q$  with respect to  $P$ ). Then

$$\lim_{n \rightarrow +\infty} \int_A (I_n - P(A_n)) \frac{dQ}{dP} dP = \lim_{n \rightarrow +\infty} (Q(A_n) - P(A_n)) = 0.$$

This means the assertion of Lemma 2.

**REMARK.** Put  $Q(A) = P(A|B)$ , where  $B$  is a fixed event of positive probability. Then  $dQ/dP = \chi_B/P(B)$  where  $\chi_B$  is the indicator of the event  $B$ .  $\chi_B/P(B)$  is square integrable. Under the conditions of Lemma 2 we have

$$\limsup_{n \rightarrow +\infty} P(A_n) = \limsup_{n \rightarrow +\infty} P(A_n|B).$$

This result can be found in [5]. The method of proof of Lemma 2 is essentially based on the proof of [5].

**LEMMA 3.** Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables having mean values 0 and finite variances  $D^2(\xi_i)$ . Let us suppose further that the sequence  $S_n = \xi_1 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) forms a martingale. Let  $k_n \leq m_n$  be two sequences of positive integers tending to infinity. Let

$$A_n = \left\{ \max_{k_n \leq j \leq m_n} \left| \sum_{k=k_n}^j \xi_k \right| \geq \lambda D(S_{m_n} - S_{k_n}) \right\}.$$

Then if  $Q$  is a probability measure, which is absolutely continuous with respect to  $P$  and the Radon–Nikodym derivative of  $Q$  with respect to  $P$  is square integrable, we have

$$\limsup_{n \rightarrow +\infty} Q \left( \max_{k_n \leq j \leq m_n} \left| \sum_{k=k_n}^j \xi_k \right| \geq \lambda D(S_{m_n} - S_{k_n}) \right) \leq \frac{1}{\lambda^2}.$$

**PROOF.**  $A_n$  and  $Q$  satisfy the conditions of Lemma 2. Thus it is enough to prove that

$$\limsup_{n \rightarrow +\infty} P(A_n) \leq \frac{1}{\lambda^2}.$$

For the proof of this assertion we use the standard method of proving the Kolmogorov's inequality for martingales. Let  $|S_j - S_{k_n}|$  be the first of those  $|S_j - S_{k_n}|$  ( $j = k_n, k_n + 1, \dots, m_n$ ) for which  $|S_j - S_{k_n}| \geq \lambda D(S_{m_n} - S_{k_n})$  if only such a  $j$  exists. Then

$$\begin{aligned} D^2(S_{m_n} - S_{k_n}) &= \int_{\Omega} (S_{m_n} - S_{k_n})^2 dP \geq \sum_{j=k_n}^{m_n} \int_{\{\nu(\omega)=j\}} [(S_j - S_{k_n})^2 + \\ &+ 2(S_j - S_{k_n})(S_{m_n} - S_j) + (S_{m_n} - S_j)^2] dP. \end{aligned}$$

Let

$$z_j(\omega) = \begin{cases} 2(S_j - S_{k_n}), & \text{if } v(\omega) = j \\ 0 & \text{if } v(\omega) \neq j. \end{cases}$$

Then the integral of the second term is

$$M(z_j(S_{m_n} - S_j)).$$

$z_j$  depends only on  $S_1, S_2, \dots, S_j$ . So we have

$$M(z_j(S_{m_n} - S_j)) = M(z_j M(S_{m_n} - S_j | S_1, S_2, \dots, S_j)) = 0,$$

because  $S_1, S_2, \dots$  forms a martingale. Consequently we have

$$\begin{aligned} D^2(S_{m_n} - S_{k_n}) &\geq \sum_{j=k_n}^{m_n} \int_{\{\nu(\omega)=j\}} (S_j - S_{k_n})^2 dP \geq \lambda^2 D^2(S_{m_n} - S_{k_n}) \sum_{j=k_n}^{m_n} P(\nu(\omega) = j) = \\ &= \lambda^2 D^2(S_{m_n} - S_{k_n}) P(A_n). \end{aligned}$$

This means the assertion of our Lemma 3.

**REMARK.** Put  $Q(A) = P(A|B)$  where  $B$  is a fixed event of positive probability. Then  $dQ/dP = \chi_B/P(B)$ , where  $\chi_B$  is the indicator of the event  $B$ . Under the conditions of Lemma 3 we have

$$\limsup_{n \rightarrow +\infty} P \left( \max_{k_n \leq j \leq m_n} \left| \sum_{k=k_n}^j \xi_k \right| \geq \lambda D(S_{m_n} - S_{k_n}) \mid B \right) = \frac{1}{\lambda^2}.$$

3. We are now in the position to formulate the first of our theorems.

**THEOREM 1.** Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables and let  $S_n = \xi_1 + \xi_2 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) be a martingale. Suppose further that conditions 1., 2. and 3. are satisfied. Let  $v_1, v_2, \dots$  be a sequence of positive integer-valued random variables for which  $v_n/n$  converges in probability to a positive random variable  $v$ . Then the sequence  $S_{v_n}/B_{v_n}$  is strongly mixing with limiting distribution  $\Phi(x)$ .

**PROOF.** For the proof the following well known lemma is used: Let  $\tau_1, \tau_2, \dots$  be a sequence of random variables converging in probability to the random variable  $\tau$ . Let  $a < b$  be continuity points of the distribution function of  $\tau$ , and consider  $A_n = \{a \leq \tau_n < b\}$ ,  $A = \{a \leq \tau < b\}$ . Then  $P(A_n \cap A) \rightarrow 0$ , if  $n \rightarrow +\infty$ , where  $A_n \cap A = A_n \bar{A} + \bar{A}_n A$  is the set theoretical symmetric difference of  $A_n$  and  $A$ .

Let  $\epsilon > 0$  be arbitrary. Put  $0 < a < b$  such that they be continuity points of the distribution function of  $v$  and

$$P(a \leq v < b) > 1 - \frac{\epsilon}{2}$$

be satisfied. Then by the above lemma there exists an index  $n_0 = n_0(\epsilon)$  such that for  $n \geq n_0$  we have

$$P \left( a \leq \frac{v_n}{n} < b \right) > 1 - \epsilon.$$

Define  $0 < \varrho < 1$  and  $K > 0$  such that

$$|\Phi(x \pm 2\varrho) - \Phi(x)| < \varepsilon, \quad 2(1 - \Phi(K)) < \varepsilon$$

hold. We have obviously  $K \rightarrow +\infty$  if  $\varepsilon \rightarrow 0$ . Put  $\delta > 0$  such that  $0 < \delta < \varepsilon$  and  $\varrho^2/\delta > K$  hold. ( $\varrho = \varrho(\varepsilon)$ ,  $K = K(\varepsilon)$ ,  $\delta = \delta(\varepsilon)$ ). Finally let us choose a subdivision of the interval  $[a, b]$  by the points  $a = a_0 < a_1 < \dots < a_k = b$  such that  $a_i$  be continuity points of the distribution function  $v$  and

$$0 < a_i - a_{i-1} < \frac{\delta a}{2}$$

hold. Then let us fix  $k$ .

We have for  $i = 1, 2, \dots, k$

$$\frac{S_{r_n}}{B_{r_n}} = \frac{S_{\{na_{i-1}\}}}{B_{\{na_{i-1}\}}} + \frac{S_{r_n} - S_{\{na_{i-1}\}}}{B_{\{na_{i-1}\}}} \cdot \frac{B_{\{na_{i-1}\}}}{B_{r_n}} + \left( \frac{B_{\{na_{i-1}\}}}{B_{r_n}} - 1 \right) \frac{S_{\{na_{i-1}\}}}{B_{\{na_{i-1}\}}}.$$

Let  $C(n, i, \varrho)$  ( $i = 1, 2, \dots, k$ ) denote the event

$$\left\{ \left| \frac{S_{r_n} - S_{\{na_{i-1}\}}}{B_{\{na_{i-1}\}}} \cdot \frac{B_{\{na_{i-1}\}}}{B_{r_n}} + \left( \frac{B_{\{na_{i-1}\}}}{B_{r_n}} - 1 \right) \frac{S_{\{na_{i-1}\}}}{B_{\{na_{i-1}\}}} \right| < 2\varrho \right\}.$$

Introduce also the following notations:

$$A_n = \left\{ a \leq \frac{v_n}{n} < b \right\}, \quad A = \{a \leq v < b\}$$

$$A_n^{(i)} = \left\{ a_{i-1} \leq \frac{v_n}{n} < a_i \right\}, \quad A^{(i)} = \{a_{i-1} \leq v < a_i\}, \quad (i = 1, 2, \dots, k)$$

Then for  $n \geq n_0$  we have

$$P(S_{r_n} < x B_{r_n}, \bar{A}_n) < \varepsilon.$$

Let  $B$  be any event. We have to investigate only the probability

$$P \left( \frac{S_{r_n}}{B_{r_n}} < x, A_n, B \right) = \sum_{i=1}^k P \left( \frac{S_{r_n}}{B_{r_n}} < x, A_n^{(i)}, B \right).$$

Now

$$\begin{aligned} P \left( \frac{S_{r_n}}{B_{r_n}} < x, A_n, B \right) &= \sum_{i=1}^k P \left( \frac{S_{r_n}}{B_{r_n}} < x, A_n^{(i)}, C(n, i, \varrho), B \right) + \\ &\quad + \sum_{i=1}^k P \left( \frac{S_{r_n}}{B_{r_n}} < x, A_n^{(i)}, \overline{C(n, i, \varrho)}, B \right). \end{aligned}$$

An easy calculation shows that

$$(1) \quad \sum_{i=1}^k P \left( \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} < x - 2\varrho, A_n^{(i)}, B \right) - \sum_{i=1}^k P(A_n^{(i)}, \overline{C(n, i, \varrho)}) - \varepsilon \leq \\ \leq P \left( \frac{S_{r_n}}{B_{r_n}} < x, B \right) \leq \\ \leq \sum_{i=1}^k P \left( \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} < x + 2\varrho, A_n^{(i)}, B \right) + \sum_{i=1}^k P(A_n^{(i)}, \overline{C(n, i, \varrho)}) + \varepsilon.$$

We have for any three events  $A$ ,  $B$  and  $C$

$$|P(AB) - P(AC)| \leq P(B \cap C).$$

On the other hand by the Lemma at the beginning of the proof we obtain

$$\sum_{i=1}^k P(A_n^{(i)} \cap A^{(i)}) \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

since  $k$  is fixed. Applying these we obtain

$$\left| \sum_{i=1}^k P \left( \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} < x \pm 2\varrho, A_n^{(i)}, B \right) - \sum_{i=1}^k P \left( \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} < x \pm 2\varrho, A^{(i)}, B \right) \right| \rightarrow 0,$$

as  $n \rightarrow +\infty$ . Since the sequence

$$\{S_{[na_{i-1}]}/B_{[na_{i-1}]}\}$$

is strongly mixing with limiting distribution  $\Phi(x)$ , we see that

$$(2) \quad \sum_{i=1}^k \lim_{n \rightarrow +\infty} P \left( \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} < x \pm 2\varrho, A^{(i)}, B \right) = \Phi(x \pm 2\varrho)P(BA).$$

Now from the choice of  $\varrho$ ,  $a$  and  $b$  it follows that

$$\Phi(x)P(B) - \varepsilon \leq \Phi(x \pm 2\varrho)P(BA) \leq \Phi(x)P(B) + \varepsilon.$$

From this and from (1) and (2) we obtain

$$\Phi(x)P(B) - 2\varepsilon - d \leq \liminf_{n \rightarrow +\infty} P \left( \frac{S_{r_n}}{B_{r_n}} < x, B \right) \leq \limsup_{n \rightarrow +\infty} P \left( \frac{S_{r_n}}{B_{r_n}} < x, B \right) \leq \\ = \Phi(x)P(B) + 2\varepsilon + d,$$

where

$$d = \limsup_{n \rightarrow +\infty} \sum_{i=1}^k P(A_n^{(i)}, \overline{C(n, i, \varrho)}).$$

Thus for the proof it remains only to prove  $d = 0$ . First we shall estimate the sum

$$(3) \quad \sum_{i=1}^k P \left( \left| \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} \right| \left| \frac{B_{[na_{i-1}]} - 1}{B_{r_n}} \right| < \varrho, A_n^{(i)} \right).$$

We have  $B_n \leq B_p$  if  $n < p$  since  $S_n$  is a martingale and so  $S_n^2$  is a semi-martingale. Further condition 2. ensures that  $B_n = \sqrt{n}c_n$  where  $\lim_{n \rightarrow +\infty} c_n = c > 0$ . Now if the event  $A_n^{(i)}$  takes place then

$$\left| \frac{B_{[na_{i-1}]} - 1}{B_{r_n}} \right| \leq \left| \frac{B_{[na_{i-1}]} - 1}{B_{[na_{i-1}]} - 1} \right| \leq \delta$$

if  $n \geq n_1(\delta) = n_1(\varepsilon)$ . So for  $n \geq n_1$  (3) becomes smaller than

$$(4) \quad \sum_{i=1}^k P \left( \left| \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} \right| > \frac{\varrho}{\delta}, A_n^{(i)} \right).$$

If we substitute again  $A_n^{(i)}$  by  $A^{(i)}$ , we commit an error of at most  $\varepsilon$  if  $n \geq n_2(\varepsilon)$ . (We suppose  $n_2 \geq n_1 \geq n_0$ .) We obtain from (4)

$$(5) \quad \sum_{i=1}^k P \left( \left| \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} \right| > \frac{\varrho}{\delta}, A_n^{(i)} \right) \leq \sum_{i=1}^k P \left( \left| \frac{S_{[na_{i-1}]}}{B_{[na_{i-1}]}} \right| > \frac{\varrho}{\delta}, A^{(i)} \right) + \varepsilon.$$

Now by the choice of  $\varrho > 0$  and  $\delta > 0$  we have  $\varrho/\delta > K$ . So, the sequence  $\{S_{[na_{i-1}]}/B_{[na_{i-1}]}\}$  being strongly mixing with limiting distribution  $\Phi(x)$ , we find an index  $n_3 = n_3(\varepsilon)$  (we suppose  $n_3 \geq n_2$ ) such that for  $n \geq n_3$  the left hand side of (5) is smaller than  $3\varepsilon$ . So (3) is smaller than  $3\varepsilon$  if  $n \geq n_3$ .

Next we turn to estimate

$$(6) \quad \sum_{i=1}^k P \left( \left| \frac{S_{r_n} - S_{[na_{i-1}]}}{B_{[na_{i-1}]}} \right| \left| \frac{B_{[na_{i-1}]} - 1}{B_{r_n}} \right| > \varrho, A_n^{(i)} \right).$$

If  $A_n^{(i)}$  takes place then (6) is smaller than

$$(7) \quad \sum_{i=1}^k P \left( \max_{[na_{i-1}] \leq j \leq [na_i]} |S_j - S_{[na_{i-1}]}| > \frac{\varrho}{1+\delta} B_{[na_{i-1}]}, A_n^{(i)} \right).$$

Applying again the substitution of the events  $A_n^{(i)}$  by the events  $A^{(i)}$  ( $i = 1, 2, \dots, k$ ) we will have for  $n \geq n_4 = n_4(\varepsilon)$  ( $n_4 \geq n_3$ )

$$(8) \quad \begin{aligned} & \sum_{i=1}^k P \left( \max_{[na_{i-1}] \leq j \leq [na_i]} |S_j - S_{[na_{i-1}]}| > \frac{\varrho}{1+\delta} B_{[na_{i-1}]}, A_n^{(i)} \right) \leq \\ & \leq \sum_{i=1}^k P \left( \max_{[na_{i-1}] \leq j \leq [na_i]} |S_j - S_{[na_{i-1}]}| > \frac{\varrho}{1+\delta} B_{[na_{i-1}]} \middle| A^{(i)} \right) P(A^{(i)}) + \varepsilon. \end{aligned}$$

The remark to Lemma 3 shows that we can find an index  $n_5 = n_5(\epsilon)$ , ( $n_5 \geq n_4$ ) such that for  $n \geq n_5$  the right hand side of (8) becomes smaller than

$$(9) \quad \sum_{i=1}^k \left| \frac{D^2(S_{[na_i]} - S_{[na_{i-1}]}) (1+\delta)^2}{\varrho^2 B_{[na_{i-1}]}^2} + \epsilon \right| P(A^{(i)}) + \epsilon \leq 1 \\ \leq \frac{(1+\delta)^2}{\varrho^2} \sum_{i=1}^k \left( \frac{B_{[na_{i-1}]}^2}{B_{[na_i]}^2} - 1 \right) P(A^{(i)}) + 2\epsilon \leq \frac{(1+\delta)^2 2\delta}{\varrho^2} \sum_{i=1}^k P(A^{(i)}) + 2\epsilon \leq \\ \leq \frac{2(1+\delta)^2}{K} + 2\epsilon.$$

Taking into account (9) and (3) we see that  $d \leq \frac{2(1+\delta)^2}{K} + 5\epsilon$ . Since  $\epsilon > 0$  was chosen arbitrarily, this means that  $d = 0$ . This proves Theorem 1.

A direct consequence of Theorem 1 is

**THEOREM 2.** Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables for which  $S_n = \xi_1 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) is a martingale. Let us suppose that the conditions 1., 2. and 3. are fulfilled. Let further  $v_n$  be a sequence of positive integer-valued random variables for which  $v_n/n$  converges in probability to a positive random variable  $v$ . Then

$$\lim_{n \rightarrow +\infty} P \left( \frac{S_{v_n}}{B_{v_n}} < x \right) = \Phi(x).$$

We formulate now another consequence of Theorem 1.

**COROLLARY 1.** Let  $Q$  be any probability measure defined on the measurable space  $\{\Omega, A\}$  which is absolutely continuous with respect to  $P$ . Then under the conditions of Theorem 1 we have

$$\lim_{n \rightarrow +\infty} Q \left( \frac{S_{v_n}}{B_{v_n}} < x \right) = \Phi(x).$$

4. It is interesting to investigate the limiting behaviour of the distribution of  $S_{v_n}$  when it is normalized by the deterministic factor  $B_{v_n}$ . In what follows we show that  $\{S_{v_n}/B_{v_n}\}$  is a stable sequence of random variables, the local density of which is  $\Phi(x/\bar{v})$ . Namely we prove the following more general

<sup>1</sup> Here we used the relation

$$D^2(S_{[na_i]} - S_{[na_{i-1}]}) = D^2(S_{[na_i]}) - D^2(S_{[na_{i-1}]}).$$

To prove this we remark that

$$M(S_{[na_i]} S_{[na_{i-1}]}) = M(S_{[na_{i-1}]} M(S_{[na_i]} | S_{[na_{i-1}]}) = M(S^2_{[na_{i-1}]}) = D^2(S_{[na_{i-1}]}),$$

by the martingale property of the sequence  $S_n$ . Now

$$D^2(S_{[na_i]} - S_{[na_{i-1}]}) = M((S_{[na_i]} - S_{[na_{i-1}]})^2) = \\ = D^2(S_{[na_i]}) + D^2(S_{[na_{i-1}]}) - 2M(S_{[na_i]} S_{[na_{i-1}]}) = D^2(S_{[na_i]}) - D^2(S_{[na_{i-1}]}).$$

**THEOREM 3.** Let  $\xi_1, \xi_2, \dots$  be a sequence of  $m$ -dependent random variables and let  $S_n = \xi_1 + \dots + \xi_n$  ( $n = 1, 2, \dots$ ) be a martingale. Suppose further that conditions 1., 2. and 3. are satisfied. Let  $v_1, v_2, \dots$  and  $\mu_1, \mu_2, \dots$  be sequences of positive integer-valued random variables for which  $v_n/n$  and  $\mu_n/n$  converge in probability to the positive random variables  $v$  and  $\mu$  respectively. Then the sequence  $\{S_{v_n}/B_{\mu_n}\}$  is stable with local density

$$\Phi\left(x\left(\frac{\mu}{v}\right)^{1/2}\right).$$

**PROOF.** Since  $B_n$  is of the form  $\sqrt{nc_n}$  where  $\lim_{n \rightarrow +\infty} c_n = c > 0$ , it can be easily seen that

$$\frac{B_{v_n}}{B_{\mu_n}}$$

converges in probability to  $\sqrt{\frac{v}{\mu}}$ . On the other hand

$$\frac{S_{v_n}}{B_{\mu_n}} = \frac{S_{v_n}}{B_{v_n}} \frac{B_{v_n}}{B_{\mu_n}},$$

where the first factor on the right hand side is a strongly mixing sequence with limiting distribution  $\Phi(x)$ , while the second member converges in probability to the random variable

$$\sqrt{\frac{v}{\mu}}.$$

Now we state without proof the following lemma (see [6], Theorem 1.).

**LEMMA 4.** Let  $\{\xi_n\}$  be a strongly mixing sequence of random variables with limiting distribution  $F(x)$ , and  $\eta_n$  be a sequence of random variables converging in probability to the random variable  $\eta$ . Let further  $g(x, y)$  be a continuous function of two variables. Then the sequence  $\{g(\xi_n, \eta_n)\}$  is stable with local density

$$\int_{\{g(x, \eta(\omega)) < z\}} dF(x).$$

Continuing the proof of our theorem we see that  $S_{v_n}/B_{v_n}$  is strongly mixing with limiting distribution  $\Phi(x)$ ,  $B_{v_n}/B_{\mu_n}$  converges in probability to  $\sqrt{\frac{v}{\mu}}$ , so the conditions of Lemma 4 are satisfied with  $g(x, y) = x \cdot y$ . Thus the sequence

$$S_{v_n}/B_{\mu_n}$$

is stable and the local density is

$$\int_{\{x \sqrt{\frac{v}{\mu}} < z\}} d\Phi(x) = \Phi\left(z \sqrt{\frac{\mu}{v}}\right).$$

So our assertion is proved.

COROLLARY 2. Putting in Theorem 3  $\mu_n = n$  we obtain that the sequence  $\{S_{\nu_n}/B_n\}$  is stable, with local density  $\Phi(z/\sqrt{\nu})$ .

COROLLARY 3. Let  $Q$  be any probability measure defined on the measurable space  $\{\Omega, \mathcal{A}\}$  which is absolutely continuous with respect to  $P$ . Then under the conditions of Theorem 3  $\{S_{\nu_n}/B_{\mu_n}\}$  is also stable on the probability space  $\{\Omega, \mathcal{A}, Q\}$  and the local density is the same as in Theorem 3.

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## ОПЕРАТОРЫ ИЗ $l_2 \rightarrow l_2$ , ИНДУЦИРОВАННЫЕ МАТРИЦАМИ

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Через  $A$ ,  $B$ , ... обозначаются бесконечные матрицы, состоящие из произвольных комплексных чисел. Если

$$(1) \quad A = \begin{bmatrix} a_{11} & \dots & a_{1n} & \dots \\ \vdots & & \vdots & \\ a_{n1} & \dots & a_{nn} & \dots \\ \vdots & & \vdots & \end{bmatrix} = [a_{ik}]_{\Gamma}$$

произвольная бесконечная матрица, то сопряженной матрицей называется матрица  $A^* = [a_{ik}^*]_{\Gamma}$ , в которой  $a_{ik}^* = \bar{a}_{ki}$ . Очевидно, что  $(A^*)^* = A$ .

Пусть

$$(2) \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \\ \ddots \end{bmatrix} = [x_k]_{\Gamma}$$

произвольный вектор-столбец, состоящий из любых комплексных чисел; мы говорим, что произведение матрицы (1) на вектор (2) — т.е. вектор  $Ax$  — определен, если ряды  $\sum_{k=1}^{\infty} a_{ik}x_k$  для всех  $i = 1, 2, \dots$  сходятся; при этом вектор  $Ax$  определяется равенством

$$Ax = y, \quad y = [y_i]_{\Gamma}, \quad \text{где} \quad y_i = \sum_{k=1}^{\infty} a_{ik}x_k.$$

При данной матрице  $A$  обозначим через  $I_{\max}(A)$  множество всех векторов  $x$  из пространства  $l_2$ , для которых вектор  $Ax$  определен и принадлежит к  $l_2$ , т.е.

$$I_{\max}(A) = \{x; x \in l_2, Ax \in l_2\}.$$

Очевидно, что  $I_{\max}(A)$  является линейным многообразием в пространстве  $l_2$  и при этом встречается и такой случай, когда  $I_{\max}(A)$  состоит только из нулевого вектора. Действительно, если обозначим

$$(3) \quad C = \begin{bmatrix} 1 & & & \cdots \\ 0 & 1 & & \cdots \\ 1 & 0 & & \cdots \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \\ \vdots & & & \ddots \end{bmatrix},$$

где ненаписанные элементы равны нулю, то легко видеть, что  $I_{\max}(C)$  состоит только из нулевого вектора.

В дальнейшем жирными буквами обозначаются линейные, вообще не ограниченные операторы. Обозначим через  $l_2 \rightarrow l_2$  множество всех линейных операторов  $A$ , определенных в некотором линейном подмножестве пространства  $l_2$  и принимающих значения также из пространства  $l_2$  т.е.

$$l_2 \rightarrow l_2 = \{A; \mathcal{D}(A) \subset l_2, \mathcal{R}(A) \subset l_2\},$$

где  $\mathcal{D}(A)$  и  $\mathcal{R}(A)$  обозначают соответственно область определения и область значений оператора  $A$ .

**ОПРЕДЕЛЕНИЕ.** Оператором, индуцированным матрицей  $A$  называется оператор  $A$ , определенный равенством

$$(4) \quad Ax = Ax, \mathcal{D}(A) = I, I \subset I_{\max}(A),$$

где  $I$  любое линейное подмножество множества  $I_{\max}(A)$ .

Очевидно, что оператор, определенный равенство (4), является оператором из  $l_2 \rightarrow l_2$ , т.е.  $A \in l_2 \rightarrow l_2$ .

Поскольку линейное подмножество  $I$  множества  $I_{\max}(A)$  может быть выбрано произвольно, то одна и та же самая матрица  $A$  может индуцировать нескольких операторов.

**ОПРЕДЕЛЕНИЕ.** Максимальным оператором, индуцированным матрицей  $A$  называется оператор  $A_{\max}$ , определенный равенством

$$A_{\max}x = Ax, \mathcal{D}(A_{\max}) = I_{\max}(A).$$

Очевидно, что  $A_{\max}$  является распространением любого оператора, индуцированного матрицей  $A$ .

В этой статье мы подробно исследуем некоторые операторы, индуцированные матрицами, удовлетворяющими некоторым весьма общим условиям. Именно, интересно выяснить условия, при которых некоторые операторы, индуцированные матрицей  $A$ , являются замкнутыми операторами или плотно определенными операторами в пространстве  $l_2$ . В том случае, когда некоторый оператор  $\mathbf{A}$ , индуцированный матрицей  $A$ , плотно определен в  $l_2$ , существует сопряженный оператор  $\mathbf{A}^*$  и можно поставить следующий вопрос: является ли оператор  $\mathbf{A}^*$  оператором, индуцированным матрицей  $A^*$ ?

1. Исследуем сначала тот случай, когда все столбцевые вектора матрицы (1) принадлежат пространству  $l_2$ , т.е. когда при обозначении

$$(5) \quad \mathbf{v}_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \\ \vdots \\ \vdots \end{bmatrix}, \quad (k = 1, 2, \dots).$$

выполняются следующие условия:

$$\|\mathbf{v}_k\|^2 = \sum_{i=1}^n |a_{ik}|^2 < +\infty, \quad (k = 1, 2, \dots).$$

Определим через  $l_0$  множество всех конечных векторов, т.е. множество таких векторов, которые имеют только конечное число координат, неравных нулю. Очевидно, что  $l_0$  является линейным многообразием пространства  $l_2$ , лежащим плотно в  $l_2$ . Определим оператор  $\mathbf{A}_0$  следующим образом:

$$(6) \quad \mathbf{A}_0 \mathbf{x} = A \mathbf{x}, \quad \mathcal{D}(\mathbf{A}_0) = l_0.$$

Покажем, что  $l_0 \subset l_{\max}(A)$  и  $\mathbf{A}_0 \in l_2 \rightarrow l_2$ , т.е.  $\mathbf{A}_0$  один из операторов, индуцированных матрицей  $A$ .

Действительно, если  $\mathbf{x} \in l_0$  произвольный вектор, имеющий вид

$$(7) \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ \vdots \end{bmatrix},$$

то принимая во внимание обозначение (5), легко видеть, что

$$\mathbf{A}_0 \mathbf{x} = A \mathbf{x} = \sum_{k=1}^n x_k \mathbf{v}_k,$$

и отсюда по нашему предположению относительно векторов  $\mathbf{v}_k$  уже следует, что  $\mathbf{A}_0 \mathbf{x} \in l_2$ .

Так как оператор  $\mathbf{A}_0$  определен плотно в  $l_2$ , то сопряженный оператор  $\mathbf{A}_0^*$  существует. В связи с этим докажем следующий результат:

**ТЕОРЕМА 1.** *Сопряженный оператор оператора  $\mathbf{A}_0$  есть максимальный оператор, индуцированный матрицей  $A^*$ , т.е.*

$$\mathbf{A}_0^* \mathbf{y} = A^* \mathbf{y}, \quad \mathcal{D}(\mathbf{A}_0^*) = l_{\max}(A^*).$$

**ДОКАЗАТЕЛЬСТВО.** Прежде всего введем следующее соглашение: если  $\mathbf{x} = [x_i]_1^n$  и  $\mathbf{y} = [y_i]_1^n$  произвольные векторы, не обязательно принадлежащие к  $l_2$ , для которых ряд  $\sum_{i=1}^{\infty} x_i \bar{y}_i$  сходится, то обозначим через  $(\mathbf{x}, \mathbf{y})$  сумму этого ряда. (Если  $\mathbf{x}$  и  $\mathbf{y} \in l_2$ , то  $(\mathbf{x}, \mathbf{y})$  является обычным скалярным произведением.)

После этого покажем, что для всех  $\mathbf{x} \in l_0$  и  $\mathbf{y} \in l_2$  выполняется равенство

$$(8) \quad (\mathbf{A}\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y}).$$

Действительно, если  $\mathbf{x}$  произвольный вектор вида (7) и  $\mathbf{y} = [y_i]_1^n \in l_2$ , то

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{k=1}^n |a_{ik} x_k \bar{y}_i| &\leq \sum_{k=1}^n |x_k| \left\{ \sum_{i=1}^{\infty} |a_{ik}|^2 \right\}^{1/2} \cdot \left\{ \sum_{i=1}^{\infty} |y_i|^2 \right\}^{1/2} = \\ &= \|\mathbf{y}\| \cdot \sum_{k=1}^n |x_k| \|\mathbf{v}_k\| < +\infty, \end{aligned}$$

откуда следует, что ряд  $\sum_{i=1}^{\infty} \sum_{k=1}^n a_{ik} x_k \bar{y}_i$  абсолютно сходящийся, таким образом перменяя порядок суммирования, получим

$$(9) \quad (\mathbf{A}\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \left( \sum_{k=1}^n a_{ik} x_k \right) \bar{y}_i = \sum_{k=1}^n x_k \left( \sum_{i=1}^{\infty} a_{ki}^* y_i \right) = (\mathbf{x}, A^*\mathbf{y}),$$

т.е. мы пришли к равенству (8).

Заметим, что хотя  $A^*\mathbf{y}$  определено для всякого  $\mathbf{y} \in l_2$ , но вообще  $A^*\mathbf{y}$  не принадлежит к  $l_2$ .

Обозначим через  $\mathbf{T}$  максимальный оператор, индуцированный матрицей  $A^*$ , т.е. оператор  $\mathbf{T}$  определяется равенством

$$\mathbf{T}\mathbf{y} = A^*\mathbf{y}, \quad \mathcal{D}(\mathbf{T}) = l_{\max}(A^*).$$

Нам нужно доказать, что  $\mathbf{T} = \mathbf{A}_0^*$ .

Пусть  $y \in \mathcal{D}(T)$  произвольный вектор, тогда равенство (8) выполняется для всех  $x \in l_0$ , и принимая во внимание определения операторов  $A_0$  и  $T$ , получим, что

$$(A_0x, y) = (x, Ty)$$

для всех  $x \in \mathcal{D}(A_0)$ , а это означает, что  $y \in \mathcal{D}(A_0^*)$  и  $A_0^*y = Ty$ , т.е. мы доказали, что  $T \subset A_0^*$ .

Наоборот, пусть  $y \in \mathcal{D}(A_0^*)$  произвольный вектор, тогда для всех  $x \in \mathcal{D}(A_0)$  выполняется равенство  $(A_0x, y) = (x, A_0^*y)$ . Так как одновременно  $y \in l_2$ , то из равенства (8) получим, что  $(A_0x, y) = (x, A^*y)$ . Вычитая полученные равенства друг из друга, и обозначая  $z = A_0^*y - A^*y$ , получим, что для всех  $x \in l_0$  выполняется равенство

$$(10) \quad (x, z) = 0.$$

Отметим, что если вектор  $z$  принадлежал бы к  $l_2$ , то из выполнения равенства (10) для всех  $x \in l_0$  и из того, что  $l_0$  лежит плотно в  $l_2$ , уже следовало бы, что  $z = 0$ . Но здесь принадлежность вектора  $z$  к пространству  $l_2$  не известна, так как хотя  $A_0^*y \in l_2$ , но не известно, что вектор  $A^*y$  принадлежит ли к  $l_2$ .

Однако покажем, что  $z = 0$ . Для этого пусть  $z = [z_i]_1^\infty$  и обозначим через  $\mathbf{h}_k$  тот вектор,  $k$ -тая координата которого равна  $z_k$ , а остальные координаты равны нулю, тогда  $\mathbf{h}_k \in l_0$  и из равенства (10) при  $x = \mathbf{h}_k$  получим, что  $(\mathbf{h}_k, z) = |z_k|^2 = 0$ , ( $k = 1, 2, \dots$ ), из которого уже следует наше утверждение.

Равенство  $z = 0$  означает, что  $A_0^*y = A^*y$ , из которого следует, что  $y \in l_{\max}(A^*)$ , т.е.  $y \in \mathcal{D}(T)$  и  $A_0^*y = Ty$ , которое означает, что  $A_0^* \subset T$ , а это вместе с утверждением  $T \subset A_0^*$  уже означает, что  $A_0^* = T$  и теорема доказана.

Из доказанной теоремы сразу вытекает следующее

**СЛЕДСТВИЕ.** Если  $A$  произвольный оператор, индуцированный матрицей  $A$  и являющийся расширением оператора  $A_0$ , то сопряженный оператор  $A^*$  является оператором, индуцированным матрицей  $A^*$ , т.е.  $A^*y = A^*y$  и при этом

$$(11) \quad \mathcal{D}(A^*) = \{y; y \in l_{\max}(A^*), (Ax, y) = (x, A^*y) \text{ для всех } x \in \mathcal{D}(A)\}.$$

Действительно, пусть  $A$  произвольный оператор, определенный равенством (4) и  $A \supset A_0$ , тогда  $A^* \subset A_0^*$ , таким образом нуждается в доказательстве только равенство (11), но это сразу следует из определения  $\mathcal{D}(A^*)$ , так как  $\mathcal{D}(A^*)$  состоит из тех  $y \in l_2$ , для которых существует вектор  $z \in l_2$  такой, что равенство  $(Ax, y) = (x, z)$  выполняется для всех  $x \in \mathcal{D}(A)$ , при этом здесь  $Ax = Ax$  и  $z = A^*y = A^*y$ , но тогда  $y \in l_{\max}(A^*)$  и отсюда равенство (11) уже следует.

Рассматривая теперь некоторое расширение оператора  $A_0$ , определим оператор  $A_1$  равенством

$$(12) \quad A_1x = Ax,$$

$$(13) \quad \mathcal{D}(A_1) = \left\{ x; x = [x_k]_1^\infty \in l_2, \sum_{k=1}^{\infty} |x_k| \|v_k\| < +\infty \right\}.$$

**ТЕОРЕМА 2.**  $\mathbf{A}_1$  является оператором, индуцированным матрицей  $A$ , т.е.  $\mathbf{A}_1 \in l_2 \rightarrow l_2$ , кроме того  $\mathbf{A}_0 \subset \mathbf{A}_1$  и  $\mathbf{A}_1^*$  есть максимальный оператор, индуцированный матрицей  $A^*$ , т.е.  $\mathbf{A}_1^* = \mathbf{A}_0^*$ .

**ДОКАЗАТЕЛЬСТВО.** Если  $\mathbf{x} = [x_k]_1^\infty \in \mathcal{D}(\mathbf{A}_1)$  и  $\mathbf{y} = [y_i]_1^\infty \in l_2$  произвольные вектора, то на основании неравенства Коши-Буняковского

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik} x_k \bar{y}_i| \leq \|\mathbf{y}\| \sum_{k=1}^{\infty} |x_k| \|\mathbf{v}_k\| < +\infty,$$

т.е. двойной ряд  $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} x_k \bar{y}_i$  абсолютно сходится, отсюда следует, что ряд

$\sum_{i=1}^{\infty} \left( \sum_{k=1}^{\infty} a_{ik} x_k \right) \bar{y}_i$  сходится при каждом  $\mathbf{x} \in \mathcal{D}(\mathbf{A}_1)$  и для всех  $\mathbf{y} \in l_2$ , а отсюда

по известной теореме Телица — Ландау следует, что  $\sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} x_k \right|^2 < +\infty$ ,

из которого следует, что при каждом  $\mathbf{x} \in \mathcal{D}(\mathbf{A}_1)$  вектор  $A\mathbf{x}$  определен и  $A\mathbf{x} \in l_2$ , а это означает, что  $\mathcal{D}(\mathbf{A}_1) \subset l_{\max}(A)$  и так как  $\mathcal{D}(\mathbf{A}_1)$  линейное многообразие пространства  $l_2$ , то оператор  $\mathbf{A}_1$ , определенный равенством (12) и (13), является оператором, индуцированным матрицей  $A$ .

Очевидно, что  $l_0 \subset \mathcal{D}(\mathbf{A}_1)$ , а отсюда уже следует, что  $\mathbf{A}_0 \subset \mathbf{A}_1$ , т.е.  $\mathbf{A}_1$  является распространением оператора  $\mathbf{A}_0$ , из которого следует, что  $\mathbf{A}_1^* \subset \mathbf{A}_0^*$ .

Докажем, что  $\mathcal{D}(\mathbf{A}_1^*) = l_{\max}(A^*)$ . Пусть  $\mathbf{x} = [x_k]_1^\infty \in \mathcal{D}(\mathbf{A}_1)$  и  $\mathbf{y} = [y_i]_1^\infty \in l_2$  произвольные вектора, тогда из вышесказанной абсолютной сходимости ряда  $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} x_k \bar{y}_i$  легко доказать — подобно равенству (9) — что

$$(14) \quad (A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y}).$$

В этом равенстве вектор  $A^*\mathbf{y}$  вообще не принадлежит к  $l_2$ , но пусть теперь,  $\mathbf{x} \in \mathcal{D}(\mathbf{A}_1)$  и  $\mathbf{y} \in l_{\max}(A)$  произвольные вектора, то  $A^*\mathbf{y} \in l_2$  и равенство (14) можно переписать в виде  $(\mathbf{A}_1 \mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$ , откуда следует что  $\mathbf{y} \in \mathcal{D}(\mathbf{A}_1^*)$  и  $\mathbf{A}_1^* \mathbf{y} = A^*\mathbf{y}$  и тем самым наше утверждение доказано. Итак мы доказали, что  $\mathbf{A}_1^*$  есть максимальный оператор, индуцированный матрицей  $A^*$ , а отсюда на основании теоремы 1 следует, что  $\mathbf{A}_1^* = \mathbf{A}_0^*$  и теорема доказана.

Применив теперь следствие теоремы 1 к максимальному оператору, определенному матрицей  $A$ , получаем следующий результат:

**ТЕОРЕМА 3.** Если столбцевые вектора матрицы  $A$  принадлежат к  $l_2$ , то оператор  $\mathbf{A}_{\max}$  определен плотно в  $l_2$  и сопряженный оператор  $\mathbf{A}_{\max}^*$  определяется равенствами

$$(15) \quad \mathbf{A}_{\max}^* \mathbf{y} = A^*\mathbf{y},$$

$$(16) \quad \mathcal{D}(\mathbf{A}_{\max}^*) = \{\mathbf{y}; \mathbf{y} \in l_{\max}(A^*), (A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y}) \text{ для всех } \mathbf{x} \in l_{\max}(A)\}.$$

**ЗАМЕЧАНИЕ.** В вышеприведенных рассуждениях мы рассматривали некоторые специальные операторы, индуцированные матрицей  $A$ , именно операторы  $\mathbf{A}_0$ ,  $\mathbf{A}_1$  и  $\mathbf{A}_{\max}$  и показали, что сопряженные операторы этих операторов являются операторами, индуцированными матрицей  $A^*$ . Остается

открытым следующий вопрос: если  $A$  любой, в пространстве  $l_2$  плотно определенный оператор, индуцированный матрицей  $A$ , то является ли оператор  $A^*$  оператором, индуцированным матрицей  $A^*$ ? Было бы интересно доказать это утверждение или показать на примере неверность этого утверждения.

Рассмотрим теперь вопрос о том, что среди операторов, индуцированных матрицей  $A$ , имеются ли в пространстве  $l_2$  плотно определенные замкнутые операторы и в частном случае оператор  $A_{\max}$  является ли замкнутым оператором?

Известно, что некоторый оператор, переводящий гильбертово пространство в себе, является замкнутым оператором тогда и только тогда, когда оператор имеет второе сопряженное, при этом второе сопряженное есть сам исходный оператор (см. напр. [1]). Но для существования второго сопряженного необходимо и достаточно, чтобы первое сопряженное было в пространстве  $l_2$  плотно определенным оператором. На основании этих утверждений легко построить матрицу  $A$  так, что соответствующий ей оператор  $A_{\max}$  не является замкнутым оператором. Действительно, рассмотрим матрицу  $C$ , определенную равенством (3) и обозначим  $A = C^*$ , тогда столбцевые вектора матрицы  $A$  принадлежат к  $l_2$ , поэтому на основании теоремы 3, оператор  $A_{\max}$ , индуцированный этой матрицей, является в пространстве  $l_2$  плотно определенным оператором, для которого  $\mathcal{D}(A_{\max}^*) \subset l_{\max}(A^*) = l_{\max}(C)$ . Но мы видели, что  $l_{\max}(C)$  состоит только из нулевого вектора, таким образом  $A_{\max}^*$  определен только при нулевом векторе и из вышеприведенных рассуждений следует, что  $A_{\max}$  не может быть замкнутым оператором. Пусть теперь  $A$  любой, в  $l_2$  плотно определенный оператор, индуцированный вышеупомянутой матрицей  $A$ . Если было бы доказано, что сопряженный оператор  $A^*$  является оператором, индуцированным матрицей  $A^*$  (см. замечание), то этим было бы доказано существование матрицы  $A$ , обладающей тем свойством, что ни один плотно определенный оператор, индуцированный этой матрицей, не является замкнутым оператором. Но вопрос о существовании такой матрицы остается пока открытым.

Резюмируя вышеприведенные рассуждения, можно сказать, что принадлежность всех столбцевых векторов матрицы  $A$  к  $l_2$  вообще еще не обеспечивает замкнутость операторов, индуцированных матрицей  $A$ .

2. Исследуем теперь тот случай, когда все строчные векторы матрицы (1) принадлежат пространству  $l_2$ , т.е. когда при обозначении

$$\mathbf{u}_i = [a_{i1}, a_{i2} \dots a_{in} \dots], \quad (i = 1, 2, \dots)$$

выполняются следующие условия:

$$\|\mathbf{u}_i\|^2 = \sum_{k=1}^{\infty} |a_{ik}|^2 < +\infty, \quad (i = 1, 2, \dots).$$

Легко видеть, что в этом случае при каждом  $\mathbf{x} \in l_2$  вектор  $A\mathbf{x}$  определен, но вообще не принадлежит к  $l_2$ . Докажем следующий результат:

**ТЕОРЕМА 4.** *Если все строчные векторы матрицы  $A$  принадлежат к  $l_2$ , то  $A_{\max}$  является замкнутым оператором.*

**ДОКАЗАТЕЛЬСТВО.** Обозначим  $B = A^*$ , тогда все столбцевые вектора матрицы  $B$  принадлежат к  $l_2$ . Определим оператор  $B_0$  на множестве всех конечных векторов, подобно тому, как в пункте 1 мы определили оператор  $A_0$  равенством (6), т.е.  $B_0x = Bx$  и  $\mathcal{D}(B_0) = l_0$ , тогда по теореме 1 сопряженный оператор  $B_0^*$  есть максимальный оператор, индуцированный матрицей  $B^*$ , т.е.  $B_0^*y = B^*y$  и  $\mathcal{D}(B_0^*) = l_{\max}(B^*)$ . Принимая во внимание, что  $B^* = A$ , получим, что  $B_0^*y = Ay$ ,  $\mathcal{D}(B_0^*) = l_{\max}(A)$ , откуда следует, что  $B_0^* = A_{\max}$  и теорема следует из того, что сопряженный оператор любого оператора является замкнутым оператором.

**ЗАМЕЧАНИЕ.** Приналежность всех строчных векторов матрицы  $A$  к  $l_2$  вообще еще не обеспечивает того, чтобы оператор  $A_{\max}$  был плотно определен в пространстве  $l_2$ , откуда следует, что  $A_{\max}^*$  вообще не существует. Действительно, если рассмотрим оператор  $C_{\max}$ , индуцированный матрицей (3), то  $\mathcal{D}(C_{\max}) = l_{\max}(C)$  состоит только из нулевого вектора.

**3.** Предположим теперь, что и столбцевые и строчные вектора матрицы  $A$  принадлежат к  $l_2$ . Отметим, что в этом случае вектора  $Ax$  и  $A^*x$  определены для каждого вектора  $x \in l_2$ , но не обязательно принадлежат к  $l_2$ .

Так как столбцевые векторы матриц  $A$  и  $A^*$  принадлежат к  $l_2$ , то из результатов пункта 1 следует, что  $l_0 \subset l_{\max}(A)$  и  $l_0 \subset l_{\max}(A^*)$ , из него следует, что линейные многообразия  $l_{\max}(A)$  и  $l_{\max}(A^*)$  плотны в пространстве  $l_2$ . Принимая во внимание теоремы 3 и 4, получим следующий результат:

**ТЕОРЕМА 5.** Если все столбцевые и строчные вектора матрицы  $A$  принадлежат к  $l_2$ , то оператор  $A_{\max}$  является замкнутым, в пространстве  $l_2$  плотно определенным оператором, для которого  $A_{\max}^*$  определяется равенствами (15) и (16), далее  $A_{\max}^{**} = A_{\max}$ .

**СЛЕДСТВИЕ 1.** Если  $A$  произвольный оператор, индуцированный матрицей  $A$ , то оператор  $A$  имеет замыкание  $\bar{A}$ , которое является также оператором, индуцированным матрицей  $A$ .

**ДОКАЗАТЕЛЬСТВО.** Для того, чтобы оператор  $A$  имел замыкание, необходимо и достаточно, чтобы оператор  $A$  имел замкнутое расширение, но если оператор  $A$  есть оператор, индуцированный матрицей  $A$ , то  $A_{\max}$  является замкнутым расширением оператора  $A$  и очевидно, что  $\bar{A} \subset A_{\max}$  и отсюда наше следствие уже следует.

**СЛЕДСТВИЕ 2.** Операторы  $A_0$  и  $A_1$  – определенные равенствами (6) и (12), (13) – имеют замыкания, при этом  $\bar{A}_0 = \bar{A}_1$  и  $\bar{A}_0$  определяется равенствами

$$(17) \quad \bar{A}_0x = Ax$$

$$(18) \quad \mathcal{D}(\bar{A}_0) = \{x; x \in l_2, (Ax, y) = (x, A^*y) \text{ для всех } y \in l_{\max}(A^*)\}.$$

**ДОКАЗАТЕЛЬСТВО.** На основании следствия 1, операторы  $A_0$  и  $A_1$  имеют замыкания  $\bar{A}_0$  и  $\bar{A}_1$ . Так как  $\mathcal{D}(\bar{A}_0^*) = l_{\max}(A^*)$ , то оператор  $\bar{A}_0^*$  плотно определен в пространстве  $l_0$  и отсюда следует, что  $\bar{A}_0^{**}$  существует.

Но по теореме 2 имеет место равенство  $A_0^* = A_1^*$ , из которого следует, что  $A_0^{**} = A_1^{**}$  и так как второе сопряженное некоторого оператора равно замыканию этого оператора, то  $\bar{A}_0 = \bar{A}_1$ . Так как  $\bar{A}_0$  является оператором, индуцированным матрицей  $A$ , то равенство (17) очевидно. Далее, по определению  $\mathcal{D}(A_0^{**})$  состоит из тех векторов  $x$  из  $I_2$ , для которых существует вектор  $z \in I_2$  такой, что равенство  $(A_0^*y, x) = (y, z)$  выполняется для всех  $y \in \mathcal{D}(A_0^*)$  и при этом  $z = A_0^{**}x$ , и так как  $A_0^{**} = \bar{A}_0$ ,  $A_0^*y = A^*y$ , то

$$\mathcal{D}(A_0) = \{x; x \in I_2, (A^*y, x) = (y, Ax) \text{ для всех } y \in I_{\max}(A^*)\},$$

из которого равенство (18) уже следует.

**ЗАМЕЧАНИЕ.** В том случае, когда матрица (1) симметрична, выведенные результаты упрощаются и о них можно найти в работах [2], [3] и также в [1].

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# ON THE DENSITY OF NON-OVERLAPPING UNIT SPHERES LYING IN A STRIP

By

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Dedicated to Prof. F. KÁRTESZI on his 60th birthday

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**1.**<sup>1</sup> In order to formulate our result we introduce the notion of the *density* of a non-overlapping system of spheres with respect to a strip in which the spheres lie.

Let  $\{S_i\}$  be a system of non-overlapping unit spheres lying in a strip  $\sigma(t)$  of thickness  $t$ , i.e. the strip is bounded by two parallel planes at a distance  $t$  apart. The *density*  $d$  of  $\{S_i\}$  with respect to the strip  $\sigma(t)$  is defined by

$$d = \lim_{R \rightarrow \infty} \frac{\sum C(R) \cap S_i}{C(R) \cap \sigma(t)},$$

where  $C(R)$  denotes a cylinder of radius  $R$  with its axis perpendicular to  $\sigma(t)$  fixed at an arbitrary point  $O$  of  $\sigma(t)$ .<sup>2</sup> It is easy to show that  $d$  does not depend on the choice of  $O$ .

Let us consider a system of non-overlapping unit spheres lying in a strip  $\sigma(t)$  of thickness  $t \leq 2 + \sqrt{2}$ , so that the spheres form two layers and in each layer the centers of the spheres constitute a rectangular net with sides  $a = 2$  and  $b = 2\sqrt{4t - t^2 - 1}$  (Fig. 1, 2) and let us denote the density of such a sphere-system with respect to  $\sigma(t)$  by  $\delta(t)$ .

The following theorem is due to MOLNÁR<sup>3</sup>.

*The density of a system of non-overlapping unit spheres in a strip of thickness  $t \leq 2 + \sqrt{2}$  is at most  $\delta(t)$ .*

*Equality can be attained for all values  $t \leq 2 + \sqrt{2}$ .*

<sup>1</sup> The first part of this article is due to MOLNÁR the second to HORVÁTH.

<sup>2</sup> We denote a domain and its measure (area or volume respectively) with the same symbol.

<sup>3</sup> See MOLNÁR [5].

Consider now a system of non-overlapping unit spheres lying in a strip  $\sigma(t)$  of thickness  $2 + \sqrt{2} \leq t \leq \frac{1}{3}\sqrt{\frac{8}{3}}$ , so that the spheres form two layers and the spheres of the same layer touch the corresponding boundary plane of the strip and in each layer the centres of the spheres constitute a equilateral quadrangular net of side 2 (Fig. 3, 4, 5). We denote the density of such a sphere-system with respect to  $\sigma(t)$  by  $\delta(t)$ .

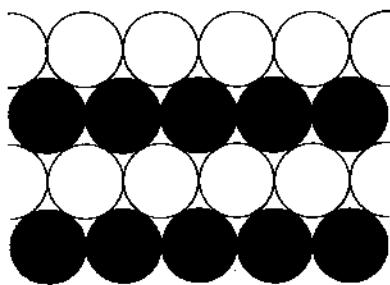


Fig. 1

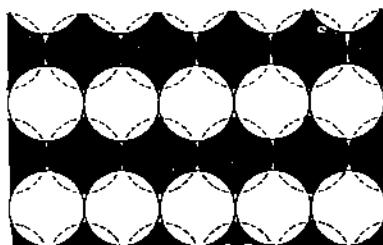


Fig. 2

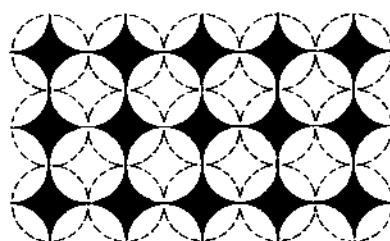


Fig. 3

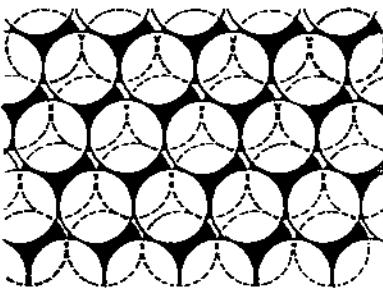


Fig. 4

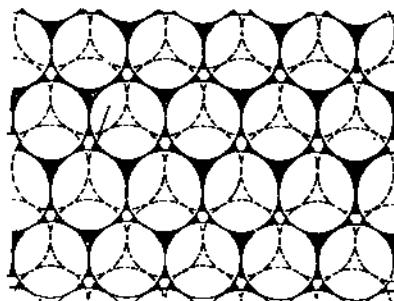


Fig. 5

The following conjecture is due to MOLNÁR.

*The density of a system of non-overlapping unit spheres in a strip of thickness  $2 + \sqrt{2} \leq t \leq \sqrt{\frac{8}{3}}$  is at most  $\delta(t)$ .*

*Equality can be attained for all values  $2 + \sqrt{2} \leq t \leq \sqrt{\frac{8}{3}}$ .*

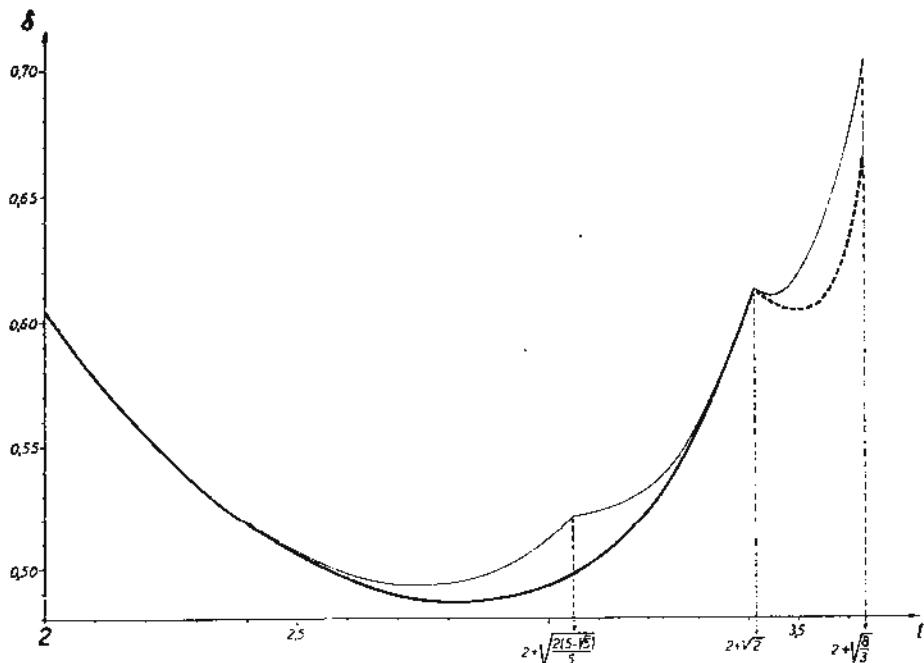


Fig. 6

Our main result is the following

**THEOREM.** *The density of a system of non-overlapping unit spheres in a strip of thickness  $t$  is at most*

$$\frac{\pi(4t-t^2)^3}{3t(5t^4-40t^3+96t^2-64t+16)\sqrt{4t-t^2-1}}, \text{ if } 2 \leq t \leq 2 + \sqrt{\frac{2(5-\sqrt{5})}{5}},$$

$$\frac{2\pi(4t-t^2)^2\sqrt{4t-t^2-1}}{3t(5t^4-40t^3+92t^2-48t+8)}, \text{ if } 2 + \sqrt{\frac{2(5-\sqrt{5})}{5}} \leq t \leq 2 + \sqrt{2},$$

$$\frac{\pi(t-2)^4}{3t(4t-t^2)(t^2-4t+3)^{\frac{3}{2}}}, \text{ if } 2 + \sqrt{2} \leq t \leq 2 + \frac{4}{\sqrt{7}},$$

$$\frac{\pi(12+4t-t^2)^4}{12\sqrt{3}t(4t-t^2)^{\frac{3}{2}}(t^2-4t+12)^3}, \text{ if } 2+\frac{4}{\sqrt{7}} \leq t \leq 2+4\sqrt[3]{3\sqrt{3}-5} \text{ and}$$

$$\frac{16\pi(12+4t-t^2)^2}{\sqrt{3}t(4t-t^2)^{\frac{1}{2}}(t^2-4t+12)^3}, \text{ if } 2+4\sqrt[3]{3\sqrt{3}-5} \leq t < 4.$$

The upper bound is precise only for  $t=2$  (Fig. 1) and  $t=2+\sqrt{2}$  (Fig. 3). For  $2 \leq t \leq 2+\sqrt{2}$  our upper bound (Fig. 6, thin line) is weaker than MOLNÁR's upper bound (Fig. 6, heavy line). For  $2+\sqrt{2} \leq t \leq \sqrt{\frac{8}{3}}$  our upper bound (thin line) lies near to the conjectured MOLNÁR's upper bound (broken heavy line).

In our theorem we use the following Lemma.

**LEMMA OF HAJÓS.<sup>4</sup>** Let  $c$  and  $C$  be two concentric circles of radii  $r$  and  $R$  respectively ( $r < R$ ). Further let a finite number of disjoint segments of the circle  $C$  be given having no interior point in common with  $c$ . The sum of the area of the segments is maximal if the segments determine a polygon  $H(r, R)$  whose sides, saving at most one, touch the circle  $c$ .

The upper bound of our theorem is exactly  $\frac{4\pi}{3H(r, R)}$ , where  $r = \frac{1}{2}\sqrt{t^2-4t+8}$  and  $R$  has the corresponding values given by our lemma.

We proceed to prove our theorem.

Let  $\{S_i\}$  be a system of non-overlapping unit spheres lying in the strip  $\sigma(t)$  of thickness  $t$ . Without loss of generality we may suppose that to the system  $\{S_i\}$  no unit sphere can be added, i.e. the system of spheres is saturated.

Obviously the centres  $\{O_i\}$  of  $\{S_i\}$  lie in a strip  $\sigma(r)$  of thickness  $r=t-2$ , bounded by two parallel planes  $\pi_1$  and  $\pi_2$ . Projecting the system  $\{S_i\}$  onto  $\pi_1$  we get a circle-system  $\{S_i^*\}$  of unit circles with the corresponding center-system  $\{C_i\}$ .

We define the density  $d^*$  of  $\{S_i^*\}$  in its plane by

$$d^* = \lim_{R \rightarrow \infty} \frac{\sum K(R) \cap S_i^*}{K(R)},$$

where  $K(R)$  denotes a circle of radius  $R$  centred at a fixed arbitrary point  $O$  of the plane. Since

$$d = \lim_{R \rightarrow \infty} \frac{\sum C(R) \cap S_i}{C(R) \cap \sigma(t)} = \lim_{R \rightarrow \infty} \frac{4}{3t} \frac{\sum K(R) \cap S_i^*}{K(R)},$$

where  $K(R)$  denotes the intersection of  $C(R)$  and  $\pi_1$ , we get  $d = \frac{4}{3t} d^*$ , i.e. the density of  $\{S_i\}$  and of its projection  $\{S_i^*\}$  are proportional. Therefore to study the density of  $\{S_i\}$  it suffices to consider the corresponding density of  $\{S_i^*\}$ .

<sup>4</sup> See MOLNÁR [3], [4], [5].

Let us consider the circlesystem  $\{S_i^*\}$  with its centre-system  $\{C_i\}$ . Of course  $C_i C_j \cong \sqrt{t^2 - 4t + 8}$ ,  $i \neq j$ .

Associating with a point  $C_i$  ( $i = 1, 2, 3, \dots$ ) the set  $D_i$  of all points  $P$  lying nearer to  $C_i$  than to any other point  $C_j$ , more precisely  $C_i P \cong C_j P$ ,  $i \neq j$ , we obtain a convex polygon  $D_i$  (DIRICHLET cell, VORONOI polygon)<sup>5</sup> (Fig. 7). It is known that the convex polygons  $\{D_i\}$  form a tessellation.<sup>6</sup>

It is easy to show that to prove our theorem it suffices to show only that  $D_i \cong H(r, R)$ , where  $r = \frac{1}{2} \sqrt{t^2 - 4t + 8}$  and

$$R = \frac{4t - t^2}{2\sqrt{4t - t^2 - 1}}, \text{ if } 2 \leq t \leq 2 + \sqrt{2};$$

$$R = \frac{(t-2)\sqrt{4t-t^2}}{2}, \text{ if } 2 + \sqrt{2} \leq t \leq 2 + \frac{4}{\sqrt{7}},$$

$$R = \frac{12 + 4t - t^2}{8\sqrt{3}}, \text{ if } 2 + \frac{4}{\sqrt{7}} \leq t < 4 \text{ is.}^7$$

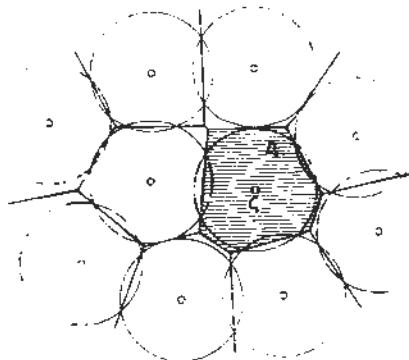


Fig. 7

In order to prove  $D_i \cong H(r, R)$  obviously it suffices to prove that  $V_i C_i \cong R_i$ , where  $V_i$  denotes any vertex of  $D_i$ .

Let  $V_i$  be any vertex of  $D_i$ . By the definition of  $D_i$  the vertex  $V_i$  is a radical point of  $S_i^*$  and at least another two circles  $S_j^*$  and  $S_k^*$ .

What is the minimum value of  $V_i C_i$ ?

<sup>5</sup> See COXETER [1], p. 53.

<sup>6</sup> See FEJES TÓTH [2], or MOLNÁR [3].

<sup>7</sup> See MOLNÁR [3], [4], [5], [6].

At first we proceed to show that we can restrict ourselves to the case when the triangle  $O_iO_jO_k$  corresponding to the triangle  $C_iC_jC_k$  is equilateral with sidelength 2.

Since  $V_iC_i = V_jC_j = V_kC_k$  we can denote the centres  $C_i, C_j, C_k$  in such a way that the greatest angle of the triangle  $O_iO_jO_k$  is  $\angle O_iO_jO_k$ . Consequently  $\angle O_iO_jO_k \geq 60^\circ$ . Displacing the spheres  $S_i$  and  $S_k$  in the direction  $C_iC_j$  and  $C_kC_j$  respectively until  $S_i, S_j$  and  $S_k$  mutually touch each other, obviously  $V_iC_i$  does not increase. Hence it suffices to consider the case when  $O_iO_j = O_jO_k = 2$ , i.e. the triangle  $O_iO_jO_k$  is an isosceles one. Let  $v_i$  be the perpendicular line to the plane  $\pi_1$  passing through  $V_i$ . If the triangle  $C_iC_jC_k$  has acute angles at  $C_j$  and  $C_k$  (Fig. 8) then rotating  $S_j$  around  $v_i$  until  $S_j$  touches  $S_k$  the value of  $V_iC_i$  does not increase and the triangle  $O_iO_jO_k$  is equilateral and its sides are equal to 2. If one of the angles at  $C_j$  or  $C_k$  is „obtuse”, for instance  $\angle C_iC_jC_k \geq 90^\circ$  (Fig. 9) then we consider the cylinder  $C$  of radius  $V_iC_i$  per-

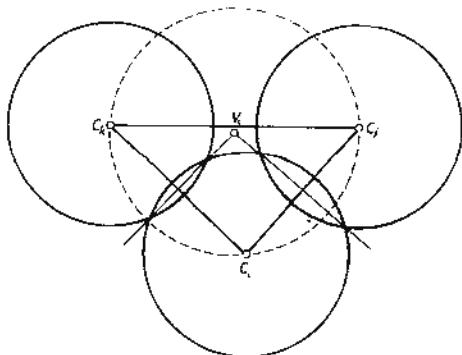


Fig. 8

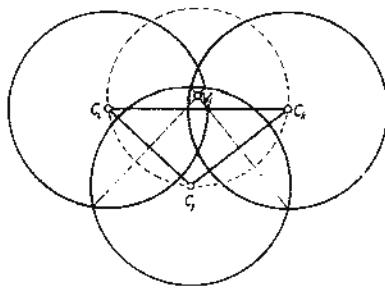


Fig. 9

pendicular to the plane  $\pi_1$  passing through  $O_i, O_j, O_k$ . Using the condition that  $O_iO_j = 2$  it is easy to show that the centre  $O_j$  can be moved on the cylinder  $C$  in the allowable strip  $\sigma(\tau)$  towards  $O_k$  until  $S_j$  touches  $S_k$ , i.e.  $O_jO_k = 2$ .

Therefore we may suppose that the triangle  $O_iO_jO_k$  corresponding to  $C_iC_jC_k$  is equilateral, with sides 2.

2. To complete the proof of our theorem it suffices to prove only the following Lemma.

**LEMMA.** *The vertices A and B of an equilateral triangle ABC, with sides 2, are lying on the boundary planes of a strip of thickness  $\tau < 2$ . Consider the circular cylinder with axis perpendicular to the boundary planes and whose surface contains the vertices of the triangle. The radius R of the cylinder is minimal if the distance of C from one of the boundary planes is*

- a) 0 if  $\tau \leq \sqrt{2}$ ,
- b)  $\frac{1}{2}(\tau \pm \sqrt{16 - 7\tau^2})$  if  $\sqrt{2} \leq \tau \leq \frac{4}{\sqrt{7}}$  and
- c)  $\frac{\tau}{2}$  if  $\frac{4}{\sqrt{7}} \leq \tau < 2$ .<sup>8</sup>

PROOF. For the sake of simplicity the statements will be proved in the following order: c), b) and a). Let  $R$  denote the radius of the cylinder above.

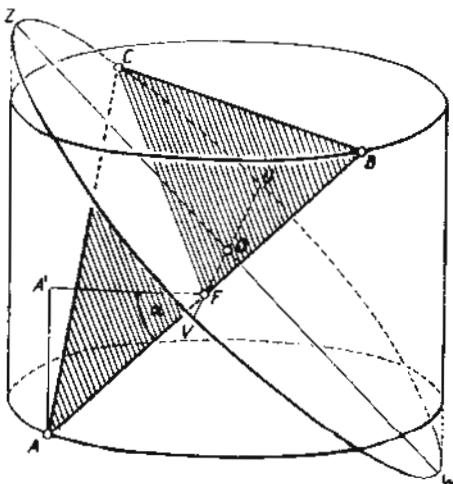


Fig. 10

$F$  denotes the midpoint of  $AB$ , and  $\alpha$  denotes the angle between  $AB$  and the boundary planes (Fig. 10). Obviously

$$(1) \quad \sin \alpha = \frac{\tau}{2}.$$

It can easily be shown that 1. the perpendicular bisecting plane of  $AB$  cuts the cylinder in an ellipse, whose axes are  $WZ = \frac{2R}{\sin \alpha}$  and  $UV = 2R$ , 2.  $F$  is on the minor axis of the ellipse and  $FC = \sqrt{3}$ .

<sup>8</sup> Of course the corresponding minimal values of  $R$  are

$$R = \frac{4 - \tau^2}{2\sqrt{3 - \tau^2}}, \quad R = \frac{\tau\sqrt{4 - \tau^2}}{2} \quad \text{and} \quad R = \frac{16 - \tau^2}{8\sqrt{3}} \quad \text{respectively.}$$

The centre of the ellipse is denoted by  $O$ , furthermore let  $A'$  be the perpendicular projection of  $A$  on the plane through  $F$  and parallel to the boundary planes. As  $FA' = \cos \alpha$ ,

$$(2) \quad FO = \sqrt{R^2 - FA'^2} = \sqrt{R^2 - \cos^2 \alpha}.$$

It is trivial that the radius  $R$  of the cylinder is minimal if, rotating  $C$  about  $F$  in the plane of the ellipse in any direction,  $C$  becomes an exterior point of the ellipse, i.e. the minimum is attained if  $C$  is the point of the ellipse furthest from  $F$ .

In the following it will be examined that, for a given value of  $\alpha$  what value of  $R$  assures that the maximal distance between  $F$  and a point of the ellipse is  $\sqrt{3}$ .

Let the axes of the ellipse be the axes of the coordinate system. In the usual parametric form the points of the ellipse are  $P\left(\frac{R}{\sin \alpha} \cos t, R \sin t\right)$ . Let the coordinates of  $F$  be  $F(0, -d)$ . Obviously

$$(3) \quad \begin{aligned} FP^2 &= \frac{R^2}{\sin^2 \alpha} \cos^2 t + R^2 \sin^2 t + 2Rd \sin t + d^2 \\ &= -R^2 \operatorname{ctg}^2 \alpha \sin^2 t + 2Rd \sin t + d^2 + \frac{R^2}{\sin^2 \alpha} = y(t). \end{aligned}$$

As  $y'(t) = 2R \cos t(d - R \operatorname{ctg}^2 \alpha \sin t)$ ,  $y'(t) = 0$  if  $\cos t = 0$  and  $\sin t = \frac{d}{R \operatorname{ctg}^2 \alpha}$  respectively, i.e. if  $t_1 = \frac{\pi}{2}$ ,  $t_2 = \frac{3\pi}{2}$  and  $t_3 = \arcsin \frac{d}{R \operatorname{ctg}^2 \alpha}$ ,  $t_4 = \pi - t_3$  respectively. In the latter case  $d \leq R \operatorname{ctg}^2 \alpha$ . On the other hand calculating the second, third and fourth derivatives,<sup>9</sup> it can be obtained that

1. if  $d > R \operatorname{ctg}^2 \alpha$ , then the maximum of (3) is at  $t_1 = \frac{\pi}{2}$ , the minimum at  $t_2 = \frac{3\pi}{2}$ ,
2. if  $d < R \operatorname{ctg}^2 \alpha$ , then the minima are at  $t_1, t_2$ , the maxima at  $t_3, t_4$ ,
3. if  $d = R \operatorname{ctg}^2 \alpha$ , then the maxima are at  $t_1 = t_3 = t_4$  and the minimum is at  $t_2$ .

Simple computation shows that if the maximum is  $\sqrt{3}$ , then

1. if  $d > R \operatorname{ctg}^2 \alpha$ , then  $R = \frac{4 - \sin^2 \alpha}{2\sqrt{3}} = \frac{16 - \tau^2}{8\sqrt{3}}$ ,
2. if  $d \leq R \operatorname{ctg}^2 \alpha$ , then  $R = \sin 2\alpha = \frac{\tau\sqrt{4 - \tau^2}}{2}$ .

<sup>9</sup>  $y''(t) = 2R(-d \sin t - R \operatorname{ctg}^2 \alpha \cos 2t)$   
 $y'''(t) = 2R(-d \cos t + 2R \operatorname{ctg}^2 \alpha \sin 2t)$   
 $y''''(t) = 2R(d \sin t + 4R \operatorname{ctg}^2 \alpha \cos 2t)$ .

Substitution of (1) and (2) into  $d \geq R \operatorname{ctg}^2 \alpha$  gives  $\tau \geq \frac{4}{\sqrt{7}}$ . As  $V$ , the endpoint of the minor axis, is furthest point of the ellipse from  $F$ , and the distance of  $V$  from a boundary plane of the strip is  $\frac{\tau}{2}$ , section c) of the lemma is proved.

Now substitute (1) and (2) into  $d \leq R \cot^2 \alpha$ . This yields  $\tau \geq \frac{4}{\sqrt{7}}$ . The distances of  $P_3$  and  $P_4$  from the bisecting plane of the strip can easily be calculated:  $P_4 P'_4 = \frac{\sqrt{16 - 7\tau^2}}{2}$ .

Clearly  $P_4 P'_4 \leq \frac{\tau}{2}$  if  $\sqrt{2} \leq \tau \leq \frac{4}{\sqrt{7}}$  and this proves b).

If  $\tau < \sqrt{2}$ , the points  $P_3$  and  $P_4$  are outside the strip. It is easily seen that the function  $y(t)$  is monotonic decreasing in  $(t_3, \frac{\pi}{2})$  and it is monotonic increasing in  $(\frac{\pi}{2}, t_4)$ , consequently those points of the ellipse will be of greatest distance from  $F$ , which are on the boundary planes. This proves section a) of the Lemma.

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## ON CONNECTED SETS OF POINTS

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RUSHTON constructed a connected set of points in the plane of diameter 2, omitting 3 points of which the remaining set does not contain two points of distance 1. ERDŐS conjectured that this example is best possible in the following sense:

*If  $M$  is a connected set in the plane of diameter  $>2$  and  $S$  is a subset of  $M$  such that  $M \setminus S$  does not contain a pair of points of distance 1, then  $|S| \leq c$ .*<sup>1</sup>

The aim of this paper is to prove this conjecture of ERDŐS.<sup>2</sup>

PROOF. Let  $x$  be a point of  $M$ . We denote by  $A_x$  the set of those points of  $M$  which are at distance 1 from  $x$ . Put  $B = \{x\} \cup A_x$ .

First we prove, that  $|B| = c$ .

LEMMA. *If  $H$  is a connected set and  $H'$  is a set such that  $|\overline{H'} \cap (H \setminus H')| \leq 1$ ,  $H' \cap (\overline{H} \setminus \overline{H'}) = \emptyset$ , then  $H \setminus H'$  is also connected.*

Put  $H \cap H' = U$ ,  $H \setminus H' = H_1 \cup H_2$ . Since  $\overline{H'} \cap (H \setminus H') = (\overline{H'} \cap H_1) \cup (\overline{H'} \cap H_2)$ , we may suppose, that  $\overline{H'} \cap H_1 = \emptyset$ . But then  $H_1 \cap H_2 \cup U = H_1 \cap H_2$  and  $\overline{H_1} \cap (H_2 \cup U) = \overline{H_1} \cap H_2$ , and this implies, that one of the sets  $H_1 \cap H_2$ ,  $\overline{H_1} \cap H_2$  is not empty. This proves the Lemma.

Now let  $p$  and  $q$  be two points of  $M$  of distance  $\rho > 2$ . Let  $K_r$  be the circle of centre  $p$  and radius  $r$ . It is enough to prove that  $K_r$  contains a point of  $B$  for any  $1 < r < \rho - 1$ .

Let us divide  $K_r$  into arcs of length  $< \frac{1}{2}$ . If such an arc contains some points of  $M$  then let us choose one of these points. Let  $x_1, x_2, \dots, x_k$  be the chosen points, and let  $C_i$  be the set of points of distance  $< 1$  from  $x_i$ . Clearly every point of  $M \cap K_r$  is contained in one of the  $C_i$ -s.

<sup>1</sup>  $|S|$  is the power of  $S$ ,  $c$  denotes continuum. The closure of a set  $H$  will be denoted by  $\overline{H}$ .

<sup>2</sup> A different proof was given independently by B. BOLLOBÁS.

If  $K_r \cap B = \emptyset$ , then the set  $\bar{C}_i \setminus C_i$  ( $1 \leq i \leq k$ ) contains at most one point of  $M$  (by the connectedness of  $M$  it does contain a point of  $M$ ; but we shall not use this fact). Using our Lemma  $k$ -times we get, that  $M \setminus \left( \bigcup_{i=1}^k C_i \right) = M'$  is connected. But  $M'$  contains no point of  $K_r$  and thus  $K_r$  separates the points  $p, q$  of  $M'$ . This is a contradiction. Thus we have proved, that  $|B| = c$ .

Now we prove the theorem. Let us suppose indirectly, that  $|S| < c$ . We know, that  $|B| = c$ , hence  $|B \setminus S| = c$ . By the definition of  $S$   $A_x \subset S$  if  $x \in B \setminus S$ . For two points there are at most two points in the plane being from both of them at distance 1. This implies that the power of  $B \setminus S$  is at most twice the power of the set of pairs  $(u, v)$   $u \in S, v \in S$ . This power is finite if  $S$  is finite and it equals to  $|S|$  if  $S$  is infinite. In both cases we have got  $|B \setminus S| < c$ , which is a contradiction. Thus  $|S| = c$ .

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