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GYÖRGY HAJÓS

(1912 – 1972)

Professor G. Hajós died the 17th of March, 1972 after a long illness. His death is an irreparable loss not only for the editorial board of our review but also for the whole mathematical life in Hungary.

SOME THEOREMS IN ADDITIVE NUMBER THEORY[1]

By

M. V. SUBBARAO and D. SURYANARAYANA[2]

University of Alberta, Edmonton, Canada

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1. Introduction

In 1922, G. H. HARDY and J. E. LITTLEWOOD [4] in paper III of their celebrated series of papers entitled "Partitio Numerorum" made a number of conjectures. Two of these (conjectures H and L, p. 609 and p. 611 of [4]) are as follows: (a) Every large number is either a square or the sum of a prime and a square (b) Every large number is either a cube or the sum of a prime and a (positive) cube. In each of these conjectures, they have stated the asymptotic expressions also for the number of representations.

The above two conjectures are still open and are made on the basis of the extended Riemann Hypothesis (e.R.H.), namely that every zero of every Dirichlet's function

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{x_q(m)}{m^s},$$

where $\chi_q(m)$ is a character (mod q) has a real part which does not exceed $1/2$ for all q and all χ_q. However, it has been shown by C. HOOLEY [5] in 1957 that every large integer is the sum of a prime and two squares (cf. [4], conjecture J, p. 610) assuming the e.R.H. and by JU. V. LINNIK (cf. [7], also cf. [8], ch. VII) in 1960 without any hypothesis. In 1929, it has been shown by G. K. STANELY (cf. [13], Theorem G) that every large integer is the sum of two primes and a square assuming the e.R.H. and by T. ESTERMANN [2] in 1937 without assuming any hypothesis. In 1968, R. J. MICH [9] has shown that "nearly every" integer n is expressible in the form $n = p + m^2$. For a precise statement of his result we refer to [9].

The problems corresponding to (a) and (b) above involving 4th or higher powers have not received much attention. The utmost that is known in this direction is the result of L. K. HUA (cf. [6], p. 179), namely that every large integer can be expressed as the sum of a prime and s k-th

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[2] On leave from Andhra University, Waltair, India.

powers of integers if $s \geq s_0 \sim (3k \log k)/2$. In this connection, we might also mention the following result proved quite recently by K. PRACHAR [12]: Given a positive integer, i, there exist constants $\eta > 1$ and $\delta > 0$ such that for large N at least δN of the positive integers up to N are not expressible in the form $p + m^i$, where p is a prime and m is a positive integer not exceeding $\eta \log N$.

In a different direction, T. ESTERMANN [1] established in 1931 that every sufficiently large integer n can be expressed as the sum of a prime and a square-free integer and that the number $T(n)$ of such representations is given asymptotically by

$$T(n) \sim \prod_{\substack{p \\ p \nmid n}} \left\{ 1 - \frac{1}{p(p-1)} \right\} \operatorname{Li} n,$$

where the product is extended over all primes p which do not divide n and

$$\operatorname{Li} n = \int_2^n \frac{dt}{\log t}.$$

In 1935, A. PAGE (cf. [11], theorem III) established an asymptotic formula for $T(n)$ with an O-estimate for the error term, namely

$$T(n) = \prod_{\substack{p \\ p \nmid n}} \left\{ 1 - \frac{1}{p(p-1)} \right\} \operatorname{Li} n + O\left(\frac{n}{\log^5 n} (\log \log n)^8 \log \log \log n \right).$$

This O-estimate has been further improved in 1936 by A. WALFISZ [17] to $O\left(\frac{n}{\log^H n}\right)$, where $H > 0$.

Let r be any integer > 1. It is trivial after ESTERMANN's [1] result that every sufficiently large integer can be expressed as the sum of a prime and a r-free integer (that is, a positive integer which is not divisible by the r-th power of any integer > 1), since every square-free integer is a r-free integer. In 1949, L. MIRSKY [10] obtained the following asymptotic formula for the number $T(r; n)$ of representations of n as the sum of a prime and a r-free integer:

$$T(r: n) = \prod_{\substack{p \\ p \nmid n}} \left\{ 1 - \frac{1}{p^{r-1}(p-1)} \right\} \operatorname{Li} n + O\left(\frac{n}{\log^H n} \right), \tag{1.1}$$

where H is any positive number and the O-estimate depends at most on r and H.

In this paper, we are concerned with the following problem: Let k and r be integers such that $1 < r < k$; and let us define an integer n to be a

(k, r)- integer, if it is of the form $n = a^k b$, where a is a positive integer and b is a r-free integer. Clearly, a (k, r)-integer is a cross between a k-th power of a positive integer and a r-free integer; and the set $Q_{k,r}$ of all (k, r)-integers includes the set S_k of all k-th power integers and the set Q_r of all r-free integers. Since every large integer is the sum of a prime and a r-free integer, it is trivial, that every large integer is the sum of a prime and a (k, r)-integer. However, let us define a proper (k, r)-integer as a (k, r)-integer which is not a r-free integer; these are integers of the form $n = a^k b$, where $a > 1$ and b is a r-free integer. The set $Q^*_{k,r}$ of such integers includes the set S_k, but excludes the set Q_r. The problem with which we are concerned in this paper is as follows: Is every sufficiently large integer n be expressible as the sum of a prime and a proper (k, r)-integer.

We answer this problem in the affirmative. In fact, we obtain an asymptotic formula for the number $T^*(k, r; n)$ of representations of n as the sum of a prime and a proper (k, r)-integer, with an error term, $O\left(\dfrac{n}{\log^H n}\right)$, where $H > 0$.

Let $T(k, r; n)$ denote the number of representations of n as the sum of a prime and a (k, r)-integer. We establish an asymptotic formula for $T(k, r; n)$, which yields (1.1) as a particular case. Further, we improve the O-term in the asymptotic formula for $T(k, r; n)$ from $O\left(\dfrac{n}{\log^H n}\right)$ to $O(ne^{-B\sqrt{\log n}})$, where B is an absolute positive constant and to $O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right)$, assuming what we call the page hypothesis (stated below) and the e.R.H., respectively. We obtain the consequent improvements in the O-term of the asymptotic formula for $T^*(k, r; n)$.

Page Hypothesis (cf. [11], p. 117). The greatest real zero σ possessed by Dirichlet's L-functions with modulus q satisfies the inequality $\sigma < 1 - \dfrac{A}{\log q}$, where A is an absolute positive constant.

A. Page believes that this hypothesis is very likely to hold (see [11], p. 117) and in fact, he has shown that there is at most one real primitive character which does not satisfy this hypothesis.

2. Prerequisites. The function $\lambda_{k,r}(n)$ introduced by one of the authors and V. C. Harris [14] in some other connection, plays an important role in our present context. $\lambda_{k,r}(n)$ is a multiplicative function defined for powers of an arbitrary prime p as follows:

$$\lambda_{k,r}(p^\alpha) = \begin{cases} 1, & \text{if } \alpha \equiv 0 \pmod{k} \\ -1, & \text{if } \alpha \equiv r \pmod{k} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For brevity, we here after write $\lambda(n)$ for $\lambda_{k,r}(n)$. It should be noted that $\lambda_{k,r}(n)$ is different from the well-known Liouville's function, which is also generally represented by the same symbol.

It has been shown (cf. [14], Theorem 3) that

$$\sum_{d/n} \lambda(d) = q_{k,r}(n), \tag{2.2}$$

where $q_{k,r}(n) = 1$ or 0 according as $n \in Q_{k,r}$ or $n \notin Q_{k,r}$.

REMARK 2.1. We note for a later use that $\lambda(n) = 0$, unless n is of the form $n = a^k b^r$, in which case $\lambda(n) = \mu(b)$, where μ is the Möbius function.

We now prove the following:

LEMMA 2.1. *For $x \geq 1$, $n \geq 1$,*

$$\sum_{\substack{d \leq x \\ (d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \prod_{\substack{p \\ p \nmid n}} \left\{ 1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r} - p^{-k}}{1-p^{-k}} \right\} + O\left(\zeta\left(\frac{k}{r}\right) x^{\frac{1}{r} - \frac{3}{4}} \right), \tag{2.3}$$

where the O-constant is independent of n, k and r, $\zeta(s)$ being the Riemann Zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$ and $\varphi(n)$ is the Euler totient function.

PROOF. Let $\varepsilon(m) = 1$ or 0 according as $m = 1$ or $m > 1$. Then the series in (2.3) would become $\sum_{d=1}^{\infty} \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$.

We have by Remark 2.1 and the fact (cf. [3], theorem 327) that $\Phi(n) > Cn^{\frac{3}{4}}$ for all $n \geq 1$ (C being a suitable positive constant),

$$\sum_{d \leq m} \left| \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)} \right| \leq \sum_{a^k b^r \leq m} \frac{1}{\varphi(a^k b^r)} < \frac{1}{C} \sum_{\substack{a,b \\ a^k b^r \leq m}} (a^k b^r)^{-\frac{3}{4}} <$$

$$< \frac{1}{C} \left(\sum_{a=1}^{\infty} a^{-\frac{3k}{4}} \right) \left(\sum_{b=1}^{\infty} b^{-\frac{3r}{4}} \right) = \frac{1}{C} \zeta\left(\frac{3k}{4}\right) \zeta\left(\frac{3r}{4}\right).$$

Hence the series $\sum_{d=1}^{\infty} \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$ is absolutely convergent.

Further, $f(d) = \frac{\lambda(d)\,\varepsilon((d,n))}{\varphi(d)}$ is a multiplicative function of d and $f(1) = 1$.

Hence the series can be expanded into an infinite product of Euler type (cf. [3], Theorem 286), so that by (2.1) we have

$$\sum_{\substack{d=1\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \prod_p \left\{1+\frac{\lambda(p)\varepsilon((p,n))}{\varphi(p)}+\frac{\lambda(p^2)\varepsilon((p^2,n))}{\varphi(p^2)}+\dots\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{\varphi(p^k)}+\frac{1}{\varphi(p^{2k})}+\dots-\frac{1}{\varphi(p^r)}-\frac{1}{\varphi(p^{r+k})}-\dots\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{p^k(1-p^{-1})}+\frac{1}{p^{2k}(1-p^{-1})}+\right.$$

$$\left.+\dots-\left[\frac{1}{p^r(1-p^{-1})}+\frac{1}{p^{r+k}(1-p^{-1})}+\dots\right]\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1+\frac{1}{1-p^{-1}}\cdot\frac{1}{p^k}\cdot\frac{1}{1-p^{-k}}-\frac{1}{1-p^{-1}}\cdot\frac{1}{p^r}\cdot\frac{1}{1-p^{-k}}\right\} =$$

$$= \prod_{\substack{p\\p\nmid n}} \left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}.$$

Also, by Remark 2.1 and $\varphi(n) > Cn^{\frac{3}{4}}$ for all $n \geq 1$, we have

$$\sum_{\substack{d>x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} \leq \sum_{a^k b^r > x} \frac{1}{\varphi(a^k b^r)} < \frac{1}{C} \sum_{\substack{a,b\\a^k b^r > x}} a^{-\frac{3k}{4}} b^{-\frac{3r}{4}} =$$

$$= \frac{1}{C} \sum_{a=1}^{\infty} a^{-\frac{3k}{4}} \sum_{b > \sqrt[r]{xa^{-k}}} b^{-\frac{3r}{4}} = O\left(\sum_{a=1}^{\infty} a^{-\frac{3k}{4}} (\sqrt[r]{xa^{-k}})^{1-\frac{3r}{4}}\right) =$$

$$= O\left(x^{\frac{1}{r}-\frac{3}{4}} \sum_{a=1}^{\infty} a^{-\frac{k}{r}}\right) = O\left(\zeta\left(\frac{k}{r}\right) x^{\frac{1}{r}-\frac{3}{4}}\right).$$

Since

$$\sum_{\substack{d\leq x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)} = \sum_{\substack{d=1\\(d,n)=1}}^{\infty} \frac{\lambda(d)}{\varphi(d)} - \sum_{\substack{d>x\\(d,n)=1}} \frac{\lambda(d)}{\varphi(d)},$$

lemma 2.1 follows.

Let $\pi(x; u, v)$ denote the number of primes $p \leq x$, $p \equiv v \pmod u$ and $(u, v) = 1$. The it is well-known that

$$\frac{1}{x}\left\{\pi(x; u, v) - \frac{1}{\varphi(u)} \operatorname{Li} x\right\} \to 0 \text{ as } x \to \infty$$

uniformly with respect to u and v.

Suppose that

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O(x\,\omega(x)), \tag{2.4}$$

where $\omega(x)$ is a monotonic decreasing function tending to zero as $x \to \infty$, and the O-constant is independent of u and v.

There exist explicit expressions for $\omega(x)$ in the literature. As a consequence of SIEGAL–WALFISZ (cf. [17], Hilfssatz 3, p. 598) theorem, we have the following result which is due to VAN DER CORPUT:

LEMMA 2.2. (cf. [16], Footnote 4, pp. 279–280). *Let H be any positive number, then*

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O\left(\frac{x}{\log^{5H} x}\right), \tag{2.5}$$

where the O-constant dependes only on H.

LEMMA 2.3. (cf. [11], (33), p. 135). *If the Page hypothesis is true, then*

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O(x e^{-5B\sqrt{\log x}}), \tag{2.6}$$

where B is an absolute positive constant and the O-constant is independent of u and v.

LEMMA 2.4. (cf. [15], Theorem 6, p. 427). *If e.R.H. is true, then for $u < x$,*

$$\pi(x; u, v) = \frac{1}{\varphi(u)} \operatorname{Li} x + O\left(x^{\frac{1}{2}} \log x\right), \tag{2.7}$$

where the O-constant is independent of u and v.

3. Main Results. First we prove the following:

THEOREM 3.1. *If $T(k, r; n)$ is the number of representations of n as the sum of a prime and a (k, r)-integer, then for $n \to \infty$,*

$$T(k, r; n) = \prod_{\substack{p \\ p \nmid n}} \left\{1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r} - p^{-k}}{1-p^{-k}}\right\} \operatorname{Li} n + O\left(\zeta\left(\frac{k}{r}\right) n [\omega(n)]^{\frac{1}{5}}\right), \tag{3.1}$$

where the O-constant is independent of n, k, r and $\omega(x)$ is the function which appears in the O-term of (2.4).

PROOF. Let y denote a certain function of n, to be chosen later, which tends to infinity with n. The O-notation below refers to the passage $n \to \infty$.

We have by (2.2),

$$\begin{aligned} T(k, r; n) &= \sum_{\substack{p \\ p+m=n}} q_{k,r}(m) = \sum_{\substack{p \\ p+m=n}} \sum_{d\delta=m} \lambda(d) = \sum_{\substack{p \\ p+d\delta=n}} \lambda(d) = \\ &= \sum_{\substack{p+d\delta \\ d \le y}} \lambda(d) + \sum_{\substack{p+d\delta=n \\ d>y}} \lambda(d) = \sum\nolimits_1 + \sum\nolimits_2, \end{aligned} \tag{3.2}$$

say. Now,

$$\sum_1 = \sum_{\substack{p+d\delta=n\\ d\le y}} \lambda(d) = \sum_{d\le y} \lambda(d) \sum_{\substack{p<n\\ p\equiv n \pmod d}} 1 = \tag{3.3}$$

$$= \sum_{\substack{d\le y\\ (d,n)=1}} \lambda(d) \sum_{\substack{p<n\\ p\equiv n \pmod d}} 1 + \sum_{\substack{d\le y\\ (d,n)>1}} \lambda(d) \sum_{\substack{p<n\\ p\equiv n \pmod d}} 1 = \sum_{11} + \sum_{12},$$

say. We now estimate $\sum_{12}$. Since $(d, n) > 1$ and $n = p + d\delta$, we have $(d, n) = p$. It is obvious that the number of d's $\le y$ with $(d, n) = p$ for some prime p, is certainly $\le y$. Hence

$$\sum_{12} = \sum_{\substack{d\le y\\ (d,n)>1}} \lambda(d) \sum_{\substack{p<n\\ p\equiv n \pmod d}} 1 = \sum_{\substack{p+d\delta=n\\ d\le y}} \lambda(d) = O(y). \tag{3.4}$$

Next, by (2.4),

(3.5)

$$\sum_{11} = \sum_{\substack{d\le y\\ (d,n)=1}} \lambda(d) \sum_{\substack{p<n\\ p\equiv n \pmod d}} 1 = \sum_{\substack{d\le y\\ (d,n)=1}} \lambda(d) \left\{ \frac{1}{\varphi(d)} \operatorname{Li} n + O(n\omega(n)) \right\} =$$

$$= \operatorname{Li} n \sum_{\substack{d\le y\\ (d,n)=1}} \frac{\lambda(d)}{\varphi(d)} + O(ny\,\omega(n)).$$

Hence, by lemma 2.1, we have

(3.6)

$$\sum_{11} = \prod_{\substack{p\\ p\nmid n}} \left\{ 1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}} \right\} \operatorname{Li} n + O\left(\zeta\left(\frac{k}{r}\right) y^{\frac{1}{r}-\frac{3}{4}} \operatorname{Li} n \right) + O(ny\omega(n)).$$

Also, by Remark 2.1,

$$|\sum_2| = \Big| \sum_{\substack{p+d\delta=n\\ d>y}} \lambda(d) \Big| = \tag{3.7}$$

$$= \Big| \sum_{\substack{p+a^k b^r \delta=n\\ a^k b^r>y}} \mu(b) \Big| \le \sum_{\substack{p+a^k b^r \delta=n\\ a^k b^r>y}} 1 \le \sum_{\substack{a^k b^r \delta<n\\ a^k b^r>y}} 1 \le \sum_{a^k b^r \delta>y} \frac{n}{a^k b^r} =$$

$$= n \sum_{a=1}^{\infty} a^{-k} \sum_{b>\sqrt[r]{ya^{-k}}} b^{-r} = O\left(n \sum_{a=1}^{\infty} a^{-k} \left(\sqrt[r]{ya^{-k}}\right)^{1-r} \right) =$$

$$= O\left(ny^{\frac{1}{r}-1} \sum_{a=1}^{\infty} a^{-\frac{k}{r}} \right) = O\left(\zeta\left(\frac{k}{r}\right) ny^{\frac{1}{r}-1} \right).$$

Hence by (3.2), (3.3), (3.4), (3.6) and (3.7), we have

$$T(k,r;n) = \prod_{\substack{p \\ p\nmid n}} \left(1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}}\right) \mathrm{Li}\, n + \tag{3.8}$$

$$+ O\left(\zeta\left(\frac{k}{r}\right) y^{\frac{1}{r}-\frac{3}{4}} \mathrm{Li}\, n\right) + O(ny\,\omega(n)) + O(y) + O\left(\zeta\left(\frac{k}{r}\right) y^{\frac{1}{r}-1}\right).$$

Now, choosing $y = [\omega(n)]^{-\frac{4}{5}}$, we see that $y \to \infty$ as $n \to \infty$, since $\omega(n) \to 0$ as $n \to \infty$. The first O-term is (3.8) becomes

$$O\left(\zeta\left(\frac{k}{r}\right) [\omega(n)]^{-\frac{4}{5}\left(\frac{1}{r}-\frac{3}{4}\right)} \mathrm{Li}\, n\right) = O\left(\zeta\left(\frac{k}{r}\right) n[\omega(n)]^{\frac{1}{5}}\right),$$

since $\mathrm{Li}\, n \sim \dfrac{n}{\log n} = O(n)$, $\omega(n)$ is monotonic decreasing and $r \geq 2$.

It is clear that the second and third O-terms in (3.8) are each $O\left(n[\omega(n)]^{\frac{1}{5}}\right)$. The fourth O-term in (3.8) is

$$O\left(\zeta\left(\frac{k}{r}\right) n[\omega(n)]^{-\frac{4}{5}\left(\frac{1}{r}-1\right)}\right) = O\left(\zeta\left(\frac{k}{r}\right) n[\omega(n)]^{\frac{1}{5}}\right),$$

since $\omega(n)$ is monotonic decreasing and $r \geq 2$.

Hence theorem 3.1 follows:

REMARK 3.1 In the evaluation of $\sum_{11}$ above, we argued that

$$\sum_{\substack{d \leq y \\ (d,n)=1}} \lambda(d) \cdot O(n\,\omega(n)) = O(ny\,\omega(n)).$$

This argument holds, provided the O-constant is independent of d. This is indeed true, in virtue of (2.5), (2.6) and (2.7) with

$$\omega(x) = \frac{1}{\log^{5H} x}, \quad \omega(x) = e^{-5B\sqrt{\log x}}$$

and

$$\omega(x) = x^{-\frac{1}{2}} \log x$$

respectively in (2.4).

In view of the above Remark and $\zeta\left(\dfrac{k}{r}\right) \leq \zeta\left(\dfrac{r+1}{r}\right)$, we have the following:

COROLLARY 3.1.

$$(3.9)\qquad T(k,r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}\mathrm{Li}\,n+O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on r and H.

COROLLARY 3.2. *If the Page hypothesis is true, then*

$$(3.10)\qquad T(k,r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}\mathrm{Li}\,n+O\left(ne^{-B\sqrt{\log n}}\right),$$

where B is an absolute positive constant and the O-constant depends at most on r.

COROLLARY 3.3. *If the e.R.H. is true, then*

$$(3.11)\qquad T(k,r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}\mathrm{Li}\,n+O\left(n^{\frac{9}{10}}\log^{\frac{1}{5}} n\right),$$

where the O-constant depends at most on r.

REMARK 3.2. We note that in the limiting case $k\to\infty$, a (k,r)-integer becomes a r-free integer.

Hence, by taking limits of (3.9), (3.10) and (3.11) as $k\to\infty$, we obtain the following:

$$(3.12)\qquad T(r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{p^{r-1}(p-1)}\right\}\mathrm{Li}\,n+O\left(\frac{n}{\log^H n}\right),$$

$$(3.13)\qquad T(r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{p^{r-1}(p-1)}\right\}\mathrm{Li}\,n+O\left(ne^{-B\sqrt{\log n}}\right),$$

$$(3.14)\qquad T(r;n)=\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{p^{r-1}(p-1)}\right\}\mathrm{Li}\,n+O\left(n^{\frac{9}{10}}\log^{\frac{1}{5}} n\right).$$

REMARK 3.3. It may be noted that (3.12) is the same as (1.1). However, (3.13) and (3.14) give better O-estimates on the assumption of the Page hypothesis and on the e.R.H. respectively.

COROLLARY 3.4. *If $T^*(k,r;n)$ is the number of representations of n as the sum of a prime and a proper (k,r)-integer, then for $n\to\infty$,*

$$(3.15)\qquad T^*(k,r;n)=$$

$$=\left[\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{1-p^{-1}}\cdot\frac{p^{-r}-p^{-k}}{1-p^{-k}}\right\}-\prod_{\substack{p\\ p\nmid n}}\left\{1-\frac{1}{p^{r-1}(p-1)}\right\}\right]\mathrm{Li}\,n+O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on r and H.

PROOF. This follows by (3.9) and (3.12), since

$$T^*(k, r; n) = T(k, r; n) - T(r; n).$$

COROLLARY 3.5. *The O-term in* (3.15) *can be replaced by* $O(ne^{-B\sqrt{\log n}})$ *and by* $O\left(n^{\frac{9}{10}} \log^{\frac{1}{5}} n\right)$, *on the assumption of the page hypothesis and on the e.R.H. respectively.*

THEOREM 3.2. *Every sufficiently large integer can be expressed as the sum of a prime and a proper* (k, r)*-integer.*

PROOF. Let

$$\alpha(k, r; p) = 1 - \frac{1}{1-p^{-1}} \cdot \frac{p^{-r}-p^{-k}}{1-p^{-k}} \quad \text{and} \quad \beta(r; p) = 1 - \frac{1}{p^{r-1}(p-1)}.$$

It is clear that $\alpha(k, r; p) > \beta(r; p)$. By (3.15), we have

$$(3.16) \qquad T^*(k, r; n) = \left[\prod_{\substack{p \\ p \nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p \nmid n}} \beta(r; p)\right] \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on r and H.

Since the main term in (3.16) is positive for every n, it is sufficient, if we prove the following: As $n \to \infty$,

$$(3.17) \qquad T^*(k, r; n) \sim \left[\prod_{\substack{p \\ p \nmid n}} \alpha(k, r; p) - \prod_{\substack{p \\ p \nmid n}} \beta(r; p)\right] \mathrm{Li}\, n.$$

Let

$$\gamma(k, r; p) = \frac{\alpha(k, r; p)}{\beta(r; p)} - 1,$$

we have

$$\beta(r, p;) \cdot \gamma(k, r; p) = \alpha(k, r; p) - \beta(r; p) =$$

$$= \frac{1}{p^r(1-p^{-1})} \cdot \frac{p^{-r}-p^{-k}}{(1-p^{-1})\cdot(1-p^{-k})} = \frac{1}{p^r(1-p^{-1})} \cdot \frac{p^r-1}{p^k-1} > \frac{1}{2p^k},$$

since $p^r - 1 > \frac{p^r}{2}$, $\frac{1}{1-p^{-1}} > 1$ and $\frac{1}{p^k-1} > \frac{1}{p^k}$.

Hence $\gamma(k, r; p) > \frac{1}{\beta(r; p)} \cdot \frac{1}{2p^k}$, so that

$$\frac{\alpha(k, r; p)}{\beta(r; p)} = 1 + \gamma(k, r; p) > 1 + \frac{1}{2p^k}.$$

We note that $\prod_p \beta(r; p)$, the product being extended over all primes p, is a convergent infinite product, and we denote it by β_r.

We have

$$\prod_{\substack{p\\ p\nmid n}} \alpha(k,r;p) - \prod_{\substack{p\\ p\nmid n}} \beta(r;p) =$$

$$= \prod_{\substack{p\\ p\nmid n}} \beta(r;p)\left[\prod_{\substack{p\\ p\nmid n}} \frac{\alpha(k,r;p)}{\beta(r;p)} - 1\right] > \prod_{\substack{p\\ p\nmid n}} \beta(r;p)\left[\prod_{\substack{p\\ p\nmid n}}\left(1+\frac{1}{2p^k}\right)-1\right] =$$

$$= \frac{\prod_{p} \beta(r;p)}{\prod_{p\mid n} \beta(r;p)}\left[\prod_{\substack{p\\ p\nmid n}}\left(1+\frac{1}{2p^k}\right)-1\right] > \beta_r\left[\prod_{\substack{p\\ p\nmid n}}\left(1+\frac{1}{2p^k}\right)-1\right],$$

since $\prod_{p\mid n} \beta(r;p) < 1$.

Hence

$$\prod_{\substack{p\\ p\nmid n}} \alpha(k,r;p) - \prod_{\substack{p\\ p\nmid n}} \beta(r;p) > \beta_r \cdot \frac{1}{2p^k},$$

where the p on the right is any prime not dividing n.

As a simple consequence of the prime number theorem in the form $\sum_{p\le x} \log p \sim x$, we see that for every sufficiently large n, there is a prime p such that $p \le 2\log n$ and $p\nmid n$. Hence for all sufficiently large n, we have

$$\prod_{\substack{p\\ p\nmid n}} \alpha(k,r;p) - \prod_{\substack{p\\ p\nmid n}} \beta(r;p) > \frac{\beta_r}{2}\cdot\frac{1}{2^k\log^k n} = \frac{\beta_r}{2^{k+1}\log^k n}.$$

This together with (3.16) gives

$$\frac{T^*(k,r;n)}{\left[\prod_{\substack{p\\ p\nmid n}} \alpha(k,r;p) - \prod_{\substack{p\\ p\nmid n}} \beta(r;p)\,\mathrm{Li}\, n\right]} = 1 + O\left(\frac{n}{\log^H n}\cdot\frac{2^{k+1}\log^k n}{\beta_r}\cdot\frac{\log n}{n}\right) =$$

$$= 1 + O\left[\frac{1}{(\log n)^{H-k-1}}\right],$$

where the O-constant is independent of n.

Hence, if $H > k+1$, then (3.17) follows. Since the number H in lemma 2.2 is an arbitrary positive number, we can take H such that $H > k+1$ for any fixed k.

Thus theorem 3.2 follows.

REMARK 3.4. On the assumption of the Page hypothesis or on the e.R.H. also, the conclusion of theorem 3.2 follows. This is rather more immediate in view of Corollary 3.5. However, we have the improved O-estimates in the asymptotic formula for $T^*(k,r;n)$ on the assumption of these hypotheses.

As particular cases of theorem 3.2, we have the following:

COROLLARY 3.6. ($r = 2$). *Every sufficiently large integer can be expressed as the sum of a prime and an integer of the form $a^k b$, where $a > 1$ and b is a square-free integer; k being any given integer ≥ 3.*

COROLLARY 3.7. ($k = 3$, $r = 2$). *Every sufficiently large integer can be expressed as the sum of a prime and an integer of the form $a^3 b$, where $a > 1$ and b is square-free.*

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ON CLOSE-TO-CONVEX FUNCTIONS OF HIGHER ORDER

By

A. W. GOODMAN

University of South Florida

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Since the introduction of close-to-convex functions by W. KAPLAN [10] in 1952, they have been the subject of numerous papers and the concept has been generalized in several directions [2, 12, 14, 21]. In this work we consider the class $\mathcal{K}(\beta)$ of close-to-convex functions of order β where $\beta \geqq 0$ is an otherwise arbitrary real number. POMMERENKE [13, 15], READE [19] and RÉNYI [21, 22] have previously investigated $\mathcal{K}(\beta)$ with $0 < \beta < 1$. If $\beta > 1$, a function in $\mathcal{K}(\beta)$ is not necessarily univalent, and this may be the reason that this case has not previously received attention. Nevertheless, the class has some interesting features.

1. Definitions. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

be regular in $E: |z| < 1$ and let

$$\varphi(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

be regular and convex in E, ($\varphi(z)$ maps E onto a convex domain). According to KAPLAN [10], $f(z)$ is said to be close-to-convex in E if there is a constant $C \neq 0$ and a convex $\varphi(z)$ such that in E

$$R \frac{Cf'(z)}{\varphi'(z)} > 0. \tag{1.3}$$

In the original treatment, and in most subsequent papers the functions f and φ are not necessarily normalized, and the constant in equation (1.3) is omitted. For our purpose, it seems better to normalize the functions as we have done in (1.1) and (1.2) and to include a multiplicative constant which is visible in (1.3).

We first use an iteration process to define close-to-convex functions of higher order. Indeed, let $\mathcal{K}(0)$ be the class of normalized functions that

are regular and convex in E, and let $\mathcal{K}(1)$ be the class of normalized functions that are regular and close-to-convex in E.

DEFINITION 1. *Let $n \geqq 2$ be an integer. We say that $f(z)$ belongs to the class $\mathcal{K}(n)$, if it is regular in E, has the normalization* (1.1), *and if there is a $C \neq 0$ and a $\varphi(z)$ in $\mathcal{K}(n-1)$ such that*

$$R \frac{Cf'(z)}{\varphi'(z)} > 0. \tag{1.4}$$

It is occasionally convenient to write the condition (1.4) in one of the equivalent forms:

$$\left| \arg \frac{Cf'(z)}{\varphi'(z)} \right| < \frac{\pi}{2}, \tag{1.5}$$

or

$$\frac{Cf'(z)}{\varphi'(z)} = p(z) = p_0 + p_1 z + \ldots + p_n z^n + \ldots, \tag{1.6}$$

where $p(z)$ is a function with positive real part in E.

THEOREM 1. *Let $f_n(z) \in \mathcal{K}(n)$, where $n \geqq 1$. Then there is a $C \neq 0$ and a $\varphi(z)$ in $\mathcal{K}(0)$ such that in E*

$$\left| \arg \frac{Cf_n'(z)}{\varphi'(z)} \right| < n \frac{\pi}{2}. \tag{1.7}$$

PROOF. We remark first that if $f_j(z)$ is in $\mathcal{K}(j)$ then $f_j'(z) \neq 0$ in E. By definition, if $f_n(z)$ is in $\mathcal{K}(n)$ then it generates a sequence of functions $f_j(z)$, $j = 0, 1, 2, \ldots, n-1$ and a sequence of constants C_j such that for $j = 1, 2, \ldots n$

$$\frac{C_j f_j'(z)}{f_{j-1}'(z)} = p_j(z), \tag{1.8}$$

where $p_j(z)$ is a function with positive real part. We form the product of the n functions in (1.8) and set $f_0(z) = \varphi(z)$. Then the cancellation of common terms gives (1.7).

THEOREM 2. *Suppose that $f(z)$, given by* (1.1), *is regular in E and that there is a function $\varphi(z)$ in $\mathcal{K}(0)$ and a constant $C \neq 0$ such that* (1.7) *holds in E. Then $\varphi(z) \in \mathcal{K}(n)$.*

In proving this theorem we will need $[f'(z)]^\alpha$ where $\alpha > 0$. Since $f'(z) \neq 0$ in E, the function $[f'(z)]^\alpha$ can always be defined so that it gives an analytic function, and throughout this paper we will always select that branch for which $[f'(0)]^\alpha = 1$. With this agreement, we set

$$f_j(z) = \int_0^z [f'(t)]^{j/n} [\varphi'(t)]^{(n-j)/n} \, dt, \tag{1.9}$$

so that $f_n(z) \equiv f(z)$, and $f_0(z) \equiv \varphi(z) \in \mathcal{K}(0)$. Further each $f_j(z)$ has the form (1.1). If $C^{1/n}$ is selected properly we find that for $j = 1, 2, \ldots, n$.

$$C^{1/n} \frac{f_j'(z)}{f_{j-1}'(z)} = C\left[\frac{f'(z)}{\varphi'(z)}\right]^{1/n} = p(z), \tag{1.10}$$

a function with positive real part in E. Hence $f_j(z)$ is in $\mathcal{K}(j)$ for $j = 1, 2, \ldots, n$.

With these two theorems in view, we can use the property (1.7) as a definition; and modify it to include arbitrary $\beta \geq 0$.

DEFINITION 2. *Let $\beta \geq 0$. We say that $f(z)$ belongs to the class $\mathcal{K}(\beta)$ if it is regular in E, has the normalization* (1.1), *and if there is a $C \neq 0$ and a $\varphi(z)$ in $\mathcal{K}(0)$ such that in E*

$$\left| \arg \frac{Cf'(z)}{\varphi'(z)} \right| \leq \beta \frac{\pi}{2}. \tag{1.11}$$

It is clear that in most situations the equality sign in (1.11) can be dropped. However, we want the class $\mathcal{K}(0)$ to consist of all normalized convex functions, and to obtain this we must permit equality in (1.11) when $\beta = 0$.

When $0 < \beta < 1$, the classes $\mathcal{K}(\beta)$ have been studied earlier [13, 15, 19, 21, 22]. If $f(z) \in \mathcal{K}(\beta)$ and $\beta \leq 1$, then $f(z)$ is close-to-convex and hence univalent in E. If $\beta = 1 + \varepsilon$, $\varepsilon > 0$, then (as the author has shown [7]) there is no bound that can be placed on the valence of $f(z)$, that will be valid for all functions in $\mathcal{K}(\beta)$.

2. Linear invariance. The concept of a linear-invariant family of functions was introduced and thoroughly investigated by POMMERENKE [16, 17].

DEFINITION 3. *Let $\mathcal{L}$ be the class of all Möbius transformations that map E onto E. A family $\mathcal{F}$ of functions is said to be a linear-invariant family if:*

(1) *Each $f(z)$ in $\mathcal{F}$ is regular in E, has the normalization* (1.1), *and $f'(z) \neq 0$ in E,*

(2) *For each $f(z)$ in $\mathcal{F}$ and each $L(z)$ in $\mathcal{L}$, the function*

$$F(z) \equiv \frac{f(L(z)) - f(L(0))}{f'(L(0))\, L'(0)} = z + A_2 z^2 + \ldots \tag{2.1}$$

is also in $\mathcal{F}$.

Each linear invariant family $\mathcal{F}$ has an order α (which may be infinite) defined by

$$\alpha = \sup_{f \in \mathcal{F}} |a_2|. \tag{2.2}$$

It is easy to find linear-invariant families. Any geometric condition that is preserved under the transformation (2.1) will serve to define a

linear-invariant family. For example the family of normalized univalent functions is a linear-invariant family. Further, any particular $f_0(z)$ that satisfies the first part of Definition 3 can be used to generate a minimal linear-invariant family $\mathcal{F}_0$ which contains f_0. One merely places in $\mathcal{F}_0$ all of the transforms of $f_0(z)$ defined by (2.1). Both READE [20] and POMMERENKE [15] have observed that if $0 \leqq \beta \leqq 1$, then $\mathcal{K}(\beta)$ is a linear-invariant family. Their method gives the more general.

THEOREM 3. *For each $\beta \geqq 0$, the family $\mathcal{K}(\beta)$ is a linear-invariant family of order $\alpha = \beta + 1$.*

PROOF. Let $f(z) \in \mathcal{K}(\beta)$ and let C and $\varphi(z)$ satisfy the conditions of Definition 2. Then

$$\Phi(z) \equiv \frac{\varphi(L(z)) - \varphi(L(0))}{\varphi'(L(0))\, L'(0)} \in \mathcal{K}(0). \tag{2.3}$$

If $F(z)$ is given by (2.1), then

$$\frac{F'(z)}{\Phi'(z)} = \frac{f'(L(z))\, L'(z)}{f'(L(0))\, L'(0)} \cdot \frac{\varphi'(L(0))\, L'(0)}{\varphi'(L(z))\, L'(z)} = \frac{\varphi'(L(0))}{f'(L(0))} \cdot \frac{f'(L(z))}{\varphi'(L(z))}. \tag{2.4}$$

Consequently, if C_1 is selected properly, then $|\arg C_1 F'(z)/\Phi'(z)| < \beta\pi/2$, and hence $F(z) \in \mathcal{K}(\beta)$.

To determine the order of the family $\mathcal{K}(\beta)$ we observe that for $\beta > 0$

$$\left[C\, \frac{f'(z)}{\varphi'(z)}\right]^{1/\beta} = p_0 + p_1 z + p_2 z^2 + \ldots \equiv p(z), \tag{2.5}$$

where $p(z)$ has positive real part in E. Raising both sides of (2.5) to the β power and comparing coefficients on both sides we find that $C = p_0^\beta$ and that $a_2 = b_2 + \beta p_1/2p_0$. By Loewner's Theorem for normalized convex functions, $|b_2| \leqq 1$, and it is well-known that $|p_1| \leqq 2|p_0|$. Hence, $|a_2| < \beta + 1$.

Finally, we consider the function

$$F_\beta(z) \equiv \frac{1}{2(\beta+1)} \left[\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right] \equiv z + \sum_{n=2}^{\infty} A_n(\beta) z^n, \tag{2.6}$$

where the quantities $A_n(\beta)$ are defined by (2.6). Using $\varphi(z) = z/(1-z)$ it is an easy matter to show that $F_\beta(z)$ is in $\mathcal{K}(\beta)$. Further $A_2(\beta) = \beta + 1$ so that the upper bound just established for $|a_2|$ is indeed the least upper bound for the class $\mathcal{K}(\beta)$.

The extremal function $F_\beta(z)$ will play an important role in the rest of the paper. It should be noted that if $\gamma < \beta$, then $F_\beta(z) \notin \mathcal{K}(\gamma)$.

In his fundamental work on linear-invariant families POMMERENKE [16] forms a universal linear-invariant family $\mathcal{U}_\alpha$ by taking the union of all linear invariant families of order $\gamma \leqq \alpha$, and for this universal family he proves a large number of theorems. Since $\mathcal{K}(\beta) \subset \mathcal{U}_{\beta+1}$, all of these theo-

rems hold for the class $\mathcal{K}(\beta)$. If the extremal function for such a theorem is in $\mathcal{K}(\beta)$, then the theorem is a best possible one for the class $\mathcal{K}(\beta)$. On the other hand, if the extremal function is in $\mathcal{U}_{\beta+1}$ but not in $\mathcal{K}(\beta)$, then a further refinement is desirable. We close this section with one theorem of the first type, and consider some theorems of the second type in the following sections.

Let $S \equiv f(E)$, contain the two points w_1 and w_2 and let V be an arbitrary curve in S that joins w_1 and w_2. By the diameter of V we understand the diameter of the point set obtained by projecting V on the plane. We set

$$d(w_1, w_2) \equiv \inf_V \operatorname{diam} V, \tag{2.7}$$

$$l(w_1, w_2) \equiv \inf_V \int_V |dw| = \inf_V (\text{length of } V), \tag{2.8}$$

and we let $d(w)$ be the radius of the largest open disk with center at w that is contained in S. With these preparations the POMMERENKE result [16, Theorem 1.1], originally proved for the family $\mathcal{U}_\alpha$, can be stated as

THEOREM 4. *Let $f(z) \in \mathcal{K}(\beta)$ and let $|z| = r < 1$. Then*

$$|\log(1-z^2) f'(z)| \leq (\beta+1) \log \frac{1+r}{1-r}, \tag{2.9}$$

$$\frac{1}{2(\beta+1)}\left[1-\left(\frac{1-r}{1+r}\right)^{\beta+1}\right] \leq d(0, f(z)) \leq \tag{2.10}$$

$$\leq l(0, f(z)) \leq \frac{1}{2(\beta+1)}\left[\left(\frac{1+r}{1-r}\right)^{\beta+1}-1\right],$$

$$\frac{1}{2(\beta+1)}\left[1-\left(\frac{1-r}{1+r}\right)^{\beta+1}\right] \leq \frac{d(0, f(z))}{|(1-z^2) f'(z)|} \leq \tag{2.11}$$

$$\leq \frac{l(0, f(z))}{|(1-z^2) f'(z)|} \leq \frac{1}{2(\beta+1)}\left[\left(\frac{1+r}{1-r}\right)^{\beta+1}-1\right],$$

$$\frac{1}{2(\beta+1)}(1-|z|^2)|f'(z)| \leq d(f(z)) \leq (1-|z|^2)|f'(z)|, \tag{2.12}$$

and

$$\frac{(1-r)^\beta}{(1+r)^{\beta+2}} \leq |f'(z)| \leq \frac{(1+r)^\beta}{(1-r)^{\beta+2}}. \tag{2.13}$$

With a little effort the reader can show that equality occurs on the right side of (2.12) for $f(z) = z/(1-rz)$, and in all the other relations the equality sign holds for $F_\beta(z)$, either at $z=r$ or $z=-r$.

3. A Rotation Theorem. Sharp bounds for $|\arg f'(z)|$, when $f(z)$ is in $\mathcal{U}_2$, are not easy to find [16, Corollary 2.1]. However, for the subclass $\mathcal{K}(\beta)$ we have

THEOREM 5. *Let $f(z)\in\mathcal{K}(\beta)$ and let $|z|=r<1$. Then*

$$-2(\beta+1)\arcsin r \leqq \arg f'(z) \leqq 2(\beta+1)\arcsin r \tag{3.1}$$

and this inequality is sharp for each r.

PROOF. From the definition of $\mathcal{K}(\beta)$, we know that $[Cf'(z)/\varphi'(z)]^{1/\beta}$ maps E into the half plane $Rw>0$. Without loss of generality we can select C so that $|C|=1$. Since $C^{1/\beta}=e^{i\gamma}$ is the image point of $z=0$, we must have $|\gamma|<\pi/2$. Consequently, $g(z)\equiv[f'(z)/\varphi'(z)]^{1/\beta}$ maps E onto the half plane $-\pi/2-\gamma<\arg w<\pi/2-\gamma$, with $g(0)=1$. We let $L(z)$ be a linear transformation that maps E onto the same half plane, with $L(0)=1$. Then the Schwarz lemma applied to $L^{-1}(g(z))$ tells us that $g(z)$ maps the disk $E_r:|z|\leqq r$, onto a set contained in $L(E_r)$. Consequently $|\arg g(z)|\leqq \leqq \max\arg L(z)$. Now $L(z)=(1+e^{-2i\gamma}z)/(1-z)$ so that $|\arg g(z)|\leqq \leqq 2\arcsin r$. Hence

$$|\arg f'(z)| \leqq \beta(2\arcsin r)+\max\arg\varphi'(z), \tag{3.2}$$

where $\varphi(z)$ is in $\mathcal{K}(0)$. But for convex functions it is well-known [2, 3, 6] that $|\arg\varphi'(z)|\leqq 2\arcsin r$. Hence (3.2) yields (3.1). It is an easy matter to show that the function

$$f_\delta(z)\equiv\frac{1}{(\beta+1)(1-e^{i\delta})}\left[\left(\frac{1+e^{i\delta}z}{1-z}\right)^{\beta+1}-1\right] \tag{3.3}$$

is in $\mathcal{K}(\beta)$ and that with suitable choice of δ and z the equality sign holds in (3.1) for this function.

According to the Noshiro–Warshawski Theorem, if $Rf'(z)>0$ in E, then $f(z)$ is univalent in E. Consequently, Theorem 5, gives

COROLLARY 1. *Let $f(z)\in\mathcal{K}(\beta)$. Then $f(z)$ is univalent in the disk $|z|<$ $<\sin[\pi/4(\beta+1)]$.*

This result is almost certainly not sharp. It seems likely that the radius of univalence for the class $\mathcal{K}(\beta)$ is $\csc\eta-\cot\eta$, where $\eta=\pi/(\beta+1)$, since this is the value for the special function $F_\beta(z)$.

In contrast, we can prove

THEOREM 6. *Suppose that $f_k(z)\in\mathcal{K}(\beta)$ and $m_k>0$, for $k=1,2,\ldots,n$, $(n\geqq 2)$. Then the function*

$$F(z)\equiv\sum_{k=1}^{n} m_k f_k(z) \tag{3.4}$$

is univalent in $|z|<\sin(\pi/4(\beta+1))\equiv r_1$, and this bound is sharp.

PROOF. By Theorem 5, we have $Rf_k'(z)>0$ for $|z|<r_1$ and hence $F(z)$ is univalent in $|z|<r_1$. To see that no $r>r_1$ will have the same property it

is sufficient to let $f_1(z) = e^{-i\theta} f_0(e^{i\delta} z)$ with suitable choice of Θ and δ, and to set $f_2(z) = \overline{f_1(\bar{z})}$ with $m_1 = m_2 = 1/2$. Then $F'(r_1) = 0$.

The radius of convexity for an arbitrary linear invariant family was determined in [16, Theorem 2.5]. However, the radius of starlikeness seems to present a more difficult problem, either for $\mathcal{U}_\alpha$ or $\mathcal{K}(\beta)$.

4. Relations Among the Classes. In order to form a vector space it is necessary to introduce an abstract sum, and an abstract multiplication by a scalar. HORNICH [9] denotes the abstract sum $f \oplus g$ by $[f, g]$ where

$$[f, g] \equiv \int_0^z f'(t)\, g'(t)\, dt. \tag{4.1}$$

Multiplication by the scalar λ, is denoted by $\lambda \times f$ where

$$\lambda \times f \equiv \int_0^z [f'(t)]^\lambda\, dt. \tag{4.2}$$

With a few additional quite natural conditions, he obtains a Banach space, and the investigation of this space and related spaces is continued in a recent work of CAMPBELL, CIMA, and PFALTZGRAFF [4].

To obtain a theorem about the combination $[\lambda \times f, \mu \times g]$ we will need

LEMMA 1. *Let $\varphi(z)$ and $\psi(z)$ be in $\mathcal{K}(0)$; i.e., normalized and convex in E. Let $u + v \leq 1$, with $u \geq 0$, and $v \geq 0$. If*

$$\Phi(z) \equiv \int_0^z [\varphi'(t)]^u [\psi'(t)]^v\, dt, \tag{4.3}$$

then $\Phi(z)$ is in $\mathcal{K}(0)$.

This lemma follows from the condition $R(1 + z\Phi''(z)/\Phi'(z)) > 0$ and since the proof is so simple, it must be well-known.

THEOREM 7. *Let $\lambda \geqq 0$, $\mu \geqq 0$, and $\lambda + \mu \geqq 1$. If $f(z)$ is in $\mathcal{K}(\beta)$ and $g(z)$ is in $\mathcal{K}(\gamma)$, then*

$$F(z) \equiv \int_0^z [f'(t)]^\lambda [g'(t)]^\mu\, dt \tag{4.4}$$

is in $\mathcal{K}(\delta)$ where $\delta = \lambda\beta + \mu\gamma + 2(\lambda + \mu - 1)$, and the value for δ is sharp.

PROOF. By hypothesis there exist C_1, C_2, $\varphi(z)$ and $\psi(z)$ in $\mathcal{K}(0)$ such that

$$\left| \arg \frac{C_1 f'(z)}{\varphi'(z)} \right| < \beta \frac{\pi}{2}, \quad \left| \arg \frac{C_2 g'(z)}{\varphi'(z)} \right| < \gamma \frac{\pi}{2}. \tag{4.5}$$

Let u and v be nonnegative numbers such that $u+v = 1$, $\lambda \geqq u$, and $\mu \geqq v$. Then, with $\Phi(z)$ defined by (4.3), and $F(z)$ by (4.4), we have

$$\arg \frac{C_1^\lambda C_2^\mu F'(z)}{\Phi'(z)} =$$

$$= \left| \arg \left[\frac{C_1 f'(z)}{\varphi'(z)} \right]^\lambda \left[\frac{C_2 g'(z)}{\psi'(z)} \right]^\mu [\varphi'(z)]^{\lambda-u} [\psi'(z)]^{\mu-v} \right| \leqq$$

$$\leqq (\lambda\beta+\mu\gamma)\frac{\pi}{2}+(\lambda-u+\mu-v)\pi = \delta\frac{\pi}{2},$$

since $\varphi(z)$ and $\psi(z)$ are convex. Hence $F(z) \in \mathcal{K}(\delta)$. To see that no smaller value for δ will suffice, we set $f(z) = F_\beta(z)$, and $g(z) = F_\gamma(z)$, with the notation (2.6), and use $\varphi(z) = \psi(z) = z/(1-z)$. Then

$$\frac{F'(z)}{\varphi'(z)} = \frac{(1+z)^{\lambda\beta+\mu\gamma}(1-z)^2}{(1-z)^{\lambda\beta+\mu\gamma+2(\lambda+\mu)}}.$$

By selecting z properly we can force $\arg [F'(z)/\varphi'(z)]$ to be arbitrarily close to $\pm[\lambda\beta+\mu\gamma+2(\lambda+\mu-1)]\pi/2$.

COROLLARY 1. *Suppose that $\lambda+\mu = 1$, $\lambda \geqq 0$, $\mu \geqq 0$, $f(z)$ is in $\mathcal{K}(\beta)$ and $g(z)$ is in $\mathcal{K}(\gamma)$. Then $F(z)$, defined by* (4.4), *is in $\mathcal{K}(\lambda\beta+\mu\gamma)$.*

COROLLARY 2. *Suppose that $\lambda+\mu = 1$, $\lambda \geqq 0$, $\mu \geqq 0$, and $f(z)$ and $g(z)$ are in $\mathcal{K}(\beta)$. Then $F(z)$ defined by* (4.4) *is in $\mathcal{K}(\beta)$.*

COROLLARY 3. *Suppose that $\varphi(z)$, $\psi(z)$ are in $\mathcal{K}(0)$, $u \geqq 0$, and $v \geqq 0$. If $\Phi(z)$ is defined by* (4.3) *and $u+v \leqq 3/2$, then $\Phi(z)$ is close-to-convex in E, and the bound $3/2$ is sharp.*

5. The coefficient problem. If β is a positive integer, then we can give sharp upper bounds for the coefficients, but if β is not an integer, then the determination of sharp upper bounds is still an open problem.

THEOREM 8. *Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}(\beta)$ and let*

$$F_\beta(z) = \frac{1}{2(\beta+1)}\left[\left(\frac{1+z}{1-z}\right)^{\beta+1}-1\right] \equiv z+\sum_{n=2}^{\infty} A_n(\beta)z^n \tag{5.1}$$

where $A_n(\beta)$ is defined by (5.1). *If β is an integer, then*

$$|a_n| \leqq A_n(\beta), \qquad n = 2, 3, \ldots \tag{5.2}$$

and the bound $A_n(\beta)$ is sharp for each $n \geqq 2$, and each positive integer β.

We note that when $\beta = 1$, the result contains the theorem of READE [18] on close-to-convex functions. RÉNYI [21] has obtained an upper bound for $|a_n|$ when $0 < \beta < 1$, but his bound is not sharp unless $n = 2$.

PROOF. We use the standard notation $f(z) \ll g(z)$ to mean that $g(z) = \sum_0^\infty b_n z^n$ and $|a_n| \leq b_n$ for $n = 0, 1, 2, \ldots$. Without loss of generality, we select C in (1.11) so that $|C| = 1$. Then $[Cf'(z)/\varphi'(z)]^{1/\beta}$ has positive real part and hence

$$\left[\frac{Cf'(z)}{\varphi'(z)}\right]^{1/\beta} \ll \frac{1+z}{1-z} = 1+2z+2z^2+\ldots \equiv 1+u. \tag{5.3}$$

Now if β is an integer, then the coefficients of $(1+u)^\beta$ are all positive or zero. It is important to note that if $\beta = m+q$, where m is an integer, and $0<q<1$, then in the expansion of $(1+u)^\beta$ the coefficient of u^{m+2} is negative, whereas for lower powers of u the coefficients are all positive. Continuing with the case in which β is an integer, we have

$$\frac{Cf'(z)}{\varphi'(z)} \ll \left(\frac{1+z}{1-z}\right)^\beta,$$

$$Cf'(z) \ll \left(\frac{1+z}{1-z}\right)^\beta \sum_{n=1}^\infty nz^{n-1} = \left(\frac{1+z}{1-z}\right)^\beta \frac{1}{(1-z)^2},$$

and since $|C| = 1$, we have

$$f'(z) \ll (1+z)^\beta/(1-z)^{\beta+2}.$$

Clearly integration preserves coefficient domination. Hence

$$f(z) \ll F_\beta(z).$$

If we reconsider the proof, when β is not an integer, we obtain

THEOREM 9. *Let $f(z) \in \mathcal{K}(\beta)$, where $[\beta] = m \geq 0$. Then the bound* (5.2) *holds for $n = 2, 3, \ldots, m+2$,* and *the bound is sharp for these values of n.*

Note that in particular we recapture the order of the linear-invariant family $\mathcal{K}(\beta)$, because for arbitrary $\beta \geq 0$, (5.2) gives $|a_2| \leq \beta+1$.

To obtain a recursion formula for computing $A_n(\beta)$, we observe the identity

$$(1-z^2)F_\beta'(z) = 2(\beta+1)F_\beta(z)+1$$

and on equating coefficients we have

$$(n+1)A_{n+1}(\beta) = 2(\beta+1)A_n(\beta)+(n-1)A_{n-1}(\beta). \tag{5.4}$$

Using (5.4), we find that

$$A_3(\beta) = 2\beta^2+4\beta+3)/3 \text{ and } A_4(\beta) = (\beta+1)(\beta^2+2\beta+3)/3.$$

The elementary inequality $|a_3| \leq A_3(\beta)$ for all $f(z)$ in $\mathcal{K}(\beta)$ still poses an open problem [4].

A direct formula for computing the bounds $A_n(\beta)$ can be found by using the binomial formula. After some simplification, we find that for $n \geqq 2$.

$$A_n(\beta) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \prod_{j=1}^{n-1} (\beta + n + 1 - k - j). \tag{5.5}$$

In this connection one may consult the paper by KIRWAN [11] who is lead to the same extremal function except that $\beta+1$ is replaced by $k/2$.

Despite the presence of negative terms in the sum on the right side of (5.5), each $A_n(\beta) > 0$, and $A_n(\beta)$ is a monotonic increasing function of β for $n \geqq 2$. This follows by induction from (5.4). Further $A_n(\beta)$ is a polynomial of degree $n-1$ in β and the leading coefficient is $2^{n-1}/n!$.

6. Length and area. Let $L(r, f)$ denote the length of the image of the circle $|z| = r$ under the function $f(z)$, and let $A(r, f)$ denote the area of the image of the disk $|z| \leqq r$ under $f(z)$. DUREN [6] proved that if $f(z)$ is in $\mathcal{K}(1)$, and $C > 0$ in (1.3) then $L(r, f) \leqq L(r, z/(1-z)^2)$. The restriction $C > 0$ gives a proper subclass of $\mathcal{K}(1)$. Later CLUNIE and DUREN [5] extended the theorem to include the full class of close-to-convex functions. DUREN [6] noted that the inequality for areas, $A(r, f) \leqq A(r, z/(1-z)^2)$ follows immediately from the known coefficient bound $|a_n| \leqq n$. Using the methods developed by CLUNIE and DUREN, we can prove

THEOREM 11. *Suppose that $f(z)$ is in $\mathcal{K}(\beta)$. If $\beta \geqq 1$, then for each r in* (0, 1)

$$L(r, f) \leqq L(r, F_\beta), \tag{6.1}$$

where $F_\beta(z)$ is defined by (2.6). *If $\beta \geqq 1/2$, then for each r in* (0, 1)

$$A(r, f) \leqq A(r, F_\beta). \tag{6.2}$$

We note that the case $0 < \beta < 1$ has been considered previously by POMMERENKE [13], and that if β is a positive integer, then (6.2) follows immediately from the coefficient bounds established in Theorem 8.

PROOF. We select C in (1.11) so that $|C| = 1$ and we put

$$\frac{Cf'(z)}{q'(z)} = [p(z)]^\beta \tag{6.3}$$

where

$$p(z) = p_0 + p_1 z + \ldots, \qquad |p_0| = 1,$$

has positive real part. Now if $p_0 = 1$, the function $p(z)$ can be represented as a Stieltjes integral

$$\int_0^{2\pi} \frac{1 + ze^{is}}{1 - ze^{is}}\, d\nu(s), \qquad \int_0^{2\pi} d\nu(s) = 1, \tag{6.4}$$

where $\nu(s)$ is nondecreasing. If $p_0 = a_0 + ib_0$, then we apply (6.4) to $P(z) \equiv \equiv [p(z) - ib_0]/a_0$ and obtain

$$(6.5) \qquad p(z) = ib_0 + a_0 \int_0^{2\pi} \frac{1+ze^{is}}{1-ze^{is}}\, d\nu(s) =$$

$$= \int_0^{2\pi} \frac{(a_0+ib_0)+z(a_0-ib_0)\,e^{is}}{1-ze^{is}}\, d\nu(s) = p_0 \int_0^{2\pi} \frac{1+ze^{i(s+\alpha)}}{1-ze^{is}}\, d\nu(s).$$

Further, of $\varphi(z)$ is a normalized convex function there is a nondecreasing $\mu(t)$ such that (see DUREN [6])

$$(6.6) \qquad \varphi'(z) = \exp\left[\int_0^{2\pi} \log(1-ze^{it})^{-2}\, d\mu(t)\right], \qquad \int_0^{2\pi} d\mu(t) = 1.$$

Let γ be a positive number (eventually we will set $\gamma = 1, 2$) and consider $|f'(z)|^\gamma$. From (6.3) and the exponential form of the arithmetic-geometric inequality, we have

$$(6.7) \qquad |f'(z)|^\gamma = |p(z)|^{\beta\gamma}\, |\varphi'(z)|^\gamma =$$

$$= \left[\int_0^{2\pi} \left|\frac{1+ze^{i(s+\alpha)}}{1-ze^{is}}\right| d\nu(s)\right]^{\beta\gamma} \left[\int_0^{2\pi} \frac{1}{|1-ze^{it}|^{2\gamma}}\, d\mu(t)\right].$$

To simplify the first integral in (6.7) we assume that $\beta\gamma > 1$, and apply the Hölder inequality where $p = \beta\gamma$, and $1/q = 1 - 1/p$. We find that

$$(6.8) \qquad \left[\int_0^{2\pi} \left|\frac{1+ze^{i(s+\alpha)}}{1-ze^{is}}\right| \cdot 1\, d\nu(s)\right]^{\beta\gamma} \leq$$

$$\leq \int_0^{2\pi} \left|\frac{1+ze^{i(s+\alpha)}}{1-ze^{is}}\right|^{\beta\gamma} d\nu(s) \left[\int_0^1 1^q\, d\nu(s)\right]^{p/q}.$$

By the normalization, the last factor is 1 and can be dropped. Then, with $z = re^{i\Theta}$, and r fixed in (0, 1)

$$(6.9) \qquad \int_0^{2\pi} |f'(z)|^\gamma\, d\Theta =$$

$$= \int_0^{2\pi} \left[\int_0^{2\pi} \left|\frac{1+ze^{i(s+\alpha)}}{1-ze^{is}}\right|^{\beta\gamma} d\nu(s)\right] \left[\int_0^{2\pi} \frac{1}{|1-ze^{it}|^{2\gamma}}\, d\mu(t)\right] d\Theta \leq$$

$$\leq \int_0^{2\pi}\int_0^{2\pi} I(\alpha, r, s, t)\, d\nu(s)\, d\mu(t),$$

where

$$(6.10)\qquad I(r,\alpha,s,t)=\int_{-\pi}^{\pi}\left|\frac{1+re^{i(s+\alpha+\Theta)}}{1-re^{i(s+\Theta)}}\right|^{\beta\gamma}\frac{1}{|1-re^{i(t+\Theta)}|^{2\gamma}}\,d\Theta.$$

It is intuitively clear that $I(r,\alpha,s,t)\leqq I(r,0,0,0)$, but we have been unable to find an elementary proof of this fact. However we can achieve our objective in the following way. Given a nonnegative measurable function $F(\Theta)$ on $[-\pi,\pi]$, let $F^*(\Theta)$ denote its symetrically decreasing rearrangement as defined in [8, p. 278]. Then according to CLUNIE and DUREN [5] we have

LEMMA 2. *If $F(\Theta)$, $G(\Theta)$, and $H(\Theta)$ are nonnegative integrable functions on $[-\pi,\pi]$, then*

$$(6.11)\qquad \int_{-\pi}^{\pi}F(\Theta)G(\Theta)H(\Theta)\,d\Theta\leqq\int_{-\pi}^{\pi}F^*(\Theta)G^*(\Theta)H^*(\Theta)\,d\Theta.$$

When this lemma is applied to (6.10) we obtain $I(r,\alpha,s,t)\leqq$ $\leqq I(r,0,0,0)$. Thus if $\beta\gamma\geqq 1$, then

$$(6.12)\qquad \int_0^{2\pi}|f(z)|^{\gamma}\,d\Theta\leqq\int_0^{2\pi}\left|\frac{(1+re^{i\Theta})^{\beta}}{(1-re^{i\Theta})^{\beta+2}}\right|^{\gamma}d\Theta.$$

Now set $\gamma=1$ and insert the factor r on both sides of (6.12). This will give (6.1), the assertion about arc length. Next set $\gamma=2$ in (6.12), introduce the factor r and integrate both sides of (6.12). This will give (6.2). Since $\beta\gamma\geqq 1$, we find that (6.2) holds if $\beta\geqq 1/2$

7. Close-to-star functions of higher order. In 1915 ALEXANDER [1] proved the beautiful theorem that if F and f are related by

$$(7.1)\qquad F(z)=zf'(z),$$

then $F(z)$ is starlike with respect to the origin if and only if $f(z)$ is convex. Since that time it has been customary to associate with each generalized class of convex functions, a second class of generalized starlike functions whose members are related by (7.1) [2]. Then each theorem in one class automatically gives a second theorem in the other class. In many cases the interplay of theorems is interesting and the results obtained in this way are often quite useful. Following the established tradition we say that $F(z)$ is close-to-star of order β, $(\beta\geqq 0)$ if there is an $f(z)$ in $\mathcal{K}(\beta)$ such that (7.1) holds, and we let $\mathcal{S}(\beta)$ denote the class of all such functions. It should be noted that READE [18, 20] has introduced essentially the same definition and the same terminology in the special case $\beta=1$.

We will not take the space to state the various theorems for the class $\mathcal{S}(\beta)$ that parallel the ones for $\mathcal{K}(\beta)$. However it is interesting to observe

that Theorem 5 gives the sharp bound for $|\arg F(z)/z|$ in the class $\mathcal{S}(\beta)$, and this has a nice geometric meaning. Indeed $\arg [F(z)/z]$ is the angular displacement of the point z under the transformation $F(z)$.

8. The geometric characterization. Let Γ be the image of $|z| = r$ under $f(z)$, where $f(z) \in \mathcal{K}(\beta)$, and let $\psi(\Theta)$ be some determination of the angle that the line tangent to Γ at $z = re^{i\Theta}$ makes with the real axis. Of course $\psi(\Theta)$ is not uniquely determined, but a selection can be made so that $\psi(\Theta+2\pi) = \psi(\Theta)+2\pi$, for all Θ, and $\psi(\Theta)$ is differentiable. Then, as is well known,

$$\frac{d\psi}{d\Theta} = R\left(1 + re^{i\Theta}\frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\right). \tag{8.1}$$

The change in ψ measures the amount of turn of the tangent line, and as Kaplan [10] has shown, the class $\mathcal{K}(1)$ is essentially characterized by the condition: $\Theta_2 > \Theta_1$ implies that $\psi(\Theta_2)-\psi(\Theta_1) > -\pi$. This result, and the proof generalize to give

Theorem 12. *Let $f(z)$ be normalized as in* (1.1). *A necessary and sufficient condition that $f(z)$ be in $\mathcal{K}(\beta)$ is that for each r in* (0, 1) *and for each pair $\Theta_2 > \Theta_1$, we have*

$$\int_{\Theta_1}^{\Theta_2} R\left(1 + re^{i\Theta}\frac{f''(re^{i\Theta})}{f'(re^{i\Theta})}\right) d\Theta > -\beta\pi. \tag{8.2}$$

In other words $\psi(\Theta_2)-\psi(\Theta_1)$, the turn of the tangent line, is always greater than $-\beta\pi$. This result explains geometrically why $f(z)$ is univalent if $\beta \leqq 1$, for in this case the "backward hairpin turn" is never sufficient for the curve to intersect itself.

To prove Theorem 12 a key lemma in the original work of Kaplan must be modified as indicated in

Lemma 3. *Let $t(\Theta)$ be a real-valued function of Θ such that*

$$t(\Theta+2\pi) = t(\Theta)+2\pi \tag{8.3}$$

for all Θ, and if $\Theta_2 > \Theta_1$ then

$$t(\Theta_2) > t(\Theta_1)-\beta\pi. \tag{8.4}$$

Then there is a function $s(\Theta)$ such that

$$s(\Theta+2\pi) = s(\Theta)+2\pi, \tag{8.5}$$

$$|s(\Theta)-t(\Theta)| \leqq \beta\frac{\pi}{2} \tag{8.6}$$

and $s(\Theta)$ is monotonic non-decreasing for all Θ.

As Rényi [21] has already observed when $0 < \beta < 1$, the proofs of Lemma 3 and Theorem 13 follow the pattern used by Kaplan [10] and are left for the reader.

After this paper was completed the author learned that James W. Noonan had considered the same class of functions and independently obtai-

ned some of the same results [On close-to convex functions of order β, to appear]. Further the coefficient bounds given in Theorems 8 and 9 can be proved for all real β and every integer $n \geqq 2$. The key to this proof is contained in the work of D. A. BRANNAN, J. G. CLUNIE and W. E. KIRWAN [*Annales Academiae Scientiarum Fennicae*, to appear], and an essential and difficult step to complete the proof is supplied D. AHARONOV, and S. FRIEDLAND [same *Annales*, to appear].

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DISCRETE CONVOLUTIONS AND LINEAR POSITIVE OPERATORS I

By

GRIGOR MOLDOVAN*

Babes–Bolyai University, Cluj, Romania

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In the approximation theory of the real functions, one of the most important theorems is the well known theorem of Weierstrass. It states that any continuous function on a compact interval, $f: [0,1] \to R$, is the limit of a sequence of polynomials uniformly converging towards this function. One of the most elegant proofs of the Weierstrass theorem was given by S. N. BERNSTEIN [4] in 1912, using the polynomials

$$(1) \qquad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad f \in C[0,1], \quad x \in [0,1]$$

today bearing his name. This fact amplified and enriched the researches in this domain with new and important results.

Later on, due to some important results belonging to T. POPOVICIU [15, 16], H. BOHMAN [5], P. P. KOROVKIN [12], concerning the convergence of some sequences of linear operators, a general theory of the linear positive operators developed. This theory has recently led to numerous results in this domain.

DEFINITION. Let A be set real functions defined and bounded on $[a, b]$ and let $L(.;x): A \to A$ be a linear operator. This operator is called linear positive if $f \geq 0$ implies $L(f;x) \geq 0$, $\forall x \in [a, b]$.

Numerous examples of linear positive operators were constructed, problems of convergence and many other interesting properties being studied in connection with them. Here we mention some operators to which we are going to refer later namely: the Bernstein operators (1) [4], BASKAKOV [3], CHENEY–SHARMA [7], JAKIMOVSKI – RAMANUJAN [10], MIRAKYAN [13] and STANCU [21].

A large part of the linear positive operators studied so far constitute generalizations of the Bernstein operator, conceived under diverse forme. Among these, many contain parameters which, if replaced by zero, reduce

* These results are part of a paper with the same title, presented at a comunication meeting of the *Faculty of Mathematics-Mechanics*, Cluj, 3 April 1970.

to the Bernstein operator (e.g. Cheney–Sharma, Jakimovski–Ramanujan and Stancu operators).

Some general methods of construction of such linear positive operators have lately boen indicated. We mention here the probabilistic methods used for this purpose, developed by D. D. STANCU [20] and V. A. BASKAKOV [2]. We have to mention here the fundamental works of A. RÉNYI concerning probabilistic method in analysis [19].

In the present paper we deal with a method, algebraic in a way, of construction of linear operators, on the space real functions defined and bounded on [0,1] interval. We are going to show how certain discrete convolutions lead to the formation of some linear positive operators. This was done with a view to giving a general method of construction of operators that generalize the Bernstein operator.

Our method allowed us to find new classes of linear positive operators. In some subsequent papers we shall study convergence and other properties of these operators. The results will be extended to the multivariate case.

We present here two of these results which are connected to the topics of this paper. Following theorem refers to the operator (34) (see belov).

THEOREM. *Let* $f, s \in C[0,1]$, $s: [0,1] \to [0,1]$ *and* $x_k^{(n)} = \frac{k}{n}$. *Let also* $s(x) = x + \omega_n(x)$, $\lim_{n\to\infty} \omega_n(x) = 0$ *uniformly.*

If

$$(1) \qquad 0 < \alpha \to 0, \ n\beta \to 0, \ \frac{n\beta}{\alpha} \to 0, \ \textit{when} \ n \to \infty,$$

or

$$(11) \qquad n\alpha \to 0, \ n\beta \to 0, \ \frac{n\beta}{\alpha} \to c (c > 0), \ \textit{when} \ n \to \infty,$$

then the sequence of operator (34), $L_n^{[s,\alpha,\beta]}$, *converges uniformly on* [0,1] *to* f.

A similar theorem takes place for $\alpha = 0$. We mention here the following algebraic property of the operator of D. D. STANCU (27) defined in [21].

THEOREM. *Let* E *be the space of real functions defined and bounded on* [0, 1] *and let* $\mathcal{P}_m (0 \le m \le n)$ *be the space of polynomialy of degree at most* n. *Then the restriction of operator* (27) $P_n^{[\alpha]}: E \to \mathcal{P}_n$ *to* $\mathcal{P}_m$ *is an injective homomorphism and* $P_n^{[\alpha]} \mathcal{P}_m = \mathcal{P}_m$, *i.e.* $P_n^{[\alpha]}$ *induces an automorphism in* $\mathcal{P}_m$.

§1. Discrete convolutions

1. In the following we are going to denote the set of the real numbers with R, and with $N = \{0, 1, 2, \ldots\}$. Let $\mathcal{P}(E)$ be the family of all the sequences $P = (P_n)_{n\in N}$ of polynomials, where n is the degree of P_n,

defined on the same set of real numbers E, namely $P_n: E \to R$, $E \subseteq R$. This family forms a linear space over R with respect to the usual operations

$$P^{(1)} + P^{(2)} = (P^{(1)} + P^{(2)})_{n \in N}, \quad \lambda P = (\lambda P_n)_{n \in N}, \quad \lambda \in R. \tag{2}$$

DEFINITION. Let $P^{(1)}, P^{(2)} \in \mathcal{P}(E)$. With the two sequences we form a sequence of functions

$$(Q_n)_{n \in N}, \quad Q_n: E \times E \to R,$$

as follows

$$Q_n(u, v) = \sum_{k=0}^{n} P_k^{(1)}(u) \cdot P_{n-k}^{(2)}(v), \quad n \in N, \quad u, v \in R. \tag{3}$$

We defined, in this way, an operation "$*$" in the set $\mathcal{P}(E)$, which we call *discrete convolution*. We denote $Q = P^{(1)} * P^{(2)}$, $Q = (Q_n)_{n \in N}$.

From new on we are going to use the shorter term *convolution* instead of discrete convolution. We are going to denote with $\mathcal{F}(E^2)$ the set of sequences of real functions defined on $E^2 = E \times E$ thus $Q = (Q_n)_{n \in N} \in \mathcal{F}(E^2)$. We indicate some special convolutive relations of the form (3).

DEFINITION. Let $\mathcal{P}_1(E) \subset \mathcal{P}(E)$ be the set of all the sequences $P = (P_n)_{n \in N}$ of polynomials that verifiy the following convolutive relation

$$P_n(u+v) = \sum_{k=0}^{n} P_k(u) \cdot P_{n-k}(v), \quad u, v \in E \text{ and } u+v \in E. \tag{4}$$

We use the term *binomial convolution* in order to denote such a convolution [18].

As we are going to see later, the set $\mathcal{P}_1(E)$ is not empty. In the case of a binomial convolution we have

$$(P * P)(u, v) = P(u, v), \quad u, v \in E \text{ and } u+v \in E$$

and if $E = R$ then u and v are arbitrary.

The relation (4) represents a functional equation. The solution of this functional equation in $\mathcal{P}(R)$ was found by T. POPOVICIU [17].

2. We are giving two examples of P_n polynomials that verify the relation (4):

$$P_n(x) = \frac{x^n}{n!}. \tag{a}$$

In this case the relation (4) become the well known formula of the Newton binomial

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}. \tag{5}$$

$$P_n(x) = \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}. \tag{b}$$

We obtain the Vandermonde formula

$$\binom{u+v}{n} = \sum_{k=0}^{n} \binom{u}{k}\binom{v}{n-k}, \tag{6}$$

$$\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}, \quad \binom{x}{0} = 1. \tag{7}$$

3. Some generalizations were given for the relation (4) which led to generalizations for the relations (9) and (10). Thus, H. A. ROTHE [9] shows that the following convolutive relation

$$P_n(u+v;y) = \sum_{k=0}^{n} P_k(u;y)\cdot P_{n-k}(v;y) \tag{8}$$

where $u, v, y \in R$ is verified by

$$P_k(x;y) = \frac{x}{x+yk}\binom{x+yk}{x} \tag{9}$$

and

$$P_k(x;y) = \frac{x}{x+yk}\cdot\frac{(x+yk)^k}{k!}. \tag{10}$$

The following convolutive relation

$$Q_n(u+v;y) = \sum_{k=0}^{n} P_k(u;y)\cdot Q_{n-k}(v;y) \tag{11}$$

is also known, where

$$Q_k(x;y) = \frac{x+yk}{x} P_k(x;y) \tag{12}$$

and which is verified by the same expresions (9) and (10).

If we put $y = 0$ in the convolutive relation (8) the solution (9) leads us to the Vandermonde formula (6) and the solution (10) leads us to the Newton formula (5).

We also mention the relation

$$\frac{p(u+v)+qun}{u+v} P_n(u+v;y) = \sum_{k=0}^{n} (p+qk)\, P_k(u;y)\cdot P_{n-k}(v;y), \tag{13}$$

where $P_k(x;y)$ are given by (9), which is called Rothe–Hagen relation. If we make $q = 0$ in relation (13) we obtain the relation (8) with the notations (9). The convolution relations (6), (8) and (13) were much studied by H. W. GOULD [8, 9], L. CARLITZ [6] and athers.

§2. Convolutive operators

1. The general problem. With the help of the relation (3) we introduced the operation

$$*: \mathcal{P}(E)\times\mathcal{P}(E) \to \mathcal{F}(E^2).$$

Now let $\Delta = (x_k^{(n)})$, $k = 0,1,\ldots,n$; $n = 0,1,\ldots$ be a matrix of real numbers belonging to a set $X\subseteq R$ and $f: X\to R$ a certain function. In connection to the function f and matrix Δ, we define the following convolution operation

$$\underset{(f,\Delta)}{*} : \mathcal{P}(E)\times\mathcal{P}(E) \to \mathcal{F}(E^2) \tag{14}$$

that makes and element from $\mathcal{F}(E^2)$ to correspond to a pair $(P^{(1)}, P^{(2)})$ of elements from $\mathcal{P}(E)$ by the following convolutive relation

$$Q_n(f; u, v) = \sum_{k=0}^{n} f(x_k^{(n)})\, P_k^{(1)}(u)\cdot P_{n-k}^{(2)}(v). \tag{15}$$

We are considering now $u: X\to E$ and $v: X\to E$, that is $u = u(x)$, $v = v(x)$, $x\in X$, two arbitrary functions.

Let us choose two elements $P^{(1)}, P^{(2)}\in\mathcal{P}(E)$. Let us note with R^X the set of all the functions $f: X\to R$, $X\subseteq R$. The convolution operation $\underset{(f,\Delta)}{*}$ will permit us to an operator

$$L_n(\,.\,; P^{(1)}, P^{(2)}; x): R^X \to R^X$$

in this way

$$L_n(f; P^{(1)}, P^{(2)}; x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)})\, P_k^{(1)}[u(x)]\cdot P_{n-k}^{(2)}[v(x)],\ x\in X, \tag{16}$$

where

$$A_n(x) = \{Q_n[1; u(x), v(x)]\}^{-1} < \infty,\ \forall x\in X.$$

In the seguel we suppose withaut mentioning it that this condition holds.

Definition. The operators of the form (16) are called by us *convolutive operators.*

It is to be noticed that the convolutive operator (16) is a linear one.

From the definition of the convolutive operators it also follows that

$$L_n(1; P^{(1)}, P^{(2)}; x) = 1,\ \forall P^{(1)}, P^{(2)}\in\mathcal{P}(E). \tag{17}$$

Definition. If from $f\geq 0$ it results that $L_n(f; P^{(1)}, P^{(2)}; x)\geq 0$, $\forall x\in$ then the operator (16) is called *positive convolutive operator*

It is to be noticed that if

$$A_n(x)\, P_k^{(1)}[u(x)] \cdot P_{n-k}^{(2)}[v(x)] \geq 0,\ k = 0,1, \ldots, n;\ n \in N,\ \forall\, x \in X$$

then the convolution (15) provides positive linear operators.

The study of the positive convolutive operators is of importance in the theory of approximation.

2. Examples of positive convolutive operators

(A) We have given two examples of binomial convolutions. As we have already seen at the proceding point, each convolution generates a convolutive operator. For the binomial formula (5) that appeared as a solution of the functional equation (4) where

$$P_n(x) = \frac{x^n}{n!} \tag{18}$$

the convolution operator (16) is transcribed in this way

$$\begin{gathered} L_n(f;x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)}) \frac{[u(x)]^k}{k!} \cdot \frac{[v(x)]^{n-k}}{(n-k)!}, \\ A_n(x) = \left\{ \frac{[u(x)+v(x)]^n}{n!} \right\}^{-1}. \end{gathered} \tag{19}$$

If $u(x) = s(x)$, $v(x) = 1 - s(x)$, $E = X = [0,1]$, then $A_n(x) = n!$ and so we have

$$B_n^{[s]}(f;x) = \sum_{k=0}^{n} f(x_k^{(n)}) \binom{n}{k} [s(x)]^k\, [1-s(x)]^{n-k}. \tag{20}$$

This operator was examined by the author in [14]. Problems of convergency and some other problems were studied in connection with it. The operator (20) constitutes a generalization of the Bernstein operator which is obtained from it for $s(x) = x$ and $x_k^{(n)} = \frac{k}{n}$. A similar case when $x_k^{(n)} = \frac{k}{n}$ was examined by D. H. Tucker [23]. The case $s(x) = x$ and $x_k^{(n)} = \frac{k+\alpha}{n+\beta}$ was thoroughly studied by D. D. Stancu [22]. In the paper [14] the case when $s(x) = x$ and $x_k^{(n)} = \frac{k+\alpha}{n}$ was considered as an example.

(B) Starting from the Vandermonde formula (6) with the help of the relation (16) we are going to construct corresponding linear positive operators.

The convolutive operator (16), in the case of a binomial, convolution where

$$P_n(x) = \binom{x}{n} = \frac{x(x-1)\ldots(x-n+1)}{n!} \tag{21}$$

is written

$$L_n(f;x) = A_n(x) \sum_{k=0}^{n} f(x_k^{(n)}) \binom{u(x)}{k} \binom{v(x)}{n-k}. \tag{22}$$

From the Vandermonde formula (6) and the definition of the convolutive operators it results that

$$A_n(x) = \left[\binom{u(x)+v(x)}{n}\right]^{-1}. \tag{23}$$

In the operator (22) we are going to introduce a parameter α, in the functions $u(x)$ and $v(x)$ we are going to replace then conveniently in order to get a linear positive operator.

If

$$u(x) = -\frac{s(x)}{\alpha}, \quad v(x) = -\frac{1-s(x)}{\alpha} \tag{24}$$

where $0 \leq s(x) \leq 1$, $x \in X = [0,1]$, $E = R$ and $\alpha > 0$ then

$$A_n(x) = \left[\binom{-\frac{1}{\alpha}}{n}\right]^{-1} = n!(-\alpha)^n (1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\,\alpha),$$

$$\binom{-\frac{s(x)}{\alpha}}{k}\binom{-\frac{1-s(k)}{\alpha}}{n-k} =$$

$$= \frac{(-\alpha)^{-n}}{k!(n-k)!} \prod_{\nu=0}^{k-1} [s(x)+\nu\alpha] \cdot \prod_{\mu=0}^{n-k-1} [1-s(x)+\mu\alpha].$$

Thus the convolutive operator (16) is written

$$P_n^{[s,\alpha]}(f;x) = \sum_{k=0}^{n} f(x_k^{(n)})\, w_{n,k}^{[s]}(x,\alpha) \tag{25}$$

where

$$w_{n,k}^{[s]}(x,\alpha) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1} [s(x)+\nu\alpha] \cdot \prod_{\mu=0}^{n-k-1} [1-s(x)+\mu\alpha].}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\,\alpha)}. \tag{26}$$

This operator appears as a generalization of the D. D. STANCU operator [21] recently considered and studied. If we take $s(x) = x$ and $x_k^{(n)} = \frac{k}{n}$ then

$$P_n^{[x,\alpha]}(f;x) = P_n^{[\alpha]}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) w_{n,k}(x,\alpha) \tag{27}$$

where

$$w_{n,k}(x,\alpha) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\cdot\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\alpha)}.$$

The D. D. STANCU operator (27) constitutes a generalization of the Bernstein operator. If $\alpha = 0$ then the operator (27) transforms into the Bernstein operator. Obviously, the same thing happens with the operator (25) if $s(x) = x$, $x_k^{(n)} = \frac{k}{n}$ and $\alpha = 0$. We also mention that the operators of MIRAKYAN [13] and BASKAKOV [3] are limit cases of the Stancu operator (27) as it is shown in [20], [21].

(C) We have seen that the polynomials (10) verify the functional equation (8). Replacing, we get a well known identity of Abel [1] which under other forms was studied by JENSEN [11].

Choosing

$$u(x) = s(x),\ v(x) = 1-s(x),\ y = \beta \text{ and } X = E = [0,1],$$

then

$$A_n(x) = \left\{\frac{u(x)+v(x)}{u(x)+v(x)+\beta n}\cdot\frac{[u(x)+v(x)+\beta n]^n}{n!}\right\}^{-1} = n!(1+\beta n)^{1-n}$$

and the operator (16) is written

$$Q_n^{[s]}(f;x) = (1+\beta n)^{1-n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} s(x)[s(x)+\beta k]^{k-1}[1-s(x)]\cdot \tag{28}$$
$$\cdot[1-s(x)+\overline{n-k}\,\beta]^{n-k-1}.$$

Here, if we put $x_k^{[n]} = \frac{k}{n}$ and $s(x) = x$, we get the Cheney – Sharma [7],

$$Q_n^{[x]}(f;x) = Q_n(f;x)$$

(29)

$$Q_n(f;x) = (1+\beta n)^{1-n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right) x(x+\beta x)^{k-1}(1-x)(1-x+\overline{n-k}\,\beta)^{n-k-1}.$$

Likewise, if

$$P_k^{(1)}[u(x)] = \frac{u(x)}{u(x)+\beta k} \cdot \frac{[u(x)+\beta k]^k}{k!}$$

and

$$P_k^{(2)}[v(x)] = \frac{[v(x)+\beta k]^k}{k!}$$

then for the operator (16) we get the expression

$$L_n(f;x) = \frac{1}{n!} A_n(x) \sum_{k=0}^{n} f(x_k^{(n)}) \binom{n}{k} u(x)[u(x)+\beta k]^{k-1} \cdot [v(x)+\overline{n-k}\,\beta)]^{n-k},$$

where, taking into consideration (11), we have

$$A_n(x) = \left\{ \frac{[u(x)+v(x)+n]^n}{n!} \right\}^{-1}$$

or if

$$u(x) = s(x),\ v(x) = 1-s(x),\ y = \beta,\ \beta \geq 0,\ X = E = [0,1]$$

then

(30)

$$P_n^{[s]}(f;x) = (1+\beta n)^{-n} \sum_{k=0}^{n} f(x_k^{(n)}) \binom{n}{k} s(x)[s(x)+\beta k]^{k-1}[1-s(x)+\overline{n-k}\,\beta]^{n-k}.$$

Now if we put $s(x) = x$, $x_k^{(n)} = \dfrac{k}{n}$ then we find again another operator of Cheney–Sharma [7], namely, $P_n^{[x]}(f;x) = P_n(f;x)$,

$$(31)\quad P_n(f;x) = (1+\beta n)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x(x+\beta k)^{k-1}(1-x+\overline{n-k}\,\beta)^{n-k}.$$

In the following paragraph we are going to refer to the operators (20), (25) and (31). These operators can be studied unitarily, from a new point of view.

§3. Three types of positive convolutive operators

1. The first type of convolutive operators. The functional equation (11), as we have already mentioned above, has the polynomials

$$P_k(x;y) = \frac{x}{x+yk} \binom{x+yk}{k},$$

as a solution. Thus the relation (11) becomes

$$\binom{u+v+yn}{x} = \sum_{k=0}^{n} \frac{u}{u+yk}\binom{u+yk}{k}\binom{v+\overline{n-k}\,y}{n-k}. \tag{32}$$

Then in the operator (16) we shall have to take

$$A_n(x) = \left\{\binom{u(x)+v(x)+yn}{n}\right\}^{-1}.$$

Let us put

$$\begin{gathered} u(x) = -\frac{s(x)}{\alpha},\ v(x) = -\frac{1-s(x)}{\alpha},\ y = -\frac{\beta}{\alpha},\ \alpha > 0,\ \beta \geq 0, \\ 0 \leq s(x) \leq 1,\ x \in X = [0,1],\ E = R. \end{gathered} \tag{33}$$

In this case the operator (16) is written so

$$L_{n,k}^{[s,\alpha,\beta]}(f;x) = \sum_{k=0}^{n} f(x_k^{(n)})\, p_{n,k}^{[s]}(x;\alpha,\beta) \tag{34}$$

where

$$p_{n,k}^{[s]}(x;\alpha,\beta) = \binom{n}{k} \cdot \frac{s(x)\prod_{\nu=1}^{k-1}[s(x)+k\beta+\nu\alpha]\cdot\prod_{\mu=0}^{n-k-1}[1-s(x)+\overline{n-k}\,\beta+\mu\alpha]}{\prod_{\sigma=0}^{n-1}(1+n\beta+\sigma\alpha)}.$$

In connection with this operator let us make some observations. First, it is obvious that

$$L_n^{[s,\alpha,\beta]}(1;x) = \sum_{k=0}^{n} p_{n,k}^{[s]}(x;\alpha,\beta) = 1.$$

Then we have the relations

$L_n^{[s,0,0]}(f;x) = B_n^{[s]}(f;x)$, $L_n^{[s,\alpha,0]}(f;x) = P_n^{[s,\alpha]}(f;x)$, $L_n^{[s,0,\beta]}(f;x) = Q_n^{[s]}(f;x)$

or if we put $s(x) = x$, $x_k^{(n)} = \dfrac{k}{n}$ then

$L_n^{[x,0,0]}(f;x) = B_n(f;x)$ the Bernstein operator (1),

$L_n^{[x,\alpha,0]}(f;x) = P_n^{[x]}(f;x)$ the Stancu operator (27),

$L_n^{[x,0,\beta]}(f;x) = Q_n(f;x)$ the Cheney – Sharma (29).

So, the operator (34) constitutes a generalization of some operators mentioned above and ultimately of the Bernstein operator.

2. The second type of convolutive operators. We are going to make use of the functional equation (convolution) (8) which is verified by the polynomials

$$P_k(x;y) = \frac{x}{x+yk}\binom{x+yk}{k}.$$

The respective relation, in this case, is transcribed so

(35)
$$\frac{u+v}{u+v+yn}\binom{u+v+yn}{n} = \sum_{k=0}^{n} \frac{u}{u+yk}\binom{u+yk}{k}\frac{v}{v+(n-k)y}\binom{v+\overline{n-k}y}{n-k}.$$

Let shall have

$$A_n(x) = \left\{\frac{u(x)+v(x)}{u(x)+v(x)+yn}\binom{u(x)+v(x)+yn}{n}\right\}^{-1}.$$

Let us now put

$$u(x) = -\frac{s(x)}{\alpha},\ v(x) = -\frac{1-s(x)}{\alpha},\ y = -\frac{\beta}{\alpha},\ \alpha > 0,\ \beta \geq 0,$$
$$0 \leq s(x) \leq 1,\ x \in X = [0,1],\ E = R,$$

then the operator (16) is written

(36)
$$\overline{L}_n^{[s,\alpha,\beta]}(f;x) = \sum_{k=0}^{n} f(x_k^{(n)})\,\overline{p}_{n,k}^{[s]}(x;\alpha,\beta)$$

where

$$\overline{p}_{n,k}^{[s]}(x;\alpha,\beta) =$$
$$= \binom{n}{k}\frac{s(x)\prod_{\nu=1}^{k-1}[s(x)+\beta k+\nu\alpha]\cdot[1-s(x)]\prod_{\mu=1}^{n-k-1}[1-s(x)+n-\overline{k}\beta+\mu\alpha]}{\prod_{\sigma=0}^{n-1}(1+n\beta+\sigma\alpha)}.$$

As in the preceding case, we are going to have some relations between this operator and some operators previously described. These relations are $\overline{L}_n^{[s,0,0]}(f;x) = B_n^{[s]}(f;x)$, $\overline{L}_n^{[s,\alpha,0]}(f;x) = P_n^{[s,\alpha]}(f;x)$, $\overline{L}_n^{[s,0,\beta]}(f;x) = P_n^{[s]}(f;x)$ and if $s(x) = x$, $x_k^{(n)} = \dfrac{k}{n}$ then

$\overline{L}_n^{[x,0,0]}(f;x) = B_n(f;x)$ the Bernstein operator (1),

$\overline{L}_n^{[x,\alpha,0]}(f;x) = P_n^{[x]}(f;x)$ the Stancu operator (27),

$\overline{L}_n^{[x,0,\beta]}(f;x) = P_n(f;x)$ the Cheney – Sharma (31).

The operator (36) also generalizes the above mentioned operators.

3. The third type of convolutive operators. Using now the convolution relation (13), where

$$P_k(x;y) = \frac{x}{x+yk}\binom{x+yk}{k}$$

we shall obtain the idenity

$$\frac{p(u+v)+qun}{u+v+yn}\binom{u+v+ny}{n} = \tag{37}$$

$$= \sum_{k=0}^{n}(p+qk)\frac{u}{u+yk}\binom{u+yk}{k}\cdot\frac{v}{v+\overline{n-k}y}\binom{v+\overline{n-k}y}{n-k}.$$

In this relation u, v, y, p, q are certian real numbers. The identity (37) is also called the identity of Rothe—Hagen. The relation (37) contains the relations (32) and (35) as particular cases. Indeed, if $p = 1$, $q = 0$ then from (37) we get the relation (35) and if $p = v+ny$, $q = -y$ then from (37) we get the relation (32). We have

$$A_n(x) = \left\{\frac{p[u(x)+v(x)]+qu(x)n}{u(x)+v(x)+ny}\binom{u(x)+v(x)+ny}{n}\right\}^{-1}$$

and taking

$$u(x) = -\frac{s(x)}{\alpha},\ v(x) = -\frac{1-s(x)}{\alpha},\ y = -\frac{\beta}{\alpha},\ \alpha > 0,\ \beta \geq 0,$$

$$0 \leq s(x) \leq 1,\ x \in X = [0,1],\ E = R,$$

the operator (16) attached to the convolution relation (38) is written

$$\overline{L}_n^{[s,\alpha,\beta]}(f;x) = \sum_{k=0}^{n} f\left(x_n^{(n)}\right)\overline{\overline{p}}_{n,k}^{[s]}(x,\alpha,\beta) \tag{38}$$

where

$$\overline{\overline{p}}_{n,k}^{[s]}(x,\alpha,\beta) = \binom{n}{k}\frac{p+qk}{p+nqs(x)}\cdot$$

$$\cdot\frac{s(x)\prod_{\nu=1}^{k-1}[s(x)+k\beta+\nu\alpha]\cdot[1-s(x)]\prod_{\mu=1}^{n-k-1}[1-s(x)+\overline{n-k}\beta+\mu\alpha]}{\prod_{\sigma=1}^{n-1}(1+n\beta+\sigma\alpha)}.$$

The operator (38) contains, as particular cases, the operators (34) and (36), but it does not constitue a generalization with new elements, different from the other two generalizations of the first and second type mentioned above.

Naturally, in this case too, we can consider $s(x) = x$ and $x_k^{(n)} = \frac{k}{n}$ and write the expression corresponding to the operator $\overline{L}_n^{[s,\alpha,\beta]}(f; x)$ and the make particularizations on the parameters α and β as well as p and q. These are written easily.

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РАСПРЕДЕЛЕНИЕ НОРМ ЦЕЛЫХ ДИВИЗОРОВ В АРИФМЕТИЧЕСКОЙ ПРОГРЕССИИ

П. ВАРБАНЕЦ и Ф. КОВАЛЬЧИК

Одесский государственный университет им. И. И. Мечникова

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Рассмотрим мнимое квадратичное поле $K = R(\sqrt{-d})$. Через K_0 обозначаем систему целых дивизоров α, соответствующих полю K, а через $N(\alpha)$ — норму дивизора α.

Целью настоящей работы является построение асимптотической формулы для числа $A(x, \varphi_1, \varphi_2; D)$ решений сравнения

$$
(1) \qquad \begin{gathered} N(\alpha) \equiv l (\mathrm{mod}\ D), \\ \alpha \in K_0, \\ N(\alpha) \leq x, \\ \varphi_1 \leq \arg \alpha \leq \varphi_2. \end{gathered}
$$

При решении этой задачи существенно используется аппарат Z-функций Гекке.

Условимся в обозначениях:

$-d_1 < 0$ — дискриминант поля K;

h — число классов дивизоров поля K;

$p > 2$ — простое рациональное число;

n — натуральное число;

g — число единиц поля K;

χ_k — характер дирихле mod p^k, $k = 1, 2, \ldots, n$;

m — целое рациональное число;

C — класс дивизоров поля K;

$\Xi_1(\alpha) = e^{gmi \arg \alpha}$;

$\Xi(\alpha) = \chi_n[N(\alpha)]\, \Xi_1(\alpha)$.

Для $\operatorname{Re} s > 1$ определяем Z-функцию Гекке

$$Z(s, \Xi, C) = \sum_{\alpha \in C} \frac{\Xi(\alpha)}{N(\alpha)^s},$$

здесь α пробегает все неассоциированные дивизоры класса C. Хорошо известно, что $Z(s, \Xi, C)$ является целой функцией, кроме случая, когда χ_n — главный характер, а $m = 0$, и тогда $Z(s, \Xi, C)$ аналитична во всей плоскости, кроме точки $s = 1$, где она имеет простой полюс с вычетом равным

$$\frac{2\pi}{g\sqrt{d_1}} \prod_{\gamma | p} \left[1 - \frac{1}{N(\gamma)}\right],$$

здесь γ пробегает все неассоциированные дивизоры из K_0, делящие p (см. [1], стр. 414).

Обозначим еще

$$Z_m(k) = \sum_{\substack{\alpha \\ N(\alpha) = k}} \Xi_1(\alpha).$$

Тогда в области $\operatorname{Re} s > 1$ имеем

$$(2) \qquad f(s) = \sum_{\substack{k=1 \\ k \equiv l (\mathrm{mod}\ p^n)}}^{\infty} \frac{Z_m(k)}{k^s} = \frac{g}{\varphi(p^n)} \sum_{\chi_n} \bar{\chi}_n(l) \sum_C Z(s, \Xi, C).$$

Ясно, что оценки сумм $\sum_{\substack{k \equiv l (\mathrm{mod}\ p^n) \\ k \leq x}} Z_m(k)$ позволяют с помощью леммы о „стаканчиках" И. М. ВИНОГРАДОВА изучить $A(x, q_1, q_2, ; p^n)$.

Для дальнейшего нам понадобится ряд вспомогательных утверждений.

ЛЕММА 1. *Пусть* $F(s) = \sum_{\substack{k=1 \\ k \equiv l (\mathrm{mod}\ D)}}^{\infty} \frac{a_k}{k^s}$ — *ряд Дирихле, сходящийся при* $\operatorname{Re} s > 1$, $a_k = O[\Phi(k)]$, *где* $\Phi(k)$ *неотрицательная неубывающая функция;* $\sum_{\substack{1 \\ k \equiv l (\mathrm{mod}\ D)}}^{\infty} \frac{a_k}{k^\sigma} = O\left(\frac{1}{\sigma - 1}\right)$ *при* $\sigma \to 1 + 0$. *Пусть еще* $x > 0$ *такое, что* $\left[x - \frac{D}{2} - l\right] = DN$, N — *натуральное*, $1 \leq l < D$; $T > 1$, $b > 1$. *Тогда*

$$\sum_{\substack{l < k \leq x \\ k \equiv l (\mathrm{mod}\ D)}} a_k = \frac{1}{2\pi i} \int_{b - iT}^{b + iT} \left[F(s) - \frac{a_l}{l^s}\right] \cdot \frac{x^s}{s} ds + O\left(\frac{x^b}{DT(b-1)}\right) +$$
$$+ O\left(\frac{x \log x\, \Phi(x)}{DT}\right).$$

Доказательство проходит также как и в лемме о частных суммах ряда Дирихле (см., напр., [2]).

ЛЕММА 2. *Пусть $Sp\,\tau$ означает след элемента $\tau \in K$ в R. Тогда*

$$E_k = \max_{\substack{l \\ (l,p)=1}} \left| \sum_{\substack{\tau \\ N(\tau)\equiv l (\mathrm{mod}\, p^k)}}^{\mathrm{mod}\, p^k} e^{2\pi i \frac{Sp\,\tau}{p^k}} \right| = O\left(p^{\frac{k}{2}}\right),$$

с абсолютной постоянной в символе „O".

ДОКАЗАТЕЛЬСТВО. Для $\tau \in K$ имеем

$$N(\tau) = a^2 + db^2,\ Sp\,\tau = 2a, \qquad \text{если } -d \equiv 2,3\ (\mathrm{mod}\ 4);$$

$$N(\tau) = \frac{(2a+b)^2 + db^2}{4},\ Sp\,\tau = 2a+b, \qquad \text{если } -d \equiv 1\ (\mathrm{mod}\ 4).$$

Так что в обоих случаях

$$E_k = \max_{\substack{l \\ (l,p)=1}} \left| \sum_{a,b}^{\mathrm{mod}\, p^k} e^{2\pi i \frac{u_1 a + u_2 b}{p^k}} \right|, \quad (u_1, u_2, p^k) = 1,$$

здесь штрих означает, что a, b удовлетворяют одному из сравнений

$$a^2 + db^2 \equiv l (\mathrm{mod}\, p^k), \quad (2a+b)^2 + db^2 \equiv 4l (\mathrm{mod}\, p^k).$$

Теперь учитывая, что решения этих сравнений допускают p-адическое описание, аналогичное тому, которое приведено в [3], мы по схеме этой работы сразу получаем $E_k = O\left(p^{\frac{k}{2}}\right)$.

ЛЕММА 3. *В области $-\varepsilon \le \sigma \le 1+\varepsilon$, $|t| \ge t_0 > 1$, справедлива оценка*

$$(3) \qquad f(s) - \frac{Z_m(l)}{l^s} \ll [(|m|+1)\,|t|]^{1+\varepsilon-\sigma} d_1^{\frac{1+\varepsilon-\sigma}{2}} p^{n\left(\frac{1+3\varepsilon+3\varepsilon^2}{2+4\varepsilon} - \frac{3+5\varepsilon}{2+4\varepsilon}\sigma\right)}.$$

Постоянная в символе $\ll$ зависит от ε.

ДОКАЗАТЕЛЬСТВО. При $\sigma = 1+\varepsilon$ имеем

$$(4) \qquad \left| f(s) - \frac{Z_m(l)}{l^s} \right| \ll \sum_{\substack{k > p^n \\ k \equiv l (\mathrm{mod}\, p^n)}} \frac{Z_m(l)}{k^{1+\varepsilon}} \ll p^{-n\left(1+\frac{\varepsilon}{2}\right)}.$$

При $\sigma = -\varepsilon$ мы используем функциональное уравнение для $Z(s, \Xi, C)$ с примитивным характером (см. [4]), при этом в силу структуры характера Ξ получаем $Z(s, \Xi, C) = Z(s, \Xi', C)$, где Ξ' — примитивный характер, соответствующий Ξ. Заметим, что определяющий модуль для Ξ' суть p^k, где $1 \le k < n$.

Таким образом,

$$f(s) = \frac{g}{\varphi(p^n)}\left\{\left[\sum_{\chi_n}{}^{*} + \ldots + \sum_{\chi_t}{}^{*}\right]\sum_{C} Z(s, \Xi', C) + Z(s, \Xi_0)\right\},$$

(здесь и везде далее знак $*$ означает суммирование по примитивным характерам mod p^k, $k = 1, \ldots, n$).

Применение функционального уравнения дает

$$f(s) = \frac{g}{\varphi(p^n)}\left\{\left[\sum_{\chi_n}{}^{*} + \ldots + \sum_{\chi_t}{}^{*}\right]\sum_{C} \Psi(s, \Xi')\, Z(1-s, \overline{\Xi}', C^{-1}) + \right.$$
$$\left. + \frac{1}{g}\prod_{\gamma | p}\left[1 - \frac{1}{N(\gamma)^s}\right]\cdot Z(s, \Xi_1)\right\}.$$

Рассмотрим сумму

$$S_k = \sum_{\chi_k}{}^{*}\, \chi_k(l)\sum_{C}\Psi(s, \Xi', C)\, Z(1-s, \overline{\Xi}', C^{-1}) =$$
$$= \left(\frac{p^k\sqrt{d_1}}{2\pi}\right)^{1-2s}\cdot\frac{\Gamma\left(\frac{|mg|}{2} + 1 - s\right)}{\Gamma\left(\frac{|mg|}{2} + s\right)} \times i^{mg} p^{-k}\, \Xi_1(p^k i\sqrt{d_1})\cdot$$
$$\cdot\frac{1}{g}\sum_{\substack{\alpha\\ \alpha\in K_0}}\frac{\overline{\Xi}_1(\alpha)}{N(\alpha)^{1-s}}\sum_{\chi_k}{}^{*}\overline{\chi}_k[N(\alpha)^l]\sum_{\substack{\tau \bmod p^k\\ \tau\in K}}\chi_k[N(\tau)]\, e^{2\pi i\, Sp\frac{\tau}{i\sqrt{d_1}\,p^k}}.$$

Учтем, что в последнем равенстве последняя внутренняя сумма обращается в нуль, если χ_k непримитивный характер mod p^k. Поэтому в сумме справа знак $*$ можно опустить. Но тогда

$$S_k = \frac{p-1}{gp}\left(\frac{p^k\sqrt{d_1}}{2\pi}\right)^{1-2s}\cdot\frac{\Gamma\left(\frac{|mg|}{2} + 1 - s\right)}{\Gamma\left(\frac{|mg|}{2} + s\right)}\cdot i^{mg}\,\Xi_1\left(ip^k\sqrt{d_1}\right)\times$$
$$\times\sum_{\substack{\alpha\\ \alpha\in K_0}}\frac{\overline{\Xi}_1(\alpha)}{N(\alpha)^{1-s}}\sum_{\substack{\tau \bmod p^k\\ N(\tau)\equiv N(\alpha)^l \pmod{p^k}}} e^{2\pi i\, Sp\frac{\tau}{i\sqrt{d_1}\,p^k}} \ll$$
$$\ll p^{k(1-2s)}[(|m|+1)\,|t|]^{1+2\varepsilon} d_1^{\frac{1}{2}+\varepsilon} E_k \ll p^{k\left(\frac{3}{2}+2\varepsilon\right)}[(|m|+1)\,|t|]^{1+2\varepsilon} d_1^{\frac{1}{2}+\varepsilon}.$$

Кроме того, при $s = -\varepsilon + it$ с помощью укороченного функционального уравнения для $Z(s, \Xi_1, C)$ получаем

$$Z(s, \Xi_1) \ll [(|m|+1)\,|t|]^{1+2\varepsilon} d_1^{\frac{1}{2}}.$$

Поэтому

$$(5)\qquad f(-\varepsilon+it)-\frac{Z_m(l)}{l^{-\varepsilon+it}} \ll \frac{1}{\varphi(p^n)}\,[(|m|+1)\,|t|]^{1+2\varepsilon} d_1^{\frac{1}{2}+\varepsilon}\, p^{n\left(\frac{3}{2}+\varepsilon\right)}.$$

Теперь из (4) – (5) по принципу Фрагмена – Линделефа получаем утверждение леммы.

ЗАМЕЧАНИЕ. По схеме доказательства леммы 3 легко показать справедливость оценки (3) и в области $|t| \leq 1$, $-\varepsilon \leq \sigma \leq 1-\varepsilon$.

ЛЕММА 4. *Пусть* $s=\sigma+it$, $-2 \leq \sigma \leq 2$, τ — *любое*, $|\arg\tau| < \frac{\pi}{2}$. *Тогда для* Z — *функции Гекке* $Z(s, \Xi, C)$ *класса идеалов* C *мнимого квадратичного поля* K *дискриминанта* $-d_1 < 0$ *с примитивным характером* $\Xi \bmod f$ *справедливо равенство*

$$\Gamma\left(\frac{|mg|}{2}+s\right) Z(s, \Xi, C = \sum_{\substack{\alpha \\ \alpha\in C}} \frac{\Xi(\alpha)}{N(\alpha)^s}\, \Gamma\left(\frac{|mg|}{2}+s, \frac{2\pi\tau N(\alpha)}{\sqrt{d_1 N(f)}}\right)+$$

$$+\varkappa(\Xi)\left(\frac{d_1 N(f)}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_{\substack{\alpha \\ \alpha\in C^{-1}}} \frac{\overline{\Xi}(\alpha)}{N(\alpha)^{1-s}}\, \Gamma\left[\frac{|mg|}{2}+1-s, \frac{2\pi\tau^{-1}(\alpha)}{\sqrt{d_1 N(f)}}\right],$$

здесь $\varkappa(\Xi)$ *имеет тот же смысл, что и в функциональном уравнении Гекке, а* $\Gamma(\alpha, \beta)$ — *неполная гамма-функция.*

В частности, если $\arg\tau = \operatorname{arc\,tg} \dfrac{2t}{|mg|+2\sigma}$,

$$x = \frac{\sqrt{d_1 N(f)\left[t^2+\left(\frac{|mg|}{2}+\sigma\right)^2\right]}}{2\pi|\tau|},\quad y = \frac{\sqrt{d_1 N(f)\left[t^2+\left(\frac{|mg|}{2}+\sigma\right)^2\right]}}{2\pi|\tau|^{-1}},$$

$$X = x\left(1+\frac{M+5}{|mg|}\log x\right),\quad Y = y\left(1+\frac{M+5}{|mg|}\log y\right),\quad M>0,$$

то при $t^2+\dfrac{|mg|^2}{4} >$ некоторой достаточно большой постоянной имеет место приближенное уравнение

$$Z(s, \Xi, C) = \sum_{\substack{\alpha\in C \\ N(\alpha)\leq X}} \frac{\Xi(\alpha)}{N(\alpha)^s}\, \Gamma_0\left[\frac{|mg|}{2}+s, \frac{2\pi\tau N(\alpha)}{\sqrt{d_1 N(f)}}\right]+$$

$$+\varkappa(\Xi)\left[\frac{d_1 N(f)}{4\pi^2}\right]^{\frac{1}{2}-s} \cdot \frac{\Gamma\left(\frac{|mg|}{2}+1-s\right)}{\Gamma\left(\frac{|mg|}{2}+s\right)} \sum_{\substack{\alpha\in C^{-1} \\ N(\alpha)\leq Y}} \frac{\overline{\Xi}(\alpha)}{N(\alpha)^{1-s}}\cdot$$

$$\cdot\,\Gamma_0\left[\frac{|mg|}{2}+1-s, \frac{2\pi\tau^{-1} N(\alpha)}{\sqrt{d_1 N(f)}}\right]+O(X^{-M})+O(Y^{-M}),$$

постоянная в символах „O" не зависит от m, t, d, τ, $N(f)$. Кроме того, $\Gamma_0(\alpha, \beta) = O(1)$ и

$$\Gamma_0\left[\frac{|mg|}{2}+s,\ \frac{2\pi\tau N(\alpha)}{\sqrt{d_1 N(f)}}\right] = e + O\left(\exp\left[-\frac{N(\alpha)-x}{x}\left(\frac{|mg|}{2}+\sigma\right)\right]\cdot\left[\frac{N(\alpha)}{x}\right]^{\frac{|mg|}{2}+\sigma}\times\right.$$

$$\left.\times\sqrt[4]{t^2+\left(\frac{|mg|}{2}+\sigma\right)^2}\left(\sqrt[4]{t^2+\left(\frac{|mg|}{2}+\sigma\right)^2}+\frac{N(\alpha)}{x}-\sqrt{t^2+\left(\frac{|mg|}{2}+\sigma\right)^2}\right]^{-1}\right),$$

а $\varrho = 1$, если $N(\alpha) \leq x$; $e = 0$, если $N(\alpha) > x$. Аналогично оценивается и

$$\Gamma_0\left[\frac{|mg|}{2}+1-s,\ \frac{2\pi\tau^{-1}N(\alpha)}{\sqrt{d_1 N(f)}}\right].$$

Эта лемма доказана в [5].

ЛЕММА 5. Справедлива оценка о среднем значении

$$\int_1^T\left|f\left(\frac{1}{2}+it\right)-\frac{Z_m(l)}{l^{\frac{1}{2}+it}}\right|^2 dt \ll \frac{(|m|+1)\cdot T^{1+2\varepsilon}}{p^{\frac{n}{2}}}d_1^{\frac{1}{2}+\varepsilon}+\frac{(|m|+1)\cdot T^{\varepsilon}}{p^n}.$$

Доказадельство проходит на классическом пути с использованием лемм 2 и 4.

Если теперь в лемме 1 в качестве a_k взять определенную нами выше функцию $Z_m(k)$, учесть, что $Z_m(k) \ll k^{\varepsilon}$, то положив $b = 1+\log x^{-1}$, мы после перенесения контура интегрирования на прямую $\sigma = \frac{1}{2}$ и применения лемм 3 и 5 получим

$$\sum_{\substack{k\equiv l(\mathrm{mod}\, p^n)\\ k\leq x\\ (l,p)=1}} Z_m(k) = E(m)\frac{2\pi x}{\varphi(p^n)\sqrt{d_1}}\prod_{\gamma/p}\left[1-\frac{1}{N(\gamma)}\right]+$$

$$+O\left(\frac{x^{1+\varepsilon}}{p^n T}\right)+O\left(\frac{(|m|+1)^{\frac{1}{2}}\,T^{\frac{1}{2}+\varepsilon}\,x^{\frac{1}{2}}}{p^{\frac{n}{4}}\,d_1^{-\frac{1}{4}-\varepsilon}}\right),$$

здесь $E(m) = 1$ при $m = 0$ и $E(m) = 0$ при $m \neq 0$.

Выбирая еще $T = x^{2\varepsilon}$, мы хорошо известным приемом получаем.

ТЕОРЕМА. При $x \gg p^{\frac{3}{2}n}$, φ_1 и φ_2 — вещественные, $0 \leq \varphi_1 - \varphi_2 \leq 2\pi$, справедлива асимптотическая формула

$$\sum_{\substack{\alpha\in K_0\\ N(\alpha)\equiv l(\mathrm{mod}\, p^n)\\ N(\alpha)\leq x\\ \varphi_1\leq\arg\alpha\leq\varphi_2}} 1 = \frac{x(\varphi_2-\varphi_1)}{\varphi(p^n)\,d_1^{\frac{1}{2}}}\prod_{\gamma/p}\left[1-\frac{1}{N(\gamma)}\right][1+o(1)]+O\left(\frac{x^{\frac{1}{2}+\varepsilon}\,d_1^{\frac{1}{4}+\varepsilon}}{p^{\frac{n}{4}}}\right). \tag{6}$$

ЗАМЕЧАНИЕ. Если поле K — одноклассное, то формула (6) определяет количество решений сравнения

$$N(\alpha) \equiv l(\mathrm{mod}\, p^n), \quad (l, p) = 1$$
$$N(\alpha) \leq x, \quad \alpha \in K,$$
$$\varphi_1 \leq \arg \alpha \leq \varphi_2,$$

что является обобщением результата из [3].

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BEMERKUNG ZUR DIVERGENZ VON FOURIERREIHEN

Von

KARL-HEINZ INDLEKOFER

Math. Seminar der J. W. Goethe Universität, Frankfurt/Main

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1. Einleitung. Es sei $f: R \to R$ eine *stetige*, 2π-periodische Funktion und $g = \bar{f}$ die dazugehörige konjugierte. Man weiß, daß $\bar{f} \in L^q[-\pi, \pi]$ für jedes $q > 0$ ist und daß die Fourierreihe $S[\bar{f}]$ von $\bar{f}$ fast überall konvergiert (vgl. [7], S. 253f und [1]). Ist $\bar{f}$ darüberhinaus stetig und $S[f]$ gleichmäßig konvergent, so gilt dasselbe auch von $S[\bar{f}]$ (vgl. [7], S. 268f). In [7], S. 300ff, beweist ZYGMUND die Existenz zweier stetiger Funktionen f und $g = \bar{f}$ mit dem Stetigkeitsmodul[1)]

$$\omega(f; \delta) = O\left(\left(\log \frac{1}{\delta/4\pi}\right)^{-1}\right) \text{ bzw. } \omega(g; \delta) = O\left(\left(\log \frac{1}{\delta/4\pi}\right)^{-1}\right),$$

so daß $S[\bar{f}]$ an mindestens einer Stelle divergiert, während $S[f]$ überall konvergiert, jedoch nicht gleichmäßig. Wir wollen in dieser Note als Anwendung des Satzes von Banach-Steinhaus folgende Ergebnisse verifizieren.

SATZ 1. *Es sei $L: (0, 2\pi] \to R^+$ eine stetige Abbildung mit folgenden Eigenschaften:*

1° *$L(\delta)$ strebt monoton (und beliebig langsam) gegen Unendlich für $\delta \searrow 0$.*

2° $L(\delta) = o\left(\log \frac{1}{\delta/4\pi}\right)$ *für* $\delta \to 0^+$.

3° $L(\delta) \cdot \left(\log \frac{1}{\delta/4\pi}\right)^{-1} \nearrow$ *für* $\delta \nearrow, \delta \in (0, 2\pi]$.

Dann gibt es überabzählbar viele, in $[-\pi, \pi]$ stetige, 2π-periodische Funktionen f mit dem Stetigkeitsmodul

$$\omega(f; \delta) = O\left(L(\delta) \cdot \left\{\log \frac{1}{\delta/4\pi}\right\}^{-1}\right),$$

1) Für $f: R \to R$, f stetig, 2π-periodisch, definiert man als *Stetigkeitsmodul* für $0 < \delta \leq 2\pi$

$$\omega(f; \delta) = \sup_{|x| \leq \pi} \sup_{|t| \leq \delta} |f(x+t) - f(x)|.$$

so daß $S[f]$ gleichmäßig konvergiert und die Menge

$$\left\{x: \sup_n |s_n(\bar{f}; x)| = \infty\right\}$$

eine in $[-\pi, \pi]$ dichte G_δ-Menge ist.[2)]

SATZ 2. *Es sei $\{\mu_n\}$ eine monoton wachsende Folge positiver Zahlen mit $\mu_n = o(\log n)$. Dann existieren überabzählbar viele, stetige, 2π-periodische Funktionen $f: R \to R$, so daß $S[f]$ gleichmäßig konvergiert und die Menge*

$$\left\{x: \sup_n \frac{|s_n \bar{f}; x)|}{\mu_n} = \infty\right\}$$

eine in $[-\pi, \pi]$ dichte G_δ-Menge ist.

Die Beweise stützen sich auf Methoden, wie sie in [4] verwendet wurden.

2. Banachräume. Es sei

$\mathfrak{B}_1 = \{f: R \to R$ *stetig, 2π-periodisch, $s_n(f; x)$ gleichmäßig konvergent*$\}$ der Vektorraum der reellwertigen, stetigen, 2π-periodischen Funktionen f, deren Fourierreihen gleichmäßig konvergieren. Dann ist $\mathfrak{B}_1$ mit der Norm

$$\|f\|_{(1)} = \sup_x |f(x)| + \sup_n \left(\sup_x |s_n(f; x)|\right) \tag{2.1}$$

ein Banachraum.

BEWEIS. Wegen

$$\sup_n \left(\sup_x |s_n(f+g; x)|\right) \leq \sup_n \left(\sup_x |s_n(f; x)|\right) + \sup_n \left(\sup_x |s_n(g; x)|\right)$$

ist $\mathfrak{B}_1$ offensichtlich ein normierter Vektorraum. Es sei nun $\{f_\nu\}$ eine Cauchyfolge von Funktionen aus $\mathfrak{B}_1$. Dann existiert wegen der Vollständigkeit des Raumes $\mathfrak{C}[-\pi, \pi]$ der in $[-\pi, \pi]$ stetigen Funktionen und des Raumes c aller konvergenten Folgen (jeweils versehen mit der üblichen sup- Norm) eine stetige Funktion f und eine Folge von stetigen Funktionen $\{F_n\}$ mit der Eigenschaft, daß es zu jedem $\varepsilon > 0$ ein $\nu_0(\varepsilon)$ gibt mit

$$\begin{aligned} &\sup_x |f_\nu(x) - f(x)| < \varepsilon \qquad \text{für } \nu \geq \nu_0, \\ &\sup_n \left(\sup_x |s_n(f_\nu; x) - F_n(x)|\right) < \varepsilon \qquad \text{für } \nu \geq \nu_0. \end{aligned} \tag{2.2}$$

[2)] Mit $s_n(\bar{f}; x)$ wird die n-te Partialsumme von $S[\bar{f}]$ an der Stelle x bezeichnet. Eine G_δ-Menge ist abzählbarer Durchschnitt offener Mengen.

Für alle $n = 0, 1, \ldots$ ist $F_n(x) \equiv s_n(f; x)$. Denn nehmen wir an, es existiere ein n_0 und ein x_0, so daß $F_{n_0}(x_0) \neq s_{n_0}(f; x_0)$ ist. Nun ist aber

$$|F_{n_0}(x) - s_{n_0}(f; x)| \leq |F_{n_0}(x) - s_{n_0}(f_\nu; x)| + |s_{n_0}(f; x) - s_{n_0}(f_\nu; x)| < 2 \cdot \varepsilon$$

für alle x und alle ν, die größer als ein genügend großes $\nu_1(\varepsilon)$ sind, was unserer Annahme widerspricht. Mit (2.2) und dem dortigen $\nu_0(\varepsilon)$ ist

$$|f(x) - s_n(f; x)| \leq |f(x) - f_{\nu_0}(x)| + |f_{\nu_0}(x) - s_n(f_{\nu_0}; x)| + \\ + |s_n(f_{\nu_0}; x) - s_n(f; x)| < 3 \cdot \varepsilon$$

für $n > n_0(\nu_0)$, und somit ist $s_n(f; x)$ gleichmäßig konvergent und $\mathfrak{B}_1$ ein Banachraum.

Ist $w : [0, 2\pi] \to R^+$ eine gegebene, stetige, auf $[0, 2\pi]$ monoton wachsende, nicht-negative Funktion mit $w(0) = 0$, so daß $w(\delta) \cdot \delta^{-1}$ monoton fällt, dann ist der Vektorraum

$$\mathfrak{B}_2 = \left\{ f; R \to R; f \textit{ stetig, } 2\pi\textit{-periodisch, } s_n(f; x) \textit{ gleichmäßig konvergent,} \right. \\ \left. \sup_{\delta \in (0, 2\pi]} \frac{\omega(f; \delta)}{w(\delta)} < \infty \right\}$$

mit der Norm

$$\|f\|_{(2)} = \sup_n \left(\sup_x |s_n(f; x)| \right) + \max \left\{ \sup_x |f(x)|, \sup_{\delta \in (0, 2\pi]} \frac{\omega(f; \delta)}{w(\delta)} \right\}$$

ein Banachraum.

Beweis. Da der Vektorraum

$$\mathfrak{X}_w = \left\{ f: R \to R; f \textit{ stetig, } 2\pi\textit{-periodisch, } \sup_{\delta \in (0, 2\pi]} \frac{\omega(f; \delta)}{w(\delta)} < \infty \right\}$$

mit der Norm

$$\|f\|_w = \max \left\{ \sup_x |f(x)|, \sup_{\delta \in (0, 2\pi]} \frac{\omega(f; \delta)}{w(\delta)} \right\}$$

ein Banachraum ist (vgl. [6], Fußnote 4)), ist die Cauchyfolge $\{f_\nu\}$ bezüglich $\|\cdot\|_{(2)}$ auch eine Cauchyfolge bezüglich $\|\cdot\|_{(1)}$ und $\|\cdot\|_w$. Die Grenzwerte seien f und g (beide stetig). Für jedes $\varepsilon > 0$ und alle $x \in [-\pi, \pi]$ ist

$$|f(x) - g(x)| \leq |f(x) - f_\nu(x)| + |f_\nu(x) - g(x)| \leq \|f - f_\nu\|_{(1)} + \|f_\nu - g\|_w < 2\varepsilon$$

für genügend großes ν, d.h. $f(x) \equiv g(x)$. Somit ist $\mathfrak{B}_2$ vollständig.

3. Beweis von Satz 1 und Satz 2. Die Beweise beruhen auf dem Satz von Banach-Steinhaus (vgl. [4], 5.8):

Sei X ein Banachraum, Y ein normierter linearer Raum und Λ_α ($\alpha \in A$) eine Familie beschränkter linearer Abbildungen von X in Y. Entweder existiert dann eine Konstante $K < \infty$ mit

$$\|\Lambda_\alpha\| \leq K \tag{3.1}$$

für alle $\alpha \in A$, oder es existiert eine dichte G_δ-Menge $G \subset X$ mit

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\|_Y = \infty \tag{3.2}$$

für alle $x \in G$.

Wir setzen

$$\Lambda_n f = s_n(\bar{f}; 0) \tag{3.3}$$

und zeigen, daß für die durch (3.3) bestimmten Abbildungen von $\mathfrak{B}_2$ in R die Bedingung (3.1) nicht erfüllt ist.

Die Λ_n sind nach der Definition von $\bar{f}$ offensichtlich *linear* und außerdem *beschränkt*, da

$$|\Lambda_n f| < c_1 \cdot n \cdot \sup_x |f(x)| \leq c_1 \cdot n \cdot \|f\|_{(2)}$$

ist. Wählt man

$$Q_m(x) = \sum_{1 \leq n \leq m} \frac{\sin nx}{n},$$

so ist $Q_m \in \mathfrak{B}_2$ für $m = 1, 2, \ldots$, und mit einer geeigneten Konstaten M gilt (vgl. [7], S. 183f)

$$\sup_m \sup_n \left(\sup_x |s_n(Q_m; x)|\right) \leq M. \tag{3.4}$$

Nun ist gleichmäßig in m, x und t

$$\Delta(m, x, t) := |Q_m(x+t) - Q_m(x)| \leq 2M. \tag{3.5}$$

Andererseits ist nach einem Satz von Bernstein (vgl. [3], S. 39f)

$$\Delta(m, x, t) = \left| \int_x^{x+t} Q'_m(y)\, dy \right| \leq m \cdot M \cdot t. \tag{3.6}$$

Dann wird

$$\begin{aligned} \|Q_m\|_{(2)} &\leq M + \max\left\{M, \sup_{\delta \in (0, 2\pi]} \frac{\sup\limits_{|x| \leq \pi} \sup\limits_{|t| \leq \delta} \Delta(m, x, t)}{w(\delta)}\right\} \leq \\ &\leq 2M + \sup_{\delta \in (0, 2\pi]} \left[\frac{1}{w(\delta)} \cdot \min(2M, m \cdot M \cdot \delta)\right] \leq \\ &\leq 2M + \frac{2M}{w(2/m)}, \end{aligned}$$

da $\delta/w(\delta)$ und $w(\delta)$ mit δ monoton wachsen. Bilden wir

$$A_m Q_m = s_m(\bar{Q}_m; 0) = 1 + \sum_{1 \leq n \leq m} n^{-1},$$

so ist

$$\|A_m\| \geq \frac{|A_m Q_m|}{\|Q_m\|_{(2)}} \geq \frac{\log m}{2M + 2M\,[w(2/m)]^{-1}}.$$

Wir wählen $w(\delta) = L(\delta) \cdot \left\{\log \frac{1}{\delta/4\pi}\right\}^{-1}$ für $\delta \in (0, 2\pi]$, $w(0) = 0$. Dies ist möglich, falls $L(\delta) = o\left(\log \frac{1}{\delta/4\pi}\right)$ für $\delta \to 0+$ ist und $L(\delta)$ für $\delta \searrow 0$ monoton gegen Unendlich wächst. Mit dieser Wahl von w folgt

$$\|A_m\| \geq c_2 \cdot L(2/m). \tag{3.8}$$

Da $L(2/m)$ für $m \to \infty$ beliebig groß wird, ist die Möglichkeit (3.1) ausgeschlossen.

Sei nun $\{r_i\}$ $(i = 1, 2, \ldots)$ eine Folge von reellen Zahlen, die dicht in $[-\pi, \pi]$ liegen. Mit $Q_m^*(x) = Q_m(x - r_i)$ erhalten wir für

$$A_n Q_m^* = s_n(\bar{Q}_m^*; r_i)$$

dieselben Abschätzungen wie in (3.8) und damit den Beweis von Satz 1 (vgl. RUDIN [4], S. 102 ff).

Zum Beweis von Satz 2 verwenden wir den Banachraum $\mathfrak{B}_1$. Mit

$$A_n Q_m^* = \frac{s_n(\bar{Q}_m^*; r_i)}{\mu_n}$$

berechnen wir analog zu (3.8)

$$\|A_m\| \geq \frac{|A_m Q_m|}{2M} \geq \frac{\log m}{2M\,\mu_m} \tag{3.9}$$

und gewinnen dadurch den Beweis von Satz 2.

4. Bemerkungen. Die obige Methode hat ihren Anwendungsbereich auch bei Fourierreihen bezüglich anderer vollständiger Orthonormalsysteme, wie z. B. des *Walsh*-Systems $\{\psi_r\}$ (vgl. etwa [2]). Mit Hilfe der Funktionen p_n und f_n von L. SCHIPP [5],[3] des Banachraums[4]

$$\mathfrak{B}_3 = \left\{ f: R \to R;\ f \textit{ stetig},\ 1\text{-}\textit{periodisch},\ \sup_{\delta \in (0,1]} \frac{\omega(f;\delta)}{w(\delta)} < \infty \right\}$$

und der Abbildungen $A_n : \mathfrak{B}_3 \to R$, die bestimmt sind durch

$$A_n f = s_n(f; x),$$

wobei $s_n(f; x)$ die n-te Partialsumme der Walsh-Fourierreihe von f an der Stelle x ist, läßt sich zeigen:

SATZ 3. *Es sei* $L : (0, 1] \to R^+$ *eine stetige Abbildung mit folgenden Eigenschaften:*

1° $L(\delta)$ *strebt monoton (und beliebig langsam) gegen Unendlich für* $\delta \searrow 0$.

2° $L(\delta) = o\left(\left(\log \frac{1}{\delta/2}\right)^{\frac{3}{4}}\right)$ *für* $\delta \to 0+$.

3° $L(\delta) \cdot \left(\log \frac{1}{\delta/2}\right)^{-\frac{3}{4}} \nearrow$ *für* $\delta \nearrow$, $\delta \in (0,1]$.

Dann gibt es überabzählbar viele, stetige, 1-periodische Funktionen $f : R \to R$ *mit dem Stetigkeitsmodul*

$$\omega(f;\delta) = O\left(L(\delta) \cdot \left(\log \frac{1}{\delta/2}\right)^{-\frac{3}{4}}\right),$$

so daß die Menge

$$\left\{x : \sup_n |s_n(f;x)| = \infty\right\}$$

eine dichte G_δ*-Menge in* $[0, 1]$ *ist.*

Auf ähnliche Art folgert man

SATZ 4. *Es sei* $\{\mu_n\}$ *eine monoton gegen Unendlich wachsende Folge positiver Zahlen mit* $\mu_n = o((\log n)^{3/4})$. *Dann existieren überabzählbar viele stetige, 1-periodische Funktionen* $f : R \to R$, *so daß die Menge*

$$\left\{x : \sup_n \frac{|s_n(f;x)|}{\mu_n} = \infty\right\}$$

eine dichte G_δ*-Menge in* $[0, 1]$ *ist.*

3)

$$p_n(x) = \begin{cases} 0 & \text{für } x \in [0, 2^{-2n^4}) \\ (-1)^j & \text{für } x \in [2^{-j}, 2^{-j+1}) \\ & \text{mit } 2(n-1)^4 + 1 \le j \le 2n^4 \\ 0 & \text{für } x \in [2^{-2(n-1)^4}, 1) \end{cases}$$

$$p_n(x+1) = p_n(x) \quad (n = 1, 2, \ldots).$$

$$f_0(x) = \begin{cases} x & \text{für } x \in [0, 1/2] \\ 1 - x & \text{für } x \in [1/2, 1], \end{cases}$$

$$f_0(x+1) = f_0(x),$$

$$f_n(x) = f_0(2^n x) \quad (n = 1, 2, \ldots).$$

4) Norm in $\mathfrak{B}_3$ ist: $\|\cdot\|_{(3)} = \max\left\{\sup_x |f(x)|, \sup_{\delta \in (0,1]} \frac{\omega(f;\delta)}{w(\delta)}\right\}$

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ON THE SOLUTIONS OF SYSTEMS OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

By

RAHMI IBRAHIM IBRAHIM ABDEL KARIM*

Cairo University, Faculty of Science, Mathematical Department Egypt

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§1. Introduction

We shall deal with the system of n non-linear differential equations

$$\mathbf{u}' = \mathbf{g}(\mathbf{u}, t) + \mathbf{h}(t) + \beta\, \mathbf{f}(t), \tag{1}$$

forr small $|\beta|$. The vectors of n components $\mathbf{g}$, $\mathbf{h}$, $\mathbf{f}$ are assumed to be continuous in t and periodic of the same period p:

$$\mathbf{g}(\mathbf{u}, t+p) = \mathbf{g}(\mathbf{u}, t),\ \mathbf{h}(t+p) = \mathbf{h}(t),\ \mathbf{f}(t+p) = \mathbf{f}(t) \tag{2}$$

Let $\mathbf{u}_0(t)$ be a given periodic solution with period p of the system of differential equations (1) for $\beta = 0$:

$$\mathbf{u}_0 = \mathbf{g}[\mathbf{u}_0(t), t] + \mathbf{h}(t),\ \mathbf{u}_0(t+p) = \mathbf{u}_0(t). \tag{3}$$

Using the substitution

$$\mathbf{x}(t) = \mathbf{u}(t) - \mathbf{u}_0(t), \tag{4}$$

we obtain the system of differential equations

$$\mathbf{x}' = \mathbf{g}(\mathbf{u}, t) - \mathbf{g}[\mathbf{u}_0(t), t] + \beta\, \mathbf{f}(t). \tag{5}$$

Under the assumption, that the vector $\mathbf{g}(\mathbf{u}, t)$, for which

$$|\mathbf{u}(t) - \mathbf{u}_0(t)| \leq \varepsilon \tag{6}$$

possesses continuous partial derivatives w. r. t. the components of $\mathbf{u}$ till the second order, the system (5) can be written — by means of Taylor's formula — in ther form

$$\mathbf{x}' = A(t)\,\mathbf{x} + \beta\, \mathbf{f}(t) + \mathbf{r}(t, \mathbf{x}). \tag{7}$$

Here we have [1]

$$A(t) = (a_{ik}(t)) = \left(\frac{\partial\, g_i[\mathbf{u}_0(t), t]}{\partial\, u_k}\right) \tag{8}$$

* The present address: IInd Chair of Analysis, Eötvös Loránd University, Budapest.

and

$$\mathbf{r}(t, \mathbf{x}) = \mathbf{g}[\mathbf{u}_0(t) + \mathbf{x}, t] - \mathbf{g}[\mathbf{u}_0(t), t] - \frac{\partial \mathbf{g}[\mathbf{u}_0(t), t]}{\partial \mathbf{u}} \cdot \mathbf{d} = \tag{9}$$

$$= \int_0^1 \mathbf{x}^T \cdot T[\mathbf{u}_0(t) + \lambda \mathbf{x}, t]\, \mathbf{x} \cdot (1 - \lambda)\, d\lambda,$$

where $\mathbf{x}^T$ denotes the transposed vector of $\mathbf{x}$, and T means the tensor

$$T[\mathbf{u}_0(t) + \lambda \mathbf{x}, t] = \begin{bmatrix} T_1[\mathbf{u}_0(t) + \lambda \mathbf{x}, t] \\ \vdots \\ T_n[\mathbf{u}_0(t) + \lambda \mathbf{x}, t] \end{bmatrix} \tag{10}$$

with

$$T_i[\mathbf{u}_0(t) + \lambda \mathbf{x}, t] = \left(\frac{\partial^2 g_i[\mathbf{u}_0(t) + \lambda \mathbf{x}, t]}{\partial u_k \, \partial u_l} \right), \quad i = 1, \ldots, n. \tag{11}$$

Clearly the matrix $A(t)$ and the tensor $T[\mathbf{u}_0(t), t]$ have the period p.

Here we shall seek solutions of (7) [or (5)] for sufficiently small ε and β_0 with

$$|\mathbf{x}(t)| \leq \varepsilon \quad \text{for} \quad |\beta| \leq \beta_0. \tag{12}$$

The linear homogeneous and adjoint system of differential equations corresponding to (7) are

$$\mathbf{y}' = A(t)\, \mathbf{y} \tag{13}$$

and

$$\mathbf{z}'(t) = -A^T(t)\, \mathbf{z} \tag{15}$$

respectively.

Refferring to [1], the following theorems hold.

THEOREM 1 (Resonance case). *If the adjoint system* (14) *has a p-periodic solution* $\mathbf{z}(t)$ *for which the relation*

$$\int_t^{t+p} \mathbf{z}^T(t)\, \mathbf{f}(t)\, dt \neq 0, \tag{15}$$

holds, then every solution $\mathbf{x}(t)$ *of* (7) *takes on — independent of the initial conditions, with increasing* t *— values whose modulus is at least of order* $\sqrt{|\beta|}$.

THEOREM 2 (Principal case). *If* (14) *possesses no p-periodic solution* $\mathbf{z}(t)$, *then* (7) *possesses, to every sufficiently small* $|\beta|$, *solutions* $\mathbf{x}(t)$ *for which*

$$|\mathbf{x}(t)| \leq \text{Const.}\ |\beta| \tag{16}$$

for all t; *e.g. the unique existing small p-periodic solution* $\mathbf{x}(t+p) = \mathbf{x}(t)$.

It remains the "exceptional case", where (14) possesses p-periodic solutions $\mathbf{z}(t)$, but for all these periodic solutions the relation

$$\int_t^{t+p} \mathbf{z}^T(t)\,\mathbf{f}(t)\,dt = 0 \tag{17}$$

holds. In this case further assumptions on the functions $g_i(\mathbf{u}, t)$ will be needed.

For this purpose, it is convenient to use in (7) the substitution

$$\mathbf{x} = \beta\,\mathbf{x}^* \quad \text{for} \quad \beta \neq 0. \tag{18}$$

We obtain [2]

$$\mathbf{x}^{*\prime} = A(t)\,\mathbf{x}^* + \mathbf{f}(t) + \beta\,\mathbf{r}^*(t, \mathbf{x}^*, \beta) \tag{19}$$

with

$$\mathbf{r}^*(t, \mathbf{x}^*, \beta) = \frac{\mathbf{g}[\mathbf{u}_0(t) + \beta\,\mathbf{x}^*, t] - \mathbf{g}[\mathbf{u}_0(t), t] - \beta\,\dfrac{\partial\,\mathbf{g}[\mathbf{u}_0(t), t]}{\partial\,\mathbf{u}}\,\mathbf{x}^*}{\beta^2}. \tag{20}$$

The term $\mathbf{r}^*(t, \mathbf{x}^*, \beta)$ and also its partial derivatives w. r. t. $\mathbf{x}^*$ till the second order are continuous functions, in particular as β tends to zero [2]. For $\beta = 0$, the system (19) is reduced to the linear system

$$\mathbf{x}^{*\prime} = A(t)\,\mathbf{x}^* + \mathbf{f}(t). \tag{21}$$

Obviously the homogeneous and adjoint equations corresponding to the system (21) are (13) and (14) respectively.

It is well known [3], that if $Y(t)$ is a fundamental matrix solution of the homogeneous system (13), then

$$Z(t) = [Y^{-1}(t)]^T \tag{22}$$

is also a fundamental matrix solution of the adjoint system (14). Let $Y(t)$ be a fundamental matrix solution of (13) such that the constant matrix (See [4])

$$P = Y^{-1}(t)\,Y(t+p) \tag{23}$$

has the normal form

$$P = \begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_s \end{bmatrix} \text{ with } P_\nu = \begin{bmatrix} 1, & p, & \dfrac{p^2}{2!}, \ldots, & \dfrac{p^{m_\nu - 1}}{(m_\nu - 1)!} \\ & 1, & p, \ldots, & \dfrac{p^{m_\nu - 2}}{(m_\nu - 2)!} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \cdot e^{\varkappa_\nu p} \quad (\nu = 1, \ldots, s), \tag{24}$$

where m_ν is the order of the submatrix P_ν. Then to every submatrix P_ν with the eigenvalue $e^{\varkappa_\nu p} = 1$, there exists exactly one p-periodic solution $\mathbf{y}_{(\nu)}(t)$ of (13) as well as one p-periodic solution $\mathbf{z}_{[\nu]}(t)$ of (14) (see [5]), where the indices (ν) and $[\nu]$ are defined by

$$(\nu) = \sum_1^{\nu-1} m_\mu + 1, \quad [\nu] = \sum_1^{\nu} m_\mu. \tag{25}$$

Let the submatrices P_ν $(\nu = 1, \ldots, s)$ of the matrix P be arranged such that

$$e^{\varkappa_\nu p} \begin{cases} = 1 & (\nu = 1, \ldots, \varrho) \\ \neq 1 & (\nu = \varrho + 1, \ldots, s). \end{cases} \tag{26}$$

Then the system (13) [(14)] has the p-periodic solutions

$$\mathbf{y}_{(\nu)}(t)\,[z_{[\nu]}(t)] \quad \text{for} \quad \nu = 1, \ldots, \varrho.$$

It is easy to prove the following lemma by using the method of variation of the constants applied on the differential equation (19) and the relations (22), (23), (2).

LEMMA. Let $\mathbf{x}^*(t, \beta, \mathbf{c})$ be the unique existing solution of the system of differential equations (19), which has the initial value at $t = 0$

$$\mathbf{x}^*(0, \beta, \mathbf{c}) = Y(o)\,\mathbf{c}. \tag{27}$$

Then the necessary and sufficient condition that this solution has the period p, is the satisfaction of the equation

$$(P^{-1} - I)\,\mathbf{c} - \int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,d\tau - \beta \int_0^p Z^T(\tau)\,\mathbf{r}^*[\tau, \mathbf{x}^*(\tau, \beta, \mathbf{c}), \beta]\,d\tau = \mathbf{0}, \tag{28}$$

where $\mathbf{I}$ is the unit matrix of order n.

If we assume, that to every sufficiently small $|\beta|$, there exists a vector $\mathbf{c}(\beta)$ with

$$\lim_{\beta \to 0} \mathbf{c}(\beta) = \mathbf{c}(o), \tag{29}$$

such that the solution $\mathbf{x}^*[t, \beta, \mathbf{c}(\beta)]$ is p-periodic, then — due to the continuous dependence of the solution on β and $\mathbf{c}$ — there exists the limit

$$\lim_{\beta \to 0} \mathbf{x}^*[t, \beta, \mathbf{c}(\beta)] = \mathbf{x}^*[t, o, \mathbf{c}(o)]. \tag{30}$$

And consequently $\mathbf{x}^*[t, o, \mathbf{c}(o)]$ is a p-periodic solution of (19) for $\beta = 0$. Further from (28), it follows for $\mathbf{c}(o)$ the relation

$$(P^{-1} - I)\,\mathbf{c}(o) - \int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,d\tau = \mathbf{0}. \tag{31}$$

Necessary and sufficient condition for the solvability of (31) is the satisfaction of the exceptional condition for the linear system (19) for $\beta = 0$:

$$\int_0^p \mathbf{z}_{[\nu]}^T(\tau)\,\mathbf{f}(\tau)\,d\tau = 0 \quad (\nu = 1, \ldots, \varrho) \tag{32}$$

[see (17)]. Assuming that the condition (32) holds, then the $[\nu]$-th equations of (28) become

$$b_{[\nu]}(\mathbf{c}, \beta) = \int_0^p \mathbf{z}_{[\nu]}^T(\tau)\,\mathbf{r}^*[\tau, \mathbf{x}^*(\tau, \beta, \mathbf{c}), \beta]\,d\tau = 0 \quad (\nu = 1, \ldots, \varrho). \tag{33}$$

Reffering to (33), it follows for $\beta \to 0$, that

$$b_{[\nu]}[\mathbf{c}(o), o] = \int_0^p \mathbf{z}_{[\nu]}^T(\tau)\,\mathbf{r}^*\{\tau, \mathbf{x}^*[\tau, o, \mathbf{c}(o)], o\}\,d\tau = 0 \quad (\nu = 1, \ldots, \varrho). \tag{34}$$

We state [2]:

THEOREM 3 (Exceptional case). *Let all the p-periodic solutions* $\mathbf{z}_{[\nu]}(t)$ $(\nu = 1, \ldots, \varrho)$ *of the system* (14) *satisfy the exceptional condition* (32), *and let* $\mathbf{c}(o)$ *be a solution of the system* (31), *such that the corresponding solution* $\mathbf{x}^*[t, o, \mathbf{c}(o)]$ *of* (19) *for* $\beta = 0$ *is p-periodic. Moreover let all the p-periodic solutions* $\mathbf{z}_{[\nu]}(t)$ *of* (14) *satisfy also the relation* (34) *and finally let the functional determinant*

$$\left| \frac{\partial\, b_{[\nu]}[\mathbf{c}(o), o]}{\partial\, c_{(\sigma)}} \right| \neq 0 \quad (\nu, \sigma = 1, \ldots, \varrho) \tag{35}$$

be different from zero. Then there exists to every sufficiently small $|\beta|$ *a vector solution* $\mathbf{c}(\beta)$ *of* (28), *such that the solution* $\mathbf{x}^*[t, \beta, \mathbf{c}(\beta)]$ *of* (19) *has the period p. Consequently, it follows the existence of a p-periodic solution* $\mathbf{x}(t, \beta) = \beta\mathbf{x}^*[t, \beta, \mathbf{c}(\beta)]$ *of* (7) *of order* $|\beta|$.

§2. Expansion in power series of the parameter

In this § it will be assumed, that the functions $g_i(\mathbf{u}, t)$ $(i = 1, \ldots, n)$ for which (6) holds are analytic functions in the variables $u_1, u_2, \ldots, u_n$ for every fixed value of t, i. e. for every $i = 1, \ldots, n$ the functions

$$g_i(\mathbf{u}, t) = g_i[\mathbf{u}_0(t) + \mathbf{x}, t]$$

for which (6) holds, can be expanded in an absolute and uniform convergent power series in the form

$$g_i[\mathbf{u}_0(t) + \mathbf{x}, t] = g_i[\mathbf{u}_0(t), t] + \sum_{k=1}^{n} \frac{\partial\, g_i[\mathbf{u}_0(t), t]}{\partial\, u_k} x_k + \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial\, u_k\, \partial\, u_l} x_k x_l + \sum_{k,l,r}^{1\ldots n} \frac{\partial^3 g_i[\mathbf{u}_0(t), t]}{\partial\, u_k\, \partial\, u_l\, \partial\, u_r} x_k x_l x_r + \ldots, \tag{36}$$

where the coefficients, that are depending on t, are continuous functions nad p-periodic. Further let $\mathbf{f}(t)$ be a continuous function with the period p.

Then we have

THEOREM 4. *Under the assumption* (32) *and the existence of a vector* $\mathbf{c}(0)$, *so that the relations* (31), (34) *and* (35) *hold, the system of differential equations* (5) *possesses a p-periodic solution* $\mathbf{x}(t, \beta)$ *which depends analytically on* β; *i. e. it holds a power-series expansion of the form*

$$\mathbf{x}(t,\beta) = \sum_0^\infty \frac{\beta^{\gamma+1}}{(\gamma+1)!}\mathbf{x}^{(\gamma)}(t), \tag{37}$$

where the functions $\mathbf{x}^{(\gamma)}(t)$ $(\gamma = 0, 1, \ldots)$ *have the period p and the series* (37) *is absolute and uniform convergent for sufficiently small values of* β.

PROOF. Setting $\mathbf{x} = \beta\mathbf{x}^*$ for $\beta \neq 0$ in (5), it becomes

$$\mathbf{x}^{*\prime} = \frac{\mathbf{g}[u_0(t)+\beta\mathbf{x}^*, t] - \mathbf{g}[u_0(t), t]}{\beta} + \mathbf{f}(t). \tag{38}$$

Or in components $(i = 1, \ldots, n)$

$$x_i^{*\prime} = \frac{g_i[\mathbf{u}_0(t)+\beta\mathbf{x}^*, t] - g_i(\mathbf{u}_0(t), t]}{\beta} + f_i(t). \tag{39}$$

Setting $\mathbf{x} = \beta\mathbf{x}^*$ also in (36), we obtain for the R. S. of (38) the expansion

$$\sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^* + \beta \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^* x_l^* + \ldots, \tag{40}$$

which is — under the condition (6) — absolute and uniform convergent. Then from the existence theorem, the solution $\mathbf{x}^*(t, \beta, \mathbf{c})$ is in the neighbourhood of $\beta = 0$ and $\mathbf{c} = \mathbf{c}(0)$ analytic in β and $\mathbf{c}$ (see e. g. [6] or [7]). Referring to the conditions (31), (34) and (35), it follows by means of the theorem on the implicit functions for analytic systems for sufficiently small $|\beta|$ the existence of a function $\mathbf{c}(\beta)$ which depends analytically on β and assumes the value $\mathbf{c}(0)$ for $\beta = 0$ ([8] & [2], §4). Hence $c(\beta)$ can be expanded in a power series

$$\mathbf{c}(\beta) = \mathbf{c}^{(0)} + \beta\mathbf{c}^{(1)} + \frac{\beta^2}{2!}\mathbf{c}^{(2)} + \ldots, \tag{41}$$

such that the solution $\mathbf{x}^*[t, \beta, \mathbf{c}(\beta)]$ of (38) has the period p. Then it holds for $\mathbf{x}^*(t, \beta, \mathbf{c})$ a power series-expansion in β

$$\mathbf{x}^*(t,\beta,\mathbf{c}) = \sum_0^\infty \frac{\beta^\gamma}{(\gamma+1)!}\mathbf{x}^{(\gamma)}(t), \tag{42}$$

where the functions $\mathbf{x}^{(\gamma)}(t)$ are continuous and p-periodic. The periodicity of $\mathbf{x}^{(\gamma)}(t)$ follows directly from the fact, that the following relation

$$\mathbf{0} = \mathbf{x}^*[t+p, \beta, \mathbf{c}(\beta)] - \mathbf{x}^*[t, \beta, \mathbf{c}(\beta)] = \sum_0^\infty \frac{\beta^\gamma}{(\gamma+1)!}[\mathbf{x}^{(\gamma)}(t+p) - \mathbf{x}^{(\gamma)}(t)]$$

holds identic in β. Setting $\mathbf{x} = \beta\mathbf{x}^*$ in (42), it follows the required power-series expansion (37).

For the determination of the functions $\mathbf{x}^{(\gamma)}(t)$ in the power-series expansion (37), we set (37) in (5). This gives ($i = 1, 2, \ldots, n$)

$$\sum_{\gamma=0}^{\infty} \frac{\beta^{\gamma+1}}{(\gamma+1)!} [x_i^{(\gamma)}(t)]' = g_i\left(\mathbf{u}_0(t) + \sum_{0}^{\infty} \frac{\beta^{\gamma+1}}{(\gamma+1)!} \mathbf{x}^{(\gamma)}(t), t\right) - g_i[\mathbf{u}_0(t), t] + \beta f_i(t) =$$

$$(43) \qquad = \beta\left(\sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(0)} + f_i(t)\right) +$$

$$+ \frac{\beta^2}{2!}\left(\sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(1)} + \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^{(0)} x_l^{(0)}\right) + \ldots .$$

Comparing the coefficients of powers of β in (43), we obtain the sequence of linear systems of differential equations ($i = 1, \ldots, n$)

$$(44) \quad \begin{cases} (44^*) \quad x_i^{(0)\prime}(t) = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(0)}(t) + f_i^{(0)}(t) \text{ with } f_i^{(0)}(t) = f_i(t) \\ (44^{**}) \quad x_i^{(1)\prime}(t) = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(1)}(t) + f_i^{(1)}(t) \\ \quad \text{with } f_i^{(1)}(t) = \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^{(0)}(t) x_l^{(0)}(t) \\ (44^{***}) \quad x_i^{(2)\prime}(t) = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(2)}(t) + \\ \qquad + 3 \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^{(0)}(t) x_l^{(1)}(t) + q^{(2)}(t) \\ \quad \text{with } q^{(2)}(t) = \sum_{k,l,r}^{1\ldots n} \frac{\partial^3 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l \partial u_r} x_k^{(0)}(t) x_l^{(0)}(t) x_r^{(0)}(t) \\ \text{i.e.} \\ \quad x_i^{(2)\prime}(t) = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(2)}(t) + f_i^{(2)}(t) \\ \quad \text{with } f_i^{(2)}(t) = 3 \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} \cdot x_k^{(0)}(t) x_l^{(1)}(t) + q^{(2)}(t) \\ (\overline{44}) \quad x_i^{(\gamma)\prime}(t) = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(\gamma)}(t) + \\ \qquad + (\gamma+1) \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^{(0)}(t) x_l^{(\gamma-1)}(t) + q_i^{(\gamma)}(t) = \\ \qquad = \sum_{k=1}^{n} \frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k} x_k^{(\gamma)}(t) + f_i^{(\gamma)}(t) \quad (\text{for } \gamma \geqq 2) \\ \quad \text{with } f_i^{(\gamma)}(t) = (\gamma+1) \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l} x_k^{(0)}(t) x_l^{(\gamma-1)}(t) + q_i^{(\gamma)}(t) . \end{cases}$$

Here the expression between the two equality signs in $(\overline{44})$ (for $\gamma \geqq 2$) consists of three terms. The first is a linear form in $\mathbf{x}^{(\gamma)}(t)$ with the matrix $A(t) = \left(\frac{\partial g_i[\mathbf{u}_0(t), t]}{\partial u_k}\right)$, which is independent on γ. The second term is a bilinear form in $\mathbf{x}^{(\gamma-1)}(t)$ and $\mathbf{x}^{(0)}(t)$, where the coefficients $\frac{\partial^2 g_i[\mathbf{u}_0(t), t]}{\partial u_k \partial u_l}$ are independent on γ. The third term $q_i^{(\gamma)}(t)$ depends only on the vectors $\mathbf{x}^{(0)}(t)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{\gamma-2}(t)$.

The system of differential equations (44) will be solved in the form

$$\mathbf{x}^{(\gamma)}(t) = Y(t)\left(\int_0^t Z^T(\tau)\,\mathbf{f}^{(\gamma)}(\tau)\,d\tau + \mathbf{c}^{(\gamma)}\right) \quad (\gamma = 0, 1, \ldots), \tag{45}$$

where $Y(t)$ is a fundamental matrix solution of the homogeneous system corresponding to (44) with the initial value (see (27))

$$\mathbf{x}^{(\gamma)}(o) = Y(o)\,\mathbf{c}^{(\gamma)}. \tag{46}$$

We go from a solution $\mathbf{x}^{(0)}(t)$ of the system of differential equations (44*), which has the period p and at the same time is determined such that the exceptional condition for the system (44**) is satisfied. This is done as it follows: From the periodicity of $\mathbf{x}^{(0)}(t)$, it follows the satisfaction of the condition (31), from which the constants

$$c_\mu^{(0)}\begin{pmatrix}\mu = 1, \ldots, n \\ \neq (\sigma) \text{ for } \sigma = 1, \ldots \varrho\end{pmatrix}$$

are uniquely determined. The remaining constants $c_{(\sigma)}^{(0)}$ (for $\sigma = 1, \ldots, \varrho$) would be determined such that the exceptional condition, which coincides with the condition (34), is satisfied for the next following equation.

The exceptional condition for the system (44**) is namely (see (20) & [2], (24))

$$b_{[\nu]}(\mathbf{c}^{(0)}) = \int_0^p \mathbf{z}_{[\nu]}^T(\tau)\,\mathbf{f}^{(1)}(\tau)\,d\tau = \tag{47}$$

$$= \int_0^p \sum_{i,k,l} z_{i,[\nu]}(\tau)\,\frac{\partial^2 g_i[\mathbf{u}_0(\tau), \tau]}{\partial u_k\,\partial u_l} \cdot x_k^{(0)}(\tau)\,x_l^{(0)}(\tau)\,d\tau = 0 \quad (\nu = 1, \ldots, \varrho).$$

Referring to (45), we obtain the simple formula

$$\frac{\partial x_l^{(0)}(\tau)}{\partial c_{(\sigma)}} = y_{l,(\sigma)}(\tau), \tag{48}$$

from which it follows (for $\nu, \sigma = 1, \ldots, \varrho$), that

$$\frac{\partial b_{[\nu]}(\mathbf{c}^{(0)})}{\partial c_{(\sigma)}} = 2\int_0^p \sum_{i,k,l} z_{i,[\nu]}(\tau)\,\frac{\partial^2 g_i[\mathbf{u}_0(\tau), \tau]}{\partial u_k\,\partial u_l}\,x_k^{(0)}(\tau)\,y_{l,(\sigma)}(\tau)\,d\tau. \tag{49}$$

The quantities (49) are the elements of the functional determinant of the L. S. of the exceptional condition (47) for the system (44**) w. r. t. the constants $c_{(\sigma)}$ $(\sigma = 1, \ldots, \varrho)$. It is only to assume that this determinant is different from zero.

After determination the p-periodic solution $\mathbf{x}^{(0)}(t)$ of (44*), we can determine a p-periodic solution $\mathbf{x}^{(1)}(t)$ of (44**) such that at the same time the exceptional condition for the system (44***) is satisfied. This gives the components of the vector $\mathbf{c}^{(1)}$ uniquely [see (45)]; the constants $c_{\mu}^{(1)}$ for $\mu \neq (\sigma)$ $(\sigma = 1, \ldots, \varrho)$ are obtained from the corresponding condition to (31):

$$(P^{-1} - I)\,\mathbf{c}^{(\gamma)} - \int_0^p Z^T(\tau)\,\mathbf{f}^{(\gamma)}(\tau)\,d\tau = \mathbf{0} \tag{50}$$

for $\gamma = 1$, and the constants $c_{(\sigma)}^{(1)}(\sigma = 1, \ldots, \varrho)$ are obtained from the exceptional condition (here $\gamma = 1$)

$$\int_0^p \mathbf{z}_{[\nu]}^T(\tau)\,\mathbf{f}^{(\gamma+1)}(\tau)\,d\tau = 0 \tag{51}$$

with $\mathbf{f}^{(\gamma+1)}(t)$ from $(\overline{44})$ with $\gamma + 1$ instead of γ. The fact that the constants $c_{(\sigma)}^{(1)}$ $(\sigma = 1, \ldots, \varrho)$ are uniquely determined, follows again from the condition (35), which we want to prove below in general.

Continuing successively in this manner and determining the solution $\mathbf{x}^{(\gamma)}(t)$ for an arbitrary γ, such that it possesses the period p and that the exceptional condition for the successive equation for $\mathbf{x}^{(\gamma+1)}(t)$ is satisfied, where the uniqueness of $\mathbf{x}^{(\gamma)}(t)$ can be always settled by means of the condition (35), we obtain the required sequence of p-periodic functions $\mathbf{x}^{(\gamma)}(t)$ for $\gamma = 0, 1, \ldots$.

We prove now the uniqueness of the functions $x^{(\gamma)}(t)$ for $\gamma = 1, 2, \ldots$. Suppose on the contrary, that there are two distinct p-periodic solutions $\mathbf{x}_1^{(\gamma)}(t)$ and $\mathbf{x}_2^{(\gamma)}(t)$ of the system $(\overline{44})$, such that — by setting each of these two solutions $\mathbf{x}_1^{(\gamma)}(t)$ and $\mathbf{x}_2^{(\gamma)}(t)$ in the next system for $\mathbf{x}^{(\gamma+1)}(t)$ — the exceptional condition for it will be satisfied. Obviously the difference of the solutions $\mathbf{x}_1^{(\gamma)}(t)$ and $\mathbf{x}_2^{(\gamma)}(t)$ of $(\overline{44})$ is a p-periodic solution $\tilde{\mathbf{y}}(t)$ of the homogeneous system corresponding to $(\overline{44})$. Therefore $\tilde{\mathbf{y}}(t)$ can be written as a linear combination of the p-periodic solutions $\mathbf{y}_{(\nu)}(t)$ for $\nu = 1, \ldots, \varrho$, i. e. in components $(l = 1, \ldots, n)$ a representation of the form

$$\tilde{y}_l(t) = \sum_{\sigma=1}^{\varrho} y_{l,(\sigma)}(t)\,\tilde{c}_{(\sigma)} \tag{52}$$

holds, where at least one of the coefficients $\tilde{c}_{(\sigma)}$ is different from zero. Setting

$$\mathbf{x}_1^{(\gamma)}(t) - \mathbf{x}_2^{(\gamma)}(t) = \tilde{\mathbf{y}}(t) \tag{53}$$

in the exceptional condition for the next system of differential equations for $\mathbf{x}^{(\gamma+1)}(t)$ and using (52), we obtain the system of equations ($r = 1, \ldots, \varrho$)

$$(54)\quad (\gamma+1)\int_0^p \sum_{i,k,l}^{1\ldots n} z_{i,[r]}(\tau)\frac{\partial^2 g_i[\mathbf{u}_0(\tau),\tau]}{\partial u_k\,\partial u_l}\,x_k^{(\sigma)}(\tau)\cdot\sum_{\sigma=1}^{\varrho} y_{l,(\sigma)}(\tau)\,\tilde{c}_{(\sigma)}\,d\tau = 0.$$

Rearranging (54), we obtain the system of homogeneous equations in the constants $\tilde{c}_{(\sigma)}$ (for $\sigma = 1, \ldots, \varrho$)

$$\sum_{\sigma=1}^{\varrho}\tilde{c}_{(\sigma)}\int_0^p \sum_{i,k,l}^{1\ldots n} z_{i,[r]}(\tau)\frac{\partial^2 g_i[\mathbf{u}_0(\tau),\tau]}{\partial u_k\,\partial u_l}\,x_k^{(\sigma)}(\tau)\cdot y_{l,(\sigma)}(\tau)\,d\tau = 0.$$

Referring to (35) [see also (49)], the coefficient matrix of this system of equations is non-singular, and therefore all the constants $\tilde{c}_{(\sigma)}$ (for $\sigma = 1, \ldots, \varrho$) are equal to zero. This leads to a contradiction with the hypothesis that the solution $\mathbf{x}^{(\gamma)}(t)$ is not uniquely determined.

We formulate the result as

THEOREM 5. *Under the assumption, that the system of differential equations* (5) *has a p-periodic solution* $\mathbf{x}(t, \beta)$ *of the form* (37), *where the series in* (37) *is absolute and uniform convergent for* $|\beta|$ *small enough, then the functions* $\mathbf{x}^{(\gamma)}(t)$ *can be determined as solutions of the sequence of linear systems of differential equations* (44) *in the following manner: One requires firstly, that* $\mathbf{x}^{(\gamma)}(t)$ *has the period p and secondly, that the next differential equations for* $\mathbf{x}^{(\gamma+1)}(t)$ *would be in the exceptional case. The solution* $\mathbf{x}^{(0)}(t)$ *is uniquely determined by means of the same conditions, and hence the successive solutions* $\mathbf{x}^{(\gamma)}(t)$ *are also uniquely determined.*

REMARK. If the homogeneous system of differential equations corresponding to (44) possesses no p-periodic solution, then all the systems of differential equations (44) are in the principal case. And consequently the systems (44) possess the unique p-periodic solutions $\mathbf{x}^{(0)}(t)$, $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$, ... (see [3]). Then by virtue of (37) the solution $\mathbf{x}(t, \beta)$ is also p-periodic and takes on, for sufficiently small $|\beta|$ — values of order $|\beta|$.

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ON THE SOLUTIONS OF SOME SPECIAL DIFFERENTIAL EQUATIONS OF THE THIRD AND FIFTH ORDER

By

RAHMI IBRAHIM IBRAHIM ABDEL KARIM

Cairo University, Faculty of Science, Mathematical Dep., Egypt

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In continuation of investigations of [1], we study the relations between the solutions of the differential equations of the third order of the form

$$(1) \qquad (py)''' + (py)' + ry = 0$$

and

$$(2) \qquad (pz)''' + (pz)' - rz = 0,$$

where $p = p(x) > 0$ and $r = r(x)$ are continuous functions of $x \in (-\infty, \infty)$. We derive a comparison theorem between two differential equations of the form (1) and the sufficient conditions for the solutions of (1) to be oscillatory or non-oscillatory in some interval (a, ∞).

Further we discuss the analogical relations between the solutions of the differential equations of the fifth order of the form

$$(3) \qquad (py)^{(V)} + ry = 0$$

and

$$(4) \qquad (pz)^{(V)} - rz = 0,$$

assuming for p and r the same assumptions stated above. At the end we deal with the sufficient conditions for the non-vanishing of the solutions of the differential equations (3) and (4).

1. The solutions u_1, u_2, u_3 of the D. E.* (1) or (2) form a fundamental system of solutions, if the determinant (Wronskian)

$$W(u_1, u_2, u_3) = \begin{vmatrix} u_1 & u_2 & u_3 \\ (pu_1)' & (pu_2)' & (pu_3)' \\ (pu_1)'' & (pu_2)'' & (pu_3)'' \end{vmatrix}$$

is different from zero at least at one point in the interval $(-\infty, \infty)$.

* Abbreviation of "differential equation".

Evidently

$$W(u_1, u_2, u_3) = W(x) = \frac{c}{p(x)}, \tag{5}$$

where c is a convenient constant depending on the choice of the solutions.

In [1], it is proved the following theorem:

If $z_1(x)$, $z_2(x)$ are two arbitrary linearly independent solutions of the D. E. (2), then the function

$$y(x) = w[z_1, z_2](x) = \begin{vmatrix} z_1(x) & z_2(x) \\ (pz_1)'(x) & (pz_2)'(x) \end{vmatrix} \tag{6}$$

is a solution of the D. E. (1). On the other hand, if $y_1(x)$, $y_2(x)$ are two arbitrary linearly independent solutions of the D. E. (1), then the function

$$z(x) = w[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ (py_1)'(x) & (py_2)'(x) \end{vmatrix} \tag{7}$$

is a solution of the D. E. (2). (See also [2].)

THEOREM 1. *Let $y(x)$ be an arbitrary solution of the* D. E. (1). *Then there exist two solutions $z_1(x)$, $z_2(x)$ of the* D. E. (2) *such that the relation* (6) *holds; and on the contrary, if $z(x)$ is an arbitrary solution of the* D. E. (2), *then there exist two solutions $y_1(x)$, $y_2(x)$ of the* D. E. (1) *such that the relation* (7) *holds.*

PROOF. We prove the first statement. Let $y(x)$ be the solution of the D. E. (1) with the property

$$y(a) = \eta_1, \ (py)'(a) = \eta_2, \ (py)''(a) = \eta_3, \ a \in (-\infty, \infty),$$

where at least one of the numbers η_1, η_2, η_3 is different from zero. Choose the initial conditions at the point a for the required solutions z_1, z_2 such that the relations

$$w[z_1, z_2](a) = \eta_1, \ \{p \cdot w[z_1, z_2]\}'(a) = \eta_2, \ \{p \cdot w[z_1, z_2]\}''(a) = \eta_3$$

are valid, and such that at least one of the numbers $z_1(a)$, $(pz_1)'(a)$, $(pz_1)''(a)$ and also one of the numbers $z_2(a)$, $(pz_2)'(a)$, $(pz_2)''(a)$ are different from zero. This is evidently possible.

Let y_1, y_2 be two linearly independent solutions of the D. E. (1) with the properties

$$y_1(a) = (py_1)'(a) = 0, \qquad (py_1)''(a) = 1,$$
$$y_2(a) = (py_2)''(a) = 0, \qquad (py_2)'(a) = 1, \qquad a \in (-\infty, \infty).$$

The band of solutions of the D. E. (1) at the point a is defined as the set of solutions $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with the property $y(a) = 0$.

The band of solutions of the D. E. (1) fulfils the D. E. of the second order

$$pw(py)'' - (pw)'(py)' + [(pw)'' + pw]\, py = 0, \tag{8}$$

where

$$w(x) = w[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ (py_1)' & (py_2)' \end{vmatrix}$$

is the solution of the D. E. (2) with the property

$$w(a) = (pw)'(a) = 0, \ (pw)''(a) \neq 0, \ w(x) \neq 0 \text{ for } x > a.$$

By means of the D. E. (8), it can be proved

THEOREM 2. *Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the* D. E. (1) *with the property* $w(x) \neq 0$, $x \in I \subset (-\infty, \infty)$. Then

$$y_3(x) = \int_a^x \frac{y_1(t)\, y_2(x) - y_1(x)\, y_2(t)}{p(t)\, w^2(t)}\, dt, \ a, x \in I$$

is also a solution of the D. E. (1), *which forms together with $y_1(x)$ and $y_2(x)$ in I a fundamental system of solutions of the* D. E. (1).

PROOF. Evidently the set of solutions $y = c_1y_1 + c_2y_2$ fulfils the D. E. (8), where $w(x) = z(x)$ is the solution of the D. E. (2). Since by differentiating (8) we obtain the D. E. (1), then the solutions of (8) are also solutions of (1). Replacing the right side of (8) by a constant $= 1$ (say), we obtain a non-homogeneous D. E. whose particular solution y_3 (say) is again a solution of the D. E. (1). Clearly a fundamental system of solutions of the homogeneous D. E. (8) is y_1, y_2. By using the method of variation of constants, we compute $y_3 = c_1(x)y_1 + c_2(x)y_2$, where c_1 and c_2 are obtained from the equations

$$c_1' y_1 + c_2' y_2 = 0, \ c_1'(py_1)' + c_2'(py_2)' = \frac{1}{pw}.$$

This gives

$$c_1 = -\int_a^x \frac{y_2(t)}{p(t)\, w^2(t)}\, dt, \ c_2 = \int_a^x \frac{y_1(t)}{p(t)\, w^2(t)}\, dt, \ a, x \in I$$

and

$$y_3(x) = \int_a^x \frac{y_1(t)\, y_2(x) - y_1(x)\, y_2(t)}{p(t)\, w^2(t)}\, dt.$$

Evidently y_3 has a double zero point at a, which is not the case by y_1 or y_2 and consequently y_1, y_2, y_3 form in I a fundamental system of solutions of the D. E. (1).

2. This § is concerned with the necessary and sufficient conditions for the elements of the fundamental system of solutions of the D. E. (1) with a zero point to have another zero point.

THEOREM 3. *The solution $y(x)$ of the* D. E. (1) *with the property*

$$y(x_1) = (py)'(x_1) = 0, \ (py)''(x_1) \neq 0, \ -\infty < x_1 < \infty \tag{9}$$

has a zero point at the point $x_2 \neq x_1$ if and only if the solution $z(x)$ of the D. E. (2) *with the property*

$$z(x_2) = (pz)'(x_2) = 0, \ (pz)''(x_2) \neq 0 \tag{10}$$

has a zero point at the point x_1.

PROOF. Let $y(x)$ be the solution of the D. E. (1) satisfying (9) and the relation $y(x_2) = 0$, $x_2 \neq x_1$ and let y_1 and y_2 be two linearly independent solutions of the D. E. (1) with the properties

$$\begin{cases} y_1(x_2) = (py_1)'(x_2) = 0, & (py_1)''(x_2) \neq 0 \\ y_2(x_2) = (py_2)''(x_2) = 0, & (py_2)'(x_2) \neq 0. \end{cases} \tag{11}$$

Evidently $y(x)$ can be written in the form $y = c_1y_1 + c_2y_2$, where c_1 and c_2 satisfy the relations

$$c_1\, y_1(x_1) + c_2\, y_2(x_1) = 0, \ c_1(py_1)'(x_1) + c_2(py_2)'(x_1) = 0$$

and c_1, c_2 do not vanish simultaneously. This implies, that

$$\begin{vmatrix} y_1(x_1) & y_2(x_1) \\ (py_1)'(x_1) & (py_2)'(x_1) \end{vmatrix} = z_1(x_1) = 0.$$

It is evident from (7) and (11), that $z(x)$ is the solution of the D. E. (2) with the property (10).

The converse statement can be proved by replacing y and x_1 by z and x_2 respectively. However another proof is analogous to the proof given in the following theorem.

THEOREM 4. *The solution $y(x)$ of the* D. E. (1) *with the property*

$$y(x_1) = (py)''(x_1) = 0, \ (py)'(x_1) \neq 0, \tag{12}$$

or

$$(py)'(x_1) = (py)''(x_1) = 0, \ y(x_1) \neq 0, \ -\infty < x_1 < \infty$$

has a zero point at $x_2 \neq x_1$ if and only if the point x_1 is the zero point of $(pz)'(x)$, or $(pz)''(x) + (pz)(x)$ respectively, where $z(x)$ is the solution of the D. E. (2) *with the property* (10).

PROOF. We shall prove the first part. Let $y(x)$ be the solution of the D. E. (1) with the property (12) and $y(x_2) = 0$, $x_2 \neq x_1$. Then $y(x)$ can be written as $y = c_1y_1 + c_2y_2$, where y_1 and y_2 have the properties (11). The constants c_1, c_2 must be determined, such that

$$c_1\, y_1(x_1) + c_2\, y_2(x_1) = 0, \ c_1(py_1)''(x_1) + c_2(py_2)''(x_1) = 0,$$

where c_1, c_2 are not both zeros. Referring to (7), it follows that $(pz)'(x_1) = 0$ where $z(x)$ is the solution of the D. E. (2) with the property (10).

Conversely, let $z(x)$ be the solution of the D. E. (2) with the property (10) and such that $(pz)'(x_1) = 0$, $x_1 \neq x_2$ and let $y(x)$ be the solution of the D. E. (1) with the property (12). We claim, that $y(x_2) = 0$.

Let y_1, y_2, y_3 form a fundamental system of solutions of the D. E. with the properties

$$(13)\qquad \begin{cases} y_1(x_2) = (py_1)'(x_2) = 0, & (py_1)''(x_2) \neq 0, \\ y_2(x_2) = (py_2)''(x_2) = 0, & (py_2)'(x_2) \neq 0, \\ (py_3)'(x_2) = (py_3)''(x_2) = 0, & y_3(x) \neq 0. \end{cases}$$

Obviously $y(x)$ can be written as

$$(14)\qquad y = c_1 y_1 + c_2 y_2 + c_3 y_3,$$

where c_1, c_2, c_3 are obtained from the equations

$$\sum_1^3 c_r y_r(x_1) = 0, \quad \sum_1^3 c_r (py_r)'(x_1) = c \neq 0, \quad \sum_1^3 c_r (py_r)''(x_1) = 0.$$

Evaluating c_3, we get

$$c_3 = -\frac{c[y_1(x_1)(py_2)''(x_1) - y_2(x_1)(py_1)''(x_1)]}{W(x_1)} = -\frac{c(pz)'(x_1)}{W(x_1)} = 0.$$

Hence it follows from (14) and (13), that $y(x_2) = 0$.

3. Let us consider also the D. E.

$$(15)\qquad (pz)''' + (pz)' + r_1 z = 0,$$

where $p = p(x) > 0$, $r_1 = r_1(x)$ are continuous functions of $x \in (-\infty, \infty)$. Then we prove the following

THEOREM 5. *Let* $0 \leq r_1(x) \leq r(x)$, $x \in (-\infty, \infty)$. *If every solution* $z(x)$ *of the* D. E. (15) *has in some interval* (a, ∞), $-\infty < a < \infty$ *infinite number of zero points* [i. e. it is oscillatory on (a, ∞)], *then every solution of the* D. E. (1) *with one zero point is also oscillatory on* (a, ∞). *Furthermore if* y *and* z *are the solutions of the differential equations* (1) *and* (15) *satisfying the initial conditions*

$$(16)\qquad y(a) = (py)'(a) = 0, \quad (py)''(a) \neq 0, \quad z(a) = (pz)'(a) = 0, \quad (pz)''(a) \neq 0,$$

then the first zero point on the right of a *of the solution* y *is not far from* a *than the first zero point on the right of* a *of the solution* z.

On the other hand, if the solution y of the D. E. (1) satisfying the initial conditions (16) has no zero point on the right of a, then the solution z of the D. E. (15) satisfying the initial conditions (16) has also no zero point on the right of a.

PROOF. Suppose that the solutions of the D. E. (15) are oscillatory on (a, ∞). Referring to [1], theorem 5, the first part of the theorem is proved, if it is shown that every solution of the D. E. (1) with a double zero point

at the point $a \in (-\infty, \infty)$ (a arbitrary) has another zero point on the right of a, not far from a than the first zero point on the right of a of the solution of the D. E. (15).

Let y and z be two solutions of the differential equations (1) and (15) satisfying (16) and let without loss of generality $(py)''(a) = (pz)''(a) > 0$. Further let $x_0 > a[x_1 > a]$ be the first zero point on the right of a of the solution $y[z]$. By means of the method of variation of constants, it can be shown that $y(x)$ can be written in the form

$$y(x) = z(x) + \int_a^x [r_1(t) - r(t)]\, p(t)\, y(t)\, W(x, t)\, dt, \tag{17}$$

where

$$W(x, t) = \begin{vmatrix} z_1(x) & z_2(x) & z_3(x) \\ z_1(t) & z_2(t) & z_3(t) \\ (pz_1)'(t) & (pz_2)'(t) & (pz_3)'(t) \end{vmatrix}$$

and z_1, z_2, z_3 form a fundamental system of solutions of (15) [3]. Obviously the function $W(x, t) \geqq 0$ for $t \leqq x \in \langle a, x_1 \rangle$. Referring to (17), it follows that $x_0 \leq x_1$.

The second part follows also by using (17).

4. In this § we derive shortly the analogical relations between the solutions of the differential equations (3) and (4).

Under the Wronskian of the fundamental system of solutions u_1, u_2, u_3, u_4, u_5 of the D. E. (3) or (4), we understand the determinant

$$W = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ (pu_1)' & (pu_2)' & (pu_3)' & (pu_4)' & (pu_5)' \\ (pu_1)'' & (pu_2)'' & (pu_3)'' & (pu_4)'' & (pu_5)'' \\ (pu_1)''' & (pu_2)''' & (pu_3)''' & (pu_4)''' & (pu_5)''' \\ (pu_1)^{(IV)} & (pu_2)^{(IV)} & (pu_3)^{(IV)} & (pu_4)^{(IV)} & (pu_5)^{(IV)} \end{vmatrix}.$$

It is $W = \dfrac{c}{p(x)}$, where c is a suitable constant.

If $u_1(x), u_2(x), u_3(x), u_4(x)$ are arbitrary linearly independent solutions of the D. E. (3) [(4)], then

$$w[u_1, u_2, u_3, u_4](x) = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ (pu_1)' & (pu_2)' & (pu_3)' & (pu_4)' \\ (pu_1)'' & (pu_2)'' & (pu_3)'' & (pu_4)'' \\ (pu_1)''' & (pu_2)''' & (pu_3)''' & (pu_4)''' \end{vmatrix} \tag{18}$$

is a solution of the D. E. (4) [(3)].

Let y_1, y_2, y_3, y_4 be linearly independent solutions of the D. E. (3). It is easy to show, that the solution $y(x) = \sum_1^4 c_r y_r(x)$ of the D. E. (3) ful-

fils the D. E. of the fourth order

$$(19)\quad pw(py)^{(IV)}-(pw)'(py)'''+(pw)''(py)''-(pw)'''(py)'+(pw)^{(IV)}py=0,$$

where $w=w[y_1, y_2, y_3, y_4](x)$ is a solution of the D. E. (4).

By making use of the D. E. (19), it can be proved that if y_1, y_2, y_3, y_4 are linearly independent solutions of the D. E. (3) satisfying the relation $w[y_1, y_2, y_3, y_4](x)=w(x)\neq 0$, $x\in I\subset(-\infty, \infty)$ with w from (18), then the function

$$y_5(x)=-\int_a^x \frac{1}{p(t)\,w^2(t)}\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x)\\ y_1(t) & y_2(t) & y_3(t) & y_4(t)\\ (py_1)'(t) & (py_2)'(t) & (py_3)'(t) & (py_4)'(t)\\ (py_1)''(t) & (py_2)''(t) & (py_3)''(t) & (py_4)''(t)\end{vmatrix}dt,$$

$a, x\in I$ is also a solution of the D. E. (3) and forms together with y_1, y_2, y_3, y_4 in I a fundamental system of solutions of the D. E. (3).

Using the procedure given in § 2, it can be proved

THEOREM 6. *The solution $y(x)$ of the* D. E. (3) *with the property $(py)^{(k)}(x_1)=0$ $(k=0, 1, 2, 3)$, $(py)^{(IV)}(x_1)\neq 0$, or $(py)^{(k)}(x_1)=0$ $(k=0, 1, 2, 4)$, $(py)'''(x_1)\neq 0$, or $(py)^{(k)}(x_1)=0$ $(k=0, 1, 3, 4)$, $(py)''(x_1)\neq 0$, or $(py)^{(k)}(x_1)=0$, $(k=0, 2, 3, 4)$, $(py)'(x_1)\neq 0$, or $(py)^{(k)}(x_1)=0$ $(k=1, 2, 3, 4)$, $y(x_1)\neq 0$ has a zero point at the point $x_2\neq x_1\in(-\infty, \infty)$ if and only if the point x_1 is a zero point of the function $z(x)$, or $(pz)'(x)$, or $(pz)''(x)$, or $(pz)'''(x)$, or $(pz)^{(IV)}(x)$ respectively, where $z(x)$ is the solution of the* D. E. (4) *with the property $(pz)^{(k)}(x_2)=0$ $(k=0, 1, 2, 3)$, $(pz)^{(IV)}(x_2)\neq 0$.*

5. We are going to prove the following

THEOREM 7. *Let $r(x)\geqq 0$, $x\in(-\infty, \infty)$. If $y(x)$ is the solution of the* D. E. (3) *with the property*

$$(20)\quad y(a)=(py)'(a)=(py)''(a)=(py)'''(a)=0,\ (py)^{(IV)}(a)\neq 0,\ -\infty<a<\infty,$$

then $y(x)$, $(py)^{(k)}(x)$ $(k=1, 2, 3, 4)$ have no zero point for $x<a$.

PROOF. Successive integration of the D. E. (3) from a to x, $-\infty<a<\infty$, $-\infty<x<\infty$ gives

$$(21)\qquad (py)'=-\int_a^x \frac{(x-t)^3}{3!}\,r(t)y(t)dt+(py)^{(IV)}(a)\frac{(x-a)^3}{3!}+$$
$$+(py)'''(a)\frac{(x-a)^2}{2!}+(py)''(a)(x-a)+(py)'(a)$$

and

$$(22)\qquad py=-\int_a^x \frac{(x-t)^4}{4!}\,r(t)\,y(t)\,dt+(py)^{(IV)}(a)\frac{(x-a)^4}{4!}+$$
$$+(py)'''(a)\frac{(x-a)^3}{3!}+(py)''(a)\frac{(x-a)^2}{2!}+(py)'(a)(x-a)+(py)(a).$$

Let $y(x)$ be the solution of the D. E. (3) with the property (20) and let without loss of generality $(py'')''(a)>0$. Further let on the contrary $y(x_1) = 0$, where $x_1 < a$ is the first zero point of y on the left of a. Then the integral identity (22) becomes

$$0 = \int_{x_1}^{a} \frac{(x_1-t)^4}{4!} r(t)\, y(t)\, dt + (py)^{(IV)}(a) \frac{(x_1-a)^4}{4!} > 0\,,$$

which is a contradiction.

Supposing that $\bar{x}_1$ is the first zero point of $(py)'(x)$ on the left of a, then the identity (21) leads to the following contradiction

$$0 = (py)'(\bar{x}_1) = \int_{\bar{x}_1}^{a} \frac{(\bar{x}_1-t)^3}{3!} r(t)\, y(t)\, dt + (py)^{(IV)}(a) \frac{(\bar{x}_1-a)^3}{3!} < 0\,.$$

Similarly it can be also shown, that $(py)''$, $(py)'''$, $(py)^{(VI)}$ have no zero point on the left side of a.

Analogously, the following theorem can be proved.

THEOREM 8. *Let $r(x) \geqq 0$, $x \in (-\infty, \infty)$. If $z(x)$ is the solution of the* D. E. (4) *with the property $z(a) = (pz)'(a) = (pz)''(a) = (pz)'''(a) = 0$, $(pz)^{(IV)}(a) \neq 0$, $-\infty < a < \infty$, then neither $z(x)$ nor the derivatives $(pz)^{(k)}(x)$ $(k = 1, 2, 3, 4)$ have zero points on the right side of a.*

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A CONSTRUCTION OF CUBIC GRAPHS, CONTAINING ORTHOGONAL EDGES

By

L. KÁSZONYI

Department of Probability Theory of the Eötvös Loránd University, Budapest

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The edges $\bar{g}$ and $\bar{h}$ of the edge-colorable cubic graph G are orthogonal, if G has no alternating circuit containing both $\bar{g}$ and $\bar{h}$ as edges.

The aim of this paper is to give a construction for such graphs. I conjecture, that this construction is in a certain sense universal.

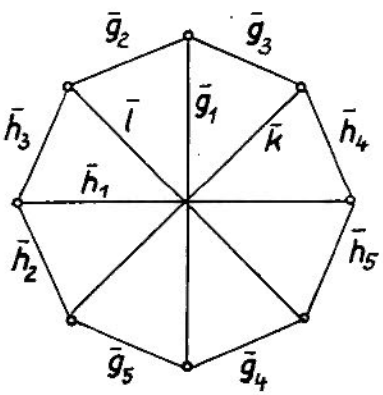

Fig. 1

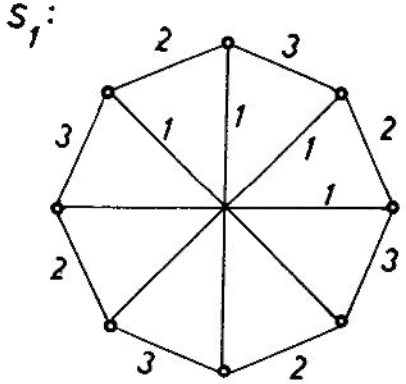

Fig. 2

Consider the graph P_0 of Fig. 1. If the colors of edges $\bar{g}_1, \bar{g}_2$ and $\bar{h}_1$ are given, we can colour the remaining edges of P_0 only in one way. Let be for example $s_1(\bar{g}_1) = 1$, $s_1(\bar{g}_2) = 2$, $s_1(\bar{h}_1) = 1$, then the colors of the edges $\bar{g}_3$, $\bar{h}_4$, $\bar{h}_3$, $\bar{h}_5$, $\bar{h}_2$, $\bar{k}$, $\bar{l}$, $\bar{g}_4$, and $\bar{g}_5$ are as follows:

$s_1(\bar{g}_3) = 3$, $s_1(\bar{h}_4) = 2$, $s_1(\bar{h}_3) = 3$, $s_1(\bar{h}_5) = 3$, $s_1(\bar{h}_2) = 2$, $s_1(\bar{k}) = 1$, $s_1(\bar{l}) = 1$, $s_1(\bar{g}_4) = 2$ and $s_1(\bar{g}_5) = 3$. (See: Fig. 2.)

Similarly, if $s_2(\bar{g}_1) = 1$, $s_2(\bar{g}_2) = 2$ and $s_2(\bar{h}_1) = 2$, then $s_2(\bar{g}_3) = 3$, $s_2(\bar{h}_4) = 1$, $s_2(\bar{k}) = 2$, $s_2(\bar{g}_5) = 3$, $s_2(\bar{g}_4) = 2$, $s_2(\bar{h}_5) = 3$, $s_2(\bar{l}) = 1$, $s_2(\bar{h}_3) = 3$, and $s_2(\bar{h}_2) = 1$ (See: Fig. 3.)

Using the symmetrie properties of P_0, we can see, that if $s_3(\overline{g}_1) = 1$, $s_3(\overline{g}_2) = 2$, and $s_3(\overline{h}_1) = 3$, then P_0 has the coloring of Fig. 4.

It is easy to see, that for every coloring s' of P_0 there is a permutation π, such, that $s'(\overline{g}_i) = \pi s_j(\overline{g}_i)$, and $s'(\overline{h}_1) = \pi s_j(\overline{h}_1)$ holds for some $j \in \{1, 2, 3\}$. It means, that $s' = \pi s_j$, thus we can get every coloring of P_0 from s_1, s_2, or s_3 by permutation of the colors.

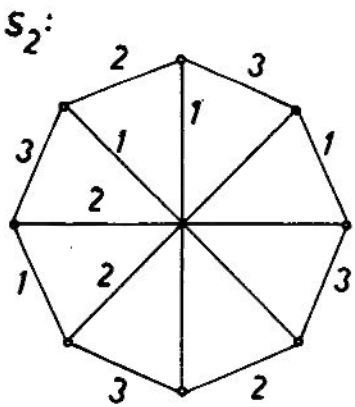

Fig. 3

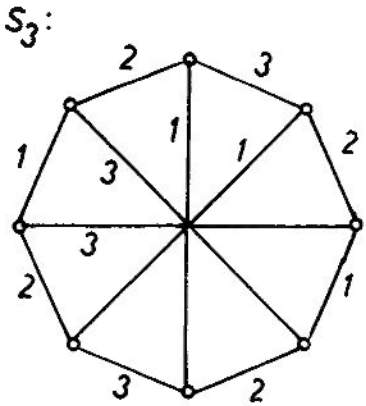

Fig. 4

It is easy to verify, that no one of the alternating circuits of the coloring s_j containes both $\overline{g}_1$ and $\overline{h}_1$ as edges. ($j = 1, 2, 3$). The permutation of the colors does not alter the set of alternating circuits, thus the edges $\overline{g}_1$ and $\overline{h}_1$ are orthogonal.

Let G be a quasi-cubic graph, and $\overline{g} = (g_1, g_2)$ being an edge of it. (A quasi-cubic graph is a graph, the points of which have valency 3 or 1.) Consider the graph G', to be got from G by deletting the edge $\overline{g}$ and joining the points g_1 and g_2 with the new points k_1 and k_2 respectivelly. The operation $G \to G'$ we will call, the cut of the edge $\overline{g}$ of G. The inverse of this operation is the following one:

Let $\overline{g}_{12} = (g_1, g_2)$ and $\overline{g}_{34} = (g_3, g_4)$ be edges of the quasy-cubic graph G', and assume, that the valences of g_2 and g_4 are 1. Let us omit the edges $\overline{g}_{12}$, and $\overline{g}_{34}$, the points g_2 and g_4 of G', and join the points g_1 and g_3 by the new edge $(g_1, g_3) = \overline{g}_{12} \vee \overline{g}_{34}$. The graph, got form G' this way is denoted by G, the operation $G' \to G$ is called connection of the edges $\overline{g}_{12}$ and $\overline{g}_{34}$ of G'. (See: Fig. 5.)

Operation 1. (O_1) (See: Fig. 6.) Let G_1 be a connected graph having orthogonal edges, and G_2 an arbitrary, connected, colorable cubic graph, which is disjunct from G_1. Let us consider the edges $\overline{g}_1$ and $\overline{g}_2$ of G_1 and of G_2, and let us cut $\overline{g}_1$ to 2 edges: $\overline{g}_{11}$ and $\overline{g}_{12}$; the edge $\overline{g}_2$ also to two edges:

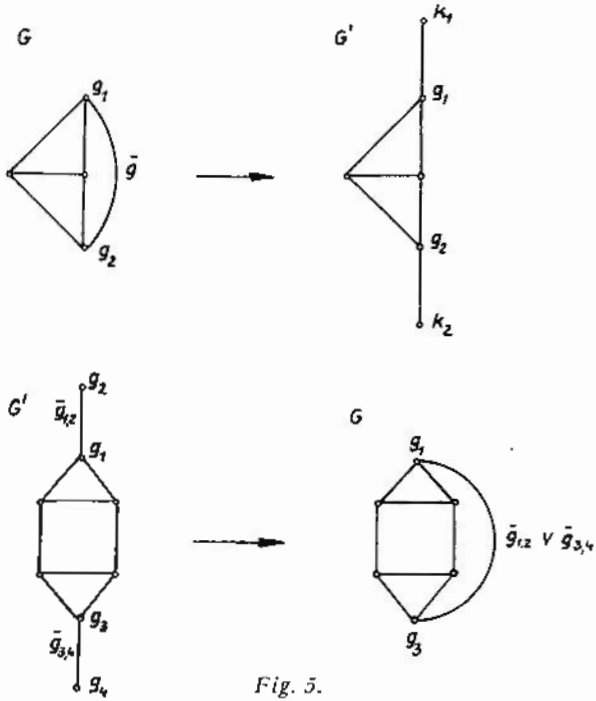
G
g_1
$\bar{g}$
g_2
G'
k_1
g_1
g_2
k_2
G'
g_2
$\bar{g}_{1,2}$
g_1
g_3
$\bar{g}_{3,4}$
g_4
G
g_1
$\bar{g}_{1,2} \vee \bar{g}_{3,4}$
g_3

Fig. 5.

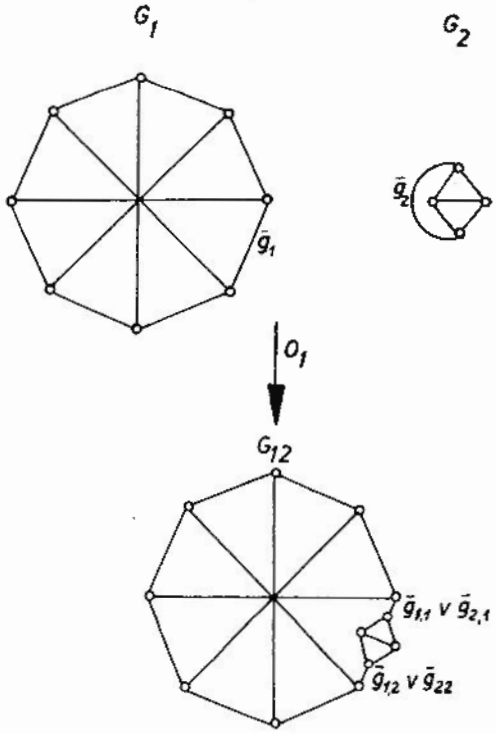
G_1
G_2
$\bar{g}_1$
$\bar{g}_2$
O_1
G_{12}
$\bar{g}_{1,1} \vee \bar{g}_{2,1}$
$\bar{g}_{1,2} \vee \bar{g}_{22}$

Fig 6.

$\overline{g}_{21}$ and $\overline{g}_{22}$. The graps got from G_1 and G_2 we denote by G_1', and G_2'. Let us consider the graph $G_1' \cup G_2'$, and connect the edge $\overline{g}_{11}$ with $\overline{g}_{21}$, and the edge $\overline{g}_{12}$ with $\overline{g}_{22}$. The resulting graph is denoted by G_{12}. If we put $\overline{g}_{11} \vee \overline{g}_{21} = \overline{g}_{11} = \overline{g}_{21}$, and $\overline{g}_{12} \vee \overline{g}_{22} = \overline{g}_{12} = \overline{g}_{22}$, then G_1' and G_2' are subgraphs of G_{12}. The set $M = \{\overline{g}_{12} \vee \overline{g}_{22}, \overline{g}_{11} \vee \overline{g}_{21}\}$ is a cut of G_{12}, that is, the number of components of the graph $G_{12} - M$ is more then that of G_{12}. It is well-known, that if C is an arbitrary circuit of G_{12}, then the number of edges of C common with M is even. Thus every alternating circuit of G_{12} containing $\overline{g}_{12} \vee \overline{g}_{22}$ as an edge, contains $\overline{g}_{11} \vee \overline{g}_{21}$ as well.

Let s be a coloring of G_{12}, and let $s(\overline{g}_{11} \vee \overline{g}_{21}) = \sigma_1$, let $C\sigma_1, \sigma_2$ and $C\sigma_1, \sigma_3$ be alternating circuits of G_{12}, containing the edge $\overline{g}_{11} \vee \overline{g}_{21}$, colored in case of s by the colors σ_1 and σ_i. $(i = 2, 3 \in \{\sigma_1, \sigma_2, \sigma_3\} = \{1, 2, 3\})$ $\overline{g}_{12} \vee \overline{g}_{21} \in$

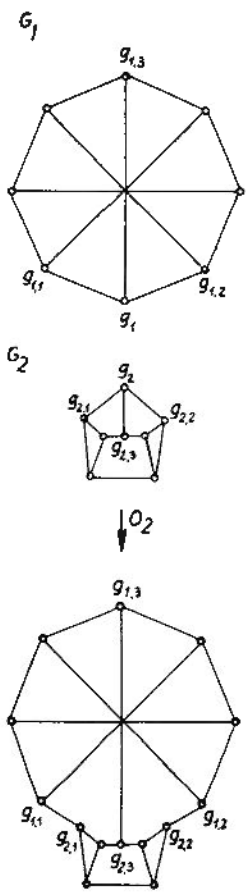

Fig. 7

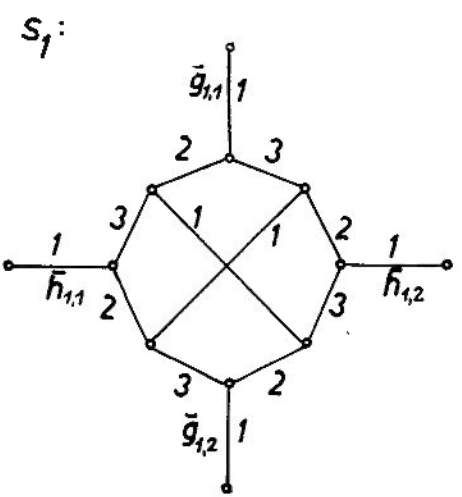

Fig. 8

$\in E(C\sigma_1, \sigma_2) \cap E(C\sigma_1, \sigma_3)$, thus $s(\overline{g}_{12} \vee \overline{g}_{22}) = \sigma_1 = s(\overline{g}_{11} \vee \overline{g}_{21})$. ($E(\cdot)$ denotes the edge-set of the graph $(\cdot)$.)

We shall prove, that G_{12} containes orthogonal edges. Let $\overline{k}$ and $\overline{l}$ be orthogonal edges of G_1, and let us assume, that $\overline{g}_1 \neq \overline{l}$. Put $\overline{x} = \overline{k}$, if $\overline{k} \neq \overline{g}_1$, and $\overline{x} = \overline{g}_{11} \vee \overline{g}_{21} = \overline{g}_{11}$, if $\overline{g}_1 = \overline{k}$. Let us assume, that G has an alternating,

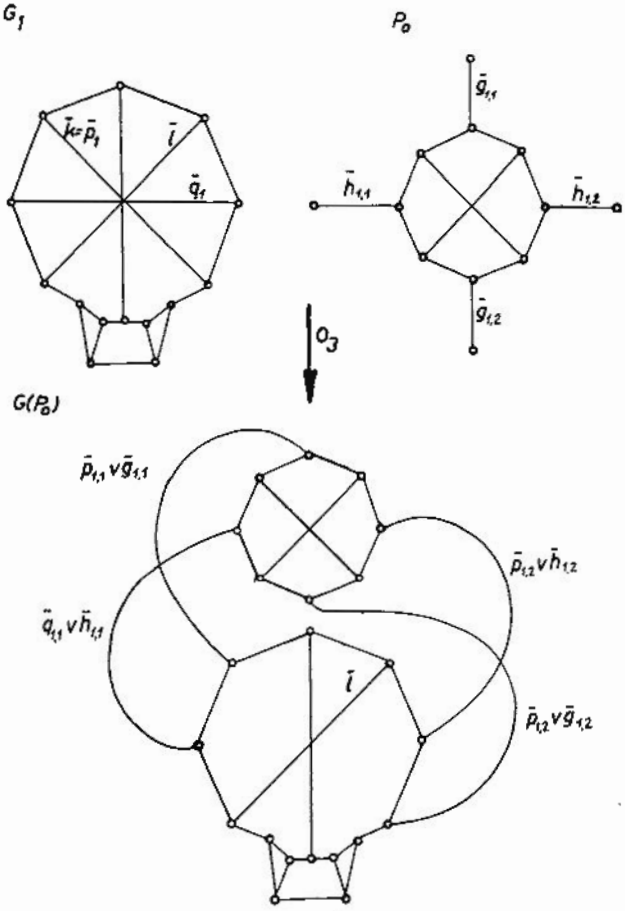

Fig. 9

circuit $C\sigma_1, \sigma_2$, wich containes both $\overline{x}$ and $\overline{l}$ as edges. Let s_1 be the coloring, by wich $C\sigma_1, \sigma_2$ is an alternating circuit, s_1 induces a coloring s_1' of G_1', for which $s_1'(\overline{g}_{11}) = s_1'(g_{12})$ holds. If we connect the edges $\overline{g}_{11}$ and $\overline{g}_{12}$ of G_1', we shall get the graph $G_1 \cdot (\overline{g}_{11} \vee \overline{g}_{12} = \overline{g}_1)$. Let us define the coloring s_1'' of G_1 as follows:

$$s_1''(\overline{g}) = \begin{cases} s_1(\overline{g}), & \text{if } \overline{g} \neq \overline{g}_1, \\ s_1'(\overline{g}_{11}) = s'(\overline{g}_{12}), & \text{if } \overline{g} = \overline{g}_1. \end{cases}$$

If $\overline{g}_{11} \vee \overline{g}_{21} \notin E(C\sigma_1, \sigma_2)$, then $C\sigma_1, \sigma_2$ is an altenrating circuit of G_1, with respect to the coloring s_1''. $C\sigma_1, \sigma_2$ containes the edges $\overline{x} = \overline{k}$ and $\overline{l}$, and this is a contradiction. If $\overline{g}_{11} \vee \overline{g}_{21} \in E(C\sigma_1, \sigma_2)$, then $\overline{g}_{12} \vee \overline{g}_{22} \in E(C\sigma_1, \sigma_2)$. Omitting the edges $\overline{g}_{11} \vee \overline{g}_{21}$ and $\overline{g}_{12} \vee \overline{g}_{22}$ of G_{12}, $C\sigma_1, \sigma_2$ falls in two pieces, the first of them being in $G_1'(C^{(1)})$, the second one in $G_2'(C^{(2)})$. The edge $\overline{g}_{11} \vee \overline{g}_{12} = \overline{g}_1$ is completing $C^{(1)}$ to an alternating circuit of G_1, containing the orthogonal edges $\overline{k}$ and l, and this is a contradiction.

Operation 2. (O_2) (See: Fig. 7.) Let G_1 be a connected graph, containing orthogonal edges, and let G_2 be a connected, colorable cubic graph; which is disjunct from G_1. Let us consider the point g_i of G_i. ($i = 1, 2$). Let us assume, that g_i is joined with the points g_{i1}, g_{i2}, and g_{i3} in G_i. Let us omit the points g_i, and the edges (g_i, g_{ij}) ($i = 1, 2, j = 1, 2, 3$), of the graph $G_1 \cup G_2$, and join g_{1j} with g_{2j}. Similarly to Operation 1, here can also see, that the graph, resulting from G_1 and G_2 in Operation 2 is containing ortogonal edges.

Operation 3. (O_3) (See: Fig. 9.)

If we cut the orthogonal edges $\overline{g}_1$ and $\overline{h}_1$ of P_0, we get the graph P_0' of Fig 8. It is easy to see, that we can get every coloring of P_0' by permuting the colors of one of the colorings s_1, s_2 and s_3, wich are illustrated on Fig. 8, 10, and 11.

These colorings have the property, that every maximal alternating path containing g_{11} ($\overline{h}_{11}$) as an edge, containes g_{12} (h_{12}) too.

Let G_1 be a cubic graph, containing the orthogonal edges k and $\overline{l}$, let us consider the edges $\overline{p}_1$ and $\overline{q}_1$ of G_1. Let us cut $\overline{p}_1$ to the two edges $\overline{p}_{11}$

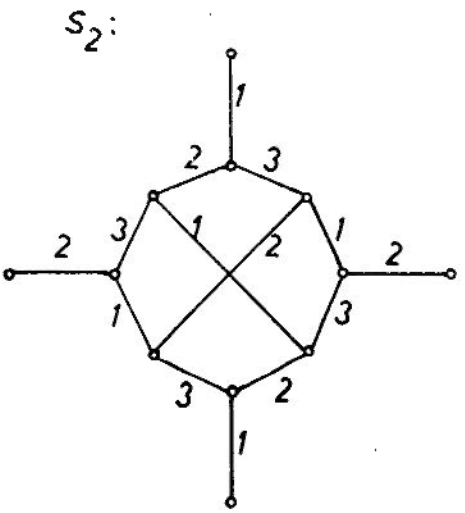

Fig. 10

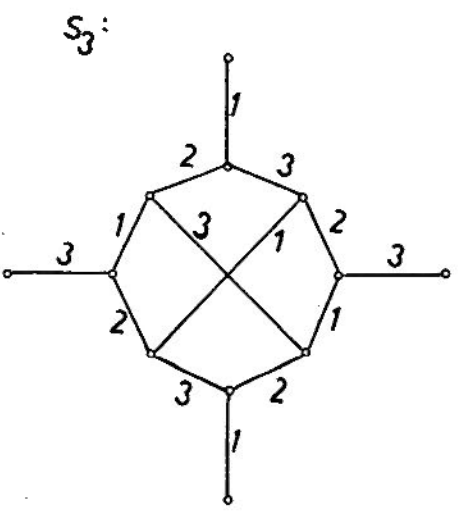

Fig. 11

and $\overline{p}_{12}$, $\overline{q}_1$ to the two edges $\overline{q}_{11}$ and $\overline{q}_{12}$, denote the resulting graph by G_1'.

We shall see, that connecting $\overline{p}_{11}$ with $\overline{g}_{11}$, $\overline{p}_{12}$ with $\overline{g}_{12}$, $\overline{q}_{11}$ with $\overline{h}_{11}$ and $\overline{q}_{21}$ with $\overline{h}_{21}$, we get a graph $G_1(P_0)$, which is containing ortogonal edges.

Assuming, that $\overline{p}_1 \neq \overline{l}$, and $\overline{q}_1 \neq \overline{k}$, let us put $\overline{x} = \overline{k}$, if $\overline{p}_1 \neq \overline{k}$, $\overline{x} = \overline{p}_{11} \vee \overline{g}_{11}$, if $\overline{p}_1 = \overline{k}_1$, $\overline{y} = \overline{l}$, if $\overline{g}_1 \neq \overline{l}$, $\overline{y} = \overline{q}_{11} \vee \overline{h}_{11}$, if $\overline{q}_1 = \overline{l}$.

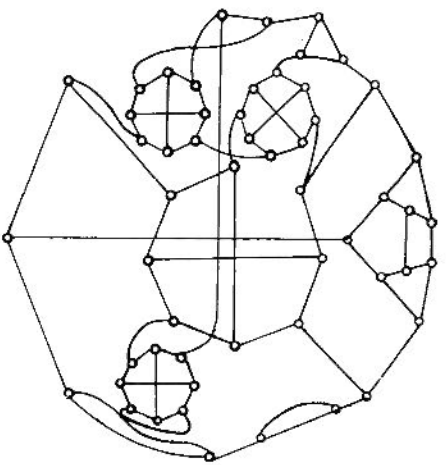

Fig. 12

Let us assume, that $G_1(P_0)$ has a coloring s, corresponding to which there exists an alternating circuit $C\sigma_1, \sigma_2$ of $G_1(P_0)$, which is containing both $\overline{x}$ and $\overline{y}$ as edges. s induces a coloring s' of G_1, which on edges distinct from $\overline{p}_1$ and $\overline{q}_1$ is equal to s, on $\overline{p}_1$ equal to $s(\overline{g}_{11} \vee \overline{p}_{11}) = s(\overline{g}_{12} \vee \overline{p}_{12})$ and on $\overline{q}_1$ to $s(\overline{h}_{11} \vee \overline{q}_{11}) = s(\overline{h}_{12} \vee \overline{q}_{12})$. It is easy to see, that using the edges $\overline{p}_1$ and $\overline{q}_1$, we may complete the pieces of $C\sigma_1, \sigma_2$ being in G_1 to an alternating circuit of G_1, which is containing $\overline{k}$ and $\overline{l}$ as edges. This contradicts to the fact, that $\overline{k}$ and $\overline{l}$ are orthogonal edges of G_1.

Starting by P_0, and using the operations O_1, O_2, and O_3, we get an infinite class of graphs, containing orthogonal edges. Such a graph is to be shown on Fig. 12.

ZUR FRAGE DER REKURSIVITÄT DER IM „ALGOL 68" VERWENDETEN ZWEISTUFIGEN GRAMMATIK

Von

RÓZSA PÉTER

I. Lehrstuhl für Analysis der Eötvös Loránd Universität, Budapest

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§1. Die Satzstruktur-Grammatik

Die Ansprüche der maschinellen Übersetzung haben die Tendenz verstärkt, die grammatischen Begriffe der Sprachen so exakt zu definieren, wie die mathematischen Begriffe definiert werden. Mit dieser Tendenz sind verschiedene mathematische Grammatiken entstanden. Hier beschäftige ich mich nur mit der sog. „Satzstruktur-Grammatik"[1] (worunter ich ihre sog. „context-free" Form verstehe), die auch im sog. „Epialgol" (früher „Metaalgol" genannt, doch im „Algol 68" wird „Metasprache" in anderem Sinn gebraucht), die zur Beschreibung der Programmierungsprache „Algol 60" dient, verwendet wird. Diese Grammatik wird durch vier endliche, nicht leere Mengen

$$T, H, P, K$$

bestimmt, wo die Elemente von T (vom „terminalen Vokabular") die konkreten Begriffe der Sprache sind; die Elemente von H (vom „Hilfsvokabular") die in der Grammatik verwendeten „Kategorienamen"; die Elemente von K gewisse ausgezeichnete (möglicherweise alle) Elemente von H; und die Elemente von P sog. „Produktionen", welche in die durch je einen Kategorienamen bezeichnete Kategorie gehörigen Elemente der Sprache „produzieren", mit anderem Wort „generieren". Genauer geschieht das folgenderweise:

Die Form einer Produktion ist (mit unwesentlicher Abweichung von der ursprünglichen Bezeichnung):

$$h := w_1 w_2 \ldots w_n$$

wo h ein Element von H (ein „Hilfsbegriff") ist, und die rechte Seite des Zeichens „:=" eine „Mischkette" aus Elementen von T und H. Eine Mischkette aus lauter zu T gehörigen Elementen wird „terminal" genannt. Das Zeichen „:=" bezeichnet die „mögliche Gleichheit"; die vorhin aufgezeichnete Produktion ist also so zu lesen: „h ist nach einer der möglichen

[1] Siehe z. B. N. Chomsky, *Syntactic Structures* ('S-Gravenhage, 1957).

Definitionen gleich $w_1 w_2 \ldots w_n$". Derselbe Hilfsbegriff kann nämlich an der linken Seite mehrerer Produktionen stehen; aber mindestens an einer linken Seite muß jeder Hilfsbegriff vorkommen.

(Eventuell kommen auch Produktionen mit leerer rechten Seite, der Form

$$h:=$$

vor. Dann kann für das „nichts" auch ein Zeichen, etwa „$\sqcup$" eingeführt und in T aufgenommen werden; das birgt die Information in sich, daß dort gewöhnlich etwas steht, jetzt aber nicht.)

Wenn für ein Vorkommen eines Hilfsbegriffes h in einer Mischkette φ die rechte Seite einer Produktion, deren linke Seite h ist, eingesetzt wird, dann sagt man, daß die entstehende Mischkette ψ durch φ „unmittelbar generiert" wird. Und eine Mischkette ψ wird durch eine Mischkette φ „generiert", wenn es eine Mischkettenfolge

$$\varphi_1 = \varphi, \varphi_2, \ldots, \varphi_r = \psi$$

gibt, worin für $i = 2, \ldots, r$ jedes φ_i durch φ_{i-1} unmittelbar generiert wird.

Die terminalen Generierten eines Kategorienamens (auch seine „Werte" genannt) bilden die mit diesem Namen bezeichnete Kategorie.

§2. Die zweistufige Grammatik

In der neuen Programmierungsprache „Algol 68" wird eine „zweistufige (Satzstruktur-) Grammatik" benutzt. Darin wird als zweite Stufe eine solche Verallgemeinerung der Satzstruktur-Grammatik verwendet, worin die Produktionenmenge unendlich ist, die Menge der ausgezeichneten Kategorienamen aber endlich bleibt. Zur Angabe der unendlich vielen Produktionen dient die sog. „Metasprache" der ersten Stufe, die durch eine Satzstruktur-Grammatik in ursprünglichem Sinn generiert wird. Genauer wird eine zweistufige Grammatik durch fünf nicht-leere endliche Mengen

$$Z, M, P, V, K$$

bestimmt, wo Z (die Menge der „Zeichen") das terminale Vokabular, M (die Menge der „Metazeichen") das Hilfsvokabular, P (die Menge der „Metaproduktionen") die Produktionenmenge der Metasprache ist, worin alle Hilfsbegriffe als „ausgezeichnete" gelten; K enthält die ausgezeichneten Kategorienamen der zweiten Stufe, und V die sog. „Vorproduktionen" enthält, aus welchen die Produktionen der zweiten Stufe entstehen. Und zwar auf folgende Weise:

Die rechtseitigen Glieder der Produktionen werden hier durch Kommata getrennt; und endliche Folgen, deren Glieder durch Kommata ge-

trennt sind, werden „Listen" genannt. Listen werde ich mit großen griechischen Buchstaben bezeichnen. So ist jede Vorproduktion der Form:

$$\vartheta := \Theta ,$$

wo ϑ eine Mischkette, und Θ eine Mischkettenliste ist. Aus einer Vorproduktion entsteht eine Produktion, wenn für jedes der darin vorkommenden Metazeichen, überall, wo es auftritt, einer seiner (allgemein unendlich vielen) Werte eingesetzt wird. Geschieht das für alle Vorproduktionen auf aller möglichen Weise, dann erhält man alle Produktionen der zweiten Stufe.

Die linken Seiten der Produktionen werden „potenzielle Kategorienamen", kürzer „Hilfsbegriffe" genannt; diese bilden das Hilfsvokabular der zweiten Stufe. Als „Kategorienamen" gelten nur endlich viele unter den potenziellen; diese bilden die Menge K.

Jene Glieder der rechten Seiten der Produktionen, die in keiner Produktion als linke Seite auftreten, werden „terminale Begriffe" genannt, und bilden das terminale Vokabular der zweiten Stufe.

So sind die linken Seiten der Produktionen Zeichenketten, und die rechten Seiten Zeichenkettenlisten, deren Glieder teils Hilfsbegriffe, teils terminale Begriffe sind. Eine Zeichenkettenliste, die aus lauter terminalen Begriffen besteht, wird „terminal" genannt.

Die weiteren Begriffe der einstufigen Grammatik lassen sich fast wörtlich übertragen:

Wenn für ein Vorkommen (als Glied) eines Hilfsbegriffes ϑ in einer Zeichenkettenliste Θ_1 die rechte Seite einer Produktion, deren linke Seite ϑ ist, eingesetzt wird, dann sagt man, daß die entstandene Zeichenkettenliste Θ_2 durch Θ_1 „unmittelbar generiert" wird. Und eine Zeichenkettenliste Θ_r wird durch eine Zeichenkettenliste Θ_1 „generiert", wenn es eine Zeichenkettenlistenfolge

$$(*) \qquad \Theta_1 : \Theta_2 ; \ldots ; \Theta_r$$

gibt, worin für $i = 2, \ldots, r$ jedes Θ_i durch Θ_{i-1} unmittelbar generiert wird.

Die terminalen Generierten eines Elementes k von K bilden die mit diesem Namen bezeichnete Kategorie.

Unter der „zweistufig generierten Sprache" wird die Zuordnung der Kategorien zu ihren Namen verstanden.

§3. Zielsetzung

Die Produktionen sind Definitionsgleichungen (im Sinne der „möglichen Gleichheit"), wobei zur Definition eines Begriffes eventuell auch dieser selbst, als ein Glied der rechten Seite verwendet wird; allgemein entstehen hier Definitionen mehrerer Begriffe mit gegenseitiger Berufung je eines Begriffes auf sich und auf die anderen. So scheinen diese Definitionen zir-

kelhaft zu sein. Sind sie nicht zirkelhaft, so handelt es sich hier um Rekursionen (genauer um simultanen Rekursionen). Der Ziel dieser Arbeit ist, die rekursiven Definitionen der Begriffe der zweistufigen Grammatiken anzugeben, soweit dies gelingt.

§4. Einiges über primitiv-rekursive Wortfunktionen

1. Die betrachteten grammatischen Gebilde sind alle Ketten gewisser Zeichen (unter „Zeichen" jetzt nicht nur „Element von Z" gemeint). Als solche können sie als Elemente einer sog. „Wortemenge" betrachtet werden.

Die Wortemengen sind Spezialfälle der „zahlenartig aufbaubaren" abstrakten Mengen, auf die ich anläßlich des Internationalen Symposiums der Grundlagen der Mathematik in Warszawa (3. IX. 1959) die Theorie der zahlentheoretischen rekursiven Funktionen verallgemeinert habe. Die natürlichen Zahlen entstehen von 0 ausgehend durch sukzessiver Anwendung der „Nachfolgerfunktion" $n+1$. In den genannten „zahlenartig aufbaubaren" Mengen übernehmen beliebigviele Elemente die Rolle von 0, und beliebigviele Funktionen die Rolle der Nachfolgerfunktion.

Eine „Wortemenge W über ein Alphabet A" – wobei A eine beliebige nicht-leere Menge ist, deren Elemente „Buchstaben" genannt werden – ist die Menge der („Worte" genannten) endlichen Ketten aus Elementen von A. Die Rolle der 0 übernimmt hier die durch Λ bezeichnete leere Kette, und die Rolle der Nachfolgerfunktion spielen hier die Anknüpfungen der einzelnen „Buchstaben" an das Ende eines Wortes; so ist hier xa für jedes $a \in A$ je eine Nachfolgerfunktion. Als „Vorgänger" eines Wortes x gelten seine zusammenhängende Teile; unter diesen werden der nach Weglassen des letzten bzw. des ersten Buchstaben übrigbleibende „Anfangsteil" bzw. „Endteil"

$$\mathrm{at}(x) \quad \text{bzw.} \quad \mathrm{et}(x)$$

seine „unmittelbare Vorgänger" genannt (da jeder der echten Vorgänger von x in wenigstens einem von beiden als Vorgänger enthalten ist).

Wegen dem Aufbau der natürlichen Zahlen ist in der Theorie der natürlichen Zahlen die sich bietende Methode zur Definition einer Funktion φ die „primitive Rekursion", wobei der Wert von φ für 0 angegeben wird, und ausserdem die Art, wie an einer Stelle φ mit Verwendung seines am unmittelbaren Vorgänger dieser Stelle angenommenen Wertes berechnet werden kann. Dementsprechend ist das Definitionschema der primitiven Rekursion in der Wortemenge, bei endlichem Alphabet, worauf ich mich hier beschränken kann:

(D)

$$f(x, x_1, \ldots, x_r) = \begin{cases} g_0(x_1, \ldots, x_r), \text{ falls } x = \Lambda \\ g(x, x_1, \ldots, x_r, f(\mathrm{at}(x), x_1, \ldots, x_r), f(\mathrm{et}(x), x_1, \ldots, x_r)) \text{ sonst,} \end{cases}$$

wo g_0 und g bereits definierte Funktionen sind.

Eine Funktion wird primitiv-rekursiv genannt, wenn sie von gewissen „Ausgangsfunktionen" ausgehend durch endlichmalige Verwendung von Substitutionen und primitiven Rekursionen definiert werden kann.

Die Hinzunahme „fiktiver Variablen", von welchen die betreffende Funktion nicht tatsächlich abhängt, wird immer gestattet.

Eine Beziehung

$$B(x_1, \ldots, x_r)$$

unter Worte $x_1, \ldots, x_r$ (die für $r = 1$ eine Eigenschaft des Wortes x_1 bedeutet) gilt als primitiv-rekursiv, wenn dazu eine primitiv-rekursive „charakteristische Funktion" angegeben werden kann, d.h. eine folgenderweise definierte primitiv-rekursive Funktion:

$$b(x_1, \ldots, x_r) = \begin{cases} \Lambda, & \text{falls } B(x_1, \ldots, x_r) \text{ besteht} \\ a_1 & \text{sonst}, \end{cases}$$

wobei a_1 ein festgewähltes Element von A ist.

2. Ohne Beweis teile ich die Folgenden mit:

In den Wortemengen ist die Nacheinandersetzung xy zweier Worte immer eine primitiv-rekursive Funktion; die Gleichheit $x=y$ und das „Vorgängersein" $x \preceq y$ sind immer primitiv-rekursive Beziehungen.

Samt einer Beziehung $B(x_1, \ldots, x_r)$ ist auch ihre Negation $\bar{B}(x_1, \ldots, x_r)$ primitiv-rekursiv. Samt $B_1(y, x_1, \ldots, x_r)$ ist auch die Beziehung

$$B_2(x_1, \ldots, x_r, x) = (\mathrm{E}y)\,[y \preceq x \;\&\; B_1(y, x_1, \ldots, x_r)]$$

primitiv-rekursiv, wobei die rechte Seite folgendes bedeutet: „Es gibt ein Vorgänger y von x, für welchen

$$B_1(y, x_1, \ldots, x_r)$$

besteht". Ohne einer Schranke (wie hier x) für y kann von Rekursivität einer solchen „es gibt"-Beziehung garnichts gesagt werden, nicht einmal im weitesten Ausbreitung des Sinnes der Rekursion, die alle bisher betrachteten Rekursionsarten umfasst.

Samt der Beziehungen $B_1(x_1, \ldots, x_r)$ und $B_2(x_1, \ldots, x_r)$ ist auch ihre Disjunktion

$$B_1(x_1, \ldots, x_r) \vee B_2(x_1, \ldots, x_r)$$

primitiv-rekursiv, wobei das Zeichen „$\vee$" als „oder" zu lesen ist (im Sinne des lateinischen „vel", das auch das Zutreffen beider Beziehungen gestattet).

Ferner ist samt $B(y, x_1, \ldots, x_r)$ auch die Funktion

$$f(x_1, \ldots, x_r, x) = \mu_y\,[y \preceq x \;\&\; B(y, x_1, \ldots, x_r)]$$

primitiv-rekursiv, welche einen bestimmten Vorgänger y von x angibt, für den

$$B(y, x_1, \ldots, x_r)$$

besteht, falls ein solches y unter den Vorgängern von x existiert; und sonst den Wert a_1 annimmt.

Sind

$$g_1(x_1, \ldots, x_r), \ldots, g_k(x_1, \ldots, x_r), g_{k+1}(x_1, \ldots, x_r)$$

primitiv-rekursive Funktionen, und

$$B_1(x_1, \ldots, x_r), \ldots, B_k(x_1, \ldots, x_r)$$

einander gegenseitig ausschließende primitiv-rekursive Bezeihungen, so ist auch die durch den folgenden Fallunterscheidungen definierte Funktion primitiv-rekursiv:

$$f(x_1, \ldots, x_r) = \begin{cases} g_1(x_1, \ldots, x_r), \text{ falls } B_1(x_1, \ldots, x_r) \text{ besteht} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_k(x_1, \ldots, x_r), \text{ falls } B_k(x_1, \ldots, x_r) \text{ besteht} \\ g_{k+1}(x_1, \ldots, x_r) \text{ sonst.} \end{cases}$$

Es gibt auch allgemeinere Rekursionsformen, und unter diesen auch solche, die auf primitive Rekursionen und Substitutionen zurückgeführt werden können. Beispiele für die letzteren sind:

1. die Wertverlaufsrekursion, wobei zur Angabe des Funktionswertes an einer von Λ verschiedenen Stelle nicht nur an unmittelbaren, sondern an beliebigen Vorgängern dieser Stelle angenommenen Werte der Funktion (beliebige Werte aus seinem vorherigen Wertverlauf) herangezogen werden;

2. Rekursionen, wobei die „Parameter" (die neben der Rekursionsvariable x auftretenden anderen Variablen, wie $x_1, \ldots, x_r$ in der Definition (D)) nicht unverändert bleiben, sondern Einsetzungen für sie erfolgen.

Die wichtige Rolle spielende „Ordnung" $o(x)$ eines Wortes x: die Anzahl seiner Buchstaben, ist eine natürliche Zahl. Daher müssen auch die natürlichen Zahlen in jeder Wortemenge vertreten werden. Die Abzählung kann durch Zuordnung des festgewählten Buchstaben a_1 zu den Buchstaben von x geschehen: daher können als Vertreter der natürlichen Zahlen

$$0, \quad 1, \quad 2, \quad 3, \ldots$$

die Worte

$$\Lambda, \quad a_1, \quad a_1a_1, \quad a_1a_1a_1, \ldots$$

gewählt werden. Nach diesem Übereinkommen ist $o(x)$ primitiv-rekursiv; x ist dann und nur dann eine natürliche Zahl, wenn

$$x = o(x)$$

gilt; und auch jede primitiv-rekursive zahlentheoretische Funktion kann durch eine primitiv-rekursive Funktion der Wortemenge vertreten werden. Genauer gilt für eine beliebige Wortemenge der

SATZ: *Es gibt zu jeder primitiv-rekursiven zahlentheoretischen Funktion φ eine primitiv-rekursive Wortfunktion f (ihr „Vertreter"), die an jeder Stelle nur von den Ordnungen der Argumente (also von natürlichen Zahlen) abhängt, und auch als Werte natürliche Zahlen annimmt; ferner so geschaffen ist, daß die für natürlichen Zahlen angenommenen Werte von f und φ übereinstimmen.*

Die Vertreter der zahlentheoretischen Funktionen — als Funktionen der Ordnungen ihrer Argumente — werde ich in der Wortemenge genau so bezeichnen, wie die betreffenden zahlentheoretischen Funktionen.

Übrigens werden nur die im folgenden Hilfsatz auftretenden beiden zahlentheoretischen Funktionen zur Verwendung kommen.

HILFSATZ aus der Zahlentheorie: Es können zwei primitiv-rekursive Funktionen

$$\sigma_1(n) \quad \text{und} \quad \sigma_2(n)$$

folgender Eigenschaft definiert werden: man kann sämtliche Paare natürlicher Zahlen derart in eine Folge ordnen, daß das n-te Glied dieser Folge das Paar

$$(\sigma_1(n), \sigma_2(n))$$

ist.

Ich bemerke noch Folgendes: Wie eine Funktion $f(n)$, deren Variable die natürlichen Zahlen durchläuft, auch in der Form f_n geschrieben, und als die unendliche Folge

$$f_0, f_1, f_2, f_3, \ldots$$

betrachtet wird, kann auch eine Wortfunktion $f(x)$, die nur von $o(x)$ abhängig ist, auch in der Form

$$f(o(x)) = f_{o(x)}$$

geschrieben, und als die unendliche Folge

$$f_\Lambda, f_{a_1}, f_{a_1 a_1}, f_{a_1 a_1 a_1}, \ldots$$

betrachtet werden.

§5. Primitiv-rekursive Definitionen für gewisse Begriffe der zweistufigen Grammatik

1. An den folgenden, sich natürlicher Weise bietenden Definitionen ist nicht immer unmittelbar zu sehen, daß sie primitiv-rekursive Wortfunktionen definieren; aber die im **§4** Mitgeteilten ermöglichen ihre Zurückführung auf primitive Rekursionen und Substitutionen. Dazu möchte ich bemerken, daß in Definitionen, wo die Angabe des Funktionswertes für ein Argument Λ zu fehlen scheint, dies implizit enthalten ist.

Für die Untersuchung der im **§2** durch die Mengen

$$Z, M, P, V, K$$

bestimmten zweistufigen Grammatik eignet sich die Wortemenge über das Alphabet Z', das die Elemente von Z, und dazu noch die Hilfszeichen ":=" und "," und ";" enthält.

Seien die Elemente von Z (die „Zeichen"):

$$z_1, z_2, \ldots, z_t$$

und sei z_1 das festgewählte Element von Z'. Dann können die charakteristischen Funktionen

$$z(x), \quad \mathrm{zk}(x), \quad \mathrm{zkl}(x), \quad \mathrm{zklf}(x)$$

der Eigenschaften

„ein Zeichen,
bzw. eine Zeichenkette,
bzw. eine Zeichenkettenliste,
bzw. eine Zeichenkettenlistenfolge zu sein"

der Reihe nach durch folgende Definitionen (die vom zweiten an auf Wertverlaufsrekursionen führen) angegeben werden:

$$z(x) = \begin{cases} \Lambda, \text{ falls } x = z_1 \vee x = z_2 \vee \ldots \vee x = z_t \\ z_1 \text{ sonst,} \end{cases}$$

$$\mathrm{zk}(x) = \begin{cases} \Lambda, \text{ falls } z(x) = \Lambda \vee (\mathrm{E}y_1)(\mathrm{E}y_2)[y_1, y_2 \preceq x \,\&\, \mathrm{zk}(y_1) = \\ \qquad = \Lambda \,\&\, z(y_2) = \Lambda \,\&\, x = y_1 y_2] \\ z_1 \text{ sonst,} \end{cases}$$

$$\mathrm{zkl}(x) = \begin{cases} \Lambda \text{ falls } \mathrm{zk}(x) = \Lambda \vee (\mathrm{E}y_1)(\mathrm{E}y_2)[y_1, y_2 \preceq x \,\&\, \mathrm{zkl}(y_1) = \\ \qquad = \Lambda \,\&\, \mathrm{zk}(y_2) = \Lambda \,\&\, x = y_1, y_2] \\ z_1 \text{ sonst,} \end{cases}$$

$$\mathrm{zklf}(x) = \begin{cases} \Lambda, \text{ falls } \mathrm{zkl}(x) = \Lambda \vee (\mathrm{E}y_1)(\mathrm{E}y_2)[y_1 y_2 \preceq x \,\&\, \mathrm{zklf}(y_1) = \\ \qquad = \Lambda \,\&\, \mathrm{zkl}(y_2) = \Lambda \,\&\, x = y_1; y_2] \\ z_1 \text{ sonst.} \end{cases}$$

Ebenso sukzessiv können die Zeichen, die Zeichenketten, die Zeichenkettenlisten und die Zeichenkettenlistenfolgen zu primitiv-rekursive Folgen geordnet werden (wobei Wiederholungen zugelassen sind):

Einheitlichkeitshalber seien die endlich vielen Zeichen in die unendliche Folge

$$z_1, z_2, \ldots, z_t, z_t, z_t, \ldots$$

geordnet, die als

$$z_{o(x)} = \begin{cases} \Lambda, & \text{falls } x = \Lambda \\ z_1, & \text{falls } o(x) = z_1 \\ z_2, & \text{falls } o(x) = z_1 z_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_t, & \text{falls } o(x) = \underbrace{z_1 z_1 \ldots z_1}_{\text{t-mal}} \end{cases}$$

definiert werden kann.

Es gilt nun allgemein, daß die aus Gliedern einer primitiv-rekursiven Folge $v_{o(x)}$ gebildeten endlichen Folgen in eine primitiv-rekursive Folge $w_{o(x)}$ geordnet werden können. Die eingliedrigen unter diesen sind ja bereits in eine primitiv-rekursive Folge $v_{o(x)}$ geordnet. Und angenommen, daß für ein $n \geq 1$ die n-gliedrigen bereits in eine primitiv-rekursive Folge

$$w_{o(x)}^{(n)}$$

geordnet sind, können die $(n+1)$-gliedrigen als

$$w_{o(x)}^{(n)} * v_{o(y)} \quad (o(x), o(y) = 1, 2, \ldots)$$

aufgeschrieben, und so gemäß des Hilfsatzes im **§4** in die primitiv-rekursive Folge

$$w_{o(x)}^{(n+1)} = w_{\sigma_1(o(x))}^{(n)} * v_{\sigma_2(o(x))}$$

geordnet werden, wobei der Stern $*$ den Trennzeichen zwischen den Gliedern (also "," oder ";") bezeichnet. Demnach ergibt sich

$$w_{o(x)}^{(o(y))}$$

durch die Definition

$$w_{o(x)}^{(o(y))} = \begin{cases} \Lambda, & \text{falls } y = \Lambda \\ v_{o(x)}, & \text{falls } o(y) = z_1 \\ w_{\sigma_1(o(x))}^{(o(\mathrm{at}(y)))} * v_{\sigma_2(o(x))} & \text{sonst} \end{cases}$$

als eine primitiv-rekursive – nur von $o(x)$ und $o(y)$ abhängige – Funktion von x und y. Sie nimmt für jedes

$$o(y) = z_1,\ z_1 z_1,\ z_1 z_1 z_1, \ldots$$

die aus Gliedern der Folge $v_{o(x)}$ gebildeten $o(y)$-gliedrigen Folgen als Werte an. All diese können nun durch abermalige Anwendung des Hilfsatzes in eine primitiv-rekursive Folge

$$w_{o(x)} = w_{\sigma_1(o(x))}^{(\sigma_2(o(x)))}$$

geordnet werden.

Bezeichne $f_{o(x)}$ die derart erhaltene primitiv-rekursive Folge sämtlicher Zeichenkettenlistenfolgen, die später zur Verwendung kommen wird.

Die charakteristischen Funktionen

$$\mathrm{lg}(x, y) \text{ und } \mathrm{fg}(x, y)$$

der Beziehungen „y ist in der Liste bzw. in der Listenfolge x eines der Glieder" können durch die folgenden Fallunterscheidungen definiert werden:

$$\mathrm{lg}(x, y) =$$

$$= \begin{cases} \Lambda, \text{ falls } \mathrm{zkl}(x) = \Lambda \,\&\, \mathrm{zk}(y) = \Lambda \,\&\, (x = y \vee (\mathrm{E}u)[u \preceq x \,\&\, x = u, y] \vee \\ \qquad \vee (\mathrm{E}u)[u \preceq x \,\&\, x = y, u] \vee (\mathrm{E}u_1)(\mathrm{E}u_2)[u_1, u_2, \preceq x \,\&\, x = u_1, y, u_2]) \\ z_1 \text{ sonst,} \end{cases}$$

und

$$\mathrm{fg}(x, y) =$$

$$= \begin{cases} \wedge, \text{ falls } \mathrm{zklf}(x) = \wedge \,\&\, \mathrm{zkl}(y) = \wedge \,\&\, (x = y \vee (\mathrm{E}u)\,[u \preceq x \,\&\, x = u; y] \vee \\ \qquad \vee (\mathrm{E}u)\,[u \preceq x \,\&\, x = y; u] \vee (\mathrm{E}u_1)\,(\mathrm{E}u_2)\,[u_1, u_2 \preceq x \,\&\, x = u_1; y; u_2]) \\ z_1 \text{ sonst.} \end{cases}$$

Das letzte Glied $\mathrm{lfg}(x)$ einer Zeichenkettenlistenfolge x ergibt sich durch die Definition:

$$\mathrm{lfg}(x) = \mu_y[y \preceq x \,\&\, \mathrm{fg}(x, y) = \wedge \,\&\, (x = y \vee (\mathrm{E}u)\,(u \preceq x \,\&\, x = u; y]))\,.$$

2. Bisher handelte es sich nur um Elemente von Z. In den folgenden Definitionen nehmen auch Elemente der anderen, unsere Grammatik bestimmenden Mengen teil.

Seien die Elemente von M (die „Metazeichen"):

$$m_1, m_2, \ldots, m_r$$

und bezeichne

$$m_i(x) \quad (i = 1, 2, \ldots, r)$$

die charakteristische Funktion der Eigenschaft: „ein Wert von m_i zu sein", d.h. „in die durch m_i bezeichnete Kategorie der durch die Satzstruktur-Grammatik (im ursrünglichen Sinn) der ersten Stufe generierten Sprache zu gehören". Von diesen habe ich in einer früheren Arbeit[2] bewiesen, daß sie – falls die Grammatik derart zirkelfrei angegeben wurde, daß kein Hilfsbegriff sich selber generiert – in der Wortemenge über das terminale Vokabular als Alphabet – das hier Z ist – primitiv-rekursiv sind. Dasselbe gilt auch in der Wortemenge über das erweiterte Alphabet Z'; man hat nur die Beziehung, bei welcher eine solche charakteristische Funktion für x als $\wedge$ definiert wurde, mit „$\& \,\mathrm{zk}(x) = \wedge$" („und x ist eine Zeichenkette") zu ergänzen.

Seien ferner die Elemente von V (die „Vorproduktionen"):

$$\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_s.$$

Aus diesen entstehen die Produktionen der zweiten Stufe durch Einsetzen von Werten der Metazeichen. Bezeichne

$$\mathrm{lp}_i(x) \quad (i = 1, 2, \ldots, s)$$

die charakteristische Funktion der Eigenschaft: „die linke Seite einer aus v_i entstandenen Produktion zu sein", und

$$\mathrm{lpr}_i(x, y) \quad (i = 1, 2, \ldots, s)$$

die charakteristische Funktion der Beziehung: x und y treten als linke

[2] R. PÉTER, Über die Rekursivität der Begriffe der mathematischen Grammatiken, *Publ. of the Math. Inst. of the Hung. Ac. of Sci.*, 8 (1963), 213–228.

bzw. rechte Seite derselben aus v_i entstandenen Produktion auf". Ist z. B. v_1 die Vorproduktion

$$z_1 m_1 m_2 := z_2 m_1 z_3 z_2, z_1 m_3$$

so ergeben sich $\mathrm{lp}_1(x)$ und $\mathrm{lpr}_1(x, y)$ mit Benutzung der Funktionen

$$m_1(x),\ m_2(x),\ m_3(x)$$

durch die Fallunterscheidungen:

$$\mathrm{lp}_1(x) = \begin{cases} \Lambda, \text{ falls } (\mathrm{E}y_1)(\mathrm{E}y_2)[y_1, y_2 \preceq x \,\&\, m_1(y_1)] = \Lambda \,\&\, m_2(y_2) = \\ \qquad = \Lambda \,\&\, x = z_1 y_1 y_2 \\ z_1 \text{ sonst} \end{cases}$$

und

$$\mathrm{lpr}_1(x, y) = \begin{cases} \Lambda, \text{ falls } (\mathrm{E}u_1)(\mathrm{E}u_2)(\mathrm{E}u_3)[u_1, u_2 \preceq x \,\&\, u_3 \preceq y \,\&\, m_1(u_1) = \\ \qquad = \Lambda \,\&\, m_2(u_2) = \Lambda \,\&\, m_3(u_3) = \Lambda \,\&\, x = z_1 u_1 u_2 \,\&\, y = \\ \qquad = z_2 u_1 z_3 z_2, z_1 u_3] \\ z_1 \text{ sonst.} \end{cases}$$

Mit diesen können die charakteristischen Funktionen

$$\mathrm{lp}(x) \text{ bzw. } \mathrm{lpr}(x, y)$$

der Eigenschaft: „die linke Seite einer Produktion zu sein" (was mit „ein potenzieller Kategoriename zu sein" gleichbedeutend ist), bzw. der Beziehung: „x und y treten als linke bzw. rechte Seite derselben Produktion auf" durch die folgenden Fallunterscheidungen definiert werden:

$$\mathrm{lp}(x) = \begin{cases} \Lambda, \text{ falls } \mathrm{lp}_1(x) = \Lambda \vee \mathrm{lp}_2(x) = \Lambda \vee \ldots \vee \mathrm{lp}_s(x) = \Lambda \\ z_1 \text{ sonst,} \end{cases}$$

$$\mathrm{lpr}(x, y) = \begin{cases} \Lambda, \text{ falls } \mathrm{lpr}_1(x, y) = \Lambda \vee \mathrm{lpr}_2(x, y) = \Lambda \vee \ldots \vee \mathrm{lpr}_s(x, y) = \Lambda \\ z_1 \text{ sonst.} \end{cases}$$

(Genau so einfach könnte auch die Eigenschaft: „ein Glied der rechten Seite einer Produktion zu sein", und damit dann der Eigenschaft: „ein terminaler Begriff zu sein" als primitiv-rekursive Funktion definiert werden; aber dies wird nicht notwendig sein.)

Eine Zeichenkettenliste kann „potenziell terminal" genannt werden, wenn keines ihrer Glieder unter den linken Seiten der Produktionen vorkommt. So kann „lp" zur folgenden Definition der charakteristischen Funktion $t(x)$ der Eigenschaft: „eine potenziell terminale Zeichenkettenliste zu sein" benutzt werden:

$$t(x) = \begin{cases} \Lambda, \text{ falls } \mathrm{zkl}(x) = \Lambda \,\&\, (\overline{\mathrm{E}y})[y \preceq x \,\&\, \mathrm{lg}(x, y) = \Lambda \,\&\, \mathrm{lp}(y) = \Lambda] \\ z_1 \text{ sonst} \end{cases}$$

(wobei $(\overline{\mathrm{E}y})$ die Negation von $(\mathrm{E}y)$, also „es gibt kein solches y" bedeutet).

Und – da bei dem unmittelbaren Generieren das Ersetzen der als Glied einer Liste vorkommenden linken Seite einer Produktion durch die rechte Seite dieser Produktion geschieht – wird „lpr" in folgender Definition der charakteristischen Funktion $\mathrm{ug}(x, y)$ der Beziehung: „y ist eine unmittelbare Generierte der Zeichenkettenliste x" verwendet:

$$\mathrm{ug}(x, y) = \begin{cases} \Lambda, \text{ falls } \mathrm{zkl}(x) = \Lambda \,\&\, (\mathrm{E}u_1)(\mathrm{E}u_2)(\mathrm{E}u_3)(\mathrm{E}u_4)[u_1, u_2, u_3 \preceq \\ \qquad \preceq x \,\&\, u_4 \preceq y \,\&\, \mathrm{lg}(x, u_2) = \Lambda \,\&\, \mathrm{lpr}(u_2, u_4) = \Lambda \,\&\, x = \\ \qquad = u_1 u_2 u_3 \,\&\, y = u_1 u_4 u_3] \\ z_1 \text{ sonst.} \end{cases}$$

Damit kann die charakteristische Funktion

$$\mathrm{gf}(x, y)$$

der Beziehung: „y ist eine mit x beginnende generierende Folge der Art $(*)$" durch die auf Wertverlaufsrekursion führende Definition

$$\mathrm{gf}(x, y) = \begin{cases} \Lambda, \text{ falls } \mathrm{zkl}(x) = \Lambda \,\&\, \mathrm{zklf}(y) = \Lambda \,\&\, (y = x \vee (\mathrm{E}u_1)(\mathrm{E}u_2)[u_1, u_2 \preceq \\ \qquad \preceq y \,\&\, \mathrm{gf}(x, u_1) = \Lambda \,\&\, \mathrm{zkl}(u_2) = \Lambda \,\&\, y = u_1; u_2 \,\& \\ \qquad \& \,\mathrm{ug}(\mathrm{lfg}(u_1), u_2) = \Lambda]) \\ z_1 \text{ sonst} \end{cases}$$

angegeben werden.

§6. Offene Frage

1. In den Bisherigen haben sich alle Begriffe, die zur exakten Fassung der grundlegenden Frage der durch eine zweistufige Grammatik generierten Sprache nötig sind, als primitiv-rekursiv erwiesen. Diese Frage lautet ja: „Wenn k ein zu K gehöriger Kategoriename ist, wie kann entschieden werden, was in die durch ihn bezeichnete Kategorie gehört?" Dies gilt für ein x, wenn x eine potenziell terminale Zeichenkettenliste ist, und es eine mit k beginnende und mit x endigende generierende Zeichenkettenlistenfolge gibt; d.h. wenn

$$t(x) = \Lambda \,\&\, (\mathrm{E}y)\,[\mathrm{gf}(k, y) = \Lambda \,\&\, \mathrm{lfg}(y) = x]$$

besteht. Wäre die charakteristische Funktion auch dieser Eigenschaft rekursiv, dann würde die durch die betrachtete Grammatik generierte Sprache als rekursiv gelten.

Für das y in diesem $(\mathrm{E}y)\,[\ldots]$ kann ich aber keine Schranke angeben. Und eine schrankenlose Beziehung $(\mathrm{E}y)\,[\ldots]$ ist allgemein in keinem Sinne rekursiv.

Es zeigt sich eine Ähnlichkeit der Generierung der betrachteten Sprache mit der Generierung der Theoreme einer axiomatischen mathematischen Theorie. Dort ist eine Formel f ein Satz der Theorie, wenn es eine Formelnfolge gibt, die mit Axiomen beginnt, und mit f endigt; wobei jedes Glied der Folge aus früheren Gliedern durch Anwendung gewisser

Schlussregeln „generiert" wird. Und allgemein ist eine mathematische Theorie nicht „rekursiv entscheidbar": die dazu gehörige „es gibt eine derartige Folge"-Beziehung ist allgemein nicht rekursiv.

Die Rekursivität des Spezialfalls „Algol 68" ist auch eine offene Frage.

„Primitiv-rekursiv abzählbar" ist eine auf der beschriebenen Art zweistufig generierte Sprache jedenfalls; in dem Sinne, dass primitiv-rekursive Funktionen definiert werden können, welche als Werte alle terminalen Generierten je eines Kategorienamens und nur diese annehmen. Mit Heranziehung der primitiv-rekursiven Folge $f_{o(x)}$, in welche in Nr. **1** des **§5** sämtliche Zeichenkettenlistenfolgen geordnet wurden, tut dies für ein $k \in K$ die derart definierte Funktion:

$$k(x) = \begin{cases} \mathrm{lfg}(f_{o(x)}), \text{ falls } t(\mathrm{lfg}(f_{o(x)})) = \Lambda \,\&\, \mathrm{gf}(k, f_{o(x)}) = \Lambda \\ \Lambda \text{ sonst.} \end{cases}$$

Im Weiterbildung vom „Algol 60" war vorstellbar, daß man sich auf eine solche zweistufige Grammatik beschränkt, die nur endlich viele terminalen Begriffe enthält.

Wie ich früher bewiesen habe[3], wäre die durch eine solche Grammatik generierte Sprache auch einstufig, durch eine Satzstruktur-Grammatik im ursprünglichen Sinn generierbar, und so — wenn nicht zirkelhaft — dann primitiv-rekursiv.

Das gilt aber nicht für jede zweistufig generierte Sprache. In der selben Arbeit habe ich dafür ein sehr einfaches Gegenbeispiel angegeben. Die betreffende zweistufig generierte Sprache, die einstufig nicht generiert werden kann, ist dabei primitiv-rekursiv.

[3] R. PÉTER, Zur zweistufigen Satzstruktur Grammatik II, *Studia Sci. Math. Hung.*, 3 (1968), 181–194.

EMPILEMENT DE CERCLES CONGRUENTS DANS UN DOMAINE SPHERIQUE CONVEXE

Par

T. MALÉ

Centre Pédagogique Supérieur, Bamako

(Reçu 24 décembre 1971)

Nous disons qu'un domaine sphérique est convexe s'il est sur une demi-sphère et si deux quelconques de ses points peuvent être joints par un arc de grand cercle contenu dans D.

Etant donné dans un domaine sphérique convexe D un système dénombrable $\{C_i\}$ de cercles disjoints, la grandeur $\frac{\Sigma C_i}{D}$ est appelée densité d'empilement de ce système de cercles par rapport à D, ou simplement densité d'empilement.[1]

Dans un article,[2] J. MOLNÁR a démontré le théorème suivant:

Soit dans un domaine sphérique convexe un système d'au moins deux cercles congruents disjoints, alors la densité d'empilement est $< \frac{\pi}{\sqrt{12}} = 0{,}9069\ldots$.[3]

Dans notre article, en nous appuyant sur des résultats de L. FEJES–TÓTH,[4] de G. KRAMMER[5] et de J. MOLNÁR,[6] nous améliorons la borne supérieure de la densité d'empilement du résultat précédent, plus précisement nous démontrons le

[1] Nous désignons par le même symbole un domaine et son aire.

[2] J. MOLNÁR, Ausfüllung und Überdeckung eines konvexen sphärischen Gebietes durch Kreise I, *Publicationes Mathematicae*, 2 (1952), 266–275.

[3] Sous sa forme initiale ce théorème est plus général car MOLNÁR utilise la notion de convexité (au sens large) suivante: un domaine sphérique est dit convexe si deux quelconques de ses points peuvent être joints par un arc de grand cercle contenu dans D.

[4] L. FEJES–TÓTH, Über dichteste Kreislagerung und dünnste Kreisüberdeckung, *Comment. Math. Helvatici*, 23 (1949), 342–349.

[5] KRAMMER, G., Megjegyzések a gömbnek körökkel való kitöltéséhez és lefedéséhez, *Mat. Lapok*, 11 (1960), 120–123.

[6] J. MOLNÁR, Collocazioni di cerchi sulla superficie di curvatura constante, *Celebrazioni archimedee del secolo XX*, Siracusa, 1961, Vol. I, 61–72.

THÉORÉME. *Soit sur une sphère de rayon R un domaine convexe dans lequel on a un système d'au moins deux cercles disjoints de rayon r, alors la densité d'empilement est*

$$< d(a) = \frac{3 \operatorname{cosec} \frac{\pi}{a} - 6}{a-6}, \qquad d(6) = \lim_{a \to 6} d(a) = \frac{\pi}{\sqrt{12}} = 0{,}9069\ldots,$$

$$\text{où } \operatorname{cosec} \frac{\pi}{a} = 2 \cos \frac{r}{R}.$$

DÉMONSTRATION. Considérons sur la sphère un domaine D convexe. Soit dans D un système $\{C_i\}$ $(i \geq 2)$[7] de cercles de centres O_i et de rayon r. Sans perdre la généralité, nous pouvons supposer $R = 1$, c'est-à-dire que la sphère est unitaire.

Soit E l'enveloppe convexe des cercles. Naturellement $E \subseteq D$, ainsi $\frac{\Sigma C_i}{D} \leq \frac{\Sigma C_i}{E}$. Nous établirons notre théorème en démontrant que $\frac{\Sigma C_i}{E} < < d(a)$.

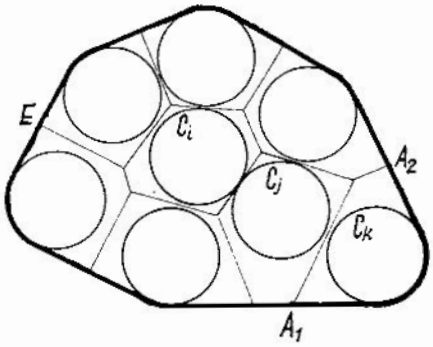

Fig. 1

Considérons le système $\{C_i\}$ de cercles et la décomposition Δ correspondante de la demi-sphère contenant D, en cellules de DIRICHLET.[8] $\Delta \cap E$ donne une décomposition de E en général en trois types de cellules S_i, S_j, S_k contenant chacune un cercle:

S_i est un polygone sphérique dont tous les sommets sont intérieurs à E;

S_j est un polygone sphérique ayant au moins un sommet sur la frontière de E;

S_k est un domaine convexe contenant un arc de cercle C_i (Fig. 1).

[7] L'hypothèse $i \geq 2$ implique naturellement $r \leq 45°$.

[8] La cellule de DIRICHLET correspondant au cercle C_i de centre O_i est le lieu des points de la demi-sphère contenant D plus proches de O_i que de tout autre centre du système.

G. L. DIRICHLET, Über die Reduction der positiven Formen mit drei unbestimmten ganzen Zahlen, *J. für die reine und angew. Math.*, **40** (1850), 209–227.

Notons par C_i, C_j et C_k les cercles du système respectivement contenus dans S_i, S_j et S_k. Pour démontrer $\frac{\Sigma C_i}{E} < d(a)$, il suffit d'établir que $\frac{C_i}{S_i} \leq d(a)$, $\frac{C_j}{S_j} \leq d(a)$ et $\frac{C_k}{S_k} < d(a)$. L'existence de S_k entraine l'inégalité stricte dans notre théorème.

a) Considérons trois cercles de rayon r ($r \leq 45°$) tangents entre eux et soit $\varrho\left(< \frac{\pi}{2}\right)$ la distance de leur centre radical au centre de l'un des trois cercles.

Les sommets de S_i sont à une distance $\geq \varrho$ du centre du cercle C_i. D'après un résultat de MOLNÁR[6]

$$\frac{C_i}{S_i} \leq d_M = \frac{3 \operatorname{cosec} \frac{\pi}{a} - 6}{[a] - 3 - \frac{6}{\pi} \operatorname{arc\,tg}\left\{\sqrt{3} \operatorname{tg} \frac{\pi}{a} \operatorname{cotg}\left(1 - \frac{[a]}{a}\right)\pi\right\}} \leq d_{F-T} =$$

$$= d(a) = \frac{3 \operatorname{cosec} \frac{\pi}{a} - 6}{a - 6},$$

$$\text{où } \operatorname{cosec} \frac{\pi}{a} = 2 \cos r.$$

b) Notons par A_1 un des sommets de S_j situé sur la frontière de E (Fig. 1) et par C_j^* le symétrique de C_j par rapport à la droite (grand cercle) frontière de E passant par A_1. Soit S_j' la cellule voisine de S_j ayant un sommet en A_1 et soit C_j' le cercle correspondant. A_1 est alors le centre radical des cercles C_j, C_j^* et C_j', c'est-à-dire $A_1O_j \geq \varrho$. En conséquence

$$\frac{C_j}{S_j} \leq d_M \leq \frac{3 \operatorname{cosec} \frac{\pi}{a} - 6}{a - 6}.$$

c) Allons d'un point commun de la frontière de C_k et de E, sur la frontière de S_k dans les deux sens opposés jusqu'à la rencontre des sommets A_1 et A_2 de S_k (Fig. 1). Avec un raisonnement analogue au cas *b)* on obtient que les angles en A_1 resp. en A_2 de l'enveloppe convexe E_k de A_1, A_2 et de C_k ne peuvent pas être obtus, ainsi

$$E_k \geq \pi - \left(2\pi - 4 \arcsin \frac{1}{\sqrt{2} \cos r}\right) \cos r,$$

avec l'égalité quand les angles de E_k en A_1 resp. en A_2 sont des angles droits.[9] Donc

$$\frac{C_k}{S_k} \leq \frac{C_k}{E_k} \leq d_1(r) = \frac{1-\cos r}{\frac{1}{2} - \left(1 - \frac{2}{\pi} \arcsin \frac{1}{\sqrt{2}\cos r}\right)\cos r}.$$

Pour achever la démonstration de notre théorème, il suffit d'établir que $d_1(r) \leq d(r)$, pour $0 \leq r \leq 45°$.

Selon les résultats de MOLNÁR[2] resp. de KRAMMER[5] $d_1(r)$ resp. $d(r)$ sont des fonctions décroissantes de r.

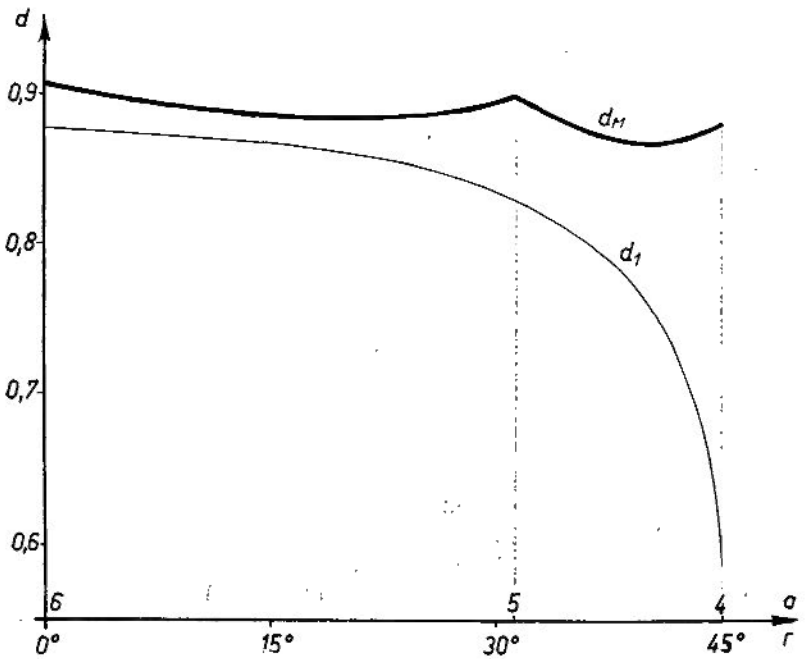

Fig. 2

Nous allons maintenant établir que $d_1(r) \leq d(r)$ pour $r \in [0,45°]$. Partageons cet intervalle en deux intervalles $[0,30°]$ et $[30°, 45°]$. Pour $r = 30°$ $(a = 5,1048\ldots)$ on a $d(a) = 0,8926\ldots$ et puisque $\lim_{r\to 0} d_1(r) = 0,8797\ldots$ on a $\lim_{r\to 0} d(r) < d(5,1048\ldots) = 0,8926\ldots$. De même pour $r = 45°$ $(a = 4)$ on a $d(4) = 0,8787\ldots$ et $d_1(r) = 0,8338\ldots$. D'où $d_1(30°) < d(4)$. D'où $d_1(r) < d(r)$ pour $0 \leq r \leq 45°$.

[9] voir FEJES-TÓTH, *Lagerungen in der Ebene auf der Kugel und im Raum*, Berlin – Göttingen – Heidelberg, 1953, p. 67.

REMARQUES. 1. Les calculs numériques montrent que

$$\frac{1-\cos r}{\frac{1}{2}-\left(1-\frac{2}{\pi}\arcsin\frac{1}{\sqrt{2}\cos r}\right)\cos r} < d_M$$

(Fig. 2). Ainsi dans notre théorème la borne $d(a)$ peut être ameliorée par d_M.

2. L'exemple suivant montre que notre théorème n'est pas valable si le domaine D est convexe au sens large. Considérons sur la sphère S le système $\{C_i\}$ de six cercles congruents disjoints de rayon 45°. D'après le théorème de FEJES–TÓTH[1] la densité d'empilement de ce système de cercles est $d(4)$. Soit C un cercle sur S tel que $C \cap \{C_i\} = \emptyset$. Naturellement $D = S - C$ est convexe au sens large, mais $\frac{\Sigma C_i}{S-C} = \frac{\Sigma C_i}{D} > d(4)$.

A NOTE ON A PAPER OF E. HLAWKA

By

I. KÁTAI

Department for Numerical Mathematic of the L. Eötvös University, Budapest

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In his paper [1] E. HLAWKA stated the following assertion. Let $\zeta_h = e^{2\pi i \varphi_h}$ $(h = 1, 2, \ldots)$ be independent random variables distributed uniformly on the unit-circle $|z| = 1$. Let

$$\omega_N(z) = \prod_{h=1}^{N} (z - \zeta_h). \tag{1}$$

Then with probability 1, holds the inequality

$$\overline{\lim_{N\to\infty}} \frac{\max_{|z|=1} |\omega_N(z)|}{e^{c(\log\log N)^2}} > 1, \tag{2}$$

c being a positive constant.

Our aim is to give an improvement of (2) which we can deduce almost immediately from the law of the iterated logarithm (see for example [2]).

E. HLAWKA deduced the inequality

$$|S_N| \overset{\text{def}}{=} \left|\sum_{h=1}^{N} \cos 2\pi \varphi_h\right| \leq 2 \log U_N \tag{3}$$

where

$$U_N = \max_{|z|=1} |\omega_N(z)|.$$

(see [1] pp. 106—107).

Take into account that $\mathrm{Re}\,\zeta_h = \cos 2\pi \eta_h$ are independent random variables, with the same distribution function, with mean value 0 and variance 1. Therefore we get

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-\frac{1}{2}} S_N = 1,$$

with probability 1. Thus we proved the following assertion:

$$\overline{\lim_{N\to\infty}} \frac{\max\limits_{|z|=1} |\omega_N(z)|}{e^{\frac{1}{2}(1-\varepsilon)(N \log\log N)^{\frac{1}{2}}}} \geq 1$$

for arbitrary constant $\varepsilon > 0$, with probability 1.

References

[1] HLAWKA, E., Interpolation analytischer Funktionen auf dem Einheitskreis, *Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an E. Landau*, VEB Deutscher Verlag der Wissenschaften, Berlin, (1968), 99–118.

[2] HARTMAN, P.–WINTNER, A., On the law of the iterated logarithm, *Amer. Journal of Math.*, **63** (1941), 169–176

ON SOME SUBSPACES OF HYPEREXTENSIONS

By

ÁKOS CSÁSZÁR

I. Department for Analysis of the L. Eötvös University, Budapest

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1. Introduction. In the paper [1], a superspace E^h, called hyperextension, was assigned to an arbitrary topological space E and an open subbase $\mathbf{S}$ in E, and various subspaces of E^h were studied. For some of them, a descriptive characterization was given up to equivalence, for others, a similar characterization was formulated for the spaces lying between E and the subspace in question. The purpose of this paper is to continue the investigations of [1] and, in particular, to give descriptive characterizations for E^h itself and another subspace E^d of E^h.

Let us first recall the definition of E^h. Let $\mathbf{S}$ be an open subbase in a topological space E with 0, $E \in \mathbf{S}$, and put

$$\mathbf{T} = \{E - S : S \in \mathbf{S}\}, \quad \mathbf{V} = \mathbf{S} \cup \mathbf{T}.$$

We consider ultra-$\mathbf{V}$-sieves in E. A sieve in E is a family $\mathbf{s}$ of subsets of E such that

$$A_1, A_2 \in \mathbf{s} \Rightarrow A_1 \cap A_2 \neq 0, \quad A \in \mathbf{s} \text{ and } A \subset A' \subset E \rightarrow A' \in \mathbf{s}.$$

For a system $\mathbf{M}$ of subsets of E, an $\mathbf{M}$-sieve $\mathbf{m}$ is a sieve such that, for $A \in \mathbf{m}$, there exists a set $M \in \mathbf{m} \cap \mathbf{M}$ with $M \subset A$. An ultra-$\mathbf{M}$-sieve is a maximal $\mathbf{M}$-sieve. Now, for $x \in E$, the sets $A \subset E$ such that $x \in V \subset A$ for a suitable $V \in \mathbf{V}$ constitute a trivial ultra-$\mathbf{V}$-sieve $\mathbf{v}(x)$. Let $E^h \supset E$ be a set such that, with the elements $x \in E^h - E$, one can associate all non-trivial ultra-$\mathbf{V}$-sieves in a one-to-one manner; let $\mathbf{v}(x)$ denote, for $x \in E^h - E$, the non-trivial ultra-$\mathbf{V}$-sieve associated with x.

Then put

$$v(A) = \{x \in E^h : A \in \mathbf{v}(x)\} \quad (A \subset E).$$

We equip E^h with the topology for which

$$\mathbf{S}^h = \{v(S) : S \in \mathbf{S}\}$$

is a subbase. Then E is a subspace of E^h, i.e. E^h is a superspace of E (but, in general, it is not an extension because E is not dense in E^h).

In [1] (4.8), the spaces lying between E and E^h were characterized, up to equivalence, in the following manner: For a superspace E' of E, in order that there exist a subspace E'' of E^h equivalent to E' (i.e. such that there is a homeomorphism h from E' onto E'' with $h(x) = x$ for $x \in E$), it is necessary and sufficient that

(1.1) E' be reduced (i.e. $x \in E' - E$ and $y \in E'$, $y \neq x$ have distinct neighbourhood filters in E'),

(1.2) there be a subbase $\mathbf{S}'$ in E' such that

$$\{S' \cap E : S' \in \mathbf{S}'\} = \mathbf{S}$$

and, putting

$$\mathbf{T}' = \{E' - S' : S' \in \mathbf{S}'\}, \quad \mathbf{V}' = \mathbf{S}' \cup \mathbf{T}', \tag{1.3}$$

the inclusion $V_1' \cap E \subset V_2' \cap E$ implies $V_1' \subset V_2'$ for $V_1', V_2' \in \mathbf{V}'$.

Moreover, for a given homeomorphism $h : E' \to E''$ where

$$E \subset E'' \subset E^h, \quad h(x) = x \text{ for } x \in E, \tag{1.4}$$

the subbase $\mathbf{S}'$ may be chosen in such a manner that (besides (1.2)) also

$$h(V') = \mathrm{r}(V' \cap E) \cap E'' \quad (V' \in \mathbf{V}') \tag{1.5}$$

be satisfied. Conversely, if (1.1) and (1.2) are true, then a subspace E'' and a homeomorphism $h : E' \to E''$ can be found satisfying (1.4) and (1.5).

Among the subspaces of E^h investigated in [1], we mention here four. The space E^d is the subspace of E^h composed of those $x \in E^h$ for which $\mathbf{v}(x)$ is centrated (i.e. $A_1, \ldots, A_n \in \mathbf{v}(x)$ implies $\bigcap_1^n A_i \neq \emptyset$). The spaces lying between E and E^d were characterized, in [1] (7.6), in the following manner: A superspace E' of E is equivalent to a subspace of E^d iff it is reduced, $\mathbf{T}$-strict (i.e. the closures $\overline{T}$ in E' of the sets $T \in \mathbf{T}$ constitute a closed subbase in E'), and strongly $\mathbf{T}$-distributive (i.e. $T_1, \ldots, T_n \in \mathbf{T}$ implies

$$\overline{\bigcap_1^n T_i} = \bigcap_1^n \overline{T}_i$$

for the closures taken in E').

Another subspace of E^h we need here is the superextension E^s. It is obtained by adding to E all points $x \in E^h - E$ for which $\mathbf{v}(x)$ is a $\mathbf{T}$-sieve (i.e., for $A \in \mathbf{v}(x)$, there exists $T \in \mathbf{v}(x) \cap \mathbf{T}$ with $T \subset A$).

The third subspace in question is E^j. We obtain E^j by taking all points $x \in E^h$ for which $\mathbf{v}(x)$ is strongly $\mathbf{S}$-centrated, i.e. if $S_i \in \mathbf{v}(x) \cap \mathbf{S}$ $(i = 1, \ldots, n)$ and $T \in \mathbf{v}(x) \cap \mathbf{T}$ imply

$$\bigcap_1^n S_i \cap T \neq 0.$$

By [1] (7.4), a superspace of E is equivalent to a subspace of E^j iff it is a reduced, **T**-strict, **T**-disjunctive (i.e. T_1, $T_2 \in \mathbf{T}$, $T_1 \cap T_2 = \emptyset$ implies $\overline{T}_1 \cap \overline{T}_2 = \emptyset$) extension.

Finally, we have to recall the definition of the subspace E^w (the Wallman-type compactification of E relative to **S**). It is obtained by adding to E those $x \in E^h$ for which $\mathbf{v}(x)$ is a centrated quasi-**T**-sieve (where $\mathbf{v}(x)$ is said to be a quasi-**T**-sieve if, for $A \in \mathbf{v}(x)$, there exists a finite number of sets $T_1, \ldots, T_n \in \mathbf{T} \cap \mathbf{v}(x)$ such that $\bigcap_1^n T_i \subset A$).

2. Characterization of the hyperextension. In order to obtain a characterization of E^h, let us introduce the following terminology. For a cover **C** of a set X, we say that **C** is *supercompact* if, from each subcover $\mathbf{C}' \subset \mathbf{C}$, a subcover $\mathbf{C}'' \subset \mathbf{C}'$ composed of at most two elements can be selected. According to the terminology used in [1], this is true iff X is supercompact relative to **C** when equipped with the topology for which **C** is a subbase. Similarly, we shall say that a cover **C** of X is *compact* if, from each subcover $\mathbf{C}' \subset \mathbf{C}$, a finite subcover $\mathbf{C}'' \subset \mathbf{C}'$ can be selected, i.e., with respect to a classical theorem of J. W. ALEXANDER, iff X is compact when equipped with the same topology as above.

Now, with the notations introduced in section 1, we can first formulate the following theorem:

(2.1) *The cover* $\mathbf{V}^h = \{\nu(V) : V \in \mathbf{V}\}$ *of the space* E^h *is supercompact.*

PROOF. If

$$E^h = \bigcup_{i \in I} \nu(V_i) \quad (V_i \in \mathbf{V}),$$

then there are two sets V_i such that they cover E. In fact, if it would not be so, then the system

$$\mathbf{A} = \{E - V_i : i \in I\} \subset \mathbf{V}$$

would be a linked system (i.e. each pair of members of **A** would have non-empty intersection), and therefore **A** would generate a **V**-sieve (consisting of all sets $B \subset E$ such that there is an $A \subset B$, $A \in \mathbf{A}$) which could be extended, in account of [1] (3.5), to an ultra-**V**-sieve **v**. By definition of E^h, there would exist $x \in E^h$ such that $\mathbf{v} = \mathbf{v}(x)$. But if $x \in \nu(V_i)$, $i \in I$, then $V_i \in \mathbf{v}(x)$ and $E - V_i \in \mathbf{A} \subset \mathbf{v}(x)$, a contradiction. Hence there are $i, j \in I$ such that $E = V_i \cup V_j$. Then

$$E^h = \nu(V_i) \cup \nu(V_j)$$

by [1] (4.2).∎

We now consider subspaces of E^h. For them, we prove:

(2.2) *For* $E \subset E' \subset E^h$, *let*

$$\mathbf{V}' = \{V^h \cap E' : V^h \in \mathbf{V}^h\} \tag{2.3}$$

be supercompact. Then $E' = E^h$.

PROOF. Assuming this equality to be false, let $x \in E^h - E'$ and $\mathbf{B} = \{V_i : i \in I\}$ be the family of those sets $V_i \in \mathbf{V}$ for which $V_i \notin \mathbf{v}(x)$, i.e. for which $E - V_i \in \mathbf{v}(x)$ by [1] (3.3). Then

$$E' \subset \bigcup_{i \in I} \nu(V_i).$$

In fact, for $y \in E'$, we have $\mathbf{v}(x) \neq \mathbf{v}(y)$ so that there are, in view of [1] (3.4), two sets $V \in \mathbf{V} \cap \mathbf{v}(x)$ and $V^* \in \mathbf{V} \cap \mathbf{v}(y)$ such that $V \cap V^* = \emptyset$. Hence there is an $i \in I$ with $V = E - V_i$, and $E - V \supset V^*$ implies $E - V \in \mathbf{v}(y)$, $V_i \in \mathbf{v}(y)$, $y \in \nu(V_i)$. Now the cover

$$\{\nu(V_i) \cap E' : i \in I\}$$

of E' is a subcover of $\mathbf{V}'$. Hence there are $i, j \in I$ such that $E' \subset \nu(V_i) \cup \nu(V_j)$, consequently $E = V_i \cup V_j$ by [1] (4.2), $(E - V_i) \cap (E - V_j) = \emptyset$. This contradicts to the fact that $E - V_i \in \mathbf{v}(x)$, $E - V_j \in \mathbf{v}(x)$. ∎

From the above results, we finally get the desired characterization of E^h:

(2.4) *A superspace E' of E is equivalent to E^h iff the conditions* (1.1) *and* (1.2) *hold and, moreover,* $\mathbf{V}'$ *is supercompact* (where $\mathbf{V}'$ is the cover introduced in (1.3)).

PROOF. By [1] (4.8), E' is equivalent to a subspace E'' of E^h iff (1.1) and (1.2) are valid, moreover, in this case, there is a homeomorphism $h : E' \to E''$ such that (1.4) and (1.5) are true. Thus h carries $\mathbf{V}'$ over

$$\mathbf{V}'' = \{V^h \cap E'' : V^h \in \mathbf{V}^h\}. \tag{2.5}$$

From (2.1) and (2.2), we know that the cover $\mathbf{V}''$ is supercompact iff $E'' = E^h$. ∎

3. Characterization of E^d. In order to characterize the subspace E^d, we shall need a lemma:

(3.1) *Let* $\mathbf{M}$ *be a system of subsets of a set E such that $M \in \mathbf{M}$ implies $E - M \in \mathbf{M}$. Then each maximal centrated* $\mathbf{M}$*-sieve is an ultra-*$\mathbf{M}$*-sieve.*

PROOF. Let $\mathbf{m}$ be a maximal centrated $\mathbf{M}$-sieve and $M_0 \in \mathbf{M}$. Either M_0 or $E - M_0$ can be added to $\mathbf{m}$ without affecting its property of being centrated; in fact, by assuming the contrary, there would exist sets $A_i \in \mathbf{m}$, $B_j \in \mathbf{m}$ such that

$$M_0 \cap \bigcap_1^p A_i = 0, \quad (E - M_0) \cap \bigcap_1^q B_j = 0$$

so that

$$\bigcap_1^p A_i \cap \bigcap_1^q B_j = \emptyset$$

which is impossible since $\mathbf{m}$ is centrated. Hence either $\mathbf{m} \cup \{M_0\}$ or $\mathbf{m} \cup \{E - M_0\}$ generates a centrated $\mathbf{M}$-sieve that must coincide with $\mathbf{m}$, so that either $M_0 \in \mathbf{m}$ or $E - M_0 \in \mathbf{m}$. By [1] (3.3), $\mathbf{m}$ is an ultra-$\mathbf{M}$-sieve. ∎

Now we can prove the following property of E^d:

(3.2) *The cover*

$$\{V^h \cap E^d : V^h \in \mathbf{V}^h\} \tag{3.3}$$

of the subspace $E^d \subset E^h$ *is compact.*

PROOF. Suppose

$$E^d \subset \bigcup_{i \in I} \nu(V_i) \quad (V_i \in \mathbf{V})$$

and assume that E cannot be covered with a finite number of V_i. Then the sets $E - V_i$ form a centrated system which generates a centrated $\mathbf{V}$-sieve in E. Clearly the family of all centrated $\mathbf{V}$-sieves is inductive so that

$$\mathbf{A} = \{E - V_i : i \in I\}$$

can be extended to a maximal centrated $\mathbf{V}$-sieve $\mathbf{v}$. By (3.1), $\mathbf{v}$ is an ultra-$\mathbf{V}$-sieve so that, by definition of E^d, there exists $x \in E^d$ such that $\mathbf{v} = \mathbf{v}(x)$. If $x \in \nu(V_i)$, $i \in I$, then $V_i \in \mathbf{v}(x)$ and $E - V_i \in \mathbf{A} \subset \mathbf{v}(x)$ which is impossible.

Consequently

$$E = \bigcup_{j=1}^{n} V_{i_j}$$

for suitable indices $i_j \in I$. It follows

$$E^d \subset \bigcup_{j=1}^{n} \nu(V_{i_j}).$$

In fact, if $x \in E^d$, $x \notin \nu(V_{i_j})$ were true for $j = 1, \ldots, n$, then $V_{i_j} \notin \mathbf{v}(x)$ would imply $E - V_{i_j} \in \mathbf{v}(x)$ by [1] (3.3), hence

$$\bigcap_{j=1}^{n} (E - V_{i_j}) = \emptyset$$

would contradict to the fact that $\mathbf{v}(x)$ is centrated. Hence, from each subcover

$$\{\nu(V_i) \cap E^d : i \in I\}$$

of (3.3), we can select a finite subcover. ∎

On the other hand, we prove:

(3.4) *If* $E \subset E' \subset E^h$ *and the cover* $\mathbf{V}'$ *in* (2.3) *is compact, then* $E' \supset E^d$.

PROOF. Assume $x \in E^d - E'$ and let $\{V_i : i \in I\}$ be the system of those sets $V_i \in \mathbf{V}$ for which $V_i \notin \mathbf{v}(x)$ i.e. $E - V_i \in \mathbf{v}(x)$. Then

$$E' \subset \bigcup_{i \in I} \nu(V_i).$$

In fact, for $y \in E'$, we have $\mathbf{v}(x) \neq \mathbf{v}(y)$, hence, by [1] (3.4), there are $V \in \mathbf{V} \cap \cap \mathbf{v}(x)$ and $V^* \in \mathbf{V} \cap \mathbf{v}(y)$ such that $V \cap V^* = \emptyset$. Hence $E - V^* \in \mathbf{v}(x)$,

$V^* = V_i$ for a suitable $i \in I$, and $y \in \nu(V_i)$. On the other hand, by choosing $V_{i_1}, \ldots, V_{i_n}$ arbitrarily,

$$\bigcap_{j=1}^{n} (E - V_{i_j}) \neq \emptyset$$

since $E - V_{i_j} \in \mathbf{v}(x)$ and $\mathbf{v}(x)$ is centrated. Therefore the sets $V_{i_1}, \ldots, V_{i_n}$ do not cover E and, by [1] (4.2), the corresponding sets $\nu(V_{i_1}), \ldots, \nu(V_{i_n})$ do not cover E', so that $\mathbf{V}'$ cannot be compact. ▌

We can now formulate the desired characterization of E^d:

(3.5) *In order that a superspace E' of E be equivalent to E^d, it is necessary and sufficient that E' be reduced,* **T**-*strict, strongly* **T**-*distributive, and that* $\mathbf{V}'$ *be compact where*

$$\mathbf{T}' = \{\bar{T} : T \in \mathbf{T}\}, \quad \mathbf{S}' = \{E' - T' : T' \in \mathbf{T}'\}, \quad \mathbf{V}' = \mathbf{S}' \cup \mathbf{T}', \tag{3.6}$$

the closure being understood in E'.

Proof. From [1] (7.6), we know that the above conditions, without the last one, hold iff E' is equivalent to a subspace E'' such that $E \subset E'' \subset \subset E^d$. A homeomorphism $h : E' \to E''$ carries a set $\bar{T} \in \mathbf{T}'$ ($T \in \mathbf{T}$) over the closure of T in E'' which coincides, by [1] (7.1), with $\nu(T) \cap E''$. Thus h carries $\mathbf{V}'$ in (3.6) over $\mathbf{V}''$ in (2.5). Now, (3.2) implies that $\mathbf{V}''$ is compact when $E'' = E^d$: conversely, by (3.4), if $\mathbf{V}''$ is compact, then $E'' \supset E^d$, i.e. $E'' = E^d$. ▌

4. Compact subspaces of E^h. We know from (3.2) that the cover $\{\nu(V) \cap E^d : V \in \mathbf{V}\}$ is compact. A fortiori, the cover $\{\nu(S) \cap E^d : S \in \mathbf{S}\}$ which is a subcover of the above cover is compact as well; in other words, the subspace E^d of E^h is compact (this is also a consequence of [1] (9.4) and (9.1)). On the other hand, the subspace E^s is compact by [1] (5.2). As a partial converse of these facts, we can prove:

(4.1) *If a subspace E' such that $E \subset E' \subset E^h$ is compact, then $E' \supset E^d \cap E^s$.*

Proof. Assume $x \in (E^d \cap E^s) - E'$, and let $\{S_i : i \in I\}$ be the family of those sets $S_i \in \mathbf{S}$ for which $S_i \notin \mathbf{v}(x)$ i.e. $E - S_i \in \mathbf{v}(x)$. Then

$$E' \subset \bigcup_{i \in I} \nu(S_i).$$

In fact, for $y \in E'$, we have $\mathbf{v}(x) \neq \mathbf{v}(y)$, hence there exist $V \in \mathbf{V} \cap \mathbf{v}(x)$ and $V^* \in \mathbf{V} \cap \mathbf{v}(y)$ such that $V \cap V^* = \emptyset$ (see [1] (3.4)). Since $x \in E^s - E' \subset \subset E^s - E$, we can suppose $V = T \in \mathbf{T}$, and then $V^* = E - T = S \in \mathbf{S}$ may be chosen. From $E - S \in \mathbf{v}(x)$, we get $S = S_i$ for a suitable $i \in I$, hence $S \in \mathbf{v}(y)$ is equivalent to $y \in \nu(S_i)$.

On the other hand, no finite system $S_{i_1}, \ldots, S_{i_n}$ can cover E ($i_j \in I$) since $E - S_{i_j} \in \mathbf{v}(x)$ and $\mathbf{v}(x)$ is centrated as a consequence of $x \in E^d$. Thus E' cannot be covered with a finite system $\nu(S_{i_1}), \ldots, \nu(S_{i_n})$, and E' is not compact. ▌

It is to be noted that, with the notations of [1],

$$E^d \cap E^s = E^w \cap E^s = E^v \cap E^s.$$

The compact subspaces of E^h containing E are precisely those subspaces E' for which $E \subset E' \subset E^h$ and the cover

$$\mathbf{S}' = \{\mathrm{v}(S) \cap E' : S \in \mathbf{S}\}$$

is compact. We know from [1] (5.2) that $E' = E^s$ is the smallest such E' for which $\mathbf{S}'$ is supercompact. It is tempting to conjecture, in view of this fact and (4.1), that $E^d \cap E^s$ is the smallest compact subspace of E^h containing E. However, we shall show by an example that, in general, no such smallest compact subspace of E^h does exist.

For this purpose, let A, B, C be three disjoint straight lines in a plane and $E = A \cup B \cup C$. Introduce the notations

$$A = \{a_x : x \in \mathbf{R}\}, \quad B = \{b_x : x \in \mathbf{R}\}, \quad C = \{c_x : x \in \mathbf{R}\}.$$

Let $\mathbf{T}$ consist of the sets $\emptyset$, E, $T = B \cup C$, and of all sets

$$T'_x = \{a_t : t \geqq x\} \cup \{b_t : t \geqq x\}$$

and

$$T''_x = \{a_t : t \geqq x\} \cup \{c_t : t \geqq x\}$$

for $x \in \mathbf{R}$. Put $\mathbf{S} = \{E - T : T \in \mathbf{T}\}$, $\mathbf{V} = \mathbf{S} \cup \mathbf{T}$, and

$$S = E - T, \quad S'_x = E - T'_x, \quad S''_x = E - T''_x.$$

The system

$$\mathbf{v} = \{S\} \cup \{T'_x : x \in \mathbf{R}\} \cup \{T''_x : x \in \mathbf{R}\}$$

is centrated and clearly generates a centrated ultra-$\mathbf{V}$-sieve. Hence $\mathbf{v} = \mathbf{v}(p)$, $p \in E^d$. Obviously $p \notin E$ since the system $\mathbf{v}$ has empty intersection.

On the other hand, the system $\mathbf{T} - \{\emptyset\}$ is a linked system so that it generates an ultra-$\mathbf{V}$-sieve $\mathbf{v}(q)$ with $q \in E^s - E$. It is also clear that q is the unique element of $E^s - E$, and $q \notin E^d$ since $T \cap T'_0 \cap T''_0 = \emptyset$ so that $\mathbf{v}(q)$ is not centrated. Hence $E^d \cap E^s = E$.

If there would exist a smallest compact subspace of E^h containing E, it should coincide with E because E^d and E^s are both compact. However, E is not compact since $\mathbf{v} - \{S\}$ is a centrated system of closed sets with empty intersection.

A necessary and sufficient condition for a smallest compact space between E and E^h to exist may be given with the help of the Wallman-type compactification E^w in the following form:

(4.2) *There is a smallest compact space E' such that $E \subset E' \subset E^h$ iff $E^w \subset E^s$, and in this case the smallest compact E' coincides with E^w.*

Proof. If $E^w \subset E^s$, then

$$E^w = E^w \cap E^s = E^d \cap E^s.$$

Since E^w is compact by [1] (9.4), this is, in view of (4.1), the smallest compact space between E and E^h.

Conversely, if E'' is the smallest compact space between E and E^h, then $E''\subset E^w$ and so, by [1] (9.1) and (9.4), $E''=E^w$. Since E^s is compact by [1] (5.2), necessarily $E''\subset E^s$, i.e. $E^w\subset E^s$. ∎

5. T-distributive superspaces. In [1], we constructed subspaces of E^h with the property that the spaces lying between E and the subspace in question are characterized, up to equivalence, by being reduced, **T**-strict extensions of E and possessing moreover the property of **T**-disjunctivity, or strong **T**-disjunctivity, or strong **T**-distributivity, respectively. In fact, to the property of **T**-disjunctivity, there corresponds the space E^j ([1] (7.4)), to strong **T**-disjunctivity, the space $E^j\cap E^v$ ([1] (7.8)), to strong **T**-distributivity, the space E^d ([1] (7.6)). It is desirable to complete these results by an analogous statement on a subspace of E^h the subspaces of which are characterized by being reduced, **T**-strict and **T**-distributive extensions of E (where E' is said to be **T**-distributive iff $T_1, T_2\in\mathbf{T}$ implies

$$\overline{T_1\cap T_2}=\overline{T}_1\cap\overline{T}_2$$

for the closures taken in E').

For this purpose, let us say that an ultra-**V**-sieve $\mathbf{v}$ is *very strongly* **S**-*centrated* if $S_i\in\mathbf{v}\cap\mathbf{S}$ $(i=1,\ldots,n)$ and $T_j\in\mathbf{v}\cap\mathbf{T}$ $(j=1,2)$ imply

$$\bigcap_1^n S_i\cap T_1\cap T_2\neq 0. \tag{5.1}$$

Let us denote by E^t the subspace of E^h composed of those points $x\in E^h$ for which $\mathbf{v}(x)$ is very strongly **S**-centrated. Clearly $E^d\subset E^t\subset E^j$.

(5.2) *A superspace E' of E is equivalent to a subspace of E^t iff it is a reduced,* **T**-*strict and* **T**-*distributive extension of E.*

Proof. If $E\subset E'\subset E^t$, $T_1, T_2\in\mathbf{T}$, $x\in\overline{T}_1\cap\overline{T}_2$ (for the closures in E'), then $x\in v(T_1)\cap v(T_2)$ since, by [1] (7.1), $\overline{T}_j=v(T_j)\cap E'$. Hence, for arbitrary $S_i\in\mathbf{v}(x)\cap\mathbf{S}$ $(i=1,\ldots,n)$, we have (5.1), consequently

$$\bigcap_1^n v(S_i)\cap T_1\cap T_2\neq 0.$$

Since the sets

$$\bigcap_1^n v(S_i)\quad (S_i\in\mathbf{v}(x)\cap\mathbf{S})$$

constitute a neighbourhood base of x in E^h, we get

$$x\in\overline{T_1\cap T_2}.$$

Thus

$$\overline{T}_1\cap\overline{T}_2\subset\overline{T_1\cap T_2}$$

and the opposite inclusion is obvious. This property remains, of course, for equivalent superspaces, and reducedness and **T**-strictness follow from $E' \subset E^j$ by [1] (7.4).

Conversely, if E' is reduced, **T**-strict and **T**-distributive, then it is also **T**-disjunctive so that, by [1] (7.4), E' is equivalent to a space E'' such that $E \subset E'' \subset E^j$. If $x \in E''$, $S_i \in \mathbf{v}(x) \cap \mathbf{S}$ $(i = 1, \ldots, n)$, $T_j \in \mathbf{v}(x) \cap \mathbf{T}$ $(j = 1, 2)$, then

$$x \in \mathrm{v}(T_1) \cap \mathrm{v}(T_2) \cap E'' = \overline{T}_1 \cap \overline{T}_2$$

for the closures in E'', hence $x \in \overline{T_1 \cap T_2}$ (E'' being **T**-distributive as well), thus

$$\bigcap_1^n \mathrm{v}(S_i) \cap T_1 \cap T_2 \neq \emptyset$$

which is the same as (5.1) so that $E'' \subset E^j$. ∎

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A REMARK ON SEMILATTICES OF GROUPS

By

S. LAJOS

Department of Mathematics, K. Marx University of Economics, Budapest

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Let S be a semigroup. Following the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1] we say that S is a semilattice of groups if it is the set-theoretical union of a family of mutually disjoint subgroups G_α $(\alpha \in A)$:

$$S = \bigcup_{\alpha \in A} G_\alpha, \tag{1}$$

such that the products $G_\alpha G_\beta$ and $G_\beta G_\alpha$ are both contained in the same G_γ $(\alpha, \beta, \gamma \in A)$. Recently the author has obtained some ideal-theoretic characterizations of semigroups that are semilattices of groups. In this short note we prove further results concerning semilattices of groups.

A new criterion is contained in the following theorem.

THEOREM 1. *A semigroup S is a semilattice of groups if and only if the condition*

$$QXQ = X \cap Q \tag{2}$$

holds for every quasi-ideal Q and every one-sided ideal X of S.

PROOF. Necessity. Let S be a semigroup which is a semilattice of groups. Then the condition

$$Q_1 \cap Q_2 = Q_1 Q_2 \tag{3}$$

holds for any couple of quasi-ideals of S (see the author [2], Theorem 2). Hence our condition (2) follows for any quasi-ideal Q and any one-sided (left or right) ideal X of S.

Sufficiency. Suppose that S is a semigroup in which the condition (2) is satisfied by every quasi-ideal Q and every one-sided ideal X of S. Then

$$SXS = X \tag{4}$$

for any one-sided ideal X of S, i.e. every one-sided ideal X of S is two-sided. On the other hand, (2) implies

$$ISI = I \tag{5}$$

for any (two-sided) ideal I of S. Hence we conclude that S is regular because every quasi-ideal of S is a two-sided ideal and vice versa. Therefore S is a regular semigroup every one-sided ideal of which is two-sided. A recent criterion of the author shows that S is a semilattice of groups (cf. [3], Theorem 3).

The following results can be proved similarly.

THEOREM 2. *A semigroup S is a semilattice of groups if and only if the condition*

$$BXB = B \cap X \tag{6}$$

holds for every bi-ideal B and every one-sided ideal X of S.

THEOREM 3. *A semigroup S is a semilattice of groups if and only if the condition*

$$BXQ = B \cap X \cap Q \tag{7}$$

holds for any bi-ideal B, any quasi-ideal Q, and any one-sided ideal X of S.

THEOREM 4. *A semigroup S is a semilattice of groups if and only if the relation*

$$Q_1XQ_2 = Q_1 \cap X \cap Q_2 \tag{8}$$

holds for any couple Q_1, Q_2 of quasi-ideals and for any one-sided X ideal X of S.

THEOREM 5. *A semigroup S is a semilattice of groups if and only if the condition*

$$B_1XB_2 = B_1 \cap X \cap B_2 \tag{9}$$

holds for every one-sided ideal X of S and every couple B_1, B_2 of bi-ideals of S.

For another characterizations of semigroups that are semilattices of groups, see the author [3]–[5].

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ON THE NONPLANARITY OF SOME CUBIC GRAPHS

By

L. KÁSZONYI

Department of Probability Theory of the Eötvös Loránd University, Budapest

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1. In this paper we mean by a graph a subset of the set consisting all of the one- and some of the two-element subsets of a given set. If G is a graph, we call the one-element sets of G vertices $[V(G)]$, the two-element sets of G edges of G $[E(G)]$.

A cubic graph is a graph, each point of which has degree 3. A cubic graph G is well-colorable, — or shortly colorable — if there is a function s, wich maps the edgeset of G to the set $\{1, 2, 3\}$ so, that if $\overline{g_1}$ and $\overline{g_2}$ are two distinct edges of G having the common endpoint g, then $s(\overline{g_1}) \neq \neq s(\overline{g_2})$.

There are non-colorable cubic graphs, for example, all cubic graphs, which are one-fold edge-connected, are such graphs. The first example for an at least two-fold connected cubic graph, which is non-colorable, was given by PETERSEN [1] (See: Fig. 1.)

DEFINITION 1. Let G be a colorable cubic graph. A circiut $C\sigma_1, \sigma_2$ of G is an alternating circuit wiht respect to the colouring s if the edges of $C\sigma_1, \sigma_2$ are coloured by s wiht the colours σ_1 and σ_2.

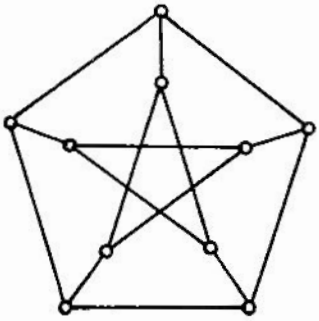

Fig. 1

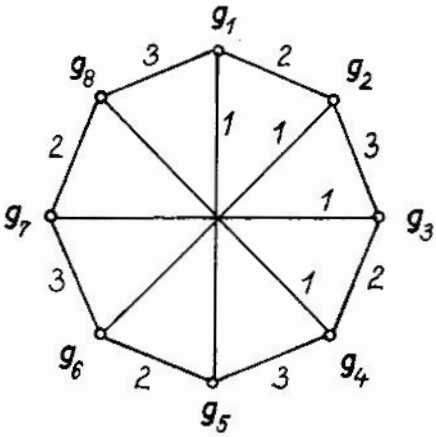

Fig. 2

If we say, that a circuit C is an alternating circuit of G, we assume, that there is a coloring s, with respect to which C is an alternating circuit.

DEFINITION 2. The edges $\overline{g}$ and $\overline{h}$ of the colorable cubic graph G are orthogonal, if there is no alternating circuit of G, which containes both $\overline{g}$ and $\overline{h}$ as an edge.

The class of graphs, which are at least two-fold connected, and have orthogonal edges, we denote by $\mathcal{E}$. Let $\mathcal{E}_0$ be the subclass of $\mathcal{E}$, consisting of the graphs, which have the minimum number of colourings.

It is easy to see, that the elements of $\mathcal{E}_0$ have 18 colorings. Such a graph is the to be seen on Fig. 2.

2. The aim of this paper is to prove, that the elements of $\mathcal{E}_0$ are non-planar.

In proving our theorem we shall use the following propositions, and definitions.

DEFINITION. The set M of edges of the graph G is a cut, if the number of the components of the graph $G \cap M$ is more than that of G. A minimal cut is a cut, which is minimal by inclusion.

Let C be an arbitrary circuit of the planar graph G, and C^* be the edge-set corresponding to C in the dual graph G^*. It is easy to see, that C^* is a minimal cut of G^*.

PROPOSITION 1. *Every at least two-fold edge-connected non-colorable graph containes a subgraph, which is topologically isomorphic to an element of* $\mathcal{E}$.

We shall sketch the proof. We prove, that every at least two-fold connected graph has an edge $\overline{g}$ such that $|M| > 3$ holds for every cut M of G containing g. What we prove, is something more: G has two such edges.

If G is at least 3-fold edge-connected — and especially if G has exactly 3 edges — then the statement is trivially thrue. Let us assume, that G is two-fold edge-connected, $|E(G)| > 3$, and that the statement is thrue for every G_i with $|E(G_i)| < |E(G)|$. G has the edges $\overline{g} = (g_1, g_2)$, and $\overline{h} = (h_1, h_2)$, such, that the graph $G \setminus \{\overline{g}, \overline{h}\}$ has two components: G_1 and G_2.

Let us consider the graphs

$$\overline{G_1} = G_1 \cup \{(g_1, h_1)\}$$

and

$$\overline{G_2} = G_2 \cup \{(g_2, h_2)\},$$

where (g_1, h_1) and (g_2, h_2) are new edges.

$|E(\overline{G_i})| < |E(G)|$ holds for $i = 1, 2$, thus by the induction hypothesis, $\overline{G_i}$ has the edge $\overline{k_i}$, wich is distinct from (g_i, h_i), and has the property that every cut M_i in G_i containing $\overline{k_i}$ has at least 3 elements. It is easy to see, that $\overline{k_1}$ and $\overline{k_2}$ has the same property in G.

Let us assume, that G is a noncolorable, two-fold edge-connected

cubic graph, and it is minimal by topological inclusion with respect to this property. $|E(G)| > 3$, because the Θ-graph is colorable.

Let $\bar{k} = (k_1, k_2)$ be an edge of G such, that if M is a cut of G, containing $\bar{k}$, then $|M| \geq 3$. Let the edges of G incident to k_i and distinct from $\bar{k}$ be $\bar{k}_{i1} = (k_i, k_{i1})$ and $\bar{k}_{i2} = (k_i, k_{i2})$. Let us consider the graph

$$\bar{G} = G \setminus (\{k_1, k_2\}) \setminus \{\bar{k}, \bar{k}_{11}, \bar{k}_{12}, \bar{k}_{21}, \bar{k}_{22}\} \cup \{(k_{11}, k_{12}), (k_{21}, k_{22})\},$$

where (k_{11}, k_{12}) and (k_{21}, k_{22}) are new edges. $\bar{G}$ is two-fold connected by the property of $\bar{k}$, and therefore colorable by the assumption for G. We shall see, that (k_{11}, k_{12}) and (k_{21}, k_{22}) are orthogonal edges of $\bar{G}$. Assume indirectly, that $\bar{G}$ has a coloring s and an alternating circuit $C\sigma_1, \sigma_2$ with respect to s, wich containes both (k_{11}, k_{12}) and (k_{21}, k_{22}) as edges.

Deleting the edges (k_{11}, k_{12}) and (k_{21}, k_{22}) of G, C falls to two pieces: to C_1 and C_2. Let us assume, that the notations are chosen on such a way, that $k_{11}, k_{21} \in V(C_1)$, and $k_{12}, k_{22} \in V(C_2)$. We define a coloring s' of G as follows:

$$s'(\bar{x}) = s(\bar{x}), \text{ if } \bar{x} \in E(\bar{G}) \text{ and } \bar{x} \notin E(C_1),$$

$$s'(\bar{x}) = \begin{cases} \sigma_1 & \text{if } \bar{x} \in E(C_1) \text{ and } s(\bar{x} = \sigma_2 \\ \sigma_2 & \text{if } \bar{x} \in E(C_1) \text{ and } s(\bar{x}) = \sigma_1 \end{cases}$$

$$s'(\bar{x}) = \sigma, \text{ if } \bar{x} = \bar{k}_{i1}, \text{ where}$$

$$\{\sigma\} = \{\sigma_1, \sigma_2\} \setminus s[(k_{i1}, k_{i2})], \quad i = 1, 2.$$

$$s'(\bar{x}) = s[(k_{i1}, k_{i2})], \text{ if } \bar{x} = \bar{k}_{i2} \text{ and}$$

$$s'(\bar{k}) = \sigma_3.$$

It is easy to verify, that s' is a well-coloring of G, and that is a contradiction by the definition of G.

If G is an arbitrary two-fold edge-connected, non-colorable cubic graph, then similarly as we got $\bar{G}$ abouve, we can find a graph G_1, wich is a proper topological subgraph of G, and it is two-fold edge-connected, and cubic. If G_1 is colorable, than the theorem is proved, and if it is not the case, then we choose a G_2 in G_1 by the same method.

The proposition is proved, because G is finite.

PROPOSITION 2. *To every G belonging to $\mathcal{E}_0$ there is a colouring s^* of G, and there are alternating circuits $X_0, X_1, X_2, \ldots, X_4$ corresponding to s^*, so that the following conditions are fullfilled:*

(I) $$\sum_{i=1}^{4} X_i = X_0,$$

(ii) $X_0 + X_i$ is a Hamilton-circuit of G, $(i = 1, 2, \ldots, 4)$,

(iii) $X_i \cap X_j = \emptyset$ for $j \equiv i+2 \pmod 4$, and for $i = 1, 2, 3, 4$.

(*iv*) $\bigcup_{0}^{4} X_i = G$

(The sum of the circuits X_i and X_j is a graph, spanned in G by the Boolean sum of the edge-sets of X_i and X_j.)

REMARK. It is easy to see, that the PETERSEN graph containes a subgraph, wich is topologicaly isomorphic to the graph P_0 of Fig. 2. The alternating circuits of P_0 corresponding to the colouring of Fig. 2. are shown on Fig. 3. It is easy to verify, that the conditions (i)–(iv) hold, too.

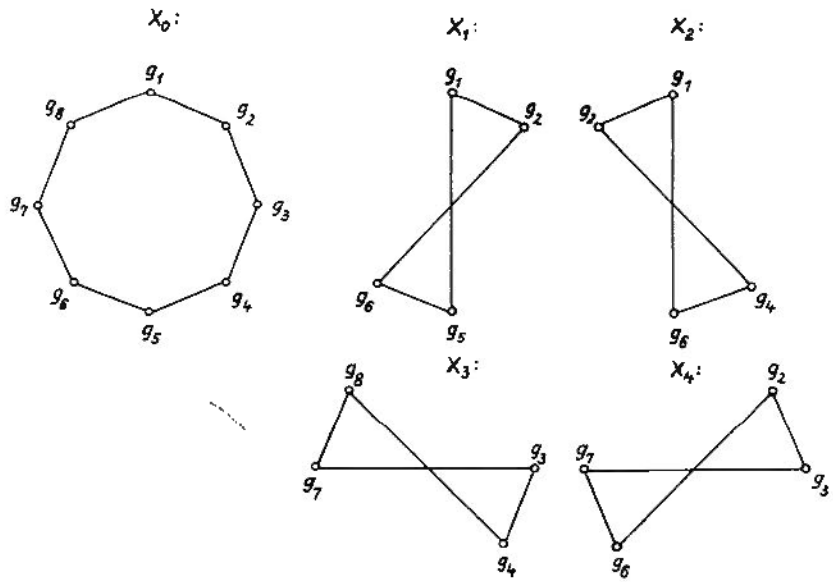

Fig. 3

Let us consider the orthogonal edges $\overline{g_1} = (g_1, g_2)$, and $\overline{h_1}$ of G, denote the edges incident to g_1 by $\overline{g_2}$ and $\overline{g_3}$.

(a) It is easy to see, that if $G \in \mathcal{E}_0$, then every alternating circuit of it containes at least one of the edges $\overline{g_1}, \overline{g_2}, \overline{g_3}, \overline{h_1}$.

(b) Using the orthogonaliti of $\overline{h_1}$ and $\overline{g_1}$ we can prove, that G has a coloring s^*, for wich $s^*(\overline{g_1}) = s^*(\overline{h_1})$ holds.

Using (a) we can prove that G has exactly five alternating circuits with respect to s^*. If we denote the two factor of G spanned in G by the edge-set $(s^*)^{-1}(\sigma_2) \cup (s^*)^{-1}(\sigma_3)$ by B_{σ_2, σ_3}, we can state, that if $x^*(\overline{g_1}) = s^*(\overline{h_1}) = \sigma_1$, then B_{σ_2, σ_3} is a Hamilton circuit of G, B_{σ_1, σ_2} and B_{σ_1, σ_3} consists of 2 circuits. Let be $B_{\sigma_2, \sigma_3} = X_0$, $B_{\sigma_1, \sigma_2} = X_1 \cup X_3$, $B_{\sigma_1, \sigma_3} = X_2 \cup X_4$, then (i), (iii) and (iv) is trivially true.

(c) Let us consider the coloring $s^{(1)}$ of G getting from s^* by changing the colors on edges of X_1. If we denote the two-factor of G spanned by the

edge-set $(s^{(1)})^{-1}(\sigma_2) \cup (s^{(1)})^{-1}(\sigma_3)$ by $B^{(1)}_{\sigma_1, \sigma_3}$, than $B^{(1)}_{\sigma_2, \sigma_3} = B_{\sigma_2, \sigma_3} + X_1 = X_0 + X_1$. Using (a), we can prove, that $B^{(1)}_{\sigma_2, \sigma_3}$ is a Hamilton circuit of G.

THEOREM 3. (Cayley) *Let $C_1, C_2, \ldots, C_i$ be a set of closed Jordan curves on the plain, no two of them having common point. Consider the graph F, the points of which are the domains of the plain determined by the C_i-s and the edges of which are pairs of domains, having a common boundary. Than F is a tree.*

Another formulation of the theorem is as follows:

THEOREM 4. *Let G be a realisation of a cubic planar graph, wich is two-fold connected in his edges and consider the two-factor X of G consisting of the components $X_1, X_2, \ldots, X_e$. Let G^* be the dual graph of G, and X_i^* the minimal cut in G^*, corresponding to X_i. Consider the graph $F^*(X_1^*, X_2^*, \ldots, X_e^*)$, the points of which are the components of $G^* - \bigcup_{i=1}^{e} X_i^*$, and the edges of which are pairs of components, joined in G^* by an edge. Than $F^*(X_1^*, \ldots, X_e^*)$ is a tree.*

We can prove Theorem 3. and Theorem 4. by induction on e. Similarly, we can prove the following theorem:

THEOREM 5. (Euler) *Let G, G^*, X_i, X_i^* and $F^*(X_1^*, \ldots, X_i^*)$ the same, as in Theorem 4. Consider the components K_j of the graph $G^* - \bigcup X_i^*$. The valency of K_j in $F^*(X_1^*, \ldots, X_e^*)$ we denote by $\varrho(K_j)$. Then*

$$e(K_j) = \nu(K_j) + \varrho(K_j) - 2\,,$$

where $e(K_j)$ denotes the number of edges, and $\nu(K_j)$ the number of vertices of K_j.

In the proof of the main theorem we shall use the following

DEFINITION. Let $\alpha = \{A_1, A_2, \ldots, A_m\}$ be a partition of the set $V(G)$ of points of the graph G, that is, $V(G) = \bigcup_{1}^{m} A_i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Let us consider the graph, the points of which are the A_i-s, and two points A_i and A_j are joined, if there is an edge of G between a point of A_i and a point of A_j.

The resulting graph we will denote by G/α, and will call the factor-graph of G with respect to the partition α.

3. THEOREM 6. *The elements of $\mathcal{E}_0$ are nonplanar.*

Let be $G \in \mathcal{E}_0$, and assume indirectly, that G is planar. Let us denote the dual graph of G by G^*. By Theorem 2., there are circuits X_0, X_1, X_2, X_3, X_4 covering G such, that the conditions (i)–(iv) are fullfilled.

Consider the circuits X_0, X_1, and X_3 of G, and the corresponding cuts X_0^*, X_1^*, and X_3 of G^*. We prove, that every edge of G is contained in $E(X_0 \cup X_1 \cup X_3)$. If this is not the case, then there is an edge $\bar{g}$ of G contained in $E(X_2)$ or in $E(X_4)$. But $X_2 \cap X_4 = \emptyset$ by (iii), hence $\bar{g} \in E(X_1 + X_2 + X_3 + X_4) = E(X_0)$, and this is a contradiction.

By Theorem 4. the graphs $F^*(X_1^*, X_3^*)$ and $F^*(X_0^*)$ are trees; the first of them has two edges, and the second has one edge.

Let us denote the points of $F^*(X_1^*, X_3^*)$ -that is, the components of $G-(X_1^* \cup X_3^*)$ – by ${}_*K_1$, ${}_*K_2$ and ${}_*K_3$ so, that the endpoints of the edges corresponding to X_1^* and X_3^* are ${}_*K_1$, ${}_*K_2$, and ${}_*K_2$, ${}_*K_3$ respectively. (See: Fig. 4.)

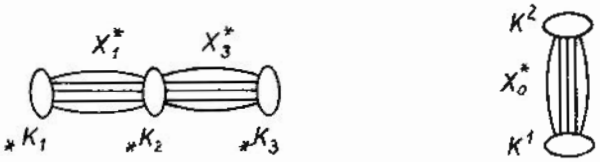

Fig. 4

Let K^1 and K^2 be the two points of $F^*(X_1^*)$, and consider the partition $\alpha_{13} = \{{}_*K_i^j;\ 1 \le i \le 3,\ 1 \le j \le 2\}$ of points of G^*, where

$${}_*K_i^j = V({}_*K_i) \cap VK^j).$$

The graph G^*/α_{13} containes no edge of the form $({}_*K_1^j, K_3^l)$, so it is isomorphic to a subgraph of the graph of Fig. 5.

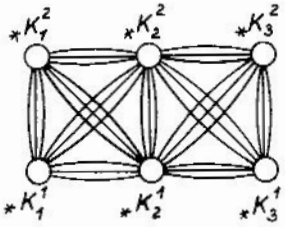

Fig. 5

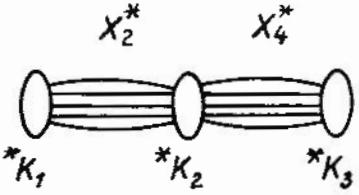

Fig. 6

Consider the circuits X_2 and X_4. The points of $F^*(X_2^*, X_4^*)$ we denote similarly as in case $F^*(X_1^*, X_3^*)$. (See: Fig. 6.) Let us put

$$\alpha_{25} = \{{}^*K_i^j;\ 1 \le i \le 3,\ 1 \le j \le 2\}$$

where ${}^*\overline{K_i^j} = V(\overline{K_i}) \cap V({}^*K^j)$, then the graph G^*/α_{24} is isomorphic to a subgraph of Δ, where Δ is the graph of Fig. 7.

We will investigate the question, what nature the connection between α_{13} and α_{24} is?

Let us consider the functions k_{13} and k_{24}, defined as follows:

$$k_{13}(q) = Q_i^j, \text{ if } q \in {}_*K_i^j$$

$$k_{24}(q) = Q_i^j, \text{ if } q \in {}^*K_i^j.$$

Using k_{13} and k_{24} the non-empty sets of α_{13} and α_{24} can be written in the form $k_{13}^{-1}(Q_i^j)$, and $k_{24}^{-1}(Q_i^j)$. We may extend the functions k_{13} and k_{24} to the set of edges of G^* in such a way, that if $\bar{q} = (q_1, q_2)$, $\bar{q} \in E(G^*)$, $q_1, q_2 \in V(G^*)$, then $k_{ij}(\bar{q}) = [k_{ij}(q_1), k_{ij}(q_2)]$. As the image of the extended functions k_{13} and k_{24} we get graphs, isomorphic to G^*/α_{13} and G^*/α_{24}.

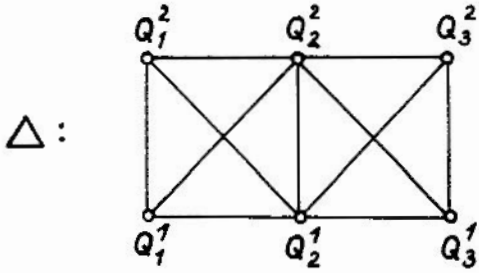

Fig. 7

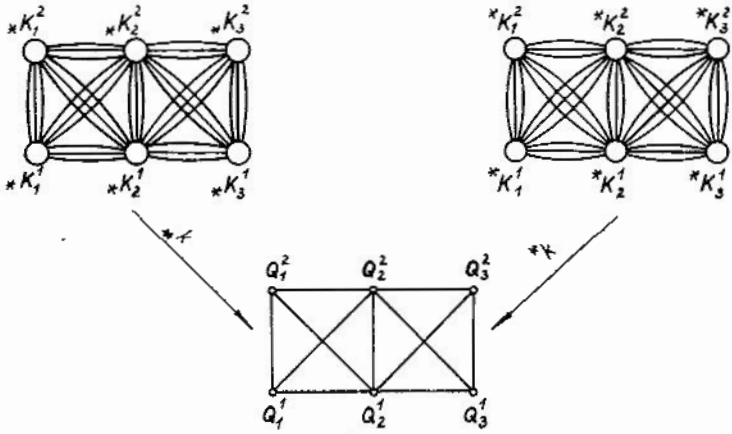

Fig. 8

Let be $\bar{q} = (q_1, q_2)$ an edge of $_*K_2$. We will see, that either $k_{13}(q_1) = k_{24}(q_1)$, or $k_{13}(q_2) = k_{24}(q_2)$. $\bar{q} \in X_0^*$, and $q \notin X_1^* \cup X_3^*$, then $\bar{q} \in X_0^* \cap (X_2^* \cup X_4^*)$. Let be for example $\bar{q} \in X_2^* \cap X_0^*$, then $k_{24}(\bar{q}) = (Q_2^1, Q_1^2)$ or $k_{24}(\bar{q}) = (Q_2^2, Q_1^1)$ and $k_{12}(\bar{q}) = (Q_2^1, Q_2^2) = [k_{13}(q_1), k_{13}(q_2)]$. Thus in the first case we have $k_{24}(q_1) = k_{13}(q_1)$, and in the second one $k_{24}(q_2) = k_{13}(q_2)$.

We may assume, that we have chosen the notations so, that $k_{13}(q_1) = k_{24}(q_1) = Q_2^1$.

We shall see, that $^*K_2^1 \simeq {}_*K_2^1$. K^1 is an even graph, that is we can devide its vertices in two classes in such a way, that there are no two vertices of K^1 in the same class, which are joined by an edge. Such classes are $_*K_2^1$ and

${}_*K_1^1 \cup {}_*K_3^1$, or ${}^*K_2^1$ and ${}_*K_1^1 \cup {}^*K_3^1$. K^1 is a connected graph, therefore these classes are well-defined, hence

$$ {}_*K_2^1 = {}^*K_2^1 \quad \text{or} \quad {}_*K_2^1 = {}^*K_1^1 \cup {}^*K_3^1. $$

Since $q_1 \in {}_*K_2^1 \cap {}^*K_2^1$, the first of them holds.

If $q_2 \in {}_*K_2^2$ then $k_{21}(q_2) = Q_1^2$, or $k_{21}(q_2) = Q_3^2$ holds, hence ${}_*K_2^2 = k_{24}^{-1}(Q_1^2) \cup k_{24}^{-1}(Q_3^2) = {}^*K_1^2 \cup {}_*K_3^2$.

Similarly ${}^*K_2^2 = {}_*K_1^2 \cup {}_*K_3^2$.

Let us consider the sets ${}_*K_1^1$, ${}_*K_2^1$ and ${}_*K_3^1$. We denote the graph, spanned by the point-set

$$ {}_*K_1^1 \cup {}_*K_2^1 \cup {}_*K_3^1 $$

by

$$ K({}_*K_1^1, {}_*K_2^1, {}_*K_3^1). $$

Using this notation we can write

$$ K^1 = K({}_*K_1^1, {}_*K_2^1, {}_*K_3^1). $$

In the same way

$$ {}^*K_2 = K({}^*K_2^1, {}^*K_2^2) = K({}_*K_2^1, {}_*K_1^2, {}_*K_3^1). $$

It is easy to verify, that the two components of the graph $G^* - (X_0^* + X_1^*)$ are

$$ L_1 = K({}_*K_1^2, {}_*K_2^1, {}_*K_3^1), $$

and

$$ L_2 = K({}_*K_1^1, {}_*K_2^2, {}_*K_3^2). $$

Similarly, the components of the graph $G - (X_0^* + X_3^*)$ are

$$ L_3 = K({}_*K_1^1, {}_*K_2^1, {}_*K_3^2), $$

and

$$ L_4 = K({}_*K_1^2, {}_*K_2^2, {}_*K_3^1). $$

By Theorem 5. the following equalities are fullfilled:

$$ (1) \qquad e(K^1) = e[K({}_*K_1^1, {}_*K_2^1)] + e[K({}_*K_2^1, {}_*K_3^1)] = |{}_*K_1^1| + |{}_*K_2^1| + |{}_*K_3^1| - 1, $$

$$ (2) \qquad e({}_*K_2) = e[K({}_*K_1^2, {}_*K_2^1)] + e[K({}_*K_2^1, {}_*K_3^2)] = |{}_*K_1^2| + |{}_*K_2^1| + |{}_*K_3^2|. $$

$$ (3) \qquad e(L_1) = e[K({}_*K_1^2, {}_*K_2^1)] + e[K({}_*K_2^1, {}_*K_3^1)] = |{}_*K_1^2| + |{}_*K_2^1| + |{}_*K_3^1| - 1, $$

$$ (4) \qquad e(L_3) = e[K({}_*K_1^1, {}_*K_2^1)] + e[K({}_*K_2^1, {}_*K_3^2)] = |{}_*K_1^1| + |{}_*K_2^1| + |{}_*K_3^2| = 1. $$

From these equalities follows, that

$$
\begin{aligned}
& e(L_1)+e(L_3)-e({}^*K_2) = \\
&= e\left[K({}_*K_1^2, {}_*K_2^1)\right]+e\left[K({}_*K_2^1, {}_*K_3^1)\right]+ \\
&+e\left[K({}_*K_1^1, {}_*K_2^1)\right]+e\left[K({}_*K_2^1, {}_*K_3^2)\right]- \\
&-e\left[K({}_*K_1^2, {}_*K_2^1)\right]-e\left[K({}_*K_2^1, {}_*K_3^2)\right] = \\
&= e\left[K({}_*K_2^1, {}_*K_3^1)\right]+e\left[K({}_*K_1^1, {}_*K_2^1)\right] = \\
&= e(K^1) = \left|{}_*K_1^2\right| + \left|{}_*K_2^1\right| + \left|{}_*K_3^1\right| - 1 + \\
&\qquad + \left|{}_*K_1^1\right| + \left|{}_*K_2^1\right| + \left|{}_*K_3^2\right| - 1 - \\
&\qquad - \left|{}_*K_1^2\right| - \left|{}_*K_2^1\right| - \left|{}_*K_3^2\right| = \\
&\qquad = \left|{}_*K_1^1\right| + \left|{}_*K_2^1\right| + \left|{}_*K_3^1\right| - 2,
\end{aligned}
$$

and this is a contradiction by (1). Thus the Theorem 6 is proved.

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ON MULTIPLICATIVE FUNCTIONS WITH CONGRUENCE PROPERTY

By

ANTAL IVÁNYI

Department of Numerical Methods and Computer Science of Eötvös Loránd University, Budapest

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We say that a function $f(n)$ defined on the set of positive integers is *multiplicative* if $(a, b) = 1$ implies

$$f(ab) = f(a)\,f(b). \tag{1}$$

If this equation holds for all pairs of positive integers, then we say that $f(n)$ is *completely multiplicative*. If in the right side of (1) is addition instead of multiplication, then $f(n)$ is *additive* resp. *completely additive* function.

For the simplicity let assume in the following that $f(n)$ is not identically zero.

Erdős [1] proved the following assertion. If an additive function $f(n)$ increases monotonically, then $f(n)$ is a constant multiple of $\log n$. It is easy to deduce from Erdős's theorem the following assertion, which was proved by Moser and Lambek [2] directly.

A. *If $f(n)$ is multiplicative, and increasing monotonically, then $f(n)$ is a power of n.*

For the multiplicative functions having only integer values, it is interesting the following theorem due to Subbarao [3].

B. *If $f(n)$ is a multiplicative function having only integer values, and satisfies the relation*

$$f(n+m) \equiv f(m) \pmod{n} \tag{2}$$

for every positive m and n, then $f(n)$ is a power of n with non-negative integer exponent: $f(n) = n^a$ ($a \geq 0$, integer).

The purpose of this paper is the examination of two problems due to Kátai. According to the first one it is possible in the right side of (2) $f(m)+f(n)$, and according to the second one in the theorem B. it is enough, if (2) is true for a fix m, and every n.

The first problem is resolved in Theorem 1. completely, and the second one for completely multiplicative functions in Theorem 2.

THEOREM 1. *If $f(n)$ is a multiplicative function with integer values, and satisfies the relation*

$$f(m+n) \equiv f(m)+f(n) \pmod{n} \tag{3}$$

for every positive m and n, then

$$f(n) = n^a,$$

where a is a positive integer.

PROOF OF THEOREM 1.

Since

$$f(mn) = f[(m-1)n+n] \equiv f[(m-1)n]+f(n) \pmod{n}, \tag{4}$$

it is easy to see that

$$f(mn) \equiv mf(n) \pmod{n}. \tag{5}$$

If we take $(m, n) = 1$, then from (5) we get

$$[f(m)-m]\,f(n) \equiv 0 \pmod{n}. \tag{6}$$

Let α be an arbitrary positive integer, p an arbitrary prime, $n = p^\alpha$, and $(m, p^\alpha) = 1$. Then

$$[f(m)-m]\,f(p^\alpha) \equiv 0 \pmod{p^\alpha},$$

and so either $p^\alpha | f(p^\alpha)$, or $p | f(m)-m$. In the latter case take $m+p^\alpha$ instead of m. Then

$$f(p^\alpha+m) \equiv m+p^\alpha \equiv m \pmod{p}. \tag{7}$$

From (3) we obtain

$$f(m+p^\alpha) \equiv f(m)+f(p^\alpha) \pmod{p^\alpha},$$

so $p | f(p^\alpha)$. Then from (5) — taking $m = n = p$ — we get $p | f(p^2)-pf(p)$, consequently $p | f(p^2)$, etc. — with induction

$$p^\alpha \mid f(p^{\alpha+1}) \quad (\alpha = 1, 2, \ldots). \tag{8}$$

Let p and q be arbitrary distinct primes, u and v be a solution of the equation $1 = xp^\alpha - yq$. Then $x = u+tq$, $y = v+tp^\alpha$ are also solutions, and there exist a t, for which $(y, q) = (x, p) = 1$. Then by (3)

$$f(xp^\alpha) = f(yq+1) \equiv f(yq)+f(1) \pmod{yq}. \tag{9}$$

If $q | f(p^\alpha)$, then by earliers and (9) we get $q | f(1) = 1$, and this can not hold. Consequently

$$q \nmid f(p^\alpha) \text{ if } q \neq p. \tag{10}$$

We say that p is an exceptional prime, if $p^\alpha | f(p^\alpha)$ for at least one α. For such a p^α we have $f(p^\alpha) = \pm p^{\alpha-1}$ [see (8) and (10)].

Suppose that there exist two exceptional primes $p<q$, let $f(p^\alpha) = \pm p^{\alpha-1}$, $f(q^\beta) = \pm q^{\beta-1}$. Taking $m = p^\alpha$ and $n = q^\beta$ in (6) we get

$$f(q^\beta)[f(p^\alpha)-p^\alpha] \equiv 0 \pmod q,$$

whence

$$f(p^\alpha)-p^\alpha \equiv 0 \pmod q$$

follows. Therefore $q \mid \pm p^{\alpha-1}-p^\alpha = -p^{\alpha-1}(p\mp 1)$. This is possible only in the case $p=2$, $q=3$. Thus we obtained that there exist at most two exceptional primes, and if there exist two, then they are 2 an 3.

Now we prove that 2 cannot be exceptional. For this take

$$f(2)f(3) = f(4+2) \equiv f(2)+f(4) \pmod 4$$

and we obtain

$$4 \mid f(2)[f(3)-1]-f(4).$$

Since $f(3)$ is odd, therefore $4 \mid f(4)$, Similarly,

$$f(2^\alpha+2\cdot 2^\alpha) \equiv f(2^\alpha)+f(2^{\alpha+1}), \pmod{2^{\alpha+1}},$$

which by using induction, gives: $2^{\alpha+1} \mid f(2^{\alpha+1})$. Consequently 2 cannot be exceptional.

Suppose that p is the exceptional (odd) prime,

$$f(p^\alpha) = \pm p^{\alpha-1} \quad (\alpha \geq 2).$$

Then

$$f(p^\alpha) \equiv f(p^\alpha-1)+f(1) \quad [\mathrm{mod}\,(p^\alpha-1)]. \tag{11}$$

Since $(p, p^\alpha-1) = 1$, therefore $p^\alpha-1$ have no exceptional divisor, and so $p^\alpha-1 \mid f(p^\alpha-1)$. Thus from (11),

$$p^\alpha-1 \mid \pm p^{\alpha-1}-1.$$

This is impossible for odd p and $\alpha \geq 2$. This contradiction gives that there no exist exceptional primes, i.e.

$$f(n) \equiv 0 \pmod n \tag{12}$$

for every n.

From (3) by (12) and SUBBARAO's theorem we get that $f(n) = n^a$ (a positive integer). Q.e.d.

THEOREM 2. *If $f(n)$ is a completely multiplicative function with integer values, and (2) holds for a fix m and every n, then $f(n) = n^a$ (a is a non-negative integer.*

PROOF OF THEOREM 2.

If $f(m) = 0$, and $f(n)$ is not zero indentically, then there exist a prime q, for which $f(q) \neq 0$, and an exponent α, for which $q^\alpha-m>0$. From (3) we have

$$f(q^{\alpha+i}) = f^{\alpha+i}(q) \equiv 0 \quad [\mathrm{mod}\,(q^{\alpha-i}-m)] \quad (i = 1, 2, \ldots),$$

and so

$$q^{\alpha+i}-m \mid f^{\alpha+i}(q) \quad (i=1,2,\ldots). \tag{13}$$

We can see that this is impossible by using a well known theorem due to G. PÓLYA, asserted that: if $\mathcal{P}$ is a finite set of primes and $n_1<n_2<\ldots$ is the ordered sequence of integers all prime divisors of which are elements of $\mathcal{P}$, then $\lim_{\nu\to\infty}(n_{\nu+1}-n_\nu)=\infty$ [4]. If we choose $\mathcal{P}$ to contain the primes q and the prime divisors of $f(q)$, we shall see that (13) cannot hold.

We get from (2) for $a\geq 2$ and $t\geq 1$ by substitution $n=(a^t-1)m$

$$(a^t-1)m \mid f(a^t m)-f(m)=f(m)[f(a^t)-1]=f(m)[f^t(a)-1]. \tag{14}$$

Let p be an arbitrary prime. Then $p^r \| f(m)$, where $r\geq 0$, and so $f(m)=p^r s$ with $(s,p)=1$. If $p\mid f(a)$, and $p\nmid a$, then $a^{\varphi(p^{r+1})}\equiv 1 \pmod{p^{r+1}}$ – where $\varphi(n)$ is the Euler-function. Hence by (14)

$$p \mid [f(a)]^{\varphi(p^{r+1})}-1. \tag{15}$$

Since $p\mid[f(a)]^{\varphi(p^{r+1})}$, by (15) $p\mid 1$, which is impossible – consequently

$$p \mid f(a) \text{ implies } p \mid a. \tag{16}$$

From (16)

$$f(a)\neq 0 \quad (a=1,2,\ldots). \tag{17}$$

Choose $t=1$, $a=p^x q$ in (14) (p and q being arbitrary distinct primes), and $f(p)=1$, then

$$(p^x q-1)m \mid f(m)[f(q)-1].$$

For sufficiently large x this gives that $f(q)=1$. Hence we obtain that

(18) if $f(p)=1$ for at least one prime p, then $f(n)\equiv 1 \quad (n=1,2,\ldots)$.

If a is a prime, then by (16) either $f(a)$ have no prime divisor $[f(a)=\pm 1, 0]$, or $f(a)$ have only a as a prime divisor $[f(a)=\pm a^{\alpha_a},\ \alpha_a\geq 1]$. By (17) $f(a)\neq 0$, by (18) $f(a)\neq 1$.

Choose $a=p$ and let t be a large odd integer in (14). Then

$$(p^t-1)m \mid f(m)[f^t(p)-1],$$

consequently $f(p)\neq -1$.

If $f(p)=-p^\alpha$ then for odd t by (14) we get

$$(p^t-1)m \mid f(m)[-p^\alpha)^t-1]=-f(m)[(p^t)^\alpha-1]-2f(m),$$

and so

$$p^t-1 \mid -2f(m).$$

For large t this is impossible. Consequently $f(p)=p^\alpha$.

Let p, q be arbitrary primes, $f(p) = p^\alpha$, $f(q) = q^\beta$, $\alpha > \beta$, and $a = pq^s$. Then from (14) by $t = 1$

$$(pq^s - 1)\, m \mid f(m)[f(p)\, f^s(q) - 1] = f(m)[p^\alpha (q^\beta)^s - 1] =$$
$$= f(m)[((pq^s)^\beta - 1 + 1)p^{\alpha-\beta} - 1] = f(m)\, p^{\alpha-\beta}[(pq^s)^\beta - 1] + (p^{\alpha-\beta} - 1)\, f(m),$$

consequently

$$pq^s - 1 \mid f(m)(p^{\alpha-\beta} - 1).$$

This is impossible for large s, consequently $\alpha = \beta$. If

$$n = \prod_{i=1}^{r} f(p_i^{\alpha_i}) \quad (r \geq 0),$$

then

$$f(n) = \prod_{i=1}^{r} f(p_i^{\alpha_i}) = \prod_{i=1}^{r} f^{\alpha_i}(p_i) = \prod_{i=1}^{r} (p_i^\alpha)^{\alpha_i} = \prod_{i=1}^{r} (p_i^{\alpha_i})^\alpha = \left(\prod_{i=1}^{r} p_i^{\alpha_i}\right)^\alpha = n^\alpha,$$

which finishes the proof of our theorem.

I am indepted to Professor I. KÁTAI for his valuable remarks.

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О ПЛОТНЕЙШЕЙ УПАКОВКИ n-МЕРНОГО ЦИЛИНДРА С ЕДИНИЧНЫМИ ШАРАМИ

Е. ХОРВАТ

Кафедра Начертательной и Проективной Геометрии Университета им. Л. Этвеша

Будапешт

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На соревновании по математике им. Миклоша Швейзера в 1962-ом году Л. ФЕЙЕШ ТОТ и А. ХЕЙПЕШ наметили следующую чадачу: „Доказываем, что отрезок a умеет вполне прободать, или касаться по крайней мере $2\left[\frac{a}{\sqrt{2}}\right]+2$ единичных шаров, не имеющих общих внутренних точек."

Л. ФЕЙЕШ ТОТ поставил обобщение этой проблемы в n-мерных эвклидовых пространствах ($n>3$). Он высказал аналогичную догадку, перерабатываемую нам следующим образом: Число единичных шаров, не имеющих внутренних точек и лежащих в n-мерном цилиндре с радиусом 2 и с высотой $h+2$ — по крайней мере $2\left[\frac{h}{\sqrt{2}}\right]+2$. Аналогичная проблема на плоскости еще неразрешенно.

Впоследствии мы обозначаем n-мерный цилиндр с радиусом r и с высотой h через $Z^n(r;h)$ и безконечный цилиндр с радиусом r через $Z^n(r)$.

В этой статье мы даем верхную оценку для числа N_n единичных шаров, не имеющих общих внутренних точек и лежащих в $Z^n(r;h)$, далее определяем плотнейшую упаковку единичных шаров в $Z^n(r)$, если $n\geqq 3$ и $1\leqq r\leqq 2$.

Вначале мы дефинируем плотность упаковки шаров в $Z^n(r)$. Пусть A — любая фиксированная точка на оси цилиндра $Z^n(r)$ и $Z^n(r;2h)$ — такой n-мерный цилиндр, симметрическая точка которого является точкой A. Величина

$$\delta_n(r)=\overline{\lim_{h\to\infty}}\,\frac{N_{2h}K_n(1)}{2h\,K_{n-1}(r)} \tag{1}$$

называется плотностью упаковки единичных шаров в $Z^n(r)$, где $K_m(r)$ является объемом m-мерного шара с радиусом r. Легко доказывать, что (1) независим от выбора точки A [1].

Доказываются следующие теоремы:

ТЕОРЕМА 1. *Пусть* $n \geqq 3$ *и* $1 \leqq r \leqq 2$. *Число единичных шаров, не имеющих общих внутренних точек и лежащих в n-мерном цилиндре* $Z^n(r; h)$ *с радиусом r и с высотой h — по крайней мере* $2\left[\frac{h-2}{\tau(r)}\right]+2$, где

$$\tau(r) = \begin{cases} 4\sqrt{2r-r^2}, & \text{если} \quad 1 \leqq r \leqq 1+\frac{\sqrt{3}}{2}, \\ 2\sqrt{4-2\sqrt{3}(r-1)}, & \text{если} \quad 1+\frac{\sqrt{3}}{2} \leqq r \leqq 1+\frac{4\sqrt{3}}{7}, \\ \sqrt{2+2\sqrt{2r-r^2}}, & \text{если} \quad 1+\frac{4\sqrt{3}}{7} \leqq r \leqq 2. \end{cases} \tag{2}$$

ТЕОРЕМА 2. *Пусть* $n \geqq 3$ *и* $1 \leqq r \leqq 2$. *Плотность плотнейшей упаковки единичных шаров в безконеном цилиндре* $Z^n(r)$ *с радиусом r —*

$$\delta_n(r) = \frac{K_n(1)}{K_{n-1}(r)} \cdot \frac{2}{\tau(r)}, \tag{3}$$

где функция $\tau(r)$ *определяется в* (2) *и* $K_m(r)$ *— объем m-мерного шара с радиусом r. В случаем плотнейшей упаковки, если* $r=1$, *то центры шаров лежат на оси цилиндра, если* $1 < r \leqq 1+\frac{\sqrt{3}}{2}$ *или* $1+\frac{\sqrt{3}}{2} < r < 2$, *то центры шаров и ось цилиндра лежат на 2-мерной, или на 3-мерной плоскостях. Если* $r=2$, *то плотнейшая упаковка может быть реализирована различным образом.*

Геометрическим значением функции $\tau(r)$ является следующее: Пусть O_i, O_j, O_K — центр трех любих шаров и O'_i, O'_j, O'_K — их перпендикулярная проекция на ось цилиндра.

$$\tau(r) = \min \cdot \max(O'_i O'_j;\ O'_j O'_k;\ O'_k O'_i). \tag{4}$$

В доказательстве теорем основную роль играет следующая лемма.

ЛЕММА. *Пусть A, B, C — точки в 3-мерном слое, ограничивающемся двумя паралельными плоскостями, далее* τ *— расстояние двух ограничивающих плоскостей и* min. $(AB, BC, CA) \geqq 2$. *Пусть* A^*, B^*, C^* *— перпендикулярная проекция точек A, B, C на одну из двух ограничивающих плоскостей и* $\varrho(\tau)$ *— минимум радиуса описанного круга треугольника* $A^*B^*C^*$. *То*

$$\varrho(\tau) = \begin{cases} \frac{4-\tau^2}{2\sqrt{3-\tau^2}}, & \text{если} \quad 0 \leqq \tau \leqq \sqrt{2}, \\ \frac{\tau\sqrt{4-\tau^2}}{2}, & \text{если} \quad \sqrt{2} \leqq \tau \leqq \frac{4}{\sqrt{7}}, \\ \frac{16-\tau^2}{8\sqrt{3}}, & \text{если} \quad \frac{4}{\sqrt{7}} \leqq \tau \leqq 2. \end{cases} \tag{5}$$

Этот минимум достигается только тогда, когда $AB = BC = CA = 2$, *точка* A *находится на одной, точка* B *на другой из двух ограничивающих плоскостей и расстояние точка* C *от одной ограничивающей плоскости по очереди* $0, \frac{1}{2}(\tau \pm \sqrt{16-7\tau^2})$, *или* $\frac{\tau}{2}$.

Доказательство леммы можно найти в [2].

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 1. Пусть $\{O_i\}$ — множество центров шаров, лежащих в цилиндре $Z^n(r; h)$ и $\{O_i'\}$ — множество их перпендикулярных проекций на ось цилиндра. Очевидно, что центры шаров лежат в цилиндре $Z^n(r-1; h-2)$ и соотношение $O_i' = O_j'$ можно только тогда, когда $r = 2$, далее $O_i' = O_j' = O_K'$ никогда реализируются. Можно обозначать центры шаров так, чтобы точка O_{j+1}' лежала на отрезке $O_j'O_{j+2}'$. Так как точки $\{O_i'\}$ лежат на отрезке длины $h-2$, из определения функции $\tau(r)$ следует, что число единичных шаров, лежащих в цилиндре $Z^n(r; h)$ — по крайней мере $2\left[\frac{h-2}{\tau(r)}\right]+2$. Из этого следует, что достаточно найти функцию $\tau(r)$.

Сперва мы определим функцию $\tau(r)$ в случае $n=3$, а потом в случае $n>3$.

1. Пусть $n=3$.

а) Если $1 \leqq r \leqq 1+\frac{\sqrt{3}}{2}$, то можно видеть, что $O_j'O_{j+1}' \geqq 2\sqrt{2r-r^2}$ и в случае равенства центры O_j, O_{j+1} и ось цилиндра лежат на одной плоскости. Подобным образом $O_{j+1}'O_{j+2}' \geqq 2\sqrt{2r-r^2}$. Из этого $O_j'O_{j+2}' \geqq 4\sqrt{2r-r^2}$ и в случае равенства центры O_j, O_{j+1}, O_{j+2} и ось цилиндра лежат на одной плоскости.

б) Пусть $1+\frac{\sqrt{3}}{2} < r \leqq 2$. Обозначим перпендикулярные проекции точек O_j, O_{j+1}, O_{j+2} на одну плоскость Π, перпендикулярную на ось цилиндра, через O_j^*, O_{j+1}^*, O_{j+2}^* и общую точку плоскости Π и оси цилиндра через K. Точки O_j^*, O_{j+1}^*, O_{j+2}^* лежат в кругу с центром K и с радиусом $r-1$.

Легко видеть, что функция $\varrho(\tau)$, определенная в (5), строго монотонная убывающая функция и в случае $\tau \geqq \sqrt{2}$ имеем $\varrho(\tau) \leqq 1$. Поэтому длина отрезка $O_j'O_{j+2}'$ минимально, если центры O_j, O_{j+1}, O_{j+2} расположатся соотвественно случаям 2 или 3 леммы. Итак из леммы следует, что

$$\tau(r) = \begin{cases} 2\sqrt{4-2\sqrt{3}(r-1)}, & \text{если} \quad 1+\frac{\sqrt{3}}{2} < r \leqq 1+\frac{4\sqrt{3}}{7}; \\ \sqrt{2+2\sqrt{2r-r^2}}, & \text{если} \quad 1+\frac{4\sqrt{3}}{7} < r \leqq 2. \end{cases}$$

Легко счислением показывать, что если дальнейшие шары расположатся соответственно случаям 2, или 3 леммы, то эти шары не имеют общих внутренних точек.

2. Пусть $n>3$ и $1\leqq r\leqq 2$. Мы покажем, что

$$O_j' O_{j+2}' \geqq \tau(r),$$

где равенство имеется только тогда, если ось цилиндра и центры O_j, O_{j+1}, O_{j+2} лежат в случае $1\leqq r\leqq 1+\frac{\sqrt{3}}{2}$ на одной 2-мерной плоскости и в случаях $1+\frac{\sqrt{3}}{2}<r\leqq 2$ на одной 3-мерной плоскости.

Пусть отрезок $O_j' O_{j+2}'$ — минимален. Покажется, что неравенство $O_j' O_{j+2}'<\tau(r)$ невозможно. Если ось цилиндра и центры O_j, O_{j+1}, O_{j+2} лежат на 3-мерной плоскости, то на основе случая 1. $O_j'O_{j+2}'=$ $=\tau(r)$. Если ось цилиндра и центры O_j, O_{j+1}, O_{j+2} лежат не на 3-мерной плоскости, то существует одна 4-мерная плоскость, на которой лежат ось и центры O_j, O_{j+1}, O_{j+2}. На этой плоскости мы проектируем ось и центры на 3-мерной плоскость, перпендикулярную на ось и обозначаем проекции оси и центров O_j, O_{j+1}, O_{j+2} через K^* и O_j^*, O_{j+1}^*, O_{j+2}^*. Точки O_j^*, O_{j+1}^*, O_{j+2}^* лежат в одном 3-мерном шаре S с центром K^* и с радиусом $r-1$. Очевидно, что плоскость, определенная точками O_j^*, O_{j+1}^*, O_{j+2}^*, не содержит точку K^*. Пересечение этой плоскости с 3-мерном шаром является кругом с центром C^* и с радиусом r^*-1, где $r^*<r$. Пусть e — прямая, переходящая через точку C^* и параллельная с осью цилиндра. Обозначаем перпендикулярные проекции центров O_j, O_{j+1}, O_{j+2} на прямую e через $\overline{O}_j$, $\overline{O}_{j+1}$, $\overline{O}_{j+2}$. Очевидно, что

$$K^* O_{j+k}' = O_{j+k}^* O_{j+k} = C^* \overline{O}_{j+k}, \quad (k=0,1,2).$$

Поэтому

$$O_j' O_{j+2}' = \overline{O}_j \overline{O}_{j+2}.$$

Центры O_j, O_{j+1}, O_{j+2} лежат в 3-мерном цилиндре $Z^3(r^*-1; h-2)$ с радиусом r^*-1 и с осью e, поэтому на основе случая 1. из-за $r^*<r$ имеем

$$O_j' O_{j+2}' \geqq \tau(r^*) > \tau(r).$$

Таким образом теорема 1. доказана.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМА 2. следует из доказательства теоремы 1.

В 3- и 4-мерных пространствах функция $\delta_n(r)$ в интервале $\left(1;\frac{2n-1}{n}\right)$ строго убывает и в интервале $\left(\frac{2n-1}{n};2\right)$ строго возрашает. В 5-, 6- и 7-мерных пространствах функция $\delta_n(r)$ строго убывает в интервалах $\left(1;\frac{2n-1}{n}\right)$ и $\left[1+\frac{\sqrt{3}}{2};\frac{(n-1)(4+2\sqrt{3})}{(2n-1)\sqrt{3}}\right]$ и строго возрашает в интерва-

лах $\left(\frac{2n-1}{n}; 1+\frac{\sqrt{3}}{2}\right)$ и $\left[\frac{(n-1)(4+2\sqrt{3})}{(2n-1)\sqrt{3}}; 2\right]$. В n-мерных пространстах ($n \geqq 8$) функция $\delta_n(r)$ строго убивает в интервале $(1; r_n)$ и строго возрашает в интервале $(r_n; 2)$, где $1+\frac{4\sqrt{3}}{7} < r_n < 2$.

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ЗАМЕЧАНИЯ К ТЕОРИИ ОБЛАКА ШАРОВ

Е. ХОРВАТ

Кафедра Начертательной и Проективной Геометрии Университета им. Л. Этвеша
Будапешт

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1. Множество $\{K_i^n\}$ n-мерных шаров образует упаковку в n-мерном эвклидавом пространстве E^n, если любие два шара множества $\{K_i^n\}$ не имеют общих внутренних точек. Упаковка называется решетчатой, если центры шаров, образующих эту упаковку, составляют решетку.

Упаковка является облаком в направлении некоторой данной k-мерной плоскости Π_k^n, если каждая k-мерная плоскость, параллельная данной k-мерной плоскости, пересекает или касается по крайней мере одного шара. Если такое множество шаров ортогонально проектируется на некоторую $(n-k)$-мерную плоскость, вполне перпендикулярную k данной k-мерной плоскости [4], то эта $(n-k)$-мерная плоскость вполне покрыта проекцией шаров. Если упаковка является облаком в направлении каждой k-мерной плоскости, то мы скажем, что она составляет темное облако относительно k-мерных плокостей.

Теперь мы определим цилиндр в пространстве E^n. Множество точек пространства E^n, отстаящих от данной $(n-k)$-мерной плоскости Π_{n-k}^n, называемой осью цилиндра, на расстояние $\leq r$, называемое радиусом цилиндра, называется n-мерным цилиндром $Z_k^{n-k}(r)$. Пересечение цилиндра $Z_k^{n-k}(r)$ с некоторой k-мерной плоскостью Π_k^n, вполне перпендикулярной k оси цилиндра Π_{n-k}^n, представляет k-мерный шар с радиусом r и с центром O. Центр O — точка пересечения плоскостей Π_{n-k}^n и Π_k^n. Этот k-мерный шар назавем основным шаром цилиндра $Z_k^{n-k}(r)$.

Слоем называется упаковка шаров в полосе между двумя параллельными гиперплоскостями в пространстве E^n и толщиной этого слоя называется расстояние между двумя гиперплоскостями. Если слой является облаком, то минимум толщины слоя называется *толщиной облака*.

В статье [5] Л. ФЕЙЕШ ТОТ определил толщину облака в пространстве E^3, если направление прямой Π_1^3 перпендикулярно к плоскостям, ограничивающим полосу. В этом случае толщина облака равна $2+\sqrt{2}$ (Рис. 1.).

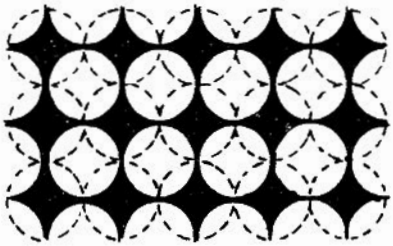

Рис. 1

А. ХЕППЕШ [7] показал, что решетчатая упаковка равными шарами в пространстве E^3 не является темным облаком. Он показал, что для каждой решетчатой упаковки равными шарами существуют прямые, параллельные с одным из трех линейно независимых направлений, и не имеющие общих точек ни с одним шаром. И. ХОРТОБАДИ [8] показал, что существуют по крайной мере три цилиндра $Z_2^1\left(\frac{3\sqrt{2}}{4}-1\right)$ с линейно независимыми положениями осей, не имеющих общих внутренних точек ни с одним шаром. Автор [9] показал существование $\binom{n}{k}$ цилиндров $Z_n^{n-k}(r)$ с разными положениями осей, не имеющих внутренних точек ни с одним шаром, где $n>3$, $2\leqq k\leqq n-1$, далее $r=\frac{3\sqrt{2}}{4}-1$, если $k=2$ и $r=\frac{\sqrt{5}}{2}-1$, если $3\leqq k\leqq n-1$.

В этой статье покажем, что в случае $4\leqq k\leqq n-1$ $r=\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}}-1=$ $=0{,}1280\ldots$ и даем верхную грань для толщины облака в пространстве E^4.

2. Доказываются следующие теоремы:

ТЕОРЕМА 1. *Пусть в пространстве E^5 задана решетчатая упаковка единичных шаров. Всегда существуют по крайней мере 3 цилиндра $Z_4^1\left(\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}}-1\right)$ с линейно независимыми положениями осей, которые не пересекают никаких шаров упаковки.*

ДОКАЗАТЕЛЬСТВО. Пусть $\mathbf{a}_5$ – наименьший вектор решетки, а $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ – такие векторы решетки, что векторы $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_1$, $\mathbf{a}_5$ представлдют основные векторы решетки в E^5. Если $|\mathbf{a}_5|>2$, то из определения вектора $\mathbf{a}_5$ следует, что шары упаковки можно отдвигать в направлении вектора $\mathbf{a}_5$ так, что $|\mathbf{a}_5|=2$ будет выполняться и шары упаковки не имеют общих витуренних точек [8].*

* Легко видеть, что в E^5 существуют по крайней мере 3 вектора, имеющих таким свойством.

Проектируем шары на некоторую гиперплоскость $\overline{\Pi}_4^5$, перпендикулярную к вектору $\mathbf{a}_5$. Проекции центра шаров определяет решетку в гиперплоскости $\overline{\Pi}_4^5$ и проекции самых шаров составляют решетчатое расположение 4-мерных шаров в гиперплоскости $\overline{\Pi}_4^5$.

Покажем, что эти 4-мерные шары не покрывают гиперплоскость $\overline{\Pi}_4^5$. Описываем через центры шаров в $\overline{\Pi}_4^5$ так называемые пустые шары [6], т.е. такие 4-мерные шары, которые не содержат центры внутри шаров и на сфере каждых пустых шаров находится по крайней мере 5 точек, не принадлежащих к одной и той же 3-мерной плоскости.

Пусть $C_1', C_2', C_3', C_4', C_0'$ — 5 точек на любой пустой сфере, не принадлежащих к одной и той же 3-мерной плоскости. Если W обьем 4-мерного параллелепипеда, определяемого векторами $\mathbf{c}_i' = \overrightarrow{C_0'C_i'}(i = 1, 2, 3, 4)$, то $\frac{W}{24}$ обьем симплекса, определяемого точками $C_0', C_1', C_2', C_3', C_4'$. Пусть R — радиус пустого шара этого симплекса. Общеизвестно, что обьем n-мерного симплекса, в вписанного в шар с радиусом R, не превосходит $\frac{(n+1)^{\frac{n+1}{2}}}{n!\, n^{\frac{n}{1}}} R^n$ [6]. Обьем махимален, если симплекс правилен. Поэтому

$$\frac{W}{24} \leqq \frac{25\sqrt{5}}{24 \cdot 16} R^4.$$

Из этого следует, что

$$R \geqq \sqrt[4]{\frac{16W}{25\sqrt{5}}}. \tag{1}$$

Из равенства $|\mathbf{a}_5| = 2$ выводит, что обьем 5-мерного параллелепипеда решетки в пространстве E^5 равен $2W$. Известно, что плотность решетчатой упаковки равными шарами в пространстве E^5 в крайнем случае — $\frac{\pi^2}{15\sqrt{2}}$ [10], поэтому

$$W \geqq 4\sqrt{2}. \tag{2}$$

Из неравенств (1) и (2) следует, что

$$R \geqq \sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}} = 1{,}1280\ldots. \tag{3}$$

В соотношении (3) знак равенства не допустим, потому что в (2) знак равенства имеется только тогда, когда плотность в пространстве E^5 равна $\frac{\pi^2}{15\sqrt{2}}$ [10], а в этом случае симплекс $C_0'C_1'C_2'C_3'C_4'$ не правилен.

Пусть C – центр пустого шара симплекса $C_0'C_1'C_2'C_3'C_4'$. 4-мерный шар с центром C и с радиусом $\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{2}}}-1$ не имеет общей точки с проекциями шаров решетчатой упаковки пространства E^5. Пусть этот 4-мерный шар – основной шар цилиндра $Z_4^1\left(\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}}-1\right)$ с положением оси паралельным к вектору $\mathbf{a}_5$ Цилиндр $Z_4^1\left(\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{2}}}-1\right)$ не имеет общей точки с шарами решетчатой упаковки в пространстве E^5.

Таким образом Теорема 1 доказана.

ТЕОРЕМА 2. *Пусть в пространстве E^n задана решетчатая упаковка единичных шаров. Существует не меньше $l\binom{m}{k}$ цилиндров $Z_n^{n-k}(r)$ с разными положениями осей, которые не имеют общих внутренних точек с шарами упаковки, где $n\geq 3$, $2\leq k\leq n-1$, далее*

$$\text{если } k=2, \text{ то } m=n,\ l=1 \text{ и } r=\frac{3\sqrt{2}}{4}-1=0{,}0606\ldots,$$

$$\text{если } k=3, \text{ то } m=n,\ l=1 \text{ и } r=\frac{\sqrt{5}}{2}-1=0{,}1180\ldots,$$

$$\text{если } 4\leq k\leq n-1, \text{ то случае } l=1,\ m=n, \text{ и } r=\frac{\sqrt{5}}{2}-1,$$

$$\text{прочем } l=3,\ m=n-1 \text{ и } r=\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}}-1=0{,}128\ldots.$$

ДОКАЗАТЕЛЬСТВО ТЕОРЕМА 2 в дословно совпадает с доказательством Теоремы 2 работы [9], если в случае $4\leq k\leq n-1$ применать результат нашей теоремы 1.

ЗАМЕЧАНИЕ. Блихфельд [1, 2, 3] определил плотнейшую решетчатую упаковку равными шарами в пространствах E^6, E^7 и E^8. С методом доказательства Теоремы 1 в этих случаях можно определить цилиндры $Z_k^1(r)$ $(k=5, 6, 7)$, не пересекающие никаких шаров упаковки. С счетом можно установить, что так полученные величины для радиуса цилиндра меньше чем $\sqrt[4]{\frac{64\sqrt{2}}{25\sqrt{5}}}-1$.

3. В следующем мы будем заниматься облаком шаров, определяемым некоторым слоем в пространстве E^4. Мы даем верхную грань для толщины облака, если направление прямой Π_1^1 перпендикулярно к 3-мерной гиперплоскости, ограничивающей полосу.

ТЕОРЕМА 3. *Если упаковка равными шарами в слое пространства E^4 составляет облако в направления прямой, перпендикулярной к 3-меной плоскости, органичивающей полосу, то толщина облака не превосходит*

$$\frac{1}{5}(10+4\sqrt{5}+2\sqrt{10}) = 5{,}05376\ldots.$$

ДОКАЗАТЕЛЬСТВО. Мы будем конструировать упаковку единичных шаров с шириной $\frac{1}{5}(10+4\sqrt{5}+2\sqrt{10})$ пространства E^4, которая представляет облако.

Пусть Π_3^4 — некоторая гиперплоскость в пространстве E^4. Рассмотрим наименее плотное решетчатое покрытие 3-мерными единичными шарами гиперплоскости Π_3^4. Мы будем исходить из пространственно центрированной кубической решетки, которая получается из точек обычной кубической решетки добавлением к ней центров всех кубов. Ребры кубов равны $\frac{4}{\sqrt{5}}$. Вершины кубической решетки сортируем в две группы A и B так, чтобы соседние вершины принадлежали к разным группам. Центры кубов составляют кубическую решетку, и их тоже сортируем подобно в две группы C и D. Точки типов B, C и D решетки переносим перпендикулярно к гиперплоскости Π_3^4 на расстояние $\frac{2}{\sqrt{5}}$, $\frac{2+2\sqrt{2}}{\sqrt{5}}$, или $\frac{4+2\sqrt{2}}{\sqrt{5}}$ соответственно в одном из двух полупространств, определяемых гиперплоскостью Π_3^4. 4-мерные единичные шары, описанные вокруг перенесенных точек типов A, B, C и D, не имеют общих внутренних точек в слое с шириной $2+\frac{1}{5}(4\sqrt{5}+2\sqrt{10})$ пространства E^4. Очевидно, что перпендикулярная проекция шаров на гиперплоскость Π_3^4 покрывает эту гиперплоскость.

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CONCERNING THE ZEROS OF SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS OF THE n-TH ORDER, WHERE n IS ODD

By

RAHMI IBRAHIM IBRAHIM ABDEL KARIM

Cairo University, Faculty of Science, Mathematical Department, Egypt

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We consider the differential equations of the n-th order of the form

$$[p(x)y^{(m)}]^{(n-m)}+q(x)y=0 \tag{1}$$

and

$$[p(x)z^{(n-m)}]^{(m)}-q(x)z=0, \tag{2}$$

where $p(x)>0$ and $q(x)$ are continuous functions of $x\in(-\infty,\infty)$, n is a positive odd integer and m and $n-m$ are positive integers. In [1] the relations between the solutions of the differential equations (1) and (2) are investigated.

In this note we deal with the zeros of solutions of differential equations of the form (1) or (2).

We state the following

THEOREM 1. *The solutions of the differential equations* (1) *or* (2) *with the properties*

(3)

$$y(a)=y'(a)=\ldots=y^{(m)}(a)=(py^{(m)})'(a)=\ldots=(py^{(m)})^{(n-m-2)}(a)=0,$$
$$(py^{(m)})^{(n-m-1)}(a)\neq 0$$

or

(4)

$$z(a)=z'(a)=\ldots=z^{(n-m)}(a)=(pz^{(n-m)})'(a)=\ldots=(pz^{(n-m)})^{(m-2)}(a)=0,$$
$$(pz^{(n-m)})^{(m-1)}(a)\neq 0,$$

$-\infty<a<\infty$ *respectively are linearly dependent.*

PROOF. Let $y(x)$ be an arbitrary solution of the differential equation (1) satisfying (3) with $(py^{(m)})^{(n-m-1)}(a)=c\neq 0$ and let $y_r(x)$ $(r=1,2,\ldots,n)$ form a fundamental system of solutions of the differential

equation (1). Writing $y(x) = \sum_1^n c_r y_r(x)$ and evaluating these constants c_r, we obtain

$$y(x) = \frac{cy^*(x)}{(py^{*(m)})^{(n-m-1)}(a)}.$$

Here

$$y^*(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1(a) & y_2(a) & \dots & y_n(a) \\ y_1'(a) & y_2'(a) & \dots & y_n'(a) \\ \vdots & \vdots & & \vdots \\ y_1^{(m)}(a) & y_2^{(m)}(a) & \dots & y_n^{(m)}(a) \\ (py_1^{(m)})'(a) & (py_2^{(m)})'(a) & \dots & (py_n^{(m)})'(a) \\ \vdots & \vdots & & \vdots \\ (py_1^{(m)})^{(n-m-2)}(a) & (py_2^{(m)})^{(n-m-2)}(a) & \dots & (py_n^{(m)})^{(n-m-2)}(a) \end{vmatrix}$$

is a solution of the differential equation (1) satisfying (3). The proof for the differential equation (2) is similar.

It can be also shown, that the solutions of the differential equation (1) with one of the properties

$$y(a) = y'(a) = \dots = (py^{(m)})^{(n-m-3)}(a) = (py^{(m)})^{(n-m-1)}(a) = 0,$$
$$\vdots \qquad\qquad (py^{(m)})^{(n-m-2)}(a) \neq 0,$$

or

$$y'(a) = y''(a) = \dots = (py^{(m)})^{(n-m-1)}(a) = 0, \; y(a) \neq 0,$$

$-\infty < a < \infty$ are linearly dependent. The analogous statement holds also for the differential equation (2).

We note, that if $n-m-i<0$ $[m-i<0]$, where i is a positive integer, then $(py^{(m)})^{(n-m-i)}$ $[(pz^{(n-m)})^{(m-i)}]$ is interpreted as $y^{(n-i)}$ $[z^{(n-i)}]$.

We consider two differential equations of the form (1), i.e. with the differential equation (1) we consider also the differential equation

$$[p(x)u^{(m)}]^{(n-m)} + Q(x)u = 0, \tag{5}$$

where $p(x)>0$ and $Q(x)$ are continuous functions of $x \in (-\infty, \infty)$.

It is easy to prove, that if $y(x)$ and $u(x)$ are two solutions of the differential equations (1) and (5) with the same initial conditions at the point a, $-\infty < a < \infty$, then the solution of the differential equation (1) can be written as

$$y(x) = u(x) + \int_a^x [Q(t) - q(t)] \frac{W(x,t)}{W(t)} y(t)\, dt = \tag{6}$$

$$= u(x) + \int_a^x [Q(t) - q(t)] p(t)\, W(x,t)\, y(t)\, dt.$$

$W(x, t)$ is a function of the form

$$(7)\qquad W(x,t) = \begin{vmatrix} u_1(x) & u_2(x) & \dots & u_n(x) \\ u_1'(t) & u_2'(t) & \dots & u_n'(t) \\ u_1''(t) & u_2''(t) & \dots & u_n''(t) \\ \vdots & \vdots & & \vdots \\ u_1^{(m)}(t) & u_2^{(m)}(t) & \dots & u_n^{(m)}(t) \\ (pu_1^{(m)})'(t) & (pu_2^{(m)})'(t) & \dots & (pu_n^{(m)})'(t) \\ \vdots & \vdots & & \vdots \\ (pu_1^{(m)})^{(n-m-2)}(t) & (pu_2^{(m)})^{(n-m-2)}(t) & \dots & (pu_n^{(m)})^{(n-m-2)}(t) \end{vmatrix},$$

where $u_1, u_2, \dots, u_n$ form a fundamental system of solutions of the differential equation (5) whose Wronskian is $W(x) = \dfrac{1}{p(x)}$.

THEOREM 2. *Let $0 \leqq Q(x) < q(x)$ $[q(x) < Q(x) \leqq 0]$, $-\infty < x < \infty$. Let $y(x)$ and $u(x)$ be two solutions of the differential equations* (1) *and* (5) *with the properties*

$$(8)\begin{cases} y(a) = y'(a) = \dots = y^{(m)}(a) = (py^{(m)})'(a) = \dots = (py^{(m)})^{(n-m-2)}(a) = 0, \\ py^{(m)})^{(n-m-1)}(a) \neq 0, \\ u(a) = u'(a) = \dots = u^{(m)}(a) = (pu^{(m)})'(a) = \dots = (pu^{(m)})^{(n-m-2)}(a) = 0, \\ (pu^{(m)})^{(n-m-1)}(a) \neq 0; \quad -\infty < a < \infty \end{cases}$$

and let $x^* > a$ $[x^* < a]$ be the first zero point on the right [left] of a of $u(x)$. Then $y(x)$ has at least one zero point in (a, x^*) $[(x^*, a)]$. Further, if $y(x)$ has no zero point on the right [left] of a, then $u(x)$ has also no zero point on the right [left] of a.

PROOF. Consider the case $0 \leqq Q(x) < q(x)$, $x \in (-\infty, \infty)$. Let $y(x)$ and $u(x)$ be solutions of the differential equations (1) and (5) satisfying (8) and such that $(py^{(m)})^{(n-m-1)}(a) = (pu^{(m)})^{(n-m-1)}(a) > 0$ (see theorem 1). By virtue of (6), we have

$$(9)\qquad y(x^*) = u(x^*) + \int_a^{x^*} [Q(t) - q(t)]\, p(t)\, W(x^*, t) y(t)\, dt,$$

where x^* is the first zero point on the right of a of $u(x)$ ([1], §2). For fixed t, the function $W(x, t) = \bar{u}(x)$ which is defined in (7) is a solution of the differential equation (5) with the properties

$$\bar{u}(t) = \bar{u}'(t) = \dots = (p\bar{u}^{(m)})^{(n-m-2)}(t) = 0,\ (p\bar{u}^{(m)})^{(n-m-1)}(t) = \frac{1}{p(t)} > 0$$

and therefore $W(x, t) \geqq 0$ for $t \leqq x \in [a, x^*]$. Hence from (9), the requirement follows.

Similarly the other statement can be proved.

REMARK. The preceding theorem holds also for two differential equations of the form (2), if the assumption $0 \leqq Q(x) < q(x)$ $[q(x) < Q(x) \leqq 0]$, $x \in (-\infty, \infty)$ is replaced by $Q(x) < q(x) \leqq 0$ $[0 \leqq q(x) < Q(x)]$, $x \in (-\infty, \infty)$ respectively and the initial conditions (8) are replaced by the analogous initial conditions as (4).

We consider now the differential equation

$$[p(x)y^{(m)}]^{(n-m)} + \lambda q(x)y = 0, \tag{10}$$

where $p(x) > 0$, $q(x) \geqq 0$ are continuous functions of $x \in (-\infty, \infty)$, λ is a parameter and $q(x) \equiv 0$ does not hold in any interval.

Then we state

THEOREM 3. *Let $-\infty < a < b < \infty$ and let $y(x)$ be the solution of the differential equation* (10) *with the property* (3). *Then there exists a parameter $\lambda > 0$ such that $y(x)$ has at least one zero point in (a, b).*

PROOF. Let $y(x)$ be the solution of the differential equation (10) satisfying (3) and with $(py^{(m)})^{(n-m-1)}(a) = 1$. Considering also the differential equation $(pY^{(m)})^{(n-m)} = 0$, whose fundamental system of solution is ($r = 1, 2, \ldots, m$; $s = 1, 2, \ldots, n-m$)

$$Y_r = \frac{(x-a)^{r-1}}{(r-1)!}, \quad Y_{m+s} = \frac{1}{(m-1)!\,(s-1)!} \int_a^x \frac{(x-t)^{m-1}(t-a)^{s-1}}{p(t)}\, dt$$

and their Wronski determinant is $W(x) = \dfrac{1}{p(x)}$ [2]. It is evident, that Y_n and y satisfy the same initial conditions at the point a. Applying the formula (6), we find

$$y(x) = \frac{1}{(m-1)!\,(n-m-1)!} \left[\int_a^x \frac{(x-t)^{m-1}(t-a)^{n-m-1}}{p(t)}\, dt - \right.$$

$$\left. - \lambda \int_a^x \left(q(t)y(t) \int_t^x \frac{(x-\tau)^{m-1}(\tau-t)^{n-m-1}}{p(\tau)}\, d\tau \right) dt \right].$$

Assuming on the contrary, that $y(x)$ has no zero point in (a, b), i.e. $y(x, \lambda) > 0$ for $x \in (a, b)$, $\lambda > 0$, a contradiction follows since the function $y(b, \lambda)$ is continuous in λ and is negative for $\lambda > 0$ large enough.

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ON PERTURBATIONS OF VAN DER POL'S EQUATION

By

I. FARKAS and M. FARKAS

Budapest University of Technology, Budapest

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In this paper the non-constant periodic solution of van der Pol's equation is considered under small periodic perturbations. It is assumed that the perturbation is "controllably periodic" i.e. its period can be chosen appropriately. The existence and uniqueness (in a certain sense) of a period and a periodic solution of the perturbed equation will be proved. Poincaré's method will be applied for the determination of the first approximation of this period and solution. Sufficient criterion will be given for the asymptotic stability of the perturbed periodic solution. The results are based on those of papers [1] and [2] and are close to and generalizations of W. S. Loud's results ([3]).

1. As it is well known ([4]) van der Pol's differential equation

$$\ddot{u}+m(u^2-1)\dot{u}+u=0, \tag{1.1}$$

where m is a positive constant and dot denotes differentiation with respect to t, has a non-constant periodic solution. This solution, will be denoted by $u_0(t)$ and its (least positive) period by τ_0. Without loss of generality it will be assumed that

$$u_0(0)=a>0, \quad \dot{u}_0(0)=0. \tag{1.2}$$

Following Loud by introducing the notations

$$\begin{aligned} x_1 &= -\dot{u}+m\left(\frac{1}{3}a^3-a\right)-m\left(\frac{1}{3}u^3-u\right) \\ x_2 &= u-a \end{aligned} \tag{1.3}$$

the differential equation (1.1) is equivalent to the system

$$\begin{aligned} \dot{x}_1 &= x_2+a \\ \dot{x}_2 &= -x_1+m\left(\frac{1}{3}a^3-a\right)-m\left[\frac{1}{3}(x_2+a)^3-(x_2+a)\right]. \end{aligned} \tag{1.4}$$

Introducing the notations

$$x = \operatorname{col}(x_1, x_2)$$

$$f(x) = \operatorname{col}\left\{x_2 + a,\ -x_1 + m\left(\frac{1}{3}a^3 - a\right) - m\left[\frac{1}{3}(x_2+a)^3 - (x_2+a)\right]\right\}$$

system (1.4) will be written in the vector form

$$\dot{x} = f(x). \tag{1.5}$$

(In what follows, the coordinates of a vector z will be denoted by z_1 and z_2). The periodic solution of (1.5) corresponding to $u_0(t)$ is

$$p(t) = \operatorname{col}\left\{-\dot{u}_0(t) + m\left(\frac{1}{3}a^3 - a\right) - m\left[\frac{1}{3}\dot{u}_0^3(t) - u_0(t)\right],\quad u_0(t) - a\right\}. \tag{1.6}$$

The first variational system of (1.5) corresponding to the solution (1.6) is

$$\dot{y} = f_x[p(t)]y, \tag{1.7}$$

where

$$f_x[p(t)] = \begin{bmatrix} 0 & 1 \\ -1 & -m[u_0^2(t) - 1] \end{bmatrix}.$$

A solution of the homogeneous linear system (1.7) is

$$\dot{p}(t) = \operatorname{col}[u_0(t),\quad \dot{u}_0(t)]. \tag{1.8}$$

If we denote by $v_0(t)$ the solution of the differential equation

$$\ddot{v} + m[u_0^2(t) - 1]\dot{v} + v = 0$$

for which

$$v_0(0) = 0,\quad \dot{v}_0(0) = \frac{1}{a} \tag{1.9}$$

holds, then $\operatorname{col}[v_0(t), \dot{v}_0(t)]$ is a second independent solution of (1.7). Let us denote the Wronskian of these solutions by $W(t)$. It follows from (1.2) and (1.9) that $W(0) = 1$, thus by Liouville's formula

$$W(t) = \exp\left\{-m\int_0^t [u_0^2(\alpha) - 1]\,d\alpha\right\}.$$

The fundamental matrix solution of (1.7) which at the point $t = 0$ is equal to the unit matrix U is

$$Y_0(t) = \begin{bmatrix} \frac{1}{a}u_0(t) & av_0(t) \\ \frac{1}{a}\dot{u}_0(t) & a\dot{v}_0(t) \end{bmatrix} \tag{1.10}$$

Since $W(t) = \det Y_0(t)$, therefore the characteristic matrix of system (1.7) is $Y_0(\tau_0)$ and

$$\det Y_0(\tau_0) = W(\tau_0) = \exp\left\{-m \int_0^{\tau_0} [u_0^2(t) - 1]\, dt\right\}. \tag{1.11}$$

The product of the characteristic multipliers of system (1.7) is equal to the latter determinant. Since (1.7) has $\dot{p}(t)$ as a non-constant periodic solution, 1 is a characteristic multiplier. Thus the second characteristic multiplier is equal to (1.11).

M. Urabe states in [5] that the second characteristic multiplier is less than one for arbitrary positive values of the parameter m. By numerical methods he actually determines the value of this multiplier for values of m less than 20. By (1.11) the statement that the second characteristic multiplier is less than 1, is equivalent to the inequality

$$\int_0^{\tau_0} [u_0^2(t) - 1]\, dt > 0. \tag{1.12}$$

2. Set

$$I_t = \{t : -\infty < t < \infty\},$$

R^2 the "phase plane" of $(u, \dot{u})$,

$$I_\mu = \{\mu : |\mu| < \alpha\} \quad \text{for some } \alpha > 0,$$
$$I_\tau = \{\tau : |\tau - \tau_0| < \beta\} \quad \text{for some } 0 < \beta < \tau_0,$$

and let $\gamma\left(\frac{t}{\tau}, u, \dot{u}, \mu, \tau\right)$ be a function analytic in $I_t \times R^2 \times I_\mu \times I_\tau$ and periodic in the variable t with period τ (for each fixed $\tau \in I_\tau$). Consider the perturbed van der Pol's equation

$$\ddot{u} + m(u^2 - 1)\dot{u} + u = \mu\, \gamma\left(\frac{t}{\tau}, u, \dot{u}, \mu, \tau\right). \tag{2.1}$$

Using the notations (1.3) the system equivalent to the differential equation (2.1) can be written in the form

$$\dot{x} = f(x) + \mu\, g\left(\frac{t}{\tau}, x, \mu, \tau\right), \tag{2.2}$$

where x and f are the same as in (1.5) and

$$g\left(\frac{t}{\tau}, x, \mu, \tau\right) = \operatorname{col}\left[-\gamma\left\{\frac{t}{\tau}, x_2 + a, -x_1 + m\left(\frac{1}{3}a^3 - a\right) - \right.\right.$$
$$\left.\left. - m\left(\frac{1}{3}(x_2 + a)^3 - (x_2 + a)\right), \mu, \tau\right\}, 0\right].$$

$x(t;\ \vartheta, h, \mu, \tau)$ denotes the solution of system (2.2) that satisfies the initial condition

$$x(\vartheta;\vartheta, h, \mu, \tau) = \mathrm{col}\,(0,\ h).$$

As it was shown in [1] this takes care of all solutions of (2.2) which have initial conditions (i.e. ϑ and h) close to those of $p(t)$, i.e. $p(0) = \mathrm{col}\,(0, 0)$, for small enough $|\mu|$, $|\vartheta|$ and $|\tau-\tau_0|$. The corresponding solution of the scalar equation (2.1) will be denoted by $u(t;\ \vartheta, a+h, \mu, \tau)$. As it is seen from (1.3)

$$u(\vartheta;\vartheta, a+h, \mu, \tau) = a+h.$$

There holds the following

THEOREM 1. *If the inequality* (1.12) *holds then each* (μ, ϑ), *for which* $|\mu|$ *and* $|\vartheta|$ *is small enough, determines uniquely a period* $\tau(\mu, \vartheta)$ *and a number* $h(\mu, \vartheta)$ *such that the solution*

$$w(t;\mu, \vartheta) = u[t;\vartheta, a+h(\mu, \vartheta), \mu, \tau(\mu, \vartheta)] \tag{2.3}$$

of the perturbed van der Pol equation (2.1) *(where also* $\tau = \tau(\mu, \vartheta)$*) is periodic with period* $\tau(\mu, \vartheta)$; *the functions* (2.3), $\tau(\mu, \vartheta)$ *and* $h(\mu, \vartheta)$ *are analytic in a neighbourhood of* $(0, 0)$; $\tau(0, 0) = \tau_0$; $h(0, 0) = 0$ *and* $w(t;\ 0, 0) = u_0(t)$.

PROOF. Since condition (1.12) implies that the second characteristic multiplier of system (1.7) is in modulus less than one, this statement is an immediate consequence of Theorem 1. of [2].

The following theorem gives in explicit form the first approximation of the period and the periodic solution of the perturbed equation in terms of the periodic solution $u_0(t)$ of the unperturbed van der Pol's equation. Note that the function $v_0(t)$ occurring in the matrix (1.10) can easily be expressed with $u_0(t)$ (at least locally) since both are (independent) solutions of the same linear second order differential equation (see [3]).

Introducing yet the notation

$$\gamma^*(t) = \gamma\left[\frac{t+\vartheta}{\tau_0}, u_0(t), \dot{u}_0(t), 0, \tau_0\right]$$

there holds

THEOREM 2. *If conditions of Theorem* 1 *are valid then the period and the periodic solution, corresponding to* (μ, ϑ) *are given by*

$$\tau(\mu, \vartheta) = \tau_0 + \mu\,\tau_1(\vartheta) + o(\mu), \tag{2.4}$$

where

$$\tau_1(\vartheta) = \int_0^{\tau_0} \frac{\gamma^*(r)}{W(r)}\left\{\dot{v}_0(r) - \frac{v_0(\tau_0)}{a[1-W(\tau_0)]}\,\dot{u}_0(r)\right\}dr, \tag{2.5}$$

and

$$w(t;\mu,\vartheta) = u_0(t-\vartheta)+\mu\left[\dot{v}_0(t-\vartheta)\int_0^{t-\vartheta}\frac{\gamma^*(r)\dot{u}_0(r)}{W(r)}\,dr-\right.$$

$$-\dot{u}_0(t-\vartheta)\int_0^{t-\vartheta}\frac{\gamma^*(r)v_0(r)}{W(r)}\,dr-$$

$$\left.-\frac{W(\tau_0)}{W(\tau_0)-1}\dot{v}_0(t-\vartheta)\int_0^{\tau_0}\frac{\gamma^*(r)\dot{u}_0(r)}{W(r)}\,dr\right]+o(\mu), \tag{2.6}$$

respectively.

PROOF. Poincaré's method, worked out for the present case in [2], will be applied. Denote by $\varphi(t;\ \mu,\ \vartheta)$ the periodic solution with period $\tau(\mu,\vartheta)$ of system (2.2) — where $\tau=\tau(\mu,\vartheta)$ — corresponding to $w(t;\mu,\vartheta)$. Accordingly

$$\dot{\varphi}(t;\mu,\vartheta)\equiv f[\varphi(t;\mu,\vartheta)]+\mu\, g\left[\frac{t}{\tau(\mu,\vartheta)},\varphi(t;\mu,\vartheta),\mu,\tau(\mu,\vartheta)\right]. \tag{2.7}$$

Introducing the new independent variable s by the transformation formula

$$t=\vartheta+s\,\tau(\mu,\vartheta)$$

and setting

$$\psi(s;\mu,\vartheta)=\varphi[\vartheta+s\,\tau(\mu,\vartheta);\mu,\vartheta],$$

$$\psi(s;\mu,\vartheta)=\sum_{k=0}^{\infty}\mu^k\psi^k(s;\vartheta),$$

$$\tau(\mu,\vartheta)=\sum_{k=0}^{\infty}\mu^k\tau_k(\vartheta),$$

we have $\psi^0(s;\vartheta)=p(s\tau_0)$ and $\tau_0(\vartheta)=\tau_0$. Expand the right and the left hand side of (2.7) by powers of μ. Equating coefficients of μ we have that

$$\frac{d\psi^1(s;\vartheta)}{ds}\equiv\tau_0\begin{bmatrix}0 & 1\\ -1 & -m[u_0^2(s\tau_0)-1]\end{bmatrix}\psi^1(s;\vartheta)+\tau_1(\vartheta)\begin{bmatrix}u_0(s\tau_0)\\ \dot{u}_0(s\tau_0)\end{bmatrix}+$$

$$+\tau_0\, g\left[s+\frac{\vartheta}{\tau_0},p(s\tau_0),0,\tau_0\right].$$

Thus $\psi^1(s;\vartheta)$ satisfies an inhomogeneous linear differential system. It is easy to see that a fundamental matrix solution, of the corresponding

homogeneous system is $Y_0(s\tau_0)$, where the matrix Y_0 is given in (1.10) and $Y_0(0) = U$, the unit matrix. Thus introducing the notations

$$\psi^1(0;\vartheta) = \mathrm{col}\,(c_1, c_2) = c,$$

$$b(r) = \tau_1(\vartheta)\,\mathrm{col}\,[u_0(r\tau_0), \dot{u}_0(r\tau_0)] + \tau_0\, g\left[r + \frac{\vartheta}{\tau_0}, p(r\tau_0), 0, \tau_0\right].$$

we have

$$\psi^1(s;\vartheta) = Y_0(s\tau_0)\,c + Y_0(s\tau_0)\int_0^s Y_0^{-1}(r\tau_0)\,b(r)\,dr.$$

From conditions satisfied by the vector solution φ follows that the first coordinate of $\psi^1(0, \vartheta)$ is zero, i.e.

$$c_1 = \psi_1^1(0;\vartheta) = 0,$$

and that $\psi^1(s;\vartheta)$ is periodic with period 1, i.e.

$$\psi^1(1;\vartheta) = \psi^1(0,\vartheta).$$

The last equation yields an algebraic linear system of equations for the unknowns c_2 and $\tau_1(\vartheta)$. By solving this system we get for $\tau_1(\vartheta)$ the expression (2.5) and for the coordinates of the vector $\psi^1(s;\vartheta)$ the following expressions

$$\psi_1^1(s;\vartheta) = s u_0(s\tau_0)\,\tau_1(\vartheta) - \frac{W(\tau_0)\,v_0(s\tau_0)}{W(\tau_0)-1}\int_0^{\tau_0}\frac{\gamma^*(r)\,\dot{u}_0(r)}{W(r)}\,dr -$$

$$- u_0(s\tau_0)\int_0^{s\tau_0}\frac{\gamma^*(r)\,v_0(r)}{W(r)}\,dr + v_0(s\tau_0)\int_0^{s\tau_0}\frac{\gamma^*(r)\,\dot{u}_0(r)}{W(r)}\,dr.$$

$$\psi_2^1(s;\vartheta) = \dot{v}_0(s\tau_0)\int_0^{s\tau_0}\frac{\gamma^*(r)\,\dot{u}_0(r)}{W(r)}\,dr - \dot{u}_0(s\tau_0)\int_0^{s\tau_0}\frac{\gamma^*(r)\,\dot{v}_0(r)}{W(r)}\,dr -$$

$$(2.8)\qquad - \frac{W(\tau_0)}{W(\tau_0)-1}\,\dot{v}_0(s\tau_0)\int_0^{\tau_0}\frac{\gamma^*(r)\,\dot{u}_0(r)}{W(r)}\,dr + s\,\dot{u}_0(s\tau_0)\,\tau_1(\vartheta).$$

Taking into account (1.3) and (1.6) we get that

$$w[\vartheta + s\tau(\mu,\vartheta), \mu, \vartheta] = a + \psi_2(s;\mu,\vartheta) = a + p_2(s\tau_0) + \mu\,\psi_2^1(s;\vartheta) + o(\mu) =$$

$$= u_0(s\tau_0) + \mu\,\psi_2^1(s;\vartheta) + o(\mu),$$

or going back to the variable t

$$w(t;\mu,\vartheta) = u_0\left(\frac{t-\vartheta}{\tau(\mu,\vartheta)}\,\tau_0\right) + \mu\,\psi_2^1\left(\frac{t-\vartheta}{\tau(\mu,\vartheta)};\vartheta\right) + o(\mu).$$

Substituting (2.8) into the last expression and expanding again everything by powers of μ we get (2.6) which was to be proved.

3. Applying Theorems 2 and 4 of [2] it is possible to give an explicit sufficient condition for the asymptotic stability of the periodic solution $w(t;\mu,\vartheta)$ of the perturbed van der Pol equation. No quantities beyond those determined in paragraph **2** are needed. The nature of the formula (1.3) which transforms the differential equation (2.1) into the system (2.2) ensures that the asymptotic stability of the solution $\varphi(t;\mu,\vartheta)$ of the system implies the asymptotic stability of the corresponding solution $w(t;\mu,\vartheta)$ of (2.1).

THEOREM 3. *If conditions of* Theorem 1 *are valid then there exist* $\varrho_1>0$, $\varrho_2>0$ *such that the periodic solution* $w(t;\mu,\vartheta)$ *of* (2.1) [*where* $\tau=\tau(\mu,\vartheta)$] *is asymptotically stable for all* (μ,ϑ) *satisfying the conditions*

$$|\mu|<\varrho_1,\quad |\vartheta|<\varrho_2$$

$$\mu\lambda'_\mu(0,0)<0$$

where

$$(3.1)\qquad \lambda'_\mu(0,0)=\int_0^{\tau_0}\frac{\gamma'_t\left[\frac{t}{\tau_0},u_0(t),\dot u_0(t),0,\tau_0\right]}{W(t)}\left[\dot v_0(t)-\frac{v_0(\tau_0)\,\dot u_0(t)}{a[1-W(\tau_0)]}\right]dt.$$

[Here γ'_t denotes the partial derivative with respect to t of the function γ occurring in (2.1)].

PROOF. For given (μ,ϑ) which are in modulus sufficiently small consider the periodic solution $\varphi(t;\mu,\vartheta)$ of (2.2) where $\tau=\tau(\mu,\vartheta)$. Denote by $\lambda(\mu,\vartheta)$ the characteristic multiplier of the first variational system of (2.2) corresponding to the solution $\varphi(t;\mu,\vartheta)$, for which $\lambda(0,0)=1$ holds. If $\lambda'_\mu(0,0)$ denotes the partial derivative with respect to μ at $(0,0)$ of the function $\lambda(\mu,\vartheta)$ our statement follows from Theorem 2 of [2] and only (3.1) is left to be proved. In proving this formula we shall make use of Theorem 4 of paper [2] and apply the notations introduced there. Thus by (1.10):

$$C_0(0)=Y_0(\tau_0)=\begin{bmatrix}1 & av_0(\tau_0)\\ 0 & a\dot v_0(\tau_0)\end{bmatrix},$$

$$W(\tau_0)=\det Y_0(\tau_0)=a\dot v_0(\tau_0),$$

$$d(\lambda;0,0)=\begin{vmatrix}1-\lambda & av_0(\tau_0)\\ 0 & a\dot v_0(\tau_0)-\lambda\end{vmatrix},$$

$$d'_\lambda(1;0,0)=1-W(\tau_0),$$

$$B(t,0)=\left[f''_{xx}[p(t)]\varphi'_\mu(t;0,0)+g'_x\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]Y_0(t)$$

where

$$\varphi'_\mu(t;0,0) = \psi^1\left(\frac{t}{\tau_0}, 0\right) - \dot{p}(t)\,\frac{t\,\tau_1(0)}{\tau_0}, \tag{3.2}$$

and

$$f''_{xx}[p(t)]\,\varphi'_\mu(t;0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -2mu_0(t)\varphi'_{2\mu}(t;0,0) \end{bmatrix}.$$

Denoting the elements of the matrix $C_0(0)$ and $C_1(0)$ by c^0_{ik} and c^1_{ik}, respectively ($i, k = 1, 2$), by the quoted Theorem of [2]

$$\lambda'_\mu(0,0) = \frac{1}{W(\tau_0)-1}\left\{\begin{vmatrix} c^1_{11} & c^1_{12} \\ c^0_{21} & c^0_{22}-1 \end{vmatrix} + \begin{vmatrix} c^0_{11}-1 & c^0_{12} \\ c^2_{21} & c^2_{22} \end{vmatrix}\right\} =$$

$$= \frac{1}{W(\tau_0)-1}\left\{c^1_{11}\,[W(\tau_0)-1] - c^1_{21}\,a v_0(\tau_0)\right\} = c^1_{11} + \frac{c^1_{21}\,a v_0(\tau_0)}{1-W(\tau_0)}.$$

After long but simple calculations and substitutions introducing the row vector

$$q(t) = \frac{1}{W(t)}\left[\dot{v}_0(t) - \frac{v_0(\tau_0)\,\dot{u}_0(t)}{a[1-W(\tau_0)]},\ -v_0(t) + \frac{v_0(\tau_0)\,u_0(t)}{a[1-W(\tau_0)]}\right]$$

we get the expression

$$\lambda'_\mu(0,0) = -\frac{a v_0(\tau_0)}{1-W(\tau_0)}\,\tau_1(0) +$$

$$+ \int_0^{\tau_0} q(t)\left\{f''_{xx}[p(t)]\varphi'_\mu(t;0,0) + g'_x\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right)\right\}\dot{p}(t)\,dt, \tag{3.3}$$

where the integrand is a scalar product.

It is easy to verify that the transpose of $q(t)$, i.e. the column vector $q^T(t)$ satisfies the differential system

$$\dot{q}^T(t) \equiv -f'^T_x[p(t)]\,q^T(t),$$

or taking the transpose of both sides the row vector $q(t)$ satisfies

$$\dot{q}(t) = -q(t)\,f'_x[p(t)]; \tag{3.4}$$

further that $q(t)$ is periodic with period τ_0. On the other hand it is easy to see that $\varphi'_\mu(t;0,0)$ satisfies the system

$$\dot{\varphi}'_\mu(t;0,0) \equiv f'_x[p(t)]\,\varphi'_\mu(t;0,0) + g\left(\frac{t}{\tau_0}, p(t), 0, \tau_0\right). \tag{3.5}$$

Now, consider the following derivative

$$\left\{q(t)\left[f'_x(p(t))\,\varphi'_\mu(t;0,0)+g\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]\right\}^{\cdot}=$$

$$=\dot{q}(t)\left[f'_x(p(t))\,\varphi'_\mu(t;0.0)+g\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]+$$

$$+q(t)\left[f''_{xx}(p(t))\,\varphi'_\mu(t;0,0)\dot{p}(t)+f'_x(p(t))\dot{\varphi}'_\mu(t;0,0)+\right.$$

$$\left.+g'_t\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)+g'_x\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\dot{p}(t)\right].$$

(Here g'_t and g'_x are the derivative with respect to t and the derivative matrix with respect to x of the vector $g\left(\frac{t}{\tau_0},x,0,\tau_0\right)$, respectively). Substituting the expressions from (3.4) and (3.5) we get

$$q(t)\left[f''_{xx}(p(t))\varphi'_\mu(t;0,0)+g'_x\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]\dot{p}(t)=$$

$$=\left\{q(t)\left[f'_x(p(t))\,\varphi'_\mu(t;0,0)+g\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]\right\}^{\cdot}-q(t)g'_t\left(\frac{t}{\tau_0},p(t),0,\tau_0\right).$$

Substituting the last expression into (3.3) we have

$$\lambda'_\mu(0,0)=-\frac{av_0(\tau_0)}{1-W(\tau_0)}\tau_1(0)+\left\{q(t)\left[f'_x(p(t))\,\varphi'_\mu(t;0,0)+\right.\right.$$

$$\left.\left.+g\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)\right]\right\}_0^{\tau_0}-\int_0^{\tau_0}q(t)\,g'_t\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)dt.$$

Taking into account the periodicity of $q(t)$, $f'_x(p(t))$ and $g\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)$ and the expression (3.2), the first and second term cancel out, thus

$$\lambda'_\mu(0,0)=-\int_0^{\tau_0}q(t)\,g'_t\left(\frac{t}{\tau_0},p(t),0,\tau_0\right)dt=$$

$$=\int_0^{\tau_0}\frac{\gamma'_t\left(\frac{t}{\tau_0},u_0(t),\dot{u}_0(t),0,\tau_0\right)}{W(t)}\left[\dot{v}_0(t)-\frac{v_0(\tau_0)\,\dot{u}_0(t)}{a[1-W(\tau_0)]}\right]dt$$

and this was to be proved.

Note that this result is in accordance with that of W. S. LOUD ([3]) who considered van der Pol's equation under perturbations with period equal to the period of the unperturbed periodic solution.

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