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SUR LES ESPACES QUASI-TOPOLOGIQUES ET LES ORDRES SEMI-TOPOGÈNES

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1. Dans sa monographie [1], Á. CSÁSZÁR montre que les espaces topologiques, les espaces uniformes et les espaces de proximité peuvent être traités d'une manière plus unitaire dans le cadre de la théorie des structures syntopogènes. Dans cette note, nous nous proposons de montrer que dans ce cadre on peut présenter aussi les espaces quasi-topologiques. Dans notre acception, un *espace quasi-topologique* c'est un espace qu'on obtient en ne retenant qu'une partie des axiomes définissant un espace topologique classique. Nous allons utiliser la caractérisation de ces espaces au moyen des fonctions du type «fermeture». En ce qui suit, nous allons utiliser la terminologie rencontrée dans les articles de P. C. HAMMER (voir par exemple [4]). Quant aux ordres semi-topogènes, nous allons employer la terminologie et les notations du livre [1]. Premièrement, nous allons rappeler certaines définitions et propriétés des fonctions d'ensemble et des ordres semitopogènes dont nous aurons besoin dans la présente note.

DÉFINITION 1.1. (1) *Un ordre semi-topogène parfait sur E est une relation $<$ définie sur l'ensemble de tous les sous-ensembles de E, qui satisfait aux axiomes suivants:*

$$(O_1) \qquad \emptyset < \emptyset, \quad E < E,$$

$$(O_2) \qquad A < B \Rightarrow A \subseteq B,$$

$$(O_3) \qquad A \subseteq A' < B' \subseteq B \Rightarrow A < B,$$

$$(P') \qquad A_\gamma < B \, (\gamma \in \Gamma \neq \emptyset) \Rightarrow \bigcup_{\gamma \in \Gamma} A_\gamma < B,$$

pour tout ensemble d'indices Γ. On désigne par $\mathcal{P}$ l'ensemble des ordres semi-topogènes parfaits sur E.

(2) *Un ordre semi-topogène parfait* $<$ *sur* E *est un ordre topogène parfait sur* E, *s'il satisfait à l'axiome suivant:*

$$(Q') \qquad A_k < B_k \ (k = 1, 2) \Rightarrow A_1 \cap A_2 < B_1 \cap B_2 .$$

On désigne par $\mathcal{P}_t$ *l'ensemble des ordres topogènes parfaits sur* E.

(3) *Un ordre semi-topogène parfait* $<$ *sur* E *est un ordre topogène biparfait si*

$$(B') \qquad A_\gamma < B_\gamma (\gamma \in \Gamma \neq \emptyset) \Rightarrow \bigcap_{\gamma \in \Gamma} A_\gamma < \bigcap_{\gamma \in \Gamma} B_\gamma .$$

On désigne par $\mathcal{P}_b$ *l'ensemble des ordres topogènes biparfaits sur* E.

(4) *Un ordre semi-topogène parfait* $<$ *sur* E *est idempotent si* $< = <^2$, *c'est-à-dire si*

$$(1) \qquad A < B \Rightarrow (\exists) C \text{ tel que } A < C < B .$$

On désigne par $\mathcal{P}_i$ *l'ensemble des ordres semi-topogènes parfaits idempotents sur* E.

(5) *Un ordre semi-topogène parfait* $<$ *sur* E *est symétrique si*

$$(S) \qquad A < B \Rightarrow cB < cA ,$$

où c *est la fonction de complémentarisation:* $cX = E - X$.
On désigne par $\mathcal{P}_s$ *l'ensemble des ordres semi-topogèenes parfaits symétriques sur* E.

L'ensemble $\mathcal{P}$ des ordres semi-topogènes parfaits sur E possède un ordre naturel $\subseteq$ défini par

$$(1.1) \qquad <_1 \subseteq <_2 \Leftrightarrow (A <_1 B \Rightarrow A <_2 B) .$$

Par rapport à cet ordre, $\mathcal{P}$ est un treillis complet dont le plus grand élément est l'ordre $<_M$ défini par

$$A <_M B \Leftrightarrow A \subseteq B$$

et les plus petit élément est l'ordre $<_m$ défini par

$$A <_m B \Leftrightarrow A = \emptyset , \text{ ou, } B = E .$$

La borne supérieure d'un enseble $\{<_\gamma / \gamma \in \Gamma \neq \emptyset\}$ d'ordres semi-topogènes parfaits est l'ordre semi-topogène parfait $(\bigcup_{\gamma \in \Gamma} <_\gamma)^P$ (voir le th. (4.4) de [1]) et la borne inférieure est l'ordre semi-topogène parfait $\bigcap_{\gamma \in \Gamma} <_\gamma$ (voir le th. (4.3) de [1]).

DÉFINITION 1.2. (1) *On dit qu'une fonction* $h : 2^E \to 2^E$ *est une fonction t-expansive sur* E, *si elle satisfait aux axiomes suivants:*

$$(E_1) \qquad h\emptyset = \emptyset ,$$

$$(E_2) \qquad A \subseteq B \Rightarrow hA \subseteq hB \ (h \text{ est isotone}) ,$$

(E_3) $\qquad A \subseteq hA$ *pour tout* $A \subseteq E$ (*h est extensive*).

On désigne par $\mathcal{E}$ *l'ensemble des fonctions t-expansives sur* E.

(2) *Une fonction t-expansive h sur E est une fonction t-fermeture de Hammer si elle est idempotente, c'est-à-dire si* $h(hA)=hA$ *pour tout* $A \subseteq E$. *On désigne par* $\mathcal{E}_i$ *l'ensemble des fonctions t-fermeture de Hammer sur* E.

(3) *On dit qu'une fonction t-expansive h sur E est additive si*

(E_4) $\qquad h(A \cup B) = hA \cup hB$ *pour tout* $A, B \subseteq E$.

On désigne par $\mathcal{E}_a$ *l'ensemble des fonctions t-expansives additives sur* E.

(4) *On dit qu'une fonction t-expansive h est totalement additive si*

(E_5) $\qquad h(\bigcup_{\gamma \in \Gamma} A_\gamma) = \bigcup_{\gamma \in \Gamma} hA_\gamma$ *pour toute famille* $\{A_\gamma / \gamma \in \Gamma \neq \emptyset\}$.

On désigne par $\mathcal{E}_{ta}$ *l'ensemble des fonctions t-expansives totalements additives sur* E.

(5) *La fonction* $g = chc$ *est la fonction adjointe de la fonction h.*

L'algèbre des ensembles et la relation d'inclusion sur 2^E conduisent à une algèbre et à une relation d'ordre sur $\mathcal{E}$. Si h, g et h_α sont des éléments de $\mathcal{E}$, alors $h \cup g$, $h \cap g$, $\bigcap_\alpha h_\alpha$, $\bigcup_\alpha h_\alpha$ sont fonctions de $\mathcal{E}$, définies par

$$(h \cup g)A = (hA) \cup (gA),\ (h \cap g)A = (hA) \cap (gA),\ (\bigcap_\alpha h_\alpha)A = \bigcap_\alpha (h_\alpha A)$$

$(\bigcup_\alpha h_\alpha)A = \bigcup_\alpha (h_\alpha A)$ (voir le th.4 de [4]). La relation d'inclusion de 2^E induit une relation d'ordre désignée par $\subseteq$ sur $\mathcal{E}$, en disant que $h \subseteq g$ si et seulement si $hA \subseteq gA$ pour tout $A \subseteq E$. Par rapport à ces opérations et à cette relation d'ordre, l'ensemble $\mathcal{E}$ est un treillis complet dont le plus petit élément est la fonction identique sur 2^E, que nous désignons par «2», et dont le plus grand élément est la fonction h_M définie par

$$h_M A = \begin{cases} \emptyset & \text{si } A = \emptyset, \\ E & \text{si } A \neq \emptyset. \end{cases}$$

On définit, encore, sur $\mathcal{E}$ l'opération de composition habituelle de h et g, où hg est la fonction donnée par $(hg)A = h(gA)$ pour tout $A \subseteq E$. Par rapport à cette opération, $\mathcal{E}$ est un demi-groupe, dont l'élément unité est e. Pour toute fonction $h \in \mathcal{E}$ nous définirons la fonction $h^{-1} : 2^E \to 2^E$ par

(1.2) $$h^{-1} B = \{x \in E / h(\{x\}) \cap B \neq \emptyset\}.$$

On voit aisément que $h^{-1} \in \mathcal{E}$. En ce qui suit, nous allons écrire $h(x)$ au lieu de $h(\{x\})$ pour tout $x \in E$. De la définition de la fonction h^{-1} il résulte que

(1.3) $$x \in h^{-1}(y) \Leftrightarrow y \in h(x$$

L'ensemble $\mathcal{E}$ étant fermé aux compositions arbitraires, il s'ensuit que $hh^{-1}, h^{-1}h \in \mathcal{E}$ et on voit que $h \subseteq hh^{-1}$. Une fonction $h \in \mathcal{E}$ qui satis-

fait à l'égalité $h=h^{-1}$, c'est-à-dire $hA=h^{-1}A$ pour tout $A\subseteq E$, s'appelle *symétrique*. Si h est une fonction totalement additive, alors h est symétrique si et seulement si elle satisfait à l'équivalence

(1.4) $$y\in h(x)\Leftrightarrow x\in h(y)$$ (voir le th. (1.4) de [6]).

2. Dans le théorème qui suit, nous montrons que les notions de fonction t-expansive et d'ordre semi-topogène parfait sont équivalentes.

THÉORÈME 2.1. (1) *Si $<$ est un ordre semi-topogène parfait sur E, alors la fonction h définie par*

$$h_< A = \{x\in E/x < V \Rightarrow V \cap A \neq \emptyset\}$$

est une fonction t-expansive.

(2) *Si h est une fonction t-expansive sur E, alors la relation $<_h$ définie par*

$$A <_h B \Leftrightarrow A \subseteq chcB$$

est un ordre semi-topogène parfait sur E.

(3) *Les applications $< \to h_<$ et $h \to <_h$ sont des bijections inverses l'une à l'autre, qui inversent les ordres des treillis $\mathcal{E}$ et $\mathcal{P}$.*

DÉMONSTRATION. On vérifie directement que la fonction $h_<$ est une fonction t-expansive (voir le th. (2.1) de [5]).

(2) La relation $<_h$ est un ordre semi-topogène, ainsi qu'il résulte du théorème (2.2) de [5]. Il reste à prouver que $<_h$ est parfait. Désignons par i la fonction adjointe de la fonction h. Compte tenu de la définition de l'ordre $<_h$, de $A_\gamma <_h B$ $(\gamma\in\Gamma\neq\emptyset)$ il résulte que $A_\gamma\subseteq iB$ $(\gamma\in\Gamma)$, d'où $\bigcup_{\gamma\in\Gamma} A_\gamma\subseteq iB$, donc $\bigcup_{\gamma\in\Gamma} A_\gamma <_h B$ et $<_h$ est parfait.

(3) Soit $< \to h_<$; $h\to <_h$. Montrons que si $h=h_<$, alors $<_h=<$. Si $A<B$ et $A\not<_h B$, alors $A\nsubseteq iB$, ou, $hcB\nsubseteq cA$, donc il existe $x\in hcB$ tel que $x\in A$. De $x\in A<B$ il résulte $x<B$, et de la définition de la fonction $h_<$ et de la relation $x\in hcB$ on déduit que $B\cap cB\neq\emptyset$, ce qui est faux. Par conséquent $A<B$ implique $A<_h B$, d'où $<\subseteq <_h$. Réciproquement, soit $A<_h B$ et $x\in A$. De $x\in A<_h B$ il résulte que $x<_h B$, donc $x\in chcB$, d'où $x\notin hcB$. De l'égalité $h=h_<$ et de la définition de la fonction $h_<$, il s'ensuit qu'il existe V tel que $x<V$ et $V\cap cB=\emptyset$, ou $V\subseteq B$ et donc $x<B$. Le point x, étant arbitraire dans A, et $<$ étant parfait, il résulte du théorème (4.1) de [1] que $A<B$ et par conséquent, $<_h\subseteq <$. Les inclusions $<\subseteq <_h$ et $<_h\subseteq <$ conduisent à l'égalité $<_h=<$. Dans le raisonnement ci-dessus, nous avons supposé que $A\neq\emptyset$, mais si $A=\emptyset$, les inclusions sont évidentes.

Soit, maintenant, $h\to <_h$; $< \to h_<$. Montrons que si $<=<_h$, alors il en résulte que $h_<=h$. Soit A un sous-ensemble arbitraire de E. S'il existe $x\in hA$ tel que $x\notin h_< A$, alors il existe $V\subseteq E$ tel que $x<V$ et $A\subseteq cV$. De ces relations et de l'égalité $<_h=<$ on déduit successivement: $x<_h V$ et $A\subseteq cV$, $x\in chcV$ et $A\subseteq cV$, $x\in chcV$ et $hA\subseteq hcV$, $x\in chcV$ et $chcV\subseteq chA$, $x\notin hA$, ce qui contredit la relation $x\in hA$. Par conséquent $hA\subseteq h_< A$. Montrons, maintenant, que $h_< A\subseteq hA$, ou, de façon équivalente, $chA\subseteq$

$\subseteq ch_< A$. Si $x \in chA = chccA = icA$, alors $x <_h cA$, d'où, en utilisant l'égalité $<_h = <$, nous obtenons $x < cA$. Mais, puisque $(cA) \cap A = \emptyset$ il résulte que $x \notin h_< A$, ou, $x \in ch_< A$, donc $h_< A \subseteq hA$. Cette inclusion et l'inclusion contraire conduisent à l'égalité $h = h_<$. Pour prouver la dernière partie du point (3), nous remarquons tout d'abord, que nous avons l'équivalence suivante

$$h_2 \subseteq h_1 \Leftrightarrow i_1 \subseteq i_2 \tag{2.2}$$

où i_1 et i_2 sont respectivement les fonctions adjointes des fonctions t-expansives h_1 et h_2. En utilisant l'équivalence (2.2), la dernière partie de l'énoncé entraîne la formulation suivante:

$$<_1 \subseteq <_2 \Leftrightarrow i_1 \subseteq i_2 \tag{2.3}$$

où i_1 et i_2 sont respectivement les fonctions adjointes des fonctions $h_{<_1}$ et $h_{<_2}$. On voit aisément que pour toute fonction $h \in \mathcal{E}$ la fonction adjointe peut être définie encore par $iA = \{x \in E / x <_h A\}$. De ces remarques il résulte que si $<_1 \subseteq <_2$, alors $i_1 A = \{x \in E / x <_1 A\} \subseteq \{x \in E / x <_2 A\} = i_2 A$, d'où $i_1 \subseteq i_2$. Inversement, si $A <_1 B$, ólors $A \subseteq i_1 B \subseteq i_2 B$, donc $A <_2 B$ et le théorème est complètement démontré.

Remarques. a) La première partie du théorème est vraie seulement dans l'hypothèse que la relation $<$ est un ordre semi-topogène. Mais, dans ce cas, la deuxième partie du théorème ne nous conduit pas, généralement, à la relation $<$ considérée initialement. Elle nous conduit au plus petit ordre semi-topogène parfait qui inclut l'ordre $<$, c'est-à-dire à la relation $<^p$ dans la terminologie de [1]. En effet, si $<$ est un ordre semi-topogène, alors la fonction t-expansive $h_<$ qui lui correspond est égale à la fonction t-expansive $h_{<^p}$ qui correspond à l'ordre semi-topogène parfait $<^p$, parce que $i_< A = \{x \in E / x < A\} = \{x \in E / x <^p A\} = i_{<^p} A$. En définissant la relation d'équivalence ϱ sur l'ensemble $\mathcal{S}$ des ordres semi-topogènes sur E en posant

$$<_1 \varrho <_2 \Leftrightarrow <_1^p = <_2^p, \tag{2.4}$$

alors à l'aide du théorème (2.1), on obtient une bijection entre l'ensemble $\mathcal{E}$ et l'ensemble quotient $\mathcal{S}/\varrho$.

b) Par l'antiisomorphisme φ déterminé dans le théorème (2.1), on obtient les correspondances:

$$e \overset{\varphi}{\to} <_M,\ h_M \overset{\varphi}{\to} <_m,\ h \cup g \overset{\varphi}{\to} <_h \cap <_g,\ h \cap g \to (<_h \cup <_g)^p.$$

c) S'il existe deux ordres semi-topogènes parfaits sur E, alors leur produit $<_1 \cdot <_2$ (voir [2]) est aussi un ordre semi-topogène parfait sur E. Si $<_h, <_g, <_{hg}$ sont les ordres semi-topogènes qui correspondent, par φ, respectivement aux fonctions t-expansives h, g et hg, alors nous avons les relations

$$<_{hg} \subseteq <_h \cap <_g,\ <_h \cdot <_g \subseteq <_h.$$

Considérons maintenant une fonction t-expansive h, la fonction h^{-1} définie par (1.2) et notons par $< = <_h$, $<^{-1} = <_{h^{-1}}$ les ordres qui correspondent par φ respectivement à h et h^{-1}.

PROPOSITION 2.1. *L'ordre semi-topogène parfait $<^{-1}$ et l'ordre semi-topogène co-parfait $<^c$ (voir [2]) sont définies au moyen de la fonction h par*

$$A <^{-1} B \Leftrightarrow A \subseteq \{x \in E / h(x) \subseteq B\} \tag{2.5}$$

$$A <^c B \Leftrightarrow hA \subseteq B\,. \tag{2.6}$$

DÉMONSTRATION. À partir de la définition de la fonction h^{-1} on voit aisément que sa fonction adjointe i^{-1} est définie par $i^{-1}A = \{x \in E / h(x) \subseteq A\}$, d'où (2.5). Pour prouver l'équivalence (2.6), nous remarquons que les affirmations suivantes sont équivalentes chacune à l'autre: $A <^c B$, $cB < cA$, $cB \subseteq icA$, $cB \subseteq chA$, $hA \subseteq B$. Donc, la proposition est complètement démontrée.

3. Ensuite, nous allons montrer qu'à chaque espèce de fonction t-expansive on peut faire correspondre par l'antiisomorphisme φ une certain espèce d'ordre semi-topogène parfait. Si h correspond par φ à $<$, alors nous dirons qu'ils sont associés.

THÉORÈME 3.1. (1) *Pour tout ordre semi-topogène parfait idempotent $<$, la fonction t-expansive associée est une fonction t-fermeture de Hammer. Inversement, l'ordre semi-topogène parfait $<_h$ associé à une fonction t-fermeture de Hammer h est idempotent, et donc $\varphi(\mathcal{E}_i) = \mathcal{P}_i$.*

(2) *Pour tout ordre topogène parfait $<$, la fonction t-expansive associée $h_<$ est additive. Inversement, l'ordre semi-topogène parfait $<_h$ associé à une fonction t-expansive additive h est un ordre topogène parfait et donc $\varphi(\mathcal{E}_a) = \mathcal{P}_t$.*

(3) *Pour tout ordre topogène biparfait $<$, la fonction t-expansive $h_<$ associée est totalement additive. Inversement, pour toute fonction t-expansive totalement additive h, l'ordre semi-topogène parfait $<_h$ associé est un ordre topogène biparfait et donc $\varphi(\mathcal{E}_{ta}) = \mathcal{P}_b$.*

DÉMONSTRATION. (1) Soit $h_<$ la fonction t-expansive associée à l'ordre semi-topogène parfait idempotent $<$. Nous allons montrer que la fonction $i_<$ adjointe est idempotente, ce qui équivant au fait que $h_<$ est idempotente (voir [5]). La fonction $i_<$, étant contractive, il résulte que $i_<^2 A = i_<(i_< A) \subseteq i_< A$. D'autre part, si $x \in i_< A$, alors $x < A$ et il existera $C \subseteq E$ tel que $x < C < A$ (parce que $<$ est idempotent). Mais, pour tout $y \in C < A$, on a $y < A$, donc $C \subseteq i_< A$. De $x < C \subseteq i_< A$ il s'ensuit que $x < i_< A$, ou, $x \in i_<^2 A$. Par conséquent $i_< A \subseteq i_<^2 A$, d'où il résulte $i_< = i_<^2$ et la première partie de (1) est établie.

Soit, maintenant, h une fonction t-fermeture de Hammer et $<_h$ l'ordre semi-topogène parfait associé. Si $A <_h B$ alors $A \subseteq iB = i^2 B = i(iB) \subseteq iB$, d'où $A < iB < B$, c'est-à-dire $<_h$ est idempotent et le point (1) est démontré.

(2)* Soit $h=h_<$ la fonction t-expansive associée à l'ordre topogène parfait $<$ et i la fonction adjointe de la fonction h. Il est évident que h est additive si et seulement si sa fonction adjointe i satisfait à la relation $i(A\cap B) = iA\cap iB$ pour tout $A, B\subseteq E$. D'une part, la fonction i étant contractive, on a $i(A\cap B)\subseteq iA\cap iB$. D'autre part, si $x\in iA\cap iB$, alors $x<A$ et $x<B$, et, en utilisant (Q') on obtient $x<A\cap B$, donc $x\in i(A\cap B)$, d'où $iA\cap iB\subseteq i(A\cap B)$, qui, avec l'inclusion contraire ci-dessus, conduit à $i(A\cap B) = iA\cap iB$. Inversement, nous considérons la fonction t-expansive additive h, sa fonction adjointe i et l'ordre semi-topogène parfait associé $<=<_h$. Si $A<B$ et $A'<B'$, on déduit successivement: $A\subseteq iB$ et $A'\subseteq iB'$, $A\cap A'\subseteq iB\cap iB' = i(B\cap B')$ (parce que h est additive), $A\cap A'<B\cap B$; d'où il s'ensuit (Q') et le point (2) est démontré.

(3) Soit $<$ un ordre topogène biparfait et $h=h_<$ la fonction t-expansive associée. On voit aisément que h est totalement additive si et seulement si sa fonction adjointe i satisfait à l'égalité $i(\bigcap_{\gamma\in\Gamma} A_\gamma) = \bigcap_{\gamma\in\Gamma}(iA_\gamma)$ pour toute famille $\{A_\gamma\subseteq E/\gamma\in\Gamma\neq\emptyset\}$. Dans les hypothèses données, nous montrons que i satisfait à cette condition. La fonction i, étant contractive, on a, d'une part

$$(3.1)\qquad i(\bigcap_{\gamma\in\Gamma} A_\gamma)\subseteq \bigcap_{\gamma\in\Gamma}(iA_\gamma) \text{ pour toute famille } \{A_\gamma\subseteq E/\gamma\in\Gamma\neq\emptyset\}.$$

Puis, si $x\in\bigcap_{\gamma\in\Gamma}(iA_\gamma)$, on obtient successivement: $x<A_\gamma$ pour tout $\gamma\in\Gamma$, $x<\bigcap_{\gamma\in\Gamma} A_\gamma$ (cf. (B')), $x\in i(\bigcap_{\gamma\in\Gamma} A_\gamma)$, donc

$$(3.2)\qquad \bigcap_{\gamma\in\Gamma}(iA_\gamma)\subseteq i(\bigcap_{\gamma\in\Gamma} A_\gamma):$$

Les inclusions (3.1) et (3.2) impliquent l'égalité $i(\bigcap_{\gamma\in\Gamma} A_\gamma) = \bigcap_{\gamma\in\Gamma}(iA_\gamma)$, d'où il résulte que h est totalement additive. Inversement, soit h une fonction t-expansive totalement additive et montrons que l'ordre semi-topogène parfait associé $<=<_h$ satisfait à l'axiome (B'). Si $A_\gamma<B_\gamma$ pour $\gamma\in\Gamma\neq\emptyset$, alors on déduit successivement: $A_\gamma\subseteq iB_\gamma$, $\bigcap_{\gamma\in\Gamma} A_\gamma\subseteq\bigcap_{\gamma\in\Gamma} iB_\gamma = i(\bigcap_{\gamma\in\Gamma} B_\gamma)$, $\bigcap_{\gamma\in\Gamma} A_\gamma<\bigcap_{\gamma\in\Gamma} B_\gamma$, c'est-à-dire (B'). Les égalités des points (1), (2) et (3) étant, maintenant, évidentes, le théorème est tout à fait démontré.

De ce theorème il résulte immédiatement le corollaire suivant:

COROLLAIRE 3.1.1. *Entre l'ensemble des fonctions fermeture de Kuratowski (les topologies classiques) et l'ensemble des ordres topogènes parfaits idempotents (les topologies de Császár) on peut établir un antiisomorphisme, à savoir la restriction de l'antiisomorphisme φ à l'ensemble $\mathcal{E}_i\cap\mathcal{E}_a$.*

Nous venons ainsi de retrouver le théorème (7.20) de [1]. Les théorèmes (5.39) et (5.44) de [1] montrent qu'il existe un antiisomorphisme χ

* Voir aussi [8], théorème (5.2).

entre l'ensemble $\mathcal{P}_b$ des ordres topogènes biparfaits sur un ensemble E et l'ensemble $\mathcal{R}$ des relations réflexives sur E. Sur l'ensemble $\mathcal{R}$ on considére l'ordre naturel définie par

$$U \subseteq U' \Leftrightarrow x\,U\,y \text{ entraîne } x\,U\,y\,. \tag{3.3}$$

Des théorèmes ci-dessus et du théorème 3.1 on obtient le théorème suivant.

THÉORÈME 3.2. *Entre l'ensemble $\mathcal{E}_{ta}$ des fonctions t-expansives totalement additives et l'ensemble $\mathcal{R}$ des relations réflexives sur E il existe un isomorphisme, à savoir $\psi = \chi \circ \varphi$, où φ et χ sont respectivement les antiisomorphismes du théorème 3.1 et celui mis en évidente ci-dessus. Par cet isomorphisme, à une fonction t-expansive totalement additive h correspond la relation réflexive $U = U_h$ définie par*

$$x\,U\,y \Leftrightarrow x \in h(y)\,. \tag{3.4}$$

Inversement, à une relation réflexive U sur E correspond par ψ^{-1} la fonction t-expansive totalement additive h définie par

$$hA = \{x \in E/(\exists)\, y \in A,\ x\,U\,y\}\,. \tag{3.5}$$

La fonction t-expansive totalement additive h et la relation réflexive U qui se correspondent par l'isomorphisme ψ seront appelées associées.

DÉMONSTRATION. La première partie du théorème est évident grâce aux remarques qui le précèdent. Il reste à établir les affirmations (3.4) et (3.5). Dans ce but considérons l'ordre topogène biparfait $< = \varphi(h)$ associé à la fonction t-expansive totalement additive h. La rélation réflexive U associée à h est donc $U = \psi(<)$, et selon le théorème (5.39) de [1] U est définie par

$$x\,U\,y \Leftrightarrow x < cy \text{ est fausse}\,. \tag{3.6}$$

Mais, de la définition de l'ordre $< = <_h = \varphi(h)$ il résulte que la relation $x < cy$ est fausse si et seulement si la relation $x \in icy$ est fausse, ou si et seulement si est vraie la relation $x \in cicy = hy$, d'où (3.4). Inversement, si $< = \chi^{-1}(U)$ est la relation topogène biparfait associée à U conformément au théorème (5.39) de [1], alors $h = \varphi^{-1}(<)$. Si on désigne par i la fonction adjointe de la fonction h, il résulte que $x \in icA$ si et seulement si $x < cA$; de la définition de l'ordre $<$ il s'ensuit que $x < cA$ est vraie si et seulement si pour tout y tel que xUy on a $y \in cA$. On en déduit que $x \in hA = cicA$ si et seulement s'il existe un point $y \in A$ tel que xUy, c'est-à-dire que $x \in hA$ si et seulement s'il existe un point $y \in A$ tel que xUy, d'où (3.5) et le théorème est démontré.

On peut établir maintenant un théorème de caractérisation des ordres semi-topogènes parfaits symétriques au moyen des fonctions t-expansives associées.

THÉORÈME 3.3. *Un ordre semi-topogène parfait $<$ est symétrique si et seulement si la fonction t-expansive associée $h=h_<$ est totalement additive et symétrique.*

DÉMONSTRATION. *La condition est nécessaire.* En vertu du théorème (5.4) de [1], l'ordre semi-topogène parfait symétrique $<$ est un ordre topogène biparfait et selon le théorème (5.43) de [1], la relation réflexive associée U est symétrique, donc xUy implique yUx. Compte tenu de l'équivalence (3.4) du théorème 3.2 on aura alors l'implication $x\in h(y)$ entraine $y\in h(x)$. De la remarque (1.4), la fonction t-expansive totalement additive h est symétrique.

La condition est suffisante. La fonction t-expansive h étant totalement additive, il résulte que l'ordre semi-topogène parfait associé $<$ est un ordre topogène biparfait (cf. le th. (3.1)). Ensuite, à partir de l'équivalence (1.4) il résulte de (3.4) que la relation réflexive associée à h (donc aussi à $<$) est symétrique et, finalement, en appliquant le théorème (5.43) de [1], l'ordre topogène biparfait $<$ est symétrique. Ceci dit, le théorème est complètement démontré.

Pour les fonctions t-expansives totalement additives symétriques on a encore la caractérisation suivante.

THÉORÈME 3.4. *Une fonction t-expansive totalement additive h est symétrique si et seulement si elle satisfait à la condition suivante:*

$$hA \subseteq B \text{ entraîne } A \subseteq iB\,, \tag{3.7}$$

où i est la fonction adjointe de la fonction h.

DÉMONSTRATION. *La condition est nécessaire.* La fonction h étant totalement additive, on a $hA = \bigcup_{x\in A} h(x)$ pour tout $A\subseteq E$. On suppose que $hA\subseteq B$ et que $A\nsubseteq iB$; il existe donc $x\in A$ tel que $x\in ciB=cchcB=hcB=$ $=\bigcup_{y\in cB} h(y)$, d'où il s'ensuit qu'il existe $y\in cB$ tel que $x\in h(y)$. De (1.4) il s'ensuit que $y\in h(x)\subseteq hA\subseteq B$, ce qui contredit la relation $y\in cB$.

La condition est suffisante. S'il existe x et y tels que $x\in h(y)$ et $y\in ch(x)$, alors $h(x)\subseteq cy$ et de (3.7) on déduit que $x\in icy=chccy=chy$, ce qui contredit la relation $x\in h(y)$.

THÉORÈME 3.5. *Entre l'ensemble des ordres semi-topogènes parfaits idempotents et symétriques sur E et l'ensemble des relations d'équivalence sur E il existe un antiisomorphisme, à savoir la restriction de l'antiisomorphisme χ à l'ensemble des ordres semi-topogènes parfaits idempotents symétriques*

DÉMONSTRATION. Si $<$ est un ordre semi-topogène parfait idempotent et symétrique, alors la fonction t-expansive associée h est idempotente, totalement additive et satisfait à la condition (3.7), donc on $a: hA=A$ ce qui entraîne $A=iA$. Grâce au théorème 1 de [7], h est la fonction saturation de la relation d'équivalence U définie par

$$x\,U\,y \Leftrightarrow x\in h(y)\,.$$

Par conséquent, la relation réflexive U associée à $<$ est une relation d'équivalence sur E. Inversement, si U est une relation d'équivalence sur E, alors la fonction t-expansive associée h, coïncide avec la fonction saturation de la relation d'équivalence U, donc elle est t-expansive, idempotente, symétrique et totalement additive. En conclusion l'ordre semi-topogène parfait associé $<$ est idempotent et symétrique, ce qui achève la démonstration.

Dans [3], GACSÁLYI a démontré qu'un ordre topogène symétrique est engendré par une relation d'équivalence si et seulement s'il satisfait à la condition

$$A < B \Rightarrow [A \subseteq H \subseteq B \text{ pour un } H \subseteq E \text{ qui vérifie la relation } H < H]. \tag{3.8}$$

Il est maintenant naturel de nous demander si la condition (3.8) est équivalente à la condition d'idempotence pour les ordres topogènes parfaits symétriques. Nous allons montrer que la réponse est affirmative, même si on renonce à la condition de symétrie.

THÉORÈME 3.6. *Un ordre semi-topogène parfait $<$ est idempotent si et seulement si la condition (3.8) est vérifiée.*

DÉMONSTRATION. Soit $<$ un ordre topogène parfait idempotent et $A<B$. Puisque $<$ est parfait, il résulte du théorème 5 de [2] qu'il existe un ensemble maximal A_M qui contient A tel que $A \subseteq A_M < B$. L'ordre $<$ étant idempotent, il existe $C \subseteq E$ tel que $A \subseteq A_M < C < B$. De la maximalité de A_M il résulte que $A_M = C$ et l'ensemble $H = A_M$ satisfait à la condition (3.8). Réciproquement, la condition (3.8) implique, évidemment, la condition d'idempotence pour $<$ et le théorème est démontré.

Nous finissons cette note par le théorème suivant, qui conduit à l'ordre semi-topogène parfait idempotent maximal, contenu dans un ordre semi-topogène parfait donné.

THÉORÈME 3.7. *Pour tout ordre semi-topogène parfait $<$ sur un ensemble E, il existe un ordre semi-topogène parfait idempotent $<^i$, maximal, contenu dans $<$. L'ordre $<^i$ est l'ordre topogène parfait associé à la fonction t-fermeture de Hammer h^{λ_0}, où h est la fonction t-expansive associée à $<$, et λ_0 est l'ordre de projection de la fonction h.*

DÉMONSTRATION. Soit h la fonction t-expansive associée à l'ordre semi-topogène parfait $<$. P. C. HAMMER a démontré dans [4] qu'il existe un seul nombre ordinal $\lambda_0 \geq 0$ tel que la fonction h^{λ_0} soit la fonction fermeture de Hammer minimale qui contient h; λ_0 est alors appelé l'ordre de projection de h. L'ordre semi-topogène parfait donné par le théorème (3.1) est l'élément idempotent maximal contenu dans $<$, ce qui démontre le théorème.

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SEMI-SIMPLE EQUATIONAL CLASSES WITH DISTRIBUTIVE CONGRUENCE LATTICES*

By

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Let $\mathcal{K}$ be an equational class. If every subdirectly irreducible in $\mathcal{K}$ is simple then $\mathcal{K}$ is said to be *semi-simple*. If for every algebra $\mathfrak{A} \in \mathcal{K}$ the congruence lattice of $\mathfrak{A}$, $\mathcal{C}(\mathfrak{A})$, is distributive then $\mathcal{K}$ is said to be *distributive;* similarly if $\mathcal{C}(\mathfrak{A})$ is permutable for all $\mathfrak{A} \in \mathcal{K}$ then $\mathcal{K}$ is said to be *permutable*. This article is concerned with distributive semi-simple equational classes. There have been a number of recent papers dealing with such classes and such properties as congruence extension, filtrality, amalgamation, functional completeness, and injectivity. This article will describe how these properties follow from rather general assumptions.

Let $\mathcal{K}'$ be a class of algebras and let $\mathcal{K}$ be the equational class generated by $\mathcal{K}'$. If $\mathcal{K}$ is distributive then a well known theorem of B. JONSSON [13] states that $\mathcal{K} = P_s HSP_p(\mathcal{K}')$ (i.e. every algebra in $\mathcal{K}$ is a subdirect product of homomorphic images of subalgebras of prime products of members of $\mathcal{K}'$). An immediate corollary is that if $\mathcal{K}'$ is a finite set of finite algebras then $\mathcal{K} = P_s HS(\mathcal{K}')$. It is interesting to note that this corollary can be proved without any reference to prime products. For this we need two lemmas; the first can be proved in much the same way as theorem 21.7 of [12] while the second is trivial.

LEMMA 1. Let $\mathcal{K}'$ be a finite set of finite algebras. If every finitely generated subalgebra of $\mathfrak{B}$ is contained in $P_s HS(\mathcal{K}')$ then $\mathfrak{B} \in P_s HS(\mathcal{K}')$.

LEMMA 2. Let $\mathcal{C}(\mathfrak{A})$ be distributive and let $\Theta_1, \ldots, \Theta_n \in \mathcal{C}(\mathfrak{A})$ with $\Theta_1 \wedge \ldots \wedge \Theta_n = \omega$. Any homomorphic image of $\mathfrak{A}$ is isomorphic to a subdirect product of homomorphic images of $\mathfrak{A}/\Theta_1, \ldots, \mathfrak{A}/\Theta_n$.

THEOREM 0. *Let $\mathcal{K}'$ be a finite set of finite algebras and let $\mathcal{K} = HSP(\mathcal{K}')$ be distributive. Then $\mathcal{K} = P_s HS(\mathcal{K}')$.*

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PROOF. Let $\mathfrak{F}_{\mathcal{K}}(n)$ be the free algebra in $\mathcal{K}$ on n free generators; note that $\mathfrak{F}_{\mathcal{K}}(n)$ is a subdirect product of members of $S(\mathcal{K}')$. Since $\mathcal{K}'$ is a finite set of finite algebras, $\mathfrak{F}_{\mathcal{K}}(n)$ is finite. Thus we may apply lemma 2 to conclude that every homomorphic image of $\mathfrak{F}_{\mathcal{K}}(n)$ is contained in $P_s HS(\mathcal{K}')$; i.e. that every finitely generated member of $\mathcal{K}$ is contained in $P_s HS(\mathcal{K}')$. Hence $\mathcal{K} = P_s HS(\mathcal{K}')$ by lemma 1.

THEOREM 1. *Let $\mathcal{K}$ be a semi-simple distributive equational class. Let $\mathfrak{A} \in \mathcal{K}$ be a subdirect product of n simple algebras but not of fewer. Then $\mathcal{C}(\mathfrak{A}) \cong 2^n$ (the boolean lattice with n atoms).*

PROOF. By assumption $\mathcal{C}(\mathfrak{A})$ has exactly n distinct dual atoms and their meet is zero. Since $\mathcal{C}(\mathfrak{A})$ is distributive it must then be boolean with exactly n atoms.

An algebra $\mathfrak{A}$ has the *congruence extension property* (c.e.p.) if for any subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ the restriction map from $\mathcal{C}(\mathfrak{A})$ to $\mathcal{C}(\mathfrak{A}')$ is onto. An equational class has c.e.p. if every algebra in it does. Theorem 1 is closely related to c.e.p. since it says that the congruences of $\mathfrak{A}$ correspond to projections onto subdirect products of its simple factors. However a semi-simple distributive $\mathcal{K}$ need not have c.e.p. since a subalgebra of a simple algebra might not be simple. A well-known example is the equational class generated by the lattice M_5.

THEOREM 2. *Let $\mathcal{K}$ be semi-simple and distributive. Let every subalgebra of a simple algebra in $\mathcal{K}$ be simple and let $\mathfrak{F}_{\mathcal{K}}(4)$ be finite. Then $\mathcal{K}$ has c.e.p.*

PROOF. Let $\mathfrak{A} \in \mathcal{K}$; by a result of A. DAY [1] we need only show that the restriction map from $\mathcal{C}(\mathfrak{A})$ to $\mathcal{C}(\mathfrak{A}')$ is onto for every four generated $\mathfrak{A}' \subseteq \mathfrak{A}$. Since $\mathfrak{F}_{\mathcal{K}}(4)$ is finite, $\mathfrak{A}'$ is finite. Thus by theorem 1 the congruences of $\mathfrak{A}'$ are the kernels of the projections of $\mathfrak{A}'$. Because every subalgebra of a simple algebra is simple, given any projection π' on $\mathfrak{A}'$ we can find a projection π on $\mathfrak{A}$ so that $\pi|\mathfrak{A}' = \pi'$; i.e. that the restriction map from $\mathcal{C}(\mathfrak{A})$ to $\mathcal{C}(\mathfrak{A}')$ is onto. Hence $\mathcal{K}$ has c.e.p.

QUESTION 1. In Theorem 2 is it necessary to assume that $\mathfrak{F}_{\mathcal{K}}(4)$ is finite?

Let $\mathfrak{L}$ be a class of algebras, $\mathfrak{A}_i \in \mathfrak{L}$ for $i \in I$, and $\mathfrak{A}$ a subdirect product of the $\mathfrak{A}_i$. $F(\Theta)$ is the filter on $\mathfrak{P}(I)$ generated by $\{J \subseteq I \mid \exists a, b \in A, a \equiv b(\Theta), J = \{i \mid a(i) = b(i)\}\}$; if F is a filter on $\mathfrak{P}(I)$ then $\mathcal{C}(F)$ is the congruence on $\mathfrak{A}$ generated by $F: a \equiv b \mathcal{C}((F))$ iff $\{i \mid a(i) = b(i) \in F\}$. Call $\Theta \in \mathcal{C}(\mathfrak{A})$ *filtral* if $\mathcal{C}(F(\Theta)) = \Theta$. $\mathfrak{L}$ is filtral if every congruence of every subdirect product of algebras from $\mathfrak{L}$ is filtral; an equational class is filtral if its class of subdirectly irreducibles is filtral. It is known that in a filtral equational class each subdirectly irreducible is simple, each subalgebra of a simple algebra is simple and c.e.p. holds (see R. MAGARI [6]). As a partial convrese we have the following theorem essentially due to G. MAZZANTI [7].

THEOREM 3. *Let $\mathcal{K}$ be semi-simple and distributive. Let every subalgebra of a simple algebra be simple and let $\mathfrak{F}_{\mathcal{K}}$ (4) be finite. Then $\mathcal{K}$ is filtral.*

PROOF. Let $\mathfrak{A}\in\mathcal{K}$ be a subdirect product of simple $\mathfrak{A}_i$. It is easily seen that we need only show that all principal congruences are filtral. Thus for any, $a, b, c, d\in A$ we must show that $c \equiv d(\Theta(a, b))$ iff $\{i|a(i) = b(i)\} \subseteq \subseteq \{i|c(i)=d(i)\}$. By Theorem 2 $\mathcal{K}$ has c.e.p. so we need only look at the subalgebra generated by $\{a, b, c, d\}$. By assumption it is finite and so by Theorem 1 and the fact that every subalgebra of a simple algebra is simple we indeed have that $c \equiv d(\Theta(a, b))$ iff $\{i|a(i) = b(i)\} \subseteq \{i|c(i) = d(i)\}$.

QUESTION 2. In Theorem 3 is it necessary to assume that $\mathfrak{F}\mathcal{K}$ (4) is finite?

As examples consider the classes $\mathcal{W}_n$, $2\leq n\leq\omega$ recently studied by E. FRIED [4]. For $2\leq n\leq\omega$ $\mathcal{W}_n$ is generated by $\mathfrak{W}_n = \langle\{0, 1, a_1, \ldots \ldots, a_{n-2}\}, \vee, \wedge\rangle$ where $\vee$ and $\wedge$ are least upper bound and greatest lower bound for the following relation: $0<a_i$, $a_i<1$, $1<0$ for $1\leq i\leq n-2$. $\mathcal{W}_\omega$ is generated by the union of the $\mathfrak{W}_n$ for $n<\omega$. It is straightforward to show that for $2\leq n\leq\omega$ $\mathcal{W}_n$ satisfies the assumptions of Theorem 3 and so is filtral and in particular has c.e.p.

Let $\mathcal{K}$ be a class of algebras and let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\in\mathcal{K}$. Let $\beta:\mathfrak{A}\to\mathfrak{B}$ and $\gamma:\mathfrak{A}\to\mathfrak{C}$ be monomorphisms. Then $\langle\mathfrak{A}; \beta, \mathfrak{B}, \gamma, \mathfrak{C}\rangle$ is an *amalgam* in $\mathcal{K}$. An amalgam $\langle\mathfrak{A}; \beta, \mathfrak{B}, \gamma, \mathfrak{C}\rangle$ can be amalgamted in $\mathcal{K}$ if there is a $\mathfrak{D}\in\mathcal{K}$ and monomorphisms $\delta:\mathfrak{B}\to\mathfrak{D}$, $\lambda:\mathfrak{C}\to\mathfrak{D}$ such that $\delta\beta=\lambda\gamma$. $\mathcal{K}$ has the *amalgamation property* (a.p.) if every amalgam in $\mathcal{K}$ can be amalgamated in $\mathcal{K}$. The next theorem is taken from [8].

THEOREM 4. *Let $\mathcal{K}$ be a semi-simple equational class with c.e.p. $\mathcal{K}$ has a.p. iff every amalgam $\langle\mathfrak{A}; \beta, \mathfrak{B}, \gamma, \mathfrak{C}\rangle$ in $\mathcal{K}$ where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are simple and $|A|\geq 2$ can be amalgamted into a simple $\mathfrak{D}\in\mathcal{K}$.*

COROLLARY 1[4]. For $2\leq n\leq\omega\mathcal{W}_n$ has a.p.

An algebra $\mathfrak{A}$ is *functionally complete* if $2\leq|A|<\omega$ and if every function on A is an algebraic function on $\mathfrak{A}$ (i.e. a polynomial on $\mathfrak{A}$ with certain variables replaced by elements of A). If $2\leq|A|<\omega$ and every function on A is a polynomial on $\mathfrak{A}$ then $\mathfrak{A}$ is *primal*. Let $\mathfrak{A}^*$ be the algebra obtained by expanding the type of $\mathfrak{A}$ to make each $a\in A$ the value of a nullary operation. Thus $\mathfrak{A}$ is functionally complete iff $\mathfrak{A}^*$ is primal. The discriminator $t(x, y, z)$ is defined by $t(x,y, z) = z$ if $x=y$ and $t(x, y, z) = x$ if $x\neq y$. A result of H. WERNER [10] states that $\mathfrak{A}$ is functionally complete iff $t(x, y, z)$ is an algebraic function on $\mathfrak{A}$.

PROPOSITION 1. Let $\mathcal{K}$ be a permutable and distributive equational class. Let $\mathfrak{A}\in\mathcal{K}$ be simple and $2\leq|A|<\omega$. Then $\mathfrak{A}$ is functionally complete.

PROOF. We must show that $\mathfrak{A}^*$ is primal. A result of A. FOSTER and A. PIXLEY [3] states that $\mathfrak{A}^*$ is primal iff $\mathcal{K}(\mathfrak{A}^*)$, the variety generated by $\mathfrak{A}^*$, is permutable and distributive, $\mathfrak{A}^*$ is simple, $\mathfrak{A}^*$ has no proper subalgebras or proper automorphisms. The first two hold for $\mathfrak{A}$ and so for $\mathfrak{A}^*$ while the last holds since every element of $\mathfrak{A}^*$ is the value of a nullary operation.

The assumptions in Proposition 1 about distributivity and permutability cannot be eliminated. If distributivity is dropped then any cyclic group of prime power order provides a counter example while if permutability is dropped then any finite simple lattice provides a counter example. However it is possible to state positive results in each of these directions. The corollary to the following theorem is due to H. WERNER [11].

THEOREM 5. *Let $\mathcal{K}$ be a permutable equaitonal class; let $\mathfrak{A}\in\mathcal{K}$ be simple and $2\leq|A|<\omega$. Then $\mathfrak{A}$ is functionally complete iff for all $n<\omega$ $\mathcal{C}(\mathfrak{A}^n)$ is boolean with exactly n atoms.*

COROLLARY 2[11]. Every finite non-trivial simple non-abelian group is functionally complete.

THEOREM 6. *Let $\mathcal{K}$ be a distributive equational class and let $\mathfrak{A}\in\mathcal{K}$ with $2\leqq|A|<\omega$. Then $\mathfrak{A}$ is functionally complete iff every finite diagonal subdirect power of $\mathfrak{A}$ (i.e. a subdirect power of $\mathfrak{A}$ containing the diagonal subalgebra) is a direct power of $\mathfrak{A}$.*

PROOF. Let $\mathfrak{A}$ be functionally complete. Thus $\mathfrak{A}^*$ is primal and so every subdirect power of $\mathfrak{A}^*$ is direct. If $\mathfrak{B}$ is a diagonal subdirect power of $\mathfrak{A}$ then by making the diagonal elements of $\mathfrak{B}$ the values of nullary operations in the obvious way, $\mathfrak{B}^*$ is a diagonal subdirect power of $\mathfrak{A}^*$. Hence every finite diagonal subdirect power of $\mathfrak{A}$ is direct. Conversely we assume that every finite diagonal subidrect power of $\mathfrak{A}$ is direct and wish to show that $\mathfrak{A}^*$ is primal. It suffices to show that $\mathfrak{F}_2$, the free algebra over $\mathfrak{A}^*$ on two generators, is isomorphic to $(\mathfrak{A}^*)^{|A|^2}$. Note that every congruence of $\mathfrak{A}$ is a diagonal subalgebra of $\mathfrak{A}^2$ and so $\mathfrak{A}$ must be simple. Thus $\mathfrak{A}^*$ is simple and so $\mathfrak{F}_2\simeq(\mathfrak{A}^*)^n$ for some n. By Theorem 1 $\mathcal{C}(\mathfrak{A}^*)=$ $=2^n$. Let x, y be the free generators of $\mathfrak{F}_2$; each of the $|A|^2$ maps from $\{x, y\}$ into $\mathfrak{A}^*$ induces a homomorphism whose kernel is a dual atom of $\mathcal{C}(\mathfrak{F}_2)$. Thus if $n<|A|^2$ then two maps of $\{x, y\}$ into $\mathfrak{A}^*$ have the same kernel. But then it is easily seen that the two maps differ by an automorphism on $\mathfrak{A}^*$. Since $\mathfrak{A}^*$ has no proper automorphisms this is impossible. Since we always have $n\leq|A|^2$ we thus have $n=|A|^2$ so that $\mathfrak{A}^*$ is primal.

COROLLARY 3[4]. $\mathfrak{W}_n$ is functionally complete for $3\leq n<\omega$.

PROOF. In [4] E. FRIED proves that every finite subdirect power of $\mathfrak{W}_n$ is direct. It was shown in [9] that $\mathfrak{W}_3$ is functionally complete.

Let $\mathcal{K}$ be a class of algebras and $\mathfrak{A}\in\mathcal{K}$. $\mathfrak{A}$ is *(weak) injective* in $\mathcal{K}$ if for any $\mathfrak{B}_1, \mathfrak{B}_2\in\mathcal{K}$ with $\mathfrak{B}_1\subseteq\mathfrak{B}_2$ and any homomorphism φ from $\mathfrak{B}_1$ (onto) to $\mathfrak{A}$ there is a homomorphism φ' from $\mathfrak{B}_2$ to $\mathfrak{A}$ extending φ.

THEOREM 7[5]. Let $\mathcal{K}$ be an equational class with c.e.p. and a.p. Then every maximal subdirectly irreducible in $\mathcal{K}$ is injective in $\mathcal{K}$. If $\mathcal{K}$ has c.e.p. but not a.p. then every maximal subdirectly irreducible in $\mathcal{K}$ is weak injective in $\mathcal{K}$.

COROLLARY 4[4]. $\mathfrak{W}_n$ is injective in $\mathcal{W}_n$ for $2\leq n\leq\omega$.

THEOREM 8[2]. Let $\mathcal{K}$ be a distributive equational class. Let $\mathfrak{A}\in\mathcal{K}$ be the only subdirectly irreducible in $\mathcal{K}$ (up to isomorphism) and let $2\leq|A|<\omega$. Then the injectives in $\mathcal{K}$ are isomorphic to the extensions of $\mathfrak{A}$ by complete boolean algebras.

THEOREM 9. *Let $\mathcal{K}$ be a distributive equational class generated by simple $\mathfrak{A}$; let $2 \leq |A| < \omega$, and let $\mathfrak{A}$ be (weak) injective in $\mathcal{K}$. Then the (weak) injectives in $\mathcal{K}$ are isomorphic to the extensions of $\mathfrak{A}$ by complete boolean algebras.*

PROOF. Note that $\mathfrak{A}^*$ is the only subdirectly irreducible in $\mathcal{K}(\mathfrak{A}^*)$. By Theorem 8 the injectives in $\mathcal{K}(\mathfrak{A}^*)$ are the extensions of $\mathfrak{A}^*$ by complete boolean algebras. Dropping the added nullaries, these are easily seen to be (weak) injective in $\mathcal{K}$. Let $\mathfrak{B} \in \mathcal{K}$ be (weak) injective. Since every subdirectly irreducible in $\mathcal{K}$ is a homomorphic image of a subalgebra of $\mathfrak{A}$, $\mathfrak{B}$ is a homomorphic image of a subalgebra of $\mathfrak{A}^I$ for some I. By the (weak) injectivity of $\mathfrak{B}$, there is a homomorphism from $\mathfrak{A}^I$ onto $\mathfrak{B}$. This homomorphism interprets the added nullary, opreations in $\mathfrak{B}$ (via the image of the diagonal subalgebra of $\mathfrak{A}^I$). Thus $\mathfrak{B}^*$ is in $\mathcal{K}(\mathfrak{A}^*)$. As $\mathfrak{B}$ is (weak) injective in $\mathcal{K}$, $\mathfrak{B}^*$ is (weak) injective in $\mathcal{K}(\mathfrak{A}^*)$. By Theorems 2 and 4 $\mathcal{K}(\mathfrak{A}^*)$ has c.e.p. and a.p. and so by Theorem 7, weak injectives in $\mathcal{K}(\mathfrak{A}^*)$ are injective. Hence $\mathfrak{B}$ is isomorphic to the extension of $\mathfrak{A}$ by a complete boolean algebra.

COROLLARY 5[4]. The injectives in $\mathcal{W}_n$ (for $2 \leq n < w$) are the extensions of $\mathfrak{W}_n$ by complete boolean algebras.

REMARK. It is easily seen that in $\mathcal{W}_\omega$ there are no non-trivial injectives or weak injectives.

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INDUCTIVE DIMENSION THEORY FOR UNIFORM SPACES

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Introduction

For a uniform space $(X, \mathcal{U})$, two inductive dimension functions, $\mathcal{U}$-Ind X and $\mathcal{U}$-ind X, are introduced. The subset theorem, a product theorem, and a Urysohn inequality for both of these functions are established. $\mathcal{U}$-Ind satisfies the countable sum theorem. $\mathcal{U}$-ind $X \leq \mathcal{U}$-Ind X, where equality holds in case X is Lindelöf.

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1. $\mathcal{U}$-open sets

In [2] we introduced and studied the concepts of $\mathcal{U}$-open and $\mathcal{U}$-closed set of a uniform space $(X, \mathcal{U})$. A subset of X is called *$\mathcal{U}$-open* if it is the inverse image of an open set of R, the real numbers, under a real, bounded uniformly continuous function. A subset of X is *$\mathcal{U}$-closed* if it is the complement of a $\mathcal{U}$-open set. For a subset A of X, $\mathcal{U}_A$ denotes the relativisation of $\mathcal{U}$ to A, and "$\mathcal{U}$-open ($\mathcal{U}$-closed) set of A" is used rather than "$\mathcal{U}_A$-open ($\mathcal{U}_A$-closed) set". B is a $\mathcal{U}$-open ($\mathcal{U}$-closed) set of A iff for some $\mathcal{U}$-open ($\mathcal{U}$-closed) set Y of X $B = A \cap Y$.

$L(\mathcal{U})$, the collection of all $\mathcal{U}$-open sets of X, is a base for the topology of X. In fact if $\mathcal{U}$ is induced by a metric, $L(\mathcal{U})$ is the topology of X; and if X is Lindelöf, then $L(\mathcal{U})$ is the collection of all cozero sets of X. In the language of [1], $L(\mathcal{U})$ is a perfectly normal σ-frame, i.e., (i) The union of a countable collection of $\mathcal{U}$-open sets is $\mathcal{U}$-open; the intersection of a finite collection of $\mathcal{U}$-open stes is $\mathcal{U}$-open, (ii) For any two $\mathcal{U}$-open sets G_1, G_2 with $G_1 \cup G_2 = X$, there are disjoint $\mathcal{U}$-open sets H_1, H_2 such that $G_1 \cup H_1 = G_2 \cup H_2 = X$ ($L(\mathcal{U})$ is normal), and (iii) A $\mathcal{U}$-open set is the union of a countable collection of $\mathcal{U}$-closed sets. The dual form of (i) and (iii) is evident while the normality of $L(\mathcal{U})$ can be expressed in any one of the following ways:

1. If F_1, F_2 are disjoint $\mathcal{U}$-closed sets, there are disjoint $\mathcal{U}$-open sets G_1, G_2 such that $F_1 \subset G_1$ and $F_2 \subset G_2$.
2. If $F \subset G$ with F $\mathcal{U}$-closed and G $\mathcal{U}$-open, there is a $\mathcal{U}$-open set H and $\mathcal{U}$-closed set $\tilde{H}$ such that $F \subset H \subset \tilde{H} \subset G$.
3. If F_1, F_2 are disjoint $\mathcal{U}$-closed sets, there are $\mathcal{U}$-open sets H_1, H_2 and disjoint $\mathcal{U}$-closed sets $\tilde{H}_1$, $\tilde{H}_2$ such that $F_1 \subset H_1 \subset \tilde{H}_1$ and $F_2 \subset \subset H_2 \subset \tilde{H}_2$.

The properties of $\mathcal{U}$-open sets that are used in this paper are readily deducible from the definitons, and (i), (ii), and (iii), and are quite akin to the properties of open sets of perfectly normal spaces. For example:

Let F be $\mathcal{U}$-closed and G $\mathcal{U}$-open. Then $F-G$ is $\mathcal{U}$-closed while $G-F$ is $\mathcal{U}$-open. If E is a $\mathcal{U}$-closed set of F, then E is $\mathcal{U}$-closed.

If H is is a $\mathcal{U}$-open set of G, then H is $\mathcal{U}$-open.

If H_1, H_2 are disjoint $\mathcal{U}$-open sets of a subset of X, there are disjoint $\mathcal{U}$-open sets G_1, G_2 of X such that $H_1 \subset G_1$ and $H_2 \subset G_2$ ([2], lemma 7).

If $x \in G$ where G is $\mathcal{U}$-open, there is a $\mathcal{U}$-open set H and a $\mathcal{U}$-closed set F with $x \in H \subset F \subset G$.

2. Large Inductive Dimension

The large inductive dimension, $\mathcal{U}$-Ind X, of a uniform space $(X, \mathcal{U})$ is defined inductively as follows. $\mathcal{U}$-Ind $X = -1$ iff X is empty. For a non-negative integer n, $\mathcal{U}$-Ind $X \leq n$ if whenever F_1, F_2 are disjoint $\mathcal{U}$-closed sets, there are disjoint $\mathcal{U}$-open sets G_1, G_2 with $F_1 \subset G_1$, $F_2 \subset G_2$ and $\mathcal{U}_Y$-Ind $Y \leq n-1$, where $Y = X - G_1 \cup G_2$. $\mathcal{U}$-Ind $X = n$ if $\mathcal{U}$-Ind $X \leq n$ and $\mathcal{U}$-Ind $X \nleq n-1$. $\mathcal{U}$-Ind $X = \infty$ if for each n, $\mathcal{U}$-Ind $X \nleq n$. Henceforth, for a subset Y of X, we shall write "$\mathcal{U}$-Ind Y" rather than "$\mathcal{U}_Y$-Ind Y". The condition that $\mathcal{U}$-Ind $X \leq n$ is equivalent to the statement that if F is $\mathcal{U}$-closed and G is $\mathcal{U}$-open with $F \subset G$, there is a $\mathcal{U}$-open set H and a $\mathcal{U}$-closed set $\tilde{H}$ with $F \subset H \subset \tilde{H} \subset G$ and $\mathcal{U}$-Ind $(\tilde{H} - H) \leq \leq n-1$. In view of [1], proposition 9, since it is trivially verified that $\mathcal{U}$-Ind $X =$ Ind $L(\mathcal{U})$, and by definition $\mathcal{U}$-dim $X \leq$ dim $L(\mathcal{U})$ (see [2]),

PROPOSITION 1. $\mathcal{U}$-dim $X \leq \mathcal{U}$-Ind X.

If $\mathcal{U}$-dim $X = 0$, every finite $\mathcal{U}$-*cover* of X, i.e., a cover consisting of $\mathcal{U}$-open sets can be refined by a finite cover of mutually disjoint $\mathcal{U}$-open and therefore $\mathcal{U}$-closed sets. This implies that $\mathcal{U}$-Ind $X = 0$. Consequently $\mathcal{U}$-dim $X = 0$ iff $\mathcal{U}$-Ind $X = 0$.

LEMMA 1. *Let A be a $\mathcal{U}$-closed set of X. Then $\mathcal{U}$-Ind $A \leq \mathcal{U}$-Ind X.*

PROOF. Assuming the result holds for $\mathcal{U}$-closed sets of spaces with $\mathcal{U}$-Ind $\leq n-1$, and $\mathcal{U}$-Ind $X \leq n$, $n \geq 0$, it suffices to show $\mathcal{U}$-Ind $A \leq n$. Let F_1, F_2 be disjoint $\mathcal{U}$-closed sets of A. Since A is $\mathcal{U}$-closed, then F_1, F_2 are disjoint $\mathcal{U}$-closed sets of X. Since $\mathcal{U}$-Ind $X \leq n$, there are disjoint $\mathcal{U}$-open sets G_1, G_2 of X with $F_1 \subset G_1$, $F_2 \subset G_2$ and $\mathcal{U}$-Ind $(X - G_1 \cup G_2) \leq \leq n-1$. Now $G_1 \cap A$, $G_2 \cap A$ are disjoint $\mathcal{U}$-open sets of A with $F_1 \subset G_1 \cap A$,

$F_2 \subset G_2 \cap A$; and by the induction hypothesis $A-(G_1 \cap A) \cup (G_2 \cap A) = A \cap (X-G_1 \cup G_2)$ has $\mathcal{U}$-Ind $\leq n-1$. Hence $\mathcal{U}$-Ind $A \leq n$.

LEMMA 2. *Let E_1, E_2, F be $\mathcal{U}$-closed sets of X with $E_1 \cap E_2 = \emptyset$ and $\mathcal{U}$-Ind $F \leq n \neq -1$. Then there are $\mathcal{U}$-closed sets Q_1, Q_2 and disjoint $\mathcal{U}$-open sets P_1, P_2 such that $E_1 \subset P_1 \subset Q_1$, $E_2 \subset P_2 \subset Q_2$, $Q_1 \cap Q_2 = F - P_1 \cup P_2$ and $\mathcal{U}$-Ind $Q_1 \cap Q_2 \leq n-1$.*

PROOF. Since $\mathcal{U}$-Ind $F \leq n$, there are disjoint $\mathcal{U}$-open sets H_1, H_2 of F with $E_1 \cap F \subset H_1$, $E_2 \cap F \subset H_2$ and $\mathcal{U}$-Ind $A \leq n-1$, where $A = F - H_1 \cup H_2$. Now A is $\mathcal{U}$-closed, $B = X - A$ is $\mathcal{U}$-open; $H_1 = (F-H_2) \cap B$, $H_2 = (F-H_1) \cap B$ are $\mathcal{U}$-closed sets of B; $E_1 \cup H_1$, $E_2 \cup H_2$ are disjoint $\mathcal{U}$-closed sets of B. By normality of $L(\mathcal{U}_B)$, there are disjoint $\mathcal{U}$-open sets P_1, P_2 of B (and hence of X), and disjoint $\mathcal{U}$-closed sets T_1, T_2 of B such that $E_i \cup H_i \subset P_i \subset T_i$, $i = 1, 2$. $Q_1 = T_1 \cup A$, $Q_2 = T_2 \cup A$ are $\mathcal{U}$-closed sets of X with the stated properties.

LEMMA 3. *Let $X = \bigcup_{k=0}^{\infty} X_k$ where X_k is $\mathcal{U}$-closed, $\mathcal{U}$-Ind $(X_{k+1} - X_k) \leq n$ for each k, and $\emptyset = X_0 \subset X_1 \subset X_2 \subset \ldots$ Then $\mathcal{U}$-Ind $X \leq n$.*

PROOF. If suffices to prove the result on the assumption that it is valid for dimension $n-1$, where $n \geq 0$. Let $Q_{1,0}$, $Q_{2,0}$ be disjotin $\mathcal{U}$-closed sets of X. Then since clearly $X_j - X_{j-1}$ is a $\mathcal{U}$-closed set of $X - X_{j-1}$ of $\mathcal{U}$-Ind$\leq n$, using lemma 2, we can inductively construct sets $P_{i,j}$, $Q_{i,j}$, $i = 1, 2$, $j = 1, 2, 3, \ldots$, such that

(1) $P_{i,j}$ is a $\mathcal{U}$-open set of $X - X_{j-1}$

(2) $Q_{i,j}$ is a $\mathcal{U}$-closed set of $X - X_{j-1}$

(3) $Q_{i,j-1} - X_{j-1} \subset P_{i,j} \subset Q_{i,j}$

(4) $P_{1,j} \cap P_{2,j} \neq \emptyset$

(5) $Q_{1,j} \cap Q_{2,j} = X_j - (X_{j-1} \cup P_{1,j} \cup P_{2,j})$ and

(6) $\mathcal{U}$-Ind $Q_{1,j} \cap Q_{2,j} \leq n-1$.

Since each X_j is $\mathcal{U}$-closed, it follows from (1) that $P_{i,j}$, $P_i = \bigcup_{j=1}^{\infty} P_{i,j}$ are $\mathcal{U}$-open sets of X, while (3) and (4) imply $Q_{1,0} \subset P_1$, $Q_{2,0} \subset P_2$ and $P_1 \cap P_2 = \emptyset$. For each $k \geq 0$, (3) and (5) imply

$$Q_k = \bigcup_{j=0}^{k} Q_{1,j} \cap Q_{2,j} = X_k - \bigcup_{j=1}^{k} P_{1,j} \cup P_{2,j}.$$

Thus each Q_k is $\mathcal{U}$-closed and since by (6) $Q_{k+1} - Q_k = Q_{1,k+1} \cap Q_{2,k+1}$ has $\mathcal{U}$-Ind$\leq n-1$, by induction hypothesis $\bigcup_{k=1}^{\infty} Q_k = X - P_1 \cup P_2$ has $\mathcal{U}$-Ind$\leq n-1$. Consequently $\mathcal{U}$-Ind $X \leq n$.

LEMMA 4. *Let G be a $\mathcal{U}$-open set of a space X with $\mathcal{U}$-Ind $X \leq n$, then $\mathcal{U}$-Ind $G \leq n$.*

PROOF. For $n=-1$ the result is trivial. We assume its validity for dimension $n-1 \geq -1$. Let E be a $\mathcal{U}$-closed and H a $\mathcal{U}$-open set of G with $E \subset H$. We may choose $\mathcal{U}$-closed sets F_k and $\mathcal{U}$-open sets G_k of X, $k = 1, 2, 3 \ldots$, with $F_k \subset G_{k+1} \subset F_{k+1} \subset G = \bigcup_{k=1}^{\infty} F_k$; also $\mathcal{U}$-closed sets E_k and $\mathcal{U}$-open sets H_k of G with $E = \bigcap_{k=1}^{\infty} E_k \subset E_{k+1} \subset H_k \subset E_k \subset H$. By Lemma 1, $\mathcal{U}$-Ind $F_{k+1} \leq n$ and hence there is a $\mathcal{U}$-open set P_k and a $\mathcal{U}$-closed set Q_k of F_{k+1} with $\mathcal{U}$-Ind $Q_k - P_k \leq n-1$ and

$$E \cap F_k \subset P_k \subset Q_k \subset H_k \cap G_{k+1}.$$

Evidenty P_k is $\mathcal{U}$-open in G_{k+1} and hence in G. Let $P = \bigcup_{k=1}^{\infty} P_k$, $Q = \bigcup_{k=1}^{\infty} Q_k$. Then $E \subset P \subset Q \subset H$, P is $\mathcal{U}$-open,

$$Q_k \subset \bigcap_{i=1}^{\infty} E_i \cup \left(\bigcup_{j<i} Q_i \right) \subset \bigcap_{i=1}^{\infty} E_i \cup Q = Q.$$

Hence $Q = \bigcap_{i<1}^{\infty} E_i \cup \left(\bigcup_{j<i} Q_i \right)$ and so it is a $\mathcal{U}$-closed set of G.

Let $Y_k = \bigcup_{i \leq k} Q_i - P_i$ and $Y = \bigcup_{k=1}^{\infty} Y_k$. Since $Y_{k+1} - Y_k$ is a $\mathcal{U}$-open subset $Q_{k+1} - P_{k+1}$, which has $\mathcal{U}$-Ind $\leq n-1$, by the induction hypothesis $\mathcal{U}$-Ind $(Y_{k+1} - Y_k) \leq n-1$. Then by Lemma 3, $\mathcal{U}$-Ind $Y \leq n-1$. Finally $Q-P$ is a $\mathcal{U}$-closed set of Y and so $\mathcal{U}$-Ind $(Q-P) \leq n-1$ (Lemma 1). Thus $\mathcal{U}$-Ind $G \leq n$.

PROPOSITION 2. *(Subset Theorem). If $Y \subset X$, then $\mathcal{U}$-Ind $Y \leq$ $\leq$ $\mathcal{U}$-Ind X.*

PROOF. For $\mathcal{U}$-Ind $X = \infty$ or -1, the result is trivially true. We suppose that $\mathcal{U}$-Ind $X \leq n$ and that the result holds for dimension $\leq n-1$. Let E_1, E_2 be disjoint $\mathcal{U}$-closed sets of Y. Then there are $\mathcal{U}$-closed sets F_1, F_2 of X with $E_1 = Y \cap F_1$ and $E_2 = Y \cap F_2$. By lemma 4, the $\mathcal{U}$-open set $G = X - F_1 \cap F_2$ has $\mathcal{U}$-Ind $\leq n$. There are then disjoint $\mathcal{U}$-open sets G_1, G_2 of G (and hence of X) with $F_1 \cap G \subset G_1$, $F_2 \cap G \subset G_2$ and $\mathcal{U}$-Ind $(G - G_1 \cup G_2) \leq n-1$. Finally, $Y \subset G$, $E_i \subset F_i \cap G \cap Y \subset G_i \cap Y$, $i = 1, 2$, and by the induction hypothesis $Y - (G_1 \cap Y) \cup (G_2 \cap Y) =$ $= Y \cap (G - G_1 \cup G_2)$ has $\mathcal{U}$-Ind $\leq n-1$. Hence $\mathcal{U}$-Ind $Y \leq n$.

PROPOSITION 3. *(Countable sum Theorem). Let $X \bigcup_{k=1}^{\infty} F_k$ with F_k $\mathcal{U}$-closed and $\mathcal{U}$-Ind $F_k \leq n$. Then $\mathcal{U}$-Ind $X \leq n$.*

PROOF. Each $X_m = \bigcup_{k \leq m} F_k$ is a $\mathcal{U}$-closed set of $X = \bigcup_{m=1}^{\infty} X_m$ and by the subset theorem $\mathcal{U}$-Ind $(X_{m+1} - X_m) \leq \mathcal{U}$-Ind $F_{m+1} \leq n$. Then by Lemma 3, $\mathcal{U}$-Ind $X \leq n$.

COROLLARY 1. *If G is $\mathcal{U}$-open in X with $\mathcal{U}$-Ind $G \leq n$ and $\mathcal{U}$-Ind $(X-G) \leq n$, then $\mathcal{U}$-Ind $X \leq n$.*

3. Small Inductive Dimension

The *small inductive dimension* $\mathcal{U}$-ind X of $(X, \mathcal{U})$ is defined inductively as follows. $\mathcal{U}$-ind $X = -1$ iff X is empty. For $n = 0, 1, 2, \ldots$, $\mathcal{U}$-ind $X \leq n$ if whenever $x \in G$ with G $\mathcal{U}$-open, there is a $\mathcal{U}$-open set H and a $\mathcal{U}$-closed set $\tilde{H}$ with $x \in H \subset \tilde{H} \subset G$ and $\mathcal{U}$-ind $(\tilde{H} - H) \leq n-1$, where for a subset Y of X we write "$\mathcal{U}$-ind Y" rather than "$\mathcal{U}_Y$-ind Y". $\mathcal{U}$-ind $X = n$ if $\mathcal{U}$-ind $X \leq n$ and $\mathcal{U}$-ind $X \not\leq n-1$. Since $L(\mathcal{U})$ is a base for the topology of X, the requirement that G is $\mathcal{U}$-open in the definition of $\mathcal{U}$-ind can be replaced by the condition that G is open.

PROPOSITION 4. (i) *If the topology of X is discrete, then $\mathcal{U}$-ind $X = 0$.*

(ii) *If $\mathcal{U}$, $\mathcal{V}$ are two uniformities compatible with a Lindelöf topology on X, then $\mathcal{U}$-ind $X = \mathcal{V}$-ind X and $\mathcal{U}$-Ind $X = \mathcal{V}$-Ind X.*

(iii) *If $\mathcal{U}$ is induced by a metric, then $\mathcal{U}$-ind $X =$ ind X and $\mathcal{U}$-Ind $X =$ Ind X.*

PROOF. This result follows from the fact that in (i) every singleton is $\mathcal{U}$-open, in (ii) $L(\mathcal{U})$, $L(\mathcal{V})$ coincide with the set of all cozero sets of X, and in (iii) $L(\mathcal{U})$ is the topology of X. Example 2 of section **6** shows that the version of (i) for $\mathcal{U}$-dim or $\mathcal{U}$-Ind is false.

Trivial proofs by induction can be given of the following results.

PROPOSITION 5. ind $X \leq \mathcal{U}$-ind $X \leq \mathcal{U}$-Ind X.

PROPOSITION 6. *For $Y \subset X$, $\mathcal{U}$-ind $Y \leq \mathcal{U}$-ind X.*

PROPOSITION 7. *If $\mathcal{U}$, $\mathcal{V}$ induce the same topology on X and $L(\mathcal{U}) \subset L(\mathcal{V})$, then $\mathcal{V}$-ind $X \leq \mathcal{U}$-ind X.*

In section **6**, it is shown that dim, $\mathcal{U}$-dim as well as Ind, $\mathcal{U}$-Ind are unrelated; and that the version of proposition 7 for $\mathcal{U}$-dim or $\mathcal{U}$-Ind is not valid.

PROPOSITION 8. *If the topology of X is Lindelöf, then $\mathcal{U}$-ind $X = \mathcal{U}$-Ind X.*

PROOF. In view of proposition 4, for a Lindelöf space X, $\mathcal{U}$-ind X and $\mathcal{U}$-Ind X depend solely on the topology of X, and therefore we may denote them by ind*X and Ind*X respectively. It is shown in [3] that ind* = Ind* for a class of spaces that includes all Lindelöf spaces.

4. Urysohn Inequalities

PROPOSITION 9. *Let* $X=X_1\cup X_2$. *Then* $\mathcal{U}$-ind $X\leq\mathcal{U}$-ind X_1+ $+\mathcal{U}$-ind X_2+1.

PROOF. Suppose $0\leq\mathcal{U}$-ind $X_1\leq n$, $0\leq\mathcal{U}$-ind $X_2\leq m$, and that the proposition holds whenever the sum of the small inductive dimensions of the summands is $\leq n+m-1$. It suffices to show $\mathcal{U}$-ind $X\leq n+m+1$.

Let $x\in G$ with G $\mathcal{U}$-open. Choose a $\mathcal{U}$-open set H and a $\mathcal{U}$-closed set F with $x\in H\subset F\subset G$. Without loss of generality assume $x\in X_1$. Since $\mathcal{U}$-ind $X_1\leq n$, there is a $\mathcal{U}$-open set A and a $\mathcal{U}$-closed set B of X_1 with $x\in A\subset B\subset H\cap X_1$ and $\mathcal{U}$-ind $B-A\leq n-1$. Let G_1, G_2 be disjoint $\mathcal{U}$-open sets of X with $A\subset G_1$ and $X_1-B\subset G_2$. Then $P=G_1\cap H$ is $\mathcal{U}$-open, $Q=$ $=(X-G_2)\cap F$ is $\mathcal{U}$-closed $x\in P\subset Q\subset F\subset G$ and $(Q-P)\cap X_1\subset B-A$. Now by the subset theorem (proposition 6) $\mathcal{U}$-ind $(Q-P)\cap X_1\leq n-1$ and $\mathcal{U}$-ind $(Q-P)\cap X_2\leq m$. Finally by the induction hypothesis $\mathcal{U}$-ind $Q-P\leq n+m$. Consequently $\mathcal{U}$-ind $X\leq n+m+1$.

PROPOSITION 10. *Let* $X=X_1\cup X_2$. *Then* $\mathcal{U}$-Ind $X\leq\mathcal{U}$-Ind X_1+ $+\mathcal{U}$-Ind X_2+1.

PROOF. Suppose $0\leq\mathcal{U}$-Ind X, $\leq n$, $0\leq\mathcal{U}$-Ind $X_2\leq m$, and that the proposition holds wherever the sum of the large inductive dimensions of the summands is $\leq n+m-1$. It suffices to show $\mathcal{U}$-Ind $X\leq n+m+1$.

Let E_1, E_2 be disjoint $\mathcal{U}$-closed sets of X. Choose $\mathcal{U}$-open sets H_1, H_2, and disjoint $\mathcal{U}$-closed sets F_1, F_2 with $E_1\subset H_1\subset F_1$ and $E_2\subset H_2\subset F_2$. Since $\mathcal{U}$-Ind $X_i\leq n$, there are disjoint $\mathcal{U}$-open sets S_1, S_2 of X_1 with $F_1\cap X_1\subset S_1$, $F_2\cap X_1\subset S_2$ and $\mathcal{U}$-Ind $X_1-S_1\cup S_2\leq n-1$. Let T_1, T_2 be disjoint $\mathcal{U}$-open sets of X with $S_1\subset T_1$ and $S_2\subset T_2$. Then $G_1=(T_1-F_2)\cup H_1$, $G_2=(T_2-F_1)\cup H_2$ are disjoint $\mathcal{U}$-open sets with $E_1\subset G_1$, $E_2\subset G_2$ and $X_1-G_1\cup G_2\subset X_1-S_1\cup S_2$. By the subset theorem, $\mathcal{U}$-Ind $X_1-G_1\cup G_2\leq$ $\leq n-1$ and $\mathcal{U}$-Ind $X_2-G_1\cup G_2\leq m$. By the induction hypothesis $\mathcal{U}$-Ind $X-G_1\cup G_2\leq n+m$. Consequently $\mathcal{U}$-Ind $X\leq n+m+1$.

5. Product Theorem

PROPOSITION 11. *Let* $(X,\mathcal{U})$, $(Y,\mathcal{V})$ *be non-trivial uniform spaces with* $X\times Y$ *Lindelöf. Then* $\mathcal{U}\times\mathcal{V}-\mathrm{Ind}\,X\times Y\leq\mathcal{U}$-Ind $X+\mathcal{V}$-Ind Y.

PROOF. It is shown in [3] that $\mathrm{Ind}^*X\times Y\leq\mathrm{Ind}^*X+\mathrm{Ind}^*Y$.

6. Examples

EXAMPLE 1. *A space* $(M,\mathcal{V})$ *with* $\dim M=\mathrm{Ind}\,M=1$ *and* $\mathcal{V}$-dim $M=\mathcal{V}$-Ind $M=0$.

DOWKER [4] has given an example of a normal Hausdorff sapce N with $\dim N=\mathrm{Ind}\,N=0$, containing a normal subspace M with $\dim M=$ $=\mathrm{Ind}\,M=1$. For the Stone-Čech compactification βN of N, $\dim\beta N=$ $=\mathrm{Ind}\,\beta N=0$. Let $\mathcal{V}$ be the unique uniformity compatible with the

topology of βN. It follows from proposition 2 of [2] and the remark following proposition 1 that $\mathcal{V}$-dim $\beta N = \mathcal{V}$-Ind $\beta N = 0$. By the subset theorem $\mathcal{V}$-Ind $M = \mathcal{V}$-dim $M = 0$. If $\mathcal{U}$ is the unique uniformity on βM, then $\mathcal{U}$-Ind $M = \mathcal{U}$-dim $M = \dim M = \dim \beta N = 1$. Thus $\mathcal{V}$-Ind $M < \mathcal{U}$-Ind M and $\mathcal{V}$-dim $M < \mathcal{U}$-dim M although $L(\mathcal{V}) \subset \subset L(\mathcal{U})$.

EXAMPLE 2. *A space* $(R^n, \mathcal{E})$ *with* $\mathcal{E}$-dim $R^n = \mathcal{E}$-Ind $R^n = n-1$ *and* $\mathcal{E}$-ind $R^n = \operatorname{ind} R^n = \dim R^n = \operatorname{Ind} R^n = 0$.

R^n denotes the n-dimensional euclidean space, d the usual metric and $\mathcal{U}$ the usual uniformity on R^n. In view of propositions 4, and 2 of [2], $\mathcal{U}$-dim $R^n = \mathcal{U}$-Ind $R^n = n$. Sets of the form

$$V(A;\varepsilon) = \{(x,y) : d(x,y) = 0, \text{ or } x \notin A, y \notin A \text{ and } d(x,y) < \varepsilon\}$$

where A is a finite subset of R^n and ε is a positive real number, form a base for a uniformity $\mathcal{E}$ on R^n with $\mathcal{U} \subset \mathcal{E}$. It follows that every $\mathcal{U}$-closed ($\mathcal{U}$-open) set is $\mathcal{E}$-closed ($\mathcal{E}$-open). For a subspace $(Y, \mathcal{V}_Y)$ of a space $(X, \mathcal{V})$ $\mathcal{V}(Y)$ will denote the set of all real, bounded, uniformly continuous functions on Y. If $X \subset R^n$ and $f \in \mathcal{E}(X)$, for each positive integer m, there is a finite set $A_m \subset R^n$ and $\varepsilon_m > 0$ such that $(x, y) \in V(A_m; \varepsilon_m)$ implies $|f(x) - f(y)| < \frac{1}{m}$, where $x, y \in X$. Then if $A = \bigcup_{m=1}^{\infty} A_m$, $f \in \mathcal{U}(X - A)$. It follows that if F_m is $\mathcal{E}$-closed and G_m is $\mathcal{E}$-open in X, $m = 1, 2, \ldots$, then for some countable set $B \subset R^n$, each $F_m - B$ is a $\mathcal{U}$-closed and each $G_m - B$ is a $\mathcal{U}$-open set of $X - B$. For $a \in R^n$, $f_a : R^n \to R$ defined by $f_a(a) = 0$ and, for $x \neq a$, $f_a(x) = 1$ belongs to $\mathcal{E}(R^n)$. Thus $\{a\} = f_a^{-1}(0) = f_a^{-1}(-1, 1)$ is both $\mathcal{E}$-open and $\mathcal{E}$-closed. Hence the topology of $(R^n, \mathcal{E})$ is discrete and $\mathcal{E}$-ind $R^n = \dim R^n = \operatorname{Ind} R^n = \operatorname{ind} R^n = 0$.

LEMMA 5. *For* $X \subset R^n$, $\mathcal{E}$-Ind $X \leq \mathcal{U}$-Ind X.

PROOF. Suppose $0 \leq \mathcal{U}$-Ind $X \leq m$ and that the result is valid for all subsets of R^n with $\mathcal{U}$-Ind $\leq m-1$. It suffices to show $\mathcal{E}$-Ind $X \leq m$. Let E_1, E_2 be disjoint $\mathcal{E}$-closed sets of X. There is a countable set $A \subset R^n$ such that $E_1 - A$, $E_2 - A$ are $\mathcal{U}$-closed sets of $X - A$. Since $\mathcal{U}$-Ind $X \leq m$, $\mathcal{U}$-Ind $(X - A) \leq m$ and so there are disjoint $\mathcal{U}$-open sets H_1, H_2 of $X - A$ with $E_1 - A \subset H_1$, $E_2 - A \subset H_2$, and $\mathcal{U}$-Ind $(X - A \cup H_1 \cup H_2) \leq m-1$. By the induction hypothesis, $\mathcal{E}$-Ind $(X - A \cup H_1 \cup H_2) \leq m-1$. Let G_1, G_2 be disjoint $\mathcal{U}$-open sets of X with $H_1 \subset G_1$ and $H_2 \subset G_2$. Then $P_1 = (G_1 - E_2) \cup (E_1 \cap A)$, $P_2 = (G_2 - E_1) \cup (A - P_1)$ are disjoint $\mathcal{E}$-open sets of X with $E_1 \subset P_1$, $E_2 \subset P_2$ and $X - P_1 \cup P_2 \subset X - A \cup H_1 \cup H_2$. By the subset theorem, $\mathcal{E}$-Ind $(X - P_1 \cup P_2) \leq m-1$. Hence $\mathcal{E}$-Ind $X \leq m$.

LEMMA 6. $\mathcal{E}$-Ind $R^n \leq n-1$.

PROOF. Let Q be a countable dense (relative to the usual topology) subset of R^n. Then $R^n - Q$ contains no $\mathcal{U}$-open subset of R^n and therefore $\mathcal{U}$-Ind $(R^n - Q) \leq n-1$ ([5], theorem IV 3). By lemma 5, $\mathcal{E}$-Ind $(R^n - Q) \leq \leq n-1$. Q is, of course, an $\mathcal{E}$-open set of R^n with $\mathcal{E}$-Ind ≤ 0. By corollary 1, $\mathcal{E}$-Ind $R^n \leq n-1$.

LEMMA 7. $\mathcal{E}$-dim $R^n \geq n-1$.

PROOF. For $n=1$ the result is evident. For $n\geq 2$, let $\{G_1, \ldots, G_k\}$ be a $\mathcal{U}$-cover of R^{n-1} and suppose $\mathcal{E}$-dim $R^n < n-1$. Then the $\mathcal{E}$-cover $\{G_1\times R, \ldots, G_k\times R\}$ of R^n is refined by an $\mathcal{E}$-cover $\{H_1, \ldots, H_m\}$ of order $<n-1$. For some countable set A, $\{H_1-A, \ldots, H_m-A\}$ is a $\mathcal{U}$-cover of R^n-A of order $<n-1$. Finally, for some $a\in R$, $R^{n-1}\times\{a\}\subset R^n-A$, and the arbitrary $\mathcal{U}$-cover $\{G_1\times\{a\}, \ldots, G_k\times\{a\}\}$ of $R^{n-1}\times\{a\}$ is refined by a $\mathcal{U}$-cover $\{H_i\cap(R^{n-1}\times\{a\}) : i = 1, \ldots, m\}$ of order $<n-1$. This contradicts the fact that $\mathcal{U}$-dim $R^{n-1} = n-1$. and shows that $\mathcal{E}$-dim $R^n\geq n-1$. *Lemmas 6 and 7, and proposition 1 imply* $\mathcal{E}$-dim $R^n = \mathcal{E}$-Ind $R^n = n-1$.

EXAMPLE 3. *A space* $(R^2, \mathcal{F})$ *with* Ind $R^2 = \dim R^2 = \operatorname{ind} R^2 = 0$ *and* $\mathcal{F}$-ind $R^2 = \mathcal{F}$-Ind $R^2 = \mathcal{F}$-dim $R^2 = 1$.

$\mathcal{F}$ is the uniformity on R^2 generated by sets of the form

$$V(A;\varepsilon) = \{(x,y) : x = y, \text{ or } x\notin A, y\notin A \text{ and } d(x,y) < \varepsilon\}$$

where A is a finite subset of $R^2-\{a\}$, a a fixed point of R^2 and ε a positive real number. Clearly, $\mathcal{U}\subset\mathcal{F}\subset\mathcal{E}$ and, for a subset X of $R^2-\{a\}$, $\mathcal{F}_X = \mathcal{E}_X$. If F is $\mathcal{F}$-closed and G is $\mathcal{F}$-open, there is a countable subset A of $R^2-\{a\}$ such that $F-A$ is a $\mathcal{U}$-closed and $G-A$ is a $\mathcal{U}$-open set of R^2-A.

LEMMA 8. $\mathcal{F}$-ind $R^2 \geq 1$.

PROOF. If $\mathcal{F}$-ind $R^2 \leq 0$, there is a set F which is both $\mathcal{F}$-open and $\mathcal{F}$-closed with $a\in F\subset\{x\in R^2 : d(x, a) < 1\}$. Then for a countable subset A of $R^2-\{a\}$, $F-A$ is both $\mathcal{U}$-open and $\mathcal{U}$-closed in R^2-A. Since A is countable, there is a straight line S through a with $S\subset R^2-A$. Now $a\in(F-A)\cap S\neq S$, and $(F-A)\cap S$ is both a $\mathcal{U}$-open and a $\mathcal{U}$-closed set of S. This contradicts the fact that S is connected and shows that $\mathcal{F}$-ind $R^2 \geq 1$.

LEMMA 9. $\mathcal{F}$-Ind $R^2 \leq 1$.

PROOF. $R^2 = \{a\}\cup(R^2-\{a\})$, $\mathcal{F}$-Ind $\{a\} = 0$ and $\mathcal{F}$-Ind $(R^2-\{a\}) = \mathcal{E}$-Ind $(R^2-\{a\}) \leq \mathcal{E}$-Ind $R^2 = 1$. By corollary 1, $\mathcal{F}$-Ind $R^2 \leq 1$.

It is now clear that $\mathcal{F}$-ind $R^2 = \mathcal{F}$-Ind $R^2 = \mathcal{F}$-dim $R^2 = 1$. We note that $\mathcal{F}$-ind $\{a\} = \mathcal{F}$-ind $(R^2-\{a\}) = 0$, and hence the countable sum theorem for the small inductive dimension of uniform spaces is not valid. That Ind $R^2 = \dim R^2 = \operatorname{ind} R^2 = 0$ follows from the fact that every closed set disjoint from $\{a\}$ is open.

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DIRECT DECOMPOSITIONS OF HOMOMORPHIC MAPPINGS OF WEAKLY ASSOCIATIVE LATTICES

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The paper [1] introduces the concept of non-associative lattice, in [2] such "lattices" are denoted as weakly associative. In this paper it is shown that homomorphic mappings of weakly associative lattices which are direct products of tournaments have some very interesting properties.

1. DEFINITION 1. By a *weakly-associative lattice* (briefly WA-*lattice*, in [1] so called T-lattice) is understood a non-void set A with two binary operations denoted by $\cap$ and $\cup$ satisfying following indentities

$$
\begin{array}{lll}
(1) & x\cap x = x & x\cup x = x \\
(2) & x\cap y = y\cap x & x\cup y = y\cup x \\
(3) & x\cap(x\cup y) = x & x\cup(x\cap y) = x \\
(4) & ((x\cap z)\cup(y\cap z))\cup z = z & ((x\cup z)\cap(y\cup z))\cap z = z
\end{array}
$$

for arbitrary $x, y, z\in A$.

It is easy to show that each lattice is a WA-lattice (but evidently not vice versa) and (1) follows from (3). In WA-lattice we can introduce relation (denoted by $\leq$) in the following way: $x\leq y$ iff $x\cap y=x$. In [2] it is proved that this is equivalent to $x\leq y$ iff $x\cap y=y$, and that this relation is reflexive, antisymmetric and for each pair $x, y\in A$ there exist z and $t\in A$ with the properties:

$x\leq z$, $y\leq z$ and from $x\leq u$, $y\leq u$ follows $z\leq u$
$t\leq x$, $t\leq y$ and from $v\leq x$, $v\leq y$ follows $v\leq t$.

Then evidently $z=x\cup y$, $t=x\cap y$.

DEFINITION 2. Let A be a WA-lattice. If for each $a, b\in A$ it holds $a\cup b=a$ or $a\cup b=b$, then A is called *tournament*. Throughout this paper by tournaments we understand only tournaments with at least two elements.

In other words, a tournament is a set with at least two elements and reflexive, antisymmetric and complete binary relation. If this relation is also transitive, the tournament is a chain (see [4]).

Let us have an index set T and a set $\{A_\tau, \tau \in T\}$ of WA-lattices. It is obvious that the direct product A of WA-lattices A_τ (symbolized by $A = \prod_{\tau \in T} A_\tau$) is a WA-lattice. Let A_τ be a tournament for each $\tau \in T$ and $e \in A$ be a fixed element. Denote pr_τ the projecion of the direct product $\prod_{\tau \in T} A_\tau$ onto the τ-th direct factor A_τ and $\mathrm{pr}_\tau e = e_\tau$. By $\overline{A}_\tau$ (or $\overline{\prod_{\tau \in T'} A_\tau}$ for $T' \subseteq T$) is denoted a sub WA-lattice of A fulfilling:

$$a \in \overline{A}_\tau \text{ iff } a \in A \text{ and } \mathrm{pr}_{\tau_1} a = e_{\tau_1} \text{ for } \tau_1 \neq \tau$$

(or $a \in \overline{\prod_{\tau \in T'} A_\tau}$ iff $a \in A$ and $\mathrm{pr}_\tau a = e_\tau$ for $\tau \in T - T'$, respectively).

It is clear that $\overline{A}_\tau$(or $\overline{\prod_{\tau \in T'} A_\tau}$) is isomorphic with A_τ (or $\prod_{\tau \in T'} A_\tau$), i.e. $\overline{A}_\tau$ is a tournament too. Evidently $\mathrm{pr}_\tau \overline{A}_\tau = A_\tau$. Denote by j_τ the *natural isomorphism* of A_τ onto $\overline{A}_\tau$ (i.e. $a \in A_\tau$, $j_\tau(a) = \bar{a} \in \overline{A}_\tau$, where $\mathrm{pr}_\tau \bar{a} = a$, $\mathrm{pr}_{\tau_1} \bar{a} = e_{\tau_1}$ for $\tau_1 \neq \tau$).

DEFINITION 3. Let A_τ, B_τ be WA-lattices, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$ and φ_τ be a mapping of A_τ into B_τ for each $\tau \in T$. The mapping φ of A into B defined by the rule

$$\mathrm{pr}_\tau \varphi(a) = \varphi_\tau(\mathrm{pr}_\tau a) \text{ for each } \tau \in T \text{ and } a \in A$$

is called the *direct product of mappings* φ_τ and is denoted by $\varphi = \prod_{\tau \in T} \varphi_\tau$.

In [3], p. 127, it is proved for an arbitrary class of algebras $\mathfrak{V}$ and $A_\tau, B_\tau \in \mathfrak{V}$ that if φ_τ is a homomorphic mapping of A_τ into B_τ (for each $\tau \in T$), then $\varphi = \prod_{\tau \in T} \varphi_\tau$ is a homomorphic mapping of $A = \prod_{\tau \in T} A_\tau$ into $B = \prod_{\tau \in T} B$. From this follows that this result is in particular valid for WA-lattices. In this paper are formulated sufficient conditions for the converse of this statement for WA-lattices and surjective homomorphisms.

DEFINITION 4. We say that a set $\mathfrak{A}$ of tournaments *satisfies the condition* (C) if $\mathfrak{A}$ is a finite set and just one of the following conditions holds:

(C1) All tournaments from $\mathfrak{A}$ have neither the least nor the greatest elements.
(C2) All tournaments from $\mathfrak{A}$ have the least element (denoted by 0).
(C3) All tournaments from $\mathfrak{A}$ have the greatest element (denoted by 1) (with respect to the relation $\leq$).

We need not index the least (or greatest) element of A_τ (if they exist) because there is no danger of misunderstanding. By 0_A (or 1_A) we denote

the element of $A = \prod_{\tau \in T} A_\tau$ (if all A_τ satisfy (C2) or (C3)) satisfying $\mathrm{pr}_\tau 0_A = 0 \in A_\tau$ (or $\mathrm{pr}_\tau 1_A = 1 \in A_\tau$, respectively).

2. LEMMA. Let A_τ, B_σ be tournaments with the least element 0 (or the greatest element 1), let $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$ and φ be a surjective homomorphic mapping of A onto B. Then $\varphi(0_A) = 0_B$ (or $\varphi(1_A) = 1_B$, respectively).

PROOF. Let $\varphi^{-1}(0_B) = V$; then for each $v \in V$ we have $\varphi(0_A) = \varphi(v \cap 0_A) = \varphi(v) \cap \varphi(0_A) = 0_B \cap \varphi(0_A) = 0_B$. Dually we obtain $\varphi(1_A) = 1_A$.

THEOREM 1. *Let A_τ, B_σ be tournaments satisfying* (C) *and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$, let φ be a surjective homomorphic mapping of the* WA-*lattice A onto* WA-*lattice B. Then we can fix an element $e \in A$ such that for each $\sigma \in S$ there exists just one $\tau_\sigma \in T$ such that $\overline{B}_\sigma = \varphi(\overline{A}_{\tau_\sigma})$.*

PROOF. First we prove the inclusion $\overline{B}_\sigma \subseteq \varphi(\overline{A}_{\tau_\sigma})$.

(1) Let A_τ, B_σ satisfy (C1). We choose an arbitrary element $e \in A$. Denote $h = \varphi(e)$, $h_\sigma = \mathrm{pr}_\sigma h$. If A_τ, B_σ satisfies (C2), we can fix $e = 0_A$, if A_τ, B_σ satisfies (C3), we can fix $e = 1_A$, $h = \varphi(e)$. Let the assumptions of the theorem be valid and let for at least one $\sigma \in S$ there does not exist $\tau_\sigma \in T$ such that $\varphi(\overline{A}_{\tau_\sigma}) \supseteq \overline{B}_\sigma$. Thus, for each subset T' of T such that $\varphi(\overline{\prod_{\tau \in T'} A_\tau}) \supseteq \overline{B}_\sigma$, card $T' \geq 2$ (such T' exists, for example $T' = T$). Accordingly, there exist $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$. If the assertion of the theorem is not true, one of the following possibilities (a), (b) have to be satisfied:

(a) there exist elements $b_1, b_2 \in \overline{B}_\sigma$, $b_1 \neq h \neq b_2$ (because card $B_\sigma \geq 2$) such that there exist $\overline{a}_1 \in \varphi^{-1}(b_1)$, $\overline{a}_2 \in \varphi^{-1}(b_2)$, $\overline{a}_1 \in A_{\tau_1}$, $\overline{a}_2 \in \overline{A}_{\tau_2}$. Clearly $\overline{a}_1 \neq e \neq \overline{a}_2$.

1° let $\overline{a}_1 > e$, $\overline{a}_2 > e$ (for A_τ, B_σ satisfying (C2) we have no other possibility), then from $\overline{A}_{\tau_1} \cap \overline{A}_{\tau_2} = \{e\}$ we obtain $h = \varphi(e) = \varphi(\overline{a}_1 \cap \overline{a}_2) = \varphi(\overline{a}_1) \cap \varphi(\overline{a}_2) = b_1 \cap b_2 = b_i \neq h$ ($i = 1$ or 2) (b_1, b_2 beeing comparable, if they belong to a tournament $\overline{B}_\sigma$). A contradiction is thus obtained.

2° let $\overline{a}_1 < e$, $\overline{a}_2 < e$ (for A_τ, B_σ satisfying (C3) we have no other possibility), then we obtain a contradiction dually as in 1°.

3° let $\overline{a}_1 < e < \overline{a}_2$ (this is possible only for (C1)). From this (and from $b_1 \neq h \neq b_2$) we have $b_1 < h < b_2$. Let us transform $e' = \overline{a}_1$, $\overline{a}'_1 = e$, $\overline{a}'_2 = \overline{a}_2$, then $b'_1 = h$, $h' = b_1$, $b'_2 = b_2$. Then we have $\overline{a}'_1 > e'$, $\overline{a}'_2 > e'$, $\varphi(\overline{a}'_1) = b'_1$, $\varphi(\overline{a}'_2) = b'_2$, $b'_1 \neq h' \neq b'_2$, which is the case 1° implying a contradiction.

(b) there exists an element $b_0 \in \overline{B}_\sigma$ $b_0 \neq h$ such that for each $a \in \varphi^{-1}(b_0)$ at least two $\tau_1, \tau_2 \in T'$, $\tau_1 \neq \tau_2$ exist such that $\mathrm{pr}_{\tau_1} a \neq e_{\tau_1}$, $\mathrm{pr}_{\tau_2} a \neq e_{\tau_2}$. Choose $a \in \varphi^{-1}(b_0)$ and denote T_0 the set of all $\tau \in T'$ for which $\mathrm{pr}_\tau a \neq e_\tau$. Let (a)

not hold, then there exists $\tau_0 \in T'$ such that $\varphi(\bar{A}_\tau) \cap \bar{B}_\sigma = \{h\}$ for each $\tau = \tau_0$, $\tau \in T'$.

I. Let, for each $\tau \in T_0$, be $\mathrm{pr}_\tau a \geq e_\tau$ and let $\varphi(\bar{a}_\tau) \in \bar{B}_\sigma$, then $b_0 = \varphi(a) = \varphi(\bigcup_{\tau \in T_0} \bar{a}_\tau) = \bigcup_{\tau \in T_0} \varphi(\bar{a}_\tau) = \varphi(\bar{a}_{\tau_0})$, because for $\tau \neq \tau_0$ is $\varphi(\bar{a}_\tau) = h$ and $\bigcup_{\tau \in T_0 - \{\tau_0\}} h$ exists and is equal to h. Therefore there exists $\bigcup_{\tau \in T} \varphi(\bar{a}_{\tau_0})$ and is equal to $\varphi(\bar{a}_{\tau_0})$, we thus arrive at a contradiction with the assumption (b), because each $a \in \varphi^{-1}(b_0)$ has two projections different from e and $\bar{a}_{\tau_0} \in \varphi^{-1}(b_0)$ does not satisfy this condition. If $\varphi(\bar{a}_{\tau'}) \notin \bar{B}_\sigma$ for any $\tau' \in T_0$, then from $h = \varphi(e) = \varphi(\bar{a}_{\tau'} \cap \bar{a}_\tau) = \varphi(\bar{a}_{\tau'}) \cap \varphi(\bar{a}_\tau)$ for each $\tau \in T$, $\tau \neq \tau'$ we obtain easy $\varphi(a) \notin \bar{B}_\sigma$, which is a contradiction again.

II. If $\mathrm{pr}_\tau a \leq e_\tau$ for each $\tau \in T_0$, then we obtain a contradiction dually as in I.

III. If $\mathrm{pr}_\tau a < e_\tau$ for some $\tau \in T_0$ and $\mathrm{pr}_\tau a > e$ for other $\tau \in T_0$, we can transform elements as in 3° and by I. we obtain a contradiction again.

Accordingly, for each $\sigma \in S$ there exists just one $\tau_\sigma \in T$ such that $\varphi(\bar{A}_{\tau_\sigma}) \supseteq \bar{B}_\sigma$. Now we prove the converse inclusion.

(2) Let there exist $\sigma_0 \in S$ such that for $\tau_{\sigma_0} \in T$ is $\varphi(\bar{A}_{\tau_{\sigma_0}}) \supseteq \bar{B}_{\sigma_0}$ and $\varphi(\bar{A}_{\tau_{\sigma_0}}) \neq \bar{B}_{\sigma_0}$. Let us denote S_0 the set of all $\sigma \in S$ for which is $\varphi(\bar{A}_{\tau_\sigma}) \supseteq \bar{B}_\sigma$. Clearly $S_0 \neq \emptyset$ because $\sigma_0 \in S_0$ and

$$\varphi(\bar{A}_{\tau_{\sigma_0}}) \supseteq \overline{\prod_{\sigma \in S_0} B_\sigma}.$$

Let $\varphi(\bar{A}_{\tau_{\sigma_0}}) \neq \overline{\prod_{\sigma \in S_0} B_\sigma}$, then there exists an element

$$c \in \varphi(\bar{A}_{\tau_{\sigma_0}}) - \overline{\prod_{\sigma \in S_0} B_\sigma}$$

such that $c \neq h$ and for any $\sigma_1 \notin S_0$ is $\mathrm{pr}_{\sigma_1} c = c_1 \neq h_{\sigma_1}$. Denote $d \in A_{\tau_{\sigma_0}}$ arbitrary element satisfying $\varphi(d) = c$ and $d_1 \in \bar{A}_{\tau_{\sigma_0}}$ arbitrary element fulfilling $\varphi(d_1) = \bar{c}_1$ (where $\mathrm{pr}_{\sigma_1} \bar{c}_1 = c_1$, $\bar{c}_1 \in \bar{B}_{\sigma_1}$). If $\tau_{\sigma_1} = \tau_{\sigma_0}$, then $\sigma_1 \in S_0$, which is a contradiction. If $\tau_{\sigma_1} \neq \tau_{\sigma_0}$, then $\bar{A}_{\tau_{\sigma_0}} \cap \bar{A}_{\tau_{\sigma_1}} = \{e\}$ and thus $h = \varphi(e) = \varphi(d \cap d_1) = c \cap \bar{c}_1 \neq h$ for $c_1 > h_{\sigma_1}$ (because $\mathrm{pr}_{\sigma_1} c = \mathrm{pr}_{\sigma_1} \bar{c}_1 = c_1$) or $h = \varphi(e) = \varphi(d \cup d_1) = c \cup \bar{c}_1 \neq h$ for $c_1 < h_{\sigma_1}$ which is a contradiction again. Thus

$$\varphi(\bar{A}_{\tau_{\sigma_0}}) = \overline{\prod_{\sigma \in S_0} B_\sigma}.$$

Let card $S_0 \geq 2$, then there exist $\sigma_1, \sigma_2 \in S_0$, $\sigma_1 \neq \sigma_2$.

(i) Let B_{σ_1}, B_{σ_2} fulfil (C1) or (C2), then we can choose $\bar{b}_1 \in \bar{B}_{\sigma_1}$, $\bar{b}_2 \in \bar{B}_{\sigma_2}$ such that $\bar{b}_1 > h$, $\bar{b}_2 > h$. Let $a_1 \in \varphi^{-1}(\bar{b}_1)$, $a_2 \in \varphi^{-1}(\bar{b}_2)$, $a_1, a_2 \in \bar{A}_{\tau_{\sigma_2}}$. Then $a_1 \neq e \neq a_2$ and $h = \bar{b}_1 \cap \bar{b}_2 = \varphi(a_1) \cap \varphi(a_2) = \varphi(a_1 \cap a_2) = \varphi(a_i) = b_i \neq h$ ($i = 1$ or 2) which is a contradiction.

(ii) Let $\overline{B}_{\sigma_1}$, $\overline{B}_{\sigma_2}$ fulfil (C3) or (C1), we obtain a contradiction by dualizing (i).

Thus card $S_0 = 1$, i.e. $S_0 = \{\sigma_0\}$. In summary the assertion of the theorem 1 has now been proved.

COROLLARY 2. The direct decomposition of WA-lattice into tournaments satisyfing (C) is unique (to whithin isomorphism).

PROOF. Let A be WA-lattice, A_τ, A_γ be families of tournaments satisfying (C), A is isomorphic with $\prod_{\tau\in T} A_\tau$ and with $\prod_{\gamma\in\Gamma} A_\gamma$. Then there exists an isomorphism φ of $\prod_{\tau\in T} A_\tau$ onto $\prod_{\gamma\in\Gamma} A_\gamma$ and, vice versa, φ^{-1} is an isomorphism of $\prod_{\gamma\in\Gamma} A_\gamma$ onto $\prod_{\tau\in T} A_\tau$. By the theorem 1 we can fix $e\in\prod_{\tau\in T} A_\tau$ such that for each $\tau\in T$ there exists just one $\gamma_\tau\in\Gamma$ satisfying $\varphi(\overline{A}_\tau) = \overline{A}_{\gamma_\tau}$, $e'\in\prod A_\gamma$ such that for each $\gamma\in\Gamma$ exists just one $\tau_\gamma\in T$ satisfying $\varphi^{-1}(\overline{A}_\gamma) = \overline{A}_{\tau_\gamma}$. Evidently we can get $e'=\varphi(e)$. Then $\tau=\tau_\gamma$, $\gamma=\gamma_\tau$ and thus card $T=$card Γ and A_τ is isomorphic to $A_{\pi(\gamma)}$, where π is a permutation of the index set Γ.

COROLLARY 3. Let A_τ, B_σ be tournaments satisfying (C), φ be a surjective homomorphic mapping of WA-lattice $A = \prod_{\tau\in T} A_\tau$ onto WA-lattice $B = \prod_{\sigma\in S} B_\sigma$. Then card $S \leq$ card T.

PROOF. For each $\sigma\in S$ there exists just one $\tau_\sigma\in T$ such that $\varphi(\overline{A}_{\tau_\sigma}) = \overline{B}_\sigma$ for any fixed $e\in A$, for $\sigma_1\neq\sigma_2$ is $\tau_{\sigma_1}\neq\tau_{\sigma_2}$, i.e. card S = card $\{\tau_\sigma, \sigma\in S\} \leq$ card T.

THEOREM 4. *Let A_τ, B_τ be tournaments satisfying* (C), $A = \prod_{\tau\in T} A_\tau$, $B = \prod_{\tau\in T} B_\tau$ *be* WA-*lattices and φ be a surjective homomorphic mapping of A onto B. Then $\varphi = \prod_{\tau\in T}\varphi_\tau$, where φ_τ is a surjective homomorphic mapping of A_τ onto $B_{\pi(\tau)}$, π is a permutation of the index set T.*

PROOF. By the theorem 1 we can choose e such that for each $\tau\in T$ there exists just one $\tau_\pi\in T$ satisfying $\varphi(\overline{A}_\tau)=B_{\tau_\pi}$. Let π be a permutation of T such that $\pi(\tau)=\tau_\pi$, let j_τ be a natural isomorphism of A_τ onto $\overline{A}_\tau$. Then we can get $\varphi_\tau = j_\tau\cdot\varphi|_{\overline{A}_\tau}\cdot \mathrm{pr}_{\pi(\tau)}$ then $\varphi_\tau(A_\tau) = B_{\pi(\tau)}$ and $\varphi = \prod_{\tau\in T}\varphi_\tau$.

CONCLUDING REMARK. The theorem 4 is the converse of the general theorem on direct products of homomorphic mappings (see [3]) formulated for surjective homomorphisms and for WA-lattices which are direct products of tournaments satisfying (C) (and thus also specially for lattices which are direct products of chains satisfying (C)).

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REMARKS ON THE PAPER OF ARAKI AND ELLIOTT "ON THE DEFINITION OF C^*-ALGEBRAS"

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H. ARAKI and G. A. ELLIOTT showed recently in [1], that the conditions $\|xy\| \leqq \|x\| \cdot \|y\|$ and $\|x^*\| = \|x\|$ follow from the other axioms for C^*-algebras (see Theorems 1 and 2 bellow). They asked also wether is it enough to assume the condition $\|x^* \cdot x\| = \|x\|^2$ ($\|x^* x\| = \|x^*\| \cdot \|x\|$) in Theorem 1 (Theorem 2) only for normal x? (see Problem in [1]).

The purpose of this note firstly to give more short proofs to the theorems mentioned above and secondly to give answer in negative to the question of ARAKI and ELLIOTT, giving a Counter example.

Let A be an algebra over the complex field. An involution on A is a map $x \to x^*$ of A onto itself such that for each x, y in A, and each complex number λ

1° $(x^*)^* = x$
2° $(x+y)^* = x^* + y^*$
3° $(\lambda x)^* = \lambda^* x^*$ (λ^* is the complex conjugate of λ)
4° $(xy)^* = y^* x^*$

A *-algebra is an algebra endowed with an involution *. The algebra A is called a Banach algebra provided there is a norm $\|\cdot\|$ under which A is a Banach space such that the following multiplicative condition is satisfied

5° $\|xy\| \leqq \|x\| \, \|y\|$

We mean under a C^*-algebra a Banach*-algebra A satisfying the condition

6° $\|x^* \cdot x\| = \|x\|^2$

The following theorems of [1] omit the condition 5° and replace in part the condition 6° with some weaker assumption in the definition of C^*-algebras.

THEOREM 1. *Let A be a *-algebra such that A is a Banach space under the norm $\|\cdot\|$ satisfying the identity for all x in A*

$$\|x^* \cdot x\| = \|x\|^2 . \tag{1}$$

Then A is a C^*-algebra.

PROOF. For all x, y in A and c, d complex numbers holds the following algebraic identity ([1])

$$4c \cdot d \cdot x \cdot y = \sum_{n=0}^{3} i^n \left(d^* y^* + (-i)^n cx\right)\left(dy + i^n c^* x^*\right) . \tag{2}$$

This gives us for x, y in A

$$\|xy\| \leqq 4 \|x^*\| \cdot \|y\| \tag{3}$$

by setting $c = \|x^*\|^{-1}$, $d = \|y\|^{-1}$ (if x and y nonzero) and using the triangle inequality and the property (1) of the norm. We have now for x, y in A

$$\|xy\| \leqq 16 \|x\| \|y\| \tag{4}$$

since by (1) if xy nonzero

$$\|xy\| = \|(xy)^* (xy)\| \cdot \|xy\|^{-1} = [\|y^* (x^* xy)\| \|x^* xy\|^{-1}] [\|x^* xy\| \|xy\|^{-1}]$$

and only (3) must be used.

Then the norm of the so called extended left regular representation defined for x in A as follows (see [3] p.2.)

$$\|x\|_1 = \sup \{\|\lambda x + xy\| : y \in A,\ \lambda \text{ complex number};\ |\lambda| + \|y\| = 1\} \tag{5}$$

satisfies multiplicative condition 5° on A as operator norm and equivalent to the original norm, since obviously the following inequalities hold for each x in A

$$\|x\| \leqq \|x\|_1 \leqq 16 \|x\| . \tag{6}$$

Thus A is a Banach*-algebra with the norm $\|\cdot\|_1$ satisfying in place of (1) the inequality for all x in A

$$\|x^* x\|_1 \geqq 16^{-2} \|x\|_1^2 . \tag{7}$$

It follows by Theorem 2 of [2] that A is a C^*-algebra with some norm $\|\cdot\|_2$ which is equivalent to the norm $\|\cdot\|_1$ (and of course to the original norm). Then by 6° the following identity holds for all x in A and natural number n

$$\|x\|_2 = \|(x^* x)^{2^n}\|_2^{2^{-(n+1)}} . \tag{8}$$

The equivalence of the norms now yields the desired property:

$$\|x\|_2 = \lim_{n\to\infty} \|(x^* x)^{2^n}\|_2^{2^{-(n+1)}} = \lim_{n\to\infty} \|(x^* x)^{2^n}\|_1^{2^{-(n+1)}} = \tag{9}$$
$$= \lim_{n\to\infty} \|(x^* x)^{2^n}\|^{2^{-(n+1)}} = \|x\|$$

for all x in A, since (8) holds for the norm $\|\cdot\|$ in view of (1) too.

The proof is complete.

THEOREM 2. *Let A be a *-algebra such that A is a Banach space under the norm $\|\cdot\|$ satisfying the identity for all x in A*

$$\|x^* \cdot x\| = \|x^*\| \cdot \|x\| \tag{10}$$

Suppose that the involution $x \to x^*$ is norm continuous.

Then A is a C^*-algebra.

PROOF. The continuity of the involution gives the existence of some positive number k such that the following inequalities hold for all x in A

$$k^{-1}\|x\| \leqq \|x^*\| \leqq k\|x\| . \tag{11}$$

So the norm $\|\cdot\|'$ defined for x in A as

$$\|x\|' = \max\{\|x^*\|, \|x\|\} \tag{12}$$

is equivalent to the norm $\|\cdot\|$ satisfying the inequalities for x in A

$$\|x\| \leqq \|x\|' \leqq (1+k)\|x\| . \tag{13}$$

The identity (2) gives us now for x, y in A

$$\|xy\|' \leqq 4(1+k)\|x\|'\|y\|' \tag{14}$$

by setting $c = (\|x\|')^{-1}$, $d = (\|y\|')^{-1}$ if x and y nonzero and using the triagle inequality and the property (10) of the norm $\|\cdot\|$, and (13). For the norm $\|\cdot\|'$ it is satisfied also for x in A in view of (10) and (13) that

$$\|x^* x\|' = \|x^* x\| = \|x^*\| \cdot \|x\| \geqq (1+k)^{-2}(\|x\|')^2 . \tag{15}$$

It follows that A is a Banach*-algebra with the norm $\|\cdot\|_1$ defined by (5) $\|\cdot\|'$ being only in place of the norm $\|\cdot\|$, such that the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_1$ is given for x in A by the inequality (replacing (6))

$$\|x\| \leqq \|x\|_1 \leqq 4(1+k)^2\|x\| . \tag{16}$$

The norm satisfies the inequality in place of (15)

$$\|x^* x\|_1 \geqq 4^{-2}(1+k)^{-4}\|x\|_1^2 . \tag{17}$$

Thus A is a C^*-algebra with some norm $\|\cdot\|_2$ equivalent to the norms $\|\cdot\|_1$ and $\|\cdot\|$ in view of Theorem 2 of [2] and as above

$$\|x\|_2 = \lim_{n\to\infty} \|(x^* x)^{2^n}\|_2^{2^{-(n+1)}} = \lim_{n\to\infty} \|(x^* x)^{2^n}\|_1^{2^{-(n+1)}} = \tag{18}$$
$$= \lim_{n\to\infty} \|(x^* x)^{2^n}\|^{2^{-(n+1)}} = \|x^* x\|^{2^{-1}}$$

for $x \in A$, since from (10) only the following identity

$$\|x^* x\| = \|(x^* x)^{2^n}\|^{2^{-n}} \tag{19}$$

follows. This means that A is a C^*-algebra with the norm defined for $x \in A$ (see (20) in [1]) by

$$\|x\|_c = \|x^* x\|^{1/2}. \tag{20}$$

Thus the first part of the proof is finished. The second part consists mainly of the fact, that the relevant inequality

$$\|x\| \leq \|x\|_c \tag{21}$$

holds for all x in A (see [1]). It follows then by (10) that for all x $\|x\|^2 \leq \leq \|x^* x\| = \|x^*\| \cdot \|x\|$ so that $\|x\| \leq \|x^*\|$ and $\|x^*\| \leq \|x^{**}\| = \|x\|$ implying finally that $\|x^*\| = \|x\|$ and so for all x in A

$$\|x\| = \|x\|_c. \tag{22}$$

The theorem is now proved.

EXAMPLE. Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and let $B(H)$ be the C^*-algebra of all bounded linear operators on H with the usuall norm $\|\cdot\|$. Then the norm defined for T in $B(H)$ by the expression

$$\|T\|_1 = \left\| \frac{T^* T + TT^*}{2} \right\|^{1/2} \tag{23}$$

is equivalent to the original norm satisfying

$$\frac{1}{\sqrt{2}} \|T\| \geq \|T\|_1 \geq \|T\|. \tag{24}$$

Then A is not a C^*-algebra with the norm $\|\cdot\|_1$ but for every normal T $\|T\|_1 = \|T\|$ and so $\|T^* T\|_1 = \|T\|_1^2$ holds.

PROOF. To show that $\|\cdot\|_1$ is a norm it must be proved only that the triangle inequality (the subadditivity) is fulfilled. Since for all selfadjoint T, $\|T\| = \sup\{|(Te, e)| : e \in H, \|e\| = 1\}$ is satisfied, and

$$((\alpha T_1 + T_2)^* (\alpha T_1 + T_2) e, e) + ((\alpha T_1 + T_2)(\alpha T_1 + T_2)^* e, e) \geq 0 \tag{25}$$

holds for every T_1, T_2 in $B(H)$; $e \in H$ and for every real number α, the inequality

$$\|T_1 T_2^* + T_2^* T_1 + T_1^* T_2 + T_2 T_1^*\|^2 \leq 4\|T_2^* T_2 + T_2 \| T_1^* T_1 + T_1 T_1^*| \tag{26}$$

holds from which the desired subadditivity now follows. Since

$$\|T^*T+TT^*\| = \sup\{|((T^*T+TT^*)e,e)| : e\in H, \|e\| = 1\}$$
$$= \sup\{\|Te\|^2+\|T^*e\|^2 : e\in H, \|e\| = 1\}$$

holds for all T in $B(H)$ and $\|T^*\| = \|T\|$, (24) obvious. Let T be the operator on H defined by following action

$$Te_n = 2^{-n}e_{n+1} \quad (n = 1, 2, \ldots). \tag{27}$$

Then $T^*e_1 = 0$ and

$$T^*e_n = 2^{-(n-1)}e_{n-1} \quad (n = 2, 3, \ldots)$$

and

$$\|T\| = \|T^*\| = 2^{-1},$$

but

$$\|T\|_1 = 2^{-1}\sqrt{5}/\sqrt{8} < \|T\|$$

which shows the statement.

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REPRESENTATIONS OF PROXIMITY LATTICES

By

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MEENAKSHI (2) has defined *proximity algebras* (as Boolean algebras with a proximity relation) and, in the course of a discussion mainly concerned with the relations between (abstract) proximity structures and uniformity structures on Boolean algebras, proved a representation theorem for a certain class of proximity algebras. In this paper we define a somewhat more general concept, a *proximity lattice*, which includes as special cases both the family of all sets, and the family of all open sets, of a proximity space; we discuss the representation problem for such lattices, with special reference to two particular representations in compact Hausdorff spaces, which are respectively appropriate to the two cases just mentioned. Our methods, which do not involve uniformity structures, are to a large extent parallel with a standard method of obtaining the compactification of an ordinary proximity space. Partly to facilitate our study of representations, and partly for its own interest, we introduce the concept of *round ideals* in a proximity lattice; these themselves form a proximity lattice of a specially wellbehaved type.

1. Definitions

Let L be a pseudo-complemented[1] (finitely) distributive lattice; that is, a lattice with a least element 0 and such that every element a of L has a *pseudo-complement* $a^\dagger$ with the property that $a \wedge b = 0$ iff $b \leq a^\dagger$. It is obvious that for all a, b, $a \geq b \Rightarrow a^\dagger \leq b^\dagger$, and that $0^\dagger$ is the greatest element of L, denoted by 1. Further, $a^{\dagger\dagger} \geq a$, $a^{\dagger\dagger\dagger} = a^\dagger$, and by distributivity $(a \vee b)^\dagger = a^\dagger \wedge b^\dagger$. An element a is called *regular* iff $a^{\dagger\dagger} = a$; L is a Boolean algebra, with $a^\dagger$ as the ordinary complement of a, iff every element is regular. It is not difficult to show (cf.[1] p. 148, where *relative* pseudo-

[1] We do not assume, as in [1] p. 147, that L is *relatively* pseudo-complemented with respect to all its members, nor that it is complete.

complementation is not needed for this purpose) that $a^{\dagger\dagger} \wedge b^{\dagger\dagger} = (a \wedge b)^{\dagger\dagger}$, $(a \wedge b)^{\dagger} = (a^{\dagger} \vee b^{\dagger})^{\dagger\dagger}$, and $(a \vee a^{\dagger})^{\dagger\dagger} = 1$.

DEFINITION 1.1. *A proximity relation on L is a relation $\ll$ with the following properties:*

P1. *For all $a \in L$, $0 \ll a$, $a \ll 1$.*
P2. $a \ll b \Rightarrow a \leq b$.
P3. (i) $(a_1 \vee a_2) \ll b$ *iff* $a_1 \ll b$ *and* $a_2 \ll b$;
(ii) $a \ll (b_1 \wedge b_2)$ *iff* $a \ll b_1$ *and* $a \ll b_2$.
P4. *If* $a \ll b$ *then* $b^{\dagger} \ll a^{\dagger}$ *and* $a^{\dagger\dagger} \ll b$.
P5. *If* $a \ll b$ *then* $\exists c$, $a \ll c$, $c \ll b$.

We write aRb iff $a \ll b^{\dagger}$; by P4 and P3 this is a symmetrical relation, and satisfies conditions similar to the axioms for 'remoteness' in a proximity space: we note that aRb iff $a^{\dagger\dagger}Rb^{\dagger\dagger}$. It is important to notice that $aRb^{\dagger}$ (i.e. $a \ll b^{\dagger\dagger}$) does not imply $a \ll b$. However (using P5 in particular) we can say that $a \ll b$ iff there exists a regular element c with $aRc^{\dagger}$ and $c \leq b$.

It might appear that we could take the relation R as fundamental, writing down axioms similar to those for a proximity space, and taking the property just mentioned as a definition of '$\ll$'. However, since c_1, c_2 regular does not imply $c_1 \vee c_2$ regular, this procedure would not yield P3 without imposing an additional, rather artificial, axiom for R.

DEFINITION 1.2. *A proximity lattice $(L, \ll)$ is a pseudo-complemented distributive lattice L together with a proximity relation on L.*

If L is a Boolean algebra, $(L, \ll)$ is clearly a proximity algebra in the sense of (2).

We list some easy examples which will illustrate our methods and some of the difficulties which can occur: in all these examples the lattice elements are sets, ordered by inclusion.

Ex. 1. $L = 2^X$, the set of all subsets of a proximity space X: obviously this is a proximity algebra.

Ex. 2. The elements of L are all the open subsets (in the induced topology) of a proximity space X. This is a sublattice of the algebra of Ex. 1 (as concerns $\vee$, $\wedge$) but the pseudo-complement is not usually the set-complement. (Since the open sets of a proximity space are in one-one relation to those of its Hausdorff quotient, there is no loss of generality, in Ex. 2, in confining ourselves to separated proximity spaces; this does not hold for Ex. 1.)

Ex. 3. The elements of L are all the *regular* open sets of a (separated) proximity space X (i.e. sets G satisfying $G = \operatorname{Int} \overline{G}$). This is a proximity algebra, but not in general a sub-algebra of that of Ex.1 nor a sub-lattice of the lattice of Ex.2, since $G_1 \vee G_2$ is not usually the set union.

Ex. 4. The elements of L are those subsets of $[0, 1]$ which are (if not empty) expressible as finite unions of non-overlapping and *non-abutting* intervals, each open relative to $(0, 1)$, with the ordinary metric proximity and topology. This is a proximity algebra and a sub-algebra of that of Ex.3.

Ex. 5. The elements of L are those open sets G of the complex plane which satisfy the following condition: for any real x, if $x \notin G$, then given any $\delta > 0$ $\exists z$, *not purely real*, such that $|z-x| < \delta$, $z \notin G$. It is easy to see that L is a complete lattice and that as in Ex.2, the pseudo-complement of G is Int (comp G), but $G_1 \vee G_2$ may differ from $G_1 \cup G_2$. The proximity relation is that associated with the ordinary metric.

Now let I be the set of all $G \in L$ such that, for some n, $z \in G \Rightarrow$ $\Rightarrow |z| < 1 - 1/n$ and $|\mathcal{I}(z)| > 1/n$. (This is, in the language of section 3 below, a *round ideal* in L.) We see that, because of the restriction on the allowable elements, sup I is $\{z;\ |z| < 1\}$. All members G of I satisfy $G \ll \sup I$, but not all elements satisfying this relation are in I.

Ex. 6. In the real plane let $A(\delta)$, $B(\delta)$ denote respectively the open rectangles

$$\{(\xi, \eta);\ 0 < \xi < \delta,\ -\delta < \eta < \delta\}$$

and

$$\{(\xi, \eta);\ -\delta < \xi < \delta,\ 0 < \eta < \delta\}.$$

Let $X = A(1) \cup B(1) \cup \{(0, 0)\}$, with the usual metric proximity as a sub-space of the plane. To define an element of L we start with any finite union E of open rectangles (with sides parallel to the axes) contained in X, and apply the following rules: if, for some $\delta > 0$, E includes $A(\delta)$ or $B(\delta)$, then the set $E_1 = E \cup \{(0, 0)\}$ is in L but E itself is not; if E includes $A(\delta) \cap B(\delta)$ for some δ (but no $A(\delta_1)$ or $B(\delta_1)$) then both E and E_1 are members of L; in all other cases E is in L but E_1 is not. The proximity relation is that induced from X. It is easily verified that L is a proximity lattice, $\vee$ and $\wedge$ being set-union and set-intersection. If $c = A(1) \cap B(1)$ then $c^{\dagger\dagger} = c \cup \{(0, 0)\}$, $c^{\dagger}$ being the interior (in X) of the complement of c.

Our aim is to represent a given proximity lattice by a collection of sets in a suitable space. For convenience, we use the word 'representation' in a rather wide sense, namely:

DEFINITION 1.3. *A representation of* $(L, \ll)$ *is a function* $i: L \to 2^X$, *where* $(X, \ll)$ *is a proximity space, satisfying:* –

Rep.1: $0 \circ i = \emptyset_X$[2].
Rep.2: $a \leq b \Rightarrow a \circ i \subset b \circ i$.
Rep.3: $a \ll b$ *(in* L*)* $\Rightarrow a \circ i \ll b \circ i$ *(in* X*).*

A representation i *will be called proper on a subset* A *of* L *iff it is one-one and order-preserving both ways (on* A *to* $A \circ i$*) and satisfies, for* a_1, a_2 *in* A*:*

Rep.4. $a_1 \ll a_2$ *iff* $a_1 \circ i \ll a_2 \circ i$.
Rep.5. $a_1 R a_2$ *iff* $a_1 \circ i R a_2 \circ i$ *(in* X*).*

We note that, even when $A = L$, we cannot usually deduce one of Rep.4, Rep.5 from the other, since we have not imposed any condition directly relating ai and $a^{\dagger}i$.

[2] As usual, $\circ$ is used to denote evaluation or composition of functions. It will be omitted when no ambiguity arises.

DEFINITION 1.4. *A representation i which is proper on A will be called fully proper on A iff it satisfies: –*

Rep.6. $(a_1 \vee a_2) \circ i = a_1 \circ i \cup a_2 \circ i$,
Rep.7. $(a_1 \wedge a_2) \circ i = a_1 \circ i \cap a_2 \circ i$.

Example 4 suggests that a non-fully-proper representation may be more natural than a fully proper one. It is easy to give a fully proper representation (using half-open intervals) of the algebra in that example. There is however no special significance in the extra points thus added, and it seems most unlikely that there is any canonical method which would yield a determinate fully proper representation which was as appropriate as the representation implied by the example itself.

2. Open and closed elements; the neighbourhood relations

We write (throughout the paper):

DEFINITION 2.1. $\lambda(a) = \{b : b \leq a\}$; $\quad l(a) = \{b : b \ll a\}$;
$\varrho(a) = \{b : b \geq a\}$; $\quad r(a) = \{b : b \gg a\}$.

($b \gg a$ means of course $a \ll b$).

If sup $l(a)$ exists, we denote it by a° (in Ex 1, and in many other cases, this is in fact the interior of a in the topology of X). We accordingly define:

DEFINITION 2.2. *An element a is called open iff a° exists and $a^\circ = a$.* Dually we define $\bar{a}$ (if it exists) and closed elements.

Open and closed elements have many but not all of the properties of open and closed sets in a space.

Using P5 it is easy to see that if a° exists then $b \ll a$ implies $b \ll a^\circ$ and hence that a° is open; dually, $\bar{a}$ is closed whenever it exists. In Ex.2 and several of the other examples every element is open: such lattices are comparatively easy to represent satisfactorily.

DEFINITION 2.3. *A simple proximity lattice is one in which every element is open.*

THEOREM 1. *In a proximity lattice L:*

(i) *If* sup $\{a_i;\ i \in I\}$ *exists and all a_i are open, then* sup a_i *is open; dually for inf of closed elements.*
(ii) *Every closed element is regular.*
(iii) *a open $\Rightarrow a^\dagger$ closed; the converse holds in a proximity algebra but not in general.*

The proofs of (i) and (ii) are obvious, using for (ii) the last part of P4. We prove (iii); if b is a lower bound of $r(a^\dagger)$ then by P4

$$y \ll a \Rightarrow y^\dagger \gg a^\dagger \Rightarrow y^\dagger \geq b \Rightarrow b^\dagger \geq y^{\dagger\dagger} \geq y.$$

If a is open this gives $b^\dagger \geq a$ and so $b \leq b^{\dagger\dagger} \leq a^\dagger$, as required. The dual argument holds if a is closed and L is an algebra.

Ex.6 shows that a closed does not always imply $a^\dagger$ open, and that a_1, a_2 open does not imply $a_1 \wedge a_2$ open. For if $a_1 = A(1) \cup \{(0, 0)\}$, $a_2 = B(1) \cup \{(0, 0)\}$ then $a_1 \wedge a_2 = (A(1) \cap B(1)) \cup \{(0,0)\}$. If $c = A(1) \cap B(1)$ then c is open, $c^\dagger$ closed, $c^{\dagger\dagger} = a_1 \wedge a_2$ not open, in fact $(a_1 \wedge a_2)^\circ = c$, while a_1 and a_2 are both open.

However, all Boolean algebras, and many pseudo-complemented lattices, are *infinitely* distributive[3], in the following sense:

(ID) *If* $a = \sup \{a_i;\ i \in I\}$ *exists, then for all* b $\sup \{b \wedge a_i;\ i \in I\}$ *exists and is* $b \wedge a$.

It is easily seen that in a proximity lattice satisfying (ID) the meet of two open elements is open and, if a°, b° exist, $(a \wedge b)^\circ = a^\circ \wedge b^\circ$. In aproximity *algebra*, we deduce by taking complements that a_1, a_2 closed implies $a_1^\dagger, a_2^\dagger$ open and $a_1 \vee a_2$ closed. This gives some information about the more general case in the light of the following theorem.

THEOREM 2. *The regular elements of a proximity lattice L form a proximity algebra* L_r *under the same order and proximity relations (strictly, their restrictions to* $L_r \times L_r$*), the lattice operations in* L_r *being*

$$a_1 \wedge_r a_2 = a_1 \wedge a_2\,,$$
$$a_1 \vee_r a_2 = (a_1 \vee a_2)^{\dagger\dagger}\,.$$

If $a^\dagger$ *is closed in L (in particular, if a is open in L) then* $a^\dagger$ *is also closed in* L_r *and* $a^{\dagger\dagger}$ *open in* L_r*; if L is simple then so is* L_r.

As is well known, it follows from the identities in **§1** that L_r is a Boolean algebra, with the given join and meet. Using the axioms for L, in particular P4, we see that the proximity axioms hold for L_r. If b (regular) is closed in L so that $b = \inf r(b)$ (in L), then using P4 and P5 we see that if $c \in r(b)$ then $\exists d$ such that $d^{\dagger\dagger} \ll c$ and $d^{\dagger\dagger} \in r(b)$; hence also $b = \inf (r(b) \cap L_r)$, so that b is closed in L_r. The rest follows from Theorem 1.

We can define two "neighbourhood" relations, each analogous to the relation $A \subset \operatorname{Int} B$ in a proximity space, in a manner which does not assume the existence of b°.

DEFINITION 2.4. *bNa (b is a neighbourhood of a) means: – for all* d, $\lambda(d) \supset l(b) \Rightarrow d \geq a$.

bN a (b is a strong neighbourhood of a) means:* $a = \sup \{a \wedge c;\ c \ll b\}$.

(If bNa is false, we write $b \not N a$.)

We state the main properties of these relations, omitting the proofs, which are straightforward.

[3] (1), p. 165.

LEMMA 2.1.

(i) $aNa \Leftrightarrow aN^*a \Leftrightarrow a$ *open.*

(ii) $a \ll b \Rightarrow bN^*a \Rightarrow bNa \Rightarrow a \leq b$.

(iii) *If* b° *exists (for some given* b*) then for all* a $bNa \Leftrightarrow a \leq b^\circ$; *if also* L *is* ID *then* $bNa \Leftrightarrow bN^*a$. *Hence, if* L *is (ordinally) complete (i.e. suprema always exist) and* ID, N *and* N^* *are equivalent; the same is true for an algebra even if it is not complete.*

(iv) *If* bNa_1 *and* bNa_2 *then* $bN(a_1 \vee a_2)$ *and similarly for* N^*. *If* bNa_1 *and* $a_2 \leq a_1$ *then* bNa_2; *this is true for* N^* *if* L *is* ID *but not in general.*

(v) b_1Na *and* $b_2 \geq b_1 \Rightarrow b_2Na$; *similarly for* N^*. *If* L *is* ID *then* b_1N^*a *and* $b_2N^*a \Rightarrow (b_1 \wedge b_2)N^*a$.

A counter-example for the last part of (iv) is supplied by Ex.6 above; this shows also that N, N^* are not in general equivalent. (In the notation of the discussion immediately following Theorem 1, $a_1N^*a_1$, $c^{\dagger\dagger} \leq a_1$, $a_1N^*c^{\dagger\dagger}$.)

It is now natural to make the following definition:

DEFINITION 2.5. *A representation* i *of a proximity lattice* L *will be called topological iff* $bNa \Rightarrow a \circ i \subset \mathrm{Int}\,(b \circ i)$ *(the topology being that induced by the proximity on the representation space), and topologically proper iff the converse also holds.*

Considering Ex. 4 above, we note that the representation given in the example is topologically proper, while the fully proper representation by half-open intervals is not. It is possible to construct a representation (which appears rather artificial) which is both fully and topologically proper, but no such representation can exist in a compact Hausdorff space. For if i is a fully proper representation in such a space, and $0 < \xi < \xi_1 < \frac{1}{2} < \eta_1 < \eta < 1$, then since $\left(\xi_1, \frac{1}{2}\right) \ll (\xi, \eta)$ we have $\left[\left(\xi_1, \frac{1}{2}\right) \circ i\right]^- \subset (\xi, \eta) \circ i$, with a similar result for $\left(\frac{1}{2}, \eta_1\right)$. Since $\left(\xi_1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \eta_1\right)$ are in proximity, we have also

$$\emptyset \neq \left[\left(\xi_1, \frac{1}{2}\right) \circ i\right]^- \cap \left[\left(\frac{1}{2}, \eta_1\right) \circ i\right]^- \subset (\xi, \eta) \circ i\,.$$

Now if i is also topological, $\left(\xi, \frac{1}{2}\right) \circ i$ and $\left(\frac{1}{2}, \eta\right) \circ i$ are open and disjoint, so neither meets the closure of the other. It follows that $(\xi, \eta) \circ i \neq \left(\xi, \frac{1}{2}\right) \circ i \cup \left(\frac{1}{2}, \eta\right) \circ i$, while $\left(\xi, \frac{1}{2}\right) \vee \left(\frac{1}{2}, \eta\right) = (\xi, \eta)$ in L — a contradiction.

(A similar argument applies to the proximity algebra of all subsets in the space of rational numbers in (0, 1).)

To conclude this section, we note that we can use the relation N (or N^*) to give a condition which in the case of an actual proximity space (Ex.1) is equivalent to saying that the proximity is separated (Hausdorffian).

DEFINITION 2.6. *A proximity lattice is said to be separated iff* $a_1 \not\leq a_2$ *implies* $\exists b$, bNa_2 *and* $b\not{N}a_1$.

(In a space X, if $x_1 \in A_1 \backslash A_2$, consider $b = X \backslash \{x_1\}$.)

3. Ideals and filters. Round ideals and round filters

We recall that a subset A of L is an *ideal* iff $a_1, a_2 \in A \Rightarrow a_1 \vee a_2 \in A$ and $a \in A \Rightarrow a \wedge b \in A$ for all b. A *dual ideal* is defined similarly; a proper dual ideal ($\neq \emptyset, L$) is often called a *filter in L* (not to be confused with a filter on L in the classical sense of a *set* of subsets). By standard Zorn arguments, any filter is contained in a maximal filter (ultrafilter) and we have:

LEMMA 3.1. *A filter F in L is an ultrafilter iff, for every* $a \in L$, *either* $a \in F$ *or* $a^\dagger \in F$.

We note that for an ultrafilter, U, $a \in U \Leftrightarrow a^{\dagger\dagger} \in U$, so that in the non-Boolean case ultrafilters do not always 'separate' distinct elements. If however $a^{\dagger\dagger} \neq b^{\dagger\dagger}$, say for example $a^{\dagger\dagger} \not\leq b^{\dagger\dagger}$ so that $a^{\dagger\dagger} \wedge b^\dagger \neq 0$, then an ultrafilter U *refining* (i.e. including) $\varrho(a^{\dagger\dagger} \wedge b^\dagger)$ will have a and $a^{\dagger\dagger}$ as members but not b or $b^{\dagger\dagger}$ since $b^\dagger \wedge b^{\dagger\dagger} = 0 \notin U$.

It is interesting to note that the condition of the lemma (applied to an ideal I), while necessary, is not usually sufficient for I to be maximal (proper). Consider, e.g. a lattice of five elements with relations $0 < a < c < 1$, $0 < b < c < 1$, $a^\dagger = b$, $b^\dagger = a$, $c^\dagger = 0$, taking $I = \{0, a\}$.

If A is any subset of L, $r(A)$ denotes as usual $\bigcup\{r(a); a \in A\}$; by P5, $r[r(A)] = r(A)$. In particular, if F is a filter, it is clear that $r(F)$ is also a filter included in F.

DEFINITION 3.1. *An ideal I is round iff* $l(I) = I$; *dually, a filter F is round iff* $r(F) = F$.

If F itself is not round, then $r(F)$ is clearly the largest round filter contained in F. Almost exactly as for proximity spaces ((3), 34–37), it is easy to prove:

LEMMA 3.2. *Any round filter is contained in some maximal round filter. If F is a round filter then each of the following is a necessary and sufficient condition that it be maximal round:*

(i) *For some ultrafilter* U, $F = r(U)$.
(ii) *For every* $a \in L$, *either* $a^\dagger \in F$ *or* $r(a) \subset F$.

For example, we prove (ii) sufficient by remarking that if F_1, F are round filters with $F_1 \supset F$, $b \in F_1 \backslash F$ then $\exists a \in F_1$, $a \ll b$; hence $a^\dagger \notin F (\subset F_1)$ while $r(a) \not\subset F$ as $b \in r(a) \backslash F$. We remark also that if F is round and $a^\dagger \in F$ then $\exists b \in F$ with $b \ll a^\dagger$, $b^\dagger \gg a^{\dagger\dagger} \geq a$, so that $r(a) \not\subset F$.

The following lemma (a kind of converse of the last) is important later. It is easily proved for proximity spaces (cf. [3], 37, Theorem 6.15), but requires some care in the general case.

LEMMA 3.3. *If $a \not\ll \not\ll b$, then there exists a maximal round filter M containing neither $a^\dagger$ nor b, so that $r(a) \subset M$.*

By Zorn's lemma, $\varrho(a)$, which does not meet $l(b)$, can be enlarged to a filter F which is maximal subject to $F \cap l(b) = \emptyset$. We show first that if $c \ll b$ and $d \in F$ then $d \wedge c^\dagger \notin l(b)$.

$$\begin{aligned} \text{For } [d \wedge (c \vee c^\dagger)]^{\dagger\dagger} &= d^{\dagger\dagger} \wedge (c \vee c^\dagger)^{\dagger\dagger} \text{ (by §1)} \\ &= d^{\dagger\dagger} \text{ since } (c \vee c^\dagger)^{\dagger\dagger} = 1. \end{aligned}$$

Now $d \wedge c \ll b$ so that if also $d \wedge c^\dagger \ll b$ we should have (by P3 and P4), $d^{\dagger\dagger} = [d \wedge (c \vee c^\dagger)]^{\dagger\dagger} \ll b$ contrary to supposition. It now follows that $c^\dagger \in F$, for if not we could add to F all elements of form $d \wedge c^\dagger$, $d \in F$, and all larger elements, without violating the condition $F \cap l(b) = \emptyset$. Next we take c_1, $c \ll c_1 \ll b$; as above for $c^\dagger$ we show that $c_1^\dagger \in F$ whence $c^\dagger (\gg c_1^\dagger) \in r(F)$. Finally, we enlarge $r(F)$ to a maximal round filter M; since $a \in \varrho(a) \subset F$, we have $r(a) \subset r(F)$ so that $a^\dagger \notin M$, also $b \notin M$ since M is round and, if $c \ll b$, c and $c^\dagger$ cannot both be in M.

4. The proximity lattice of round ideals

DEFINITION 4.1. *If I_1, I_2 are round ideals, we write $I_1 \ll I_2$ iff there exists $a_2 \in I_2$ such that $I_1 \subset \lambda(a_2)$ (and hence, I_2 being round, there exists $b_2 \in I_2$ such that $I_1 \subset l(b_2)$).*

THEOREM 3. *The non-empty round ideals, ordered by inclusion, form a simple complete* ID *proximity lattice L_{RI}.*

It is almost obvious that if I_1, I_2 are round ideals then so also is their set-intersection $I_1 \cap I_2$, which is therefore $I_1 \wedge I_2$. (Alternatively, it is easy to see that $I_1 \wedge I_2 = \{a_1 \wedge a_2;\ a_1 \in I_1,\ a_2 \in I_2\}$.) If $\{I_j;\ j \in J\}$ is any set of round ideals, we form the set of all elements of the form $a = \sup \{a_r;\ 1 \leq r \leq n\}$ where $a_r \in I_{j(r)}$, say, for each r, n being finite but arbitrary. This again is a round ideal, as is easily seen, and is therefore $\sup \{I_j;\ j \in J\}$, since any round ideal including all I_j must contain all elements of the above form. Using the second formula for $I_1 \wedge I_2$ we see that L_{RI} is distributive, in fact ID. Its least member is clearly $\{0\}$ and greatest member L itself. We next say that the pseudo-complement $I^\dagger$ exists and is given by $\{b;\ \exists c \gg b,\ I \subset \lambda(c^\dagger)\}$; this is an ideal because if $b_1 \vee b_2 \ll c_1 \vee c_2$ say, $\lambda[(c_1 \vee c_2)^\dagger] = \lambda(c_1^\dagger \wedge c_2^\dagger) = \lambda(c_1^\dagger) \cap \lambda(c_2^\dagger)$, and is round by P5. It meets

I only in 0, and includes every *round* ideal with this property, since $I \subset \lambda(c^\dagger) \Leftrightarrow c \wedge d = 0$, all $d \in I$.

The axioms P1, P2 are clearly satisfied by our definition, and P3 (i) holds because if $I_1 \subset \lambda(a_3)$, $I_2 \subset \lambda(b_3)$, a_3 and b_3 in I_3, then by construction $I_1 \vee I_2 \subset \lambda(a_3 \vee b_3)$, while P3 (ii) needs only $\lambda(a_2 \wedge a_3) = \lambda(a_2) \cap \lambda(a_3)$. We consider P4 and P5. Let $I_1 \ll I_2$ and $I_1 \subset \lambda(a_2)$, $a_2 \ll b_2$, $b_2 \in I_2$. Then $l(b_2)$ is itself a round ideal and a_2, b_2 show that $I_1 \ll l(b_2) \ll I_2$; this proves P5. From the formula above for $I^\dagger$ we see that $a \ll a_2^\dagger \Rightarrow a \in I_1^\dagger$, whence $I_1^\dagger \supset l(a_2^\dagger)$ and in particular $b_2^\dagger \in I_1^\dagger$, while $b_2 \in I_2$ gives $I_2^\dagger \subset l(b_2^\dagger)$ since $b_2 \leq c^\dagger \Leftrightarrow c \leq b_2^\dagger$; thus $b_2^\dagger$ shows that $I_2^\dagger \ll I_1^\dagger$. Finally, $b_2^\dagger \in I_1^\dagger$ gives similarly $I_1^{\dagger\dagger} \subset l(b_2^{\dagger\dagger})$, but as I_2 is round $b_2^{\dagger\dagger}$ must also be in I_2, as $c_2 \gg b_2$, c_2 in I_2, implies $c_2 \gg b_2^{\dagger\dagger}$. Thus $I_1^{\dagger\dagger} \ll I_2$ as required.

We note next that $a \in I_1$ implies $\exists b \in I_1$, $a \in l(b) \subset \lambda(b) \subset I_1$. Thus $I_1 = \bigcup\{l(b);\ b \in I_1\}$ and as each $l(b)$ is a round ideal, with $l(b) \ll I_1$, we must have $I_1 = \sup\{I;\ I \ll I_1\}$ so that L_{RI} is simple; this completes the proof.

We have just shown that the ideals of the form $l(b)$ form a 'base' for L_{RI} in a sense analogous to that occurring in the term 'base of a topology'. By a slight extension of the argument used above in proving P5, we see that these ideals $l(b)$ also form a 'proximity base' for L_{RI} in the sense that if $I_1 \ll I_2$ then there exist b, c with $I_1 \leq l(b) \ll l(c) \leq I_2$ (cf. [3], 47, Def. 8.2).

We may consider the operator l as mapping L into L_{RI}; when thinking in this way we shall write $a \circ l$ instead of $l(a)$. That is, $l(a)$ is used to denote the set $\{b : b \ll a\}$ as a subset of L, but $a \circ l$ to denote the same set as an element of L_{RI}, though complete consistency is unattainable. We now state the most important properties of this mapping.

THEOREM 4. *The map l preserves the relations '$\leq$' and '$\ll$', and $(a \wedge b) \circ l = a \circ l \wedge b \circ l$. The restriction of l to the open elements of L is one-one and satisfies $a \circ l \leq b \circ l$ iff $a \leq b$, $a \circ l \ll b \circ l$ iff $a \ll b$. For all a, $a^\dagger \circ l \leq (a \circ l)^\dagger$, and if $a_1 \ll a_2$ then $(a_1 \circ l)^{\dagger\dagger} \leq (a_1^{\dagger\dagger} \circ l)^{\dagger\dagger} \leq (a_1^\dagger \circ l)^\dagger \ll a_2 \circ l$. A necessary and sufficient condition that $(a \circ l)^\dagger = a^\dagger \circ l$ is that, for all c, $l(a) \subset l(c^\dagger) \Rightarrow l(c) \subset l(a^\dagger)$. If I is a round ideal and $a = \sup I$ exists then $I^\dagger = a^\dagger \circ l$; in particular, $(a \circ l)^\dagger = a^\dagger \circ l$ if a is open, or a regular and open in L_r.*

Most of these statements follow at once from the definitions. If a is open and $a \circ l \ll b \circ l$, then $\exists c$, $c \in l(b)$, $l(a) \subset \lambda(c)$, so that $a = \sup l(a)$ satisfies $a \leq c \ll b$ (whether or not b is also open). For any a, $a \circ l \wedge a^\dagger \circ l = (a \wedge a^\dagger) \circ l = 0 \circ l = 0_{RI}$, hence $a^\dagger \circ l \leq (a \circ l)^\dagger$. If now $a_1 \ll a_2$, $\exists a_3$, a_4, $a_1 \ll a_3 \ll a_4 \ll a_2$, $a_1^\dagger \gg a_3^\dagger$. Thus $d \in (a_1^\dagger \circ l)^\dagger \Rightarrow d \wedge a_3^\dagger = 0$ ($a_3^\dagger$ being a member of $l(a_1^\dagger)$) so that $d \leq a_3^{\dagger\dagger} \ll a_4$; it follows that $(a_1^\dagger \circ l)^\dagger \ll a_2 \circ l$; while $(a_1 \circ l)^{\dagger\dagger} \leq (a_1^{\dagger\dagger} \circ l)^{\dagger\dagger}$; because $a_1 \leq a_1^{\dagger\dagger}$.

Now suppose that $a^\dagger \circ l = (a \circ l)^\dagger$ for some a, and that $l(a) \subset l(c^\dagger)$ in L. If $b \in l(c)$ then, from the formula for $I^\dagger$ given in proving Theorem 3, $b \in (a \circ l)^\dagger = a^\dagger \circ l$, i.e. $b \in l(a^\dagger)$ in L; the converse (sufficiency) is proved similarly.

Finally, suppose that $a = \sup I$ and $b \in I^\dagger$, so that $b \wedge c = 0$, $c \leq b^\dagger$, for all c of I. Then $a \leq b^\dagger$, $a^\dagger \geq b^{\dagger\dagger} \geq b$. As $I^\dagger$ is round we can replace b by $b_1 \gg b$ and obtain $I^\dagger \leq a^\dagger \circ l$; the converse is obvious as $I \cap l(a^\dagger) = \{0\}$ or, in L_{RI}, $I \wedge a^\dagger \circ l = 0$. If a is open, we can put $I = a \circ l$ in the result just proved (but note that Ex.5 shows that $a = \sup I$ does not of itself imply $I = a \circ l$). Again, if a is regular and open in L_r, we can apply the same argument to any regular element $b^{\dagger\dagger}$ of $(a \circ l)^\dagger$, using only regular elements c of $a \circ l$ and working for the moment in L_r, thus obtaining $b^{\dagger\dagger} \leq a^\dagger$. But, if b is any element of $(a \circ l)^\dagger$, since this ideal is round it also contains $b^{\dagger\dagger}$ so that as before $(a \circ l)^\dagger \leq a^\dagger \circ l$.

We remark that there are obviously dual forms of Theorems 3 and 4 concerning the lattice of round filters, L_{RF}, but as we do not need any detailed study of this lattice we do not investigate how far the duality is exact.

As L_{RI} is itself a proximity lattice, we can form $(L_{RI})_{RI}$ and $(L_{RI})_{RF}$. The first of these is easily seen to be isomorphic with L_{RI} itself. Rather more interestingly, there also exists a natural isomorphism between L_{RF} and $(L_{RI})_{RF}$.

THEOREM 5. *The mappings* $l_1 : L_{RI} \to (L_{RI})_{RI}$ *and* $l_2 : L_{RF} \to (L_{RI})_{RF}$, *defined by*

$$I_0 \circ l_1 = \{I;\ I \ll I_0\}$$

and

$$F_0 \circ l_2 = \{I;\ \exists a \in F_0,\ a \circ l \leq I\}$$

are one-one onto mappings and are order-preserving (with respect to inclusion) both ways.

COROLLARY. l_2 *maps the maximal round filters in* L *onto the maximal round filters in* L_{RI}.

Using the methods and results of Theorems 3 and 4, it is easy to see that $I_0 \circ l_1$ is indeed a round ideal and $F_0 \circ l_2$ a round filter, and that both mappings are order-preserving in the wide sense. The map l_1 is one-one because L_{RI} is simple so that $I_0 = \sup \{I;\ I \ll I_0\}$ for every I_0. If now I^* is given in $(L_{RI})_{RI}$ we put $I_0 = \cup \{I;\ I \in I^*\}$ and show that I_0 (a union of subsets of L) is a round ideal such that $I_0 \circ l_1 = I^*$; thus l_1 is onto and its inverse, being given by the formula quoted, is also order-preserving. The details are left to the reader.

Suppose now that $F_1 \circ l_2 \subset F_2 \circ l_2$; take any $a \in F_1$ and note that certainly $a \circ l \in F_1 \circ l_2$. Hence $\exists b, b \in F_2$, such that $l(b) \subset l(a)$ in L; since F_2 is round there exists $c \in F_2 \cap l(b)$, so that $a \gg c$ and therefore $a \in F_2$. That is, $F_1 \circ l_2 \subset F_2 \circ l_2$ implies $F_1 \subset F_2$, so that l_2 is one-one and l_2^{-1} order-preserving. Finally, given any F^* in $(L_{RI})_{RF}$ we put $F_0 = \{a;\ a \circ l \in F^*\}$ and show that, F^* being round, F_0 also is round in L and satisfies $F_0 \circ l_2 = F^*$ so that l_2 is onto; the details are again omitted.

5. Representation in the space of ultrafilters

The obvious method of trying to obtain a representation of L is to form (as in the well-known representation of Boolean algebras) the set X_0 of all ultrafilters in L, with the map f given by $a \circ f = \{U;\ a \in U,\ U \in X_0\}$. The axioms Rep 1 and Rep 2 clearly hold, and we ensure Rep 3 by the following definition.

DEFINITION 5.1. *If A, B are subsets of X_0 then $A \ll B$ means: — $\exists a, b \in L$ such that $a \ll b$ and $A \subset a \circ f$, $b \circ f \subset B$.*

It easily follows from the properties of ultrafilters that $(a \wedge b) \circ f = a \circ f \cap b \circ f$, $a^\dagger \circ f = X_0 \backslash a \circ f$, and $(a \vee b) \circ f = a \circ f \cup b \circ f$, whence $(X_0, \ll)$ is indeed a proximity space, and $a \circ f \ll b \circ f$ iff $a \ll b^{\dagger\dagger}$.

The topology induced on X_0 by its proximity is given by $U_0 \in \operatorname{Int} B$ iff $\exists a, b,\ a \ll b,\ a \in U_0,\ b \circ f \subset B$; we can deduce that a base of open sets is $\{b \circ g;\ b \in L\}$ where $b \circ g = \{U;\ \exists a \in U,\ a \ll b\} = \{U;\ b \in r(U)\}$. It is not difficult to prove that this makes X_0 a compact regular space; we omit the proof as we shall be proving a similar result for the closely associated space X defined in the next section.

If (and only if) L is an algebra, the representation is one-one and coincides with that defined in (2) (the definition itself in (2) is meaningful for any proximity algebra, and coincides with the definition of our representation). Our remarks above prove that Rep 4, Rep 6 and Rep 7 all hold, whence Rep 5 follows easily since $a^{\dagger\dagger} = a$ and $a^\dagger \circ f = X_0 \backslash a \circ f$. In spite of these excellent qualities, the representation has the grave drawback that X_0 is not Hausdorff. As we shall see, when we take the natural step of reducing X_0 to a Hausdorff space by taking the quotient with respect to the relation $U_1 \sim U_2$ iff $\{U_1\} \ll A \Leftrightarrow \{U_2\} \ll A$ all A, the representation, carried into the quotient space, loses many of its properties.

6. Two representations in the space of maximal round filters

In forming the quotient $X_0/\sim$, it is sufficient to let the set A, in the formula above, run through a base of open sets. In our space, we easily get

$$U_1 \sim U_2 \text{ iff } r(U_1) = r(U_2).$$

Just as in the compactification of a proximity space, we may therefore take as the elements of $X = X_0/\sim$ not the actual equivalence classes but the filters $r(U)$, i.e., the maximal round filters in L. The map corresponding to f is then h (say) defined by

$$a \circ h = \{M \in X;\ r(a) \subset M\},$$

for $r(a) \subset M \Leftrightarrow \exists U,\ a \in U,\ r(U) = M$, since if $r(a) \subset M$ then $a^\dagger \notin M$ and $\exists U \supset M,\ a \in U$. We find it more convenient to define first the topology of X rather than the proximity; corresponding to the sets $\{b \circ g;\ b \in L\}$ in X_0 we take, as a base of open sets for X, $\{b \circ j : b \in L\}$ where $b \circ j = \{M;\ b \in M\}$.

We note that, by section 3, $a \circ h = X \backslash a^\dagger \circ j$; thus every set $a \circ h$ is closed and $a^\dagger = b^\dagger \Rightarrow a \circ h = b \circ h$. We now have two maps h, j of L into the same space X. If L is an algebra, each is deducible from the other by taking complements, and neither has any obvious advantage. In general, however, since a determines $a^\dagger$ but not conversely, j is likely to be a better representation than h in that it identifies fewer elements. (This is not surprising, since the open sets of a topological space form a pseudo-complemented lattice while the closed sets usually do not.) In the particular case of Ex.1, where the maximal round filters correspond exactly to those of proximity space theory, $a^\dagger h$ gives the closure of the image of $a^\dagger$ in the compactification of the space, and aj its (open) complement, which will include the image of the interior of a. In general, therefore, we may expect that in some sense aj will represent an 'interior' of a and ah a 'closure'; our theorems are largely concerned with illustrating this viewpoint.

LEMMA 6.1. *The space X is compact Hausdorff, and thus has the (unique) proximity consistent with its topology, given by $A \ll B$ iff $\overline{A} \subset \operatorname{Int} B$.*

The proof is again analogous to one often used in proximity space compactification. Let M_1, M_2 be distinct elements of X and $a \in M_1 \backslash M_2$ (by choice of notation). We can find b, c in the round filter M_1 with $c \ll b \leq b^{\dagger\dagger} \ll a$, $c^\dagger \gg b^\dagger \gg a^\dagger$. By lemma 3.2, as $b^{\dagger\dagger} \notin M_2$ we have $c^\dagger \in r(b^\dagger) \subset M_2$; thus $M_2 \in c^\dagger \circ j$, $M_1 \in c \circ j$ and these are disjoint open sets in X.

To prove compactness, consider any ultrafilter $\mathcal{U}$, in the ordinary sense, on the space X, and write $F = \{e \in L; e \circ j \in \mathcal{U}\}$, $R = r(F)$ in L. Let a, b be any elements of L with $a \ll b$. If $b \notin R$, choose c, d with $b \gg c^{\dagger\dagger} \gg \gg d \gg a$. Then $b \notin r(F) \Rightarrow c^{\dagger\dagger} \notin F \Rightarrow c^{\dagger\dagger} \circ j \notin \mathcal{U}$; now by lemma 3.2 $X \backslash c^{\dagger\dagger} \circ j \subset d^\dagger \circ j$. Since $\mathcal{U}$ is an ultrafilter this set is in $\mathcal{U}$ and so $d^\dagger \in F$, $a^\dagger \in R$. Thus R, which is clearly a round filter, is maximal round by lemma 3.2. If $e \circ j$ is any basic open neighbourhood of R in X, $\exists e_1, e_1 \in F$, $e_1 \ll e$, so that $e_1 \circ j$, *a fortiori* $e \circ j$, is in $\mathcal{U}$. That is, $\mathcal{U} \to R$ in the topology of X, so that X is compact.

We now state the simpler properties of the map j; properties of h can usually be deduced from the fact that $a^{\dagger\dagger} h = ah = X \backslash a^\dagger j$.

THEOREM 6. (i) $a \leq b \Rightarrow a \circ j \subset b \circ j$.
(ii) $(a \wedge b) \circ j = a \circ j \cap b \circ j$.
(iii) $a \circ j = [l(a)] \circ j$ (*i.e.* $\bigcup \{b \circ j : b \ll a\}$).
(iv) $(a \circ j)^- \subset a \circ h$.
(v) $a \ll b$ *iff* $a \circ h \subset b \circ j$.
(vi) $a \circ j \subset b \circ j$ *iff* $l(a) \subset l(b)$; *in particular,*

$a \circ j \cap b \circ j = \emptyset$ (in X) iff $l(a \wedge b) = l(a) \cap l(b) = \{0\}$. Thus j is a representation of L in X (not necessarily proper).

COROLLARY. *The restriction of j to the open elements of L is one-one, and j is one-one on L iff L is simple.*

The proofs of (i)–(iii) are trivial, while (iv) follows from the fact that $X\backslash a\circ h = a^{\dagger}\circ j$, open, and $a^{\dagger}j\cap aj = (a^{\dagger}\wedge a)j = 0j = \emptyset$. From the definitions, $a\ll b \Rightarrow a\circ h\subset b\circ j$, while the converse follows from lemmas 3.2 and 3.3, for if $a\not\ll b$, $\exists M$ with $r(a)\subset M$, $b\notin M$. If $l(a)\subset l(b)$ then $aj\subset bj$ by (iii), but if $l(a)\not\subset l(b)$, $\exists c$, $c\ll a$, $c\not\ll b$, so, again by lemmas 3.2 and 3.3, $\exists M$, $b\notin M$, $a\in r(c)\subset M$, whence (vi) follows.

If now a, b are distinct open elements then $l(a)\neq l(b)$ so $aj\neq bj$. If however a is not open, $\exists b$, $a\wedge b\neq a$, $l(a)\subset\lambda(a)\cap\lambda(b) = \lambda(a\wedge b)$; as $l(a)$ is round we see that $l(a) = l(a\cap b)$ so $aj = (a\wedge b)j$, $a\neq a\wedge b$. Thus j is one-one iff L is simple.

Our next result connects the mapping j with the theory of round ideals. Let X_2, j_2 be the space and map associated with L_{RI} as X, j are with L. Since X, X_2 are subsets of L_{RF}, $(L_{RI})_{RF}$ we can by Theorem 5 consider (the restriction of) l_2 as a one-one map of X onto X_2. If I is a round ideal in L, $I\circ j\circ l_2$ will denote (l_2 being one-one) the set $\{x_2; x_2\circ l_2^{-1}\in I\circ j\}$ in X_2, where $I\circ j$ as usual means $\bigcup\{a\circ j; a\in I\}$. We note also that as I is round $(I) = I$ (where I is considered as a set in L.

THEOREM 7. (i) *$a\circ j\circ l_2 = a\circ l\circ j_2$ for all a; hence if I is a round ideal $I\circ j\circ l_2 = I\circ l\circ j_2 = I\circ j_2$.*

(ii) *$l_2|X$ is a homeomorphism of X with X_2.*

(iii) *j_2 is a complete lattice isomorphism of L_{RI} with the lattice L_G of all the open sets of X_2 (ordered by inclusion); in particular, $X_2\backslash(I\circ j_2)^- = I^{\dagger}\circ j_2$.*

(iv) *$I_1\ll I_2$ iff $I_1\circ j_2\ll I_2\circ j_2$ in X_2, i.e., iff $(I_1\circ j_2)^-\subset I_2\circ j_2$: thus j_2 is a proximity-lattice isomorphism.*

COROLLARY 1. *j_2 is a proximity-lattice isomorphism of the proximity algebra of regular elements of L_{RI} with the proximity algebra of all regular open sets ($G = \operatorname{Int}\overline{G}$) of X_2.*

COROLLARY 2. *For any proximity lattice L, the restriction of j to the open elements is strictly order-preserving and proximity preserving in both directions.*

If $a\in M$ then by definition $al\in Ml_2$ and so $Ml_2\in alj_2$ if $M\in aj$. Conversely, any point of alj_2 can by Theorem 5, Corollary, be expressed as Ml_2, where $al\in Ml_2$ so that $\exists b\in M$, $bl\leq al$. As M is round, $\exists c\in M\cap l(b)$, whence $c\in l(a)$ so $a\in M$, $M\in aj$. This proves (i).

If $M_0\in G$, G open in X, then $\exists a$, $M_0\in aj\subset G$ so that $M_0l_2\in ajl_2\subset Gl_2$. But $ajl_2 = alj_2$ is a basic open set of X_2, so as M_0 is arbitrary Gl_2 must be open in X_2. Conversely if G_2 is open in X_2 any point of G_2 can be expressed as M_0l_2. Then $\exists I$, $M_0l_2\in Ij_2\subset G_2$; that is, $I\in M_0l_2$ and $\exists a\in M_0$, $al\leq I$, whence $ajl_2 = alj_2\subset Ij_2\subset G_2$, so that $M_0\in aj\subset G_2l_2^{-1}$. Thus $G_2l_2^{-1}$ is open and l_2, being one-one, is a homeomorphism.

To prove (iii) it is now sufficient to show that every open set G_0 of X is of the form $I_0 j$ for some round ideal I_0 in L; for then, by (i) and (ii), j_2 is onto L_G, but by Theorem 6 (vi) and Corollary, j_2 is one-one and order-preserving both ways as L_{RI} is simple; it is therefore a complete lattice-isomorphism.

Given G_0, write $I_0 = \bigcup\{l(a); aj \subset G_0\}$. We show that $b_1, b_2 \in I_0 \Rightarrow b_1 \vee b_2 \in I_0$; I_0 is therefore a round ideal as the other defining conditions are obviously satisfied. Let $b_i \ll a_i$, $a_i j \subset G_0$, $i = 1, 2$; choose c_i, $b_i \ll c_i \ll a_i$. If $c_1 \vee c_2 \in M$ then $c_1^\dagger, c_2^\dagger$ cannot both be in M as $(c_1 \vee c_2) \wedge c_1^\dagger \wedge c_2^\dagger = 0$; by lemma 3.2, one at least of a_1, a_2 is in M, so that $M \in a_1 j \cup a_2 j \subset G_0$. That is, $(c_1 \vee c_2) j \subset G_0$, and $b_1 \vee b_2 \in l(c_1 \vee c_2) \subset I_0$. If now $M_0 \in G_0$ then $\exists a$, $M_0 \in aj \subset G_0$; as M_0 is round $\exists b \in l(a) \cap M_0$ so that $M_0 \in bj \subset I_0 j$; the converse $(I_0 j \subset G_0)$ is trivial.

We note that in L_G we have $G^\dagger = \mathrm{Int}\,(X_2 \backslash G)$, $G^{\dagger\dagger} = \mathrm{Int}\,(\bar{G})$. Thus given I_0 we can find I_1 with $I_1 j_2 = X_2 \backslash (I_0 j_2)^- = (I_0 j_2)^\dagger$. By the isomorphism I_1 is the largest element of L_{RI} satisfying $I_1 \wedge I_0 = 0_{RI}$, i.e. $I_1 = I_0^\dagger$.

Finally, if $I_1 \ll I_2$ then by Theorem 6, (iv) and (v), $(I_1 j_2)^- \subset I_2 j_2$ so that $I_1 j_2 \ll I_2 j_2$ in X_2, while if $I_1 \ll I_2$ is false there exists by lemma 3.3 an element of X_2 which is in neither $I_2 j_2$ nor $I_1^\dagger j_2 = X_2 \backslash (I_1 j_2)^-$, so that $(I_1 j_2)^- \subset I_2 j_2$ is also false.

This completes the proof. Corollary 1 follows at once from the formula for $G^{\dagger\dagger}$. For Corollary 2, we remark that, l_2 being one-one, we have $aj = alj_2 l_2^{-1}$. If we restrict ourselves to the open elements of L, the maps l, j_2, l_2 and their inverses are all one-one and strictly order-preserving and proximity-preserving, so that the same is true of j and j^{-1}. (Here j, j_2, and l_2^{-1} are considered as maps into the lattices of open sets of X or X_2 rather than into the spaces themselves.)

From Theorems 4 to 7 it is easy to deduce further properties of the map j, particularly as applied to open elements.

THEOREM 8. (i) *$(aj)^- = ah$ iff $(al)^\dagger = a^\dagger l$. The conditions* Rep 4 *and* Rep 5 *are satisfied by j over the set of all such elements. (In particular, by Theorem* 6, *Corollary, j is proper over the set of open elements.)*

COROLLARY. *If b is closed, then $bj = \mathrm{Int}\,(bh)$ and hence is a regular open set.*

(ii) *If a is open and $a^{\dagger\dagger}$ closed then aj is a regular open set iff $al = a^{\dagger\dagger} l$. If L is simple, then aj is a regular open set iff a is regular; $j|L_r$ is a proximity-lattice isomorphism of L_r with a sub-algebra of the proximity algebra (as in* Ex.3*) of all regular open sets of X. If L is also order-complete then $L_r j$ contains all such sets.*

The first statement is an immediate consequence of Theorem 6, (iii), (vi) with Theorem 7 (i), (ii), (iii) since $ah = X \backslash a^\dagger j$; we easily obtain $[X \backslash (aj)^-] l_2 = (al)^\dagger j_2$ which is $(a^\dagger j) l_2$ iff $(al)^\dagger = a^\dagger l$. If a_1, a_2 satisfy this condition then Rep 4 reduces to Theorem 6 (v), while

$$a_1 j \, R \, a_2 j \Leftrightarrow (a_1 j)^- \cap (a_2 j)^- = \emptyset \Leftrightarrow a_1 h \cap a_2 h = \emptyset \Leftrightarrow a_1 h \subset a_2^\dagger j$$

and, again by 6 (v), this gives Rep 5. We note that in general j is not one-one over the set of all such elements, but is so over the subset of open elements. For the Corollary, we use b closed $\Rightarrow b^\dagger$ open in L_r (Theorems 1, 2) so $(b^\dagger \circ l)^\dagger = b \circ l$ by Theorem 4, since b being closed is regular. It remains only to put $a = b^\dagger$ and take complements in X.

If a is open and $a^{\dagger\dagger}$ closed then putting $b = a^{\dagger\dagger}$ we have $(al)^{\dagger\dagger} = (a^\dagger l)^\dagger = bl$ by Theorem 4, since $a^\dagger$ is open in L_r. By Theorem 7, $blj_2 = (alj_2)^{\dagger\dagger}$ in the lattice L_G, so applying l_2^{-1} we have $bj = (ajl_2)^{\dagger\dagger} l_2^{-1} = (aj)^{\dagger\dagger}$ in the (isomorphic) lattice of open sets of X. Thus aj is regular open iff $aj = a^{\dagger\dagger}j$, i.e. iff $al = a^{\dagger\dagger}l$. If L is simple, $a^{\dagger\dagger}$ is always closed (Theorem 1) and $al = a^{\dagger\dagger}l$ iff $a = a^{\dagger\dagger}$. We already know that j is one-one (L simple), that j, j^{-1} preserve order and proximity, and that $(a_1 \wedge a_2)j = a_1 j \cap a_2 j$. Now $(a_1 \vee a_2)^{\dagger\dagger} j = (a_1^\dagger \wedge a_2^\dagger)^\dagger j = [(a_1 j)^\dagger \cap (a_2 j)^\dagger]^\dagger = (a_1 j \cup a_2 j)^{\dagger\dagger}$, the smallest *regular* open set containing $a_1 j$ and $a_2 j$. Thus $L_r j$ is a subalgebra of the algebra of regular open sets, though not usually a sublattice of the lattice of all open sets.

Finally, any regular open set G of X is (uniquely) expressible as Ij, I regular in L_{RI} (Theorem 7). If L is simple and complete, let $a = \sup I$; since I is round, $I \leq a \circ l$. But by Theorem 4 $I^\dagger = a^\dagger l$, whence $I = I^{\dagger\dagger} = a^{\dagger\dagger} l \geq al$. That is, $I = al$, $Ij = alj = aj$, so by the above results $a \in L_r$ and $G \in L_r j$.

7. A representation particularly appropriate for proximity algebras

The representation j studied above, while appropriate for simple proximity lattices, is clearly a very blunt instrument in general, e.g. for an algebra arising from the standard case of Ex.1. By the use of the neighbourhood relation we can obtain a better (more nearly proper) representation, which is particularly satisfactory for proximity algebras which fulfil two special conditions. One of these is the separation condition of section **2**, while the other is an existence condition, satisfied in particular by all algebras arising as in Ex.1.

DEFINITION 7.1. *With L, j, X as before, we define the map $k: L \to 2^X$, by $a \circ k = \bigcap\{b \circ j;\ bNa\}$. If $a \neq 0$ implies $a \circ k \neq \emptyset$, we say that L satisfies condition E (the 'existence' condition).*

(It can be shown that, even for proximity algebras properly representable in Hausdorff proximity spaces, condition E does not follow from the axioms.)

We could define similarly a map k^* using N^*; most of the results below can be obtained in a very similar way for N^*, but we do not discuss this case in detail. From lemma 2.1, together with the fact that $ah = \bigcap\{bj;\ b \gg a\}$, we have at once, for all L:

LEMMA 7.1. *For all a, $aj \subset ak \subset ah$; if a is open then $ak = aj$.*

COROLLARY. *k is a topological representation of L in X.*

For k is order-preserving by lemma 2.1 (iv), and $a \ll b \Rightarrow ak \subset ah \ll \ll bj \subset bk$ (in X). We note that it can well happen that (e.g. in a simple algebra such as that of Ex.3) a, $a^\dagger$ are both open and closed, and $aj = ak \neq \neq ah$; hence a closed does not imply $ak = ah$.

Since bj is open and $bj \subset bk$, bNa implies $ak \subset bj \subset \operatorname{Int} bk$, so that k is topological.

LEMMA 7.2. *If L is separated then $aj = \operatorname{Int} ak$.*

As $\{cj;\ c \in L\}$ is a base of open sets, it suffices to show that $cj \subset ak \Rightarrow \Rightarrow cj \subset aj$. Let then $M_0 \in cj \subset ak$; $c_1 \in M_0$, $c_1 \ll c$. If $c_1 \not\leq a$ then $\exists b$, bNa, $b И c_1$: *a fortiori* $c_1 \nless \nless b$. By lemma 3.3 $\exists M$, $r(c_1) \subset M$, $b \notin M$, from which $c \in M$, $M \in cj \subset ak$, $M \notin bj$, a contradiction. Thus in fact $c_1 \leq a$ and $M_0 \in c_1 j \subset aj$, as required.

LEMMA 7.3. *If L satisfies condition E then $(ak)^- = ah = X \backslash a^\dagger j$.*

It is sufficient to show that $cj \cap ak = \emptyset$ implies $cj \subset a^\dagger j$. As before, let c, $c_1 \in M_0$, $c_1 \ll c$, and suppose $c_1 \not\leq a^\dagger$. Then, by E, $(c_1 \wedge a)k \neq \emptyset$. Since cNc_1 we have $(c_1 \wedge a)k \subset cj$, so $(c_1 \wedge a)k \subset cj \cap ak = \emptyset$. This contradiction proves that $c_1 \leq a^\dagger$ and so $M_0 \in a^\dagger j$.

LEMMA 7.4. *If L satisfies condition E, and a_0 is an element such that $a_1 < a_0 \Rightarrow a_1^{\dagger\dagger} \not\leq a_0$, then $b И a_0 \Rightarrow a_0 k \not\subset bj$. Hence, if L is also separated, $a_0 \not\leq a$ implies $a_0 k \not\subset ak$.*

Since $b И a_0$, $\exists d$, $d \not\leq a_0$, such that $l(b) \subset \lambda(d)$. Let $a_1 = a_0 \wedge d < a_0$. Then by supposition $a_2 = a_0 \wedge a_1^\dagger \neq 0$, whence $a_2 k \neq \emptyset$. If $M \in a_2 k$ and $b \in M$, then $\exists b_1$, b_2 in M, $b_2 \ll b_1 \ll b$. Hence $b_1 \leq d$, $b_1 \wedge a_0 \leq a_1$, so $b_1 \wedge a_2 \leq a_1 \wedge a_1^\dagger = 0$, and $a_2 \leq b_1^\dagger \ll b_2^\dagger$. But this gives $b_2^\dagger N a_2$, whence $M \in a_2 k \subset b_2^\dagger j$, impossible as b_2 and $b_2^\dagger$ cannot both be in M. Thus $a_2 k \cap bj = \emptyset$ and so, as $a_2 k \subset a_0 k$, $a_0 k \not\subset bj$.

If now L is also separated, and $a_0 \not\leq a$, then $\exists b$, bNa, $b И a_0$ so that $ak \subset bj$, $a_0 k \not\subset bj$ and hence $a_0 k \not\subset ak$.

Combining our lemmas, and restricting ourselves to a proximity algebra so that the special condition of lemma 7.4 is certainly satisfied, we obtain:

THEOREM 9. *If L is a proximity algebra, then k is one-one iff L satisfies condition E and is separated. Then k is a proper and topologically proper representation of L.*

The conditions are obviously necessary; if they are satisfied lemma 7.4 shows that k is one-one, strictly order-preserving both ways and, with lemma 7.2 , that k is topologically proper. By Theorem 6 (v) with lemmas 7.2 and 7.3, $a \ll b \Leftrightarrow ah \subset bj \Leftrightarrow (ak)^- \subset \operatorname{Int} bk \Leftrightarrow ak \ll bk$ in X; this is Rep 4; while as $b^\dagger j = X \backslash b \circ h$ we obtain Rep 5 similarly: in fact $a \ll b^\dagger \Leftrightarrow (ak)^- \subset \subset X \backslash (bk)^-$ so that $aRb \Leftrightarrow (ak)^- \cap (bk)^- = \emptyset \Leftrightarrow akRbk$ in X.

We remark that the discussion at the end of section 2 shows that, even for simple proximity algebras for which Theorem 9 holds, k is not

usually fully proper. Indeed, even for the algebra of all subsets of a separated proximity space (Ex.1), we may have $ak \cap a^{\dagger}k \neq \emptyset$, as is shown by the following example.

EXAMPLE 7. Let X be the space of ordinals $\leq \omega_2$ (the first ordinal of cardinal $> \aleph_1$) with its natural compact Hausdorff interval-topology and the associated proximity, and X_1 the sub-space obtained by omitting ω_2 (with the induced proximity). Let A_1 be the set of all ordinals in X_1 obtainable as limits of strictly increasing enumerable sequences, B_1 the set of those similarly obtainable by sequences of length ω_1. Any set E not bounded above in X_1 contains sequences of both types so that $\bar{E} \cap A_1$ and $\bar{E} \cap B_1$ are non-empty. Thus $A_1 \subset \operatorname{Int} C$ and $B_1 \subset \operatorname{Int} D$ (in X_1) imply that both C and D include a residual segment of X_1. It easily follows that, when we consider 2^{X_1} as a proximity algebra, there exists a maximal round filter containing all C, D with CNA_1, DNB_1 so that $A_1k \cap A_2k \neq \emptyset$ (the space of maximal round filters is indeed a copy of X).

This last anomaly may easily be removed, while retaining topological propriety, for any algebra for which Theorem 9 holds. For if $a \neq 0$ then lemma 7.4 gives $ak \not\subset a^{\dagger}k$, so if we define k_1 by $a \circ k_1 = a \circ k \cap (X \backslash a^{\dagger} \circ k)$, which is clearly order-preserving, then the following condition is satisfied: condition $E_1 : a \neq 0 \Rightarrow a \circ k_1 \neq \emptyset$. Examination of the proofs of lemmas 7.1 to 7.4 above shows that they all (and hence Theorem 9) remain valid if k is replaced by k_1.

Two further questions naturally arise. Is it always possible, by a further modification of k, to obtain a fully proper (but not usually topological) representation i in X, retaining the most important properties of k, in particular, such that $aj \subset ai$? Secondly, can we, by restricting k or k_1 to some subspace of X, obtain a fully proper and topologically proper representation in a space X_1 which is Hausdorff but usually non-compact? I have not been able to answer either of these questions. We can however see that any general method for answering the first question seems bound to be non-constructive, since for any fully proper representation i of an algebra L and any given point x of $1i$ the set $\{a; x \in ai\}$ must be an ultrafilter in L, and if L has no atoms there are no trivial ultrafilters.

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SUBDIRECT IRREDUCIBLE WEAKLY ASSOCIATIVE LATTICES WITH CONGRUENCE EXTENSION PROPERTY

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Introduction. The most effective method of investigation for Universal Algebras is the use equational classes. However, numerous questions, which are useful in the investigation, are strange to this method. The most important among these are the *Amalgamation Property* and the *Congruence Extencion Property* (further on CEP).

It is well-known that the only equational class of lattices where CEP does hold is the class of distributive lattices. The introduction of the *Weakly Associative Lattices* (further on WAL) (see, e.g., in [2] or in [9]) has given us several equational classes where CEP does hold (see [3] and [5]).

The aim of this paper is to describe those subdirectly irreducible WAL-s which satisfy CEP. Then, we shall give identities such that in the class of WAL-s satisfying these identities exactly those WAL-s satisfy CEP which have been given in [3].

Preliminaries. A WAL is a relational system with a reflexive and antisymmetrical relation $\leqq$ where each pair of elements has both least upper bound (lub) and greatest lower bound (glb).

Denoted by $a \vee b$ and by $a \wedge b$ the lub and the glb of the elements a and b, respectively, we are in the position to define the WAL-s by these operations, as follows. $\mathfrak{A} = \langle A; \vee, \wedge \rangle$ is a WAL if the two binary operations $\vee$ and $\wedge$ are idempotent and commutatives, the both absorption laws hold, and

$$[(a \vee c) \wedge (b \vee c)] \wedge c = [(a \wedge c) \vee (b \wedge c)] \vee c = c$$

are valid (see [2] and [9]).

Let $\mathfrak{B}$ be a subalgebra of an algebra $\mathfrak{A}$ and Θ be a congruence relation on $\mathfrak{A}$. If the meet of the congruence-classes of Θ by $\mathfrak{B}$ are the classes of a congruence relation Φ on $\mathfrak{B}$ then Θ is called an extension of Φ to $\mathfrak{A}$. $\mathfrak{A}$ is said to satisfy CEP whenever each congruence relation on any sub-

algebra of $\mathfrak{A}$ has an extension to $\mathfrak{A}$. An equational class satisfies CEP, by definition, if all elements of this class satisfy CEP.

CEP, **generally, for** WAL-**s.** To determine, in an equational class, the connection between algebras satisfying a given property and equational subclasses one has to answer three questions:

1. Does all subdirectly irreducible component of an algebra satisfying the given property satisfy this property, too?
2. Does all subdirect products of subdirect irreducible algebras satisfying the given property satisfy this property, too?
3. Which are those subdirect irreducible algebras that satisfy the given property?

If the answer of question 1 is yes then the class of all algebras satisfying the given property is contained in the equational subclass generated by the subdirectly irreducible ones.

If the answer of question 2 is yes then the class of all algebras satisfying the given property containes the equational subclass generated by the subdirectly irreducible ones.

If both of the two answers are yes then the property is an "equational" one.

This shows that the answer of question 3 is always essential.

In our case the given equational class is the class of all WAL-s and the given property is CEP.

We shall give a complete answer of question 3 and a partial answer of question 2.

A WAL $\mathfrak{A}$ is said to have the *unique bound property* (further on UBP) if any two different elements of $\mathfrak{A}$ have exactly one common upper bound and exactly one common lower bound.

THEOREM 1. *A* WAL $\mathfrak{A}$ *has* UBP *if and only if it is subdirectly irreducible and it satisfies* CEP.

PROOF. Let us suppose, firstly, that $\mathfrak{A}$ is a subdirectly irreducible WAL satisfying CEP. Being $\mathfrak{A}$ subdirectly irreducible it has a smallest non-trivial congruence-relation Φ, i.e., there are different elements which belongs to the same conrguence-class C of Φ. The idempotency implies that there are elements $a<b$ in C.

Let, now, $c\in\mathfrak{A}$ such that both $a\leqq c$ and $b\leqq c$. Then a, b, c form a subalgebra of $\mathfrak{A}$ in which the sets $\{a\}$ and $\{b, c\}$ are cosets of a congruence-relation. This congruence-relation has an extension Ψ to $\mathfrak{A}$, by CEP. Since $a\neq b(\Psi)$, $a\equiv b(\Phi)$ and being Φ the smallest non-trivial congruence-relation on $\mathfrak{A}$ therefore Ψ must be the trivial ones, i.e., by $b\equiv c(\Psi)$, we have $c=b$. Dually, $d\leqq a$ and $d\leqq b$ imply $d=a$.

Let, now, $b\leqq c$ be supposed for an element c of $\mathfrak{A}$. $b=b\wedge c\equiv c(\Phi)$ implies $a'=a\wedge c\in C$ and $b=b\vee b\equiv a'\vee b(\Phi)$ implies $b'=a'\vee b\in C$. Clearly, $a'\leqq b'$, $b\leqq b'$, and $v\leqq c$, for each $v\in\{a', b', b\}$. $b=b'=a'$ would imply $b=$ $=b'=a'=a\wedge c\leqq a$ contradicting $a<b$. In case $b<b'$ $(\in C)$ the relations $b\leqq c$

and $b' \leqq c$ imply $b' = c$ and in case $a' < b'$ $(\in C)$ the relations $a' \leqq c$ and $b' \leqq c$ imply $b' = c$, as it just has been proven. In both cases $c = b'$ belongs to C. Dually, $d \in C$ follows from $d \leqq a$, for each $d \in \mathfrak{A}$.

Let, finally, u be any element of $\mathfrak{A}$. $b \leqq b \vee u$, and $a \wedge u \leqq a$ imply that both $b \vee u$ and $a \wedge u$ belong to C. Hence, by $u = (a \wedge u) \vee u \equiv b \vee u(\Phi)$, $u \in C$, i.e., all elements of $\mathfrak{A}$ belong to the class C.

Let us consider a lower bound z of x and y different from $x \wedge y$ $(x, y, z \in \mathfrak{A})$. Since $z < x \wedge y$ and both of them belong to C, $z \leqq x$ and $x \wedge y \leqq x$ imply $x \wedge y = x$. The equality of $x \wedge y$ and y is implied similarly. Thus, $x = y$, i.e., the only lower bound of different elements is their meet. The UBP is proven, using duality.

Let, conversely, the UBP be supposed for $\mathfrak{A}$. Let, further, Φ be a congruence on $\mathfrak{A}$ such that there are different elements a and b in $\mathfrak{A}$ with $a \equiv b(\Phi)$. We may suppose, without the loss of generality, that $a < b$ holds.

For $x \in \mathfrak{A}$, $a = x \vee b \geqq b$ contradicts $a < b$, i.e., a and $x \vee b$ are different. In case $x \leqq a$, the relation $x \leqq x \vee b$ implies $x = (x \vee b) \wedge a$, by UBP. Thus, $x = (x \vee b) \wedge a \equiv (x \vee a) \wedge a = a(\Phi)$ holds. Dually, $b \leqq y$ implies $y \equiv a(\Phi)$, too. So, for any $u \in \mathfrak{A}$, we have, using $a \wedge u \leqq a$ and $b \leqq b \vee u$, $u = u \vee (u \wedge a) \equiv u \vee b \equiv$ $\equiv a(\Phi)$, i.e., $\mathfrak{A}$ is simple, hence, subdirectly irreducible.

Having each subalgebra of $\mathfrak{A}$ also the UBP each of them is simple, too. Since the trivial congruence-relations have always extension to $\mathfrak{A}$, this algebra satisfies also CEP.

It is worth formulating the result we have been just found:

COROLLARY. *If the* WAL $\mathfrak{A}$ *has the* UBP *then each subalgebra of* $\mathfrak{A}$ *is simple.*

One needs, for the description of the extension of congruences, the description of the extension of the principal congruences (see [1]). One can do this by telling when are the elements c and d congruent by $\Theta(a, b)$ (see, e.g., [6] for the notations). This has been done in [3] and in [5] by giving three equivalent conditions. We are going to prove the equivalence of two such conditions, generally, for WAL-s having UBP.

THEOREM 2. *Let* $\mathfrak{A}$ *be a* WAL *having the* UBP. *For* $a \leqq b$ *and* x *in* $\mathfrak{A}$ *we define*

$$x' = (a \vee x) \vee b \quad \textit{and} \quad x^+ = a \wedge (x \wedge b).$$

Then, for $c \leqq d$ *in* $\mathfrak{A}$, $\quad c \equiv d(\Theta(a, b))$ *if and only if*

$$\textit{both } c' \wedge c'' = d' \wedge d'' \quad \textit{and} \quad c^+ \vee c^{++} = d^+ \vee d^{++}$$

are valid.

PROOF. Let, first, $a \neq b$ be supposed. Then, by Theorem 1, $c \equiv$ $\equiv d(\Theta(a, b))$ is always fulfilled. This means, that in this case we have to prove that for each $x \in \mathfrak{A}$ both $x' \wedge x''$ and $x^+ \vee x^{++}$ are independent from x. It is enough to prove this statement, using duality, only for $x' \wedge x''$.

In case $x' = b$, $b' = (a \vee b) \vee b = b$ implies $x' \wedge x'' = b \wedge b = b$.

Let, now, $y'=y$, for some $y\in\mathfrak{A}$. Then, $y=y'=(a\vee y)\vee b\leqq a\vee y\leqq y$ implies $y=a\vee y$, i.e., $a\leqq y$. On the other hand $b\leqq(a\vee y)\vee b=y'=y$ is valid. Then, using UBP, $a\neq b$ implies $y=a\vee b=b$. Hence, $x'\neq b$ implies $x'\neq x''$, i.e., from $y'\geqq b$ and from the UBP follows $x'\wedge x''=b$.

Now, let us suppose that $a=b$. Then, $c\equiv d(\Theta(a, b))$ is possible only if $c=d$. If, for $c\leqq d$ in $\mathfrak{A}$, $c'\wedge c''=d'\wedge d''$ and $c^{+}\vee c^{++}=d^{+}\vee d^{++}$ hold we shall prove $c=d$.

The condition $a=b$ implies $x'\wedge x''=a\vee x$ and $x^{+}\vee x^{++}=a\wedge x$, for all $x\in\mathfrak{A}$. These imply, for $u=a\wedge c=a\wedge d$ and for $v=a\vee c=a\vee d$, the relations $u\leqq c\leqq v$ and $u\leqq d\leqq v$. Hence, we have for $c\neq d$, by UBP, $u=c\wedge d=c$ and $v=c\vee d=d$, i.e., $c=a\wedge d=a\wedge(a\vee c)=a=a\vee(a\wedge d)=a\vee c=d$, which is a contradiction.

Remark. The existence of a polinomial $p(x_1, x_2, x_3, x_4, x_5)$ with $c=p(a, a, b, c, d)$ and $d=p(b, a, b, c, d)$ if and only if $c\equiv d(\Theta(a, b))$ $(a\leqq b,\ c\leqq d)$ for WAL-s having UBP would imply that the answer of question 2 is yes. The polinomial defined in [5] has, in some special cases, this property.

Construction of WAL-s **having** UBP. Theorem 1 gives us a complete description of WAL-s having UBP. However, it would be possible that there would not exist such WAL-s. That it is not the case it has been shown in [3] but it also would be possible that those WAL-s would given us all examples. We are going to prove that there are other examples, too. We shall give a general method constructing WAL-s having UBP.

A relational system with a reflexive and antisymmetrical relation $\leqq$ will be called partial WAL. It has been proven in [4] that each partial WAL is, indeed, a relative subalgebra of some WAL. We shall say that a partial WAL $\mathfrak{A}$ has the UBP if any two different elements of $\mathfrak{A}$ have no more then one common upper bound and no more then one common lower bound.

We shall prove:

Theorem 3. *There exist to each partial* WAL $\mathfrak{A}$ *having* UBP *a* WAL $\mathfrak{A}'$ *containing* $\mathfrak{A}$ *and having* UBP, *too.*

Remark. Theorem 3 guarantees that it is nearly no matter how one begins the construction of a WAL having UBP.

To prove Theorem 3 we need two construction-steps. Let $\mathfrak{A}$ be any partial WAL having UBP. Let us choose different elements a and b in $\mathfrak{A}$ such that no lower bound of b is less then or equal to an upper bound of a. Then, we define a partial WAL $\mathfrak{A}(a, b)$ adding the one more relation $a<b$ to the relations on $\mathfrak{A}$.

Proposition 1. $\mathfrak{A}(a, b)$ *has the* UBP.

Proof. Let $x\leqq u$, $y\leqq u$, $x\leqq v$, and $y\leqq v$ in $\mathfrak{A}(a, b)$, for the different elements x and y. If all of these four relations belong to $\mathfrak{A}$ then $u=v$ is implied, in advance. If it is not the case then we may suppose $x=a$ and $u=b$, by symmetry. This is, however, impossible since in this case b would

have a lower bound (namely y) which would be less then or equal to an upper bound of a (namely v) contradicting to the construction of $\mathfrak{A}(a, b)$. The duality completes the proof.

Now, we define $\mathfrak{A}_\gamma$ for cardinals $\gamma \leqq \alpha$, for a fixed cardinal α. If $\gamma = \beta + 1$, then we define $\mathfrak{A}_\gamma = \mathfrak{A}_\beta(a_\gamma, b_\gamma)$, for some a_γ, b_γ which satisfy the condition. If $\gamma = \beta + 1$ holds for no β then we define $a \leqq b$ in $\mathfrak{A}_\gamma$ if and only if $a \leqq b$ is valid in $\mathfrak{A}_\beta$, for some $\beta < \gamma$. All of these $\mathfrak{A}_\gamma$ will be called *inner extensions* of $\mathfrak{A}$.

We have, obviously,

PROPOSITION 2. *Each inner extension of a partial* WAL $\mathfrak{A}$ *having* UBP *is, itself, a partial* WAL *and it also has* UBP.

Let, now, $\mathfrak{A}$ a partial WAL having UBP. Let us choose a set $\hat{A}$ consisting of some subsets A_ν of the underlying set A of $\mathfrak{A}$ such that:

1. If the elements a and b of $\mathfrak{A}$ have no common upper bound in $\mathfrak{A}$ then they belong exactly to one of the A_ν.
2. The pairs of elements of a given A_ν have no common upper bound in $\mathfrak{A}$. (We may choose, e.g., exactly the pairs of such a_ν and b_ν.)

We define $\mathfrak{A}^*$ as follows:

1. The underlying set A^* of $\mathfrak{A}^*$ is the disjoint union of A and $\hat{A}$.
2. Let, for $a, b \in A$, $a \leqq b$ in $\mathfrak{A}^*$ if and only if

$$a \leqq b \text{ in } \mathfrak{A},$$

for $a \in A$, $A_\nu \in \hat{A}$ $a \leqq A_\nu$ in $\mathfrak{A}^*$ if and only if

$$a \in A_\nu,$$

no other pairs of elements of $\mathfrak{A}^*$ is comparable.

It is routin to verify:

PROPOSITION 3. $\mathfrak{A}^*$ *is a partial* WAL *having* UBP.

One can define, dually, $\mathfrak{A}_*$ where each pair of elements of $\mathfrak{A}$ has common lower bound.

We shall call $\mathfrak{A}^+ = (\mathfrak{A}^*)_*$ an outer extension of $\mathfrak{A}$. Now, let $\mathfrak{A}^{(1)}$ define by $\mathfrak{A}^{(1)} = (\overline{\mathfrak{A}})^+$ for some inner extension $\overline{\mathfrak{A}}$ of $\mathfrak{A}$. $\mathfrak{A}^{(1)}$ is, of course, not uniquely determined. We define, for each natural number n, $\mathfrak{A}^{(n)}$ by $\mathfrak{A}^{(n)} = (\mathfrak{A}^{(n-1)})^{(1)}$ and $\mathfrak{A}'$ as the direct union of all $\mathfrak{A}^{(n)}$.

Propositions 2 and 3 guarantee that $\mathfrak{A}^{(1)}$ is a partial WAL having UBP whenever $\mathfrak{A}$ has. Thus, $\mathfrak{A}'$ also has, clearly, UBP, moreover $\mathfrak{A}'$ is not only a partial WAL but it is a WAL proving Theorem 3.

REMARK. One can say that this is the only possibility to construct a WAL having UBP. Let, namely, $\mathfrak{A}$ be such a WAL with the underlying set A. Then $\mathfrak{A}_0$ with the underlying set A and with the relations $a \leqq a$ is,

clearly, a partial WAL having UBP and $\mathfrak{A}$ is, obviously, an inner extension of $\mathfrak{A}_0$.

It has been shown, in papers [3] and [5], that the finite algebras have been given in these papers have over the CEP a direct decomposition. We refer to Proposition 11 in [3] yielding this result:

PROPOSITION 11 IN [3]. *Let $\mathfrak{A}$ be a subdirect product of the algebras $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Let us assume that there exist a family f_λ and g_μ of polynomials ($\lambda \in \Lambda$, $\mu \in M$) such that*

(i) $f_\lambda(x, y, y, \ldots) = x$ *and* $g_\mu(x, y, y) = y$
holds in all $\mathfrak{A}_i$;

(ii) *there exist, in each $\mathfrak{A}_i$, to the elements a and b elements $a_1, \ldots$ and $\lambda \in \Lambda$ such that $f_\lambda(a, a_1, \ldots) = b$ and to an element a and to different elements b and c a $\mu \in M$ such that $g_\mu(a, b, c) = a$, whenever $i = 1, \ldots, n$.*

Then any subdirect product of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is isomorphic to a direct product of some of them.

THEOREM 4. *A* WAL $\mathfrak{A}$ *has the property listed above if and only if* $|\mathfrak{A}| > 2$ *and* $\mathfrak{A}$ *has* UBP.

PROOF. It has been proven, indirectly in [5], that the twoelements WAL has not this property. Let us suppose that the given polynomial exist in $\mathfrak{A}$ and let c be an upper bound of the different elements a and b. If $a \vee b \neq c$ then, for a suitable $g(x, y, z)$ we would have $g(a, a \vee b, c) = a$ (or $g(b, a \vee b, c) = b$). Let us map this three-elements chain to the two-elements one, by $\varphi(a) = 0$ and $\varphi(a \vee b) = \varphi(c) = 1$. Then we would have $0 = \varphi(a) = \varphi(g(a, a \vee b, c)) = g(0, 1, 1) = \varphi(g(a, c, c)) = \varphi(c) = 1$ which is a contradiction. Hence, using duality, $\mathfrak{A}$ must have UBP.

Now, let $\mathfrak{A}$ have UBP and let $|\mathfrak{A}| > 2$. We are going to give the polynomial f_λ and g_μ. Let, first $t(x, y, z) = ((x \vee y) \vee z) \wedge x$ and $f(x, y, z, u, v) = \bar{t}(t(x, y, z), u, v)$ with $\bar{t}(x, u, v) = ((x \wedge u) \wedge v) \vee x$. Since $t(x, y, y) = \bar{t}(x, y, y) = x$, we have $\bar{t}(t(x, y, y), y, y) = t(x, y, y) = x$.

Let, now, a and b be given elements in $\mathfrak{A}$. Since $\mathfrak{A}$ has, clearly, no greatest element there exist $a_1 > a$. Let $a_2 = (a \wedge b) \vee a_1$. Let, further, $a_3 < a \wedge b$ and $a_4 = a_3 \wedge b$. Then, we have $t(a, a_1, a_2) = ((a \vee a_1) \vee a_2) \wedge a = (a_1 \vee a_2) \wedge a = a_2 \wedge a$. Since $a < a_1 \leqq a_2$ implies $a \neq a_2$ and $a \wedge b$ is a lower bound of both of them, from UBP follows $a_2 \wedge a = a \wedge b$. Dually, we have $t(a \wedge b, a_3, a_4) = b$, proving $\bar{t}(\bar{t}(a, a_1, a_2), a_3, a_4) = b$.

Finally, we define the polynomials

$$h_1(x, y, z) = (x \wedge y) \vee \big(((x \wedge z) \vee y) \wedge z\big) \text{ and}$$
$$h_2(x, y, z) = (x \vee z) \wedge \big(((x \vee y) \wedge z) \vee y\big), \text{ further,}$$
$$g_1(x, y, z) = h_1(x, y \vee z, y) \qquad g_2(x, y, z) = h_1(x, y \vee z, z)$$
$$g_3(x, y, z) = h_2(x, y \vee z, y) \qquad g_4(x, y, z) = h_2(x, y \vee z, z).$$

Clearly, $h_1(x, y, y) = h_2(x, y, y) = y$, implying $g_i(x, y, y) = y$, for each i.

Now, let $b \neq c$, i.e., either $b \vee c > b$ or $b \vee c > c$ be supposed. We may suppose, without the loss of generality, $b > c$. Thus, we must consider only $h_1(a, b, c)$ and $h_2(a, b, c)$ with $b > c$. $c < b \leqq (a \wedge c) \vee b$ implies that $c \neq (a \wedge c) \vee b$. Since $a \wedge c$ is a lower bound both of c and of $(a \wedge c) \vee b$, we have, by UBP, $((a \wedge c) \vee b) \wedge c = a \wedge c$, i.e., $h_1(a, b, c) = (a \wedge b) \vee (a \wedge c)$. In case $a \wedge b \neq a \wedge c$ UBP implies $h_1(a, b, c) = a$. Otherwise, $a \wedge b \leqq c$ implies, by UBP, $a \wedge b = c$, i.e., $c \leqq a$. Dually, $h_2(a, b, c) = a$, unless $a \leqq b$. Then, UBP implies either $c = a$ or $b = a$ when $h_1(a, b, c) = h_2(a, b, c) = a$ is implied by the absorption law.

Since every subalgebra of a WAL having UBP has UBP, too, we arrive, using Proposition 11 in [3], at

THEOREM 5. *Any finite algebra of the equational class generated by* WAL-*s having* UBP *is a direct product of a distributive lattice and of some* WAL-*s having* UBP.

Combining Theorem 5 with Tehorem 6 in [8], we have:

THEOREM 6. *Each finite* WAL *having* UBP *is functionally complete.*

CEP, **for special classes of** WAL-**s.** The large number of the WAL-s having UBP makes impossible to take in the situation in the equational class generated by these WAL-s. Therefore, we are intend to consider UBP only in a smaller class of WAL-s which is, however, wide enough to contain all the significant WAL-s.

To narrow down the possibilities to get WAL-s having UBP we should need a method to declair, somehow, that two elements in such a WAL must be comparable. Since we are interested in equational classes, the only method we can use is to give identities.

Let us consider two polinomial p and q defined for WAL-s. Supposing either $p \leqq q$ or $q \leqq p$ for some subdirect irreducible WAL-s this imply only $p \wedge q \leqq p \vee q$ in the equational class generated by them. However, the relation $p \wedge q \leqq p \vee q$ for an equational class of WAL-s does not, generally, imply either $p \leqq q$ or $q \leqq p$ for the subdirect irreducible elements of this class. Nevertheless, this implication is valid for WAL-s having UBP.

PROPOSITION 4. *Let $p \wedge q \leqq p \vee q$ fulfilled in a class of* WAL-*s defined by equations. If $\mathfrak{A}$ is a* WAL *belonging to this class and having* UBP *then either $p \leqq q$ or $q \leqq p$ is valid in $\mathfrak{A}$.*

PROOF. q is, clearly, a common upper bound of $p \wedge q$ and of p. Thus, by UBP, $p \wedge q \neq p$ implies $p \vee q = (p \wedge q) \vee q = p$, i.e., we have either $p = p \wedge q$ or $p = p \vee q$ proving the statement.

We shall use the notation $p \sim q$ instead of $p \wedge q \leqq p \vee q$ which is, clearly, equivalent to the identity $p \vee q = (p \wedge q) \vee (p \vee q)$.

Such kind of identities hold always for lattices and for tournaments (see, e.g., [2]). Indeed, $p \wedge q \leqq p \vee q$ is implied by associativity for lattices and by comparability of any two elements for tournaments. This mean that the classes we shall construct will be wide enough.

The first step of specialization of the class of WAL-s will be done by the identity $x \wedge y \sim x \vee y$. This class is, however, too wide for our sake. One can, namely, define partial WAL-s where for incomparable elements $x \vee y < x \wedge y$ is valid, whenever both hands exist. Then, one can construct, with a small modification, a WAL which contains the given partial WAL, has UBP and, moreover, satisfies the condition above. This means that we can construct such WAL-s freely.

The final specialization will be done in the following manner. Let $\mathbf{W}^*$ denote the class of WAL-s satisfying the following three identities:

(i)	$x \wedge y \sim x \vee y$	
(ii)	$x \wedge y \sim y \wedge z$	for $y \leqq x \vee z$
(iii)	$x \vee y \sim y \vee z$	for $y \geqq x \wedge z$.

It is obvious that the duality holds in $\mathbf{W}^*$.

PROPOSITION 5. *Let $\mathfrak{A} \in \mathbf{W}^*$ having* UBP. *Then, for incomparable elements a and b in $\mathfrak{A}$ the relation $a \vee b < a \wedge b$ holds.*

PROOF. (i) implies, by Proposition 4, either $a \wedge b \leqq a \vee b$ or $a \vee b < a \wedge b$. In the first case from $a \wedge b \leqq a \leqq a \vee b$ follows, using UBP, either $a \leqq b$ or $b \leqq a$. That is, the incomparability implies the given relation.

PROPOSITION 6. *Let $\mathfrak{A}$ be as in Proposition* 5 *and a, b, c incomparable elements of $\mathfrak{A}$ having a common upper (lower) bound u (v). Then, they have, also, a common lower (upper) bound $v(u)$ such that*

$$u = a \vee b = b \vee c = c \vee a \quad \textit{and} \quad v = a \wedge b = b \wedge c = c \wedge a \ \textit{hold.}$$

PROOF. It is enough to deal, by duality, only with that case when u is given. Then, $u = a \vee b = b \vee c = c \vee a$ is an immediate consequence of UBP. So, (ii) and Proposition 4 imply the comparability of $a \wedge b$ and $b \wedge c$ We may suppose, by symmetry, $a \wedge b \leqq b \wedge c$. Thus, we have, using Proposition 5, $u = a \vee b < a \wedge b$ and $u = b \vee c < b \wedge c$ proving that u is an upper bound of both $a \wedge b$ and $b \wedge c$ different from their meet which is $a \wedge b$. Hence, $a \wedge b = b \wedge c$, by UBP. The symmetry of the elements a, b, and c completes the proof.

We shall prove that in $\mathbf{W}^*$ all subdirect product of WAL-s having UBP satisfy CEP. Firstly, we shall give those WAL-s in $\mathbf{W}^*$ which have UBP.

Let A be any set of elements and we define a WAL, denoted by W_A, as follows. W_A consist of all elements of A and two more elements denoted by 0 and by 1. Moreover, we define $1 < 0$ and $0 < a < 1$ for all $a \in A$. It has been proven in [3] that W_A is subdirect irreducible and satisfies CEP for countable A. It is easy to verify this property for all A, since W_A has, clearly, UBP. One can, also, easily prove that all W_A belong to $\mathbf{W}^*$. Let, further, $W^{(7)}$ consist of all cosets modulo 7 such that $j > i$ if and only if the coset $j - i$ contains either 1 or 5. The verification of the necessery statements are, also, obvious (see, e.g., [3]).

THEOREM 7. *An algebra* $\mathfrak{A} \in \mathbf{W}^*$ *has* UBP *if and only if it is isomorphic either to* $W^{(7)}$ *or to one of* W_A.

PROOF. Let us, first, suppose that in $\mathfrak{A}$ the relation $x<y<z$ implies $z<x$. If $\mathfrak{A}$ containes comparable elements only then it is, clearly, isomorphic to a W_A where $|A| \leqq 1$. Let, now, the existence of incomparable elements a and b in $\mathfrak{A}$ be supposed. Then, by Proposition 5, defined 1 and 0 by $1=a\vee b$ and $0=a\wedge b$ we arrive to $1<0$. Let us consider any $u<0$ in $\mathfrak{A}$. Then, $u<0<a$ and $u<0<b$ imply $a<u$ and $b<u$, respectively, by our assumption. Thus, UBP implies $u = 1$. Dually, $1<u$ is only in case $u=0$ possible. Let, now, $0<v<u$. This implies, by our assumption, $u<0$, $u=1$. Let us consider an element $c\in\mathfrak{A}$ different from both 0 and 1. Then we have $v=c\wedge 1<1$ which implies, combining with $1<0$, $0<v$. So, $v<c$ would imply $c=1$, i.e., $v=c$. Hence, we have for all $c\in\mathfrak{A}$ different from both 0 and 1 that $0<c<1$ proving that $\mathfrak{A}=W_A$ where A denotes the set of all elements of $\mathfrak{A}$ different from both 0 and 1.

In the further case $\mathfrak{A}$ containes elements $a<b<c$ such that $c<a$ does not hold. Since $a<c$ is by UBP impossible, a and c must be incomparable. For the sake of convenience we shall denote the elements of $\mathfrak{A}$ by the cosets modulo 7. Then we have $2<0<5$ with incomparable 2 and 5. Let us denote $3=2\vee 5$ and $4=2\wedge 5$. 3 and 4 are different and $3<4$, by (i). Obviously, 0 is different from both 3 and 4. We get, using UBP, $0\wedge 3=2$ and $0\vee 4=5$. For $1=0\vee 3$ and for $6=0\wedge 4$ condition (i) implies $1<2$ and $5<6$. Since $0<x<2$ is for no x in $\{0, 2, 3, 4, 5\}$ possible, 1 does not belong to the set $\{0, 2, 3, 4, 5\}$. The proof of $6\notin\{0, 2, 3, 4, 5\}$ is similar. The difference of 1 and 6 is implied by $6<0<1$. Hence, we have:

$$0<1<2<3<4<5<6<0 \quad \text{and} \quad 6<4<2<0<5<3<1.$$

To show that these elements form a subalgebra isomorphic to $W^{(7)}$ we have to prove $1<6$. For $u=2\wedge 6$, we have, by (i), $0=2\vee 6<u$. Being $u\geqq 0=1\wedge 5$, (iii) implies $1\vee u=u\vee 5$ and this latter one is equal to 6, by UBP. However, $1\vee u=6$ means, exactly, $1<6$.

It is left to prove that $\mathfrak{A}$ has no more elements. Let us suppose, on the contrary, that there is an element $u\in\mathfrak{A}$, different from the given elements. Then we may suppose, that $\mathfrak{A}$ has an element which is comparable with one of the seven elements. By symmetry and by duality we may suppose $u<0$. Then, by (iii), u, 6, and 2 have the same meet, i.e., $1<u$. Let us consider the elements $x=5\wedge u$ and $y=5\vee u$. By (i) $y<x$; being $5\neq u$. The method which gave us $1<u$ guarantees $6<x$ and $y<4$. Thus, x and 4 have two different lower bound, namely y and 6, contradicting the UBP.

Further results on W*. We have to mention that (i), (ii) and (iii) hold also, for the following five-elements WAL $\{0, 1, a, b, c\}$ where 0 is lower bound of all elements 1 is upper bound of all elements and, moreover, $a<b<c$ is valid. The equational class of WAL-s generated by this WAL covers the class of all distributive lattices. Thus, we have that this class is also contained in $\mathbf{W}^*$.

Let us mention that **W*** does not contain all WAL-s. For example, it does not contain the free WAL on two generators (see [4]) since (i) does not hold, e.g., for the generators.

It is obvious that the condition of Theorem 2 in [8] does hold for WAL-s listed in Theorem 4, i.e., the equational class generated by them satisfy CEP. This gives us the following result.

THEOREM 8. *The equational subclass of* **W*** *generated by those* WAL-*s which have* UBP *satisfies* CEP.

Problems

1. Let $\mathfrak{A}$ be a WAL satisfying CEP. Does each subdirectly irreducible subdirect component of $\mathfrak{A}$ satisfy CEP, too?

2. Let $\mathfrak{A}_i$ be WAL-s having UBP. Does each subdirect product (or, equivalently, the direct product) of them satisfy CEP?

3. May one answer any of the above problems affirmatively at least in an equational subclass which is wide enough? (Problem 2. was solved, affirmatively, by E. FRIED, G. GRÄTZER and R. W. QUACKENBUSH in 1975.)

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ON THE ASYMPTOTIC PROPERTIES OF DIFFUSION PROCESSES

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Let $\xi(t)$ be a real diffusion process which is a solution of the stochastic Ito differential equation

$$d\xi(t) = a(\xi(t))\,dt + dw(t) \tag{1}$$

where $w(t)$ is the Wiener process. It is known [1] that a solution $\xi(t)$ of this equation exists if, for example, the drift coefficient $a(x)$ satisfies the following conditions

$$|a(x)| \leq C(1+x^2)^{1/2}, \quad |a(x_1)-a(x_2)| < C\,|x_1-x_2|$$

($C>0$ is a suitable constant).

The proof of the following theorem can also be found in [1]: If $a(x)=-a(-x)$, $a(x)\sim\alpha x^\beta$ as x tends to $+\infty$ and

(i) $\alpha>0$, $\beta=1$ then

$$\lim_{t\to\infty}\frac{\ln|\xi(t)|}{t}=\alpha$$

with probability one [p. 127],

(ii) $\alpha>0$, $|\beta|<1$ then

$$\lim_{t\to\infty}\frac{|\xi(t)|^{1-\beta}}{t}=\alpha(1-\beta)$$

with probability one [p. 125, 127, 133]

(iii) $\alpha\in(-\infty,+\infty)$ $\beta<-1$ then $\xi(t)$ has "similar asymptotic properties" as $w(t)$, e.g.

$$\lim_{t\to\infty}P(\xi(t)<x\sqrt{t})=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{z^2}{2}}\,dz$$

[p. 155].

In this paper we discuss the case $\beta = -1$.

THEOREM. *If* $a(x) = -a(-x)$,

$$\lim_{t\to\infty} x\,a(x) = \alpha, \quad \gamma = \alpha + \frac{1}{2} > 0,$$

and $\xi_\alpha(t)$ *is a solution of* (1), *then*

$$(A) \qquad \lim_{t\to\infty} P\left(\xi_\alpha(t) < x\sqrt{t}\right) = 2^{-\gamma}\Gamma(\gamma)^{-1} \int_{-\infty}^{x} |z|^{2\gamma-1} e^{-\frac{z^2}{2}}\, dz$$

and

$$(B) \qquad \overline{\lim_{t\to\infty}} \frac{|\xi_\alpha(t)|}{\sqrt{2t \ln\ln t}} = 1$$

with probability one.

REMARK. Applying the theorems mentioned in [2] (p. 269 and 273) it is easy to see that $\xi_\alpha(t)$ is not recurrent if $\alpha > \frac{1}{2}$ and the expectation of the return time (τ) is finite if $\alpha < -\frac{1}{2}$ and it is infinite if $\alpha > -\frac{1}{2}$. [It is also possible to prove that

$$E(\tau^\delta) < +\infty \quad \text{if } \alpha < \frac{1}{2} - \delta$$

and

$$E(\tau^\delta) = +\infty \quad \text{if } \alpha > \frac{1}{2} - \delta \quad \left(\alpha < \frac{1}{2}\right),$$

but in this paper we don't prove this statement.]

For the sake of simplicity we prove the following discrete version of our theorem:

Let $\xi_\alpha(n)$ a random walk whose state space is the set of integer numbers, $n = 0, 1, 2, \ldots$. Suppose that $|\xi_\alpha(n+1) - \xi^\alpha(n)| \equiv 1$ and the sequence of transition probabilities

$$p_k = P\left(\xi_\alpha(n+1) - \xi_\alpha(n) = 1 \mid \xi_\alpha(n) = k\right)$$

is given.

If $0 < p_k = \frac{1}{2}[1 + a(k)] < 1$, where $a(k) = -a(-k)$,

$$\lim_{k\to\infty} k\,a(k) = \alpha, \quad \gamma = \alpha + \frac{1}{2} > 0,$$

then

$$(A) \qquad \lim_{n\to\infty} P\left(\xi_\alpha(n) < x\sqrt{n}\right) = 2^{-\gamma}\Gamma(\gamma)^{-1} \int_{-\infty}^{x} |z|^{2\gamma-1} e^{-\frac{z^2}{2}} dz$$

and

$$(B) \qquad \overline{\lim_{n\to\infty}} \frac{|\xi_\alpha(n)|}{\sqrt{2n \ln\ln n}} = 1$$

with probability one.

LEMMA. If $\alpha > -\frac{1}{2}$ then

$$E\left(\xi_\alpha(n)^{2m}\right) = n^m \prod_{i=1}^{m} (2\alpha+2i-1) + o(n^m)$$

$$m = 1, 2, \ldots .$$

PROOF.

$$E\left(\xi_\alpha^2(n+1)\right) = E\left(E\left(\xi_\alpha(n)+\xi_\alpha(n+1)-\xi_\alpha(n)\right)^2 | \xi_\alpha(n)\right) =$$

$$= E\left(\xi_\alpha^2(n)\right) + 2E\left(\xi_\alpha(n)\, a\left(\xi_\alpha(n)\right)\right) + 1 \quad n = 0, 1, 2, \ldots .$$

Summing these equation and using the relation $\lim_{k\to\infty} ka(k) = \alpha$ and the following (above mentioned) fact: $E(\tau) = \infty$ if

$$\alpha > -\frac{1}{2} \text{ i.e. } \sum_{i=1}^{n} P\left(|\xi_\alpha(i)| < C\right) = o(n) \text{ if } \alpha > -\frac{1}{2}$$

and $C > 0$ is and arbitrary constant we get

$$E\left(\xi_\alpha^2(n)\right) = (2\alpha+1)\, n + o(n).$$

The proof is similar if $m > 1$.

PROOF OF THE THEOREM. Applying the lemma and the method of moments we can easily see that the distribution of $n^{-1}\xi_\alpha^2(n)$ tends to a Γ-distribution, which proves (A).

Finally (B) is a consequence of the lemma and [3].

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LAGERUNGEN INKONGRUENTER KUGELN I.

Von

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(Eingegangen am 6. Februar 1974)

HADWIGER hat in [3] die Kugel Γ von minimalem Radius bestimmt, in der die $k \leqq 6$ Einheitskugeln zu lagern sind. In dieser Arbeit untersuchen wir die Lagerung von $n \leqq 6$ kongruenten Kugeln mit $r > 1$ und $k \leqq 6-n$ Einheitskugeln mit den folgenden Voraussetzungen:

1. Wenn die Kugelmittelpunkte auf der Oberfläche einer Kugel $\mathbf{G}$ von minimalem Radius sind. Wir bezeichnen so eine Lagerung mit $|n, k|$, und den dazugehörigen Radius mit $R|n, k|$.

2. Enthält eine Kugel Γ von minimalem Radius die Kugeln dieses Systems. Wir bezeichnen so die Lagerung mit $[n, k]$, und den dazugehörigen Radius mit $P[n, k]$.

Die Lagerung von Kugeln mit verschiedenen Radien erscheint in der Molekulargeometrie folgendenmaßen [2]: Seien in einem Molekül auf der äußeren, sogenannten Valenzschale des Zentriatoms $n+k$ Elektronpaare. Der Raumanspruch der in der chemischen Bindung teilnehmenden Elektronpaare $X_1, X_2, \ldots, X_k$ ist kleiner als der der nicht-bindenden Elektronpaare $E_1, E_2, \ldots, E_n$. Bei erster Näherung können diese Elektronpaare wegen der Abstoßungs-Wirkung zwischen den Elektronpaaren und des verschiedenen Raumanspruchs als nicht übereinandergreifende Kugeln mit Radius 1 und $r > 1$ betrachtet werden. Es ist vermutlich, daß die gleichgewichtige Geometrie und die strukturalischen Parameter eines Moleküls im Falle der Kugellagerung $|n, k|$ oder $[n, k]$ konform werden.* In diesen Modellen ist die Anzieungskraft zwischen dem Kern und den Elektronen durch die Minimalisierung des Radius der Kugeln $\mathbf{G}'$ bzw. Γ' repräsentiert

Nach Einführung einiger Hilfssätze und Bezeichnungen können zwei zusammenfassende Sätze aufgestellt werden, die alle Fälle der Lagerungen $|n, k|$ und $[n, k]$ enthalten, wenn $n+k \leqq 6$ ist. Im allgemeinen ist der Beweis eines Falles $[n, k]$ dem Beweis des entsprechenden Falles $|n, k|$

* B. CSÁKVÁRI hat diese Hypothese aufgestellt.

ähnlich, darum beweisen wir beide Sätze für die einzelnen Werte von n nacheinander, oder wir berufen uns auf dem Gang dieses Beweises. In mehreren Fällen wird die langwierige Beschreibung durch eine Figur ersetzt. Die sich aus den im Satz befindlichen Graphen, Figuren, bzw. dem Text eindeutig ergebenden Rechnungen werden auch nicht angeführt.

HILFSSATZ H.1. Sind die durch die n Punkte bestimmten Mindestabstände in einer Kugel von Mittelpunkt O und Radius R nicht kleiner als R, so wächst der Abstand jeglicher Punkte $B \neq 0$ von den Punkten der Punktmenge, falls B sich auf der Halbgerade OB bewegt (B' S. Fig. 1).

BEWEIS. Sei T ein beliebige Punkt der Punktmenge. Im Dreieck OTB ist

$$OT \leqq R \leqq BT,$$

darum ist

$$OBT \sphericalangle < 90°.$$

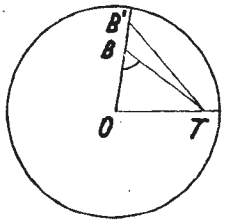


Fig. 1.

So ist im Dreieck TBB' $B'T$ die längste Seite, d.h. $B'T > BT$.

Bei dem Beweisen des Hauptsatzes werden wir auch folgende drei Hilfsätze anwenden. Ihr Beweis ist einfach, darum wird davon abgesehen.

HILFSSATZ H.2. Sind die Mittelpunkte zweier sich schneidenden Kreise auf derselben Seite der durch die Schnittpunkte bestimmten Gerade, so enthält der Kreis mit kleinerem Radius den kürzeren Bogen zwischen den Schnittpunkten des Kreises mit größerem Radius.

HILFSSATZ H.3. Die Menge der Mittelpunkte der bei angegebenen $2d$ nicht kürzeren Sehnen einer Kugel mit Mittelpunkt O und Radius R, bilden einen Kugelkörper mit Mittelpunkt O und Radius $\sqrt{R^2-d^2}$.

HILFSSATZ H.4. Der Umfang der einem Kreis einbeschriebenen Vierecke ist im Falle des Quadrats maximal.

Wir bezeichnen die Mittelpunkte der Kugeln $\mathbf{X}_i$, $\mathbf{E}_j$ mit X_i, E_j. So ist die sich auf die Lagerung beziehende Voranssetzung die folgende: $X_iX_j \geqq 2$; $X_iE_j \geqq r+1$; $E_iE_j \geqq 2r$ für jeglichen i, j, wenn $i \neq j$ ist. Zu jeder Lagerung ordnen wir – wie weiter unten folgt – einen Graphen zu: Die Punkte des Graphen sind die Mittelpunkte $X_1, X_2, \ldots, X_k$ und $E_1, E_2, \ldots$ $\ldots, E_n$. Berühren sich die Kugeln, so verbinden wir ihre Mittelpunkte folgendenmaßen:

, wenn $X_iX_j = 2$,

, wenn $E_iE_j = 2r$,

, wenn $X_iE_j = r+1$.

In dieser Arbeit gilt bei den Lagerungen in den Kugeln für die nicht isolierten Punkte auch, daß X_i auf der Oberfläche der Kugel Γ_{P-1} mit Radius $P-1$ und E_j auf der Oberfläche der Kugel Γ_{P-r} mit Radius $P-r$ sind. Darum folgt daraus die Identität der Konstruktionen nicht, wenn die Graphen einer $[n, k]$ und $|n, k|$ Lagerung identisch sind. Die Identität der Konstruktionen wird der Bezeichnung $[n, k] = |n, k|$ ausgedrückt. In Satz 2. wird also der Graph der Lagerung nur dann angegeben, wenn dieser von der entsprechenden Konstruktion $|n, k|$ abweicht.

Im Weiteren bezeichnen wir mit r_k den Radius des um das reguläre k-Eck – dessen Seiten 2 Einheiten beträgt – umbeschriebenen Kreises, wo wir einen Punkt ($k=1$) und auch eine Strecke ($k=2$) als reguläres k-Eck auffassen.

SATZ 1. *Wenn die Mittelpunkte der nicht übereinandergreifenden n kongruenten Kugeln mit Radius $r>1$, und der $k \leqq 6-n$ Kugeln mit Einheitsradius auf der Oberfläche einer Kugel sind, so ist der Minimalradius $R|n, k|$ dieser Kugel, und der Graph der zugehörigen Lagerung der folgende:*

1. 0. $|0, k|$.

$$R|0,1| = r_1 = 0\,.$$

$$R|0,2| = r_2 = 1\,.$$

$$R|0,3| = r_3 = \frac{2}{\sqrt{3}} \sim 1{,}1547\,.$$

$$R|0,4| = \sqrt{\frac{3}{2}} \sim 1{,}2247\,.$$

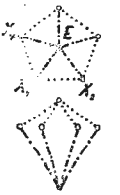
$$R|0,5| = \sqrt{2} \sim 1{,}4142\,.$$

$$R|0,6| = \sqrt{2}\,.$$

1. 1. $|1, k|$.

Wenn $k \leqq 4$ und $r>1$ oder $k=5$ und $r \geqq \sqrt{5}$, dann

$$R = \frac{(r+1)^2}{2\sqrt{(r+1)^2 - r_k^2}} \text{ ist.}$$



$$R|1\,(r < \sqrt{5}), 5| = \frac{1}{2}\sqrt{4+(r+1)^2}\,.$$

1. 2. $|2, k|$.

$$R|2(r \leqq 1+\sqrt{2}), 1| = \frac{(r+1)^2}{2\sqrt{(r+1)^2-r^2}}.$$

$$R|2(r \leqq 1+\sqrt{2}), 2| = \sqrt{r^2+\frac{(r^2-2r-1)^2}{8r}}.$$

$$R|2(r \leqq 1+\sqrt{2}), 3| =$$

$$= \sqrt{\frac{a+\sqrt{a^2-4(2r^2+3r-1)(r+1)^4(r-1)(r^2+6r+1)}}{8(2r^2+3r-1)}},$$

wo $a = r^5+9r^4+13r^3+r^2+6r+2$ *ist.*

$R|2(r \leqq 1+\sqrt{2}), 4|$ *ist die Lösung folgender Gleichung:*

$$32R^6(r+1)^2-8R^4A+4R^2B-C+4\sqrt{(R^2-1)(R^2-r^2)}\cdot$$
$$\cdot(8R^4(r+1)^2-2R^2D+E) = 0,$$

wo

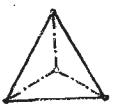
$$A = 5r^4+16r^3+30r^2+16r+5,$$
$$B = 3r^6+16r^5+57r^4+56r^3+57r^2+16r+3,$$
$$C = r^8+8r^7+32r^6+72r^5+158r^4+72r^3+32r^2+8r+1,$$
$$D = 3r^4+12r^3+26r^2+12r+3,$$
$$E = r^6+6r^5+23r^4+36r^3+23r^2+6r+1$$

ist.

$$R|2(r > 1+\sqrt{2}), k \leqq 7| = r.$$

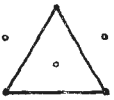
1. 3. $|3, k|$.

$$R\left|3\left(r \leqq \frac{2\sqrt{6}+3}{5}\right), 1\right| = \frac{(r+1)^2}{2\sqrt{(r+1)^2-r_3^2r^2}}.$$



$$R\left|3\left(r > \frac{2\sqrt{6}+3}{5}\right), k \leqq 2, 1\right| =$$

$$= R\left|3\left(r > 1+\sqrt{\frac{8}{5}}\right), 3\right| = \frac{2}{\sqrt{3}}r.$$

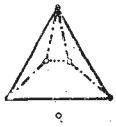


$$R\left|3\left(r \leqq \frac{2\sqrt{6}+3}{5}\right), 2\right| = \frac{r+1}{\sqrt{2}}.$$

$$R\left|3\left(r \leqq 1+\sqrt{\frac{8}{5}}\right), 3\right| =$$

$$= \frac{1}{2}\sqrt{\frac{(3r^2+2r+3)(r^2+6r+1)}{-r^2+10r-1}}.$$

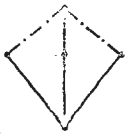
Die $|3,3|$ *Lagerung ist auch auf folgende Weise erfüllbar, wenn* $1{,}8892 < r < 1+\sqrt{\frac{8}{5}}$ *gilt.*



1. 4. $|4, k|$.

$$R|4\,(r \leqq 1+\sqrt{2}),\ 1| = R|4\,(r \leqq 1+\sqrt{2}),\ 2| =$$

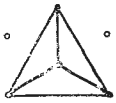
$$= \frac{\sqrt{(r+1)^2+4r^2}}{2}.$$



$k = 1$

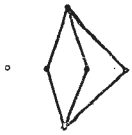
$k = 2$

$$R|4\,(r > 1+\sqrt{2}),\ k \leqq 4| = r\sqrt{\frac{3}{2}}.$$



1. 5. $|5, 1|$.

$$R|5,1| = R|6,0| = rR|0,6| = r\sqrt{2}.$$



SATZ 2. *Wenn eine Kugel die nicht übereinandergreifenden* n *kongruenten Kugeln mit Radius* $r > 1$, *und die* $k \leqq 6-n$ *Kugeln mit Einheitsradius enthält, so ist der Minimalradius* $P\,[n, k]$ *dieser Kugel, und der Graph der dazugehörigen Lagerungen wie folgt:*

2. 0. $[0, k]$.

$$P\,[0, k] = R|0, k| + 1, \qquad [0, k] = |0, k|.$$

2. 1. $[1, k]$.

Wenn $k \leqq 4$ *und* $r > 1$ *oder* $k = 5$ *und*

$$r \geqq b = \frac{(3+\sqrt{5})\left(\sqrt{2}+\sqrt{5+\sqrt{5}}\right)}{\sqrt{2}\left(3+\sqrt{5}+\sqrt{2(5+\sqrt{5})}\right)} \sim 1{,}6808, \textit{ dann}$$

$$P = r\,\frac{2(r+1)-r_k^2+2\sqrt{(r+1)^2-r_k^2}}{4r-r_k^2}, \qquad [1,k] = |1,k|\,.$$

$P\,[1\,(r < b), 5]$ *ist die Lösung der folgenden Gleichung:*

$P^3-(r+2)\,P^2+r\,P+r = 0\,.$

2. 2. $[2, k]$.

$P\,[2\,(r \leqq 1{,}5), 1] =$

$$= \frac{r+2+2\sqrt{2r+1}}{4-r}, \qquad [2\,(r \leqq 1{,}5), 1] = |2,1|.$$

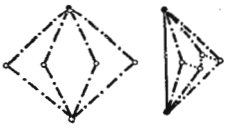
$P\,[2\,(r \leqq 1{,}5), 2] =$

$$= r\,\frac{r+1+\sqrt{6r}}{-r^2+4r-1}, \qquad [2\,(r \leqq 1{,}5), 2] = |2,2|.$$

$P\,[2\,(r \leqq 1{,}5), 3]$ *ist die Lösung der folgenden Gleichung:*

$P^3\,(r^2-3r+1)+P^2\,(4-r)\,r^2-P\,(2r-1)\,r^2-r^3 = 0.$

$P\,[2\,(r \leqq 1{,}5), 4] = \dfrac{r+1+\sqrt{(r+1)^2+4r}}{2}$. *Zwei Kugelsysteme sind möglich:*



$P\,[2\,(r > 1{,}5), k \leqq 6] = 2r\,,$ $\qquad [2\,(r > 1{,}5), k \leqq 6] =$

$= |2\,(r > 1+\sqrt{2}), k \leqq 7|\,.$

2. 3. $[3, k]$.

$P\,[3\,(r \leqq 3-\sqrt{3}), 1] =$

$$= \frac{r+3+\sqrt{-3r^2+18r+9}}{6-2r}, \qquad [3\,(r \leqq 3-\sqrt{3}), 1] = |3,1|.$$

$$P[3(r > 3-\sqrt{3}), 1] = \\ = P[3(r > 3-\sqrt{3}), 2] = \\ = P[3(r > 5-2\sqrt{3}), 3] = \\ = r\left(1+\frac{2}{\sqrt{3}}\right),$$

$$[3(r > 3-\sqrt{3}), 1] = \\ = [3(r > 3-\sqrt{3}), 2] = \\ = [3(r > 5-2\sqrt{3}), 3] = \\ = \left|3\left(r > \frac{2\sqrt{6}+3}{5}\right), 1\right|.$$

$$P[3(r \leqq 3-\sqrt{3}), 2] = \\ = \frac{r+1+\sqrt{(r+1)^2+4r}}{2}.$$

$$P[3(\leqq 5-2\sqrt{3}), 3] = \\ = r\,\frac{r+1+\sqrt{-r^2+10r-1}}{-r^2+4r-1},$$

$$[3(r \leqq 5-2\sqrt{3}), 3] = \\ = \left|3\left(r \leqq 1+\sqrt{\frac{8}{5}}\right), 3\right|$$

bzw.

$$[3(1{,}3945 < r \leqq 5-2\sqrt{3}), 3] = \\ = \left|3\left(1{,}8892 < r \leqq 1+\sqrt{\frac{8}{5}}\right), 3\right|.$$

2. 4. $[4, k]$.

$P[4(r \leqq 4-\sqrt{6}), 1] = P[4(r \leqq 4-\sqrt{6}), 2]$ *ist die Lösung folgender Gleichung:*

$$P^3 - P^2(2r+1) + Pr + r^2 = 0.$$

$k = 1$

$k = 2$

$$P[4(r > 4-\sqrt{6}), k \leqq 4] = \\ = r\left(1+\sqrt{\frac{3}{2}}\right),$$

$$[4(r > 4-\sqrt{6}), k \leqq 4] = \\ = |4(r > 1+\sqrt{2}), k \leqq 4|.$$

2. 5. $[5, 1]$.

$$P[5,1] = P[6,0] = rP[0,6] = \\ = r(1+\sqrt{2}),$$

$$[5,1] = |5,1|.$$

BEWEIS.

1–2. 0. $|0, k| = [0, k]$.

Wenn der Radius, der in [3] bewiesenen $[0, k]$ Lagerung um 1 vermindert wird, so gelangen wir zur entsprechenden $|0, k|$, denn $R|0, k| < P[0, k]-1$ kann offensichtlich nicht bestehen.

1. 1. $|1, k|$.

$|1, k|$-Fall wird in zwei Teilen bewiesen. Wenn $k \leqq 4$ oder $k=5$ und $r > \sqrt{5}$ (Punkt a), bzw. $k=5$ und $1 < r \leqq \sqrt{5}$ (Punkt b).

a) Wir betrachten die Pyramide $X_1X_2 \ldots X_kE$, wo $X_iX_{i+1} = 2$ und $X_iE = r+1$ $(i = 1, 2, \ldots, k;\ k+1 = 1)$ ist. E' liegt auf der Umkugel von Pyramide mit dem Punkt E gegenüber. Neben obigen Voraussetzungen ist $X_iE' < 2$. Wir zeigen, daß diese Pyramide die beste $|1, k|$ Lagerung ergibt.

Sei der Mittelpunkt E ein Punkt der Oberfläche der Kugel $\mathbf{G}'$, deren Radius $R' \leqq R$, wo R der Radius der in unserem Satz figurierenden $\mathbf{G}$ Kugel ist. Die Kugel mit Mittelpunkt E und Radius $r+1$ bedeckt von der Kugel $\mathbf{G}'$ wegen $EX_i = r+1 > 2 > X_iE'$ und $R' \leqq R$ einen Teil, der größer als eine Halbkugel ist. Sei h der Schnittkreis (S. Fig. 2). Teilen wir

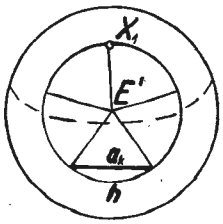


Fig. 2.

die übriggebliebende Kugelkappe auf gleiche Teile k mit Großkreisen die durch E' gehen, so erhalten wir auf dem Grenzkreis h die Spitzen eines $a_k \leqq 2$ seitigen regulären k-Ecks. Von den Punkten X_i liegt mindestens einer auf $h\,(X_1)$, sonst könnte R' nicht minimal sein. Im Falle $R' < R$ können $X_1, X_2, \ldots, X_k$ nicht auf der Kugelkappe liegen, da wegen $a_k < 2$ und $E'X_i < 2$ die um X_i umbeschriebene Kugel mit Radius 2 zwei Gebiete enthält. Im Falle $R' = R$, wenn wir einen von den Mittelpunkten X_i nicht auf der Spitze des regulären k-Ecks wählen, so wird durch die um X_i umbeschriebene Kugel von Radius 2 das entsprechende Gebiet auch vollständig bedeckt, darum kann jeder der weiteren Punkte nicht auf der Kugelkappe sein.

Aus der Rechnung ergibt sich, daß

$$R\,|1,k| = \frac{(r+1)^2}{2\sqrt{(r+1)^2-r_k^2}}$$

ist.

b) Sei $k=5$ und $1<r\leqq\sqrt{5}$. In diesem Falle ist

$$R\,|1\,(r\leqq\sqrt{5}),5| = \sqrt{1+\left(\frac{r+1}{2}\right)^2}.$$

Die dazugehörige Anordnung ist eine Doppelpyramide mit viereckiger Grundfläche. Das Grundviereck ist nur im Falle $r=1$ eindeutig (regulärer Oktaeder), sonst können sich die Mittelpunkte der Einheitskugeln auf dem Umkreis des Vierecks bewegen. Ist $r=\sqrt{5}$, so kann auch eine sechste Einheitskugel angebracht werden, d.h.

$$R\,|1\,(\sqrt{5}),5| = R\,|1\,(\sqrt{5}),6| = \sqrt{\frac{5+\sqrt{5}}{2}}.$$

Wir betrachten einem Punkt E auf der Oberfläche einer Kugel $\mathbf{G}'$, deren Radius $R'\leqq\sqrt{1+\left(\frac{r+1}{2}\right)^2}$ ist. Die Kugel mit dem Mittelpunkt E und dem Radius $r+1$ schneidet aus der Kugel $\mathbf{G}'$ wegen $1<r\leqq\sqrt{5}$ einen Kreis mit dem Radius $\varrho\leqq r_5$ heraus, der ein Gebiet größer als eine Halbkugel von $\mathbf{G}'$ bedeckt. Weitere Teile des Beweises sind dem vorhergehenden ähnlich. Im Falle $r<5$ sind die Mittelpunkte X_1, X_2, X_3, X_4 auf dem Grenzkreis, und X_5, E gegenüberliegende Punkte von $\mathbf{G}'$.

2.1. $[1,k]$.

$[1,k]$ Anordnungsbeweis kann den Grenzen entsprechend die im Satz vorkommen, ähnlich dem Beweis der Lagerung $|1,k|$ gelöst werden, darum gehen wir auf diese einzelheiten nicht ein.

1.2. $|2,k|$.

a) Sei $1<r<1+\sqrt{2}$.

Zuerst wird gezeigt, daß wenn $k\leqq 4$, so ist

$$r < R\,|2\,(r<1+\sqrt{2}),k| < \frac{r+1}{\sqrt{2}}.$$

$R'=r$ kann nicht sein, da in diesem Falle E_1, E_2 gegenüberliegende Punkte von $\mathbf{G}'$ sind, und dazu, daß die um E_i umbeschriebene Kugel von Radius $r+1$ die gesamte Oberfläche $\mathbf{G}'$ nicht bedecken soll, muß

$$r+1 \leqq R'\sqrt{2} = r\sqrt{2}, \text{ d.h. } r \geqq 1+\sqrt{2}$$

bestehen.

Wir zeigen, daß der Radiuswert $R' = \frac{r+1}{\sqrt{2}}$ vorausgesetzt $r > 1+\sqrt{2}$ immer vermindert werden kann. Daraus ergibt sich, daß im Falle eines Minimalradius zur Sehne von der Länge $r+1$ der Kugel **G** ein Zentriwinkel größer als 90° gehört. Seien die Punkte E_1, E_2 Diagonalpunkte der Kugel **G**′ mit Radius $\frac{r+1}{\sqrt{2}}$. Wegen $r<1+\sqrt{2}$ ist $E_1E_2>2r$.

Ist $k\leqq 3$, so können wegen $r+1>2$ auf dem Bogen des Kreises mit Radius $\frac{r+1}{\sqrt{2}}$, der kleiner als ein Halbkreis ist, wenigstens drei Mittelpunkte der Einheitskugeln liegen (S. Fig. 3). Die Kugeln $\mathbf{E}_1$, $\mathbf{E}_2$ können in diesem Falle gegeneinander gerollt werden, wo die Punkte E_1, E_2 ins Innere vom **G**′ geraten und so R' hernach vermindert werden kann.

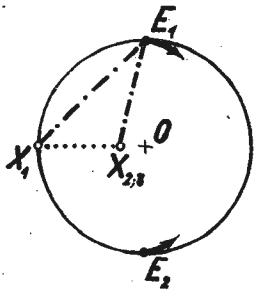


Fig. 3.

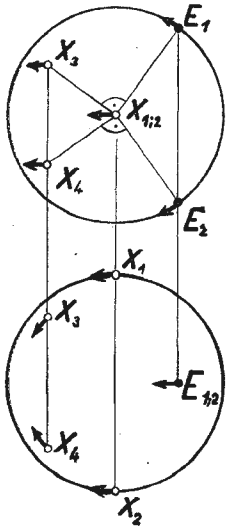


Fig. 4.

Ist $k=4$, so zeigt Fig. 4 die orthogonale Projektion auf zwei Bildebenen einer Lagerung $|2, 4|$, wo $R' = \frac{r+1}{\sqrt{2}}$, $E_1E_2 = 2r$, $E_1X_1=E_1X_2= =E_1X_3=E_2X_1=E_2X_2=E_2X_4=r+1$, $X_1X_3=X_2X_4=2$ ist. Mit der Anwendung H.3. kann durch eine kurze Rechnung beweisen werden, daß $X_3X_4>2$ ist. So kann durch die Bewegung in Richtung der Pfeile ein Polieder $X_1X_2X_3X_4E_1E_2$ erlangt werden, für dessen Kanten: $X_iX_j>2$, $X_iE_j>r+1$ und $E_1E_2>2r$ besteht, also der Radius R' mit einer Verkleinerung von Mittelpunkt O vermindert werden kann.

Nun können wir zum Beweis des Teiles *a)* von $|2, k|$ übergehen. Seien die Punkte E_1, E_2 auf der Oberfläche der Kugel $\mathbf{G}''$ mit einem Radius $R'' \leqq R$, wo $E_1E_2 \geqq 2r$ ist (S. Fig. 5). Wegen $R'' \leqq R < \frac{r+1}{\sqrt{2}}$ ist in $\mathbf{G}''$ der zur Sehne $r+1$ gehörige Zentriwinkel $> 90°$. So lassen die um E_i umbeschriebenen Kugeln mit Radius $r+1$ von $\mathbf{G}''$ ein „sphärisches Zweieckartiges" Gebiet $\mathbf{K}$ frei, dessen Spitzen X_1, X_2 sind, wo $X_1X_2 < 2R''$ und die Mittelpunkte der Grenzkreise die E_i gegenüberliegenden E_i' Punkte sind, ihr Radius $< R''$ und ihr Bogen $< 90°$, der Winkel ihrer Ebenen $180° - E_1OE_2 \sphericalangle < 90°$ ist, darum ist E_i' ein äußerer Punkt des Gebietes $\mathbf{K}$. Mit Anwendung des Hilfssatzes H.2. kann bewiesen werden, daß wenn wir den Mittelpunkt irgendeiner Einheitskugel k nicht im entsprechenden Punkt des im Satz erwänten Graphen wählen, oder $R'' < R$, oder $E_1E_2 > 2r$ ist, so kann die Lagerung $|2, k|$ nicht durchgeführt werden.

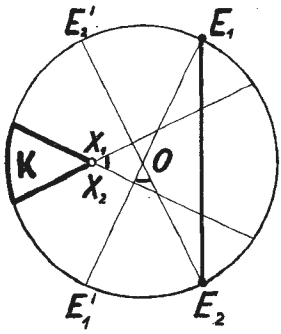


Fig. 5.

b) Wenn $r > 1 + \sqrt{2}$ ist, so können auf dem Großkreis der Kugel $\mathbf{G}$ mit Radius $R = r$ die Mittelpunkte von mindestens 7 Einheitskugeln Platz finden $(r_7 < 1 + \sqrt{2})$, die keine gemeinsamen Punkte mit den in die Polen gelagerten Kugeln mit Radius r haben $(R\sqrt{2} = r\sqrt{2} > r + 1)$. Die Mittelpunkte der beiden Kugeln mit Radius r können aber auf einer Sphäre mit Radius kleiner als r angebracht werden.

2.2. $[2, k]$.

Ist $r < 1{,}5$ und $k \leqq 3$, so kann der Beweis dem entsprechenden Falle $|2, k|$ ähnlich durchgeführt werden. Aus dem Mittelpunkt der Γ Kugel mit Radius P erscheint jede Strecke $r+1$, deren Endpunkte auf der Oberfläche der Kugeln mit Radius $P-1$ und $P-r$ sind, unter stumpfem Winkel.

Ist $r<1{,}5$ und $k=4$, so ist die Kugelpackung nicht eindeutig.

a) Seien die Mittelpunkte E_1, E_2 gegenüberliegende Punkte der Kugel mit Radius

$$P[2(r<1{,}5),4]-1=\frac{r+1+\sqrt{(r+1)^2+4r}}{2}-1\,.$$

In diesem Falle können sich die Mittelpunkte X_i in der Ebene des auf E_1E_2 senkrechten Großkreises bewegen, denn $P-1>2$, wenn $r>1$ ist. Keine Kugel $\mathbf{X}_i$ kann in $\mathbf{\Gamma}$ gelagert werden, wenn wir den Radius $\mathbf{\Gamma}$ vermindern, denn eine um E_i umbeschriebene Kugel mit Radius $r+1$ bedeckt ein Gebiet größer als die Halbkugel von der Kugel mit Mittelpunkt Ω und Radius $P'-1$ ($\mathbf{\Gamma}_{P'-1}$).

b) Auch mit der Bewegung von E_1, E_2 kann der Radius nicht vermindert werden. Nämlich ist $P-1<2$, wenn $r<1{,}5$. Darum berühren die Kugeln der Lagerung die Kugel $\mathbf{\Gamma}$, wegen H.1. $\Omega E_i=P-r$ und $\Omega X_i=$ $=P-1$ ist. Die Anordnung nach Fig. 6 kann bloß im Falle $P'=P$ verwirklicht werden, sonst geraten die Punkte des „Viereck-artigen" Gebietes $\mathbf{N}$ einander um 2 Einheiten näher (H.3).

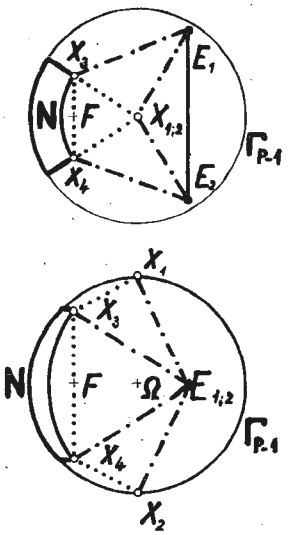


Fig. 6.

Ist $r\geqq 1{,}5$, können in die Kugel mit Minimalradius $P=2r$, die die zwei Kugeln mit Radius r enthält, mindestens noch 6 Einheitskugeln gelagert werden, denn

$$P-1=2r-1\geqq 2=r_6 \quad\text{und}\quad (P-1)^2+(P-r)^2=5r^2-4r+1\geqq(r+1)^2\,.$$

1.3. $|3, k|$.

Ist $k = 1$ und $r < \frac{2\sqrt{6}+3}{5}$, d.h. im Umkreis des $2r$ seitigen regulären Dreiecks der zur Sehne $r+1$ gehörige Zentriwinkel strumpf ist, so ist die Lagerung eindeutig. Der Beweis $|1, k|$ kann dem Fall *a)* ähnlich durchgeführt werden, wo der Mittelpunkt X der einzigen Einheitskugel die Rolle von E übernimmt.

Ist $r \geqq \frac{2\sqrt{6}+3}{5}$, bildet E_i auf dem Großkreis von **G** ein reguläres Dreieck, und der Mittelpunkt der Einheitskugel kann auf beiden Seiten der Ebene $E_1E_2E_3$ im Pol, bzw. im dessen Nähe gelagert werden. R kann wegen $|0, 3|$ nicht verringert werden.

$$R\left|3\left(r \geqq \frac{2\sqrt{6}+3}{5}\right), 1\right| = R\left|3\left(r \geqq \frac{2\sqrt{6}+3}{5}\right), 2\right| = \frac{2r\sqrt{3}}{3}.$$

Der Beweis des übrigen Teiles vom Falle $k=2$ ist noch Untersuchung $|3, 3|$ einfacher, darum folgt dieser Beweis später.

$$k = 3 \text{ und } r < 1 + \sqrt{\frac{8}{5}} \sim 2{,}2649.$$

Sei E_1, E_2 auf der Sphäre **G**′, wo $R' \leqq R|3, 3|$ und $E_1E_2 \geqq 2r$. Ist $k \leqq 3$, so ist der Radius der Lagerung $|3, k|$ bestimmt kleiner als der Umkugelradius eines $2r$ seitigen regulären Oktaeders, so ist der zur Sehne $2r$ gehörige Zentriwinkel in der Kugel mit Radius $R|3, k|$ größer als 90°.

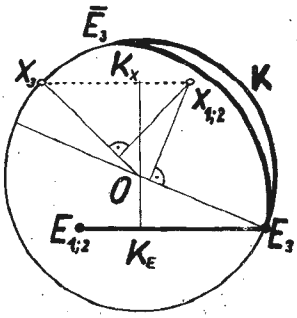


Fig. 7.

Die Kugeln mit den Mittelpunkten E_1, E_2, und mit dem Radius $2r$ lassen von **G**′ ein „Zweieck-artiges" Gebiet **K** frei, das kleiner als eine Viertelkugel ist (S. Fig. 7). Ist $R' < R$ oder $E_1E_2 > 2r$, so wird **K** kleiner. Wenn wir eine Spitze von **K** für E_3 nehmen, und um die Mittelpunkte E_i

Kugeln mit Radius $r+1$ umschreiben, so erhalten wir bloß im Falle $E_1E_2 = 2r$ und $R'=R$ auf einer Seite der Ebene $E_1E_2E_3$ ein freigebliebenes „Dreieck" $X_1X_2X_3$, dessen Spitzen-Entfernung 2 Einheiten beträgt, und jegliche andere Punktpaare einander näher als 2 Einheiten sind. Bewegen wir E_3 in **K**, so kommt E_3 auch X_1 und X_2 zugleich näher, denn die Kugeln mit den Mittelpunkten X_1 bzw. X_2 enthalten **K**, da – z.B. auch der Kreis k_{E_1} mit Mittelpunkt E_1' der E_1 gegenüberliegt, und **K** abgrenzt, und auch der Kreis k_{X_1} mit Mittelpunkt X_1 über die Punkte E_2 und E_3 geht, sich also auf dem im Frage stehenden Bogen nicht schneiden können.

k_{X_1} enthält X_2, wegen $X_1X_2 = 2<r+1$, wo $E_1X_2 = r+1<2r$ ist, d.h. k_{X_1} enthält den zu **K** gehörigen Bogen von k_{E_1}.

Also kann auf einer Seite der Ebene $E_1E_2E_3$ im Falle $R'<R$ bloß ein Punkt von X_i sein.

Mit Vergrößerung von r verringert sich die Entfernung d_1 von Mittelpunkt O zur Ebene $E_1E_2E_3$. Für die Werte $r>1{,}8892$ kann auf die durch die Ebene $E_1E_2E_3$ aus **G** abgeschnittene kleinere Kugelkappe auch X_4 Platz haben, d.h.

$$R\,|3{,}3| = R\,|3{,}4|\,.$$

Ist $d_1 = 0$, so ist $r = 1+\sqrt{\frac{8}{5}}$. In diesem Falle können auf beiden Seiten der Ebene $E_1E_2E_3$ je 3 Mittelpunkte von Einheitzkugeln Platz haben, so ist

$$R\,|3{,}3| = R\,|3, k \leqq 6| = \frac{2r}{\sqrt{3}}.$$

Sei $k = 2$ und $r < \frac{2\sqrt{6}+3}{5}$. Wir betrachten von der Lagerung $|3{,}3|$ das symmetrische Trapez $X_1X_2E_1E_2$ (S. Fig. 7). Sein Umfang ist$4(r+1)$. Wegen des Hilfssatzes H.4 gehört zum Schenkel ein Zentriwinkel kleiner als 90° in um das Trapez umbeschriebenen kleinen Kugelkreis. In der Kugel mit Radius $R|3, 3|$ gehört darum zur Sehne $r+1$ ein noch mehr spitzer Winkel.

In der Kugel mit dem Radius $R\,|3{,}2| = \frac{r+1}{\sqrt{2}}$ gehört zur Sehne $r+1$ ein Zentriwinkel von 90°. So gilt $R|3, 2|<R|3, 3|$. Beim Beweis der Lagerung $|3, 3|$ aber konnten wir sehen, daß neben obigen r Werten, wenn $R'<R|3, 3|$, auf der Kugel **G'** auf einer Seite der Ebene $E_1E_2E_3$ nur ein X_i Punkt Platz haben kann. Die Punkte E_i müssen wir also im Falle $k=2$ auf dem Großkreis einer Kugel mit Radius $\frac{r+1}{\sqrt{2}}$ wählen. Wegen $r<$ $<\frac{2\sqrt{6}+3}{5}$ können sich die Punkte E_i im Verhältnis zueinander bewegen.

2.3. [3, k].

Sei $k=1$. Wir betrachten die Entsprechende Kunstruktion |3, 1| umgebende Γ Kugel, deren Mittelpunkt Ω und Radius P ist. Im Falle $r<$ $<3-\sqrt{3}$ ist $X\Omega E_i \sphericalangle > 90°$, d.h. es bestehen die beim Beweis der Lagerung |3, 1| angewandten Voraussetzungen.

Im Falle $r \geqq 3-\sqrt{3}$ bilden E_1, E_2, E_3 auf der Ebene eines Großkreises von Γ ein $2r$ seitiges reguläres Dreieck.

$$P[3,1] = P[3,2] = r\left(1+\frac{2}{\sqrt{2}}\right).$$

Auch jetzt befassen wir uns zuerst mit der Lagerung [3, 3]. Der Mittelpunkt Ω der das Kugelsystem |3, 3| enthaltenden Kugel Γ, wird bei $1<r<5-2\sqrt{3}$ der innere Punkt der Streche $K_E K_X$ (S. Fig. 8). Da ist $\frac{2r}{\sqrt{3}}<P-r<R<P-1$. Weiterhin ist die Funktion $R(r)$ streng monoton zunehmend, so ist

$$R(r) < R(5-2\sqrt{3}) \sim 1{,}91 < 2 = \min X_i X_j.$$

Darum kann wegen H.1. festgestellt werden, daß die Punkte nicht im Inneren der Kugel Γ_{P-r} mit Mittelpunkt Ω und mit Radius $P-r$ sind.

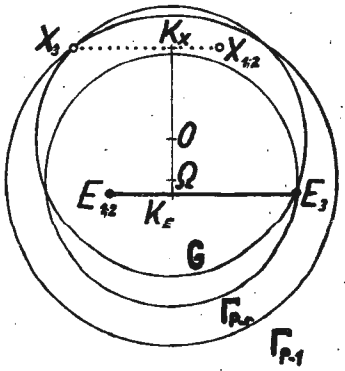


Fig. 8.

Sei

$$P' \leqq P, \quad E_1 E_2 \geqq 2r, \quad \Omega E_1 = \Omega E_2 = P'-r.$$

Die um die Mittelpunkte E_1, E_2 umbeschriebenen Kugeln mit Radius $2r$ lassen von der Oberfläche der Kugel $\Gamma_{P'-r}$ einen „Zweieck-artigen" Teil **K** frei. (Wegen $P'-r \leqq P-r<R$ ist $E_1\Omega E_2 \sphericalangle > 90°$.) Sei E_3 eine Spitze von **K**. Sei **T** die Union der um die Mittelpunkte E_i umbeschriebenen

Kugeln mit Radius $r+1$. Die Punkte X_i können also in $\Gamma_{P'-1}-\mathbf{T}$ sein. Der Radius des um die „Spitzen" von $\Gamma_{P'-1}-\mathbf{T}$ umbeschriebenen Kreises ist $\leqq r_3 = \frac{2}{\sqrt{3}}$, wo die Gleichheit nur im Falle $E_1E_2 = 2r$ und $P'=P$ gilt. Wir betrachten die durch die Punkte E_1, E_2, E_3 gehende Kugel $\mathbf{G}$ mit Radius $R(r)$, deren Mittelpunkt O im gleichen Halbraum mit Ω ist. Der Radius des gemeinsamen Kugelschnitts von $\mathbf{G}$ und Γ_{P-1} ist nicht kleiner als r_3, also enthält $\mathbf{G}$ den Raumteil $\Gamma_{P-1}-\mathbf{T}$, d.h. im Falle eines Minimalradius P sind die Punkte X_i wegen H.1. an der Grenze von Γ_{P-1}. Projizieren wir die Punkte E_i aus Ω auf Γ_{P-1} (E_i^*), so kann der entsprechende Teil des Beweises $|3, 3|$ mit dem Punktsystem $X_1, X_2, X_3, E_1^*, E_2^*, E_3^*$ neben den Abstandsbedingungen 2, $(r+1)^*$, $(2r)^*$ wiederholen.

Sei $k=2$ und $r \leqq 3-\sqrt{3}$. Der Beweis ist bei Anwendung von $[3, 3]$ analog mit $|3, 2|$. Die Lagerung $[3, 3]$ kann z.B. $X_1\Omega E_3 \sphericalangle < 90°$ sein (dies kann rechnerisch beweisen werden), darum gilt $P[3, 3] > P[3, 2]$. Die Anordnung $[3, 2]$ ist eine trigonale Doppelpyramide die gestreckter als bei $|3, 2|$ ist, denn $P-1 = \Omega X_i > \Omega E_j = P-r$.

$$P[2(r<1{,}5), 4] = P[3(r<3-\sqrt{3}), 2] = \frac{r+1+\sqrt{(r+1)^2+4r}}{2}.$$

1.4. $|4, k|$.

Seien X_1 und E_1 die Endpunkte des Durchmessers von $\mathbf{G}$, $X_1E_2 = r+1$ und $E_1E_2 = 2r$. Der Radius von $\mathbf{G}$ beträgt dann

$$R = \frac{\sqrt{(r+1)^2+4r^2}}{2} < r\sqrt{2}.$$

Die durch E_2 gehende auf E_1X_1 senkrechte Ebene schneidet aus $\mathbf{G}$ einen Kreis mit Mittelpunkt K und mit Radius E_2K heraus, für den im Intervall

$$1 < r < 1+\sqrt{2} \quad r\sqrt{2} > E_2K > \frac{2r}{\sqrt{3}}$$

gilt, d.h. mindenstens 3 große Kugeln ohne sich zu berühren zu lagern sind.

In einer Kugel $\mathbf{G}'$ mit Radius $R' \leqq R < r\sqrt{2}$ können die Mittelpunkte E_1, E_2, E_3, E_4 nicht in einer Ebene liegen.

Da die Punkte auf der Sphäre $\mathbf{G}'$ liegen, ist X_i der äußere Punkt des Tetraeders $E_1E_2E_3E_4$. Z.B. trennt die Ebene $E_1E_2E_3$ die Punkte X_1 und E_1. Die Kugel mit Mittelpunkt E_1 und Radius $2r$ bedeckt mehr als die Halbkugel von $\mathbf{G}'$, der Radius des Schnittkreises k ist aber nicht größer als der Wert des obigen E_2K. Seien A, B, C, D die Punkte des Kreises k so, daß $AB=BC=r+1$ ist und $AD=CD$ ist. Wegen $KA<R' \leqq R$ ist $AD<2r$. Sei X der E_1 gegenüberliegende Punkt, $XA \leqq r+1$ ist, Gleichheit besteht nur im Falle $R'=R$, also nur da kann der Kugelmittelpunkt X_1 in X ge-

wählt werden. Ist $R'<R$, oder X_1 nicht identisch mit X, so bedeckt die um X_1 umbeschriebene Kugel mit Radius $r+1$ das Gebiet $AXCB$ der Kugelkappe, dadurch können die Mittelpunkte E_2, E_3, E_4 nicht am freien Platz sein. Wegen $E_2K > \frac{2r}{\sqrt{3}}$ ist die Lage der Punkte E_2, E_3, E_4 auf dem Kreis k nicht fix. Bewegen wir E_3, E_1 auf dem Kreis k so, daß $E_2E_3 = E_2E_4 = 2r$ sei (S. Fig. 9). Da kann der dem E_2 gegenüberliegende Punkt wegen seiner Symmetrie auf die Ebene OE_3E_4 als X_2 betrachtet werden ($X_1X_2 = 2r > 2$). Also ist

$$R\,|4,1| = R\,|4,2| = \frac{\sqrt{(r+1)^2+4r^2}}{2}.$$

Ist $r \geqq 1+\sqrt{2}$, so kann jeder den Spitzen E_i gegenüber liegende Punkt des regulären $2r$ kantigen Tetraeders, bzw. seine Umgebung für X_i gewählt werden.

$$R\,|4\,(r \geqq 1+\sqrt{2}), k \leqq 4| = r\sqrt{\frac{3}{2}}.$$

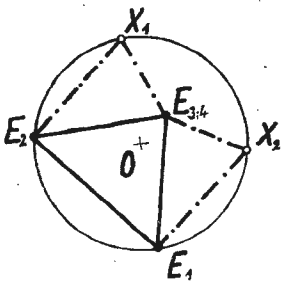


Fig. 9.

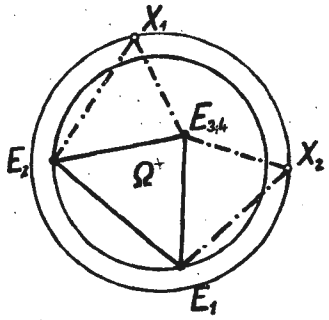


Fig. 10.

2.4. $[4, k]$.

Sei $r<4-\sqrt{6}$, $P' \leqq P$ und X_1 ein innerer – oder Grenzpunkt der Kugel $\Gamma_{P'-1}$ mit Mittelpunkt Ω und Radius $P'-1$. Die Punkte E_i können Punkte der Kugel $\Gamma_{P'-r}$ mit Mittelpunkt Ω und Radius $P'-r$ sein, die nicht im Inneren der Kugel mit Mittelpunkt X_1 und Radius $r+1$ liegen. Es kann beweisen werden, daß $P'-r<r+1$ ist, so genügt es wegen H.1. die Grenzpunkte zu untersuchen. Ähnlich dem Falle $|4, k|$ können bei $r \leqq 4-\sqrt{6}$ die Punkte E_1, E_2, E_3, E_4 nur dann auf obiger Kugelkappe liegen, wenn $P'=P$ und X_1 auf der Sphäre Γ_{P-1} sind (S. Fig. 10).

Ist $r \geqq 4-\sqrt{6}$, so ist eine bessere Lagerung als [4, 0] nicht möglich, d.h.

$$P[4(r \geqq 4-\sqrt{6}), k \leqq 4] = r\left(1+\sqrt{\frac{3}{2}}\right)$$

ist.

1 – 2.5. [5, 1] und [5, 1].

$$R|0,5| = R|0,6| = P[0,5]-1 = P[0,6]-1 = \sqrt{2},$$

darum kann eine Spitze des $2r$ kantigen Oktaeders als Mittelpunkt einer Einheitskugel betrachten werden. Also im Falle $r>1$ gelangen wir zu einem instabilen System, dessen Radius

$$R|5,1| = R|6,0| = P[5,1]-r = P[6,0]-r = r\sqrt{2}$$

ist.

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ÜBER SCHWACH MULTIPLIKATIVE SYSTEME

Von

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Herrn Prof. G. ALEXITS zum 75. Geburtstag gewidmet

Es sei (X, S, μ) ein Maßraum mit endlichem, positivem μ-Maß und $F = \{f_n : n \in \mathbf{N}\} \subset L_\mu(X)$ ein Funktionensystem mit

$$M_n = \operatorname{vrai}\max_{x \in X} |f_n(x)| < \infty \quad (n \in \mathbf{N}); \quad \mathbf{N} = \{0, 1, 2, \ldots\}.$$

Es bezeichne $G = \{g_n : n \in \mathbf{N}\}$ das Produktsystem des Funktionensystems F: das bedeutet, daß für $n \in \mathbf{N}$ mit der dyadischen Darstellung

$$n = \sum_{i=0}^{\infty} n_i 2^i \quad (n_i = 0, 1)$$

$$g_n = \prod_{i=0}^{\infty} (f_i/M_i)^{n_i} \tag{1}$$

gestzt wird. Das System F heißt *stark multiplikativ*, wenn G ein Orthogonalsystem ist, *multiplikativ*, wenn $\int_X g_n \, d\mu = 0$ $(n = 1, 2, \ldots)$ ist; und *schwach multiplikativ*, wenn für sein Produktsystem G die Beziehung gilt:

$$\sum_{n=0}^{\infty} \left| \int_X g_n \, d\mu \right| = A < \infty . \tag{2}$$

Diese Begriffe wurden von G. ALEXITS [1], [2] eingeführt. Das Rademachersche System $R = \{r_n : n \in \mathbf{N}\}$ ist ein stark multiplikatives System und das Produktsystem des Systems R ist das Walshsche Orthogonalsystem $W = \{w_n : n \in \mathbf{N}\}$.

Wir führen den folgende Begriff ein: Das System F heißt *p-schwach multiplikativ* $(1 \leq p \leq \infty)$, wenn für das Produktsystem

$$\left\| \sum_{k=0}^{2^n-1} A_k w_k \right\|_p \leq A \; (n \in \mathbf{N})^1 \tag{3}$$

gilt, wobei $A_k = \int_X g_k \, d\mu \; (k \in \mathbf{N})$ ist.

Es ist klar, daß jede schwach multiplikative System für $1 \leq p \leq \infty$ auch p-schwach multiplikativ ist.

Wir beweisen, daß die Bedingung (3) von der Anordnung des Systems F unabhängig ist. Es bezeichne $\pi : \mathbf{N} \to \mathbf{N}$ eine Anordnung der Menge $\mathbf{N}$. Mit Hilfe dieser Anordnung definieren wir eine Anordnung $T = T_\pi : \mathbf{N} \to \mathbf{N}$ von $\mathbf{N}$ folgenderweise:

$$T(n) = \sum_{k=0}^{\infty} n_k 2^{\pi(k)} \quad \left(n = \sum_{k=0}^{\infty} n_k 2^k \right).$$

Es ist klar, daß wir damit eine Anordnung $T : \mathbf{N} \to \mathbf{N}$ definiert haben und daß

$$\overline{G} = T\,G = \{\bar{g}_n = g_{T(n)} : n \in \mathbf{N}\}$$

das Produktsystem des Systems

$$\pi F = \{\bar{f}_n = f_{\pi(n)} : n \in \mathbf{N}\}$$

ist.

Die Unabhängigkeit der Bedingung (3) von der Anordnung des Systems F kann nun folgenderweise formuliert werden:

SATZ 1. *Ist πF eine Anordnung des Systems F und ist*

$$\overline{G} = \{\bar{g}_n = g_{T(n)} : n \in \mathbf{N}\}$$

das Produktsystem von πF, so gilt für $1 \leq p \leq \infty$.

$$\sup_n \left\| \sum_{k=0}^{2^n-1} A_{T(k)} w_k \right\|_p \leq \sup_n \left\| \sum_{k=0}^{2^n-1} A_k w_k \right\|_p .$$

G. ALEXITS und A. SHARMA haben den folgende Satz bewiesen [3]: Ist F ein multiplikatives System mit $M_n = 1 \; (n \in \mathbf{N})$, so ist die Reihe $\sum c_n f_n$ unter der Bedingung $\sum c_n^2 < \infty$ fast überall konvergent.

In [2] wurde dieser Satz folgenderweise verschärft: Ist F ein beschränktes, schwach multiplikatives System, so konvergiert die Reihe $\sum c_n f_n$ unter der Bedingung $\sum c_n^2 < \infty$ fast überall.

In dieser Arbeit wird dieser Satz folgenderweise verallgemeinert:

SATZ 2. *Ist F ein beschränktes, p-schwach multiplikatives System mit $1 < p < \infty$ und $M_n = O(1)$, so folgt aus $\sum a_n^2 < \infty$ die Konvergenz fast überall der Reihe $\sum a_n f_n$ bei jeder Anordnung ihrer Glieder.*

[1] $\| \; \|_p$ bezeichnet die Norm in $L^p[0, 1]$ $(1 \leq p \leq \infty)$.

Es sei $a = (a_0, a_1, a_2, \ldots)$ eine reelle Zahlenfolge und für $x \in X$, $t \in [0, 1]$ und $n = 1, 2, \ldots$ führen wir die folgenden Bezeichnungen ein:

$$(4)\qquad \begin{aligned} S_n(a)(t) &= \sum_{k=0}^{n-1} a_k w_k(t), \quad G_n(a)(x) = \sum_{k=0}^{n-1} a_k g_k(x), \\ S^*(a)(t) &= \sup_n |S_n(a)(t)|, \quad G^*(a)(x) = \sup_n |G_n(a)(x)|. \end{aligned}$$

Wir werden zeigen, daß Satz 2 eine unmittelbare Folge der nachstehende Behauptung ist:

SATZ 3. *Ist F ein p-schwach multiplikatives System mit $1 \leq p \leq \infty$ und $a = (a_0, a_1, \ldots)$ eine beliebige Zahlenfolge, so gilt*

$$(5)\qquad \|G^*(a)\|_{X,1} \leq A \|S^*(a)\|_q,$$

wobei $1/p + 1/q = 1$ ist.[2]

Aus Satz 2 ergibt sich leicht der folgende

SATZ 4. *Ist F ein p-schwach multiplikatives System mit $2 \leq p \leq \infty$ und ist $\sum a_n^2 < \infty$, so konvergiert die Reihe $\sum a_n g_n$ fast überall.*

Satz 3 ist eine Verallgemeinerung des Satzes 1 der Arbeit [4]

Beweis der Sätze

BEWEIS DES SATZES 1. Es bezeichne $S_n(f)$ die n-the Partialsumme der Walsh-Fourierreihe der Funktion $f \in L[0, 1]$:

$$S_n(f; t) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(t) \left(\hat{f}(k) = \int_0^1 f(t) w_k(t)\, dt;\ n = 1, 2, \ldots \right).$$

Es läßt sich zeigen (siehe [5]), daß für $f \in L\ [0, 1]$

$$(6)\qquad \|S_{2^n}(f)\|_p \leq \|f\|_p (1 \leq p \leq \infty); \quad \|\sup_n |S_{2^n}(f)|\|_p \leq B_p \|f\|_p (1 < p \leq \infty)$$

gilt.

In [6] wurde bewiesen, daß es eine (im Lebesguesche Sinne) meßbare Transformation $T^* : [0, 1] \to [0, 1]$ gibt, die abgesehen von einer abzählberer Menge ein-eindeutig ist, und die der Bedingung

$$(7)\qquad \begin{aligned} &\int_0^1 (f \circ T^*)(t)\, dt = \int_0^1 f(t)\, dt \quad (f \in L[0, 1]); \quad w_{T(k)}(T^*(t)) = w_k(t) \\ &\qquad\qquad (k \in \mathbf{N},\ t \in [0, 1]) \end{aligned}$$

genügt.

[2] $\| \ \|_{X,p}$ bezeichnet die Norm in $L_\mu^p(x)$.

Es sei nun $n \in \mathbf{N}$ beliebig und $m \in \mathbf{N}$ so gewählt, daß

$$\{T(k) : k = 0, 1, \ldots, 2^n - 1\} \subset \{0, 1, \ldots, 2^m - 1\}$$

gilt und setzen wir

$$F_m = \sum_{k=0}^{2^m-1} A_k w_k .$$

Dan auf Grund von (7) ergibt sich

$$(F_m \circ T^*)\hat{}(k) = \int_0^1 (F_m \circ T^*)(t) w_k(t)\, dt = \int_0^1 F_m(T^*(t)) w_{T(k)}(T^*(t))\, dt =$$

$$= \int_0^1 F_m(t) w_{T(k)}(t)\, dt = \hat{F}_m(T(k)) = A_{T(k)} .$$

Daraus und aus (6) folgt

$$\left\| \sum_{k=0}^{2^n-1} A_{T(k)} w_k \right\|_p = \|S_{2^n}(F_m \circ T^*)\|_p \leq \|F_m \circ T^*\|_p =$$

$$= \|F_m\|_p = \left\| \sum_{k=0}^{2^m-1} A_k w_k \right\|_p .$$

Aus dieser Beziehung ergibt sich unmittelbar die Behauptung des Satzes 1.

Beweis des Satzes 3. Wir führen die „Kernfunktionen"

$$D_n(x, t) = \sum_{k=0}^{n-1} g_k(x) w_k(t) \quad (t \in [0, 1];\ x \in X;\ n = 1, 2, \ldots)$$

ein. Aus der Definition von G und W folgt

$$D_{2^k}(x, t) = \prod_{n=0}^{k-1} (1 + r_n(t) f_n(x)/M_n) \geq 0 \quad (t \in [0, 1];\ x \in X). \tag{8}$$

Auf Grund der Orthogonalität des Walshschen Systems folgt, daß für $2^k \geq n$

$$G_n(a)(x) = \int_0^1 S_n(a)(t) D_{2^k}(x, t)\, dt$$

gilt, woraus sich

$$\sup_{n \leq 2^k} |G_n(a)(x)| \leq \int_0^1 \sup_{n \leq 2^k} |S_n(a)(t)|\, D_{2^k}(x, t)\, dt \quad (x \in X,\ k \in \mathbf{N})$$

ergibt. Daraus, durch Anwendung der Hölderschen Ungleichung auf Grund von (3) und (8) folgt

$$\int_X \sup_{n\leq 2^k} |G_n(a)(x)|\,dx \leq \int_X \int_0^1 \sup_{n\leq 2^k} |S_n(a)(t)|\, D_{2^k}(x,t)\,dt =$$

$$= \int_0^1 \sup_{n\leq 2^k} |S_n(a)(t)| \left(\int_X D_{2^k}(x,t)\,d\mu(x)\right) dt \leq$$

$$\leq \left\{\int_0^1 \sup_{n\leq 2^k} |S_n(a)(t)|^q\,dt\right\}^{1/q} \cdot \left\|\sum_{n=0}^{2^k-1} A_n w_n\right\|_p \leq$$

$$\leq A\cdot\left\{\int_0^1 \sup_{n\leq 2^k} |S_n(a)(t)|^q\,dt\right\}^{1/q}.$$

Aus dieser Ungleichung für $k\to\infty$ erhalten wir die Behauptung des Satzes 3.

Beweis des Satzes 4. Es sei $a^n = (0, 0, \ldots, 0, a_n, a_{n+1}, \ldots)$. Auf Grund der Bedingung $\sum a_n^2 < \infty$ läßt sich aus der Folge (a^n) eine Teilfolge (a^{n_k}) mit

$$\sum_{k=0}^{\infty} \|a^{n_k}\|_2 < \infty$$

herausgreifen. Ist $n_k \leq n < n_{k+1}$ so gilt

$$\Theta_n(x) = \sup_{m\geq n} |G_m(a)(x) - G_n(a)(x)| \leq$$

$$\leq 2 \sup_{m\geq n_k} |G_m(a)(x) - G_{n_k}(a)(x)| = 2G^*(a^{n_k})(x).$$

Da nach einem Satz von P. Sjölin ([7] Satz B) (mit einer Konstante B)

$$\|S^*(a^m)\|_2 \leq B\left\{\sum_{k=m}^{\infty} a_k^2\right\}^{1/2}$$

gilt,[3] deshalb folgt aus Satz 3, daß für $1/p + 1/q = 1$ $(q\leq 2)$

$$\int_X G^*(a^{n_k})(x)\,dx \leq A\,\|S^*(a^{n_k})\|_q \leq A\,\|S^*(a^{n_k})\|_2 \leq$$

$$\leq AB\left\{\sum_{l=n_k}^{\infty} a_l^2\right\}^{1/2}$$

[3] $\|a\|_2 = \left\{\sum_{k=0}^{\infty} a_k^2\right\}^{1/2}$ $(a = (a_0, a_1, \ldots))$.

besteht. Daraus folgt endlich

$$\sum_{k=0}^{\infty} \int_X G^*(a^{n_k})(x)\,dx \leq AB \sum_{k=0}^{\infty} \|a^{n_k}\|_2 < \infty$$

und so ist $\lim_{k\to\infty} \Theta_k(x) = 0$ fast überall in X.

Damit haben wir Satz 4 bewiesen.

Beweis des Satzes 2. Aus Satz 1 folgt, daß die Bedingung (3) von der Anordnung des Systems F unabhängig ist. Deshalb genügt es zu zeigen, daß aus der Bedingung (3) und aus $\sum a_n^2 < \infty$ die Konvergenz der Reihe $\sum a_n f_n$ μ fast überall folgt.

Offensichtlich gibt es eine Funktion $f \in L^2[0, 1]$ für welche

$$\left\| f - \sum_{k=0}^{n} a_k M_k w_{2^k} \right\|_2 \to 0 \quad (n \to \infty)$$

gilt. Auf Grund der Khintschinsche Ungleichung besteht

$$\|f\|_q \leq O(1) \left\{ \sum_{k=0}^{\infty} a_k^2 \right\}^{1/2} \quad (0 < q < \infty).$$

Aus Satz 3 und aus (6) folgt

$$\left\| \sup_n \left| \sum_{k=0}^{n} a_k f_k \right| \right\|_{X,1} = \left\| \sup_n \left| \sum_{k=0}^{n} a_k M_k g_{2^k} \right| \right\|_{X,1} \leq$$

$$\leq A \left\| \sup_n \left| \sum_{k=0}^{n} a_k M_k w_{2^k} \right| \right\|_q =$$

$$= A \left\| \sup_n |S_{2^n}(f)| \right\|_q \leq \|f\|_q \leq O(1) \left\{ \sum_{k=0}^{\infty} a_k^2 \right\}^{1/2}.$$

Daraus ergibt sich die Behauptung des Satzes ähnlich, wie im Satz 4.

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ОБ АБСОЛЮТНОЙ Р-СУММИРУЕМОСТИ ФУНКЦИОНАЛЬНЫХ РЯДОВ ПОЧТИ ВСЮДУ

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1. Пусть $\varphi = \{\varphi_k\}$ — такая система измеримых, конечных почти всюду на $e = [a, b]$ функций, чтобы ряд

$$\sum_{k=0}^{\infty} \xi_k \varphi_k(t) \tag{1}$$

для всех $x = \{\xi_k\} \in l^p$ $(1 < p < \infty)$ сходился по мере на e.

Говорят, что ряд (1) почти всюду на e является абсолютно суммируемым треугольным методом $A = (\bar{\alpha}_{nk})$ т.е. $|A|$-суммируемым почти всюду на e, если почти всюду на e

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \bar{\alpha}_{nk} \xi_k \varphi_k(t) \right| < \infty . \tag{2}$$

В настоящей статье мы исследуем $|A|$-суммируемость почти всюду на e рядов (1) для $x \in l^p$ в основном в случае, когда A — метод Риса P, задаеваемый треугольной матрицей для преобразования ряда в ряд

$$\bar{\alpha}_{nk} = P_{k-1} p_n P_n^{-1} P_{n-1}^{-1} ,$$

где $0 < P_n \uparrow$ и $p_n = P_n - P_{n-1}$. Через $P(z)$ мы обозначаем монотонно возрастающую дифференцируемую функцию, для которой $P(n) = P_n$, а через Q — обртаную к P функцию.

Отметим, что исследованием $|P|$-суммируемости рядов (1) при ортонормированной системе φ и $p=2$ посвящено много работ. Так, например, АЛЕКСИЧ, КРАЛИК и МОРИЦ в работах [6], [7] доказали, в частности, следующее утверждение

ТЕОРЕМА А. Ортогональный ряд (1) является почти всюду на e $|P|$-суммируемым для тех $x \in l^2$, для которых

$$\sum_{m=1}^{\infty} A_m < \infty ,$$

где

$$A_m = \left\{ \sum_{\nu_m+1}^{\nu_{m+1}} \xi_k^2 \right\}^{1/2}, \quad \text{а} \quad \nu_m = [Q(2^m)].$$

Воспользуясь теоремой А, ГРЕПАЧЕВСКАЯ доказала в [2], в частности, следующее утверждение.

ТЕОРЕМА В. Пусть функция $P(z)$ удовлетворяет следующим условиям:

1° $P'(z)/P(z)$ убывает монотонно при $z \to \infty$,

2° $P(z+1) \leqq \alpha P(z)$ $(1 < \alpha < 2)$ для достаточно больших z.

Ортогональный ряд (1) почти всюду на e является $|P|$-суммируемым для тех $x \in l^2$, для которых

$$\sum_{n=1}^{\infty} P'(n) P^{-1}(n) E_n \lg^{-1/2} P(n) < \infty,$$

где

$$E_n = \left\{ \sum_{k=n}^{\infty} \xi_k^2 \right\}^{1/2}.$$

Мы докажем следующие теоремы.

ТЕОРЕМА 1. *Если ряд* (1) *для всех* $x \in l^p$ $(1 < p < \infty)$ *сходится по мере на* e, *то он почти всюду на* e *является* $|P|$-*суттируемым для тех* $x \in l^p$, *для которых*

$$\sum_{m=1}^{\infty} A_m^{(p)} < \infty, \tag{3}$$

где

$$A_m^{(p)} = \left\{ \sum_{\nu_m+1}^{\nu_{m+1}} |\xi_k|^p \right\}^{1/p}.$$

ТЕОРЕМА 2. *Если функция* $P(z)$ *удовлетворяет условиям* 1° *и* 2° *теоремы* B *и ряд* (1) *для всех* $x \in l^p$ $(1 < p < \infty)$ *сходится по мере на* e, *то он почти свюду на* e *является* $|P|$-*суммируемым для тех* $x \in l^p$, *для которых*

$$\sum_{n=1}^{\infty} P'(n) P^{-1}(n) E_n^{(p)} \lg^{-1/p} P(n) < \infty, \tag{4}$$

где

$$E_n^{(p)} = \left\{ \sum_{k=n}^{\infty} |\xi_k|^p \right\}^{1/p}.$$

Очевидно, что теоремы A и B являются частными случаями теорем 1 и 2.

Мы докажем еще следующее утверждение.

ТЕОРЕМА 3. *Пусть ряд* (1) *для всех* $x \in l^p$ $(1 < p < \infty)$ *сходится по мере на* e. *Пусть далее для* $\omega_k \uparrow \infty$

1° $\sum_{k=0}^{\infty} |\alpha_{kk}| \, \omega_k^{1/(1-p)} < \infty$,

2° $\left| \dfrac{\bar{\alpha}_{nk} \, \omega_n^{1/p-1}}{\alpha_{nn}} \right|^{1/(p-1)}$ *не возрастает по* n *для* $k = 0, 1, 2, \ldots$,

3° $\sum_{n=k}^{\infty} |\bar{\alpha}_{nk}| \leq N$.

Тогда ряд (1) *почти всюду на* e *является* $|A|$*-суммируемым для тех* $x \in l^p$, *для которых*

$$\sum_{k=0}^{\infty} |\xi_k|^p \, \omega_k < \infty . \tag{5}$$

При ортогональных системах и $p=2$ получаем в случае $A=P$ отсюда теорему 5 из [1].

2. При доказательстве теорем 1—3 мы воспользуемся следующими результатами.

ЛЕММА 1. ([3], стр. 158). Если ряд (1) для всех $x \in l^p$ $(1 < p < \infty)$ сходится по мере на e, то для каждого $\varepsilon > 0$ найдутся измеримое подмножество $T_\varepsilon \subset e$ с $\operatorname{mes} T_\varepsilon > b - a - \varepsilon$ и постоянное $M_\varepsilon > 0$ такие, что для всех чисел $\{c_n\}$

$$\int_{T_\varepsilon} \left| \sum_{k=0}^{n} c_k \varphi_k(t) \right| dt \leq M_\varepsilon \left(\sum_{k=0}^{n} |c_k|^p \right)^{1/p} .$$

ЛЕММА 2. ([4], стр. 142). Для того, чтобы измеримая функция g была конечной почти всюду на e, необходимо и достаточно, чтобы для каждого $\varepsilon > 0$ нашлось такое измеримое подмножество $T_\varepsilon \subset e$ с $\operatorname{mes} T_\varepsilon > b - a - \varepsilon$, что

$$\int_{T_\varepsilon} |g(t)| \, dt < \infty .$$

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 1. В силу леммы 2 и теоремы Леви мы должны для сходимости ряда (2) показать, что для каждого $\varepsilon > 0$ найдутся измеримое подмножество $T_\varepsilon \subset e$ с $\operatorname{mes} T_\varepsilon > b - a - \varepsilon$ и постоянное $N_\varepsilon > 0$ такие, чтобы

$$B_m \equiv \sum_{n=0}^{m} \int_{T_\varepsilon} \left| \sum_{k=0}^{n} \bar{\alpha}_{nk} \xi_k \varphi_k(t) \right| dt \leq N_\varepsilon .$$

Но в силу леммы 1 имеем, что

$$B_m \leq M_\varepsilon \sum_{n=0}^{m} p_n P_n^{-1} P_{n-1}^{-1} \left(\sum_{k=0}^{n} P_{k-1}^p \, |\xi_k|^p \right)^{1/p} . \tag{6}$$

Пусть $m_0(n)$ обозначает число, для которого $\nu_{m_0(n)} < n \leq \nu_{m_0(n)+1}$.

Учитывая монотонность последовательности $\{P_k\}$, получаем из неравенства (6), что

$$B_m \leq M_\varepsilon \sum_{n=0}^{m} p_n P_n^{-1} P_{n-1}^{-1} \left(\sum_{m=0}^{m_0(n)} \sum_{k=\nu_m+1}^{\nu_{m+1}} P_{k-1}^p |\xi_k|^p \right)^{1/p} \leq$$

$$\leq M_\varepsilon \sum_{n=0}^{m} p_n P_n^{-1} P_{n-1}^{-1} \sum_{m=0}^{m_0(n)} P_{\nu_{m+1}-1} A_m^{(p)} . \tag{7}$$

Определим число $\bar{n}$ при помощи неравенства $m_0(\bar{n}-1) < m \leq m_0(\bar{n})$. Тогда учитывая, что

$$P_{\bar{n}} \geq P_{\nu m_0(n)+1} \geq P\left(Q\left(2^{m_0(\bar{n})}\right)\right) = 2^{m_0(\bar{n})} \geq 2^m ,$$

получаем из неравенства (7) после изменения порядка суммирования, что

$$B_m \leq M_\varepsilon \sum_{m=0}^{\infty} A_m^{(p)} P_{\nu_{m+1}-1} \sum_{m_0(n) \geq m} p_n P_n^{-1} P_{n-1}^{-1} \leq$$

$$\leq M_\varepsilon \sum_{m=0}^{\infty} A_m^{(p)} P_{\nu_{m+1}-1} P_{\bar{n}}^{-1} \leq 2M_\varepsilon \sum_{m=0}^{\infty} A_m^{(p)} .$$

Из сходимости ряда (3) вытекает существование постоянной $N_\varepsilon > 0$ такой, чтобы

$$B_m \leq N_\varepsilon .$$

Теорема 1 доказана.

Доказательство Теоремы 2. Покажем, что из условия (4) следует условие (3). На самом деле, так как

$$B \equiv \sum_{m=1}^{\infty} P'(n) P^{-1}(n) E_n^{(p)} \lg^{-1/p} P(n) \geq$$

$$\geq \sum_{m=1}^{\infty} (m+1)^{-1/p} E_{\nu_{m+1}}^{(p)} \sum_{n=\nu_m+1}^{\nu_{m+1}} P'(n) P^{-1}(n) ,$$

то из оценки (см. [2] стр. 740)

$$\sum_{n=\nu_m+1}^{\nu_{m+1}} P'(n) P^{-1}(n) \geq \ln 2 \left[1 - \lg \frac{P(\nu_m+1)}{P(\nu_m)} \right] = M_1 > 0 ,$$

получаем, что

$$B \geq M_1 \sum_{m=1}^{\infty} (m+1)^{-1/p} \left\{ \sum_{k=m+1}^{\infty} \left[A_k^{(p)} \right]^p \right\}^{1/p} .$$

Положив теперь в неравенстве (см. [5], теорема 345)

$$\sum_{m=1}^{\infty} |z_m|^{1/p} \leq M_2 \sum_{m=1}^{\infty} m^{-1/p} \left(\sum_{n=m}^{\infty} |z_n| \right)^{1/p}$$

$|z_k| = [A_k^{(p)}]^p$, получим, что

$$B \geq M_1 M_2 \sum_{m=1}^{\infty} A_m^{(p)}.$$

Теорема 2 доказана.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 3. Точно так, как в доказательстве теоремы 1, мы убеждаемся в том, что надо доказать неравенство

$$C_m \equiv \sum_{n=0}^{m} \left(\sum_{k=0}^{n} |\bar{\alpha}_{nk}|^p |\xi_k|^p \right)^{1/p} \leq M. \tag{8}$$

Но

$$C_m \leq \left\{ \sum_{n=0}^{m} |\alpha_{nn}| \, \omega_n^{1/(1-p)} \right\}^{1/q} \left\{ \sum_{n=0}^{m} |\alpha_{nn}|^{1-p} \omega_n \sum_{k=0}^{n} |\bar{\alpha}_{nk}|^p |\xi_k|^p \right\}^{1/p},$$

вследствие чего из условия 1° вытекает существование постоянной $N_1 > 0$ такой, что после изменения порядка суммирования в правом множителе будем иметь

$$C_m \leq N_1 \left\{ \sum_{k=0}^{m} |\xi_k|^p \sum_{n=k}^{m} |\bar{\alpha}_{nk}|^p \omega_n |\alpha_{nn}|^{1-p} \right\}^{1/p}.$$

Учитывая условие 2°, получаем, что

$$C_m \leq N_1 \left\{ \sum_{k=0}^{m} |\xi_k|^p \omega_k \sum_{n=k}^{m} |\bar{\alpha}_{nk}| \right\}^{1/p},$$

вследствие чего из условия 3° и (5) вытекает неравенство (8). Теорема 3 доказана.

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SUBALGEBRA SYSTEMS OF ALGEBRAS WITH FINITE AND INFINITE, REGULAR AND IRREGULAR ARITY

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1. Introduction

This paper investigates the subalgebra system of algebras. The interest is concentrated about, how the arity of the algebra (supremum of the arity of the operations) influences the structure of the subalgebra system or vice versa. We allow finite as well as infinite arity; and when the arity is a non-regular cardinal, we do not substitute it with its least regular upper bound, but try to list all the consequences of its exact value (the consequences that the arity is, e.g., exactly four and not three).

The structure of the subalgebra system is investigated on two successive levels of abstraction: firstly, forgetting about the operations and investigating the closed-set system, and secondly, forgetting about the concrete sets and investigating their lattices, in abstract.

The interrelations between the arity and the lattice properties have been investigated in detail in [1] and [2] gives a brief up-to-date summary, but both of them have dealt with only the case of regular cardinals. The usual proof of the representation theorem of m-algebraic lattices uses the system of ideals of a subsemilattice and fails for non-regular cardinals. We have simplified this proof (the ideals and the sub-semilattice turn out to be a dispensable detour) and so it became applicable to non-regular cardinals, too. See theorem 1. in this paper.

We also put much emphasis on the closed-set system, which we feel to be a key point on the way between the algebra and the lattice. Closed-set systems have been investigated in detail, e.g., in [3], but only for finitary and unary algebras; more precisely the arities 2 and $\aleph_0$ have been investigated. Our investigation of closed-set systems is exhaustive w.r.t. the properties introduced in this paper, that is we give a complete axiom system for closed-set systems w.r.t. these properties (which are closely connected to the arity of the possible corresponding algebras, and the properties of the lattice). This completeness result is announced in theorem 4, which might be called an independence theorem.

We would also like to draw attention to the regular cardinal 2, the role of which has not been stressed so far, though the result in [1] applied to this cardinal explains some pecularities of unary algebras.

2. Closed-set systems and lattices

Let $\mathcal{A}$ be a family of sets. $\mathcal{A}$ is a *closed-set system* if it is closed under intersection, i.e., for any subfamily $H \subseteq \mathcal{A}$ we have $\bigcap H \in \mathcal{A}$. In case $H = \emptyset$ we take $\bigcap \emptyset = A$. The elements of $\mathcal{A}$ are called the closed sets of the system. There is a one-to-one correspondence between closed-set systems and closure operations. The closure of a set X is denoted by $[X]$. Actually, $[X]$ is the smallest element of $\mathcal{A}$ containing X. If we want to distinguish the closed-set system in which the closure is taken we write $[X]_{\mathcal{A}}$. Sometimes it will be more convenient to define the corresponding closure operation instead of $\mathcal{A}$. The script capitals, like $\mathcal{A}$, will always stand for a closed-set system, and the corresponding capital A will stand for $\bigcup \mathcal{A}$.

Note, that we did not exclude the empty set, that is, $\{\emptyset\}$ is also a closed-set system.

DEFINITION. A closed-set system is called m-gen if any element, generated by any set, can also be generated by a subset of the same set having less than m elements, i.e.,

$$x \in [X] \Rightarrow (\exists\, Y \subseteq_m X)\, x \in [Y] \text{ for any } X \text{ and } x,$$

where

$$Y \subseteq_m X \text{ means that } Y \subseteq X \text{ and } |Y| < m.$$

We shall extensively use $\subseteq_m$ as a shorthand for "is a subset of ... having less than m elements". (Fig. 1.)

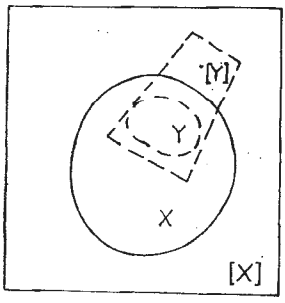

Fig. 1.

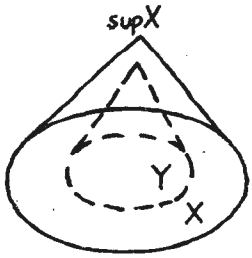

Fig. 2.

DEFINITION. A complete lattice is called m-com if every element is the supremum of some set of m-compact elements, where an element x of a lattice is m-compact if, for any set X, $x \leq \sup X$ implies that $x \leq \sup Y$ for some $Y \subseteq_m X$. (Fig. 2.)

THEOREM 1. *Let m be any cardinal. Then,*

(i) *for any m-gen closed-set system $\mathcal{A}$ the lattice $\langle \mathcal{A}, \subseteq \rangle$ is m-com,*
(ii) *for any m-com lattice $\mathcal{L}$ there is an m-gen closed-set system $\mathcal{A}$ such that $\mathcal{L} \cong \langle \mathcal{A}, \subseteq \rangle$.*

PROOF. (i) is trivial, since the closed-sets in $\mathcal{A}$ generated by one element are m-compact elements of $\langle \mathcal{A}, \subseteq \rangle$, for any m (and any closed set is the union of its closed subsets generated by one element).

(ii) given an m-com lattice $\mathcal{L}$ we construct the closed-set system $\mathcal{A}$ by defining a closure operation on the set A of m-compact elements of $\mathcal{L}$: for any $X \subseteq A$ we define $[X] \overset{\text{def}}{=} \{x \in A : x \leq \sup X\}$, where the sup is taken in $\mathcal{L}$. (Fig. 3.) This is a closure operation since $\mathcal{L}$ is complete, $\mathcal{A}$ is m-gen because all the elements of A are m-compact, and the homomorphism, from $\mathcal{L}$ onto $\langle \mathcal{A}, \subseteq \rangle$, givingly $\{x \in A : x \leq y\}$ for every y, is an isomorphism because $\mathcal{L}$ is m-com.

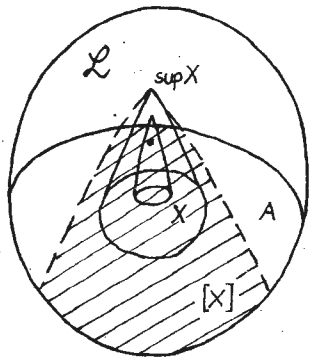

Fig. 3.

Note, that the m-compact elements do not necessarily form a subsemilattice. ▮

We remark that theorem 1. implies that every 2-com lattice is completely distributive.

DEFINITION. A set X is called m-closed (w.r.t.$\mathcal{A}$) if it contains the closure of each of its subsets having less than m elements, that is:

$$X \text{ is } m\text{-closed if } (\forall Y \subseteq_m X)[Y] \subseteq X .$$

We define an m-alg closed-set system to be a closed-set system in which a set is closed iff it is m-closed. That is:

$$\mathcal{A} \text{ is } m\text{-alg if } X \in \mathcal{A} \Leftrightarrow (\forall Y \subseteq_m X)[Y] \subseteq X .$$

DEFINITION. A partially ordered set is m-directed if any of its subsets having less than m elements has an upper bound.

A closed-set system is m-inductive if the union of any m-directed family of closed sets is also closed, that is:

$\mathcal{A}$ is m-inductive if $(\forall\, m\text{-directed } H \subseteq \mathcal{A}) \bigcup H \in \mathcal{A}$ where
H is m-directed if $(\forall X \subseteq_m H)(\exists Y \in H) \bigcup X \subseteq Y$.

In our investigation of closed-set systems and subalgebra-systems the property of being m-inductive will play a secondary role. Actually, the concept "m-inductive" is an auxiliary concept, to treat the inductions arising in connection with the regular cardinals.

DEFINITION. A cardinal m is not *regular* iff any set of cardinality m can be written as the union of fewer than m sets each of cardinality less than m.

Regular cardinals will play an important role in our discussion, e.g., they explain why the unary algebras behave in an essentially different way from the finitary algebras, or the finitary ones from the infinitary ones.

For example 0, 1, 2, $\aleph_0$, $\aleph_1$ are regular cardinals, all the other finite numbers are not.

THEOREM 2.

(i) *for any cardinal m: the property of being m-gen implies that of being m-alg which further implies that of being m-inductive*

(ii) *for any regular cardinal $m > 1$: the above properties coincide.* (Fig. 4.)

$$\begin{tikzcd}
 & \mathcal{A} \text{ is } m\text{-alg} \arrow[d, Rightarrow, bend left] \\
 & \mathcal{A} \text{ is } m\text{-gen} \arrow[u, Rightarrow] \arrow[d, Rightarrow, bend left] \\
\text{in case } 1 < m \text{ is regular} & \mathcal{A} \text{ is } m\text{-ind} \arrow[u, Rightarrow, dashed] \arrow[u, Rightarrow, bend left, dashed]
\end{tikzcd}$$

Fig. 4.

PROOF. (i) m-gen implies m-alg, because for any m-closed set:

$$\underset{\mathcal{A} \text{ is } m\text{-gen}}{x \in [X] \Rightarrow} (\exists Y \subseteq_m X)\, x \in [Y] \underset{X \text{ is } m\text{-closed}}{\subseteq} X$$

m-alg implies m-inductive because the union of any m-directed family of closed sets is always m-closed.

(ii) If $1 < m$ is regular, then m-inductive implies m-gen because, if m is regular, then $H \overset{\text{def}}{=} \{[Y] : Y \subseteq_m X\}$ is m-directed for any X and any closed-set system, and in addition, if $m > 1$, then $\bigcup H \supseteq X$. ∎

Later, we shall modify our concepts, so as to get rid of the restriction $m > 1$.

NOTATION. We use $\overleftarrow{\alpha}$ to denote the smallest cardinal β for which any $\beta \leq \xi < \alpha$ is not regular, that is:

$$\overleftarrow{\alpha} \overset{\text{def}}{=} \bigcap \{\beta : \beta \leq \xi < \alpha \Rightarrow \xi \text{ is not regular}\}.$$

THEOREM 3. *For any cardinals $\overleftarrow{\alpha} \leq \beta < \alpha$ the property of being α-inductive coincides with the property of being β inductive.*

PROOF. It is enough to prove that any $\overleftarrow{\alpha}$-directed set is also α-directed.

The proof goes by induction: let us suppose that $\eta \leq \alpha$ is such that for all $\xi < \eta$ any $\overleftarrow{\alpha}$-directed set is also ξ-directed.

We shall show that this hypothesis implies the same for η.

Let H be $\overleftarrow{\alpha}$-directed and $X \subseteq_\eta H$. $\xi \overset{\text{def}}{=} |X|$. It is enough to consider the case $\overleftarrow{\alpha} \leq \xi < \eta$. Thus X has a decomposition $X = \bigcup_{i<\zeta} X_i$ where ξ, $|X_i| < \xi$ for all $i < \zeta$.

By the induction hypothesis H is ξ-directed and so every X_i has an upper bound and also these upper bounds have a common upper bound. This upper bound is an upper bound of X as well and so H is η-directed. ∎

Now, we define four functions to caracterise the closed-set systems.

DEFINITION.

gen $\mathcal{A}$	is the smallest cardinal α for which $\mathcal{A}$	is α-gen,
alg $\mathcal{A}$	is the smallest cardinal α for which $\mathcal{A}$	is α-alg,
ind $\mathcal{A}$	is the smallest cardinal α for which $\mathcal{A}$	is α-inductive,
com $\mathcal{A}$	is the smallest cardinal α for which $\langle \mathcal{A}, \subseteq \rangle$	is α-com.

That is e.g.:

$$gen\,\mathcal{A} \overset{\text{def}}{=} \bigcap \{\alpha : \mathcal{A} \text{ is } \alpha\text{-gen}\}.$$

Since any closed-set system $\mathcal{A}$ is surely $|A|^+$-gen, the four functions, introduced above, are everywhere defined. By theorems 1, 2 and 3, for any closed-set system we have

$$ind\,\mathcal{A} = \overleftarrow{gen\,\mathcal{A}} \leq alg\,\mathcal{A} \leq gen\,\mathcal{A} \geq com\,\mathcal{A}, \text{ if } gen\,\mathcal{A} > 1.$$

The next theorem states, this is all what can be said about these functions: any statement, which is not a consequence of the above ones, does not hold for all $\mathcal{A}$. The only exceptions are the cardinals 0 and 1, an exhaustive description of which is given after the following theorem.

We may omit *ind* from our further investigations since *ind* is determined by *gen* as well as by *alg*.

The *characteristic* of $\mathcal{A}$ is defined as the triple

$$x\,\mathcal{A} \overset{\text{def}}{=} \langle com\,\mathcal{A}, alg\,\mathcal{A}, gen\,\mathcal{A} \rangle.$$

THEOREM 4. *For any triple of cardinals* $\alpha, \beta, \gamma > 1$, *such that* $\alpha \leq \beta \leq \alpha \geq \gamma$, *there exists a closed-set system of characteristic* $\langle \gamma, \beta, \alpha \rangle$.

To prove this theorem we need some definitions and lemmas.

DEFINITION. Let $\mathcal{A}_i$, $i < \zeta$ be a family of closed-set systems with mutually disjoint greatest elements.

We define $\bigcup_{i<\zeta}^{+} \mathcal{A}_i$ as a closed-set system on the set $\bigcup_{i<\zeta} A_i$. The closed-sets of $\bigcup_{i<\zeta}^{+} \mathcal{A}_i$ are those X-s, whose intersection with A_i is closed, for each $i < \zeta$. That is:

$$\bigcup_{i<\zeta}^{+} \mathcal{A}_i \overset{\text{def}}{=} \{X : (\forall i < \zeta) X \cap A_i \in \mathcal{A}_i\}.$$

Now, $\bigcup_{i<\zeta}^{+} \mathcal{A}_i$ is easily seen to be a closed-set system itself.

LEMMA 1. for any $\zeta > 0$:

$$gen \bigcup_{i<\zeta}^{+} \mathcal{A}_i = \bigcup_{i<\zeta} gen\, \mathcal{A}_i,$$

$$alg \bigcup_{i<\zeta}^{+} \mathcal{A}_i = \bigcup_{i<\zeta} alg\, \mathcal{A}_i,$$

$$com \bigcup_{i<\zeta}^{+} \mathcal{A}_i = \bigcup_{i<\zeta} com\, \mathcal{A}_i.$$

PROOF. It is easy to see that

(i) $[X]_{\bigcup_{i<\zeta}^{+} \mathcal{A}_i} = \bigcup_{i<\zeta} [X \cap A_i]_{\mathcal{A}_i}$

(ii) X is a δ-compact element of $\langle \mathcal{A}_i, \subseteq \rangle$ iff X is a δ-compact element of $\langle \bigcup_{i<\zeta}^{+} \mathcal{A}_i, \subseteq \rangle$ for any $X \in \mathcal{A}_i$.

Thus, the lemma follows from the next observations:

$$\bigcup_{i<\zeta}^{+} \mathcal{A}_i \text{ is } \delta\text{-gen} \quad \text{iff } (\forall i < \zeta)\, \mathcal{A}_i \text{ is } \delta\text{-gen}$$

$$\bigcup_{i<\zeta}^{+} \mathcal{A}_i \text{ is } \delta\text{-alg} \quad \text{iff } (\forall i < \zeta)\, \mathcal{A}_i \text{ is } \delta\text{-alg}$$

$$\bigcup_{i<\zeta}^{+} \mathcal{A}_i \text{ is } \delta\text{-com} \quad \text{iff } (\forall i < \zeta)\, \mathcal{A}_i \text{ is } \delta\text{-com}$$

which are consequences of (i) and (ii). ∎

Now we introduce a tool to change the value of *alg* without disturbing the values of *gen* and *com* too much.

DEFINITION. Let $\mathcal{A}$ be a closed-set system, $\mathcal{B} \subseteq \mathcal{A}$ and p a one-one mapping of $\mathcal{B}$ into a set disjoint from A.

Now we define a new closed-set system $\mathcal{A}[\mathcal{B}]$ on the set $A \cup p(\mathcal{B})$ as follows:

$$\mathcal{A}[\mathcal{B}] \overset{\text{def}}{=} \{X \cup \{p(Y) : X \supseteq Y \in \mathcal{B}\} : X \in \mathcal{A}\}.$$

Usually we refer to $X \cup \{p(Y) : X \supseteq Y \in \mathcal{B}\}$ as X' and to $\mathcal{A}[\mathcal{B}]$ as $\mathcal{A}'$ for convenience. $\mathcal{B}'$ is the family of closed-sets in $\mathcal{A}'$ corresponndig to $\mathcal{B} \cdot \mathcal{B}'$ is also referred to as the *change set* of the construction.

LEMMA 2. Let $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}', X'$ be defined as above, and $|A| = \alpha$.

(i) $\mathcal{A}'$ is a closed-set system, and $\langle \mathcal{A}', \subseteq \rangle \cong \langle \mathcal{A}, \subseteq \rangle$.

(ii) Every set X has a subset with fewer than α^+ elements the closure of which – taken in $\mathcal{A}'$ – contains the original set X, that is:

$$(\forall X \subseteq A')(\exists Y \subseteq_{\alpha^+} X)[Y]_{\mathcal{A}'} = [X]_{\mathcal{A}'}.$$

(iii) If δ and η have the following property:

"for all $\delta \leq \xi < \eta$, any subset of A' of cardinality ξ can be obtained as $\bigcup_{i<\zeta} Y_i$ such that $\zeta < \xi$, $|Y_i| < \xi$ and $[Y_i]_{\mathcal{A}'} \in \mathcal{B}'$ for all $i < \zeta$" then for any $X \subseteq A'$ the hypothesis, that X is δ-closed w.r.t. $\mathcal{A}'$ implies that X is also η-closed w.r.t. $\mathcal{A}'$.

PROOF. (i) Is obvious.

(ii) Let $X \subseteq A'$ be arbitrary. We have to prove the existence of a $z \subseteq_{\alpha^+} X$ such that $[z] \supseteq X$.

To do this we introduce $\overline{X}$ as

$$\overline{X} \overset{\text{def}}{=} \bigcup \{Y : p(Y) \in X\} \cup (X \cap A)$$

and define a function k from $\overline{X}$ back to X by choice:

$k(x) \overset{\text{def}}{=} x$, if $x \in X \cap A$, and we choose

$k(x) \overset{\text{def}}{=} p(Y) \in X$ otherwise, such that $x \in Y$.

(The construction of $\overline{X}$ guarantees the existence of such a Y for every $x \in \overline{X} \setminus X$.)

Now, defining z as the k-image of $\overline{X}$ we can satisfy the requirements, for:

$z \subseteq_{\alpha^+} X$ since $\overline{X} \subseteq A$ and $|A| < \alpha^+$

$[z] \supseteq X$ since $X \cap A \subseteq z$ and for any $p(Y) \in X$ we have $Y \subseteq \overline{X}$, and k has the property that for any $\overline{Y} \subseteq \overline{X}$ the closure of the k-image of Y contains Y and so also contains $p(Y)$. Thus we have $p(Y) \in z$.

(iii) Let δ and η satisfy the property prescribed in (iii).

The proof goes by induction: Let us suppose that $\xi \leq \eta$ and for all $z \leq \xi$ the conclusion of (iii) holds, that is: the property of being δ-closed implies the property of being z-closed.

From this hypothesis we prove that the conclusion of (iii) holds also for ξ.

Let $X \subseteq A'$ be δ-closed in $\mathcal{A}'$ and $Y \subseteq_\xi X$. $z \overset{\text{def}}{=} |Y|$.

$z < \delta$ implies $[Y] \subseteq X$ since X is δ-closed,

$z \geq \delta$ implies $\delta \leq z < \xi \leq \eta$, and so Y has a decomposition $Y = \bigcup_{i<\zeta} Y_i$ satisfying the requirements in (iii). Hence, we have $|Y_i| < z < \xi$ and $[Y_i] \in \mathcal{B}'$. From the induction-hypothesis follows that X is z-closed and so $p(Y_i) \in [Y_i] \in \mathcal{B}'$. Since $\{p(Y_i) : i < \zeta\} \subseteq_z X$ we have $X \supseteq [\{p(Y_i) : : i < \zeta\}] \supseteq \bigcup_{i<\zeta} Y_i = Y$.

Thus X is ξ-closed in $\mathcal{A}'$, and this completes the induction. ∎

LEMMA 3. Let $\alpha, \beta > 1$ be two cardinals satisfying $\overleftarrow{\alpha^+} \leq \beta^+ \leq \alpha^+$. Then:

(i) there exists an $\mathcal{A}$ of characteristic $\langle 2, \overleftarrow{\alpha^+}, \alpha^+ \rangle$,
(ii) there exists an $\mathcal{A}$ of characteristic $\langle 2, \beta^+, \alpha^+ \rangle$,
(iii) there exists an $\mathcal{A}$ of characteristic $\langle \alpha^+, \overleftarrow{\alpha^+}, \alpha^+ \rangle$,
(iv) there exists an $\mathcal{A}$ of characteristic $\langle 2, 2, 2 \rangle$.

To prove lemma 3. we need four statements.

Let $|Z| = \alpha$ and $k \notin Z$. Now we construct three closed-set systems, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as follows:

$$\mathcal{C}_1 \overset{\text{def}}{=} \{Y \subseteq Z : |Y| < \alpha\} \cup \{Z \cup \{k\}, \{k\}\}.$$

$$\mathcal{A} \overset{\text{def}}{=} Sb\, Z\, [Sb\, Z], \text{ where } Sb\, Z \overset{\text{def}}{=} \{X : X \subseteq Z\}.$$

$$\mathcal{B} \overset{\text{def}}{=} Sb\, Z\, [\{Y \subseteq Z : |Y| \geq \beta\}].$$

$$\mathcal{C} \overset{\text{def}}{=} \mathcal{C}_1\, [\mathcal{C}_1].$$

STATEMENT 1: The conclusion of lemma 2. (ii) holds if we substitute $\mathcal{A}$ by $\mathcal{C}$ not only for $|C_1|^+$ but also for α^+.
That is

$$(\forall X \subseteq C)(\exists Y \subseteq_{\alpha^+} X)[Y] = [X].$$

PROOF. For an arbitrary $X \subseteq C$ we have two cases:

Case 1. $\{k, p(\{k\}), p(C)\}$ is not disjoint from X. If $X \subseteq \{k, p(\{k\}), p(C)\}$ then X is clearly generated by one element. If $X \nsubseteq \{k, p(\{k\}), p(C)\}$ then any $x \in X \setminus \{k, p(\{k\}), p(C)\}$ together with some element of $\{k, p(\{k\}), p(C)\}$ generates C, which is identical with $[X]$. $\alpha^+ > 2$ proves case 1.

Case 2. $\{k, p(\{k\}), p(C)\}$ is disjoint from X. The proof of lemma 2. (ii) may be applied since $\overline{X} \subseteq Z$, and $|Z| = \alpha$.

This completes the proof of St.1.

STATEMENT 2. $gen\,\mathcal{A} = gen\,\mathcal{B} = gen\,\mathcal{C} = \alpha^+$.

PROOF. a) By lemma 2. (ii) $\mathcal{A}$, $\mathcal{B}$, and by St.1. $\mathcal{C}$ are α^+-gen.

b) $(\forall Y \subseteq_\alpha Z)[Y] = Y'$ implies $(\forall Y \subseteq_\alpha Z)\, p(Z) \notin [Y]$. Thus from $p(Z) \in [Z]$ follows that neither $\mathcal{A}$ nor $\mathcal{B}$ are α-gen.

The proof is similar for $\mathcal{C}$ substituting $p(Z)$ by $p(C)$ everywhere in the proof.

STATEMENT 3. $com\,\mathcal{A} = com\,\mathcal{B} = 2$ and $com\,\mathcal{C} = \alpha^+$.

PROOF. $\mathcal{C}_1$ is not α-com since $\{k\} \subseteq \sup\{\{x\} : x \in Z\}$ but no subset of $\{\{x\} : x \in Z\}$ with less than α elements contains $\{k\}$.

$\mathcal{C}_1$ is α^+-com by St.1., and so the fact that $com\,\mathcal{C}_1 = \alpha^+$ and $com\,SbZ = 2$ imply St.3., by lemma 2. (i).

STATEMENT 4. $alg\,\mathcal{A} = alg\,\mathcal{C} = \overleftarrow{\alpha^+}$ and $alg\,\mathcal{B} = \beta^+$.

PROOF. a) We shall show that the conditions of lemma 2. (iii) are satisfied, for $\eta = \alpha^+$ and for $\delta = \overleftarrow{\alpha^+}$ in case $\mathcal{A}$ and $\mathcal{C}$ and for $\delta = \beta^+$ in case $\mathcal{B}$.

The statement is obvious for $\mathcal{A}$ and $\mathcal{C}$, since every closed set is an element of the change set. For $\mathcal{B}$, we have to show that for any $\beta^+ \leq \xi < \alpha^+$, $X \subseteq B$ and $|X| = \xi$ there exists a decomposition $X = \bigcup_{i<\xi} X_i$ for which $[X_i]$ is an element of the change set and the required cardinality conditions are also satisfied. Being ξ not regular, there exists a decomposition $X = \bigcup_{i<\xi} X_i$ satisfying the cardinality conditions.

Since $|X| = \xi > \beta$, every X_i can be extended to $X_i \subseteq Y_i \subseteq X$, $|Y_i| < \beta$ i.e., the decomposition $X = \bigcup_{i<\xi} Y_i$ satisfies the conditions of lemma 2. (iii) Thus by lemma 2. (iii) $\mathcal{A}$ and $\mathcal{C}$ are α^+-alg and $\mathcal{B}$ is β^+-alg (since all of them are α^+-alg by St.3.)

b) $\mathcal{B}$ is not β-alg since Z is β-closed but it is not a closed-ste of $\mathcal{B}$. $\mathcal{A}$ and $\mathcal{C}$ cannot be of less than α^+-alg because m-alg implies m-gen for each regular $m > 1$.

PROOF OF LEMMA 3. $\mathcal{A}$ satisfies (i), $\mathcal{B}$ satisfies (ii), $\mathcal{C}$ satisfies (iii) and $\{Z, \emptyset\}$ satisfies (iv).

PROOF OF THEOREM 4. Let $\langle \gamma, \beta, \alpha \rangle \leq \langle \gamma', \beta', \alpha' \rangle$ means $\gamma \leq \gamma'$, $\beta \leq \beta'$ and $\alpha \leq \alpha'$. We consider a triple $\langle \gamma, \beta, \alpha \rangle$ satisfying the conditions of the theorem. Let F be a choice-function mapping each triple $\langle \gamma_1, \beta_1, \alpha_1 \rangle \leq \langle \gamma, \beta, \alpha \rangle$ to a closed-set system as follows:

(i) $\chi F(\langle \gamma_1, \beta_1, \alpha_1 \rangle) = \langle \gamma_1, \beta_1, \alpha_1 \rangle$ if such a closed-set system does exist $\chi F(\langle \gamma_1, \beta_1, \alpha_1 \rangle) = \langle 2, 2, 2 \rangle$ otherwise.

(ii) the greatest elements of the values of F are mutually disjoint.

Such an F does exists.
We construct the needed closed-set system from the values of F:

$$\mathcal{A} \overset{\text{def}}{=} \bigcup^{+}_{\langle \gamma_1, \beta_1, \alpha_1 \rangle \leq \langle \gamma, \beta, \alpha \rangle} F(\langle \gamma_1 \beta_1 \alpha_1 \rangle)$$

Now, we prove that $\mathcal{A}$ is of characteristic $\langle \gamma, \beta, \alpha \rangle$: By lemma 1. the caracteristic of $\mathcal{A}$ is not greater than $\langle \gamma, \beta, \alpha \rangle$, and it cannot be smaller either because, for any $\alpha_1 < \alpha$, $\alpha \leq \beta_1 < \beta$, $\gamma_1 < \gamma$, we have:

$$\langle 2, \overleftarrow{\alpha_1^+}, \alpha_1^+ \rangle \leq \langle \gamma, \beta, \alpha \rangle$$
$$\langle 2, \beta_1^+, \alpha_1^+ \rangle \leq \langle \gamma, \beta, \alpha \rangle$$
$$\langle \gamma_1^+, \overleftarrow{\gamma_1^+}, \gamma_1^+ \rangle \leq \langle \gamma, \beta, \alpha \rangle.$$

To any triple $\langle \gamma', \beta', \alpha' \rangle$ on the left hand of each of these inequalities there exists, by lemma 3, a closed-set system with characteristic $\langle \gamma', \beta', \alpha' \rangle$. Thus by lemma 1. gen $\mathcal{A} \neq \alpha_1$, *alg* $\mathcal{A} \neq \beta_1$, *com* $\mathcal{A} \neq \gamma_1$. We have immediately *gen* $\mathcal{A} = \alpha$ and *com* $\mathcal{A} = \gamma$. Thus $\overleftarrow{\alpha} \leq$ *alg* $\mathcal{A} \leq \beta$ and so *alg* $\mathcal{A} = \beta$. ∎

By the theorem above we have a complete axiom system for the closed-set systems, which are not 1-gen w.r.t. *com*, *ind*, *alg*, *gen*. To give a similar complete description of *all* closed-set systems, we omit the auxiliary function *ind* and introduce a new, modified version of *com* and *gen*:

DEFINITION. A complete lattice is 1-com′ iff it is a 1-com Boolean-algebra. For $m > 1$, m-com′ coincides with m-com. No lattice is 0-com′. (Note, that we *had to* exclude 0 since, for any 1-alg closed-set system, there is a lattice-isomorphic 0-alg one.)

A closed-set system $\mathcal{A}$ is m-gen′ if for any X, $x \in [X] \setminus X$ implies $(\exists Y \subseteq_m X)\, x \in [Y]$.

THEOREM 5. *Theorems* 1, 2, 4 *remain valid if, we replace all the restrictions like* $m > 1$ *with* $m > 0$, *and substitute com′, gen′ for com and gen (and of course omit ind). Let us mention that in case* $m > 1$, *com′ and gen′ coincide with com and gen. Theorem 2. remains valid if we omit the restriction* $m > 0$ *as well.*

We omit here the proof to avoid lengthiness, but remark that the proof we have for this theorem concentrates on closure operations instead of closed-set systems, and avoids case distinctions just as well as the proofs of theorems 1–4. did. ∎

Instead of theorem 5. we could augment theorem 1–4., to obtain a complete description of all closed-set system, by stating that:

$\mathcal{A}$ is 1-gen iff $\mathcal{A} = \{A\}$

$\mathcal{A}$ is 0-gen iff $\mathcal{A} = \{\emptyset\}$

$\mathcal{A}$ is 1-alg iff $(\exists B)\mathcal{A} = \{X : B \subseteq X \subseteq A\}$

$\mathcal{A}$ is 0-alg iff $\mathcal{A} = SbA$

$\mathcal{A}$ is 1-ind iff $\mathcal{A}$ is 0-ind iff $\mathcal{A}$ is a complete closed-set system (see e.g. [3]).

iff $\mathcal{A}$ is 2-ind

$\langle \mathcal{A}, \subseteq \rangle$ is 1-com iff $\langle \mathcal{A}, \subseteq \rangle$ is 0-com iff $|\mathcal{A}| = 1$.

These follow from the definitions of *alg*, *gen* e.t.c. As a consequence, we have that, if the characteristic of $\mathcal{A}$ is smaller then or incomparable with $\langle 2, 2, 2 \rangle$ then it is $\langle 0, 0, 0 \rangle$, $\langle 0, 1, 1 \rangle$, $\langle 2, 0, 2 \rangle$ or $\langle 2, 1, 2 \rangle$.

Note, that *ind* behaves at 0 and 1 as if they were not regular: 0-ind $\not\to$ 0-gen, 1-ind $\not\to$ 1-gen and 0-ind = 1-ind = 2-ind.

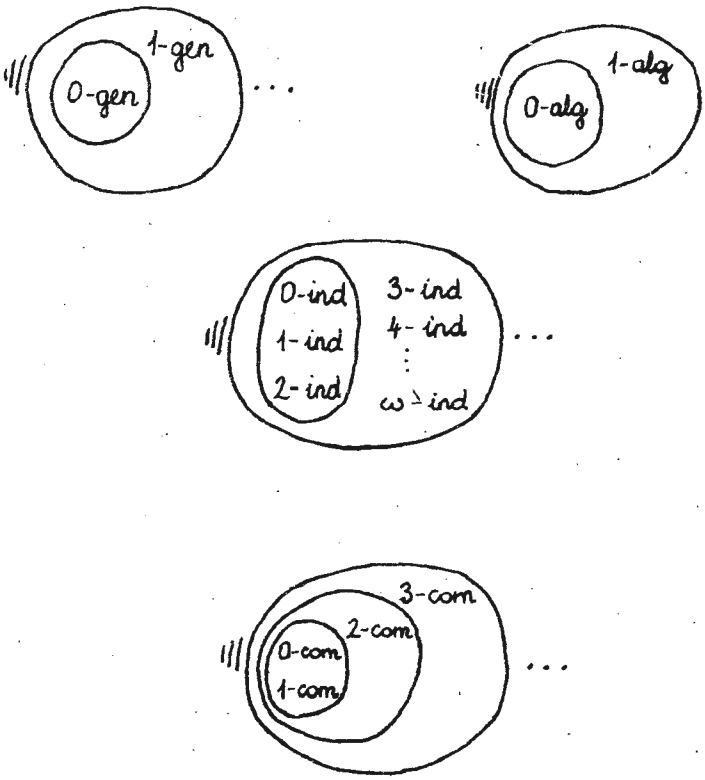

Fig. 5.

Summing up the consequences of theorem 4. and the above remarks:

(i) The range of the functions gen and alg is the class of all cardinals, the range of com is the same excluded the cardinal 1, and the range of ind is the class of the cardinals of the form $\vec{\alpha}$ except 1 and 2 (supposing we disregard the restriction in the definition of ind, that is, if we redefine *ind* $\mathcal{A}$ as the smallest cardinal α for which $\mathcal{A}$ is α-inductive).

We illustrate this in the following figure: (Fig. 5.)

(ii) By fixing m arbitrarily, the relations between the four properties are:

in case $m=0$ or m is not regular, (Fig. 6.)

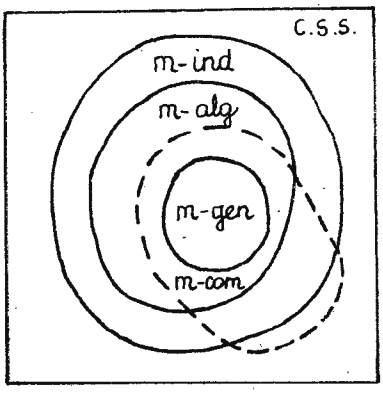

Fig. 6.

in case $m=1$, (Fig. 7.)

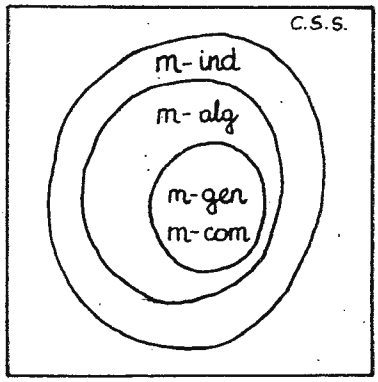

Fig. 7.

in case $1<m$ is regular, (Fig. 8.)

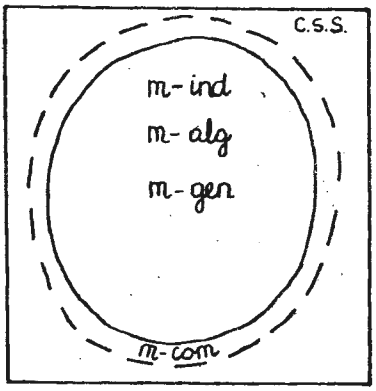

Fig. 8.

and for the modified concepts: m is regular (0-com is not defined), (Fig. 9.) m is not regular. (Fig. 10.)

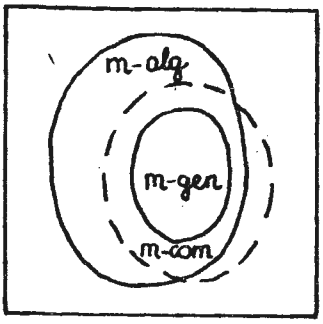

Fig. 9.

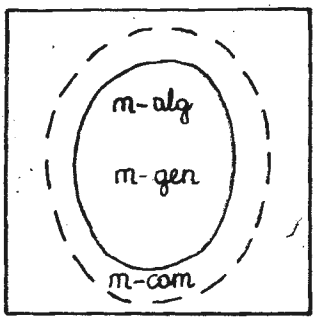

Fig. 10.

(iii) The lattice mirrors precisely the property *gen*, moreover the property *com* (based on the lattice) still mirrors *gen*; in the following sense: for any pair of cardinals $n<m \neq 1$:

$$(\exists m\text{-gen}\,\mathcal{A})(/\exists\, n\text{-gen}\,\mathcal{B})\langle\mathcal{A}, \subseteq\rangle \cong \langle\mathcal{B}, \subseteq\rangle$$

that is the lattice $\langle\mathcal{A}, \subseteq\rangle$ shows that $\mathcal{A}$ is not n-gen.

Note, that the restriction $m\neq 1$ can not be avoided by using *gen'*. The proof of lemma 2. can be used to derive that no lattice-property can precisely mirror the property *alg*:

THEOREM 6. *For all cardinals n and m satisfying $\overleftarrow{m} \leq n \leq m$ we have*:

$$(\forall\, m\text{-alg}\,\mathcal{A})(\exists\, n\text{-alg}\,\mathcal{B})\langle\mathcal{A}, \subseteq\rangle \cong \langle\mathcal{B}, \subseteq\rangle\,.$$

PROOF. By lemma 2. (i) and (iii) the theorem is immediate, since we can choose $\mathcal{A}[\mathcal{A}]$ for $\mathcal{B}$. ∎

Let us note, that theorem 4. also implies that the condition $\overleftarrow{m} \leq n \leq m$ cannot be omitted in theorem 6.

3. Applications to algebras

Now, we turn our attention to algebras and apply the results obtained so far to them. There will be a close correspondence between algebras and closed-set systems, and as a consequence of this an indirect correspondence between algebras and lattices. An algebra-property will correspond to m-alg, and another to m-gen.

We consider cardinals as special ordinals. The properties, defined below, have sense for any ordinal m, but, since we are interested in the connections between algebras and closed-set systems, we restrict ourselves to cardinals.

Let α be an ordinal and A be a set. If $f : {}^{\alpha}A \to A$, then we say that f is an *α-ary operation* on A.

$\mathfrak{A} = \langle A, F \rangle$ is an *algebra* if F is a family of operations on the set A. $S\,\mathfrak{A}$ denotes the set of closed sets of $\mathfrak{A}$, that is, the subets of A which are closed under the operations of $\mathfrak{A}$. Note, that $S\,\mathfrak{A}$ is a closed-set system. Since we have considered $\{\emptyset\}$ to be a closed-set system, we also consider $\langle \emptyset, F \rangle$ to be an algebra.

DEFINITION. For any cardinal m, we say that $\mathfrak{A}$ *is* m-ary if for any operation f of $\mathfrak{A}$, being α-ary implies $\alpha < m$. An algebra $\mathfrak{A}$ is *transitive* if to any triple $\langle p, X, y \rangle$ consisting of a polinomial-function p, a set $X \subseteq A$, and an element $y \in A$ such that p maps an argument chosen from the suitable power of X to y, there exists an operation f of $\mathfrak{A}$ for which the triple $\langle f, X, y \rangle$ has the same property.

We say, that $\mathfrak{A}$ is m-tra if it is m-ary and transitive.

We can define the modified versions of the above properties similarly as *gen'* was defined (for similar purposes):

$\mathfrak{A}$ is *transitive'* if for any triple consisting of a polinomial-function p, a set $X \subseteq A$ and an element $y \in A \setminus X$ such that ... the definition goes on precisely as above. $\mathfrak{A}$ is m-tra' if it is m-ary and transitive'.

Note that m-tra coincides with m-tra' if $m > 1$ just as it was the case with *gen'* and *com'*.

THEOREM 7. *For any cardinal* m:

(i)	$\mathfrak{A}$ *is* m-ary	*implies that* $S\,\mathfrak{A}$ *is* m-alg
	$\mathcal{A}$ *is* m-alg	*implies that* $(\exists\, m\text{-ary}\ \mathfrak{A})\mathcal{A} = S\,\mathfrak{A}$,
(ii)	$\mathfrak{A}$ *is* m-tra	*implies that* $S\,\mathfrak{A}$ *is* m-gen
	$\mathcal{A}$ *is* m-gen	*implies that* $(\exists\, m\text{-tra}\ \mathfrak{A})\mathcal{A} = S\,\mathfrak{A}$,
(iii)	$\mathfrak{A}$ *is* m-tra'	*implies that* $S\,\mathfrak{A}$ *is* m-gen
	$\mathcal{A}$ *is* m-gen'	*implies that* $(\exists\, m\text{-tra}'\ \mathfrak{A})\mathcal{A} = S\,\mathfrak{A}$.

PROOF. The first statements of (i) and (ii) are obvious.

Now, let $\mathcal{A}$ be m-alg. We construct to $\mathcal{A}$ an algebra $\mathfrak{A}_m$ on A: for all $\xi < m$, and $c \in A$, f_c^ξ is a ξ-ary operation of $\mathfrak{A}_m$ defined as:

$$f_c^\xi(\langle y_0, \ldots, y_\zeta, \ldots \rangle_{\zeta<\xi}) \overset{\text{def}}{=} \begin{cases} c \text{ if } c \in [\{y_i : i < \xi\}] \\ y_0 \text{ otherwise} \end{cases}$$

$\{f_c^\xi : \xi < m, c \in A\}$ are all the operations of $\mathfrak{A}_m$.

Clearly, $\mathfrak{A}_m$ is m-ary. It is easy to see, that

$$(\forall Y \subseteq_m A)[Y]_{\mathcal{A}} = [Y]_{S\mathfrak{A}_m}.$$

This immediately gives that

if $\mathcal{A}$ is m-alg, then $\mathcal{A} = S\,\mathfrak{A}_m$, and
if $\mathcal{A}$ is m-gen, then in addition $\mathfrak{A}_m$ is transitive, and so it is m-tra.
if $\mathcal{A}$ is m-gen', then in addition $\mathfrak{A}_m$ is transitive'.

Now we shall see that, if we forget about the operations of the algebra and we have only the closed-set system, from the properties of this latter we can reconstruct certain non-trivial properties of the operations. (Note, that we have not succeeded to do so in the closed-set system – lattice relation.) Theorem 1. and 7. are representation theorems but our next observation will be more concrete.

By using theorem 7. the only information we can obtain from an algebra by investigating its closed-set system is the following:

If $S\,\mathfrak{A}$ is m-alg, then there exists an m-ary algebra $\mathfrak{B}$, such that $S\,\mathfrak{A} = S\,\mathfrak{B}$. Now, we state more, namely, that we can construct the algebra $\mathfrak{B}$ from the algebra $\mathfrak{A}$ in a rather simple manner by taking the ξ-ary polinomials with $\xi < m$ as operations and nothing else. This algebra will be denoted by $\mathcal{P}l_{\mathfrak{A}}(m)$. This is not only an m-ary algebra having $S\,\mathfrak{A}$ as closed-sets, but this algebra is also transitive whenever $S\,\mathfrak{A}$ is m-gen.

THEOREM 8.

(i) $S\,\mathfrak{A}$ is m-alg iff $S\,\mathfrak{A} = S\,\mathcal{P}l_{\mathfrak{A}}(m)$
(ii) $S\,\mathfrak{A}$ is m-gen iff $\mathcal{P}l_{\mathfrak{A}}(m)$ is m-tra and $S\,\mathfrak{A} = S\,\mathcal{P}l_{\mathfrak{A}}(m)$

PROOF. (i) the only non-trivial part is that $S\,\mathfrak{A}$ being m-alg implies that $S\,\mathfrak{A} \supseteq S\,\mathcal{P}l_{\mathfrak{A}}(m)$. $X \notin S\,\mathfrak{A}$ implies $(\exists Y \subseteq_m X)\,[Y] \nsubseteq X$ and so there is a polinomial with arguments in Y and value not in X. This further implies the existence of a $|Y|$-ary polinomial with the same property, and so $X \notin S\,\mathcal{P}l_{\mathfrak{A}}(m)$.

(ii) It is enough to show that, if $S\,\mathfrak{A}$ is m-gen, then $\mathcal{P}l_{\mathfrak{A}}(m)$ is transitive. Let $S\,\mathfrak{A}$ be m-gen, and $x \in [X]$ arbitrary.

By, the definition of m-gen there is a $Y \subseteq_m X$ such that $x \in [Y]$, and so there is a polinomial which can be given arguments from Y such that it yields x as value. Therefore there is also a $|Y|$-ary polinomial with the same property, and so $|Y| < m$ shows that $\mathcal{P}l_{\mathfrak{A}}(m)$ is transitive.

Let us mention that the theorem holds for *tra'* and *gen'* as well.

Note, that the 1-ary algebras are exactly those which have only constants as operations, 1-tra algebras are in addition minimal, the 0-ary algebras are the algebras without operations and the 0-tra algebra is $\langle \emptyset, \emptyset \rangle$. Of course, the 1-tra' and 0-tra' algebras coincide with the 1-ary and 0-ary ones, respectively.

4. Summing-up

We can draw-up in a diagram the main correspondences between the properties of algebras, closed-set systems and lattices studied in this paper.

In the following diagram the implication symbol $\overset{\exists}{\Leftarrow}$ means representability in the sense of theorems 1 or 7. For any cardinal m, the following diagram holds: (Fig. 11.)

$$\begin{array}{ccccc} m\text{-}ary & \overset{\exists}{\Leftrightarrow} & m\text{-}alg & & \\ & & \Uparrow \Downarrow & \text{if } 1 < m \text{ is regular} & \\ m\text{-}tra & \overset{\exists}{\Leftrightarrow} & m\text{-}gen & \overset{\exists}{\Leftrightarrow} & m\text{-}com \end{array}$$

Fig. 11.

Additional correspondences:

(i) between algebras and closed-set systems (theorem 8)

(ii) an independence result for the closed-set system and lattice properties (theorem 4)

The concepts *alg*, *gen*, etc. behave irregularly at 0 and 1. These irregularities were avoided by the modified concepts *alg'*, *gen'*, etc. It remains an open problem to find a more natural definition for com'.

Here we list some of the avoided irregularities:

the restriction in the above diagram $1 < m$;

the similar restriction in theorem 4 $\alpha, \beta, \gamma > 1$;

m-alg $\not\to$ m-gen for the regular cardinals 0 and 1; com does not distinguish 1-ary and 2-ary though this distinction can be lattice represented (we do not care about the distinction between 0-ary and 1-ary since this cannot be lattice represented).

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ON INTERPOLATION ON THE ROOTS OF JACOBI POLYNOMIALS

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1. Consider the interpolating polynomial

$$L_n^{(\alpha,\beta)}(f,x) = \sum_{k=1}^{n} f(x_k)\, l_k^2(x) \quad (\alpha,\beta > -1)$$

of degree $2n-2$ of $f(x)$, where $x_k = x_{kn}^{(\alpha,\beta)}$ $(k = 1, \ldots, n)$ are the zeros of the Jacobi polinomial $P_n^{(\alpha,\beta)}(x)$ which form an orthogonal system of polynomials on $[-1, +1]$ with respect to the weight function

$$w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta \quad (-1 < x < 1;\ \alpha,\beta > -1),$$

$$l_k(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_k)(x-x_k)},$$

and the normalization is such that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

We shall prove the following

THEOREM. *Let $f(x)$ be a continuous function in $[-1, +1]$ then*

$$(1-x)^{\alpha+\frac{3}{2}} \cdot (1+x)^{\beta+\frac{3}{2}} \left| f(x) - L_n^{(\alpha,\beta)}(f,x) \right| \leq$$

$$\leq C\left\{\omega\left(f; \frac{\log n}{n}\right) + \|f\|\, n^a \log n\right\} \quad (-1 \leq x \leq 1)$$

where

$$a = \begin{cases} -1 & \text{if } q = \min(\alpha, \beta) \geq -\dfrac{1}{2} \\ -2q-2 & \text{if } q = \min(\alpha, \beta) < -\dfrac{1}{2} \end{cases}$$

and $\|f\| = \max\limits_{-1 \leq x \leq 1} |f(x)|$, *further* C *depends only on* α *and* β.

COROLLARY 1. If $f(x)$ continuous in $[-1, +1]$ then $L_n^{(\alpha,\beta)}(f, x)$ tends to $f(x)$ in every point x $(-1<x<1)$ and uniformly in $[-1+\varepsilon, +1-\varepsilon]$ $(\varepsilon>0)$.

COROLLARY 2.

$$\lim_{n\to\infty} \sum_{k=1}^{n} l_k^2(x) = 1 \quad (-1 < x < 1;\ \alpha, \beta > -1). \tag{1}$$

The convergence of $\{L_n^{(\alpha,\alpha)}(f, x)\}$ $(\alpha \geq 0)$ to $f(x)$ has been studied by J. BALÁZS [1]. The convergence of $L_n^{(\alpha,\beta)}(f, x)$ $(-1<\alpha, \beta<0)$ to $f(x)$ follows from a general theorem of G. GRÜNWALD [2] because, as it is well known, the roots of $P_n^{(\alpha,\beta)}(x)$ $(-1<\alpha, \beta<0)$ form a normal point system on $[-1, +1]$.

2. We need some lemmas.

LEMMA 1.

$$H(x) \overset{\text{def}}{=} (1-x)^{\alpha+\frac{3}{2}} \cdot (1+x)^{\beta+\frac{3}{2}} \left| 1 - \sum_{k=1}^{n} l_k^2(x) \right| \leq \tag{2}$$

$$\leq \begin{cases} c_1 n^{-1} \log n & \text{if } q = \min(\alpha, \beta) \geq -\dfrac{1}{2} \\ c_2 n^{-2q-2} \log n & \text{if } q = \min(\alpha, \beta) < -\dfrac{1}{2} \end{cases} \quad (-1 \leq x \leq 1;\ \alpha, \beta > -1),$$

where c_1, c_2 *(and later on* $c_3 \ldots$*) are positive constants depending only on* α *and* β.

PROOF. We need some relations for $P_n^{(\beta,\alpha)}(x)$.

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \quad ([3], (4.1.3)), \tag{3}$$

$$\max_{0 \leq x \leq 1} |P_n^{(\alpha,\beta)}(x)| \sim n^{-\frac{1}{2}} \left(\alpha < -\frac{1}{2}\right) \quad ([3], \text{pp. } 167.), \tag{4}$$

$$\lim_{n\to\infty} M_n n^{\frac{1}{2}} = \text{const.} \left(\alpha \geq -\frac{1}{2}\right) \quad ([3], (7.32.9)), \tag{5}$$

where

$$M_n \overset{\text{def}}{=} \max_{0 \leq \Theta < \frac{\pi}{2}} \left| \left(\sin \frac{\Theta}{2}\right)^{\alpha+\frac{1}{2}} \cdot \left(\cos \frac{\Theta}{2}\right)^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos \Theta) \right|$$

(6) $$\Theta_k = \frac{1}{n}(k\pi + O(1)) \quad ([3],(8.9.1)),$$

where $0 < \Theta_1 < \ldots < \Theta_n < \pi$ are the zeros of $P_n^{(\alpha,\beta)}(\cos \Theta)$ $(\alpha, \beta > -1)$ and $O(1)$ depends only on α and β.

(7) $$\left\{\frac{d}{dx} P_n^{(\alpha,\beta)}(x)\right\}_{x=\cos\Theta_k} \sim k^{-\alpha-\frac{3}{2}} \cdot n^{\alpha+2} \left(0 < \Theta_k \leq \frac{\pi}{2}\right) \quad ([3],(8.9.2)),$$

where $\alpha, \beta > -1$, depends only on α and β.

(8) $$\left\{\frac{d}{dx} P_n^{(\alpha,\beta)}(x)\right\}_{x=\cos\Theta} =$$

$$= \begin{cases} \Theta^{-\alpha-\frac{3}{2}} \cdot O\left(n^{\frac{1}{2}}\right) & \text{if } c\,n^{-1} \leq \Theta \leq \frac{\pi}{2} \\ O(n^{2+\alpha}) & \text{if } 0 \leq \Theta \leq c\,n^{-1} \end{cases} \quad (\alpha, \beta > -1), \ ([3],(7.32.10)).$$

It is well known that

(9) $$\sum_{k=1}^{n} h_k(x) = \sum_{k=1}^{n} \left(1 - \frac{P_n^{(\alpha,\beta)''}(x_k)}{P_n^{(\alpha,\beta)'}(x_k)}(x - x_k)\right) l_k^2(x) =$$

$$= \sum_{k=1}^{n} \left(1 - \frac{(\alpha+\beta+2)x_k + \alpha - \beta}{1 - x_k^2}(x - x_k)\right) l_k^2(x) \equiv 1.$$

From (2) and (9) by $x = \cos \Theta$ we get

(10) $$|H(\cos\Theta)| \leq c_3 (1 - \cos\Theta)^{\alpha+\frac{3}{2}} \cdot (1 + \cos\Theta)^{\beta+\frac{3}{2}} \cdot$$

$$\cdot \sum_{k+1}^{n} \frac{|\cos\Theta - \cos\Theta_k|}{\sin^2\Theta_k} l_k^2(\cos\Theta) = \sum_{|\Theta-\Theta_k| \leq n^{-1}}{}_k + \sum_{|\Theta-\Theta_k| > n^{-1}}{}_k = S_1 + S_2.$$

We consider S_1 in the case of

$$0 \leq \Theta \leq \frac{2}{n}, \quad \frac{2}{n} < \Theta \leq \frac{\pi}{2}, \quad \frac{\pi}{2} < \Theta \leq \pi - \frac{2}{n}, \quad \pi - \frac{2}{n} < \Theta \leq \pi$$

respectively but we shall deal with the first and second case only, because mean value the others are similar. By the Lagrange mean value theorem we get

$$S_1 = c_3 (1-\cos\Theta)^{\alpha+\frac{3}{2}} \cdot (1+\cos\Theta)^{\beta+\frac{3}{2}} \cdot \sum_{\substack{k \\ |\Theta-\Theta_k|\leq n^{-1}}} \frac{|\cos\Theta-\cos\Theta_k|}{\sin^2\Theta_k} \cdot$$

$$\cdot \left\{ \frac{P_n^{(\alpha,\beta)}(\cos\Theta) - P_n^{(\alpha,\beta)}(\cos\Theta_k)}{\cos\Theta-\cos\Theta_k} \right\} \{P_n^{(\alpha,\beta)'}(\cos\Theta_k)\}^{-2} =$$

$$= c_3 (1-\cos\Theta)^{\alpha+\frac{3}{2}} \cdot (1+\cos\Theta)^{\beta+\frac{3}{2}} \sum_{\substack{k \\ |\Theta-\Theta_k|\leq n^{-1}}} \frac{|\cos\Theta-\cos\Theta_k|}{\sin^2\Theta_k} \cdot$$

$$\cdot \{P_n^{(\alpha,\beta)'}(\cos\tau_k)\}^2 \{P_n^{(\alpha,\beta)'}(\cos\Theta_k)\}^{-2}, \tag{11}$$

where τ_k is between Θ and Θ_k.

From (11), (3), (6), (7), (8) follows in the case of $0 \leq \Theta \leq \frac{2}{n}$

$$S_1 \leq c_4 \left(\frac{1}{n}\right)^{2\alpha+3} \cdot n^{-1} \cdot n^{2\alpha+4} \sum_{\substack{k \\ |\Theta-\Theta_k|\leq n^{-1}}} \frac{1}{\sin^2\Theta_k} \{P_n^{(\alpha,\beta)'}(\cos\Theta_k)\}^{-2} \leq$$

$$\leq \begin{cases} c_5 n^{-1} & \text{if } q = \min(\alpha,\beta) \geq -\frac{1}{2} \\ c_6 n^{-2\alpha-2} & \text{if } q = \min(\alpha,\beta) < -\frac{1}{2} \end{cases}, \tag{12}$$

in the case of $\frac{2}{n} < \Theta \leq \frac{\pi}{2}$

$$S_1 \leq c_7 \Theta^{2\alpha+3} \cdot \left(\Theta - \frac{1}{n}\right)^{-2\alpha-3} \cdot n^{-1} \cdot n \sum_{\substack{k \\ |\Theta-\Theta_k|\leq n^{-1}}} \frac{1}{\sin^2\Theta_k} \{P_n^{(\alpha,\beta)'}(\cos\Theta_k)\}^{-2} \leq$$

$$\leq \begin{cases} c_8 n^{-1} & \text{if } q = \min(\alpha,\beta) \leq -\frac{1}{2} \\ c_9 n^{-2q-2} & \text{if } q = \min(\alpha,\beta) < -\frac{1}{2} \end{cases}. \tag{13}$$

We have used that the sum contains finitely many member only. From (10), (3), (4), (5), (6), (7) and by $1-\cos\Theta = 2\sin^2\frac{\Theta}{2}$, $1+\cos\Theta = 2\cos^2\frac{\Theta}{2}$ we have

$$S_2 \leq c_{10}\left(\sin\frac{\Theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\Theta}{2}\right)^{2\beta+1}\{P_n^{(\alpha,\beta)}(\cos\Theta)\}^2\cdot$$

$$\cdot\sum_{|\Theta-\Theta_k|>n^{-1}}{}_k\left\{\frac{n^2}{k^2}(k^{2\alpha+3}n^{-2\alpha-4}k^{2\beta+3}n^{-2\beta-4})\right\}\leq$$

$$\leq\begin{cases}c_{11}n^{-1}\log n \text{ if } q=\min(\alpha,\beta)\geq-\frac{1}{2}\\ c_{12}n^{-2q-2}\log n \text{ if } q=\min(\alpha,\beta)<-\frac{1}{2}\end{cases}. \tag{14}$$

From (10), (12), (13), (14) follows (2).

These methods have been used in [3] in estimating the Lebesgue constant of Lagrange interpolation for Jacobi abscissas.

LEMMA 2.

$$(1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\sum_{k=1}^{n}|x-x_k|\,l_k^2(x)\leq c_{13}n^{-1}\log n$$

$$(-1\leq x\leq 1;\ \alpha,\beta>-1;\ n=1,2\ldots). \tag{15}$$

PROOF. Similar to the proof of (1) so we omit it.

3. Now we shall prove our theorem. Evidently

$$(1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\left|f(x)-L_n^{(\alpha,\beta)}(f,x)\right|\equiv \tag{16}$$

$$\equiv(1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\left|\sum_{k=1}^{n}(f(x)-f(x_k))\,l_k^2(x)+f(x)\left(1-\sum_{k=1}^{n}l_k^2(x)\right)\right|\leq$$

$$\leq(1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\left\{\sum_{k=1}^{n}|f(x)-f(x_k)|\,l_k^2(x)+|f(x)|\cdot\left|1-\sum_{k=1}^{n}l_k^2(x)\right|\right\}.$$

Using the well known relation $\omega(f,\lambda\delta)\leq(\lambda+1)\omega(f,\delta)$ for the continuity modulus of $f(x)$ on $[-1,+1]$ we obtain

$$|f(x)-f(x_k)|\leq\omega\left(f;\left\{\frac{\log n}{n}\right\}\left\{\frac{n}{\log n}|x-x_k|\right\}\right)\leq$$

$$\leq\omega\left(f;\frac{\log n}{n}\right)\cdot\left(\frac{n}{\log n}|x-x_k|+1\right), \tag{17}$$

so we get

$$(18)\qquad (1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\sum_{k=1}^{n}|f(x)-f(x_k)|\,l_k^2(x)\,\omega \leq \left(f;\frac{\log n}{n}\right)\cdot$$

$$\cdot\left\{\frac{n}{\log n}(1-x)^{\alpha+\frac{3}{2}}(1+x)^{\beta+\frac{3}{2}}\sum_{k=1}^{n}|x-x_k|\,l_k^2(x)+(1-x)^{\alpha+\frac{3}{2}}(1+x)^{\beta+\frac{3}{2}}\sum_{k=1}^{n}l_k^2(x)\right\}.$$

We have from (16), (2), (15), (18)

$$(1-x)^{\alpha+\frac{3}{2}}\cdot(1+x)^{\beta+\frac{3}{2}}\left|f(x)-L_n^{(\alpha,\beta)}(f,x)\right| \leq$$

$$\leq c_{14}\left\{\omega\left(f;\frac{\log n}{n}\right)+\|f\|\cdot n^a\log^2 n\right\}$$

where

$$a=\begin{cases}-1 & \text{if } q=\min(\alpha,\beta)\geq -\frac{1}{2}\\ -2q-2 & \text{if } q=\min(\alpha,\beta)<-\frac{1}{2}\end{cases},$$

from which follows our theorem.

4. Finally I should like to state the following.

PROPLEMS.

1. Characterize the weight functions for which (1) is fulfilled.

2. Another open question is if there exists a weight function in $[-1, 1]$ for which (1) is fulfilled uniformly in $[-1, +1]$ but the fundamental points do not form normal point system.

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О СУММЕ СТЕПЕНИ РАССТОЯНИЙ

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Известно, что из всех выпуклых n-угольников, лежащих в данном круге, наибольшие периметр и площадь имеет правильный n-угольник, вписанный в данную окружность, далее, из всех выпуклых n-угольников, заключающих в себе данную окружность, наименьшие периметр и площадь имеет правильный n-угольник, описанный вокруг окружности [1]. Л. ФЕЙЕШ ТОТ [2] занимался вопросом, когда будет максимальна на эвклидавой плоскости сумма расстояний между точками $P_1, P_2, \ldots, P_n$, лежащими в данном круге и когда будет минимальна сумма возвратной величины расстояний между точками $P_1, P_2, \ldots, P_n$, лежащими на данной окружности. Он доказал, что максимум, или минимум даются вершинами правильных n-угольников, вписанных в окружности.

В этой работе мы понимаем под поверхностью постоянной кривизны эвклидаву и гиперболическую плоскости, или единичную сферу. Мы обобщаем предыдущие результаты. Мы докажем следующие теоремы:

ТЕОРЕМА 1. *Пусть $\nu \leqq 1$ и $\nu \neq 0$ данное вещественное число. На поверхности постоянной кривизны для суммы ν-степени расстояний между n разными точкамы, лежащими на данной окружности, вершины правильного n-угольника, вписанного в данную окружность, дают максимум, если $0 < \nu \leqq 1$ и минимум, если $\nu < 0$.*

ТЕОРЕМА 2. *На поверхности постоянной кривизны сумма ν-степени сторон выпуклых n-угольников, лежащих в данном круге, у правильного n-угольника, вписанного в окружность, будет максимальна, если $0 < \nu \leqq 1$ и минимальна, если $\nu < 0$.*

ТЕОРЕМА 3. *На поверхности постоянной кривизны для суммы ν-степени сторон выпуклых n-угольников, заключающих в себе данную окружность, правильный n-угольник, описанный вокруг окружности дают минимум, если $\nu \geqq 1$.*

Для доказательства теорем нам нужно следующие известные леммы:

ЛЕММА 1. Пусть C — точки деления пополам одной из двух дуг, ограниченных данными точками A и B на одной окружности поверхности постоянной кривизны, а P — произвольная точка дуги, ограниченной точками B и C и не содержащей точку A (рис. 1.). Если P двигается к точке C на дуге, тогда сумма $AP+BP$ строго возрашается.

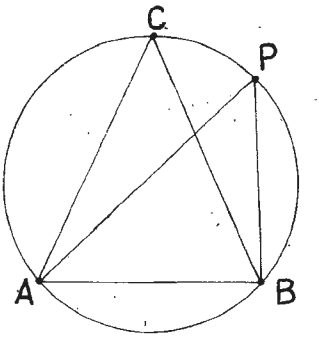

Рис. 1.

(Доказательство см. в [3].)

ЛЕММА 2. Пусть p и q — две такие касательные данной окружности $к$ и на них P и Q — точки касания на поверхности постоянной кривизны, что существует к окуржности k касательная s, пересекающаяся касательные p и q в одной из двух полуплоскостей, ограниченных прямой PQ. Пусть P_1 и Q_1 — точки пересечения касательной s с касательными p и q (рис. 2.). Сумма $PP_1+P_1Q+Q_1Q$ минимальна, если касательная s перпендикулярна к биссектрисе угла $POQ\sphericalangle$, где O — центр окружности.

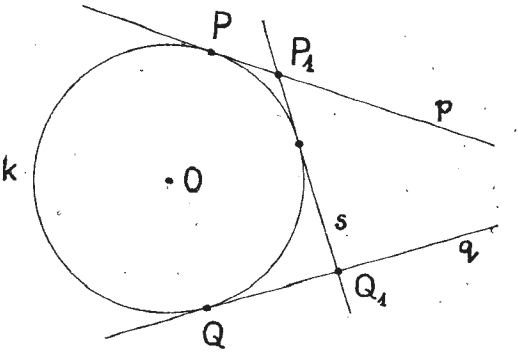

Рис. 2.

(Доказательство Леммы 2 подобно к доказательству леммы из [1], если вместо эвклидовой плоскости мы рассматриваем произвольную поверхность постоянной кривизны.)

ЛЕММА 3. Степенным средним порядка ν данных n положительных чисел $a_1, a_2, \ldots, a_n$ называется число

$$S(\nu) = \left(\frac{\sum_{i=1}^{n} a_i^{\nu}}{n} \right)^{\frac{1}{\nu}}, \tag{1}$$

где $\nu \neq 0$ вещественное число. Функция $S(\nu)$ строго возрашающая.

(Доказательство см. в [4].)

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 1. Пусть $P_1, P_2, \ldots, P_n$ — точки на данной окружности в циклическом порядке, далее

$$S_n^{\nu} = \sum_{\substack{i,j=1 \\ i<j}}^{n} P_i P_j^{\nu} \tag{2}$$

и

$$s_k^{\nu} = \sum_{i=1}^{n} P_i P_{i+k}^{\nu}, \quad (P_{n+j} = P_j) \tag{3}$$

где $k<n$ натуральное число. Из леммы 1. следует, что s_k^1 выпуклая функция от углов $P_i O P_{i+k}$ $(i = 1, 2, \ldots, n)$. (s_k^1 состоит из одного, или больше многоугольников в зависимости от того, делится n на k, или нет.) Поэтому

$$s_k^1 \leqq \sum_{i=1}^{n} Q_i Q_{i+k} = n q_k, \quad (Q_{m+j} = Q_j) \tag{4}$$

где точки $Q_1, Q_2, \ldots, Q_n$ — вершины правильного n-угольника, вписанного в окружность.

Из (4) и леммы 3. следует, что

$$\left(\frac{s_k^{\nu}}{n} \right)^{\frac{1}{\nu}} \leqq \frac{s_k^1}{n} \leqq q_k. \tag{5}$$

Так в случае $0<\nu\leqq 1$

$$s_k^{\nu} \leqq n q_k^{\nu}, \tag{6}$$

и в случае $\nu<0$

$$s_k^{\nu} \geqq n q_k^{\nu}. \tag{7}$$

Так как $s_k^\nu = s_{n-k}^\nu$, то

$$2 \cdot S_n^\nu = \sum_{k=1}^{n-1} s_k^\nu .$$

Из (6) в случае $0 < \nu \leqq 1$ следует, что

$$S_n^\nu \leqq n \sum_{k=1}^{n-1} q_k^\nu , \tag{8}$$

и из (7) в случае $\nu < 0$

$$S_n^\nu \geqq n \sum_{k=1}^{n-1} q_k^\nu . \tag{9}$$

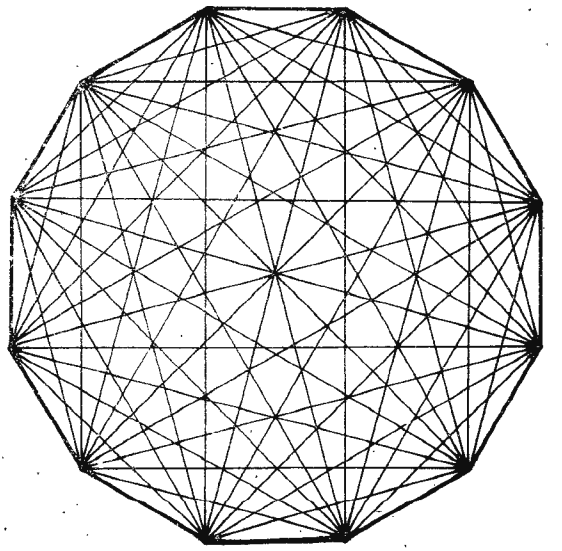

Рис. 3.

В (4), поэтому в (5), (6), (7), (8) и (9) имеем равенство тогда и только тогда, когда точки $P_1, P_2, \ldots, P_n$ вершины правильного n-угольника, вписанного в окружность.

Таким образом теорема 1. доказана.

ЗАМЕЧАНИЕ. Если $\nu = 1$, то можно предположить, что точки $P_1, P_2, \ldots, P_n$ лежат в данном круге, но максимум функции S_n^1 дается, если точки $P_1, P_2, \ldots, P_n$ вершины правильного n-угольника, вписанного в данную окружность.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 2. В случае $\nu = 1$ теорема 2. следует из леммы 1.

Пусть $p_1, p_2, \ldots, p_n$ — стороны выпуклого n-угольника, лежащего в данном круге. По лемме 3.

$$\left(\frac{\sum_{i=1}^{n} p_i^{\nu}}{n}\right)^{\frac{1}{\nu}} \leqq \frac{\sum_{i=1}^{n} p_i}{n} \leqq q, \tag{10}$$

где $\nu \leqq 1$ $(\nu \neq 0)$ вещественное число и q — длина стороны правильного n-угольника, вписанного в окружность. Из этого следует, что в случае $0 < \nu \leqq 1$

$$\sum_{\nu=1}^{n} p_i^{\nu} \leqq n q^{\nu}, \tag{11}$$

и в случае $\nu < 0$

$$\sum_{\nu=1}^{n} p_i^{\nu} \geqq n q^{\nu}. \tag{12}$$

В (10), (11), и (12) имеем равенство в том и только в том случае, если $p_1 = p_2 = \ldots = p_n = q$, т. е. n-угольник правилен.

Теорема 2 доказана.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 3. мы не будем вдаваться в подробности. Исходя из леммы 2., оно вполне согласуется с доказательством теоремы 2.

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ÜBER EBENE HYPERBOLISCHE MOSAIKE

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In dieser Arbeit beschäftigen wir uns mit der Frage, wie ein vollständiger Überblick sämtlicher aus kongruenten Dreiecken bestehenden hyperbolischen Mosaike gegeben werden kann. Sodann geben wir einen vollständigen Überblick aller aus kongruenten nichtregulären gleichwinkligen Vierecken bestehenden Mosaike der hyperbolischen Ebene. Die benachbarten Elemente der in unseren Untersuchungen vorkommenden Mosaike verknüpfen sich längs ihrer kongruenten Seiten.[1]

1. Wir geben einen vollständigen Überblick nur solcher aus kongruenten Dreiecken bestehenden hyperbolischen Mosaike, die jede Kongruenzabbildung, die irgend zwei Elemente eines Mosaiks aufeinander abbildet, in sich selbst überführt. Solche Mosaike heißen homogen.

Ist ein Dreieckmosaik homogen, so können die zu irgendwelchem Element gehörigen, mit ihm je eine gemeinsame Seite besitzenden benachbarten Elemente nur nach den folgenden vier Möglichkeiten abgeleitet werden:

1. Man spiegelt das Elementardreieck an die Seiten.
2. Man spiegelt das Elementardreieck an die Seitenmittelpunkte.
3. Man spiegelt das Elementardreieck an einen Seitenmittelpunkt und an die anderen beiden Seiten.
4. Man spiegelt das Elementardreieck an eine Seite und an die Mittelpunkte der anderen beiden Seiten.

Wir betrachten die Seiten der Mosaikelemente als voneinander verschieden, im Falle, wenn sie gleich sind, zweks ihrer Unterscheidung bezeichnet man sie mit verschiedenen Buchstaben.

Die notwendige Annahme ist für die Existenz solcher Dreieckmosaike, daß man um ein Dreieck ihm kongruente Dreiecke – mit der Berücksichti-

[1] Herr Professor L. Fejes Tóth hat mich – in seiner Beurteilung über meine Dissertation – an die Untersuchung der hyperbolischen Dreieckmosaike angeregt.

gung auf 1.—4. — lückenlos fügen könne. Man erkennt leicht, daß diese notwendigen Annahmen entsprechend den vier obenerwänten Fällen auf folgende Weise versammelt werden können:

Seien a, b, c die Seiten und α, β, γ die entsprechenden gegenüberliegenden Winkel eines Dreieckes, das zur Konstruktion eines homogenen Mosaiks zugrunde liegt.

(i) Wenn wir das Mosaik aus einem vorgegebenen Dreieck a, b, c durch Spiegelungen an die Seiten konstruieren wollen, so müssen $\alpha = \frac{\pi}{k}, \beta = \frac{\pi}{l}, \gamma = \frac{\pi}{m}$ ($k, l, m \geqq 2$ ganze Zahl) sein und natürlich besteht die Ungleichung

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1 .$$

(ii) Wenn wir das Mosaik aus einem vorgegebenen Dreieck a, b, c durch Spiegelungen an die Seitenmittelpunkte konstruieren wollen, so muß die folgende Gleichung (für $s > 2$ ganze Zahl) bestehen:

$$\alpha + \beta + \gamma = \frac{2\pi}{s} .$$

(iii) Wenn wir das Mosaik aus einem vorgegebenen Dreieck a, b, c durch Spiegelungen an die Mittelpunkte der a-Seiten und an die b- bzw. c-Seiten konstruieren wollen, so müssen $\alpha = \frac{2\pi}{n}, \beta + \gamma = \frac{\pi}{k}$ ($n \geqq 3$, $k \geqq 2$ ganze Zahl) und $2k + n < nk$ sein.

(iv) Wenn wir das Mosaik aus einem vorgegebenen Dreieck a, b, c durch Spiegelungen an die a-Seiten und an die Mittelpunkte der b- bzw. c-Seiten konstruieren wollen, so muß die folgende Gleichung (für $k > 1$ ganze Zahl) bestehen:

$$\alpha + \beta + \gamma = \frac{\pi}{k} .$$

Wir zeigen, daß die in (i)—(iv) gegebenen Annahmen zur Konstruktion der homogenen Dreieckmosaike auch hinreichend sind.

Nennt man im folgenden zwei Dreiecke bei den obenerwähnten Fügungen in einem allgemeineren Sinne benachbart, falls sie mindestens einen Randpunkt (also entweder einen Scheitel oder eine Seite) gemein haben.

Betrachten wir ein vorgegebenes Dreieck (als Anfangsdreieck bei dem Bau eines Mosaiks) und die ihm benachbarten Dreiecke. Diese Dreiecke haben freie Seiten, die nicht zu zwei Dreiecken gehören. Die freie Seiten bilden einen abgeschlossenen Streckenzug (also ein Vieleck), der das vor-

gegebene Dreieck im inneren hat. Wir sagen, das diese Dreiecke einen Gürtel (später wird als erster Gürtel genannt) um das vorgegebene Dreieck bilden (Fig. 1 im Falle (i)). Man kann aus einem beliebigen Randpunkt des vorgegebenen Dreieckes zu der Gürtelbegrenzung eine Verbindunsstrecke ziehen, die entweder im inneren eines einzigen, im Gürtel liegenden Dreieckes oder auf dem gemeinsamen Rand zweier Dreiecke vom Gürtel geht.

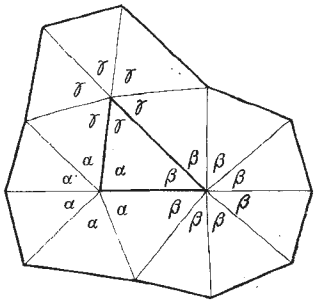

Fig. 1.

Betrachten wir nun alle Dreiecke, die den im ersten Gürtel liegenden Dreiecken benachbart sind und im äußeren den ersten Gürtelbegrenzung liegen. Diese Dreiecke bilden den zweiten Gürtel und die freien Seiten von ihnen begrenzen den Gürtel durch einen abgeschlossenen Streckenzug. Man kann aus einem beliebigen Punkt der ersten Gürtelbegrenzung zur zweiten Gürtelbegrenzung eine Verbindungsstrecke ziehen, die entweder im inneren eines einzigen, im zweiten Gürtel liegenden Dreieckes oder auf dem gemeinsamen Rand zweier Dreiecke vom zweiten Gürtel geht.

Die Bildung der Gürtel kann ähnlicher Weise unbegrenzt fortgesetzt werden. Die n-ten Gürtelbegrenzung hat im inneren die Gürtelbegrenzungen g_i (für $i<n$), und man kann aus einem beliebigen Punkt der $(n-1)$-ten Gürtelbegrenzung zur n-ten Gürtelbegrenzung eine Verbindungsstrecke ziehen, die entweder im inneren eines enzigen, im n-ten Gürtel liegenden Dreieckes oder auf dem gemeinsamen Rand zweier solcher Dreiecke geht.

Die Ebene wird durch diese Dreiecke bzw. diese Gürtel schlicht und lückenlos bedeckt. Es soll noch dazu beweisen, daß jeder Punkt der Ebene auf solche Weise überdeckt werden kann. Falls ein Punkt P in der Ebene wäre, der nach der Konstruktion beliebiger vieler Gürtel unbedeckt bleibt, so mußte die Strecke OP unendlich viele Gürtelbegrenzungen bzw. Elementardreiecke schneiden, wo der Punkt O im Anfangsdreieck der Mosaikkonstruktion beliebig ist. Daraus folgt, daß die Menge der Schnittpunkte einen Häufungspunkt besitze, dessen Umgebungen mit willkürlichen Radien Punkte aus unendlich vielen Elementardreiecken hätten, was

aber unmöglich ist. Folglich gibt es kein Punkt, der in der Mosaikkonstruktion unbedeckt bleiben kann.

Damit haben wir den vollständigen Überblick der homogenen Dreieckmosaike gegeben.

BEMERKUNG. Herr I. DOMINYÁK hat angemerkt, daß die Existenz aller halbregelmäßiger Mosaike $(2k, 2l, 2m)$ in der hyperbolischen Ebene aus der Mosaikkonstruktion nach der in (i) gegebenen Annahme folgt. Man erkennt leicht, daß die Menge aller halbregelmäßiger Mosaike (u, v, w) und die vorige Menge der Mosaike $(2k, 2l, 2m)$ übereinstimmen sollen.[2]

2. Im folgenden beschäftigen wir uns mit dem vollständigen Überblick jener Mosaike, deren Elemente kongruente gleichwinklige Vierecke sind. Vor allem soll es bemerkt sein, daß der vollständige Überblick regulärer Viereckmosaike aus jener Ungleichung

$$\frac{1}{p}+\frac{1}{q}<\frac{1}{2} \quad (p, q > 0, \text{ ganze Zahl})$$

folgt, die die regulären Mosaike $\{p, q\}$ der hyperbolischen Ebene charakteriesiert.[3]

Nun untersuchen wir die nichtregulären gleichwinkligen Vierecke. Es ist leicht einzusehen, daß die gegenüberliegenden Seiten eines solchen Viereckes paarweise gleich sind. Der Mosaikkonstruktion halber betrachten wir die regulären Lagerungen kongruenter Abstandslinien.[4]

Wir zerlegen die Ebene durch n (von einem Punkt C ausgehende, miteinander den Winkel $\frac{2\pi}{n}$ einschließende) Halbgeraden in kongruente Teile ($n \geqq 3$). Auf jeder Halbgeraden betrachten wir je einen Punkt mit der Entfernung d von C, so daß $d > \Delta\left(\frac{\pi}{n}\right)$ sei, wo $\Delta\left(\frac{\pi}{n}\right)$ das zum Winkel $\frac{\pi}{n}$ gehörige Parallellot bedeutet. Wenn man zu jeder Halbgeraden durch den auf derselben vorher bestimmten Punkt je eine senkrechte Gerade zieht, so haben die zu zwei benachbarten Halbgeraden gehörenden Senkrechten je ein gemeinsames Lot. Zu einem solchen gemeinsamen Lot gehört je eine Abstandslinie in jedem Teil der Ebene, die die Halbgeraden (mit dem Anfangspunkt C) in den fixierten Punkten jeweils berühren und folglich paarweise miteinander einen Berührungspunkt haben. Man kann von den Punkten der Halbgeraden, deren Abstand von C gleich $2d$ ist,

[2] S. die Bedeutung des Simbols (u, v, w) in [1] S. 18.

[3] S. [2] S. 94–96.

[4] In dieser Arbeit verstehen wir unter einer Abstandslinie (Äquidistante) die Gesamtheit derjenigen Punkte der Ebene, die von den Punkten einer Geraden (Grundlinie) gleichen Abstand haben und auf einer Seite von der Grundlinie liegen. Die beiden kongruenten Äquidistanten auf verschiedenen Seiten von der Grundlinie bilden ein Paar von Abstandslinien.

zu den benachbarten Abstandslinien Tangenten ziehen, wobei die Länge der Tangentenstrecken ebenfalls d ist. Von diesen Punkten können noch $(n-3)$ Halbgeraden so gezogen werden, daß diese Halbgeraden paarweise miteinander bzw. mit den Tangenten den Winkel $\frac{2\pi}{n}$ einschließen. In diese Winkel kann je eine mit den vorigen Abstandslinien kongruente Äquidistante auf ähnliche Weise wie vorher gelegt werden. Dieser Prozeß kann unendlich oft fortgesetzt werden, und so ergeben sich unendlich viele Abstandslinien, um jede Abstandslinie je ein unendliches Tangentenpolygon von gleichen Seiten und Winkeln. (Fig. 2 zeigt, falls $n=4$ ist, die Konstruktion eines solchen unendlichen Tangentenpolygons im Poincaréschen Kreismodell.)

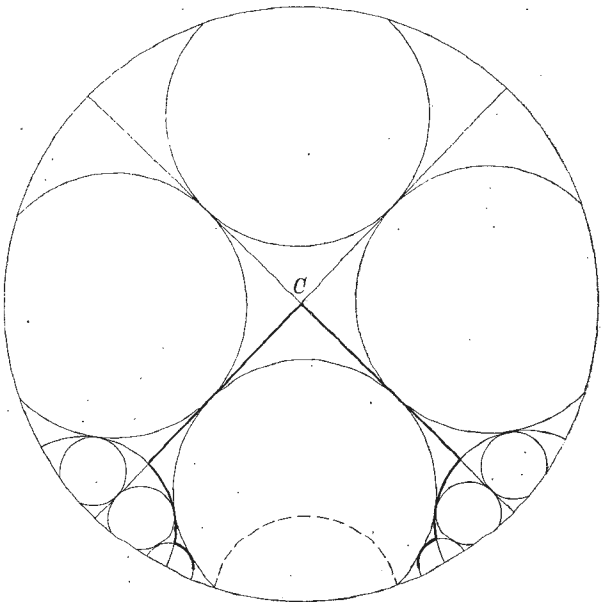

Fig. 2.

Spiegeln wir gleichzeitig diese Konfiguration an die Grundlinien der vorkommenden Abstandslinien. Nach den Spiegelungen kommen weitere Grundlinien vor, die die neuere Konfiguration begrenzen. In ähnlicher Weise wie vorhin können gleichzeitige Spiegelungen an diese Grundlinien durchgeführt werden, und dieses Spiegelungsverfahren kann unbegrenzt fortgesetzt werden. So bekommen wir eine reguläre Ausfüllung der hyperbolischen Ebene durch die Paare von Abstandslinien.

Nun betrachten wir ein Paar von Abstandslinien und ihre bezüglich der Grundlinie symmetrischen unendlichen Tangentenpolygone. Die Ver-

bindungsstrecken der symmetrischen Ecken zerlegen diesen Teil der Ebene auf kongruente gleichwinklige Vierecke. Wir bekommen ein hyperbolisches Mosaik aus kongruenten gleichwinkligen Vierecken, falls man solche Zerlegungen für alle Tangentenpolygone der vorig erhaltene Ausfüllung macht. Es folgt aus der Konstruktion, daß die Mosaikelemente sich lückenlos verknüpfen. Man erkennt auf ähnlicher Weise, wie bei der Konstruktion der Dreieckmosaike, daß jeder Punkt der Ebene durch Mosaikelemente überdeckt werden kann. Man kann ein solches Mosaik unendlich manigfaltig auch in demjenigen Falle verwirklichen, wenn n eine vorgegebene ganze Zahl ist.

Es ist leicht einzusehen, daß die reguläre Mosaike $\{4, 2k\}$ auch auf diese Weise konstruiert werden können. Wir bemerken noch, daß die folgende Tatsache aus dieser Ableitung der Mosaike $\{4, 2k\}$ folgt: Die Kreise einer zu den Mosaiken $\{4, 2k\}$ gehörigen Kreisausfüllung sind so eingelagert, daß die Mittelpunkte der Kreise je eine äquidistante Punktreihe auf den Grundlinien der Abstandslinien bilden, und so ergibt sich eine solche Kreisausfüllung als die Gesamtheit der Serien der Kreise, wo die Kreise je eines Paar von Abstandslinien und sich selbst auf der Grundlinie berühren.

Jetzt beweisen wir, daß es ausschlißlich die in dieser Konstruktion gegebenen Mosaike aus kongruenten nichtregulären gleichwinkligen Vierecken gibt. In der Tat ist es möglich die gleichwinkligen kongruenten Vierecke um einen Punkt (als Scheitel) lückenlos einlagern, falls die Winkel der Vierecke die Größe $\frac{\pi}{n}$ $(n \geqq 3)$ haben. Wenn aber wir eine unendliche Serie aus kongruenten gleichwinkligen (nichtregulären) Vierecken durch die längs ihrer gleichen Seiten passierenden Hinzufügungen bilden (Fig. 3), so die anderen Seiten der Vierecke gehören zu zwei Streckenzügen, deren benachbarten Seiten den Winkel $\frac{2\pi}{n}$ einschließen. Diese Streckenzüge sind die Tangentenpolygone eines Paares von Abstandslinien, deren Grundlinie das gemeinsame Lot aller in den Hinzufügungen vorkommenden Seiten ist. Daraus folgt unsere Behauptung:

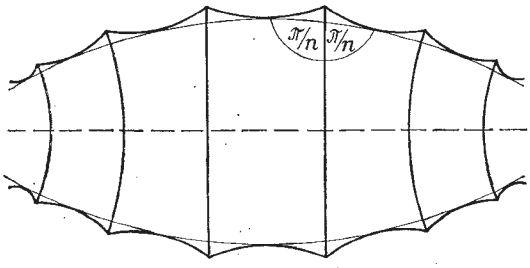

Fig. 3.

In der hyperbolischen Ebene können Mosaike aus kongruenten nichtregulären gleichwinkligen Vierecken dann und nur dann gebildet werden, wenn die Seiten der Vierecke den Winkel $\frac{\pi}{n}$ $(n \geqq 3)$ einschließen.

Damit haben wir den vollständigen Überblick der hyperbolischen gleichwinkligen Viereckmosaike gegeben.

3. Wir geben noch ein Beispiel für ein nichthomogenes Dreieckmosaik der hyperbolischen Ebene. Betrachten wir die gleichwinkligen Vierecke, aus denen Mosaike aufgebaut werden können. Die Diagonalen eines solchen gleichwinkligen Viereckes zerlegen das auf je zwei kongruente Teile. Man kann ein Dreieckmosaik aus solchen Dreiecken in der im Abschnitt 1. (iii) gegebenen Weise konstruieren. Zwei benachbarte Dreiecke, die aufeinander durch eine Spiegelung an einen Seitenmittelpunkt abgebildet werden können, bilden einen Viereck, es gibt also auch ein Viereckmosaik nach der vorigen Mosaikkonstruktion, wobei je eine Diagonale jeder Elementarvierecke gegeben sind. Falls man die Diagonale nur eines Viereckes mit der anderen Diagonale vertauscht, so bekommen wir auch ein Dreieckmosaik aus kongruenten Dreiecken, das aber offenbar nichthomogen ist.

Literatur

[1] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum* (Berlin – Göttingen – Heidelberg, 1953).
[2] L. Fejes Tóth, *Reguläre Figuren* (Budapest, 1965).

FUNCTION CLASSES, COMPACTIFICATIONS, REALCOMPACTIFICATIONS

By

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0. A classical problem of extension theory of topological spaces is to produce all Hausdorff compactifications of a given completely regular Hausdorff space. There are several known methods for the solution of this problem: it can be done with the help of proximity relations, or direction structures, or precompact uniformities, or function algebras. The latter method is of greatest importance for our present purpose; the essential idea leading to it is the following.

Let X be a completely regular Hausdorff space and Y a compact Hausdorff space containing X as a dense subspace. Consider the set $C(Y)$ of all continuous real-valued functions on Y and the class Φ of the restrictions to X of the elements $g \in C(Y)$. It is easily seen that Φ satisfies the following conditions:

(a) $\Phi \subset C^*(X)$ where $C^*(X)$ denotes the set of bounded, continuous, real-valued functions on X,

(b) The cozero-sets of the elements of Φ constitute a base for the topology of X (the cozero-set of f is the set $X - Z(f) = \{x \in X : f(x) \neq 0\}$,

(c) The limit of a uniformly convergent sequence of functions from Φ belongs to Φ,

(d) Each constant belongs to Φ,

(e) Φ is a ring under pointwise addition and multiplication.

Conversely, it can be shown that if Φ is a class of functions defined on X and satisfying the conditions (a) to (e) then there exists a compact Hausdorff space Y containing X as a dense subspace such that Φ is precisely the set of the restrictions to X of the elements of $C(Y)$. Moreover, Y is uniquely determined by Φ up to equivalence; more precisely, if Y and Y' are two compact Hausdorff spaces containing X as a dense subspace, and if we denote by Φ and Φ' the classes

$$\Phi = \{g|X : g \in C(Y)\}, \quad \Phi' = \{g|X : g \in C(Y')\},$$

then $\Phi \subset \Phi'$ iff there is a continuous mapping $h : Y' \to Y$ such that $h(x) = x$ for $x \in X$.

The above results are well-known and it is also probably known that condition (e) can be replaced by other suitable conditions, e.g. by the condition that Φ is a vector lattice (i.e. a vector space and a lattice under pointwise operations). However, it is probably of some interest to show that (e) can be replaced also by the condition that Φ is a *subtractive lattice* (i.e. a lattice under pointwise operations such that $f, g \in \Phi$ implies $f-g \in \Phi$), or an *affine lattice* (i.e. a lattice under pointwise operations satisfying $af+b \in \Phi$ for $f \in \Phi$, $a, b \in \mathbf{R}$), or as a common generalization of both, a *semi-affine lattice* (i.e. a lattice under pointwise operations such that $f \in \Phi$ implies $af+b \in \Phi$ for $b \in \mathbf{R}$, $a \in A$ where A is a subset of $\mathbf{R}$ unbounded both from above and from below).

A very natural generalization of the above problematic is to look for a construction of all Hausdorff realcompactifications of a given space X by considering the restrictions to X of the functions in $C(Y)$ for a realcompactification Y. The question is to find suitable conditions, concerning the class Φ of these restrictions, corresponding to the conditions (a) to (e) for the case of compactifications. The second and third parts of this paper are devoted to this subject.

The main idea of the fourth part is the following. The construction of the compactifications of X by means of precompact uniformities can be brought into connection with the construction by means of subclasses of $C^*(X)$ by associating with a precompact uniformity on X the class of all uniformly continuous real-valued functions with respect to the given uniformity, and conversely, by associating with a subset Φ of $C^*(X)$, satisfying the conditions (a) to (e), the precompact uniformity induced by Φ (i.e. the coarsest uniformity with respect to which each element of Φ is uniformly continuous); then the compactification corresponding to Φ can be obtained as the completion of X equipped with the uniformity associated with Φ. In the fourth part, we look for a class of uniformities which plays a similar role, with respect to realcompactifications, to that one of precompact uniformities with respect to compactifications.

1. Let X be a set and Φ a class of real-valued functions defined on X. Φ is said to be a *semi-affine lattice* if

(1.1) $f, g \in \Phi$ implies $\max(f, g) \in \Phi$, $\min(f, g) \in \Phi$,

(1.2) $f \in \Phi$, $c \in \mathbf{R}$ implies $f+c \in \Phi$,

(1.3) there is a set $A \subset \mathbf{R}$ unbounded both from above and from below such that $0 \in A$ and $f \in \Phi$ implies $cf \in \Phi$ for $c \in A$.

Now the following theorem is well-known and easy to prove:

(1.4) *Let Y be a compact Hausdorff space and X a dense subspace of Y. The class Φ of the restrictions to X of the elements of $C(Y)$ satisfies the following conditions:*

(1.5) $\Phi \subset C^*(X)$,

(1.6) *The cozero-sets of the elements of Φ constitute a base for the topology of X,*

(1.7) *The limit of a uniformly convergent sequence of functions from Φ belongs to Φ,*

(1.8) *Φ contains all constants,*

(1.9) *Φ is a semi-affine lattice.*

PROOF. (1.5), (1.8), (1.9) are obvious (the latter e.g. with $A=\mathbf{R}$), (1.6) follows from the complete regularity of Y, and (1.7) is the consequence of the fact that, if $f_n = g_n|X$ with $g_n \in C(Y)$ is a uniformly convergent sequence, then g_n also converges uniformly in view of

$$\sup\{|f_n(x)-f_m(x)| : x\in X\} = \sup\{|g_n(y)-g_m(y)| : y\in Y\}. \blacksquare$$

Let us now prove the following converse of (1.4):

(1.10) *Let X be a completely regular Hausdorff space and Φ a class of functions defined on X and satisfying the conditions* (1.5) *to* (1.9). *Then there exists a compact Hausdorff space Y such that X is a dense subspace of Y and Φ consists of the restrictions to X of the elements of $C(Y)$.*

PROOF. Let us consider the product space

$$E = \underset{f\in\Phi}{\times} E_f$$

where E_f denotes a compact interval $[a_f, b_f]$ containing the range of f. Define the mapping $h : X \to E$ by

$$\pi_f(h(x)) = f(x)$$

where $\pi_f : E \to E_f$ is the projection of E onto E_f. Clearly h is continuous and one-to-one in account of (1.6). Moreover, the inverse mapping $h^{-1} : h(X) \to X$ is continuous as well since, for $z_0 \in h(X)$, $x_0 = h^{-1}(z_0)$ and a neighbourhood V of x_0, there is a function $f\in\Phi$ such that $f(x_0) \neq 0$ and $f(x) = 0$ for $x \in X - V$. Clearly the set

$$W = \{z\in h(X) : \pi_f(z) \neq 0\}$$

is a neighbourhood of z_0 in $h(X)$ satisfying $h^{-1}(W) \subset V$.

Hence h is a homeomorphism from X onto the subspace $Z = h(X)$ of E. We identify the space X with the subspace Z by replacing $x\in X$ by $h(x)\in Z$; then $f\in\Phi$ goes over to $f\circ h^{-1} = \pi_f|Z$. The set Φ' of these functions clearly satisfies the conditions (1.7) to (1.9). If Y denotes the closure of Z in E, then Y is a compact Hausdorff space containing Z as a dense subspace, and each function $f\circ h^{-1}\in\Phi'$ has a continuous extension over Y namely $\pi_f|Y$.

The set Φ'' of the functions $\pi_f|Y$ is obviously a semi-affine lattice of continuous functions defined on Y, contains all constants, and separates the points of Y. On account of Theorem 4 of the paper [4], every element of $C(Y)$ is the limit of a uniformly convergent sequence taken from Φ''.

By (1.7), the restriction to Z of an arbitrary element of $C(Y)$ belongs to Φ'. In view of the identification of X and Z, the space Y satisfies all requirements of the theorem. ▌

Theorem (1.10) is completed by the following well-known unicity theorem:

(1.11) *Let Y_1 and Y_2 be two compact Hausdorff spaces both containing a space X as a dense subspace. Denote by Φ_1 and Φ_2 the set of the restrictions to X of the elements of $C(Y_1)$ and $C(Y_2)$ respectively. Then $\Phi_1 \subset \Phi_2$ iff there exists a continuous mapping $h : Y_2 \to Y_1$ such that*

$$h(x) = x \text{ for } x \in X. \tag{1.12}$$

$\Phi_1 = \Phi_2$ iff there is a homeomorphism h from Y_2 onto Y_1 satisfying (1.12).

PROOF. If $h : Y_2 \to Y_1$ is continuous and fulfils (1.12), then $f \in \Phi_1$ implies $f = g|X$, $g \in C(Y_1)$, hence

$$f = g \circ h|X, \; g \circ h \in C(Y_2)$$

and $f \in \Phi_2$. Conversely, if $\Phi_1 \subset \Phi_2$, then we can define a ring homomorphism $\varphi : C(Y_1) \to C(Y_2)$ by the formula

$$\varphi(g_1) = g_2 \text{ iff } g_2|X = g_1|X.$$

Clearly $\varphi(1) = 1$ so that there exists, by Theorem 10.6 of [5], a continuous mapping $h : Y_2 \to Y_1$ such that

$$\varphi(g_1) = g_1 \circ h \text{ for } g_1 \in C(Y_1).$$

Then $x \in X$ implies

$$g_1(x) = \varphi(g_1)(x) = g_1(h(x))$$

for each $g_1 \in C(Y_1)$. Since $C(Y_1)$ separates the points of Y_1, necessarily $h(x) = x$.

If $h : Y_2 \to Y_1$ is a homeomorphism satisfying (1.12), then $\Phi_1 \subset \Phi_2$ and $\Phi_2 \subset \Phi_1$ simultaneously. Conversely, if $\Phi_1 = \Phi_2$, then there exist two continuous mappings

$$h : Y_2 \to Y_1 \text{ and } k : Y_1 \to Y_2$$

such that

$$h(x) = x \text{ and } k(x) = x \text{ for } x \in X.$$

Then both

$$k \circ h : Y_2 \to Y_2 \text{ and } h \circ k : Y_1 \to Y_1$$

are continuous extensions of the identical mapping of X. Hence $k = h^{-1}$ and h is a homeomorphism from Y_2 onto Y_1. ▌

2. We turn to the problem of the realcompactifications, and begin with the definition and study of the properties needed for this purpose.

Let Φ be a set of real-valued functions defined on a set X. A filter $\mathfrak{S}$ in X is said to be a *Φ-filter* if every $f \in \Phi$ *converges on* $\mathfrak{S}$, i.e. if the filter base

$$f(\mathfrak{S}) = \{f(S) : S \in \mathfrak{S}\}$$

is convergent in $\mathbf{R}$, for every $f \in \Phi$. The function class Φ is said to be *filter-closed* if a real-valued function g defined on X belongs to Φ whenever it converges on each Φ-filter.

The following theorem shows that the concept of a filter-closed function class is related to our present problem:

(2.1) *Let Y be a topological space and X a dense subspace of Y. Then the class Φ of the restrictions to X of the elements of $C(Y)$ is filter-closed.*

Proof. Assume that a function $g : X \to \mathbf{R}$ converges on every Φ-filter. Then $g(\mathfrak{S})$ is convergent whenever $\mathfrak{S}$ is the trace in X of the neighbourhood filter of a point $y \in Y$, since $f(\mathfrak{S})$ clearly converges for $f \in \Phi$, so that $\mathfrak{S}$ is a Φ-filter. Hence, by a classical theorem ([1], § 8, n° 5, Théorème 1), g can be continuously extended over Y and $g \in \Phi$. ∎

We shall also consider another related property. Let Φ be a class of functions $f : X \to \mathbf{R}$. For an arbitrary index set I, let $f_i \in \Phi$ for $i \in I$,

$$E = \underset{i \in I}{\times} E_i \text{ where } E_i = \mathbf{R} \text{ for } i \in I, \tag{2.2}$$

define $h : X \to E$ by

$$\pi_i(h(x)) = f_i(x) \quad (x \in X,\ i \in I) \tag{2.3}$$

where π_i is the projection of E onto E_i, and let Y be the closure of $h(X)$ in E. The class Φ is said to be *composition-closed* if, for any choice of I and the functions $f_i \in \Phi$, and for every continuous function $k : Y \to \mathbf{R}$, the composition $k \circ h$ belongs to Φ.

The property of being composition-closed is obviously a generalization of that of being a ring, or an algebra, or a vector lattice, etc. On the other hand, it is equivalent to the filter-closed property:

(2.4) *A filter-closed class is composition-closed and vice versa.*

Proof. Let $I, f_i \in \Phi$, E, h, Y, k be as in the definition. Then $k \circ h : X \to \mathbf{R}$ converges on every Φ-filter. In fact, if $\mathfrak{S}$ is a Φ-filter, then

$$f_i(\mathfrak{S}) \to z_i \in \mathbf{R} \quad \text{for} \quad i \in I,$$

hence

$$h(\mathfrak{S}) \to z \in E \text{ where } \pi_i(z) = z_i.$$

Clearly $z \in Y$, thus $(k \circ h)(\mathfrak{S}) \to k(z)$. Hence, if Φ is filter-closed, then $k \circ h \in \Phi$.

Conversely, assume that Φ is composition-closed, and let $g : X \to \mathbf{R}$ be convergent on every Φ-filter. Choose an index set I such that

$$\Phi = \{f_i : i \in I\}$$

and construct E, h, Y as in the definition. For $z \in Y$, let $\mathfrak{v}$ denote the neighbourhood filter of z in Y and $\mathfrak{t}$ the trace of $\mathfrak{v}$ in $h(X)$. Let

$$\mathfrak{s} = h^{-1}(\mathfrak{t}) = \{h^{-1}(T) : T \in \mathfrak{t}\}.$$

Then $\mathfrak{s}$ is the base of a Φ-filter in X. In fact, for any $f_i \in \Phi$, $S \in \mathfrak{s}$, we have

$$f_i(S) = \pi_i(h(S)) = \pi_i(T)$$

with $T \in \mathfrak{t}$, hence

$$f_i(\mathfrak{s}) \to z_i = \pi_i(z).$$

Therefore $g(\mathfrak{s})$ converges in $\mathbf{R}$. Define $k(z)$ equal to the limit of $g(\mathfrak{s})$.

The function $k : Y \to \mathbf{R}$ is continuous on Y. In fact, for $z \in Y$ and $\varepsilon > 0$, there is a $V \in \mathfrak{v}$ open in Y such that $g(S)$ is contained in the ε-neighbourhood of $k(z)$ for

$$S = h^{-1}(V \cap h(X)).$$

If $z' \in V$, then S belongs to the filter $h^{-1}(\mathfrak{t}')$ where $\mathfrak{t}'$ is the trace of the neighbourhood filter $\mathfrak{v}'$ of z' hence $k(z')$ is in the closure of $g(S)$ and

$$|k(z') - k(z)| \leqq \varepsilon.$$

By hypothesis, $k \circ h \in \Phi$. This implies $g \in \Phi$. In fact, if $x \in X$, $z = h(x)$, then x is contained in each element of $\mathfrak{s}$, hence

$$g(\mathfrak{s}) \to g(x) = k(h(x))$$

and $g = k \circ h$. ∎

A property very similar to that of composition-closedness was introduced by J. R. Isbell in the paper [7]. A function class Φ is said to be *weakly composition-closed* if, with the same meaning of I, E, h as above, $k \circ h \in \Phi$ whenever k is a continuous real-valued function on E (and not only on Y). The following example shows that the distinction is not purely formal:

(2.5) *Let Φ denote the first Baire class on the real line. Then Φ is weakly composition-closed without being composition-closed.*

Proof. Let Y be the union of the straight lines $y = 0$ and $y = \frac{1}{n}$ ($n \in \mathbf{N}$) in the plane, equipped with the following topology. For $p = \left(x_0, \frac{1}{n}\right)$, a neighbourhood base of p consists of the sets

$$V(x_0, n, \varepsilon) = \left\{\left(x, \frac{1}{n}\right) : x_0 - \varepsilon < x < x_0 + \varepsilon\right\} \quad (\varepsilon > 0).$$

For $p = (x_0, 0)$, the elements of a neighbourhood base are of the form

$$\{p\} \cup \bigcup_{n=n_0}^{\infty} V(x_0, n, \varepsilon_n)$$

where $n_0 \in \mathbf{N}$ and the numbers $\varepsilon_n > 0$ $(n \geqq n_0)$ are arbitrary. These neighbourhood bases clearly define a topology on Y.

On the line $y = 0$, the functions of the first Baire class (with respect to the usual Euclidean topology) coincide with the restrictions of the elements of $C(Y)$. In fact, if $g \in C(Y)$, then $f_n(x) = g\left(x, \frac{1}{n}\right)$ is continuous on $\mathbf{R}$ with the usual topology and clearly $f_n(x) \to f(x) = g(x, 0)$. Conversely, if $f_n \in C(\mathbf{R})$, $f_n(x) \to f(x)$ for $x \in \mathbf{R}$, then the function g defined by

$$g(x, 0) = f(x), \quad g\left(x, \frac{1}{n}\right) = f_n(x)$$

belongs to $C(Y)$ since, for $x_0 \in \mathbf{R}$, $\varepsilon > 0$, there is $n_0 \in \mathbf{N}$ such that $n \geqq n_0$ implies

$$|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{2},$$

and then there are numbers $\varepsilon_n > 0$ $(n \geqq n_0)$ such that $x_0 - \varepsilon_n < x < x_0 + \varepsilon_n$ implies

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{2};$$

hence

$$|g(p) - g(x_0, 0)| < \varepsilon$$

for p being in a neighbourhood of $(x_0, 0)$.

This result implies that Φ is weakly composition-closed. In fact (cf. [7], p. 114), if

$$f_i \in \Phi, \; f_i(x) = g_i(x, 0), \; g_i \in C(Y) \qquad (i \in I),$$

E is defined by (2.2), and $k: E \to \mathbf{R}$ is continuous, then $g = k \circ h \in C(Y)$ for the mapping h defined, similarly to (2.3), by

$$\pi_i(h(p) = g_i(p) \qquad (i \in I, \; p \in Y)$$

Hence

$$f \in \Phi \text{ for } f(x) = g(x, 0), \; f = k \circ h'$$

where h' is defined by

$$\pi_i(h'(x)) = f_i(x) \qquad (i \in I, \; x \in \mathbf{R}).$$

On the other hand, Φ is not composition-closed. By (2.4), it suffices to show that it is not filter-closed. In fact, let $\mathfrak{F}$ be a Φ-filter. Then $f(\mathfrak{F})$ must converge for $f(x) = x$, hence $\mathfrak{F} \to x_0$ with respect to the usual topology of the real line. Now $f(\mathfrak{F})$ converges for

$$f(x) = \frac{1}{x - x_0} \quad (x \neq x_0), \quad f(x_0) = 0$$

(since clearly $f \in \Phi$) which is only possible if $\mathfrak{F}$ coincides with the trivial ultrafilter associated with x_0. But then $f(\mathfrak{F})$ converges for any real-valued function f so that Φ is not filter-closed. ∎

The above example enables us to answer a question posed by ISBELL ([7], p. 114) by the negative. It was asked whether a weakly composition-closed class Φ on a set X is necessarily of the form $C(X)$ for a suitable topology on X, under the restriction that Φ contains the limits of uniformly convergent sequences taken from Φ and $f \in \Phi$, $f(x) \neq 0$ $(x \in X)$ implies $1/f \in \Phi$. Now the class Φ in (2.5) clearly satisfies all these conditions but is not of the form $C(X)$ for any topology on X in consequence of (2.1). (The first example of this kind was given in [3].)

The characterization of the classes of the form $C(X)$ can be done with the help of another similar condition. In fact, let us say that a class Φ of real-valued functions defined on X is *strongly composition-closed* if, whenever $f_i \in \Phi$, $i \in I$ for an index set I, E and h are defined again by (2.2) and (2.3), and $k : h(X) \to \mathbf{R}$ is continuous, then $k \circ h \in \Phi$. Clearly a strongly composition-closed class is composition-closed, and a composition-closed class is weakly composition-closed. Now we have the following theorem:

(2.6) *Let Φ be a class of real-valued functions defined on a set X. There exists a topology on X such that $\Phi = C(X)$ iff Φ is strongly composition-closed.*

PROOF. If $\Phi = C(X)$ for some topology, and $f_i \in C(X)$ for $i \in I$, then $h : X \to E$ is continuous, hence $k \circ h \in C(X)$ whenever $k : h(X) \to \mathbf{R}$ is continuous. Conversely, assume that Φ is strongly composition-closed. Choose an index set I such that

$$\Phi = \{f_i : i \in I\}$$

and construct E and h by (2.2) and (2.3). Equip X with the initial topology corresponding to $h : X \to E$. Then clearly $f_i \in C(X)$ for each $i \in I$. On the other hand, if $g \in C(X)$, let us define $k : h(X) \to \mathbf{R}$ by

$$k(h(x)) = g(x) \quad (x \in X).$$

This definition is unambiguous because, if $x, y \in X$, $h(x) = h(y)$, then x and y have the same neighbourhoods in X, hence $g(x) = g(y)$. Moreover, k is continuous, since

$$z_0 \in h(X),\ z_0 = h(x_0),\ \varepsilon > 0$$

implies the existence of an open neighbourhood V of z_0 such that

$$|g(x)-g(x_0)| < \varepsilon \text{ for } x \in h^{-1}(V),$$

hence

$$|k(z)-k(z_0)| < \varepsilon \text{ for } z \in V \cap h(X).$$

Therefore $k \circ h \in \Phi$, that is to say, $g \in \Phi$. ∎

Another characterization of classes of the form $C(X)$ is given in [4], Theorem 9.

It is easy to see that a composition-closed class need not be strongly composition-closed. For this purpose, let Φ be the class of all continuous functions f on $\mathbf{R}$ such that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x)$$

exists and is finite. Then Φ consists of the restrictions to $\mathbf{R}$ of the continuous functions on the one-point compactification of $\mathbf{R}$, hence Φ is composition-closed by (2.1) and (2.4). On the other hand, $f \in \Phi$ for $f(x) = \exp(-x^2)$ but $1/f \notin \Phi$, so that Φ is not strongly composition-closed.

We can introduce further similar properties by making restrictions on the cardinality of the index set I. Hence we say that Φ is *finitely* or *countably strongly composition-closed* if, whenever $f_i \in \Phi$ for $i \in I$ and I is finite or countable, respectively, E and h are defined by (2.2) and (2.3), finally $k : h(X) \to \mathbf{R}$ is continuous, then $k \circ h \in \Phi$. Similarly, Φ is said to be *finitely composition-closed* if the same condition holds, for finite I, with the modification that k is a continuous real-valued function on the closure Y of $h(X)$. The weak variant does not lead to a new property in the finite case because the normality of $E = \mathbf{R}^n$ implies that a continuous $k : Y \to \mathbf{R}$ can be extended to a continuous $k^* : E \to \mathbf{R}$. (For this property, see [7].) The same remark applies, of course, for the countable case, but here we can say even more:

(2.7) *Let Φ be a class of real-valued functions on X. For $f_i \in \Phi$ $(i \in I)$, consider E and h as in* (2.2) *and* (2.3), *and $Y = \overline{h(X)}$. Then the following statements are equivalent:*

(2.8) *Φ is weakly composition-closed,*
(2.9) *$k \circ h \in \Phi$ whenever I is countable and $k : E \to \mathbf{R}$ is continuous,*
(2.10) *$k \circ h \in \Phi$ whenever I is countable and $k : Y \to \mathbf{R}$ is continuous.*

Remark. The property (2.9) was studied in [6].

Proof. (2.8) $\Rightarrow$ (2.9) is obvious, and (2.9) $\Rightarrow$ (2.10) follows from the normality of E in the countable case. (2.10) $\Rightarrow$ (2.9) is obvious again.

Now suppose that (2.9) is true, and consider an arbitrary index set I and a continuous $k : E \to \mathbf{R}$. Let $b_n : \mathbf{R} \to \mathbf{R}$ be, for $n \in \mathbf{N}$, the function defined by

$$b_n(t) = \begin{cases} t & \text{if } -n \leqq t \leqq n, \\ n & \text{if } t \geqq n, \\ -n & \text{if } t \leqq -n. \end{cases}$$

Now define $r_n : E \to K_n$, where

$$K_n = \underset{i \in I}{\times} K_{ni} \quad K_{ni} = [-n, n] \text{ for } i \in I,$$

by

$$\pi_i(r_n(z)) = b_n(\pi_i(z))$$

and $k_n : E \to \mathbf{R}$ by

$$k_n(z) = k(r_n(z)) \quad (z \in E).$$

Since $\pi_i(r_n(z)) \to \pi_i(z)$ for $i \in I$ and $n \to \infty$, we have $r_n(z) \to z$ and $k_n(z) \to k(z)$.

The functions k_n are uniformly continuous with respect to the product uniformity of E. In fact, by the compactness of K_n, one finds, to a given $\varepsilon > 0$, a finite subset $I_0 \subset I$ and a $\delta > 0$ such that

$$|k(z) - k(u)| < \varepsilon \text{ whenever } z, u \in K_n$$

and

$$|\pi_i(z) - \pi_i(u)| < \delta \text{ for } i \in I_0.$$

Now clearly

$$z, u \in E, \ |\pi_i(z) - \pi_i(u)| < \delta \text{ for } i \in I_0$$

implies

$$r_n(z), r_n(u) \in K_n \text{ and } |\pi_i(r_n(z)) - \pi_i(r_n(u))| < \delta$$

for $i \in I_0$, hence

$$|k_n(z) - k_n(u)| < \varepsilon.$$

Now k_n being uniformly continuous, there exists, for a given $m \in \mathbf{N}$, a finite set $I_{nm} \subset I$ such that

$$z, u \in E, \ \pi_i(z) = \pi_i(u) \text{ for } i \in I_{nm}$$

implies

$$|k_n(z) - k_n(u)| < \frac{1}{m}.$$

For the countable set

$$I' = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} I_{nm}$$

we have therefore

$$k_n(z) = k_n(u) \text{ whenever } n \in \mathbf{N} \text{ and } \pi_i(z) = \pi_i(u) \text{ for } i \in I'.$$

Hence, by $k_n(z) \to k(z)$ and $k_n(u) \to k(u)$,

$$k(z) = k(u) \text{ whenever } \pi_i(z) = \pi_i(u) \text{ for } i \in I'.*$$

Now define

$$E' = \underset{i \in I'}{\times} E_i,\ h' : X \to E',\ \pi_i(h'(x)) = f_i(x)\ (i \in I'),$$

$$g : E' \to E,\ \pi_i(g(z')) = \pi_i(z') \text{ for } i \in I',$$

$$\pi_i(g(z')) = 0 \text{ for } i \in I - I',$$

$$k' : E' \to \mathbf{R},\ k'(z') = k(g(z')).$$

The mapping g is continuous, hence, by (2.9), the function $k' \circ h'$ belongs to Φ. On the other hand,

$$k'(h'(x)) = k(g(h'(x))) = k(h(x))$$

since

$$\pi_i(g(h'(x))) = \pi_i(h(x)) \text{ for } x \in I'.$$

Hence $k' \circ h' = k \circ h$ and $k \circ h \in \Phi$. ∎

The properties considered above have therefore the following interrelations (where CC stands for composition-closed):

$$\begin{array}{ccc}
\text{strongly CC} & \Rightarrow & \text{CC} \\
\Downarrow & & \Downarrow \\
\text{countably strongly CC} & \Rightarrow & \text{weakly CC} \\
\Downarrow & & \Downarrow \\
\text{finitely strongly CC} & \Rightarrow & \text{finitely CC}
\end{array}$$

We know that the implication in the first line cannot be reversed. Moreover, the above example of a composition-closed but not strongly composition-closed class is not even finitely strongly composition-closed.

Now consider the following example: let X be the subspace of the plane composed of all straight lines $y = n$ ($n \in \mathbf{N}$) with Euclidean topology, and let Φ be the class of all continuous functions f such that, for some n_0 depending on f, the function f is constant on each line $y = n$ for $n \geqq n_0$ (cf. [7], p. 103). Φ is obviously finitely strongly composition-closed. However, Φ is not weakly composition-closed. In fact, the function

$$f(x, n) = \frac{1}{n} \sin x$$

* The fact that a continuous function on E depends only on countably many coordinates is, of course, well-known and can be obtained from much more general results ([2], [3], [8]). However, these general theorems require essentially deeper tools so that the elementary proof above is not superfluous perhaps.

does not belong to Φ, but it is clearly the limit of a uniformly convergent sequence taken from Φ. Now the following theorem is valid:

(2.11) *A weakly composition-closed class contains the limits of the uniformly convergent sequences taken from it.*

PROOF. Let Φ be weakly composition-closed on X, $f_i \in \Phi$ $(i \in \mathbf{N})$, E and h defined by (2.2) and (2.3) for $I = \mathbf{N}$, and suppose that $f_i \to g$ uniformly on X. Let Y be the closure of $h(X)$. If $z \in h(X)$, $z = h(x)$, $x \in X$, define $k(z) = g(x)$. This is possible because $h(x) = h(y)$ implies $f_i(x) = f_i(y)$ for $i \in \mathbf{N}$, hence $g(x) = g(y)$. Now if $z \in Y$, $\varepsilon > 0$, select $n \in \mathbf{N}$ such that

$$|f_n(x) - g(x)| < \frac{\varepsilon}{4} \quad (x \in X),$$

and let V be the neighbourhood of z composed of the points z' satisfying

$$|\pi_n(z') - \pi_n(z)| < \frac{\varepsilon}{4}.$$

Let $x, y \in X$ be chosen with $h(x), h(y) \in V$. Then

$$|k(h(x)) - k(h(y))| = |g(x) - g(y) \leqq$$

$$\leqq |g(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - g(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$

since

$$|f_n(x) - f_n(y)| \leqq |f_n(x) - \pi_n(z)| + |\pi_n(z) - f_n(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Thus k can be continuously extended over Y, and $k \circ h = g$ is obvious. By theorem (2.7), in particular, in account of the implication (2.8) $\Rightarrow$ $\Rightarrow$ (2.10), then $g \in \Phi$. ∎

Finally, a countably strongly composition-closed class need not be composition-closed. In fact, let X be the plane equipped with the product of the Euclidean topology on the x-axis and the discrete topology on the y-axis, and let Φ be the class of all continuous functions on X which are constant on all but a countable number of straight lines parallel to the x-axis (the value of the constant depending on y). Φ is strongly countably composition-closed. However, it is not composition-closed because it is not filter-closed (cf. (2.4)). In fact, let $\mathfrak{F}$ be a Φ-filter. The function $f(x, y) = y$ belongs to Φ, hence $f(\mathfrak{F})$ has to converge, so that

$$\pi_y(\mathfrak{F}) \to y_0.$$

On the other hand, the function

$$g(x,y) = \begin{cases} \dfrac{1}{y-y_0} & \text{if } y \neq y_0, \\ 0 & \text{if } y = y_0 \end{cases}$$

belongs to Φ as well, and the convergence of $g(\mathfrak{s})$ implies $\{y_0\} \in \pi_y(\mathfrak{s})$. Let $\mathfrak{s}_1$ denote the trace of $\mathfrak{s}$ on the line $y = y_0$; then $h(\mathfrak{s}_1)$ converges where

$$h(x,y) = \begin{cases} x & \text{if } y = y_0, \\ 0 & \text{if } y \neq y_0, \end{cases}$$

and $\mathfrak{s}_1 \to (x_0, y_0)$ for some $x_0 \in \mathbf{R}$. In other words, $\mathfrak{s} \to (x_0, y_0)$ with respect to the topology of X. Therefore any continuous function on X converges on every Φ-filter.

3. Let us turn to the application of the above results to the construction of realcompactifications. First we can state:

(3.1) *Let Y be a completely regular Hausdorff space and X a dense subspace of Y. The class Φ of the restrictions to X of the elements of $C(Y)$ satisfies the following conditions:*

(3.2) $\Phi \subset C(X)$,

(3.3) *The cozero-sets of the elements of Φ constitute a base for X,*

(3.4) *Φ is composition-closed.*

PROOF. (2.1) and (2.4). ∎

The converse is easily obtained as well:

(3.5) *Let X be a completely regular Hausdorff space, and Φ a class of functions satisfying* (3.2) *to* (3.4). *Then there is a realcompact, completely regular Hausdorff space Y such that X is a dense subspace of Y and Φ consists of the restrictions to X of the elements of $C(Y)$.*

PROOF. Let $\Phi = \{f_i : i \in I\}$, E and h defined by (2.2) and (2.3), and Y be the closure of $h(X)$ in E. The continuous mapping h is now, in account of (3.3), a homeomorphism from X onto $h(X)$ (cf. the proof of (1.4)). Hence we can identify X with $Z = h(X)$, and $f_i \in \Phi$ will correspond to

$$f_i \circ h^{-1} = \pi_i | Z.$$

The space Y is closed in a product of real lines, hence realcompact, and contains Z as a dense subspace. Each function $\pi_i | Z$ has, of course, a continuous extension over Y. Conversely, if $g \in C(Y)$, then $g \circ h \in \Phi$ by hypothesis, hence $g \circ h = f_i$ for some $i \in I$, and

$$g | Z = \pi_i | Z. \ ∎$$

The proof of the following unicity theorem is the same as that of (1.11):

(3.6) *Let Y_1 and Y_2 be two realcompact, completely regular Hausdorff spaces both containing a space X as a dense subspace. Denote by Φ_1 and Φ_2 the set of the restrictions to X of the elements of $C(Y_1)$ and $C(Y_2)$ respectively. Then $\Phi_1 \subset \Phi_2$ iff there exists a continuous mapping $h: Y_2 \to Y_1$ such that*

$$h(x) = x \quad \text{for} \quad x \in X. \tag{3.7}$$

$\Phi_1 = \Phi_2$ iff there is a homeomorphism h from Y_2 onto Y_1 satisfying (3.7). ∎

It is to notice that the mapping $h: Y_2 \to Y_1$ in (1.11) is necessarily surjective since $h(Y_2)$ is compact and contains the dense subset $X \subset Y_1$. On the contrary, a similar statement does not hold in (3.6). In fact, let X be the real interval (a, b), $Y_1 = [a, b]$, $Y_2 = [a, b)$. Clearly $\Phi_1 \subset \Phi_2$ but no mapping $h: Y_2 \to Y_1$ satisfying (3.7) can be surjective.

If Φ is a class of real-valued functions defined on a topological space X, let us call a *compactification (realcompactification) associated with Φ* a compact Hausdorff space (realcompact, completely regular Hausdorff space) Y containing X as a dense subspace such that Φ consists of the restrictions to X of the elements of $C(Y)$. Hence the conditions (1.5) to (1.9) and (3.2) to (3.4), respectively, are necessary and sufficient for the existence of a compactification or realcompactification associated with Φ.

Let us remark that the two kinds of extension are connected with each other in the following way:

(3.8) *Let Y be a realcompactification of a space X, associated with a class Φ, and let Z be the Čech–Stone compactification of Y. Then Z is a compactification associated with Φ^* where Φ^* is the class composed of all bounded functions in Φ.*

PROOF. Z is a compact Hausdorff space containing X as a dense subspace. If $h \in C(Z)$, then $h|X$ is bounded and is the restriction of $h|Y \in C(Y)$, hence $h|X \in \Phi^*$. Conversely, if $f \in \Phi^*$, then there is $g \in C(Y)$ such that $f = g|X$, and g is bounded too. Hence $g = h|Y$ with $h \in C(Z)$, and $f = h|X$. ∎

4. Let us now examine the connection of the above results with the theory of uniform spaces.

If Φ is a class of real-valued functions on a set X, let $\mathcal{U}(\Phi)$ denote the uniformity induced by Φ, i.e. the coarsest uniformity for which each $f \in \Phi$ is uniformly continuous. It is induced by the family of all pseudometrics

$$\sigma_f(x, y) = |f(x) - f(y)| \quad (f \in \Phi).$$

A uniformity $\mathcal{U}$ on X is said to be a *c-uniformity* if it is of the form $\mathcal{U}(\Phi)$ with some suitable Φ.

Let us denote, for a uniformity $\mathcal{U}$, by $C(\mathcal{U})$ the class of all real-valued functions uniformly continuous with respect to $\mathcal{U}$. If $\mathcal{U}$ is a c-uniformity then clearly $\mathcal{U} = \mathcal{U}(C(\mathcal{U}))$.

It is well-known that, if a uniformity is induced by a family of pseudometrics, then a filter $\mathfrak{F}$ is Cauchy iff, for each pseudo-metric σ in the family

and each $\varepsilon > 0$, there is a set in $\mathcal{S}$ with σ-diameter less than ε. Hence we have:

(4.1) *A filter in X is a Φ-filter iff it is Cauchy with respect to $\mathcal{U}(\Phi)$.* ▮

If $A \subset X$ then the uniformity $\mathcal{U}(\Phi)$ induces on A the uniformity $\mathcal{U}(\Phi')$ where

$$\Phi' = \{f|A : f \in \Phi\}.$$

If X is a topological space, $\Phi \subset C(X)$ and the cozero-sets of the elements of Φ constitute a base for X, then $\mathcal{U}(\Phi)$ induces the topology of X. In particular, in a completely regular space X, $\mathcal{U}(C(X))$ is compatible with the topology of X. It is well-known that a completely regular Hausdorff space is realcompact iff $\mathcal{U}(C(X))$ is complete (see [5], 15.14).

Now it is easy to prove:

(4.2) *Let Y be a realcompactification of X associated with the class Φ. Then Y, equipped with the uniformity $\mathcal{U}(C(Y))$, is the completion of the uniform space X equipped with $\mathcal{U}(\Phi)$.*

PROOF. $\mathcal{U}(C(Y))$ induces on X the uniformity $\mathcal{U}(\Phi)$, it is complete and compatible with the topology of Y, hence separated, moreover X is dense in Y. ▮

(4.3) *Under the hypotheses of* (4.2), $\Phi = C(\mathcal{U}(\Phi))$.

PROOF. Clearly $C(Y) = C\big(\mathcal{U}(C(Y))\big)$, and a real-valued function has a uniformly continuous extension over the completion of a uniform space iff it is uniformly continuous itself. ▮

In view of (3.1), (3.5), (3.6), (4.2), and (4.3), we have a complete analogy with the construction of compactifications by means of precompact uniformities as soon as we can find all those uniformities which are compatible with a given completely regular Hausdorff space X and are, moreover, of the form $\mathcal{U}(\Phi)$ with a composition-closed (or filter-closed, cf. (2.4)) class Φ. In fact, if Φ is composition-closed and $\mathcal{U}(\Phi)$ is compatible with X, then $\Phi \subset C(X)$ and the cozero-sets of the functions f clearly constitute a base for X:

$$\{x : |f_i(x) - f_i(x_0)| < \varepsilon \quad (i = 1, \ldots, n)\}$$

is the cozero-set of

$$g = \varepsilon - \min\big(\varepsilon, \max(|f_1 - f_1(x_0)|, \ldots, |f_n - f_n(x_0)|)\big),$$

and $g \in \Phi$ whenever $f_1, \ldots, f_n \in \Phi$.

On account of (4.3), we have to look for a condition which, for a c-uniformity $\mathcal{U}$, assures that $C(\mathcal{U})$ is filter-closed. The following definitions have the purpose to enable us to formulate such a condition.

Let us recall that a uniformity is said to be *fine* if it is the finest uniformity compatible with its topology. This is the case iff each continuous mapping of the given space into any uniform space is uniformly continuous.

Now let us say that a uniformity is *realfine* if each continuous real-valued function is uniformly continuous.

(4.4) *A fine uniformity is realfine. In particular, a compact or a discrete uniformity is realfine.* ∎

(4.5) *A compatible uniformity of a topological space X is realfine iff it is finer than $\mathcal{U}(C(X))$.* ∎

A realfine uniformity need not be fine. In fact, let X be an uncountable discrete topological space. Then the uniformity $\mathcal{U}(C(X))$ is realfine by (4.5) without being fine since it is strictly coarser than the discrete uniformity. (A c-uniformity, i.e. a uniformity of the form $\mathcal{U}(\Phi)$ is necessarily separable; a uniformity $\mathcal{U}$ is said to be *separable* if, for a given surrounding $U \in \mathcal{U}$, there is a countable cover of the space whose members A_i satisfy $A_i \times A_i \subset U$.)

A uniformity will be called *prerealfine* if a real-valued function is uniformly continuous whenever it is convergent on each Cauchy filter. The expression "prerealfine" is motivated by the following theorem:

(4.6) *A uniform space is prerealfine iff its completion is realfine.*

Proof. Let Y be the completion of X. If $g : Y \to \mathbf{R}$ is continuous, then $g|X$ is convergent on each Cauchy filter in X. Hence, if X is prerealfine, then $g|X$ is uniformly continuous, and it has a uniformly continuous extension over Y, clearly coinciding with g.

Conversely, if Y is realfine, let $f : X \to \mathbf{R}$ be convergent on each Cauchy filter in X. Then f has a continuous extension $g : Y \to \mathbf{R}$. Since g is uniformly continuous, the same holds for f. ∎

In particular:

(4.7) *A precompact uniformity is prerealfine.* ∎

Since a function which converges on each Cauchy filter is necessarily continuous, we also have:

(4.8) *A realfine uniformity is prerealfine.* ∎

(4.8) is not superfluous because a realfine uniformity need not be complete. In fact, if X is a completely regular Hausdorff space which is pseudo-compact without being compact, then $\mathcal{U}(C(X))$ is realfine by (4.5) but not complete since X is not realcompact. More precisely:

(4.9) *A uniform space is realfine iff it is C-embedded in its completion and the latter is realfine.*

Proof. If X is a realfine uniform space, then its completion Y is realfine by (4.8) and (4.6). Moreover, X is C-embedded in Y (i.e. each $f \in C(X)$ has an extension $g \in C(Y)$) since each element of $C(X)$ is uniformly

continuous. Conversely, if X is C-embedded in Y and Y is realfine, then each $f \in C(X)$ has a uniformly continuous extension $g : Y \to \mathbf{R}$ and is uniformly continuous itself. ∎

The following theorem furnishes the solution of our problem:

(4.10) *A uniformity can be written in the form $\mathcal{U}(\Phi)$ with a filter-closed class Φ iff it is a prerealfine c-uniformity.*

PROOF. Assume $\mathcal{U} = \mathcal{U}(\Phi)$ and Φ is a filter-closed class on X. Then $\mathcal{U}$ is plainly a c-uniformity and, if $f : X \to \mathbf{R}$ converges on each Cauchy filter with respect to $\mathcal{U}$, then f converges on each Φ-filter by (4.1), hence $f \in \Phi$ and f is uniformly continuous. Thus $\mathcal{U}$ is prerealfine.

Conversely, suppose that $\mathcal{U}$ is a prerealfine c-uniformity. Then $\mathcal{U} = \mathcal{U}(\Phi')$ with some class Φ'. Let Φ be the class of those $f : X \to \mathbf{R}$ that converge on each Φ'-filter. Clearly $\Phi' \subset \Phi$, hence $\mathcal{U}(\Phi')$ is coarser than $\mathcal{U}(\Phi)$. On the other hand, each $f \in \Phi$ converges on each Cauchy filter by (4.1), hence is uniformly continuous by hypothesis, so that $\mathcal{U}(\Phi)$ is coarser than $\mathcal{U} = \mathcal{U}(\Phi')$. Therefore $\mathcal{U} = \mathcal{U}(\Phi) = \mathcal{U}(\Phi')$. Now if $f : X \to \mathbf{R}$, converges on each Φ-filter, then it converges on each Φ'-filter by (4.1), and is contained in Φ. Hence Φ is filter-closed. ∎

A prerealfine uniformity need not be a c-uniformity: the discrete uniformity on an uncountable space is fine, hence prerealfine by (4.4) and (4.8), but not a c-uniformity since it is not separable.

The following theorem sums up our considerations:

(4.11) *Let X be a completely regular Hausdorff space. We obtain all completely regular, Hausdorff realcompactifications of X by looking for all compatible prerealfine c-uniformities on X and by taking the completions of X equipped with the uniformities in question. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two such uniformities and Y_1, Y_2 the respective completions, then $\mathcal{U}_1$ is coarser than $\mathcal{U}_2$ iff there is a continuous mapping $h : Y_2 \to Y_1$ such that*

$$h(x) = x \text{ for } x \in X. \tag{4.12}$$

$\mathcal{U}_1 = \mathcal{U}_2$ iff there is a homeomorphism from Y_2 onto Y_1 satisfying (4.12).

PROOF. If Y is a completely regular, Hausdorff realcompactification of X, then it is associated, by (3.1) and (2.4), with a filter-closed class Φ on X. On account of (4.2), Y is the completion of X equipped with $\mathcal{U}(\Phi)$. By (4.10), $\mathcal{U}(\Phi)$ is a prerealfine c-uniformity and we know (cf. the proof of (4.2)) that $\mathcal{U}(\Phi)$ is compatible with X.

Conversely, if $\mathcal{U}$ is a prerealfine c-uniformity compatible with X, then $\mathcal{U} = \mathcal{U}(\Phi)$ with a filter-closed Φ by (4.10). Now Φ fulfils the conditions (3.2) to (3.4) (cf. the remarks after (4.3)). Hence there is a realcompactification Y associated with Φ, and it is the completion of X equipped with $\mathcal{U}(\Phi)$ in view of (4.2).

If $\mathcal{U}_1 = \mathcal{U}(\Phi_1)$, $\mathcal{U}_2 = \mathcal{U}(\Phi_2)$ are compatible with X, Φ_1 and Φ_2 filter-closed, then $\Phi_1 = C(\mathcal{U}_1)$, $\Phi_2 = C(\mathcal{U}_2)$ by (4.3). Hence $\mathcal{U}_1$ is coarser than $\mathcal{U}_2$ iff $\Phi_1 \subset \Phi_2$, and the remaining statements follow from (3.6). ∎

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SUR LE PROBLÈME DE TAMMES

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TAMMES [30][1] en étudiant la distribution des orifices sur les grains de pollen de forme sphérique il a observé que pour une espèce donnée, la surface de grain de pollen est approximativement proportionelle au nombre des orifices. Il a soulevé le problème suivant connu sous le nom de problème de TAMMES:

Quel est l'arrangement de n points sur la sphère ayant la proprieté que la distance minimale entre ces points soit maximale?

Considérons sur la sphère un système fini de points et soit d la distance minimale entre les points. Si au tour de chaque point du système on place un cercle de diamètre d, on obtient un système de cercles congruents disjoints, soit un empilement de cercles. L'empilement de cercles est appellé optimal si le rayon des cercles congruents disjoints est maximal. Le problème de TAMMES se peut énoncer aussi sous la forme suivante:[2]

Determiner sur la sphère les empilements optimaux de n cercles congruents.

La littérature de ce problème est assez riche. Sont connues les solutions du problème pour $2 \leq n \leq 12$ et $n=24$.[3] Pour autres valeurs de n ils existent d'une part des bornes supérieurs dues à L. FEJES TÓTH [5], [6], [7], à MOLNÁR [19], [20], [21] et à ROBINSON [25], resp. une borne inférieure due à MOLNÁR [18], d'autre part des empilements supposée optimaux pour $n = 13-23, 26, 27, 30-33, 36, 39, 44, 46-48, 59, 60, 80, 110, 119, 120, 122$. La méthode des bandes sphériques de MOLNÁR [18], utilisable pour un n quelconque, n'étant pas crée pour un but d'optimisation,

[1] Pour le nombre en crochets se reporter à la bibliographie.

[2] Pour autres énonces du problème ou pour des problèmes analogues v. L. FEJES TÓTH [7], p. 123. Nous signalons ici la généralisation interessante du problème de TAMMES due à MOLNÁR [22], [23]. Cette généralisation a des applications par ex. concernant la structure des petits virus (v. MOLNÁR [24]).

[3] v. tableau 2.

en cas d'un n donné on le peut améliorer facilement. Le principe de GOLDBERG [11], [13] en considérant des arrangements axialement symétriques d'ordre 3, 4 et 5 est en général suitable. Le premier principe constructif (le principe de spirale) basé sur l'idée de spirale de KARABINTA [16] vient d'apparaitre dans l'article de KARABINTA et SZÉKELY [17]. Ce principe dans sa forme originelle n'est pas utilisable pour un n quelconque.

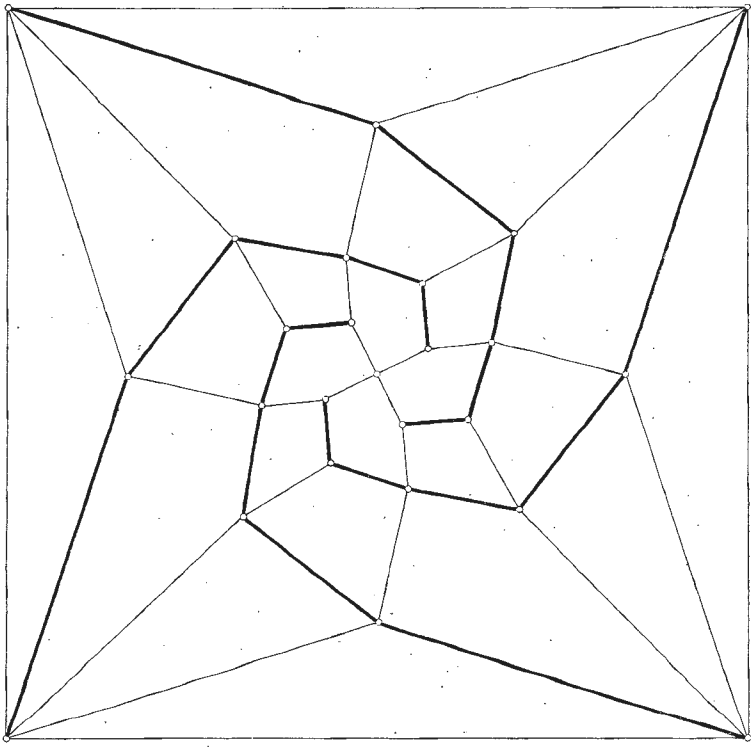

Fig. 1. (25)

Dans notre article nous généralisons le principe de spirale de telle manière que le principe modifié nous nous amène aux résultats totalement nouveaux pour $n = 29, 34, 38, 50, 53-58$ (v. tableau 2) et on obtient aussi des améliorations par rapport aux résultats antérieurs pour $n = 25, 28, 35, 37, 40-42, 45, 51, 52, 59, 60$ (v. tableau 1).[4]

[4] Dans cet article nous nous bornons aux valeurs de $n \leq 60$. Pour des valeurs supérieures de n les recherches sont en train d'avancer. Le tableau 2 contienne les empilements supposés optimaux.

Parmi nos empilements supposés optimaux, interessants sont en particulier les empilements pour $n = 59$ et $n = 60$ qui représentent des contreexemples concernant une supposition de ROBINSON [27].[5] En effet ROBINSON en caractérisant par une proprieté extremale le système de

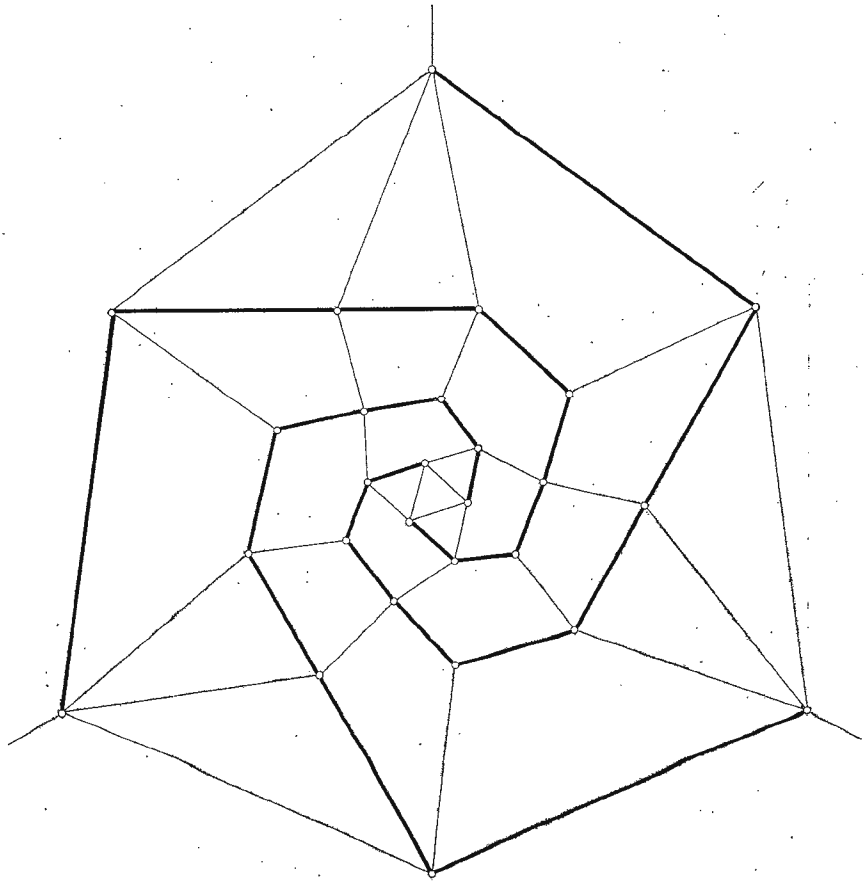

Fig. 2. (28)

[5] Selon une remarque de KATALIN BORNEMISSZA [1] le résultat de GOLDBERG [10] pour $n=19$ est erroné.

Nous mentionons ici que l'article de STROHMAJER [29] a une caractère sommaire, l'auteur des empilements supposés optimaux pour $n=18$ et 52 est D. KÓLYA (v. [29] p. 51).

60 sommets du polyedre archimedien {3, 3, 3, 3, 5},[6] a soulevé la supposition que ce système de points ayant la distance minimale entre ses points de 26°49′17″ nous fournit aussi la solution du problème de TAMMES pour $n = 59$ et $n = 60$. Mais dans notre constructions pour $n = 59$ et $n = 60$ la distance minimale entre les points est de 27°0′37,7″ > 26°49′17″, ainsi la supposition de ROBINSON est fausse.

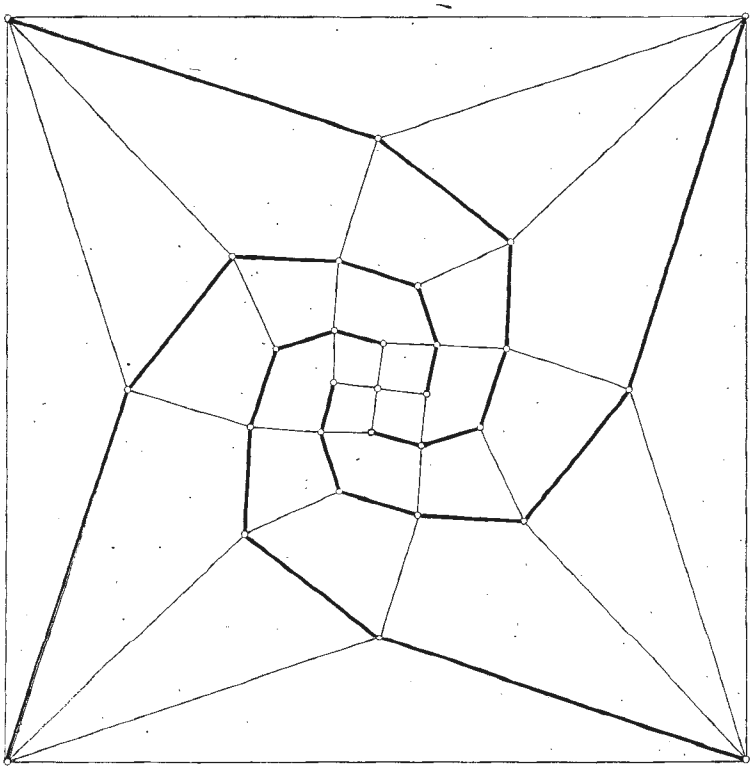

Fig. 3. (29)

[6] La premiere caractérisation du système des sommets du polyèdre {3, 3, 3, 3, 5} par une proprieté extremale est due à MOLNÁR [22], [23].

La première partie de notre article contienne la description de notre principe modifié de spirale, tandis que la deuxième partie contienne des remarques concernant les calculs numériques et les empilements supposés optimaux.

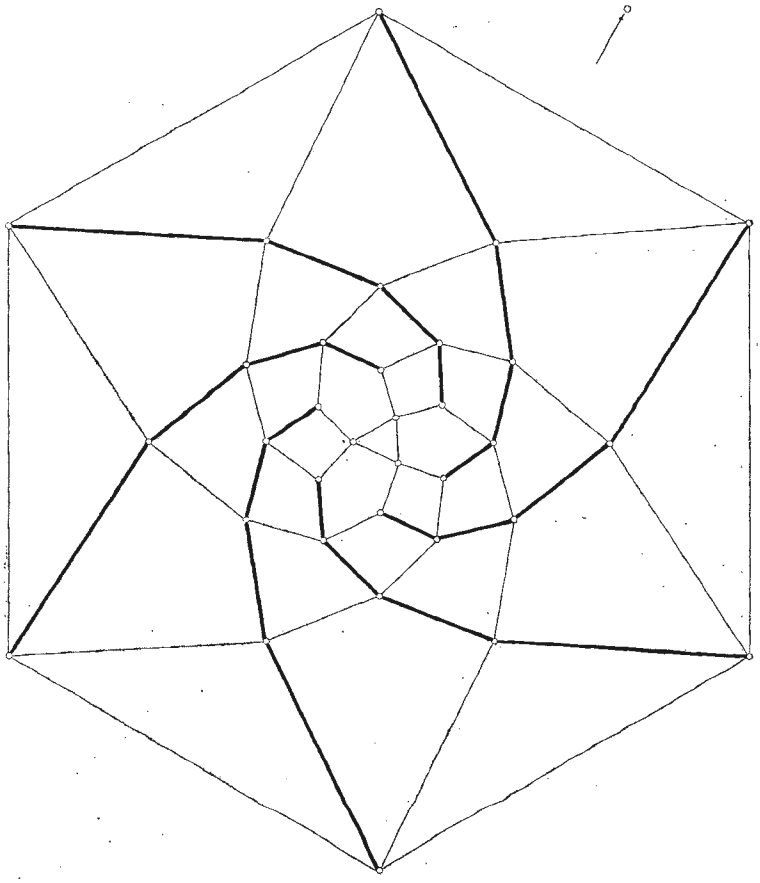

Fig. 4. (34)

1. La construction des empilements correspondants au principe modifié de spirale est la suivante:

Soit d le diamètre du cercle du système considéré et soit $O_1, O_2, \ldots O_k$ ($k = 3, 4, 5, 6, 7$) les sommets d'un polygone regulier sphérique de coté d. Au tour de $O_1, O_2, \ldots, O_k$ nous considérons les cercles correspondants $C_1, C_2, \ldots, C_k$ de diametre d. Ces cercles représentent les cercles de départ des branches de spirale. Si sur toutes les branches de spirale sont déjà arrangés les l-èmes ($l \geq 1$) cercles, un cercle consecutif, soit $(l+1)$-ème, est pris de façon suivant:

Sur une branche donnée de spirale au l-ème cercle, si est possible nous considérons un cercle C tangent de diametre d qui soit tangent aussi au moins à une cercle du branche consecutive de spirale, mais qui n'a pas de points interieurs communs avec les cercles déjà considérés. Si le système

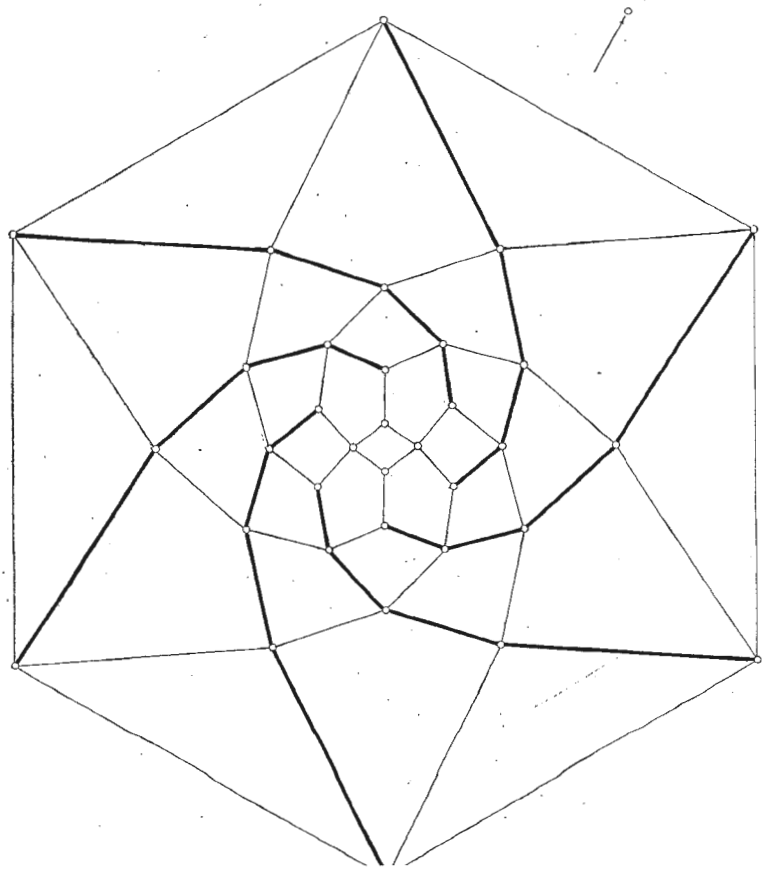

Fig. 5. (35)

$\{C\}$ de cercles qu'on peut obtenir par des rotations consecutives de C d'angle $\frac{2\pi}{k}$, autour du centre N (pôle nord) du polygon $O_1O_2\ldots O_k$, a des cercles disjoints, on considère les cercles de ce système $\{C\}$ comme les $(l+1)$-èmes cercles des branches de spirale.

Si le diamètre d du cercle est choisi de tel façon, que les derniers k cercles sont tangents, alors le système des cercles considerés donne un empilement qu', on suppose optimal pour $n=km$, si $k\leq 5$. Si $k\geq 6$ l'empilement supposé optimal est de type $n=km+2$, puisque au tour de N (pôle nord) et de S (pôle sud) on peut placer encore deux cercles, qui n'ont pas de point interieur commun avec le système de cercles considéré. Dans notre tableaux les empilements de ce type seront notés par km si $k = 3, 4, 5$ resp. par 1, km, 1 si $k\geq 6$.

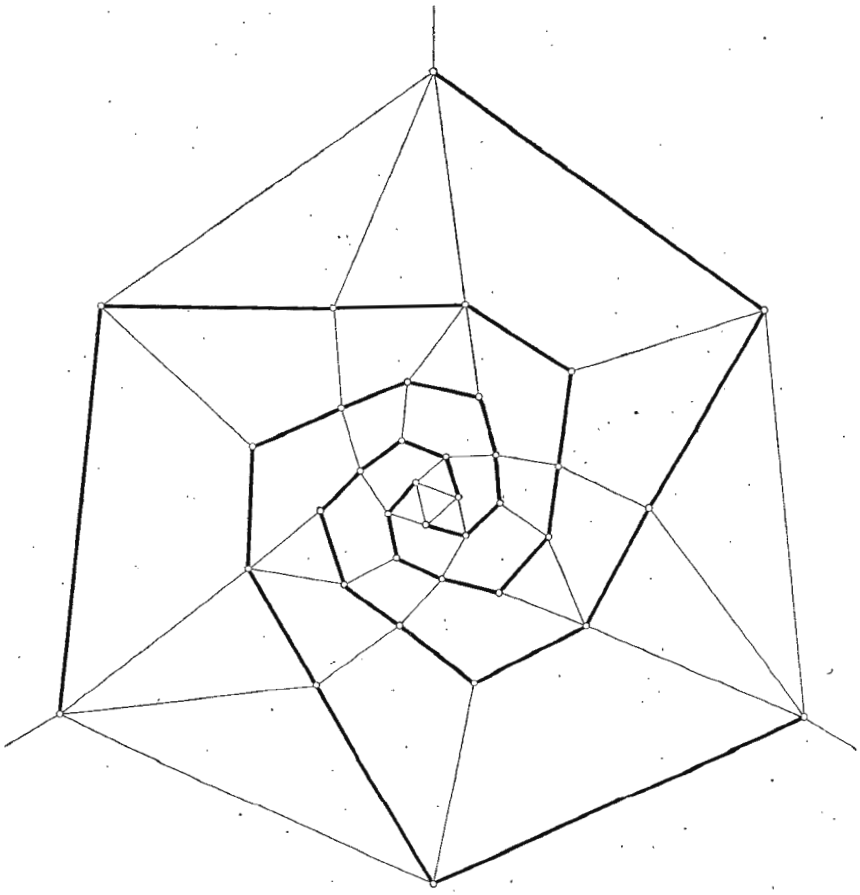

Fig. 6. (37)

Si $k = 3, 4, 5$ et le diamètre d est choisi de tel façon que S (le pôle sud) est à la distance d de centres des cercles de la dernière operation effectuée sur les branches de spirale, dans ce cas on peut encore placer un cercle au tour de S. Cet empilement est de type $km+1$.

Comme nous voyons dans la construction des spirales les deux pôles N et S ne jouent pas toujours un rôle symetrique, car le principe de spirale "condense" vers le pôle nord. Par conséquence, en général on obtient des arrangements différents si les centres des cercles de départ sont placés sur les sommets d'un polygon regulier inscrit dans un cercle de rayon d au tour du pôle nord. Dans ce cas nous pouvons placer un cercle de diamètre

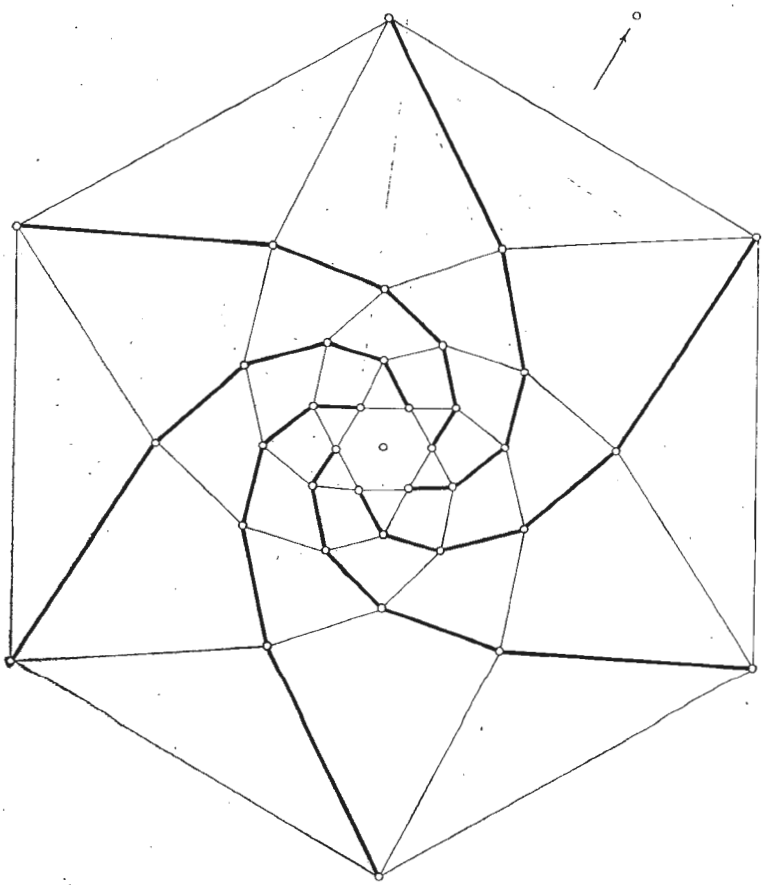

Fig. 7. (38)

d aussi au tour de N, mais ce cercle on ne le calcul pas aux branches de spirale. Les empilements obtenus de tel façon sont de type $1+km$ resp. $1+km+1$.

En plusieurs cas etait preférable de choisir le diamètre d de telle manière, que dans la place libre au tour de S soit placé encore 3 ou quatre cercles (v. les fig. 5, 8).[7]

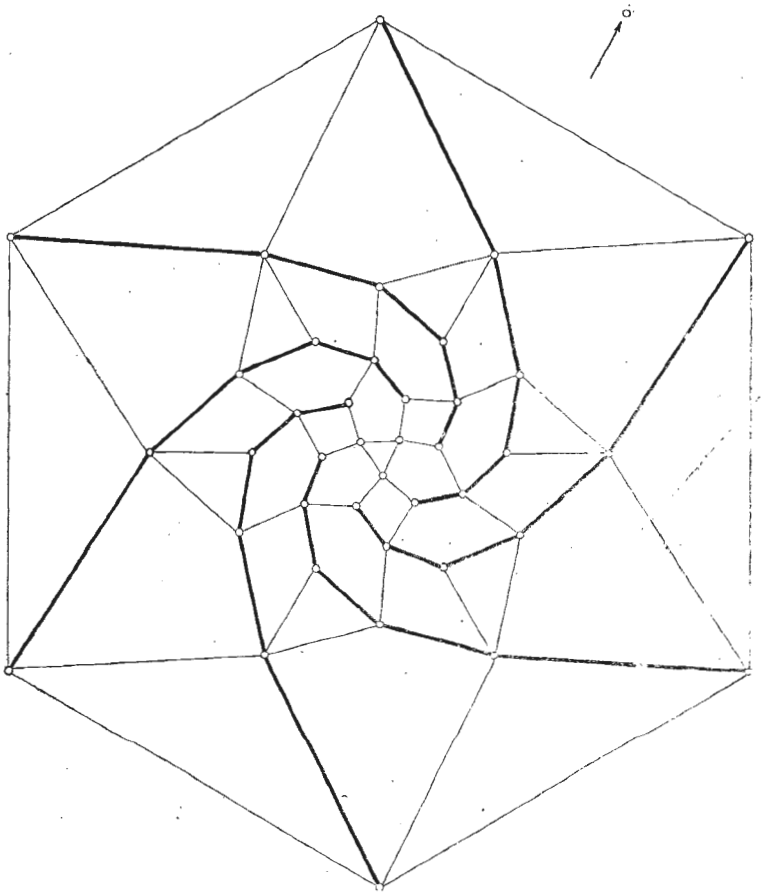

Fig. 8. (40)

[7] Les illustrations de notre article contiennent les projections stèrèographiques des centres de cercles, par rapport à N. Les segments de droite joignent les centres de cercles tangents entre eux. Les lignes polygonales épais contiennent les centres de cercles de la même spirale.

2. Remarques. 1. Les calculs numériques[8] montrent que dans l'intervalle $12 < n \leq 60$ à l'aide de notre principe modifié de spirale soit on retrouve la majorité des empilements antérieurs, soit on obtient des meilleurs empilements (v. tableau 1), soit on trouve des nouveaux résultats. Mais il y a aussi des cas ($n = 10, 14, 17, 19, 20, 22, 33, 46-48$) qui n'ont

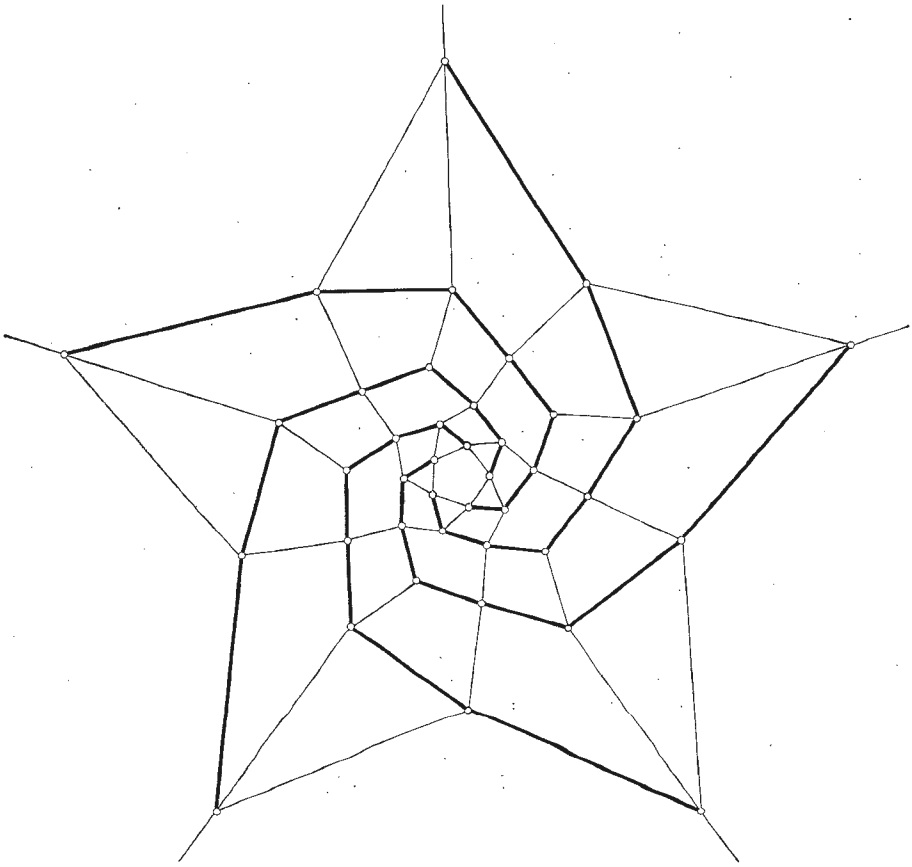

Fig. 9. (41)

pas une structure de spirale et qui nous avertissent que le principe de spirale ne fournit pas toujours les empilements optimaux, en autres mots, le problème de TAMMES en général reste encore ouvert aussi dans le sens de la supposition de la solution.

2. En notant par d_n le diamètre du cercle d'un empilement optimal de n cercles, s'avère interessant le problème de determiner les n pour les

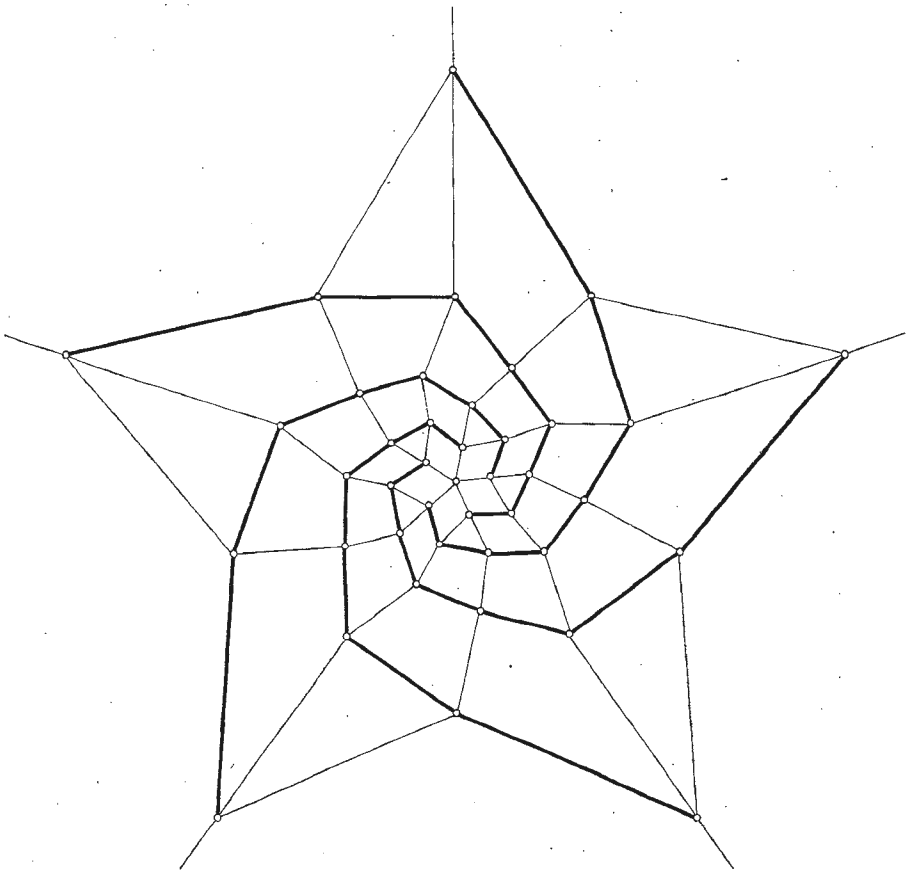

Fig. 10. (42)

[8] Les calculs ont été effectués par l'ordinateur ODRA 1304 de l'Université L. Eötvös. Nous remercions l'aide precieuse obtenue de la Chaire de Mathématique Numérique et Mécanique dirigée par I. KÁTAI et l'aide technique de Z. HARMATHY et de L. NYIRÁDI.

Concernant la précisité de calcul nous signalons le suivant: Le problème de base concernant la précisité de calcul est que les erreurs dans la détermination des nouveaux cercles sur les branches des spirales, peuvent conduire au changement brusque de la distance minimale. D'aprés les calculs, à l'aide des coordonnées polaires que nous avons eu à notre disposition nous avons detérminé la distance minimale independant du rayon du premier cercle. Ainsi le résultat du calcul de controle est chargé seulement des erreurs d'une seule formule trigonometrique sphérique de cosinus, d'ont l'erreur – compte tenu de l'erreur de représentation graphique de l'ordinateur – est inférieur a 0,02″. Nous mentionons que le rapport avec la valeur initiale en général était essentiellement meilleur. Notre tableaux contiennent naturellement les valeurs obtenues déjà par les calculs de controle.

quels $d_{n-1} = d_n$. Selon MOLNÁR beaucoup de n (non seulement les valeurs 6, 12, 24, 48, 60, 120 citées par ROBINSON [27]) satisfont la relation $d_{n-1} = d_n$ et ne sont pas exclus des cas pour les quels $d_{n-i} = d_n$, ou $i > 1$. Peut être c'est 48 le premier pour lequel $i = 2$.

En fin nous remercions l'aide précieuse donnée par J. MOLNÁR.

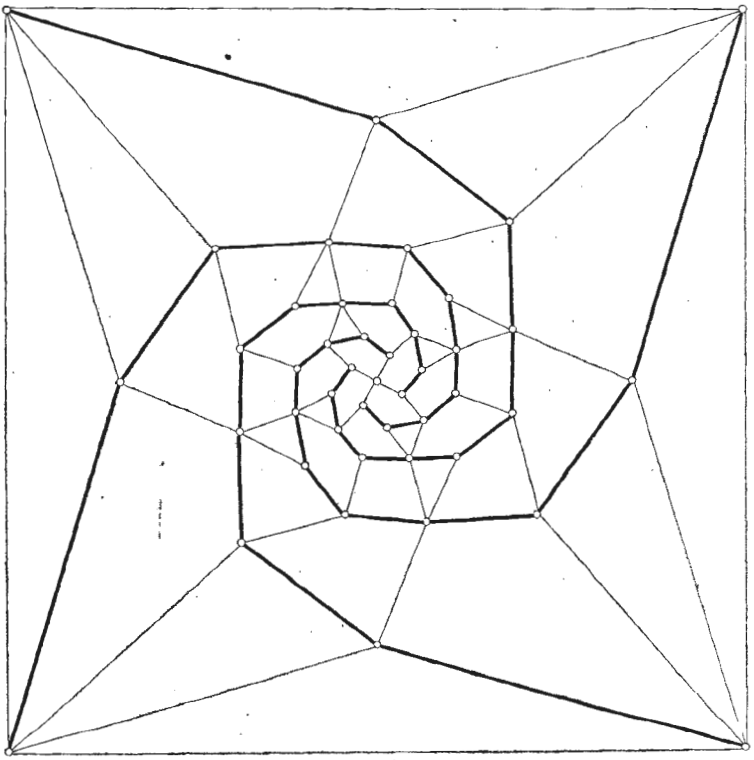

Fig. 11. (45)

Tableau 1

n	diamètre (nouvel arrangement)	type du graphe (nouvel arrangement)	Fig.	diamètre ancien arrangement	auteur, article
25	41°28′15,4″	$4m+1$	1	41°24′	JUCOVIČ [14]
28	39°2′28,3″	$1+3m$	2	38°47′	GOLDBERG [13]
35	34°51′26,4″	1; $6m$; 4	5	34°41′5,2″	KARABINTA – SZÉKELY [17]
37	34°16′6,0″	$1+3m$	6	33°30′	GOLDBERG [11]
40	32°57′24,8″	1; $6m$; 3	8	32°49′29,1″	KARABINTA – SZÉKELY [17]
41	32°38′34,4″	$1+5m$	9	32°19′	GOLDBERG [11]
42	32°30′22,9″	$1+5m+1$	10	32°10′20,6″	KARABINTA [16]
45	31°13′29,3″	$4m+1$	11	31°12′12,4″	KARABINTA – SZÉKELY [17]
51	29°18′13,7″	$5m+1$	13	28°53′50,3″	KARABINTA – SZÉKELY [17]
52	29°3′25,2″	$1+5m+1$	14	28°46′	KÓLYA [29][5]
60	27°37,7″	$3m$	16	26°45′17″	ROBINSON [27]

Tableau 2

n	diamètre	type du graphe	auteur, article
2	180°		
3	120°	$3m$	FEJES TÓTH [5]
4	109°28′16″	$3m+1$	FEJES TÓTH [5]
5	90°	$1+3m+1$	FEJES TÓTH [5]
6	90°	$1+4m+1$	FEJES TÓTH [5]
7	77°51′58″	$3m+1$	SCHÜTTE – VAN DER WAERDEN [28]
8	74°51′31″	$4m$	SCHÜTTE – VAN DER WAERDEN [28]
9	70°31′44″	$3m$	SCHÜTTE – VAN DER WAERDEN [28]
10	66°19′	0[9]	DANZER [4]
11	63°26′5,8″	$1+5m$	DANZER [3]
12	63°26′5,8″	$3m_1 = 1+5m_2+1$	FEJES TÓTH [5]
13	57°8′12,1″	$4m+1$	SCHÜTTE [28]
14	55°40′	0	SCHÜTTE [28]
15	53°39′28,2″	$3m$	SCHÜTTE – VAN DER WAERDEN [28]
16	52°14′39,7″	$4m$	SCHÜTTE – VAN DER WAERDEN [28]
17	51°5′	0	KARABINTA [16]
18	49°33′5,7″	$1+4m+1$	KÓLYA [29][5],
19	–		[10], [1][5]
20	47°25′52″	0	VAN DER WAERDEN [31]
21	45°33′26,2″	$3m$	KARABINTA – SZÉKELY [17]
22	45°1′	0	GOLDBERG [13]
23	43°41′26,7″	$4m+3$	ROBINSON [25]
24	43°41′26,7″	$3m_1 = 4m_2$	ROBINSON [25]
25	41°28′15,4″	$4m+1$	*[10]

[9] Le graphe de type 0 signifile que l'empilement correspondant n'a pas une structure de spirale.
[10] Les empilements notés par * représentent des nouveaux resultats.

n	diamètre	type du graphe	auteur, article
26	41°1′34,1″	$5m-1$	GOLDBERG [11]
27	40°40′39,2″	$1+5m+1$	GOLDBERG [11]
28	39°2′28,3″	$1+3m$	*
29	38°12′10,3″	$4m+1$	*
30	38°9′39,2″	$5m$	GOLDBERG [11]
31	37°42′35,3″	$1+5m$	GOLDBERG [11]
32	37°25′50,9″	$1+5m+1$	GOLDBERG [11]
33	36°15′32″	0	GOLDBERG [12]
34	35°23′18,8″	1; 6 m; 3	*
35	34°51′26,4″	1; 6m; 4	*
36	34°41′5,2″	$3m$	KARABINTA [16]
37	34°16′6,0″	$1+3m$	*
38	34°15′2,3″	1; 6m; 1	*
39	33°22′36,5″	$3m$	KARABINTA [16]
40	32°57′24,8″	1; 6m; 3	*
41	32°38′34,4″	$1+5m$	*
42	32°30′22,9″	$1+5m+1$	*
43	–		
44	31°57′37,4″	$4m$	KARABINTA [16]
45	31°13′29,3″	$4m+1$	*
46	–		
47	30°46′	0	ROBINSON [27]
48	30°46′	0	ROBINSON [27]
49	–		
50	29°45′10,6″	1; 6m; 1	*
51	29°18′13,7″	$5m+1$	*
52	29°3′25,2″	$1+5m+1$	*
53	28°24′53,7″	$4m+1$	*
54	28°16′32,6″	$1+4m+1$	*
55	27°48′56,8″	$5k$	*
56	27°47′28,3″	1; 6m; 1	*
57	–		
58	27°32′43,7″	1; 7m; 1	*
59	27°0′37,7″	$3m+2$	*
60	27°0′37,7″	$3m$	*

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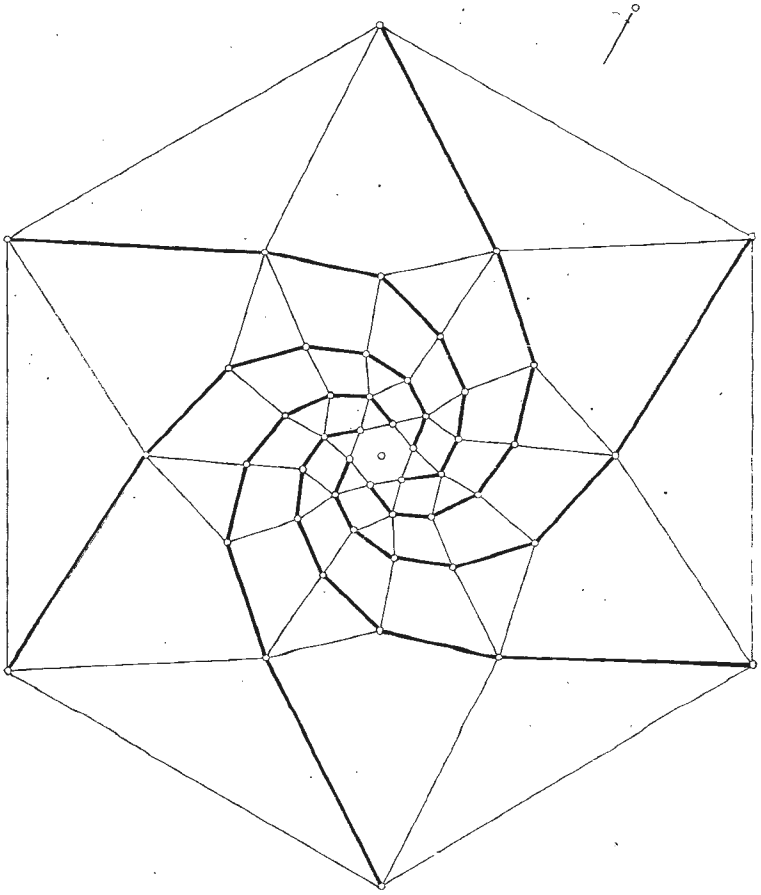

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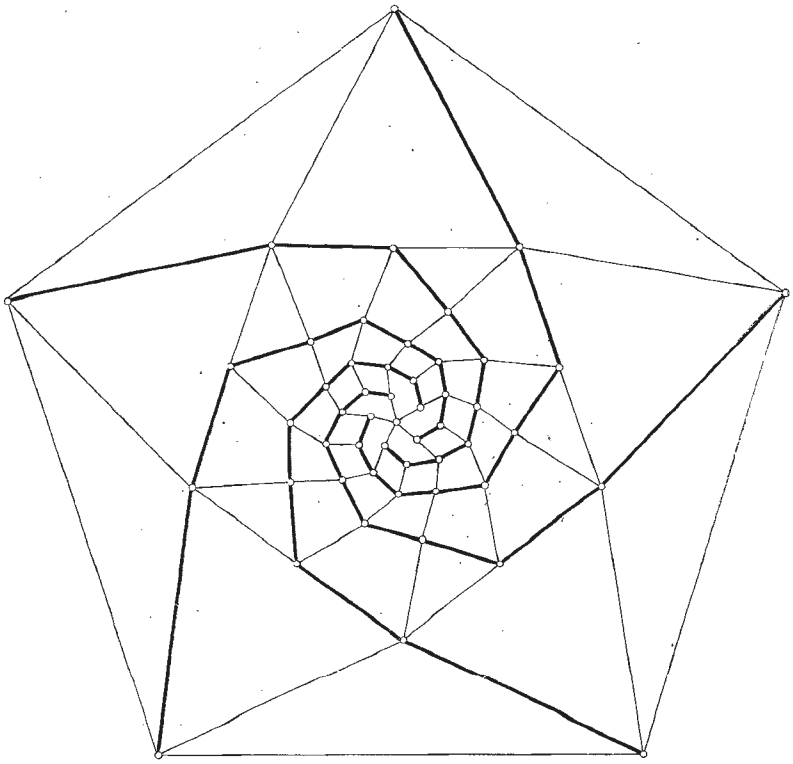

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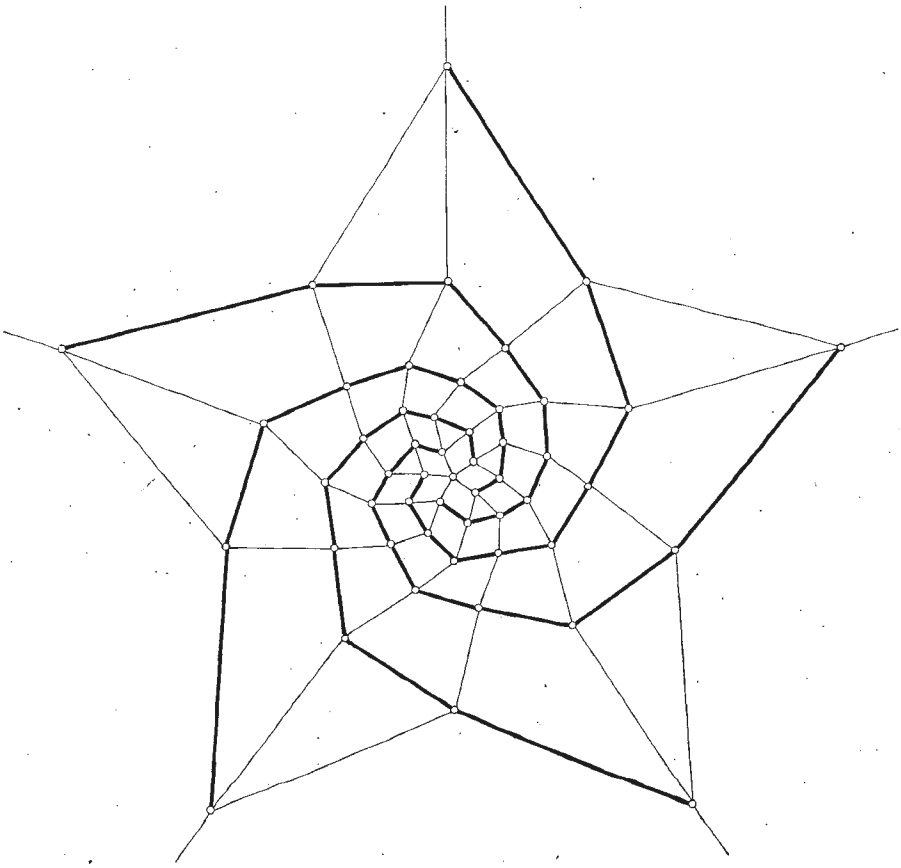

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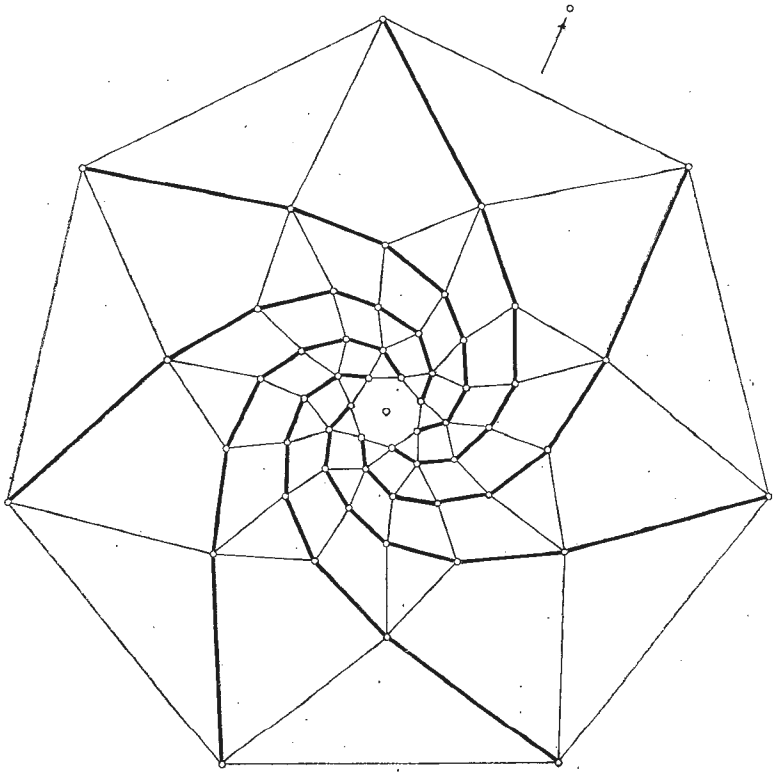

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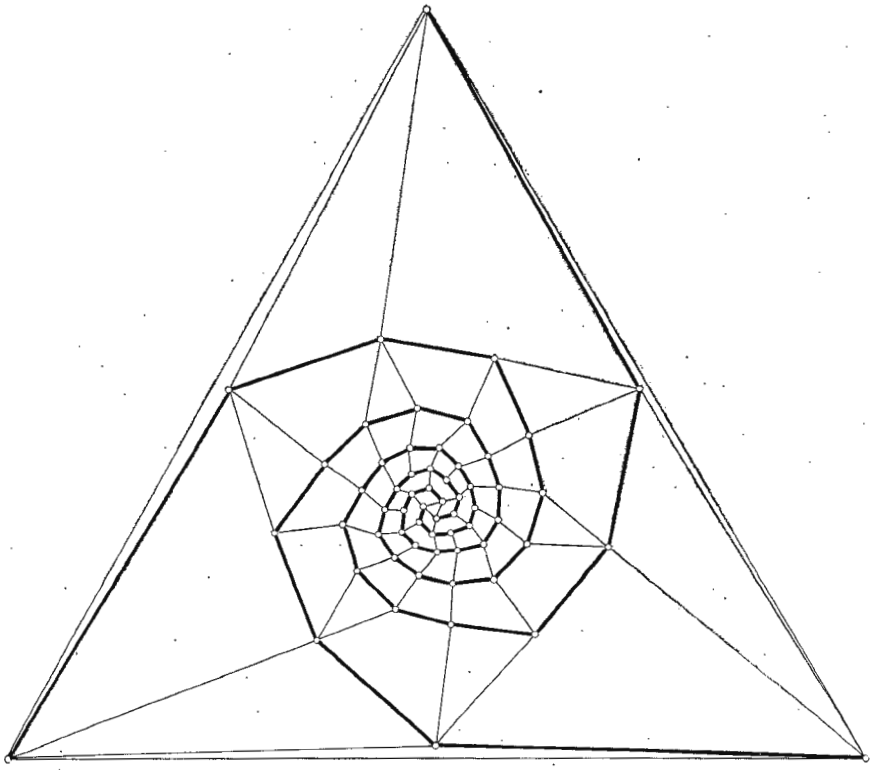

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ON LINEAR DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS OF INFINITE ORDER

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Since entire functions of finite order play a considerable role in this work we begin with stating some known facts concerning these functions.

Let $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ mean an analytic function on the entire complex plane, and $M(r) = \max_{0 \leqq \varphi \leqq 2\pi} |F(re^{i\varphi})|$.

If
$$\varrho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$
is finite, then $F(z)$ is said to be of finite order ϱ, otherwise of infinite order If ϱ is finite then clearly $\varrho \geqq 0$ by Liouville's theorem. If $F(z)$ is of finite order ϱ then
$$\sigma = \limsup_{r \to \infty} \frac{\log M(r)}{r^{\varrho}}$$
is called its type, again $0 \leqq \sigma \leqq \infty$. If $0 < \sigma < \infty$ ($\sigma = 0$, $\sigma = \infty$) then $F(z)$ is said to be of normal (minimal, maximal type of order ϱ (GELFOND [1]).

To show the meaning of the order clearer we mention that if $F(z)$ is of order ϱ, this means that to every $\delta > 0$ it can be found an $r_0 = r_0(\delta)$ so that
$$|F(z)| < \exp(|z|^{\varrho+\delta})$$
when $|z| \geqq r_0$ and at the same time
$$|F(z)| > \exp(|z|^{\varrho-\delta})$$
at least for one sequence $\{z_n\}$ where $|z_n| \to \infty$.

If the function is of order ϱ and its type is σ, then this means that σ is the smallest number satisfying $|F(z)| < \exp(\sigma|z|^{\varrho}$ for $|z|$ large enough.

By $I(\varrho, \sigma)$ we denote the class of entire functions of order less than ϱ, or of order ϱ and type at most σ. Obviously this class is a linear space if we define the addition and the multiplication by complex scalars in the usual way.

Let's consider the space $I(1, \sigma)$ where $0<\sigma<\infty$. Suppose that $F(z) \in I(1, \sigma)$. In this case we can introduce a function

$$\varphi(\xi) = \frac{1}{\xi} \int_0^\infty F\left(\frac{\omega}{\xi}\right) e^{-\omega} \, d\omega$$

called the Borel transform of $F(z)$. Using the definition of the order and type we can show that in case $\varrho<1$ the integral converges absolutely and the convergence is uniform outside every closed disk $|\xi| \leqq r$; $r>0$, and if $\varrho=1$ then the integral converges absolutely and uniformly outside the disk $|\xi| \leqq \sigma_1$ where $\sigma_1>\sigma$.

So if $|\xi|>\sigma_1$ then we obtain by a term by term integration of F:

$$\varphi(\xi) = \sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} .$$

Since this formula holds for every $\sigma_1>\sigma$ it is true for every $|\xi|>\sigma$.

We can get directly the radius of the disk outside which $\varphi(\xi)$ is analytic. (Gelfond [1]).

The application of the Borel function is very useful in the case of equations of infinite order with constant coefficients. Now we shall use the concept of this function for investigating some special kinds of equations of infinite order with polynomial coefficients.

1. Consider the equation

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{kn} z^k y^{(k)}(\alpha_{kn} z) = f(z),$$

a generalization of Euler's differential equation. This equation was investil gated by an elementary method by K. SEITZ (see [2]). The fundamenta idea of his method was that he assumed the solution to be of the form

$$\sum_{p=0}^{\infty} \frac{f_p}{B_p} z^p .$$

This was substituted into the equation and in this way he determined the unknown values B_p.

Now let $f(z) = \sum_{p=0}^{\infty} f_p z^p$ be an entire function, and we suppose the existence of a constant P for which

$$|f_p| \leqq p^p; \; p > 0.$$

Here we shall give a method applicable for a larger class of equations — for example as we remarked for the equations of infinite order with constant coefficients — and giving a more natural way of solving. With this method, however, we can find solutions only in the class of entire functions.

Let us seek the solution $y(z) = \sum_{p=0}^{\infty} \frac{y_p}{p!} z^p$ from $I(1, \sigma)$. We assume that σ is finite, its value will be exactly given later. Let $\varphi(\xi)$ be the Borel transform of the function $y(z)$. Integrating term by term one can easily show the validity of the formula

$$y(z) = \frac{1}{2\pi i} \int_{|\xi|=\sigma_1} \varphi(\xi) e^{z\xi} d\xi, \text{ where } \sigma_1 > \sigma. \tag{1}$$

Differentiating k times with respect to z we obtain

$$y^{(k)}(z) = \frac{1}{2\pi i} \int_{|\xi|=\sigma_1} \varphi(\xi) \xi^k e^{z\xi} d\xi. \tag{2}$$

Let us denote the circle $|\xi| = \sigma_1$ by C and substitute $y^{(k)}(z)$ from (2) into the equation:

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{kn} z^k \frac{1}{2\pi i} \int_C \varphi(\xi) \xi^k e^{\alpha_{kn} z\xi} d\xi = f(z).$$

Taking the power series expansion of $e^{\alpha_{kn} z\xi}$ we can interchange the order of the summation and integration, since the series is uniformly convergent:

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_{kn} \alpha_{kn}^l \frac{1}{l!} \frac{1}{2\pi i} \int_C \varphi(\xi) \xi^{k+l} d\xi^{k+l} = f(z). \tag{3}$$

Notice that here

$$\frac{1}{2\pi i} \int_C \varphi(\xi) \xi^{k+l} d\xi = y_{k+l}$$

since the left hand side is just the coefficient of the term of power $-(k+l)-1$ in the Laurent series expansion of $\varphi(\xi)$ around 0. Assuming uniform convergence in (3) we have using the notation $k+l = p$:

$$\sum_{p=0}^{\infty} \sum_{k=0}^{p} \sum_{n=0}^{\infty} a_{kn} \alpha_{kn}^{p-n} \frac{1}{(p-k)!} y_p z^p = \sum_{p=0}^{\infty} f_p z^p$$

from where one can see that choosing

$$y_p = \frac{f_p}{\sum_{k=0}^{p} \sum_{n=0}^{\infty} a_{kn} \alpha_{kn}^{p-k} \frac{1}{(p-k)!}}$$

the equality is attained. Let's denote the denominator by A_p. If A_p was finite it would guarantee the proper convergences supposed above. Therefore we prescribe the following conditions:

$$|\alpha_{kn}| \leqq q^n \text{ and } |a_{kn}| \leqq N_k q^{*n} \text{ where } 0 < q < 1;\ 0 < q^* < 1;\ N_k > 0.$$

Thus

$$|A_p| \leqq \sum_{k=0}^{p} \sum_{n=0}^{\infty} |a_{kn}| \cdot |\alpha_{kn}|^{p-k} \frac{1}{(p-k)!} \leqq$$

$$\leqq \sum_{k=0}^{p} \sum_{n=0}^{\infty} \frac{1}{(p-k)!} N_k (q^* q^{p-k})^n = \sum_{k=0}^{p} \frac{N_k}{(p-k)!} \frac{1}{1-q^* q^{p-k}}$$

which is finite. Moreover it will be comfortable to suppose the existence of the positive quantity Q satisfying

$$|A_p| \geqq Q^p. \tag{4}$$

Consider

$$y(z) = \sum_{p=0}^{\infty} \frac{f_p}{p!\, A_p} z^p.$$

We recall a well known result from the theory of entire functions:

$$\sum_{n=0}^{\infty} x_n z^n \in I(\varrho, \sigma) \quad (0 < \varrho < \infty,\ 0 < \sigma < \infty)$$

if and only if

$$\limsup_{n \to \infty} \{n!^{1/\varrho} |x_n)|^{1/n} \leqq (\varrho\, \sigma)^{1/\varrho}$$

(see [3]).

Applying this condition together with our conditions imposed above we get for $y(z)$:

$$\limsup_{p \to \infty} \left\{ p! \left| \frac{f_p}{p!\, A_p} \right| \right\}^{1/p} = \limsup_{p \to \infty} \left\{ \left| \frac{f_p}{A_p} \right| \right\}^{1/p} \leqq \frac{P}{Q}.$$

Thus indeed $y(z) \in I(1, \sigma)$ with the choice $\sigma = \dfrac{P}{Q} < \infty$.

2. Now we shall solve the following integro-differential equation of infinite order:

$$\sum_{k=0}^{\infty} \int_0^a A_k(s)\, y^{(k)}(sz)\, ds\, z^k = f(z),$$

where $A_k(s)$ is continuous on $[0, a]$, and $A_k(s) \geqq S^k k^k$ if $s \in [0, a]$ and $k \geqq 0$. Furthermore let the coefficients of $\sum_{p=0}^{\infty} f_p z^p$ satisfy the inequalities

$$|f_p| \leqq K^* K^p \quad /K^*, K > 0/. \tag{5}$$

Our method will be useful in this case, too. Again we seek the solution in $I(1, \sigma)$, (1) and (2) are valid. Let us put (2) into the equation:

$$\sum_{k=0}^{\infty} \int_0^a A_k(s) \frac{1}{2\pi i} \int_C \varphi(\xi)\, \xi^k e^{zs\xi}\, d\xi\, z^k\, ds = f(z).$$

Using the Taylor series expansion of $e^{zs\xi}$ we can again interchange the summation and the integration:

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_0^a \int_C \varphi(\xi)\, \xi^{n+k}\, d\xi\, A_k(s)\, s^n \frac{1}{n!} z^{n+k}\, ds = f(z).$$

Applying the remark in section 1 we have

$$y_{n+k} = \frac{1}{2\pi i} \int_C \varphi(\xi)\, \xi^{n+k}\, d\xi$$

and thus

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \int_0^a A_k(s)\, s^n\, ds \frac{y_{n+k}}{n!} z^{n+k} = f(z).$$

We assume the absolute and uniform convergence on the left hand side and sum with respect to $n+k = p$:

$$\sum_{p=0}^{\infty} \sum_{n=0}^{p} \int_0^a A_{p-n}(s)\, s^n\, ds \frac{1}{n!} y_p z^p = \sum_{p=0}^{\infty} f_p z^p.$$

Choosing

$$y_p = \frac{f_p}{\sum_{n=0}^{p} \int_0^a A_{p-n}(s)\, s^n\, ds \frac{1}{n!}}$$

equatily is attained. Let's denote the denominator by B_p. We state that the convergence will be the immediate consequence of our conditions imposed upon $A_k(s)$ and f_p, moreover $y \in I(1, \sigma)$, $\sigma > 0$. To prove this we need the following estimation:

$$|B_p| \geqq \sum_{n=0}^{p} \int_0^a s^{p-n}(p-n)^{p-n} s^n \, ds \frac{1}{n!} =$$

$$= \sum_{n=0}^{p} (p-n)^{p-n} \frac{a^{p+1}}{(p+1)\, n!} > p^p \frac{a^{p+1}}{p-1}$$

whence

$$\limsup_{p\to\infty} \left\{ p! \left| \frac{f_p}{p!\, B_p} \right| \right\}^{1/p} = \limsup_{p\to\infty} \left\{ \left| \frac{f_p}{B_p} \right| \right\}^{1/p} \leqq$$

$$\leqq \limsup_{p\to\infty} \left\{ \frac{K^* K^p}{p^p \dfrac{a^{p+1}}{p+1}} \right\}^{1/p} = 0$$

so $y(z)$ satisfies the necessary and sufficient condition of belonging to $I(1, \sigma)$, for every $0 < \sigma < \infty$.

REMARK 1. This method enables us to prove a uniqueness theorem for the equation $\sum_{n=0}^{\infty} a_n z^n y^{(n)}(z) = f(z)$. The necessary conditions and the whole theorem can be found by an elementary method, too. So here the proof will be omitted.

REMARK 2. YU. F. KOROBEINIK, [4] got a uniqueness theorem for differential equations of infinite order with polynomial coefficients but in his investigations the degree of the polynomial before the n-th derivative of the unknown function is less than n.

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INTERPOLATION ON THE ROOTS OF LAGUERRE POLYNOMIALS

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Let

$$-\infty \leq a < x_1 < x_2 < \ldots < x_n < b \leq +\infty \tag{1}$$

be n arbitrary points in (a, b) and consider the interpolating process

$$R_n(f, x) \overset{\text{def}}{=} \sum_{k=1}^{n} f(x_k) \left(\frac{\omega(x)}{\omega'(x_k)(x-x_k)} \right)^2 = \sum_{k=1}^{n} f(x_k) \cdot l_k^2(x)$$

$$\left(\omega(x) \overset{\text{def}}{=} \prod_{k=1}^{n} (x-x_k) \right). \tag{2}$$

This was investigated by FEJÉR ([2]) for Jacobi abscissas with parameters $-1<\alpha, \beta<0$ and $\alpha=\beta=0$, by GRÜNWALD ([4]) for normal point systems in $(-1, 1)$, by BALÁZS and TURÁN ([1]) for the roots of Hermite polynomials in $(-\infty, \infty)$, by JOÓ ([5]) for Jacobi abscissas with arbitrary parameters $\alpha, \beta > -1$.

In this paper we shall deal with this process in the case of $a=0$, $b=\infty$, $L_n^{(\alpha)}(x_k) = 0$ $(k = 1, \ldots, n)$, where $L_n^{(\alpha)}(x)$ denotes the n-th Laguerre polynomial with parameter α $(\alpha > -1)$. We shall prove the following theorems.

THEOREM 1. *If $\alpha > -1$, $f(x)$ is continuous in $[0, \infty)$, $f(x) = 0(e^x x^{-\alpha-\varepsilon})$ $(x \geqq 1, x \to \infty, \varepsilon > 0)$, $R_n^{(\alpha)}(f, x)$ is the interpolating polynomial* (2) *belonging to $f(x)$ in the case of $a=0$, $b=\infty$, $L_n^{(\alpha)}(x_k)=0$ $(k = 1, \ldots, n)$ then*

$$|R_n^{(\alpha)}(f, x) - f(x)| \leq C(\|f\|_{[0,d]} + \|f\|^*) n^{-1/4} + C\,\omega(f; 0, 2d; n^{-1/4})$$

$$(0 < \varepsilon' \leq x \leq d,\ d > 1,\ \varepsilon' > 0 > \alpha > -1) \tag{3}$$

where

$$\|f\| \overset{\text{def}}{=} \max_{[0,d]} \; \max_{0 \leq x \leq d} |f(x)|, \quad \|f\|^* \overset{\text{def}}{=} 2 \sup_{1 \leq x < \infty} \left(|f(x)|/e^x x^{-\alpha-\varepsilon}\right),$$

$$\omega(f; a, b; \delta) \overset{\text{def}}{=} \sup_{\substack{|x-y| \leq \delta \\ x, y \in [a,b]}} \{|f(x)-f(y)|\}$$

is the continuity modulus in $[a, b]$ *of the continuous function* $f(x)$, $\varepsilon' > 0$, $d > 1$ *are arbitrary fixed numbers, the C-s remain independent of* x *and* n *but depends on* ε *and* d.

COROLLARY. Under the conditions of theorem 1, $R_n^{(\alpha)}(f, x)$ tends to $f(x)$ uniformly in $[\varepsilon', d]$.

THEOREM 2. *For Jacobi and Laguerre abscissas, the relations*

$$\lim_{n\to\infty} \sum_{k=1}^{n} l_k^2(x) = \begin{cases} \dfrac{1}{-\alpha} & \text{if} \quad x = 1, \quad -1 < \alpha < 0, \quad \beta > -1 \\ +\infty & \text{if} \quad x = 1, \quad \alpha \geq 0, \quad \beta > -1 \\ 1 & \text{if} \quad -1 < x < 1, \quad \alpha, \beta > -1, \\ \dfrac{1}{-\beta} & \text{if} \quad x = 1, \quad -1 < \beta < 0, \quad \alpha > -1 \\ +\infty & \text{if} \quad x = 1, \quad \beta \geq 0, \quad \alpha > -1 \end{cases}, \tag{4}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} l_k^2(x) = \begin{cases} \dfrac{1}{-\alpha} & \text{if} \quad x = 0, \quad -1 < \alpha < 0 \\ +\infty & \text{if} \quad x = 0, \quad \alpha \geq 0 \\ 1 & \text{if} \quad x > 0, \quad \alpha > -1 \end{cases} \tag{5}$$

hold resp.

REMARK. (4) was proved by FEJÉR ([2]) on another way, for the special cases $-1 < \alpha, \beta < 0$ and, $\alpha = \beta = 0$.

For the proof of theorem 1 we need some lemmas.

LEMMA 1. *For Laguerre abscissas we have*

$$\sum_{k=1}^{n} \frac{e^{x_k}}{e^x} \frac{x^{\alpha+1}}{x_k^{\alpha+1}} l_k^2(x) \leq 1 \quad (x \geq 0;\ \alpha > -1;\ n = 0, 1, 2 \ldots) \tag{6}$$

further

$$\left(\frac{e^x}{x^\gamma}\right)^{(2n)} > 0 \quad (x > 0;\ \gamma > 0\ \ n = 0, 1, 2 \ldots). \tag{7}$$

PROOF. See Joó [6].

LEMMA 2.

$$\left|1-\sum_{k=1}^{n} l_k^2(x)\right| \leq Cn^{1/4} \quad (0<\varepsilon' \leq x \leq d,\ \alpha > -1). \tag{8}$$

PROOF. By the formula (see [9])

$$\sum_{k=1}^{n}\left[1-\frac{x_k-\alpha-1}{x_k}(x-x_k)\right] l_k^2(x) \equiv 1 \tag{9}$$

we have

$$\left|1-\sum_{k=1}^{n} l_k^2(x)\right| \leq \sum_{k=1}^{n} \frac{x_k+|\alpha+1|}{x_k}|x-x_k|\, l_k^2(x) =$$

$$= \sum_{|x-x_k|\leq n^{1/1}}{}_k + \sum_{|x-x_k|>n^{1/1}}{}_k = A+B.$$

By (6), using the following estimates:

$$L_n^{(\alpha)}(x) = 0\,(n^a)\left(a=\max\left(\frac{\alpha}{2}-\frac{1}{4},\alpha\right);\ 0\leq x\leq d;\ \alpha>-1\right) \tag{10}$$

(see [9], (7.6.11))

$$\lim_{n\to\infty} n^{-\alpha/2+1/4}\cdot M_n = \text{const.}\left(\alpha \geq -\frac{1}{2}\right) \tag{11}$$

(see [9] Theorem 7.6.5) where

$$M_n \overset{\text{def}}{=} \max_{0<x<d} e^{-x/2}\, x^{\alpha/2+1/4}\left|L_n^{(\alpha)}(x)\right|,$$

$$\sum_{k=1}^{n} x_k^{m-1}\{L_n^{(\alpha)\prime}(x_k)\}^{-2} = \frac{\Gamma(m+1)\,\Gamma(m+\alpha+1)}{\Gamma(m+\alpha+1)} \quad (0\leq m\leq 2n-1) \tag{12}$$

(see [9], (14.7.5)) we have

$$A \leq Cn^{-1/4}\sum_{k=1}^{n}\frac{e^{x_k}}{e^x}\,\frac{x^{\alpha+1}}{x_k^{\alpha+1}}\, l_k^2(x) \leq Cx^{-1/4},$$

$$B \leq n^{1/4} L_n^{(\alpha)}(x)^2\left\{\sum_{k=1}^{n}\left(L_n^{(\alpha)}(x_k)\right)^{-2} + |\alpha+1|\sum_{k=1}^{n} x_k^{-1}\left(L_n^{(\alpha)\prime}(x_k)\right)^{-2}\right\} \leq Cn^{-1/4}.$$

So (8) follows from our estimates. We obtained

$$\sum_{k=1}^{n}|x-x_k|\, l_k^2(x) \leq Cn^{-1/4} \quad (0<\varepsilon' \leq x \leq d;\ \alpha>-1) \tag{13}$$

too.

LEMMA 3.

$$\sum_{\substack{k\\ x_k>2d}} e^{x_k} x_k^{-\alpha-\varepsilon} l_k^2(x) \leq Cn^{-1/2} \quad (0<\varepsilon' \leq x \leq d,\ \alpha > -1). \tag{14}$$

PROOF. Similarly to the proof of Lemma III. 1.5 in FREUD [3], taking into consideration (7) we can obtain

$$\sum_{k=1}^{n} \frac{e^{x_k}}{(x_k+1)^{1+\alpha+\varepsilon}} \frac{1}{x_k L_n^{(\alpha)\prime}(x_k)^2} \leq \tag{15}$$

$$\leq \left(\int_0^\infty \frac{x^\alpha}{(x+1)^{1+\alpha+\varepsilon}}\,dx\right) \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \quad (\alpha > -1;\ \varepsilon > 0),$$

which by (10), (11) gives (14), using the trivial inequality

$$\sum_{\substack{k\\ x_k>2d}} e^{x_k} x_k^{-\alpha-\varepsilon} l_k^2(x) = L_n^{(\alpha)}(x)^2 \sum_{\substack{k\\ x_k>2d}} \frac{e^{x_k}}{x_k^{2+\alpha+\delta}} \left(\frac{x_k}{x-x_k}\right)^2 \{L_n^{(\alpha)\prime}(x_k)\}^{-2} \leq$$

$$\leq 4 L_n^{(\alpha)}(x)^2 \sum_{\substack{k\\ x_k>2d}} \frac{e^{x_k}}{x_k^{2+\alpha+\varepsilon}} \frac{1}{L_n^{(\alpha)\prime}(x)^2} \quad (0<\varepsilon' \leq x \leq d).$$

Now we shall prove theorem 1. We have

$$|R_n^{(\alpha)}(f,x)-f(x)| = \left|\sum_{k=1}^{n} [f(x)-f(x_k)]\, l_k^2(x) + f(x)\left(1-\sum_{k=1}^{n} l_k^2(x)\right)\right| \leq$$

$$\leq \sum_{\substack{k\\ x_k\leq 2d}} |f(x)-f(x_k)|\, l_k^2(x) + \sum_{\substack{k\\ x_k>2d}} (|f(x)|+|f(x_k)|)\, l_k^2(x) +$$

$$+ |f(x)| \cdot \left|1-\sum_{k=1}^{n} l_k^2(x)\right|.$$

From the well known relation $\omega(f;\, a,\, b;\, \lambda\delta) \leq (\lambda+1)\omega(f;\, a,\, b;\, \delta)$ we obtain

$$|f(x)-f(x_k)| \leq \omega(f;\, 0,\, 2d;\, |x-x_k|) \leq \omega(f;\, 0,\, 2d;\, n^{-1/4})\{n^{1/4}\,|x-x_k|+1\},$$
$$(0 \leq x_k,\ x \leq 2d)$$

so (8), (13), (14) gives (3) from the inequality

$$|R_n^{(\alpha)}(f,x)-f(x)| \leq \omega(f;\, 0,\, 2d;\, n^{-1/4}) \cdot$$

$$\cdot \left\{n^{1/4} \sum_{k=1}^{n} |x-x_k|\, l_k^2(x) + \sum_{k=1}^{n} l_k^2(x)\right\} +$$

$$+ \|f\|^* \sum_{\substack{k\\ x_k>2d}} e^{x_k} x_k^{-\alpha-\varepsilon} l_k^2(x) + \|f\|_{[0,\,d]} \cdot \left|1-\sum_{k=1}^{n} l_k^2(x)\right|.$$

For proving (5) we remark that this follows from theorem 1 in the case of $x>0$, and from the identity

$$\sum_{k=1}^{n} l_k^2(0) = \frac{1}{-\alpha}\left\{1-\binom{n+\alpha}{n}^2 \sum_{k=1}^{n} x_k^{-1}\left(L_n^{(\alpha)}(x_k)\right)^{-2}\right\} =$$

$$= \frac{1}{-\alpha}\left\{1-\binom{n+\alpha}{n}\right\} \tag{16}$$

in the case of $x=0$, $\alpha\neq 0$. [This identity easily follows from (9) and (12).] For $x=0$, $\alpha=0$ we obtain from (16)

$$\sum_{k=1}^{n} l_k^2(0) = \sum_{k=1}^{n} \frac{1}{k} \to \infty \quad (n\to\infty).$$

So the proof of (5) is complete. In the case $\alpha\neq 0$ (4) follows in a similar way from the result of [5] and from the identity

$$\sum_{k=1}^{n} l_k^2(1) = \frac{1}{-\alpha}\left\{1-(1+\beta)\binom{n+\alpha}{n}^2 \sum_{k=1}^{n} (1-x_k^2)^{-1}\left(P_n^{(\alpha,\beta)'}(x_k)\right)^{-2}\right\} =$$

$$= \frac{1}{-\alpha}\left\{1-\binom{n+\alpha}{n}^2 \frac{1+\beta}{2^{\alpha+\beta+1}}\cdot\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}dx\right\}. \tag{17}$$

(This follows from SZEGŐ [9], (15.1.6), (15.3.1), (14.1.11).) In the case of $\alpha=0$ (4) follows from the result of [5] and from (17) similarly to the case of Laguerre polynomials with $\alpha=0$. So the proof of theorem 2 is complete.

(4) and (5) show that the process (2) at the endpoints generally does not converge. J. SZABADOS (see [7], [8]) dealt with Hermite-Fejér interpolation for Jacobi and Laguerre abscissas respectively. He proved convergence at the endpoints making certain assumptions for the function $f(x)$. His method may be used for our process too.

We have for Jacobi and Laguerre abscissas

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\, l_k^2(1) = \begin{cases} \dfrac{1}{-\alpha} f(1) & (-1<\alpha<0;\ \beta>-1) \\ \pm\infty & (\alpha\geq 0;\ f(1)\gtrless 0) \end{cases} \tag{18}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\, l_k^2(0) = \begin{cases} \dfrac{1}{-\alpha} f(0) & (-1<\alpha<0) \\ \pm\infty & (\alpha\geq 0;\ f(0)\gtrless 0) \end{cases} \tag{19}$$

if $f(x)$ is continuous function in $[-1, 1]$ or $[0, \infty)$ resp. and $f(x) = 0(e^{\gamma x})$ $(x \to \infty; \gamma < 1)$ in the second case. These follows from (4), (5), SZEGŐ [9] p. 351, FREUD [3] p. 94, and by estimating $f(x)$ by step function.

At last I should to thank Professor J. SZABADOS for his valuable remarks.

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