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J. MOLNÁR, R. PÉTER, F. SCHIPP,
T. SCHMIDT, GY. SOÓS, J. SURÁNYI, P. SZÁSZ, I. VINCE

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RÓZSA PÉTER

(1905 – 1977)

Professor R. PÉTER died the 17th of February, 1977 after a very long and grave illness. Her death is an irreparable loss not only for the editorial board of our review but also for the international mathematical science and especially for the scientific life in Hungary.

THE NUMBER OF SEMI-k-FREE DIVISORS OF AN INTEGER

By

D. SURYANARAYANA and V. SIVA RAMA PRASAD

Andhra University, Waltair and A. M. Jain College, Madras

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1. Introduction. Let k be a fixed integer ≥ 2. A positive integer n is called semi-k-free, if the multiplicity of each prime factor of n is not equal to k; or equivalently, if n is not divisible *unitarily* by the k-th power of any prime. By a unitary divisor, we mean as usual, a divisor $d>0$ of n such that $\left(d, \frac{n}{d}\right) = 1$. The integer 1 is also considered to be semi-k-free. The concept of a semi-k-free integer has been introduced by the first author [3]. Let Q_k^{s*} denote the set of semi-k-free integers. A divisor $d>0$ of the positive integer n is called a *semi-k-free divisor* of n if $d \in Q_k^{s*}$. Let $\tau_{(k)}^{s*}(n)$ denote the number of semi-k-free divisors of n.

In this paper we prove the following:

THEOREM 1. *For* $x \geq 3$,

$$\sum_{n \leq x} \tau_{(k)}^{s*}(n) = \alpha_{(k)}^{s*} x \left(\log \gamma + 2y - 1 + \sum_p \frac{(kp-k-1)\log p}{(p^{k+1}-p+1)}\right) + \Delta_{(k)}^{s*}(x), \tag{1.1}$$

where

$$\Delta_{(k)}^{s*}(x) = O(x^{1/k} \exp\{-A \log^{3/5} x (\log\log x)^{-1/5}\}) \text{ or } O(x^{\alpha}),$$

according as $k = 2, 3$ *or* $k \geq 4$; *A being a positive constant,* γ *is Euler's constant,* α *is the number which appears in the Dirichlet divisor problem* (2.19) *and* $\alpha_{(k)}^{s*}$ *is the constant given by* (2.1).

THEOREM 2. *If the Riemann hypothesis is true, then for* $x \geq 3$, *the error term* $\Delta_{(k)}^{s*}(x)$ *in* (1.1) *is given by*

$$\Delta_{(k)}^{s*}(x) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}} \omega(x)\right)$$

or $O(x^{\alpha})$, *according as* $k = 2, 3$ *or* $k \geq 4$; *where*

$$\omega(x) = \exp\{A \log x (\log \log x)^{-1}\},$$

A being a positive constant and α *is the number which appears in the Dirichlet divisor problem* (2.19).

2. Preliminaries. Let $\mu(n)$ and $\varphi(n)$ respectively denote the Möbius function and the Euler totient function. Let $\sigma_s^*(n)$ denote the sum of the s-th powers of the square-free divisors of n. It is known (cf. [3], lemma 2) that

$$\alpha_{(k)}^{s*} \equiv \sum_{m=1}^{\infty} \frac{\mu(m)\varphi(m)}{m^{k+1}} = \prod_p \left(1 - \frac{1}{p^k} + \frac{1}{p^{k+1}}\right), \tag{2.1}$$

where the product is extended over all primes p.

It can be easily shown by using standard arguments that

$$\sum_{m \leqq x} \frac{\sigma_{-s}^*(m)}{m^u} = O(x^{1-u}), \text{ for } s > 0 \text{ and } 0 \leqq u < 1. \tag{2.2}$$

We use the following lemmas:

LEMMA 2.1 (cf. [5], lemma 3.5). For $x \geqq 3$ and for every $\varepsilon > 0$,

$$M_n(x) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m) = O(\sigma_{-1+\varepsilon}^*(n)\, x\, \delta(x), \tag{2.3}$$

where the O-constant is independent of n and x and $\delta(x)$ is given by

$$\delta(x) = \exp\{-A \log^{3/5} x (\log \log x)^{-1/5}\}, \tag{2.4}$$

A being a positive constant.

REMARK. Here after, all the constants implied by the O-symbols are independent of n and x.

LEMMA 2.2. (cf. [4], lemma 2.3, (2.8)). For $x \geqq 3$ and for every $\varepsilon > 0$,

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m)\, m = O(\sigma_{-1+\varepsilon}^*(n)\, x^2\, \delta(x)). \tag{2.5}$$

LEMMA 2.3. For $x \geqq 3$ and for every $\varepsilon > 0$,

$$N_n(x) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m)\varphi(m) = O(\sigma_{-1+\varepsilon}^*(n)\, x^2\, \delta(x)). \tag{2.6}$$

PROOF. Since $\varphi(n) = \sum_{d\delta=n} \mu(d)\delta$ (cf. [1], (16.3.1)), we have by (2.5).

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m)\varphi(m) = \sum_{\substack{d\delta \leqq x \\ (d\delta,n)=1}} \mu(d\delta)\mu(d)\delta = \sum_{\substack{d\delta \leqq x \\ (d\delta,n)=1 \\ (d,\delta)=1}} \mu^2(d)\mu(\delta)\delta =$$

$$= \sum_{\substack{d \leqq x \\ (d,n)=1}} \mu^2(d) \sum_{\substack{\delta \leqq \frac{x}{d} \\ (\delta,dn)=1}} \mu(\delta)\delta =$$

$$= O\left(\sum_{\substack{d \leqq x \\ (d,n)=1}} \mu^2(d)\,\sigma^*_{-1+\varepsilon}(dn)\cdot\frac{x^2}{d^2}\,\delta\left(\frac{x}{d}\right)\right) =$$

$$= O\left(\sigma^*_{-1+\varepsilon}(n)\,x^2 \sum_{d \leqq x} \frac{\sigma^*_{-1+\varepsilon}(d)}{d^2}\,\delta\left(\frac{x}{d}\right)\right).$$

We note that $x^\varepsilon\,\delta(x)$ is monotonic increasing for every $\varepsilon>0$, so that we have

$$\sum_{d \leqq x} \frac{\sigma^*_{-1+\varepsilon}(d)}{d^2}\,\delta\left(\frac{x}{d}\right) = \frac{1}{x^\varepsilon}\sum_{d \leqq x} \frac{\sigma^*_{-1+\varepsilon}(d)}{d^{2-\varepsilon}}\left(\frac{x}{d}\right)^\varepsilon \delta\left(\frac{x}{d}\right) \leqq$$

$$\leqq \frac{1}{x^\varepsilon}\cdot x^\varepsilon\,\delta(x)\sum_{d \leqq x} \frac{\sigma^*_{-1+\varepsilon}(d)}{d^{2-\varepsilon}} \leqq \delta(x)\sum_{d \leqq x}\frac{\tau(d)}{d^{2-\varepsilon}} = O(\delta(x)),$$

since $\tau(n) = O(n^\varepsilon)$ for every $\varepsilon>0$ (cf. [1], Theorem 315); where $\tau(n)$ is the number of divisors of n.

Hence the lemma follows.

LEMMA 2.4. For $x \geqq 3$, $s>2$ and for every $\varepsilon>0$,

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu(m)\varphi(m)}{m^s} = O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{s-2}}\right). \tag{2.7}$$

PROOF. Putting $f(m) = \dfrac{1}{m^s}$, it can be easily shown that

$$f(m+1)-f(m) = O\left(\frac{1}{m^{s+1}}\right).$$

By partial summation and lemma 2.3, we have

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu(m)\varphi(m)}{m^s} = -N_n(x)f([x]+1) - \sum_{m>x} N_n(m)\{f(m+1)-f(m)\} =$$

$$= O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{s-2}}\right) + O\left(\sum_{m>x}\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(m)}{m^{s-1}}\right) =$$

$$= O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{s-2}} + O\left(\sigma^*_{-1+\varepsilon}(n)\,\delta(x)\sum_{m>x}\frac{1}{m^{s-1}}\right),\right.$$

since $\delta(x)$ is monotonic decreasing, so that (2.7) follows in virtue of the fact that

$$\sum_{m>x} \frac{1}{m^{s-1}} = O\left(\frac{1}{x^{s-2}}\right).$$

As a particular case of (2.7) for $n = 1$, we have

$$\sum_{m>x} \frac{\mu(m)\,\varphi(m)}{m^s} = O\left(\frac{\delta(x)}{m^{s-2}}\right). \tag{2.8}$$

LEMMA 2.5. For $x \geq 3$, $s > 2$ and for every $\varepsilon > 0$,

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu(m)\,\varphi(m)\log m}{m^s} = O\left[\frac{\sigma^*_{-1+\varepsilon}(x)\,\delta(x)\log x}{x^{s-2}}\right]. \tag{2.9}$$

PROOF. Putting $g(m) = \dfrac{\log m}{m^s}$, it can be easily shown that

$$g(m+1) - g(m) = O\left(\frac{\log m}{m^{s+1}}\right).$$

By partial summation, lemma 2.3 and making use of the argument adopted in the proof of lemma 2.4, we get (2.9).

As a particular case of (2.9) for $n = 1$, we have

$$\sum_{m>x} \frac{\mu(m)\,\varphi(m)\log m}{m^s} = O\left(\frac{\delta(x)\log x}{x^{s-2}}\right). \tag{2.10}$$

LEMMA 2.6. For $s > 2$,

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)\,\varphi(m)}{m^s} = \prod_p \left(1 - \frac{p-1}{p^s}\right) \cdot \prod_{p|n} \left(1 - \frac{p-1}{p^s}\right)^{-1}. \tag{2.11}$$

PROOF. Let $e(m) = 1$ or 0 according as $m = 1$ or $m > 1$. Then the series on the left would become

$$\sum_{m=1}^{\infty} \frac{\mu(m)\,\varphi(m)\,e((m,n))}{m^s}.$$

This series is absolutely convergent for $s > 2$ and the general term is a multiplicative function of m. Hence the series can be expanded into an infinite product of Euler type (cf. [1], Theorem 286), so that we have

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty} \frac{\mu(m)\,\varphi(m)}{m^s} = \prod_{\substack{p\\p\nmid n}} \left(1-\frac{\varphi(p)}{p^s}\right) = \prod_{\substack{p\\p\nmid n}} \left(1-\frac{p-1}{p^s}\right) =$$

$$= \prod_p \left(1-\frac{p-1}{p^s}\right)\cdot \prod_{p\mid n} \left(1-\frac{p-1}{p^s}\right)^{-1}.$$

Hence the lemma follows.

As a particular case of (2.11) for $n = 1$, we have the following:

$$\sum_{m=1}^{\infty} \frac{\mu(m)\,\varphi(m)}{m^s} = \prod_p \left(1-\frac{1}{p^{s-1}}+\frac{1}{p^s}\right) \quad \text{for} \quad s>2. \tag{2.12}$$

LEMMA 2.7. For $s>2$,

$$\sum_{m=1}^{\infty} \frac{\mu(m)\,\varphi(m)\log m}{m^s} = -\prod_p \left(1-\frac{1}{p^{s-1}}+\frac{1}{p^s}\right) \sum_p \frac{(p-1)\log p}{(p^s-p+1)}. \tag{2.13}$$

PROOF. This series is uniformly convergent for $s\geq 2+\varepsilon>2$ and so by termwise differentiation of the series in (2.12) with respect to s, we get (2.13). For finding the derivative of the right hand side expression of (2.12) with respect to s, we write

$$f(s) = \prod_p \left(1-\frac{p-1}{p^s}\right).$$

Then

$$\log f(s) = \sum_p \log\left(1-\frac{p-1}{p^s}\right),$$

so that

$$\frac{f'(s)}{f(s)} = \sum_p \frac{(p-1)\log p}{(p^s-p+1)},$$

and this gives

$$f'(s) = \prod_p \left(1-\frac{1}{p^{s-1}}+\frac{1}{p^s}\right) \sum_p \frac{(p-1)\log p}{(p^s-p+1)}.$$

Hence the lemma follows.

As a consequence of (2.8) and (2.12) for $s = k+1$, we have the following in virtue of (2.1):

$$\sum_{m\leq x} \frac{\mu(m)\,\varphi(m)}{m^{k+1}} = \alpha_{(k)}^{s*} + O\left(\frac{\delta(x)}{x^{k-1}}\right). \tag{2.14}$$

As a consequence of (2.10) and (2.13) for $s = k+1$, we have the following in virtue of (2.1):

$$\sum_{m\leq x} \frac{\mu(m)\,\varphi(m)\log m}{m^{k+1}} = -\alpha_{(k)}^{s*} \sum_p \frac{(p-1)\log p}{(p^{k+1}-p+1)} + O\left(\frac{\delta(x)\log x}{x^{k-1}}\right). \tag{2.15}$$

As a consequence of (2.7) and (2.11) for $s = k+1$, we have the following invirtue of (2.1):

$$(2.16)\quad \sum_{\substack{m\leq x\\(m,n)=1}} \frac{\mu(m)\,\varphi(m)}{m^{k+1}} = \alpha_{(k)}^{s*} \prod_{p|n}\left(1-\frac{p-1}{p^{k+1}}\right)^{-1} + O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\,\delta(x)}{x^{k-1}}\right).$$

LEMMA 2.8. (cf. [5], Theorem 4.1.). If $\tau(m;\,n)$ is the number of divisors of m which are prime to n, then for $x\geq 2$,

$$(2.17)\quad \sum_{m\leq x} \tau(m;n) = \frac{\varphi(n)\,x}{n}\left(\log x + 2\gamma - 1 + \alpha(n)\right) + O\left(\sigma^*_{-\alpha}(n)\,x^{\alpha}\right),$$

where γ is Euler's constant, $\alpha(n)$ is given by

$$(2.18)\qquad \alpha(n) = -\frac{n}{\varphi(n)} \sum_{d|n} \frac{\mu(d)\log d}{d} = \sum_{p|n} \frac{\log p}{p-1}$$

and α is the number which appears in the Dirichlet's divisor problem, namely

$$(2.19)\qquad \sum_{m\leq x} \tau(m) = x\left(\log x + 2\gamma - 1\right) + O\left(x^{\alpha}\right).$$

It is known that $\frac{1}{4} < \alpha < \frac{1}{3}$ (cf. [1], p. 272). The best result known till now is due to G. A. KOLESNIK (cf. [2], p. 28), who proved that the error term in (2.19) is $O(x^{346/1067+\varepsilon})$ for every $\varepsilon > 0$. There is a conjecture that $\alpha = \frac{1}{4} + \varepsilon$.

LEMMA 2.9. For $x\geq 3$,

$$(2.20)\quad \sum_{m\leq x} \frac{\mu(m)\,\varphi(m)\,\alpha(m)}{m^{k+1}} = -\alpha_{(k)}^{s*} \sum_{p} \frac{\log p}{(p^{k+1}-p+1)} + O\left(\frac{\delta(x)\log x}{x^{k-1}}\right).$$

PROOF. By (2.18) and (2.16), we have

$$\sum_{m\leq x} \frac{\mu(m)\,\varphi(m)\,\alpha(m)}{m^{k+1}} = \sum_{m\leq x} \frac{\mu(m)\,\varphi(m)}{m^{k+1}} \sum_{p|m} \frac{\log p}{p-1} =$$

$$= \sum_{pu\leq x} \frac{\mu(pu)\,\varphi(pu)\log p}{p^{k+1}\,u^{k+1}\,(p-1)} = \sum_{\substack{pu\leq x\\(p,u)=1}} \frac{\mu(p)\,\mu(u)\,\varphi(p)\,\varphi(u)\log p}{p^{k+1}\,u^{k+1}\,(p-1)} =$$

$$= -\sum_{p\leq x} \frac{\log p}{p^{k+1}} \sum_{\substack{u\leq x/p\\(u,p)=1}} \frac{\mu(u)\,\varphi(u)}{u^{k+1}} =$$

$$= -\sum_{p\leq x}\frac{\log p}{p^{k+1}}\left\{\alpha_{(k)}^{s*}\cdot\frac{p^{k+1}}{(p^{k+1}-p+1)}+O\left(\frac{\delta\left(\frac{x}{p}\right)}{\left(\frac{x}{p}\right)^{k-1}}\right)\right\}=$$

$$= -\alpha_{(k)}^{s*}\sum_{p}\frac{\log p}{(p^{k+1}-p+1)}+\alpha_{k}^{s*}\sum_{p>x}\frac{\log p}{(p^{k+1}-p+1)}+$$

$$+O\left(\frac{1}{x^{k-1}}\sum_{p\leq x}\frac{\log p}{p^2}\delta\left(\frac{x}{p}\right)\right). \tag{2.21}$$

By (2.1), we have

$$\alpha_{(k)}^{s*}<1 \quad\text{and}\quad (p^{k+1}-p+1)>\frac{p^{k+1}}{2},$$

so that the second term in (2.21) is

$$O\left(\sum_{p>x}\frac{\log p}{p^{k+1}}\right)=O\left(\frac{\log x}{x^k}\right)=O\left(\frac{\delta(x)\log x}{x^{k-1}}\right).$$

Also, the O-term (2.21) is

$$O\left(\frac{1}{x^k}\sum_{p\leq x}\frac{\log p}{p}\cdot\left(\frac{x}{p}\right)\delta\left(\frac{x}{p}\right)\right)=O\left(\frac{x\,\delta(x)}{x^k}\sum_{p\leq x}\frac{\log p}{p}\right)=O\left(\frac{\delta(x)\log x}{x^k}\right),$$

since

$$\sum_{p\leq x}\frac{\log p}{p}=O(\log x)$$

(cf. [1], Theorem 425).

Hence the lemma follows.

LEMMA 2.10. For $x\geq 3$ and for every $\varepsilon>0$,

$$\sum_{n\leq x}\mu(n)\,\tau(m;n)=O(\chi(m)\,x\,\delta(x)),\quad\text{where}\quad \sum_{d|m}4^{\nu(d)} \tag{2.22}$$

$\nu(m)$ being the number of distinct prime factors of m.

PROOF. Let

$$M_{(n)}(x)=\sum_{\substack{m\leq x\\ n|m}}\mu(m).$$

Then we have

$$\sum_{d|n}\mu(d)\,M_d(x)=\sum_{d|n}\mu(d)\sum_{\substack{(m,d)=1\\ m\leq x}}\mu(m)=\sum_{\substack{m\leq x\\ d|n\\ (m,d)=1}}\mu(d)\,\mu(m)=$$

$$=\sum_{m\leq x}\mu(m)\sum_{\substack{d|n\\ (d,m)=1}}\mu(d).$$

If n is square-free, then it is easy to show that

$$\sum_{\substack{d|n \\ (d,m)=1}} \mu(d) = 1 \quad \text{or} \quad 0,$$

according as $n|m$ or $n\nmid m$.
Hence, if n is square-free, we have

$$\sum_{d|n} \mu(d) M_d(x) = \sum_{\substack{m\leq x \\ n|m}} \mu(m) = M_{(n)}(x).$$

Hence, if n is square-free, then by (2.3), we have

$$\begin{aligned} M_{(n)}(x) &= O\Big(\sum_{d|n} \mu^2(d)\, \sigma^*_{-1+\varepsilon}(d)\, x\, \delta(x)\Big) = \\ &= O\Big(x\, \delta(x) \prod_{p|n} \{1+\sigma^*_{-1+\varepsilon}(p)\}\Big) = \\ &= O\Big(x\, \delta(x) \prod_{p|n} (3)\Big) = O\left(3^{\nu(n)}\, x\, \delta(x)\right). \end{aligned} \tag{2.23}$$

Now,

$$\sum_{n\leq x} \mu(n)\, \tau(m;n) = \sum_{n\leq x} \mu(n) \sum_{\substack{d\,\delta=m \\ (d,n)=1}} 1 = \sum_{n\leq x} \mu(n) \sum_{d\,\delta=m} \sum_{r|(d,n)} \mu(r) =$$

$$= \sum_{\substack{n\leq x \\ rs\,\delta=m \\ r|n}} \mu(n)\,\mu(r) = \sum_{rs|m} \mu(r) \sum_{\substack{n\leq x \\ r|n}} \mu(n) = \sum_{rs|m} \mu(r)\, M_{(r)}(x).$$

Hence, for square-free r, applying (2.23), we get

$$\sum_{n\leq x} \mu(n)\, \tau(m;n) = O\Big(x\, \delta(x) \sum_{rs|m} \mu^2(r)\, 3^{\nu(r)}\Big),$$

Now, the lemma follows, since

$$\begin{aligned} \sum_{rs|m} \mu^2(r)\, 3^{\nu(r)} &= \sum_{u|m} \sum_{r|u} \mu^2(r)\, 3^{\nu(r)} = \sum_{u|m} \Big\{\prod_{p|u} (1+3)\Big\} = \\ &= \sum_{u|m} 4^{\nu(u)} = \chi(m). \end{aligned}$$

LEMMA 2.11. (cf. [5], lemma 5.2). If the Riemann hypothesis is true, then for $x\geq 3$ and for every $\varepsilon>0$,

$$M_n(x) \equiv \sum_{\substack{m\leq x \\ (m,n)=1}} \mu(m) = O\big(\sigma^*_{-1/2+\varepsilon}(x)\, x^{1/2}\, \omega(x)\big), \tag{2.24}$$

where

$$\omega(x) = \exp\{A \log x\, (\log\log x)^{-1}\}, \tag{2.25}$$

A being a positive constant.

LEMMA 2.12. (cf. [4], lemma 2.3. (2.9.)). If the Riemann hypothesis is true, then for $x \geq 3$ and for every $\varepsilon > 0$,

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu(m)\, m = O\left(\sigma^*_{-1/2+\varepsilon}(n)\, x^{3/2}\, \omega(x)\right). \tag{2.26}$$

LEMMA 2.13. If the Rimeann hypothesis is true, then for $x \geq 3$ and for every $\varepsilon > 0$,

$$N_n(x) \equiv \sum_{\substack{m \leq x \\ (m,n)=1}} \mu(m)\, \varphi(m) = O\left(\sigma^*_{-1/2+\varepsilon}(n)\, x^{3/2}\, \omega(x)\right). \tag{2.27}$$

PROOF. Following the same argument adopted in lemma 2.3 and making use of lemma 2.12 instead of lemma 2.2, we get this lemma. We have only to replace $\sigma^*_{-1+\varepsilon}(n)\ \delta(x)$ in lemma 2.3 by

$$\sigma^*_{-1/2+\varepsilon}(n)\, x^{-1/2}\, \omega(x).$$

LEMMA 2.14. If the Riemann hypothesis true, then for $x \geq 3$, $s > 2$ and for every $\varepsilon > 0$,

$$\sum_{\substack{m > x \\ (m,n)=1}} \frac{\mu(m)\, \varphi(m)}{m^s} = O\left(\frac{\sigma^*_{-1/2+\varepsilon}(n)\, \omega(x)}{x^{s-3/2}}\right). \tag{2.28}$$

PROOF. Following the same argument adopted in lemma 2.4 and making use of lemma 2.13 instead of lemma 2.3, we get this lemma. We have only to replace $\sigma^*_{-1+\varepsilon}(n)\ \delta(x)$ in lemma 2.4 by

$$\sigma^*_{-1/2+\varepsilon}(n)\, x^{-1/2}\, \omega(x).$$

LEMMA 2.15. If the Riemann hypothesis is true, then for $x \geq 3$, $s > 2$ and for every $\varepsilon > 0$,

$$\sum_{m > x} \frac{\mu(m)\, \varphi(m) \log m}{m^s} = O\left(\frac{\sigma^*_{-1/2+\varepsilon}(n)\, \omega(x) \log x}{x^{s-3/2}}\right). \tag{2.29}$$

PROOF. Following the same argument adopted in lemma 2.5 and making use of lemma 2.13 instead of lemma 2.3, we get this lemma. We have only to replace $\sigma^*_{-1+\varepsilon}(n)\ \delta(x)$ in lemma 2.5 by

$$\sigma^*_{-1/2+\varepsilon}(n)\, x^{-1/2}\, \omega(x).$$

The results corresponding to (2.14), (2.15) and (2.16) in case the Riemann hypothesis is true are given by the following:

$$\sum_{m \leq x} \frac{\mu(m)\, \varphi(m)}{m^{k+1}} = \alpha^{s*}_{(k)} + O\left(\frac{\omega(x)}{x^{k-1/2}}\right), \tag{2.30}$$

$$\sum_{m \leq x} \frac{\mu(m)\, \varphi(m) \log m}{m^{k+1}} = -\alpha^{s*}_{(k)} \sum_{p} \frac{(p-1)\log p}{(p^{k+1}-p+1)} + O\left(\frac{\omega(x) \log x}{x^{k-1/2}}\right), \tag{2.31}$$

$$(2.32)\quad \sum_{\substack{m\leq x\\(m,n)=1}} \frac{\mu(m)\,\varphi(m)}{m^{k+1}} = \alpha_{(k)}^{s*} \prod_{p|n}\left(1-\frac{p-1}{p^{k+1}}\right)^{-1} + O\left(\frac{\sigma^*_{-1/2+\varepsilon}(n)\,\omega(x)}{x^{k-1/2}}\right).$$

LEMMA 2.16. If the Riemann hypothesis is true, then for $x\geq 3$,

$$(2.33)\quad \sum_{m\leq x} \frac{\mu(m)\,\varphi(m)\,\alpha(m)}{m^{k+1}} = -\alpha_{(k)}^{s*} \sum_{p} \frac{\log p}{(p^{k+1}-p+1)} + O\left(\frac{\omega(x)\log x}{x^{k-1/2}}\right).$$

PROOF. Following the same argument adopted in lemma 2.9 and making use of (2.32) instead of (2.16), we get this lemma. We have only to replace $\delta(x)$ in lemma 2.9 by $x^{-1/2}\,\omega(x)$.

LEMMA 2.17. If the Riemann hypothesis is true, then for $x\geq 3$ and for every $\varepsilon>0$,

$$(2.34)\quad \sum_{n\leq x} \mu(n)\,\tau(m;n) = O(\chi(m)\,x^{1/2}\,\omega(x)).$$

PROOF. Following the same argument adopted in lemma 2.10 and making use of (2.24) instead of (2.3), we get this lemma. We have only to replace $\delta(x)$ in lemma 2.10 by $x^{-1/2}\,\omega(x)$.

3. Proof of theorem 1. Let $q_k^{s*}(n)$ denote the characteristic function of the set of semi-k-free integers. It has been shown (cf. [3]) that

$$q_k^{s*}(n) = \sum_{\substack{d^k\delta=n\\(d,\delta)=1}} \mu(d).$$

Hence

$$\tau_{(k)}^{s*}(n) = \sum_{rs=n} q_k^{s*}(r) = \sum_{rs=n}\ \sum_{\substack{d^k\delta=r\\(d,\delta)=1}} \mu(d) = \sum_{\substack{d^k\delta s=n\\(d,\delta)=1}} \mu(d) =$$

$$= \sum_{d^k u=n} \mu(d) \sum_{\substack{\delta s=u\\(\delta,d)=1}} 1 = \sum_{d^k u=n} \mu(d)\,\tau(u;d).$$

Hence

$$(3.1)\quad \sum_{n\leq x} \tau_{(k)}^{s*}(n) = \sum_{n\leq x}\ \sum_{d^k u=n} \mu(d)\,\tau(u;d) = \sum_{d^k u\leq x} \mu(d)\,\tau(u;d),$$

the summation on the right being taken over all ordered pairs (d, u) such that $d^k\,u\leq x$.

Let $z = x^{1/k}$. Further, let $0<\varrho = \varrho(x)<1$, where the function $\varrho(x)$ will be suitably chosen later.

Now, if $d^k u\leq x$, then both $d>\varrho z$ and $u>\varrho^{-k}$ can not simultaneously hold good, and so from (3.1), we have

$$\sum_{n\leq x} \tau_{(k)}^{s*}(n) = \sum_{\substack{d^k u\leq x\\ d\leq\delta z}} \mu(d)\,\tau(u;d) + \sum_{\substack{d^k u\leq x\\ u\leq\varrho^{-k}}} \mu(d)\,\tau(u;d) - \sum_{\substack{d\leq\varrho z\\ u\leq\varrho^{-k}}} \mu(d)\,\tau(u;d) =$$

$$(3.2)\quad = S_1+S_2-S_3,$$

say.

By (2.17), we have

$$S_1 = \sum_{\substack{d^k u \leq x \\ d \leq \varrho z}} \mu(d)\,\tau(u;d) = \sum_{d \leq \varrho z} \mu(d) \sum_{u \leq x/d^k} \tau(u;d) =$$

$$= \sum_{d \leq \varrho z} \mu(d) \left\{ \frac{\varphi(d)}{d} \cdot \frac{x}{d^k} \left(\log \frac{x}{d^k} + 2\gamma - 1 + \alpha(d) \right) + O\left(\sigma^*_{-\alpha}(d) \cdot \frac{x^\alpha}{d^{k\alpha}} \right) \right\} =$$

$$= x(\log x + 2\gamma - 1) \sum_{d \leq \varrho z} \frac{\mu(d)\,\varphi(d)}{d^{k+1}} - kx \sum_{d \leq \varrho z} \frac{\mu(d)\,\varphi(d) \log d}{d^{k+1}} +$$

$$+ x \sum_{d \leq \varrho z} \frac{\mu(d)\,\varphi(d)\,\alpha(d)}{d^{k+1}} + E^{s*}_{(k)}(x),$$

where

$$E^{s*}_{(k)}(x) = O\left(x^\alpha \cdot \sum_{d \leq \varrho z} \frac{\sigma^*_{-\alpha}(d)}{d^{k\alpha}} \right).$$

If $k = 2$ or 3, then since $\frac{1}{4} < \alpha < \frac{1}{3}$, we have $k\alpha < 1$, so that by (2.2),

$$E^{s*}_{(k)}(x) = O(x^\alpha (\varrho z)^{1-k\alpha}) = O(\varrho^{1-k\alpha} z);$$

and if $k \geq 4$, then $k\alpha > 1$, so that $E^{s*}_{(k)}(x) = O(x^\alpha)$.

Hence we have

$$E^{s*}_{(k)}(x) = O(\varrho^{1-k\alpha} z) \quad \text{or} \quad O(x^\alpha), \tag{3.4}$$

according as $k = 2, 3$ or $k \geq 4$.

Now, by (3.3), (2.14), (2.15) and (2.20), we have

$$S_1 = x(\log x + 2\gamma - 1) \left\{ \alpha^{s*}_{(k)} + O\left(\frac{\delta(\varrho z)}{(\varrho z)^{k-1}} \right) \right\} -$$

$$- kx \left\{ -\alpha^{s*}_{(k)} \sum_p \frac{(p-1)\log p}{(p^{k+1} - p + 1)} + O\left(\frac{\delta(\varrho z) \log(\varrho z)}{(\varrho z)^{k-1}} \right) \right\} +$$

$$+ x \left\{ -\alpha^{s*}_{(k)} \sum_p \frac{\log p}{(p^{k+1} - p + 1)} + O\left(\frac{\delta(\varrho z) \log(\varrho z)}{(\varrho z)^{k-1}} \right) \right\} + E^{s*}_{(k)}(x) =$$

$$= \alpha^{s*}_{(k)} x \left(\log x + 2\gamma - 1 + \sum_p \frac{(kp - k - 1)\log p}{(p^{k+1} - p + 1)} \right) +$$

$$+ O(\varrho^{1-k} z\, \delta(\varrho z) \log z) + E^{s*}_{(k)}(x). \tag{3.5}$$

We have by (2.22),

$$S_2 = \sum_{\substack{d^k u \leq x \\ u \leq \varrho^{-k}}} \mu(d)\,\tau(u;d) = \sum_{u \leq \varrho^{-k}} \sum_{d \leq \sqrt[k]{x/u}} \mu(d)\,\tau(u;d) =$$

$$= O\left(\sum_{u \leq \varrho^{-k}} \chi(u) \left(\sqrt[k]{\frac{x}{u}} \right) \delta\left(\sqrt[k]{\frac{x}{u}} \right) \right).$$

Since $\delta(x)$ is monotonic decreasing and $\left(\sqrt[k]{\frac{x}{u}}\right) \geq \varrho z$, we have

$$\delta\left(\sqrt[k]{\frac{x}{u}}\right) \leq \delta(\varrho z).$$

Also,

$$\sum_{n \leq x} \chi(n) = O(x \log^4 x).$$

Hence

$$S_2 = O\left(z\,\delta(\varrho z) \sum_{u \leq \varrho^{-k}} \chi(u)\, u^{-1/k}\right) = O\left(z\,\delta(\varrho z)\,(\varrho^{-k})^{-1/k} \log^4(\varrho^{-k})\right) =$$

$$= O\left(\varrho^{1-k} z\,\delta(\varrho z) \log^4\left(\frac{1}{\varrho}\right)\right). \tag{3.6}$$

Also, we have by (2.22),

$$S_3 = \sum_{\substack{d \leq \varrho z \\ u \leq \varrho^{-k}}} \mu(d)\,\tau(u;d) = \sum_{u \leq \varrho^{-k}} \sum_{d \leq \varrho z} \mu(d)\,\tau(u;d) =$$

$$= O\left(\sum_{u \leq \varrho^{-k}} \chi(u)\,\varrho z\,\delta(\varrho z)\right) = O\left(\varrho z\,\delta(\varrho z)\,\varrho^{-k} \log^4(\varrho^{-k})\right) =$$

$$= O\left(\varrho^{1-k} z\,\delta(\varrho z) \log^4\left(\frac{1}{\varrho}\right)\right). \tag{3.7}$$

Hence, by (3.2), (3.5), (3.6) and (3.7), we have

$$\sum_{m \leq x} \tau^{s*}_{(k)}(n) = \alpha^{s*}_{(k)}\, x \left(\log x + 2\gamma - 1 + \sum_{p} \frac{(kp-k-1)\log p}{(p^{k+1}-p+1)}\right) +$$

$$+ O\left(\varrho^{1-k} z\,\varrho(\varrho z) \log z\right) + O\left(\varrho^{1-k} z\,\delta(\varrho z) \log^4\left(\frac{1}{\varrho}\right)\right) + E^{s*}_{(k)}(x). \tag{3.8}$$

Now, we choose

$$\varrho = \varrho(x) = \{\delta(x^{1/2k})\}^{1/k}, \tag{3.9}$$

and write

$$f(x) = \log^{3/5}(x^{1/2k})\,\{\log\log(x^{1/2k})\}^{-1/5} = \left(\frac{1}{2k}\right)^{3/5} U^{3/5}\,(V - \log 2k)^{-1/5}, \tag{3.10}$$

where $U = \log x$ and $V = \log\log x$.

(3.11) For $V \geq 2 \log 2k$, that is, $U \geq 4k^2$, $x \geq \exp(4k^2)$, we have

$$V^{1/5} \leq (V - \log 2k)^{-1/5} \leq \left(\frac{V}{2}\right)^{-1/5}$$

and therefore

$$\frac{1}{2} k^{-3/5} U^{3/5} V^{-1/5} \leq f(x) \leq k^{-3/5} U^{3/5} V^{-1/5}. \tag{3.12}$$

(3.13) We assume without loss of generality that the constant A in (2.4) is less than 1.

By (3.9), (2.4) and (3.10), we have

$$\varrho = \exp\left\{-\frac{A}{k} f(x)\right\}. \tag{3.14}$$

By (3.11), we have

$$k^{-8/5} U^{3/5} V^{-1/5} \leq \frac{U}{2k}.$$

Hence, by (3.12), (3.13), (3.14) and the above,

$$\varrho \geq \exp(-Ak^{-8/5} U^{3/5} V^{-1/5} \geq \exp(-k^{-8/5} U^{3/5} V^{-1/5}) \geq$$

$$\geq \exp\left(-\frac{U}{2k}\right) = \exp\left(-\frac{\log x}{2k}\right),$$

so that $\varrho \geq x^{-1/2k}$.

Hence

$$\log\left(\frac{1}{\varrho}\right) \leq \log(\sqrt{z}) = O(\log x) \quad \text{and} \quad \varrho z \geq x^{1/2k}. \tag{3.15}$$

Since $\varrho(x)$ is monotonic decreasing, $\delta(\varrho z) \leq \delta(x^{1/2k}) = \varrho^k$, so that by (3.12) and (3.14), we have

$$\varrho^{1-k} \delta(\varrho z) \leq \varrho \leq \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\}. \tag{3.16}$$

Hence, by (3.15) and (3.16), the first and second O-terms of (3.8) are

$$O\left(x^{1/k} \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\} \log^4 x\right).$$

Hence if $\Delta_{(k)}^{s*}(x)$ denotes the error term in the asymptotic formula (3.8), then we have

$$\Delta_{(k)}^{s*}(x) = O\left(x^{1/k} \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\} \log^4 x\right) + E_{(k)}^{s*}(x). \tag{3.17}$$

Case $k=2$ or 3. In this case, we have $0 < 1 - k\alpha < 1$, since $\frac{1}{4} < \alpha < \frac{1}{3}$. By (3.14) and (3.12), we have

$$\varrho^{1-k\alpha} = \exp\left\{-\frac{A(1-k\alpha)}{k} f(x)\right\} \leq \exp\left\{-\frac{A(1-k\alpha)}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\},$$

so that by (3.4),

$$E^{s*}_{(k)}(x) = O\left(x^{1/k} \exp\left\{-\frac{A(1-k\alpha)}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\}\right).$$

Again, since $0<1-k\alpha<1$, the first O-term in (3.17) is also of the above order of $E^{s*}_{(k)}(x)$. Hence

$$\Delta^{s*}_{(k)}(x) = O\left(x^{1/k} \exp\{-B \log^{3/5} x (\log\log x)^{-1/5}\}\right), \tag{3.18}$$

where B is a positive constant.

Case $k \geq 4$. In this case, by (3.4), $E^{s*}_{(k)}(x) = O(x^{\alpha})$ and the first O-term in (3.17) is $O(x^{1/k}) = O(x^{1/4}) = O(x^{\alpha})$ Hence $\Delta^{s*}_{(k)}(x) = O(x^{\alpha})$.

Hence theorem 1 follows.

4. Proof of theorem 2. Following the same procedure adooted in theorem 1 and making use of (2.30), (2.31), (2.33) and (2.34) instead of (2.14), (2.15), (2.20) and (2.22), we get the following instead of (3.8):

$$\begin{aligned}\sum_{n\leq x} \tau^{s*}_{(k)}(n) = \alpha^{s*}_{(k)} x \left(\log x + 2\gamma - 1 + \sum_p \frac{(kp-k-1)\log p}{(p^{k+1}-p+1)}\right) + \\ + O\left(\varrho^{1/2-k} z^{1/2} \omega(\varrho z) \log z\right) + \\ + O\left(\varrho^{1/2-k} z^{1/2} \omega(\varrho z) \log^4\left(\frac{1}{\varrho}\right)\right) + E^{s*}_{(k)}(x).\end{aligned} \tag{4.1}$$

Now, Chosing

$$\varrho = z^{-\frac{1}{1+2k(1-\alpha)}},$$

we see that $0<\varrho<1, \frac{1}{\varrho}<z$, so that $\log\left(\frac{1}{\varrho}\right)<\log z$ and

$$\varrho^{1/2-k} z^{1/2} = \varrho^{1-k\alpha} z = x^{\frac{2-\alpha}{1+2k(1-\alpha)}}.$$

Since $\omega(x)$ is monotonic increasing, we have $\omega(\varrho z)<\omega(z)$.

Also, by (2.25), we see that $\omega(x^{1/k}) \log^4 x = O(\omega(x))$. Hence, if $\Delta^{s*}_{(k)}(x)$ dentoes the error term in the asymptotic formula (4.1), then

$$\Delta^{s*}_{(k)}(x) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}} \omega(x)\right) + E^{s*}_{(k)}(x). \tag{4.2}$$

Case $k = 2$ or 3. In this case, by (3.4), we have

$$E^{s*}_{(k)}(x) = O(\varrho^{1-k\alpha} z) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}}\right).$$

Hence by (4.2), theorem 2 follows in this case.

Case $k \geq 4$. In this case, by (3.4), we have $E_{(k)}^{s*}(x) = O(x^{\alpha})$.

Also, since $k \geq 4$ and $\frac{1}{4} < \alpha < \frac{1}{3}$, we have

$$\frac{2-\alpha}{1+2k(1-\alpha)} \leq \frac{2-\alpha}{9-8\alpha} < \alpha .$$

Since $\omega(x) = O(x^{\varepsilon})$ for every $\varepsilon > 0$, taking

$$\varepsilon = \frac{1}{2}\left\{\alpha - \frac{2-\alpha}{9-8\alpha}\right\},$$

we see that the first O-term in (4.2) is

$$O\left(x^{\frac{\alpha}{2}+\frac{2-\alpha}{2(9-8\alpha)}}\right) = O(x^{\alpha}) .$$

Hence theorem 2 follows in this case also.

Thus theorem 2 is completely proved.

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THE CHAIN OF RIGHT IDEALS IN RINGS AND SEMIGROUPS

By

BEDŘICH PONDĚLÍČEK

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Among the non-empty subsets of a ring (semigroup) A we define a relation by

$$P \leqq Q \leftrightarrow PQ = P = QP .$$

The purpose of this note is two-fold. Firstly we prove the following characterization of division rings.

THEOREM 1. *An associative ring A is a division ring if and only if the right ideals of A form a chain with respect to the relation $\leqq$.*

Before the proof we mention that in [1] S. LAJOS has characterized the strongly regular rings as rings all right ideals of which form a semilattice under the multiplication of subsets. (An associative ring A is strongly regular if $a \in a^2A$ for each $a \in A$.) Further by [2] the right ideals, left ideals, quasi-ideals and biideals coincide in each strongly regular ring.

PROOF. Let the set $R(A)$ of all right ideals of A forms a chain. Then $R(A)$ is a semilattice under the multiplication of subsets. By [1] Theorem 1, A is a strongly regular ring.

We first show that the idempotents in A form a chain. Let e, f be idempotents of A. Then $R(e) \leqq R(f)$ or $R(f) \leqq R(e)$, where by $R(x)$ we denote the principal right ideal generated by $x \in A$. Suppose (without loss of generality) that $R(e) \leqq R(f)$. Then $e \in R(e) = R(f)\,R(e) = R(e)\,R(f)$ and so we have $e \in fA \cap Af$ which implies $e = fu = vf$ for some $u, v \in A$. Hence $fe = f(fu) = fu = e = vf = (vf)f = ef$. Thus $e \leqq f$.

Now we shall prove that A contains exactly two idempotents. By way of contradiction, we assume there exist idempotents e and f in A such that $e \neq 0 \neq f \neq e$ and $e \leqq f$. Put $g = f - e$. Then we have $g^2 = (f-e)^2 = f^2 - fe - ef + e^2 = f - e = g$ and so g is an idempotent. Since the idempotents in A form a chain, we have $e \leqq g$ or $g \leqq e$. If $e \leqq g$, then $e = eg = e(f-e) = ef - e^2 = 0$. This contradicts the fact that $e \neq 0$. If $g \leqq e$, then $g = eg = 0$ and so $f = e$, a contradiction. Thus, A is a regular

ring containing two idempotents and so by [3] Proposition 2, A is a division ring.

The converse statement is trivial.

Taking into account the remarks done before the poof in the Theorem 1 one can write left ideal, quasi-ideal and biideal for right ideal, respectively.

We can pose the analoguous problem for semigroups. Is it possible to characterize the groups similarly?

THEOREM 2. *A semigroup S is a chain of right simple semigroups if and only if the right ideals of S form a chain with respect to the relation $\leqq$.*

PROOF. Let the set $R(S)$ of all right ideals of a semigroup S forms a chain. Then $R(S)$ is a semilattice under the multiplication of subsets. It follows from [4] that S is a semilattice of right simple semigroups. We shall prove that this semilattice $G(S)$ is a chain. Evidently, S is the union of a family of disjoint right simple semigroups $G_i\,(i\in I)$ such that the product $G_i G_k$ and $G_k G_i$ are both contained in the same G_j for every couple i, k in I. Let $G_i, G_k \in G(S)$, $i, k \in I$. Then $G_i S, G_k S \in R(S)$ and so $G_i S \leqq G_k S$ or $G_k S \leqq G_i S$. Suppose (without loss of generality) that $G_i S \leqq G_k S$. Then $G_i S = G_k S\, G_i S \subset G_k S$ and thus

$$G_i = G_i G_i \subset G_k S = \bigcup_{j\in I} G_k G_j .$$

Hence $G_i = G_k G_l$ for some $l \in I$. Therefore $G_k G_i = G_i$.

Conversely, assume that a semigroup S is a chain of right simple semigroups. By [4], the intersection of any two right ideals of S is equal to their product. We shall prove that $R(S)$ is a chain under the set inclusion. Since S is the union of disjoint right simple semigroups $G_i\,(i\in I)$, we have for any right ideal R of S, $R = \bigcup_{j\in J} G_j$, where $J \subset I$. Let $R_1, R_2 \in R(S)$. By way of contradiction, we assume $R_1 \not\subset R_2$ and $R_2 \not\subset R_1$. Then there exist i, k in I such that $G_i \subset R_1 - R_2$ and $G_k \subset R_2 - R_1$. Thus $G_i G_k \subset G_i$ or $G_i G_k \subset G_k$. Suppose that $G_i G_k \subset G_i$. Then $G_i G_k \subset R_1 R_2$ and so $G_i \subset R_1 R_2 = R_1 \cap R_2 \subset R_2$, a contradiction.

THEOREM 3. *A semigroup S is a chain of groups if and only if the right ideals of S and the left ideals of S form chains with respect to the relation $\leqq$.*

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SEPARATION PROPERTIES OF DISK PLANES

By

F. S. CATER

Portland State University, Portland, Oregon U.S.A.

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1. Introduction

Separation axioms and some other topological properties discussed in general topology are closely related to set theory and cardinal numbers. In this spirit, we invent a certain kind of topological space called a "disk plane" which is determined by a given subset of the real line. The separation axioms satisfied by this space are determined by properties of this set. For example, the disk plane is normal if and only if the set contains no uncountable subset that is closed in the Euclidean topology (Theorem 1). Likewise the disk plane is metrizable if and only if the set is countable. The continuum hypothesis also plays a role in determining the properties of disk planes. If we assume the continuum hypothesis, any completely normal disk plane is metrizable.

A disk plane is a generalization of a standard example (attributed independently to NIEMYTZKI and MOORE) of a topological space that is regular but not normal.

Much of our work is dependent on Lemma 1, and the proof of Lemma 1 rests ultimately on certain geometric properties of the circle. This illustrates again how point set topology borrows from other branches of mathematics.

Finally, we present some partial results on homeomorphisms of one disk plane onto another. This will also involve a Lipschitz-like condition (Theorem 9). In our conclusion, we will review some questions about disk planes not resolved in this work, that can be the subject of further study.

2. Disk Planes

In all that follows, X will denote the upper half of the Euclidean plane R^2 including the x-axis: $X = \{(x, y) \in R^2 : y \geqq 0\}$. The x-axis is denoted by R: $R = \{(x, y) \in R^2 : y = 0\}$. We let E denote some nonvoid subset of R. Now we define a topology on X as follows. For each $x \in E$ and each open disk U in X tangent to R at x, let $U \cup \{x\}$ be an open set. The topology

on X is generated by all the Euclidean open sets together with all sets of the form $U \cup \{x\}$ just described. The resulting topological space will be called the *disk plane* $\{X, E\}$. If $E = R$, then the disk plane $\{X, R\}$ is known variously as *balloon space*, the *Niemytzki plane* and the *Moore plane*. Balloon space is one of the standard examples of a regular Hausdorff space that is not normal. Every disk plane $\{X, E\}$ is obviously Hausdorff (no matter what set E is), and every topological space mentioned in this paper will be Hausdorff.

By the *Bing subspace* of the disk plane $\{X, E\}$, we mean the subspace formed by deleting from X all the points in $R \backslash E$. This subspace will sometimes be denoted $\{X, E\}_1$. We will often find that the disk plane has different topological properties than its Bing subspace. This subspace is inspired by Example E of [1].

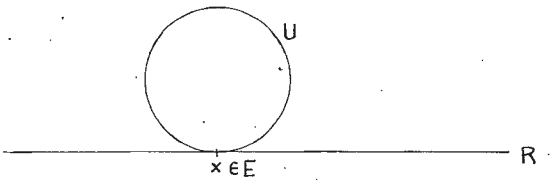

Fig. 1.

In this paper we will examine how properties of the set E relate to various degrees of separation of the disk plane $\{X, E\}$. We review here the kinds of separation we will consider.

1. A space is *regular* if for any closed set A and any point $x \notin A$, x and A have disjoint nbds.
2. A space is *completely regular* if for any closed set A and any point $x \notin A$, there is a continuous real valued mapping f on the space such that $f(x) = 0$ and $f(A) = 1$.
3. A space is *normal* if for any disjoint closed sets A and B, A and B have disjoint nbds.
4. A space is *completely normal* if for any separated sets A and B (this means $A \cap \bar{B} = \bar{A} \cap B = \emptyset$), A and B have disjoint nbds.
5. A space is *perfectly normal* if it is normal and each closed set is a G_δ-set.
6. A space is *metrizable* if there is a metric on its underlying set which induces its topology in the familiar way.

It is known that each of these conditions implies the one before it. For convenience, we call a nbd. of x of the form $U \cup \{x\}$ described above a *basic* nbd. when $x \in E$. If $x \in X \backslash E$, we call the intersection of X with any open disk centered at x, a *basic* nbd. of x. By a nbd. of any point or set y, we mean an open set containing y. (Thus all nbds. are open.)

We will often refer to a subset Y of X as a metric space in its own right under the Euclidean metric. We say that Y is *first category* if Y is the union of countably many closed nowhere dense subsets of Y, and Y

is *second category* otherwise. A subtle point: Y may be a second category metric space when Y is a first category subset of X in the sense that Y is a subset of the union of countably many closed nowhere dense subsets of X. Consider $Y = R$ for example.

A word about our notation: if x and y are points or subsets of X, we let $|x, y|$ denote the Euclidean distance between x and y. For example,

$$|R, X\backslash R| = 0.$$

Of course, the only distance we will mention is Euclidean distance. When we refer to open sets, closed sets, closures of sets in $\{X, E\}$, etc., we will usually have the disk plane topology in mind. When we have the Euclidean topology in mind, we will say so.

We immediately dispose of the property completely regular.

PROPOSITION. Any disk plane is completely regular.

PROOF. Let p be a point and let S be a closed set such that $p \notin S$. If $p \notin E$, then just define $f(x) = |x, p|/(|x, p| + |x, S|)$, and the rest is obvious.

Now suppose $p \in E$. Let $p \in U \subseteq V$ where U and V are basic nbds of p as shown in figure 2, and $S \cap V = \emptyset$. Define $f(x) = 0$ for all $x \in U$ (in particular $f(p) = 0$), and for $x \in X\backslash U$, define

$$f(x) = |x, U|/(|x, U| + |x, X\backslash V|).$$

Then f is continuous on $\{X, E\}$ and $f(S) = 1$. ∎

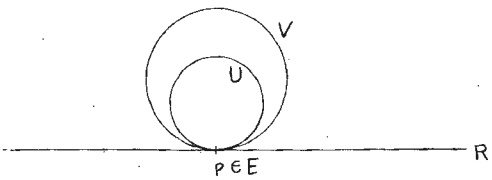

Fig. 2.

We come to the lemmas which are crucial in determining the degree of separation of a disk plane.

LEMMA 1. Let $Q \subseteq X\backslash R$. For each $q \in Q$, let U_q be the open disk with center q and radius $|q, R|/19$, and let $S = \bigcup_{q \in Q} U_q$. Then

$$S \cap E = \overline{Q} \cap E.$$

PROOF. It suffices to prove that $\overline{S} \cap E \subseteq \overline{Q} \cap E$ since the reverse inclusion is clear. We argue by contradiction. Suppose $t \notin \overline{Q} \cap E$ but $t \in \overline{S} \cap E$. Since t is not in the closure of any U_q, it follows (from the radii of the U_q) that there is a sequence of points $q_n \in Q$ such that $\lim_{n \to \infty} |t, q_n| = 0$ and t lies in the closure of $\bigcup_n U_{q_n}$.

Now let I be a basic nbd. of t of radius r such that $I \cap Q = \emptyset$, and let J be the basic nbd. of t of radius $(2r)/3$. By passing to a subsequence

if necessary, we may suppose, without loss of generality, that $J \cap U_{q_n} \neq \emptyset$ for each n.

For each n, let a_n be a boundary point of circle J (other than t) lying in U_{q_n}. The figure 3 is self explanatory and figure 4 is an enlargement of the part of figure 3 that interests us. Let R be the x-axis, t be the origin $(0, 0)$, $(0, r)$ be the center of circle I, $(0, (2r)/3)$ be the center of circle J, $a_n = (x_n, y_n)$, $c_n = (x_n, y'_n)$, $d_n = (x_n, 0)$.

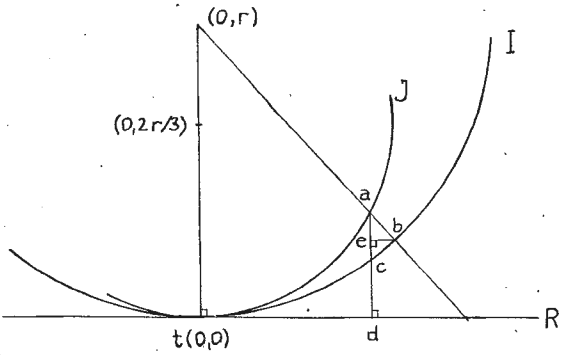

Fig. 3.

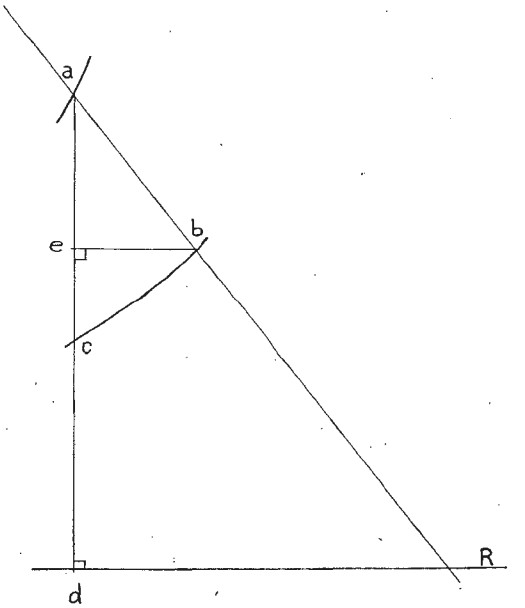

Fig. 4.

Then $|c_n, d_n|/|a_n, d_n| = y'_n/y_n = ((4r/3)-y_n)/(2r-y'_n)$, where the second equation follows from the equations of circles I and J, and

(1) $$\lim_{n\to\infty} |c_n, d_n|/|a_n, d_n| = 2/3.$$

It follows from (1) that

(2) $$\lim_{n\to\infty} |a_n, c_n|/|a_n, d_n| = 1/3.$$

But as $n\to\infty$, the central angle in circle I tends to 0, so the angles $a_n\, b_n\, e_n$ and $a_n\, c_n\, b_n$ each tend to $\pi/2$, and

(3) $$\lim_{n\to\infty} |a_n, b_n|/|a_n, c_n| = 1.$$

We combine (2) and (3) to obtain

(4) $$\lim_{n\to\infty} |a_n, b_n|/|a_n, d_n| = 1/3.$$

But $|a_n, d_n| > |q_n, R| - |q_n, R|/19$ since a_n lies in the circle with center q_n and radius $|q_n, R|/19$. So

(5) $$|a_n, d_n| > 18|q_n, R|/19.$$

Thus

(6) $$\limsup_{n\to\infty} |a_n, b_n|/|q_n, R| \geq 6/19.$$

There exist indices n for which $|a_n, b_n| > 2|q_n, R|/19$, and since $|a_n, q_n| < |q_n, R|/19$, we see that q_n lies in I for these n. This conflicts with the choice of the nbd. I.

LEMMA 2. Let A, B be separated subsets of the disk plane $\{X, E\}$; that is, $\overline{A}\cap B = A\cap\overline{B} = \emptyset$. Suppose the sets $A\cap E$ and $B\cap E$ have disjoint nbds. Then A and B have disjoint nbds.

PROOF. Each member of $B\backslash E$ has positive distance from the set A. For each $b\in B\backslash E$, let U_b be an open disk with center b and radius $<$ min $(|b, A|, |b, R|)/19$ if $b\notin R$, and let U_b be an open disk with center b and radius $<|b, A|/2$ which does not meet any basic nbd. of radius 1 of any point in $A\cap E$ if $b\in R$. Then by LEMMA 1, the closure of $\bigcup_{b\in B\backslash E} U_b$ is disjoint from $A\cap E$. Clearly the closure of $\bigcup_{b\in B\backslash E} U_b$ is disjoint from $A\backslash E$. Thus $B\backslash E$ has a nbd. whose closure is disjoint from A. By reversing the roles of A and B, we see that $B\cap E$ has a nbd. V whose closure is disjoint from $A\backslash E$. By hypothesis, $B\cap E$ has a nbd. W whose closure is disjoint from $A\cap E$. Hence $V\cap W$ is a nbd. of $B\cap E$ whose closure is disjoint from A. Since $B\backslash E$ and $B\cap E$ have such nbds., so has B. ∎

We now give a necessary and sufficient condition on E that $\{X, E\}$ be normal. At this juncture it is well to observe that a subset F of E is closed if and only if the Euclidean closure of F is a subset of E. The verification is left to the reader.

LEMMA 3. The disk plane $\{X, E\}$ is normal if and only if every closed subset of E is an F_σ-set in the Euclidean topology.

PROOF. Suppose $\{X, E\}$ is normal. Let A be any closed subset of E. Then $R\backslash A$ is also closed. Since $\{X, E\}$ is normal, A and $R\backslash A$ have disjoint nbds. U and V respectively. For each $a\in A$, let U_a be a basic nbd. of a such that $a\in U_a\subseteq U$ and for each $b\in R\backslash A$ let V_b be a basic nbd. of b such that $b\in V_b\subseteq V$. Now $U_a\cap V_b = \emptyset$ for any $a\in A$, $b\in R\backslash A$. For each integer $n>0$, put $A_n = \{a\in A$: radius $U_a>1/n\}$. Then $A = \bigcup_n A_n$. For each $b\in R\backslash A$, $|b, A_n|>0$; for otherwise V_b would intersect U_a for some $a\in A_n$. Hence the Euclidean closure of A_n is a subset of A, and so A is the union of countably many closed sets in the Euclidean topology.

Now suppose that every closed subset of E is an F_σ-set in the Euclidean topology. Let A and B be closed disjoint subsets of the disk plane $\{X, E\}$. By Lemma 2, it suffices to prove that $A\cap E$ and $B\cap E$ have disjoint nbds. For each $x\in(A\cap R)\backslash E$ the distance from x to B is positive. Thus there is a basic nbd. U_x of x of radius $<\frac{1}{2}|x, B|$ such that the closure of U_x is disjoint from any basic nbd. of radius 1 of any point in $B\cap E$. Likewise each $y\in(B\cap R)\backslash E$ has a basic nbd. V_y such that closure of V_y is disjoint from $A\cup(\bigcup_{x\in(A\cap R)\backslash E} U_x)$ and from any basuc nbd. of radius 1 of any point in $A\cap E$. Thus

$$A\cup(\bigcup_{x\in(A\cap R)\backslash E} U_x) \quad \text{and} \quad B\cup(\bigcup_{y\in(B\cap R)\backslash E} V_y)$$

are disjoint sets. It suffices, then, to prove that

$$(A\cap E)\backslash\bigcup_{x\in(A\cap R)\backslash E} U_x = F \quad \text{and} \quad (B\cap E)\backslash\bigcup_{y\in(B\cap R)\backslash E} V_y = G$$

have disjoint nbds.

Now any cluster point of F lies in $A\cap R$ but not in $\bigcup_{x\in(A\cap R)\backslash E} U_x$. It follows that F is a closed set. Likewise G is closed. By assumption, there exist closed sets P_n and Q_n in the Euclidean topology such that $F = \bigcup_{n=1}^{\infty} P_n$ and $G = \bigcup_{n=1}^{\infty} Q_n$. Of course every P_i is disjoint from every Q_j.

For each $x\in P_1$, let N_{1x} be a basic nbd. of x of radius <1. Then (since $P_1\cap Q_1 = \emptyset$) for each $y\in Q_1$ there is a basic nbd. M_{1y} of y of radius <1 such that $N_{1x}\cap M_{1y} = \emptyset$ for all $x\in P_1$, $y\in Q_1$. Then (since $P_2\cap Q_1 = \emptyset$) for each $x\in P_2$ there is a basic nbd. N_{2x} of x of radius <1 such that $N_{2x}\cap M_{1y} = \emptyset$ for all $x\in P_2$, $y\in Q_1$. Then (since $(P_1\cup P_2)\cap Q_2 = \emptyset$) for each $y\in Q_2$ there is a basic nbd. M_{2y} of y of radius <1 such that $N_{ix}\cap M_{2y} = \emptyset$ $(i = 1,2)$ for all $x\in P_1\cup P_2$, $y\in Q_2$. Then (since $P_3\cap(Q_1\cup Q_2) = \emptyset$) for each $x\in P_3$ there is a basic nbd. N_{3x} of x of radius <1 such that $N_{3x}\cap M_{jy} = \emptyset$ $(j = 1,2)$ for all $x\in P_3$, $y\in Q_1\cup Q_2$. ... We proceed in this manner to construct a nbd. $N_n = \bigcup_{x\in P_n} N_{nx}$ of each P_n and a nbd. $M_n = \bigcup_{y\in Q_n} M_{ny}$ of each Q_n such that $N_i\cap M_j = \emptyset$ for all indices i and j. Hence $\bigcup_i N_i$ and $\bigcup_j M_j$ are disjoint nbds. of F and G. ∎

Lemma 3 can be used to prove a sharper result. In Theorem 1 we will obtain another condition equivalent to the disk plane $\{X, E\}$ being normal.

THEOREM 1. *The disk plane $\{X, E\}$ is normal if and only if E has no uncountable subset which is closed in the Euclidean topology.*

PROOF. Suppose E has no uncountable subset which is closed in the Euclidean topology. Let F be a closed subset of E in the disk plane $\{X, E\}$. Then the Euclidean closure G of F is also a subset of E, and by assumption, G must be countable. Thus F is countable, and F is an F_σ-set in the Euclidean topology. By Lemma 3, $\{X, E\}$ is normal.

Now suppose $\{X, E\}$ is normal. We argue by contradiction. Assume that F is an uncountable subset of E that is closed in the Euclidean topology. Then any subset of F is closed in the disk plane $\{X, E\}$. For each subset P of F, choose an open set U_P such that $P \subseteq U_P$ and the closure of U_P does not intersect $F \backslash P$. (The existence of U_P follows from normality since P and $F \backslash P$ are closed.) Let S denote the countable set of all points in X both of whose coordinates are nonzero rational numbers. Then

$$P \to S \cap U_P$$

is a mapping of the power set of F into the power set of S. But this mapping is one-to-one; for if P_1 and P_2 are distinct subsets with $P_2 \nsubseteq P_1$, say, then some $x \in P_2 \backslash P_1$ has a nbd. U such that $U \subseteq U_{P_2}$, $U \cap U_{P_1} = \emptyset$, $S \cap U \neq \emptyset$ and $S \cap U_{P_2} \neq S \cap U_{P_1}$. But $|S| = \aleph_0$ and $|F| = 2^{\aleph_0}$ because F contains a nonvoid perfect subset in the Euclidean topology. It follows that $2^{|F|} \leq 2^{|S|}$, which is impossible. ∎

Thus in particular, balloon space is not normal because R is uncountable. It also has been well-known that if F is a closed discrete subset of a topological space Y which has a dense subset S such that $|F| = 2^{|S|}$, then Y is not normal. This is proved by a cardinality argument essentially like the second part of the proof of Theorem 1.

Lemma 3 can now be improved as follows. The disk plane $\{X, E\}$ is normal if and only if every closed subset of E is countable. This works just as well when "closed subset" is replaced by "Euclidean closed subset."

Having disposed of complete regularity and normality, we turn to complete normality.

THEOREM 2. *The disk plane $\{X, E\}$ is completely normal if and only if every subset of E is the intersection of E with an F_σ-set in the Euclidean topology.*

PROOF. Suppose every subset of E is the intersection of E with an F_σ-set in the Euclidean topology. Let A and B be separated subsets of $\{X, E\}$; $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. To show that A and B have disjoint nbds., it suffices to show that $A \cap E = F$ and $B \cap E = G$ have disjoint nbds. By hypothesis, there exist closed sets S_n and T_n in the Euclidean topology, such that

$$F = E \cap (\cup_{n=1}^{\infty} S_n) = \cup_{n=1}^{\infty} (S_n \cap E),$$

$$G = E \cap (\cup_{n=1}^{\infty} T_n) = \cup_{n=1}^{\infty} (T_n \cap E).$$

Now put $S_n \cap E = P_n$ and $T_n \cap E = Q_n$ for each n. Then $F = \cup_n P_n$ and $G = \cup_n Q_n$. The distance between any P_i and any point in any Q_j

is positive, and the distance between any Q_j and any point in any P_i is positive. Thus each $x \in P_i$ has a nbd. disjoint from all basic nbds. of radius 1 of all points in Q_j. Likewise each $y \in Q_j$ has a basic nbd. disjoint from all basic nbds. of radius 1 of all points in P_i. The proof that F and G have disjoint nbds. is precisely like the last paragraph of the proof of Lemma 3.

Now suppose that $\{X, E\}$ is completely normal. Let F be any subset of E. Then F and $E \backslash F$ are separated sets, and by hypothesis there exist open sets U and V such that $F \subseteq U$, $E \backslash F \subseteq V$ and $U \cap V = \emptyset$. For each $x \in F$, let U_x be a basic nbd. of x such that $x \in U_x \subseteq U$, and for each $y \in E \backslash F$, let V_y be a basic nbd. of y such that $y \in V_y \subseteq V$. Then $U_x \cap V_y = \emptyset$ for any $x \in F$, $y \in E \backslash F$. Let $P_n = \{x \in F\colon \text{radius } U_x > 1/n\}$ for each $n > 0$. Then the Euclidean closure $\overline{P_n}$ of P_n is disjoint from $E \backslash F$; otherwise some V_y would meet some U_x $(x \in F, y \in E \backslash F)$. Thus

$$F = \bigcup_n (\overline{P_n} \cap E) = E \cap (\bigcup_n \overline{P_n}) . \blacksquare$$

Now we bring category into the picture.

COROLLARY 1. Suppose E contains a subset F which is a second category metric space, dense in itself, under the Euclidean metric. Then the disk plane $\{X, E\}$ is not completely normal.

PROOF. Let G be a countable dense subset of the metric space F. If $\{X, E\}$ is completely normal, then by Theorem 2, $F \backslash G$ is the union of countably many nowhere dense closed subsets of F, and F is first category. This is impossible.

Again we bring cardinality into the picture.

COROLLARY 2. If $|E| = 2^{\aleph_0}$, then $\{X, E\}$ is not completely normal.

PROOF. If $|E| = 2^{\aleph_0}$, then the number of subsets of E is $2^{2^{\aleph_0}}$. But the number of F_σ-sets in the Euclidean topology is only $2^{\aleph_0}$. The rest is clear. $\blacksquare$

We can also use cardinality to show that normality is not equivalent to metrizability for disk planes.

EXAMPLE 1. There exists a disk plane $\{X, E\}$ that is normal but not metrizable.

PROOF. Either $\aleph_1 < 2^{\aleph_0}$ or $\aleph_1 = 2^{\aleph_0}$.

Case I. If $\aleph_1 < 2^{\aleph_0}$ just let E be any subset of R satisfying $|E| = \aleph_1$. Then $\{X, E\}$ is not second countable, but it is separable. Hence $\{X, E\}$ is not metrizable. But the only closed subsets of E in the Euclidean topology are countable, because uncountable closed sets have cardinality $2^{\aleph_0}$. Thus any Euclidean closed subset of E is countable, so $\{X, E\}$ is normal by Theorem 1 and the remarks following it.

Case II. Now suppose $\aleph_1 = 2^{\aleph_0}$. Let Ω denote the well ordered set of all countable ordinal numbers. Since the set $\mathcal{S}$ of all closed nowhere dense subsets of R in the Euclidean topology has cardinality $\aleph_1$, there is a bijective mapping $\alpha \mapsto S_\alpha$ of Ω onto $\mathcal{S}$. Let $\Gamma \subseteq \Omega$ be the set of all $\gamma \in \Omega$ such that $S_\gamma \nsubseteq \bigcup_{\alpha<\gamma} S_\alpha$. Then Γ is cofinal in Ω because R is not the union of countably many sets in $\mathcal{S}$. So $|\Gamma| = \aleph_1$. For each $\gamma \in \Gamma$, choose an element

$x_\gamma \in S_\gamma \setminus \bigcup_{\alpha<\gamma} S_\alpha$. It follows that each member of $\mathcal{S}$ contains at most countably many of the elements in the set

$$E = \{x_\gamma\}_{\gamma\in\Gamma}.$$

We claim that no Euclidean closed subset F of E is uncountable. Suppose, on the contrary, that F is such a set. Then F has a nonvoid perfect subset G in the Euclidean topology, and G in turn has a nonvoid, perfect nowhere dense subset S_γ. But $|S_\gamma| = \aleph_1$ and $S_\gamma \subseteq E$. Thus S_γ contains uncountably many points in E, contrary to the construction of E.

That $\{X, E\}$ is normal but not metrizable follows from the same argument used in Case I. ∎

It is interesting that Example 1 can be carried through whether the continuum hypothesis $\aleph_1 = 2^{\aleph_0}$ is assumed or not. However, the construction is different in the two cases.

It is easy to show that if the Bing subspace $\{X, E\}_1$ is normal, then each subset of E is the intersection of E with some F_σ-set in the Euclidean topology. The proof is essentially like the second part of the proof of Theorem 2. Thus $\{X, E\}$ is completely normal if and only if its Bing subspace $\{X, E\}_1$ is normal. (Compare with [1, Example E].)

3. Paracompact Spaces

By a *paracompact* space we mean a topological space for which every open covering has an open locally finite refinement. It is known [2, p. 339] that any paracompact Hausdorff space is normal. We find that for disk planes the converse is also true.

THEOREM 3. *A disk plane $\{X, E\}$ is paracompact if and only if $\{X, E\}$ is normal.*

PROOF. It suffices to prove that $\{X, E\}$ is paracompact if $\{X, E\}$ is normal. Assume that $\{X, E\}$ is normal and let $\{U_\alpha\}$ be an open covering of $\{X, E\}$. We must prove that $\{U_\alpha\}$ has an open locally finite refinement. By passing to a refinement if necessary, we may assume without loss of generality that each U_α is a basic nbd. of some point in X. Let Y denote the union of all the U_α that are basic nbds. of points in $X\setminus E$. Then $R\setminus Y = E\setminus Y$ is a Euclidean closed subset of E. By Theorem 1, $E\setminus Y$ is countable. Let $x_1, x_2, x_3, \ldots$ be the points in $E\setminus Y$ enumerated. (If $E\setminus Y$ is void, then $\{U_\alpha\}$ has a refinement composed of Euclidean open sets which in turn has the desired open locally finite refinement because any metric space is paracompact [5, p. 160].) By induction we select for each n a basic nbd. V_n of x_n such that the V_n are pairwise disjoint, each V_n is a subset of some U_α and the radius of V_n tends to 0 as $n\to\infty$. For each n let S_n denote the closure of the basic nbd. of x_n with radius = 1/2 radius of V_n, and let T_n denote the closure of the basic nbd. of x_n with radius = 1/4 radius of V_n.

Now $X\setminus \bigcup_n T_n$ is covered by a family of Euclidean open sets each of which is a subset of some U_α, and this covering has a Euclidean open

locally finite refinement because any metric space is paracompact. Call this refinement $\{Q_\beta\}$. Also $\cup_n S_n$ is a Euclidean closed set. For each β put

$$P_\beta = Q_\beta \setminus \cup_n S_n .$$

It follows that the family $\{P_\beta\} \cup \{V_n\}$ is an open covering of $\{X, E\}$ that is a locally finite refinement of $\{U_\alpha\}$. Note that each point in the set $\cup_n T_n$ has a nbd. that meets at most one member of this refinement. ∎

The situation for the Bing subspace $\{X, E\}_1$ is quite different.

THEOREM 4. *The Bing subspace $\{X, E\}_1$ is paracompact if and only if E is countable.*

PROOF. If E is countable, it is easy to see that $\{X, E\}_1$ has a countable basis and is regular; so $\{X, E\}_1$ is metrizable and hence paracompact.

Let E be uncountable. Take any open covering of $\{X, E\}_1$ composed of $X \setminus R$ together with a basic nbd. of each point in E. Let $\{U_\alpha\}$ be any open refinement of this covering. For each $x \in E$, let V_x be a basic nbd. of x which is a subset of some U_α. For each $n > 0$, put $W_n = \{x \in E$: radius $V_x > 1/n\}$. Then $E = \cup_n W_n$ and some W_n is uncountable. Say W_N is uncountable. Let t be a condensation point in the real line R of the set W_N. Then the point $(t, 1/n)$ lies in uncountably many V_x and uncountably many U_α. Hence $\{X, E\}_1$ is not paracompact. ∎

In Example 1 we produced an uncountable set $E \subseteq R$ which contains no uncountable Euclidean closed subset of R. Thus $\{X, E\}$ is paracompact but $\{X, E\}_1$ is not. We have an example of a first countable separable paracompact Hausdorff space with a nonparacompact subspace. Indeed the subspace has an open covering such that for any open refinement of it, some point lies in uncountably many sets in the refinement. If we assume the continuum hypothesis, the subspace is not normal.

4. Developable Spaces

By a *development* of a topological space S we mean a sequence of open coverings $\{U_{1\alpha}\}, \{U_{2\alpha}\}, \{U_{3\alpha}\}, \ldots$ such that for each closed set A of S and each point $x \in S \setminus A$, there exists an index n such that the star of x with respect to $\{U_{n\alpha}\}$ is disjoint from A [6]. We say that the space S is *developable* if it has a development. Of course any subspace of a developable space must be developable. Moreover, any metric space is developable; just let $\{U_{n\alpha}\}$ consist of all the open spheres of radius $\leq 1/n$. We identify the developable disk planes.

THEOREM 5. *A disk plane $\{X, E\}$ is developable if and only if E is an F_σ-set in the Euclidean topology.*

PROOF. Suppose E is an F_σ-set in the Euclidean topology. Say E is the union of an expanding sequence of Euclidean closed sets, $S_1 \subseteq S_2 \subseteq \subseteq S_3 \subseteq \ldots$. Let $\{U_{n\alpha}\}$ be the set of all basic nbds. of points of radius $\leq 1/n$ that do not intersect both S_n and $R \setminus S_n$ and do not intersect S_n in more than one point. This is clearly a development of $\{X, E\}$.

Now suppose that $\{X, E\}$ is developable and $\{U_{n\alpha}\}$ is a development of $\{X, E\}$. For each $n>0$, let $S_n = E\backslash$(star of $R\backslash E$ with respect to $\{U_{n\alpha}\}$). Since $R\backslash E$ is closed, each $x\in E$ lies in some S_n. Morever, each S_n is closed. Hence the Euclidean closure T_n of S_n is a subset of E. Finally, $E = \bigcup_n T_n$ is a Euclidean F_σ-set. ▌

But the Bing subspace $\{X, E\}_1$ is developable for any set E; let $\{U_{n\alpha}\}$ consist of all the basic nbds. of radius $\leq 1/n$ that do not intersect R in more than one point. Now let E be the set of points in R with irrational x coordinate. Then $\{X, E\}_1$ is developable and $\{X, E\}$ is not. This is an example of a first countable, separable, completely regular nondevelopable space S with a countable subset T such that the subspace $S\backslash T$ is developable.

On the other hand, it is not hard to show that if we delete finitely many points from a first countable nondevelopable T_1 space, the resulting subspace is also nondevelopable.

Before we discuss metric spaces we find another property equivalent to those in Theorem 5.

LEMMA 4. For any disk plane $\{X, E\}$ the following are equivalent.

(1) E is an F_σ-set in the Euclidean topology.
(2) $\{X, E\}$ is developable.
(3) Every closed set in $\{X, E\}$ is a G_δ-set in $\{X, E\}$.

PROOF. (1)$\Leftrightarrow$(2). Theorem 5.

PROOF. (1)$\Rightarrow$(3). Assume (1). Say $E = \bigcup_n A_n$ where each A_n is Euclidean closed. Let G be any closed set in $\{X, E\}$. Let V_n be the union of all basic nbds. of radius $1/n$ of all points in $G\cap E$. Let

$$U_n = \{x\in X : |x, G| < 1/n \quad \text{and} \quad x\notin A_n\cup\ldots\cup A_2\cup A_1\}.$$

Then U_n and V_n are open in $\{X, E\}$, and $G\subseteq U_n\cup V_n$ for all n. So

$$G\subseteq \bigcap_n (U_n\cup V_n).$$

Take any $x\in\bigcap_n(U_n\cup V_n)$. Either $x\in V_n$ for infinitely many n or $x\in U_n$ for infinitely many n. In the former case, $x\in G\cap E$, and in the latter case $|x, G| = 0$ and $x\notin E$. In any case, $x\in G$. Thus

$$G = \bigcap_n (U_n\cup V_n).$$

PROOF. (3)$\Rightarrow$(1). Assume (3). Then $R\backslash E$ is closed and is a G_δ-set. Say $R\backslash E = \bigcap_n U_n$ where each U_n is open. But there clearly exists a Euclidean open set V_n for each n, such that $R\backslash E\subseteq V_n\subseteq U_n$. Then $R\backslash E = \bigcap_n V_n$. Finally

$$E = \bigcup_n (R\backslash V_n)$$

and E is an F_σ-subset of R in the Euclidean topology. ▌

We say that a space is *collectionwise normal* if the sets in any discrete family of closed sets have pairwise disjoint nbds. We find that for Bing subspaces, collectionwise normal is equivalent to metrizable.

THEOREM 6. *For a disk plane $\{X, E\}$ the following are equivalent.*

(1) *$\{X, E\}$ is perfectly normal.*
(2) *E is countable.*
(3) *$\{X, E\}$ is second countable.*
(4) *$\{X, E\}$ is metrizable.*

PROOF (1)$\Rightarrow$(2). Assume (1). By Lemma 4, E is an F_σ-subset of R in the Euclidean topology. But $\{X, E\}$ is also normal since it is perfectly normal. By Theorem 1, every Euclidean closed subset of E is countable. It follows that E is countable.

PROOF (2)$\Rightarrow$(3). Each point in E has a countable basis for its nbd. system. The rest is clear.

PROOF (3)$\Rightarrow$(4). Any second countable regular Hausdorff space is metrizable.

PROOF (4)$\Rightarrow$(1). Any metrizable space is perfectly normal. ▌

Now suppose that the Bing subspace $\{X, E\}_1$ is collectionwise normal. The points in E form a discrete family of sets in $\{X, E\}_1$. Then for each $x \in E$, there is a basic nbd. V_x of x such that the sets V_x are pairwise disjoint. For each $n > 0$, let $E_n = \{x \in E$: radius $V_x > 1/n\}$. So $E = \bigcup_n E_n$. But clearly each set E_n must be discrete in the Euclidean topology and be countable. Thus E is countable. We will use this fact in Corollary 5.

We conclude this section with some corollaries.

COROLLARY 3. Assume that $2^{\aleph_0} < 2^{\aleph_1}$. (This would follow for example from the continuum hypothesis, $2^{\aleph_0} = \aleph_1$.) Then any completely normal disk plane is metrizable.

PROOF. Suppose that E is uncountable; that is, $|E| \geq \aleph_1$. Then the number of subsets of E exceeds $2^{\aleph_0}$ by hypothesis. But the number of F_σ-subsets of R in the Euclidean topology is only $2^{\aleph_0}$, so it follows from Theorem 2 that $\{X, E\}$ is not completely normal. ▌

COROLLARY 4. The Bing subspace $\{X, E\}_1$ has the property that every closed subset is a G_δ-set. (Compare with Lemma 4.)

PROOF. Let S be any closed subset of $\{X, E\}_1$. Fix an integer $n > 0$. For each $x \in S \setminus E$, let V_x denote the basic nbd. of x with radius $|x, E|/n$, and for each $x \in S \cap E$, let V_x denote the basic nbd. of x with radius $1/n$. Put $U_n = \bigcup_x V_x$. It follows that $S = \bigcap_n U_n$. ▌

COROLLARY 5. For a Bing subspace $\{X, E\}_1$, the following are equivalent.

(1) E is countable.
(2) $\{X, E\}_1$ is second countable.
(3) $\{X, E\}_1$ is metrizable.
(4) $\{X, E\}_1$ is collectionwise normal.

The proof is almost exactly the proof of Theorem 6, so we leave it to the reader.

We note that $\{X, E\}$ must be metrizable if $\{X, E\}_1$ is metrizable.

Another kind of topological space that is sometimes discussed is a *countably paracompact* space [3]. We say that S is countably paracompact if every countably open covering of S has an open locally finite refinement.

COROLLARY 6. For a disk plane $\{X, E\}$ the following are equivalent.

(1) $\{X, E\}$ is countably paracompact.
(2) $\{X, E\}$ is normal.
(3) $\{X, E\}$ is paracompact.

PROOF. (2)$\Leftrightarrow$(3) is Theorem 3. (3)$\Rightarrow$(1) is clear. It remains only to prove that (1)$\Rightarrow$(2). We proceed by contradiction. Assume $\{X, E\}$ is countably paracompact but not normal. Then E contains an uncountable Euclidean closed set which in turn contains a nonvoid Euclidean perfect set A. Then A is also a complete metric space under the Euclidean metric. Let $(a_n) \subseteq A$ be a sequence of distinct points in A which is dense in the metric space A. Let $\{U_n\}$ be the countable open covering of X where $U_0 = X \setminus \{a_n : n \geq 1\}$, and U_n is a basic nbd. of a_n for each $n \geq 1$. Let V_n be an open set containing a_n in some open locally finite refinement of $\{U_n\}$.

For each $x \in A$ let W_x be a basic nbd. of x intersecting at most finitely many V_n. For each integer $m > 0$, let $A_m = \{x \in A$: radius $W_x > 1/m\}$. Then $A = \bigcup_m A_m$ and some A_m is second category in the metric space A. Call it A_p. For each integer $m > 0$, let $B_m = \{x \in A_p : W_x \cap (\bigcup_{k \geq 0} V_{m+k}) = \emptyset\}$. Then $A_p = \bigcup_m B_m$ and some B_m is second category in the metric space A. Call it B_q. Then there is some interval J such that $A \cap J \neq \emptyset$ and $B_q \cap J$ is dense in the open set $A \cap J$ relative to the metric space A. Likewise $\{a_n : n \geq 1\} \cap J$ is dense in $A \cap J$. There exists an a_{q+k} $(k \geq 0)$ lying in J and clearly V_{q+k} intersects W_x for some $x \in B_q$. This is contrary to the choices of A_p and B_q. ∎

5. Homeomorphisms

We now turn to homeomorphisms of one disk plane $\{X, E\}$ onto another disk plane $\{X, E_0\}$. It is interesting that any such homeomorphism must also be a homeomorphism with respect to the Eulidean topology. Of course both $\{X, E\}$ and $\{X, E_0\}$ have topologies finer than the Euclidean topology.

THEOREM 7. *Let $\{X, E\}$ and $\{X, E_0\}$ be disk planes. Then any homeomorphism f of $\{X, E\}$ onto $\{X, E_0\}$ is also a homeomorphism with respect to the Euclidean topology, and $fE = E_0$.*

PROOF. Take any $p \in X$ and any $\varepsilon > 0$. We can enclose p in the interior of a square I of diameter $< \varepsilon$ as shown in figure 5. The boundary of I is evidently compact, and so is f(bdry I). But $f(p) \notin f$(bdry I) and f(bdry I) $\subseteq \bigcup_{n=1}^{\infty} \{q \in X : |q, f(p)| > 1/n\}$. So f(bdry I) $\subseteq \{q \in X : |q, f(p)| > 1/n\}$ for some n and f(bdry I) is bounded away from $f(p)$. Thus we can enclose $f(p)$ in the interior of a square J such that no point in J lies in f(bdry I).

We claim that $f^{-1}J \subseteq I$. Suppose, on the contrary, that $q \in f^{-1}J$ and $q \notin I$. Then $fq \in J$ and there is a connected set S containing fp and fq and

$S \subseteq J$. Thus $f^{-1}S$ is a connected set containing p and q, so some point q' in $f^{-1}S$ lies in bdry I. Then $fq' \in f(\text{bdry } I) \cap J$ which is impossible. Then $f^{-1}J \subseteq I$ and clearly there is a $\delta > 0$ such that $|t, fp| < \delta$ implies $|f^{-1}t, p| < \varepsilon$. Thus f^{-1} is continuous at fp relative to the Euclidean topology. Finally f^{-1} and likewise f are continuous relative to the Euclidean topology.

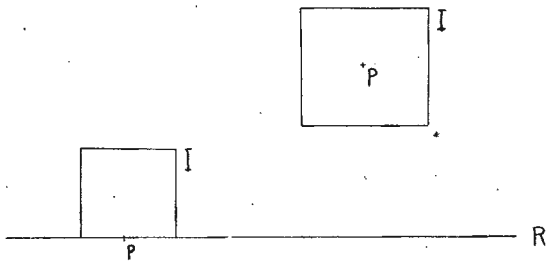

Fig. 5.

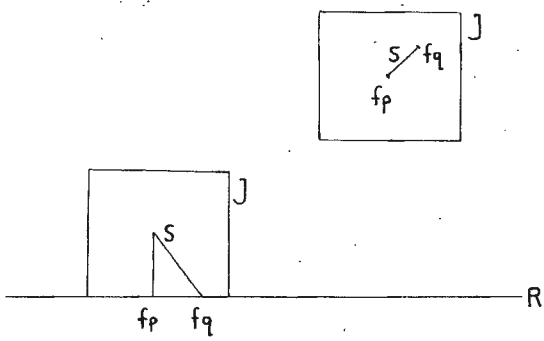

Fig. 6.

Now suppose that $p \in E$. Let V be a basic nbd. of p and let U be any open set such that $p \in U \subseteq V$. Then any line segment joining any point in $R \backslash \{p\}$ to any point in $U \backslash \{p\}$ must contain a boundary point of U. By joining points in U close enough to p, to points in R close enough to p, we see that there are points in the boundary of U arbitrarily close to p. Now bdry $U \subseteq \bigcup_{n=1}^{\infty} \{q \in X: |p, q| > 1/n\}$, but bdry U is not a subset of any of the open sets $\{q \in X: |p, q| > 1/n\}$. Thus bdry U is not compact. We conclude that p does not have a basis for its nbd. system composed of open sets with compact boundaries. On the other hand, the only points in $\{X, E_0\}$ with this property are points in E_0. Thus $fp \in E_0$. Hence $fE \subseteq E_0$, and similarly $f^{-1} E_0 \subseteq E$. ∎

However the converse of Theorem 7 is false as we now see.

EXAMPLE 2. There is a homeomorphism of X onto X relative to the Euclidean topology which is not a homeomorphism of balloon space onto itself.

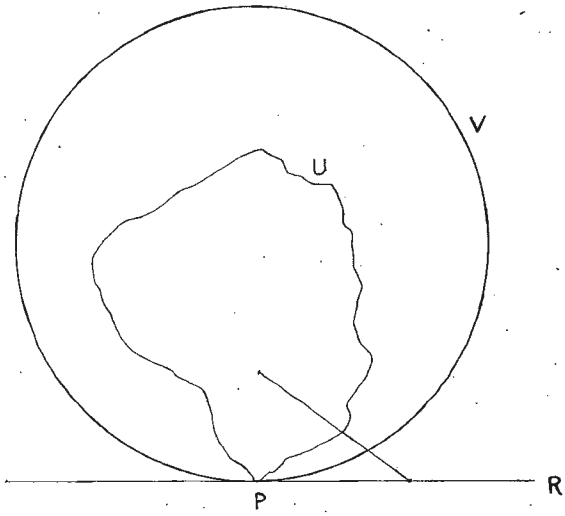

Fig. 7.

PROOF. For $0 \leq y \leq 1$ let $g(y) = (2y - y^2)^{1/2}$, and for $y \geq 1$, let $g(y) = 1$. Thus g is a continuous mapping of the nonnegative real axis onto $[0,1]$.

Now for each point (x, y) in X, let $f(x, y) = (x + g(y), y)$. Then f is obviously bicontinuous relative to the Euclidean topology. Let $Y = \{(0, y) \in X : y > 0\}$. Then $(0,0)$ is in the closure of Y and $f(0,0) = (0,0)$ is not in the closure of $fY = \{(g(y), y) \in X : y > 0\}$. Note that the basic nbd. of $(0,0)$ of radius 1 does not meet fY. So f is not a continuous mapping of balloon space onto itself. It likewise follows that f^{-1} is not a continuous mapping of balloon space onto itself. ∎

In Theorems 8 and 9 we will find a sufficient condition that a Euclidean homeomorphism f of X onto X be a disk plane homeomorphism. The property is that f satisfy a Lipschitz-like condition between points in R and points in $X \backslash R$.

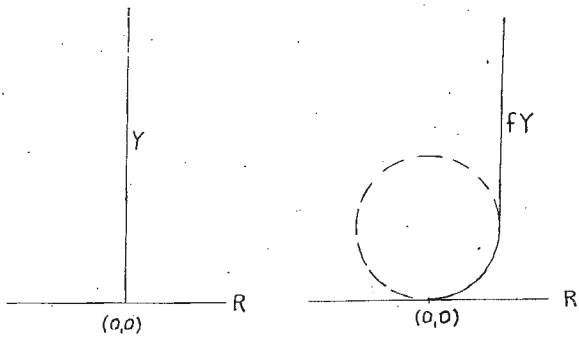

Fig. 8.

THEOREM 8. *Let E and E_0 be nonvoid subsets of R and suppose f is a bijective mapping of X onto X such that $fR = R$ and $fE = E_0$. Suppose there exists a number $M>0$ such that for any $w\in R$ and any $z\in X\backslash R$,*

$$|w, z|/M \leq |f(w), f(z)| \leq M|w, z| .$$

Then the mapping f: $\{X, E\}\mapsto\{X, E_0\}$ and the mapping f^{-1}: $\{X, E_0\}\mapsto\{X, E\}$ are continuous at each point of R relative to the disk plane topologies.

PROOF. First suppose that $p\in R\backslash E$. Then for any $q\in R$, there exists $t\in X\backslash R$ such that $|p, t|<|p, q|$ and $|q, t|<|p, q|$. Then

$$|f(p), f(q)| \leq |f(p), f(t)| + |f(t), f(q)| \leq M|p, t| + M|t, q| \leq 2M|p, q|$$

and it is clear that f is continuous at p. So f is continuous at all points of $R\backslash E$, and likewise f^{-1} is continuous at all points of $R\backslash E_0$.

Now suppose that $p\in E$. We argue by contradiction. Assume that f is not continuous at p. Since $\{X, E\}$ and $\{X, E_0\}$ are first countable spaces, there exists a sequence $(p_n)\subseteq X$ such that $p_n\to p$ but $f(p_n)\not\to f(p)$. For convenience, let p'_n denote $f(p_n)$. Then $p_n\to p$ and $p'_n\not\to p'$. Indeed we can suppose, without loss of generality, that there is a basic nbd. U of p' such that $p'_n\notin U$ for all n and $p_n\in X\backslash R$ for all n. Let $p_n = (x_n, y_n)$ and $p'_n = (x'_n, y'_n)$. We can assume, without loss of generality, that $p = p' = (0,0)$ by translating if necessary.

Now $\sqrt{x_n^2+y_n^2}\to 0$ and it follows from the hypothesis that $\sqrt{x_n'^2+y_n'^2}\to 0$. Hence $y_n\to 0$ and $y'_n\to 0$. Since $p'_n\notin U$, we havethat $y'_n/\sqrt{x_n'^2+y_n'^2}\to 0$. Then

$$(y_n/\sqrt{x_n^2+y_n^2})\,(y'_n/\sqrt{x_n'^2+y_n'^2})^{-1} = \tag{1}$$
$$= (y_n/y'_n)\,(\sqrt{x_n'^2+y_n'^2}/\sqrt{x_n^2+y_n^2}) \leq M(y_n/y'_n) .$$

But y'_n is the distance from (x'_n, y'_n) to $(x'_n, 0)$ and y_n cannot exceed the distance from (x_n, y_n) to $f^{-1}(x_n, 0)$. From (1) it follows that

$$(y_n/\sqrt{x_n^2+y_n^2})\,(y'_n/\sqrt{x_n'^2+y_n'^2})^{-1} \leq M^2 . \tag{2}$$

It follows that $y_n/\sqrt{x_n^2+y_n^2}\to 0$, and hence $|x_n|/\sqrt{x_n^2+y_n^2}\to 1$. But

$$|x'_n|/\sqrt{x_n'^2+y_n'^2}\to 1$$

also, so

$$(|x_n|/\sqrt{x_n^2+y_n^2})\,(|x'_n|/\sqrt{x_n'^2+y_n'^2})^{-1} = (|x_n|/|x'_n|)\,(\sqrt{x_n'^2+y_n'^2}/\sqrt{x_n^2+y_n^2})\to 1 . \tag{3}$$

It follows from (3) that for large enough n,

$$1/2M \leq |x'_n|/|x_n| \leq 2M . \tag{4}$$

We assume, without loss of generality, that (4) holds for all n.

Let U have radius r. Let V be a basic nbd. of (0,0) with radius $s<r$ such that $s<r/4M^3$. We assume, without loss of generality, that $(x_n, y_n)\in V$ and $0<y_n<s$ and $0<y'_n<r$ for all n. Then we have that

(5) $$x_n^2 \leq s^2-(s-y_n)^2 = 2sy_n-y_n^2 .$$

Likewise $(x'_n, y'_n) \notin U$, so

(6) $$x_n'^2 \geq r^2-(r-y'_n)^2 = 2ry'_n-y_n'^2 .$$

(Consult figure 9.)

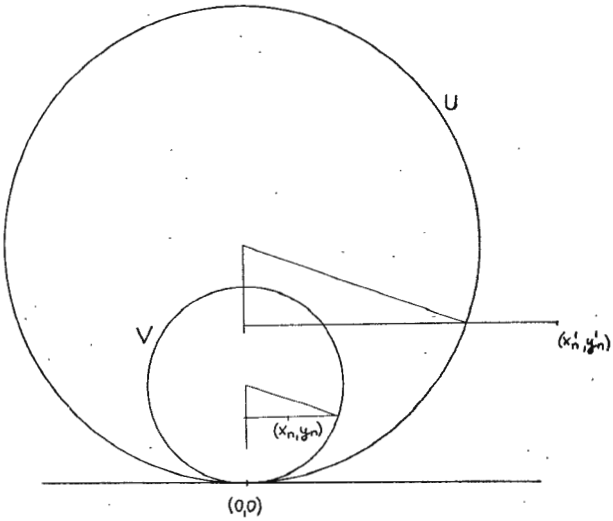

Fig. 9.

Thus

(7) $$x_n^2/x_n'^2 \leq (y_n/y'_n)\left((2s-y_n)/(2r-y'_n)\right),$$

and

(8) $$(x_n^2/x_n'^2)\left((2r-y'_n)/(2s-y_n)\right) \leq y_n/y'_n .$$

So

(9) $$\liminf_{n\to\infty} y_n/y'_n \geq (r/s)\liminf_{n\to\infty}(x_n^2/x_n'^2) \geq (1/4M^2)(r/s) > M .$$

Thus there is an index N with $y_N/y'_N > M$, and

(10) $$y'_N < y_N/M .$$

But y'_N is the distance from (x'_N, y'_N) to $(x'_N, 0)$, and y_N cannot exceed the distance from (x_N, y_N) to $f^{-1}(x'_N, 0)$. Finally

(11) $$|(x'_N, y'_N), (x'_N, 0)| < |f^{-1}(x'_N, y'_N), f^{-1}(x'_N, 0)|/M$$

which conflicts with the hypothesis.

So f is continuous at each point in E. Likewise f^{-1} must be continuous at each point in E_0. ∎

We also state

THEOREM 9. *If f is a homeomorphism of X onto X relative to the Euclidean topology satisfying the conditions of Theorem 8, then f is a homeomorphism of the disk plane $\{X, E\}$ onto the disk plane $\{X, E_0\}$.*

The proof is left to the reader.

6. Conclusion

We have determined which disk planes satisfy each of the separation axioms listed in Section 2.* We also determined which disk planes are countably paracompact, paracompact, and developable. In sections 3 and 4 we used the Bing subspace of a disk plane to produce examples of a topological space with a subspace satisfying an unusual combination of properties.

YOUNGLOVE [7] observed that when E is a Euclidean G_δ-subset of R, the Bing subspace $\{X, E\}_1$ is countably paracompact if and only if E is countable. Compare with our Corollary 6.

We conclude with some questions not answered in this paper that could be the subject of further study.

1. It can be shown, as in the proof of Corollary 6, that $\{X, E\}$ is normal if $\{X, E\}_1$ is countably paracompact. Is the converse true?
2. If the answer to 1 is no, find a necessary and sufficient condition for $\{X, E\}_1$ to be countably paracompact.
3. Is the converse of Corollary 3 true? If every completely normal disk plane is metrizable, must $2^{\aleph_0} < 2^{\aleph_1}$?
4. The condition given in Theorems 8 and 9 is unfortunately not necessary for f to be a disk plane homeomorphism. (This is easily shown by example.) Is there a "nice" way to characterize those Euclidean homeomorphisms of X onto X that are also homeomorphisms of balloon space onto itself?

7. Postscript

The construction in Example 1 inspired the following set theoretic constructions that admittedly have nothing to do with the topology of disk planes. We attach these digressions for their own sake.

DIGRESSION 1. Suppose $\aleph_1 = 2^{\aleph_0}$. Let $\mathcal{S}$ be a family of $\leq \aleph_1$ second category subsets of the real line R. Then there exists a sequence of pairwise disjoint sets $E_1, E_2, \ldots, E_n, \ldots$ such that $|S \cap E_n| = \aleph_1$ for each index n and each $S \in \mathcal{S}$, and $|T \cap E_n| \leq \aleph_0$ for each index n and each first category set T.

PROOF. Let Ω denote the well ordered set of all countable ordinal numbers. Of course $|\Omega| = \aleph_1$. Since $\aleph_1 |\mathcal{S}| = \aleph_1$, there exists a surjective

*The author wishes to express his thanks to Prof. BOGNÁR MÁTYÁS for his valuable suggestions.

mapping $\alpha \mapsto S_\alpha$ of Ω onto $\mathcal{S}$ such that for each $S \in \mathcal{S}$, $S = S_\alpha$ for $\aleph_1$ indices α. Likewise there is a bijective mapping $\alpha \mapsto T_\alpha$ of Ω onto the set of all closed nowhere dense subsets of R.

By transfinite induction, we select for each $\alpha \in \Omega$, a sequence of distinct points $x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n}, \ldots$ in $S_\alpha \setminus \bigcup_{\beta \leq \alpha} T_\beta$ such that $x_{\alpha n} \neq x_{\beta m}$ for all $\beta < \alpha$ and all n, m. This construction is possible because S_α is not the union of countably many first category sets. For each n, put $E_n = \{x_{\alpha n}\}_{\alpha \in \Omega}$. Clearly $|T_\alpha \cap E_n| \leq \aleph_0$ for all $\alpha \in \Omega$ and all n; hence $|T \cap E_n| \leq \aleph_0$ for all first category sets T and all n. Moreover, $|S \cap E_n| = \aleph_1$ since $x_{\alpha n} \in S_\alpha = S$ for $\aleph_1$ indices α and any $S \in \mathcal{S}$. ∎

DIGRESSION 2. Suppose $\aleph_1 = 2^{\aleph_0}$. There exists a family $\mathcal{S}_0$ of $\aleph_2$ subsets of R such that

(1) for any distinct sets A, B in $\mathcal{S}_0$, all the sets $A \cap B$, $(R \setminus A) \cap (R \setminus B)$, $(R \setminus A) \cap B$, $A \cap (R \setminus B)$, are second category
and

(2) for any first category set T and any $S \in \mathcal{S}_0$, $|T \cap S| \leq \aleph_0$.

PROOF. Let Γ denote the family of all families γ of subsets of R satisfying (1) and (2). To show that there exists a nonvoid such γ, let $\mathcal{S} = \{R\}$ in Digression 1, and note that $\{E_1\}$ satisfies (1) vacuously and also satisfies (2). Also $R \setminus E_1$ is second category since $E_2 \subseteq R \setminus E_1$.

Partially order Γ by inclusion. By Zorn's axiom, there is a maximal family in Γ.

Now suppose that γ is a member of Γ and $|\gamma| \leq \aleph_1$. Let $\mathcal{S}$ be the family $\{S : S \in \gamma \text{ or } R \setminus S \in \gamma\}$. Let E_1 and E_2 be the sets given by Digression 1. For any $S \in \mathcal{S}$, we have

$$|E_1 \cap S| = |E_2 \cap S| = |E_1 \cap (R \setminus S)| = |E_2 \cap (R \setminus S)| = \aleph_1,$$

so $E_1 \cap S$, $E_2 \cap S$, $E_1 \cap (R \setminus S)$, and $E_2 \cap (R \setminus S)$ are second category. But $E_2 \subseteq R \setminus E_1$. It follows that $\gamma \cup \{E_1\}$ satisfies (1) and (2) and lies in Γ, so γ is not a maximal member of Γ. Finally, any maximal member of Γ has cardinality $\geq \aleph_2$. ∎

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EIN ANALOGON EINES SATZES VON F. SZÁSZ

Von

DINH VAN HUYNH

Math, Inst. Hanoi, Vietnam

(Eingegangen am 2. September 1976)

Unter einem Ring verstehen wir stets einen assoziativen Ring.

Bezüglich der Untersuchung der hinerichenden Bedingung der Existenz eines Rechtseinselementes in einem Ring hat F. Szász [3] einen Satz bewiesen, der eine Verallgemeinerung einer Behauptung von Ch. Hopkins darstellt. Das Ziel dieser Note ist es, ein Analogon dieses Satzes von F. Szász (mit einer einfacheren Methode) zu beweisen, indem wir statt der Existenz des Einselementes von dem Faktorring R/J des Ringes R nach dem JACOBSONschen Radikal J von R die Existenz eines Rechtseinselementes von R/J voraussetzen, was F. Szász in [3] vorgeschlagen hat. Es gilt also der

Satz. *Ist R ein Ring mit dem* JACOBSON*schen Radikal J, der die folgenden Eigenschaften besitzt:*

1. *der Faktorring R/J hat ein Rechtseinselement;*
2. *es gilt die Minimalbedingung für die in J liegenden Rechtsideale von R,*

so gibt es in R Linksideale L_1 und L_2 mit $R = L_1 + L_2$ und $L_1 \cap L_2 = (0)$, wobei L_1 ein Ring mit einem Rechtseinselement und L_2 ein nilpotenter artinscher Ring ist.

Beweis. Es sei R ein Ring mit den Bedingungen des Satzes. Wegen 2. können wir wie in [2] (S. 151–152) zeigen, daß das JACOBSONsche Radikal J von R nilpotent ist. Es sei $\bar{e}$ ein Rechtseinselement von R/J. Da J nilpotent ist, gibt es in R ein idempotenter Vertreter e von $\bar{e}$, d.h. $e^2 = e$ und $e \in \bar{e}$ ([1], S. 54). Mit den Bezeichnungen

$$L_1 = Re \overset{\text{def}}{=} \{re \mid \forall\ r \in R\}$$

und

$$L_2 = R(1-e) \overset{\text{def.}}{=} \{r - re \mid \forall\ r \in R\}$$

gilt $L_1 \cap L_2 = (0)$ und $R = L_1 + L_2$, wobei L_1 und L_2 Linksideale von R sind und L_1 ein Rechtseinselement e besitzt. Da $\bar{e}$ Rechtseinselement von R/J ist, gilt offenbar $L_2 \leq J$. Wir sollen noch zeigen, daß die Minimalbedingung für Rechtsideale von L_2 gilt, d.h. L_2 artinsch ist. Diese Behauptung beweisen wir durch vollständige Induktion nach dem Nilpotenzgrad k von dem Radikal J von R. Für $k = 1$ ist $J = (0)$, die Behauptung ist also trivial. Die Behauptung sei für alle solche Ringe bewiesen, die den Bedingungen des Satzes genügen und deren Radikal einen Nilpotenzgrad besitzt, der kleiner als k ist.

Da $J^k = (0)$ und $M \overset{\text{def}}{=} J^{k-1} \neq 0$ ist, ist der Nilpotenzgrad des Radikals von $\bar{R} \overset{\text{def}}{=} R/M$ gleich $k-1$. $\bar{R}$ erfüllt offenbar alle Bedingungen des Satzes. Wir können also die Induktionsannahme für $\bar{R}$ anwenden. Es sei $\bar{e}$ das Bild von e bei dem natürlichen Homomorphismus von R auf $\bar{R}$. Dann sind

$$\bar{L}_1 \overset{\text{def}}{=} \{\bar{r}\bar{e} \mid \forall \bar{r} \in \bar{R}\}$$

und

$$\bar{L}_2 \overset{\text{def}}{=} \{\bar{r} - \bar{r}\bar{e} \mid \forall \bar{r} \in \bar{R}\}$$

Bilder von L_1 bzw. L_2 bei diesem Homomorphismus und es gilt $\bar{R} = \bar{L}_1 \oplus \bar{L}_2$. Nach Induktionsannahme ist $\bar{L}_2$ artinsch. Wegen Satz 6.1 aus [2] (S. 142) und der Isomorphie

$$\bar{L}_2 = (L_2 + M)/M \cong L_2/(L_2 \cap M)$$

genügt es zu zeigen, daß $A \overset{\text{def}}{=} L_2 \cap M$ als Ideal von L_2 artinsch ist. Es sei

$$(1) \qquad A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots$$

eine absteigende Kette von Rechtsidealen A_i von A. Dann ist

$$(2) \qquad A_1 + A_1 R \supseteq \ldots \supseteq A_n + A_n R \supseteq A_{n+1} + A_{n+1} R \supseteq \ldots$$

eine absteigende Kette von in J liegenden Rechtsidealen von R. Es sei $x \in A_i$ und $x \notin A_{i+1}$ für ein i. Dann gilt $x \in A_i + A_i R$. Wäre auch $x \in A_{i+1} + A_{i+1} R$, so würde $x \in A_{i+1}$ sich ergeben. Dabei wurde die folgende Tatsache benutzt: Es gilt

$$A_{i+1} + A_{i+1} R = A_{i+1} + A_{i+1}(Re + J) = A_{i+1} + A_{i+1} Re.$$

Ist

$$x = a + bre \, (a, b \in A_{i+1}, r \in R), \text{ d.h. } x \in A_{i+1} + A_{i+1} R,$$

so gilt $0 = xe = ae + bre^2 = bre$, woraus $x = a \in A_{i+1}$ folgt. Daher haben wir $A_i + A_i R \supset A_{i+1} + A_{i+1} R$, wenn $A_i \supset A_{i+1}$ gilt. Wegen 2. muß (2) und somit auch (1) nach endlich vielen Gliedern abbrechen, d.h. A ist artinsch.

Damit is der Beweis des Satzes erbracht.

Die folgende Folgerung folgt unmittelbar aus dem Satz und der Tatsache, dass jeder nilpotente artinsche Ring ein Torsionsring ist.

FOLGERUNG. Ein Ring R mit den Eigenschaften 1. und 2. des Satzes besitzt ein Rechtseinselement, wenn R eine der zwei fogenden Bedingungen erfüllt:

(a) R enthält kein von Null verschiedenes nilpotentes artinsches Linksideal;

(b) das JACOBSONsche Radikal J von R ist torsionsfrei.

Insbesondere enthält jeder torsionsfreie artinsche Ring stets ein Rechtseinselement.

Literatur

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ПРИМЕНЕНИЕ МЕТОДА УСРЕДНЕНИЯ К ОДНОЙ ДВУХТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ СИСТЕМ ИНТЕГРО – ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА ФРЕДГОЛЬМА НЕРАЗРЕШЕННЫХ ОТНОСИТЕЛЬНО ПРОИЗВОДНОЙ

Д. Д. БАЙНОВ
Пловдивский университет имени П. Хилендарского

С. Д. МИЛУШЕВА
Высший машинно – Электротехнический институт имени В. И. Ленина

София

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В работах [1], [2] был обоснован метод усреднения для решения двухточечной краевой задачи для систем обыкновенных дифференциальных уравнений. В настоящей работе этот метод применяется для решения нелинейной двухточечной краевой задачи для систем интегро – дифференциальных уравнений типа Фредгольма неразрешенных относительно производной.

Надо отметить, что метод усреднения для задачи Коши для систем интегро – дифференциальных уравнений был обоснован в работах [3] – [6].

Пусть для системы

$$\dot{x}(t) = \varepsilon X\left(t, x(t), y(t), \dot{x}(t), \dot{y}(t), \int_0^1 \varphi(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s))\, ds\right)$$

(1)

$$\dot{y}(t) = Y\left(t, x(t), y(t), \dot{x}(t), \dot{y}(t), \int_0^1 \varphi_1(t, s, x(s), \dot{x}(s), \dot{y}(s))\, ds\right),$$

где

$$x = (x^{(1)}, \ldots, x^{(n)}), \quad y = (y^{(1)}, \ldots, y^{(m)}), \quad X = (X^{(1)}, \ldots, X^{(n)}),$$

$$Y = (Y^{(1)}, \ldots, Y^{(m)}), \quad \varphi = (\varphi^{(1)}, \ldots, \varphi^{(q)}), \quad \varphi_1 = (\varphi_1^{(1)}, \ldots, \varphi_1^{(r)}),$$

а $\varepsilon > 0$ – малый параметр, задано краевое условие

(2)
$$x(0) = x^{\circ}, \quad R[\lambda, y(0), y(T)] = 0,$$

$$\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}), \quad R = (R^{(1)}, \ldots, R^{(m)}), \quad \lambda \in \Lambda \subset R_m,$$

$$T = L\varepsilon^{-1}, \quad L = \text{const} > 0.$$

Наряду с системой (1) рассмотрим вырожденную ($\varepsilon = 0$) по отношению к ней систему

$$x = \text{const}, \quad \dot{x} = \text{const}, \tag{3}$$

$$\dot{y}(t) = Y\left(t, x, y(t), \dot{x}, \dot{y}(t), \int_0^1 \varphi_1(t, s, x, y(s), \dot{x}, \dot{y}(s))\, ds\right),$$

с краевым условием

$$R[\lambda, y(0), y(T)] = 0. \tag{4}$$

Пусть вдоль интегральных кривых $y = \psi(t, x, \dot{x}, \lambda, T)$ краевой задачи (3), (4), где λ рассматривается как векторный параметр, существует независящее от параметра λ среднее значение

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X\left(t, x, \psi(t, x, \dot{x}, \lambda, T), \dot{x}, \frac{\partial}{\partial t}\psi(t, x, \dot{x}, \lambda, T)\right) dt,$$

$$\int_0^1 \varphi\left(\left(t, s, x, \psi(s, x, \dot{x}\, \lambda, T), \dot{x}, \frac{\partial}{\partial s}\psi(s, x, \dot{x}, \lambda, T)\right) ds\right) dt = \overline{X}(x, \dot{x}). \tag{5}$$

Тогда усредненным уравнением первого приближения для медленных переменных $x(t)$ системы (1) будем называть уравнение

$$\dot{\xi}(t) = \varepsilon\, \overline{X}(\xi(t), \dot{\xi}(t)) \tag{6}$$

с начальным условием

$$\xi(0) = x^0. \tag{7}$$

Отметим, что если $x = (x^{(1)}, \ldots, x^{(n)})$ и $A = (a_{ij})_{n,m}$, то по определению

$$\|x\| = \left[\sum_{i=1}^{n} (x^{(i)})^2\right]^{1/2}, \quad \|A\| = \left[\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}^2\right]^{1/2}.$$

Докажем теорему о близости компонент $x(t)$ решения $\{x(t), y(t)\}$ краевой задачи (1), (2) и решения задачи Коши (6), (7).

Теорема 1. Пусть:

1. *Функции $\varphi(t, s, x, y, u, v)$ и $\varphi_1(t, s, x, y, u, v)$ определены, непрерывны и равномерно ограничены вместе со своими частными производными по t для всех $t \geq 0$, $s \geq 0$, $(x, y, u, v) \in G$, где G некоторая открытая область $2(n+m)$-мерного пространства переменных x, y, u, v.*

Функция $X(t, x, y, u, v, z)$ определена, непрерывна и равномерно ограничена вместе со своими частными производными первого порядка для всех $t \geq 0$ и $(x, y, u, v, z) \in G \times G_1$, где G_1-некоторая открытая область пространства R_q.

Функция $Y(t, x, y, u, v, w)$ определена, непрерывна и равномерно ограничена вместе со своими частными производными первого порядка для

всех $t \geq 0$ *и* $(x, y, u, v, w) \in G \times G_2$, *где* G_2-*некоторая открытая область пространства* R_r.

2. *Существуют равномерно ограниченные в соответствующих проекциях области* $G^* = \{t \geq 0,\ s \geq 0,\ G \times G_1 \times G_2\}$ *обратные матрицы*

$$\left(I_1 - \varepsilon \frac{\partial X}{\partial u}\right)^{-1}, \quad \left(I_2 - \frac{\partial Y}{\partial v}\right)^{-1}, \quad \left[I_2 - \frac{\partial Y}{\partial v} - \varepsilon \frac{\partial Y}{\partial u}\left(I_1 - \varepsilon \frac{\partial X}{\partial u}\right)^{-1} \frac{\partial X}{\partial v}\right]^{-1},$$

где I_1 *и* I_2-*соответственно* n *и* m-*мерные единичные матрицы.*

3. *Через каждую точку области* $G^*(t, x, y, u)$ ($G^*(t, x, y, u)$-*проекция области* G^* *в пространстве переменных* t, x, y, u) *проходит единственная интегральная кривая краевой задачи* (3), (4), *соответствующая некоторому значению параметра* λ, *причем это решение определено для всех значений* $t \in [0, \infty)$ *и при этих значениях* t *целиком лежит в* $G^*(x, y, u)$.

Пусть указанные решения краевой задачи (3), (4) *задаются в форме*

$$y = \psi(t, x, \dot{x}, \lambda, T) \tag{8}$$

где $\psi - m$ — *мерная, непрерывная по совокупности всех переменных вектор — функция.*

4. *Для всех* $(x, u, \lambda) \in G^*(x, u) \times \Lambda$ *существует независящий от параметра* λ *предел*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X\left(t, x, \psi(t, x, u, \lambda, T), u, \frac{\partial}{\partial t} \psi(t, x, u, \lambda, T)\right) dt,$$

$$\int_0^1 \varphi\left(\left(t, s, x, \psi(s, x, u, \lambda, T), u, \frac{\partial}{\partial s} \psi(s, x, u, \lambda, T)\right) ds\right) dt = \overline{X}(x, u), \tag{9}$$

причем предельный переход в (9) *происходит равномерно относительно совокупности* $(x, u, \lambda) \in G^*(x, u) \times \Lambda$. *Среднее значение* $\overline{X}(x, u)$-*определена, непрерывна и равномерно ограничена функция для всех* $(x, u) \in G^*(x, u)$ *и удовлетворяет в* $G^*(x, u)$ *неравенством*

$$\|\overline{X}(x, u) - \overline{X}(x', u')\| \leq c(\|x - x'\| + \|u - u'\|),$$

где c-*положительная постоянная.*

5. *Краевая задача* (1), (2) *имеет единственное решение* $\{x(t), y(t)\}$ *удовлетворяющее условием*

$$x(0) = x^\circ, \quad R[\lambda, y(0)\, y(T)] = 0,$$

где под λ *подразумевается некоторое фиксированное значение параметра* λ. *Решение* $\{x(t), y(t)\}$ *определено на отрезке* $[0, T]$ *при всех значениях* $\varepsilon \in (0, \varepsilon^*]$ *и не выходит из некоторой открытой подобласти* $G^{*1}(x, y) \subset G^*(x, y)$ *лежащей целиком внутри* $G^*(x, y)$ *вместе со своей границей.*

6. *Решение усредненного уравнения*

$$\dot{\xi}(t) = \varepsilon \overline{X}(\xi(t), \dot{\xi}(t)) \tag{10}$$

удовлетворяющее начальному условию $\xi(0) = x^0$ *существует, однозначно определено при* $t \geq 0$ *для всех значениях* $\varepsilon \in (0, \varepsilon^*]$ *и кривая* $\{\xi(t), \dot{\xi}(t)\}$ *соответствующая этому решению на интервале* $0 \leq t < \infty$ *не выходит из области* $G^*(x, u)$.

7. *Уравнение в частным производных*

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial v}\left(I_2 - \frac{\partial Y}{\partial v}\right)^{-1}\left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial \varphi_1}{\partial t} ds\right) =$$

$$= X(t, x, y, u, v, z) - \overline{X}(x, u), \tag{11}$$

$$W|_{t=0} = 0, \quad x = const, \quad u = const$$

обладает некоторым решением $W = W(t, x, y, u, v)$, *которое определено и равномерно ограничено вместе со своими частными производными первого порядка во всей области* G^*.

Тогда, если $\{x(t), y(t)\}$ *решение краевой задачи* (1), (2), *а* $\xi(t)$-*решение задачи Коши* (6), (7), *то для любого сколь угодно малого* $\delta > 0$, *можно указать такое* $\varepsilon^0 > 0$, *что при* $0 < \varepsilon \leq \varepsilon^0$ *на отрезке* $0 \leq t \leq T$ *будет выполняться неравенство*

$$\|x(t) - \xi(t)\| < \delta.$$

ДОКАЗАТЕЛЬСТВО. Обозначим через $\eta(t)$ выражение

$$\eta(t) = x(t) - \varepsilon W(t, x(t), y(t), \dot{x}(t), \dot{y}(t)), \tag{12}$$

где $\{x(t), y(t)\}$-решение краевой задачи (1), (2), а W-непрерывно дифференцируемое решение (11).

Оценим разность $\dot{\eta}(t) - \dot{\xi}(t)$, где $\xi(t)$-решение усредненного уравнения (10) с начальным условием $\xi(0) = x^0$.

Имеем

$$\dot{\eta}(t) - \dot{\xi}(t) = \dot{x}(t) - \tag{13}$$

$$-\varepsilon\left\{\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}\dot{x}(t) + \frac{\partial W}{\partial y}\dot{y}(t) + \frac{\partial W}{\partial \dot{x}}\ddot{x}(t) + \frac{\partial W}{\partial \dot{y}}\ddot{y}(t) + \overline{X}(\xi(t), \dot{\xi}(t))\right\}.$$

Для определения $\ddot{x}(t)$ и $\ddot{y}(t)$ продифференцируем систему (1):

$$\ddot{x}(t) = \varepsilon\left\{\frac{\partial X}{\partial t} + \frac{\partial X}{\partial x}\dot{x}(t) + \frac{\partial X}{\partial y}\dot{y}(t) + \frac{\partial X}{\partial \dot{x}}\ddot{x}(t) + \frac{\partial X}{\partial \dot{y}}\ddot{y}(t) + \right.$$

$$\left. + \frac{\partial X}{\partial z}\int_0^1 \frac{\partial}{\partial t}\varphi(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s))\, ds\right\},$$

$$\ddot{y}(t) = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x}\dot{x}(t) + \frac{\partial Y}{\partial y}\dot{y}(t) + \frac{\partial Y}{\partial \dot{x}}\ddot{x}(t) + \frac{\partial Y}{\partial \dot{y}}\ddot{y}(t) +$$

$$+ \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial}{\partial t}\varphi_1(t, s, x(s), y(s), \dot{x}(s), \dot{y}(s))\, ds.$$

Отсюда

$$\left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)\ddot{x}(t) - \varepsilon\frac{\partial X}{\partial \dot{y}}\ddot{y}(t) =$$

$$= \varepsilon\left\{\frac{\partial X}{\partial t} + \frac{\partial X}{\partial y}Y + \frac{\partial X}{\partial z}\int_0^1 \frac{\partial \varphi}{\partial t}ds\right\} + \varepsilon^2\frac{\partial X}{\partial x}X, \tag{14}$$

$$-\frac{\partial Y}{\partial \dot{x}}\ddot{x}(t) + \left(I_2 - \frac{\partial Y}{\partial \dot{y}}\right)\ddot{y}(t) = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y}Y + \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial \varphi_1}{\partial x}ds + \varepsilon\frac{\partial Y}{\partial x}X.$$

Решение системы (14) относительно $\ddot{x}(t)$ и $\ddot{y}(t)$ представляем в виде

$$\ddot{x}(t) = \varepsilon P,$$

$$\ddot{y}(t) = \left(I_2 - \frac{\partial Y}{\partial \dot{y}}\right)^{-1}\left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y}Y + \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial \varphi_1}{\partial t}ds\right) + \varepsilon Q, \tag{15}$$

где

$$P = \left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)^{-1}\left\{I_1 + \varepsilon\frac{\partial X}{\partial \dot{y}}\left[I_2 - \frac{\partial Y}{\partial \dot{y}} - \varepsilon\frac{\partial Y}{\partial \dot{x}}\left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)^{-1}\frac{\partial X}{\partial \dot{y}}\right]^{-1}\cdot\right.$$

$$\left.\cdot\frac{\partial Y}{\partial \dot{x}}\left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)^{-1}\right\}\left[\frac{\partial X}{\partial t} + \frac{\partial X}{\partial y}Y + \frac{\partial X}{\partial z}\int_0^1 \frac{\partial \varphi}{\partial t}ds + \varepsilon\frac{\partial X}{\partial x}X\right] +$$

$$+\left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)^{-1}\frac{\partial X}{\partial \dot{y}}\left[I_2 - \frac{\partial Y}{\partial \dot{y}} - \varepsilon\frac{\partial Y}{\partial \dot{x}}\left(I_1 - \varepsilon\frac{\partial X}{\partial \dot{x}}\right)^{-1}\frac{\partial X}{\partial \dot{y}}\right]^{-1}\cdot$$

$$\cdot\left[\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y}Y + \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial \varphi_1}{\partial t}ds + \varepsilon\frac{\partial Y}{\partial x}X\right],$$

$$Q=\left(I_2-\frac{\partial Y}{\partial \dot y}\right)^{-1}\left\{\frac{\partial Y}{\partial x}X+\frac{\partial Y}{\partial \dot x}\left(I_1-\varepsilon\frac{\partial X}{\partial \dot x}\right)^{-1}\cdot\left[\frac{\partial X}{\partial t}+\frac{\partial X}{\partial y}Y+\right.\right.$$

$$\left.\left.+\frac{\partial X}{\partial z}\int_0^1\frac{\partial \varphi}{\partial t}ds+\varepsilon\frac{\partial X}{x\partial}X\right]\right\}+\frac{O(\varepsilon)}{\varepsilon}\left\{\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial y}Y+\frac{\partial Y}{\partial w}\int_0^1\frac{\partial \varphi_1}{\partial t}ds+\right.$$

$$+\varepsilon\frac{\partial Y}{\partial x}X+\varepsilon\frac{\partial Y}{\partial \dot x}\left(I_1-\varepsilon\frac{\partial X}{\partial \dot x}\right)^{-1}\cdot\left[\frac{\partial X}{\partial t}+\frac{\partial X}{\partial y}Y+\frac{\partial X}{\partial z}\int_0^1\frac{\partial \varphi}{\partial t}ds+\right.$$

$$\left.\left.+\varepsilon\frac{\partial X}{\partial x}X\right]\right\}.$$

Подставляя (15) в (13) получаем

$$\dot\eta(t)-\dot\xi(t)=-\varepsilon\left[\frac{\partial W}{\partial t}+\frac{\partial W}{\partial y}Y+\frac{\partial W}{\partial \dot y}\left(I_2-\frac{\partial Y}{\partial \dot y}\right)^{-1}\left(\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial y}Y+\right.\right.$$

$$\left.\left.+\frac{\partial Y}{\partial w}\int_0^1\frac{\partial \varphi_1}{\partial t}ds\right)-X(t,x,y,\dot x,\dot y,z)+\overline{X}(x,\dot x)\right]+\varepsilon[\overline{X}(\eta,\dot\eta)-\overline{X}(\xi,\dot\xi)]+$$

$$+\varepsilon[\overline{X}(x,\dot x)-\overline{X}(\eta,\dot\eta)]-\varepsilon^2\left(\frac{\partial W}{\partial x}X+\frac{\partial W}{\partial \dot x}P+\frac{\partial W}{\partial \dot y}Q\right). \tag{16}$$

В силу условий теоремы

$$\|\dot\eta(t)-\dot\xi(t)\|\le\varepsilon c[\|\eta(t)-\xi(t)\|+\|\dot\eta(t)-\dot\xi(t)\|+\|\eta(t)-x(t)\|+\|\dot\eta(t)-\dot x(t)\|]+\varepsilon^2 M, \tag{17}$$

где

$$\left\|\frac{\partial W}{\partial x}X+\frac{\partial W}{\partial \dot x}P+\frac{\partial W}{\partial \dot y}Q\right\|\le M=\text{const}. \tag{18}$$

Имея ввиду, что

$$\|\eta(t)-x(t)\|=\varepsilon\|W\|\le\varepsilon M_1, \tag{19}$$

$$\|\dot\eta(t)-\dot x(t)\|=\varepsilon\|\dot W\|\le\varepsilon M_2, \tag{20}$$

где M_1 и M_2-положительные постоянные, получаем

$$\|\dot\eta(t)-\dot\xi(t)\|\le\frac{\varepsilon c}{1-\varepsilon c}\|\eta(t)-\xi(t)\|+\frac{\varepsilon^2(cM_1+cM_2+M)}{1-\varepsilon c},$$

$$[\eta(t)-\xi(t)]|_{t=0}=0. \tag{21}$$

Отсюда следует, что

$$\|\eta(t)-\xi(t)\| \leq \frac{\varepsilon^2(c\,M_1+c\,M_2+M)}{1-\varepsilon\,c}\int_0^t \exp\{\varepsilon\,c(1-\varepsilon\,c)^{-1}(t-\tau)\}\,d\,\tau =$$

$$= \varepsilon\,c^{-1}[\exp\{\varepsilon\,c(1-\varepsilon\,c)^{-1}t\}-1]\,(c\,M_1+c\,M_2+M). \tag{22}$$

На отрезке $0 \leq t \leq T$ имеем оценку

$$\|\eta(t)-\xi(t)\| \leq \varepsilon\,c^{-1}(c\,M_1+c\,M_2+M)\,[\exp\{c\,L(1-\varepsilon\,c)^{-1}\}-1]. \tag{23}$$

Учитывая выражение для $\eta(t)$ находим

$$\|x(t)-\xi(t)\| \leq \|x(t)-\eta(t)\|+\|\eta(t)-\xi(t)\| \leq \varepsilon\|W\|+ \tag{24}$$

$$+\varepsilon\,c^{-1}(c\,M_1+c\,M_2+M)\,[\exp\{c\,L(1-\varepsilon\,c)^{-1}\}-1].$$

Так как функция W ограничена, то выражение, стоящее в правой части неравенства (24) при достаточно малом $\varepsilon(0<\varepsilon\leq\varepsilon^0,\ \varepsilon^0\leq\varepsilon^*)$ может быть сделано сколь угодно малым, т. е., справедливо неравенство

$$\|x(t)-\xi(t)\|<\delta.$$

Этим теорема I доказана.

Переидем теперь к обоснованию второго варианта метода усреднения.

Пусть вдоль интегральных кривых $y=\psi(t,x,\dot{x},\lambda,T)$ краевой задачи (3), (4), где λ рассматривается как векторный параметр, существуют независящие от λ средние значения

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X\left(t,x,\psi(t,x,\dot{x},\lambda,T),\dot{x},\frac{\partial}{\partial t}\psi(t,x,\dot{x},\lambda,T),z\right)dt=\overline{X}(x,\dot{x},z), \tag{25}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi\left(t,s,x,\psi(s,x,\dot{x},\lambda,T),\dot{x},\frac{\partial}{\partial s}\psi(s,x,\dot{x},\lambda,T)\right)ds=\overline{\varphi}(t,x,\dot{x}). \tag{26}$$

Тогда усредненным уравнением первого приближения для медленных переменных $x(t)$ решения $\{x(t),y(t)\}$ краевой задачи (1), (2) назовем уравнение

$$\dot{\xi}(t)=\varepsilon\,\overline{X}\left(\xi(t),\dot{\xi}(t),\int_0^1\overline{\varphi}(t,\xi(s),\dot{\xi}(s))\,ds\right) \tag{27}$$

с начальным условием

$$\xi(0)=x^0. \tag{28}$$

Докажем теорему о близости компонент $x(t)$ решения $\{x(t), y(t)\}$ краевой задачи (1), (2) и решения $\xi(t)$ задачи Коши (27), (28).

Теорема 2. *Пуст:*

1. *Выполнены условия* 1, 2, 3 *и* 5 *теоремы* 1.

2. *Функция* $\varphi(t, s, x, y, u, v)$ *удовлетворяет в области* $G^*(t, s, x, y, u, v)$ *неравенству*

$$\|\varphi(t, s, x, y, u, v) - \varphi(t, s, x', y, u', v)\| \leqq \sigma_1(t, s)\,[\|x - x'\| + \|u - u'\|],$$

где функция $\sigma_1(t, s)$ *со своей стороны удовлетворяет условию*

$$\int_0^1 \sigma_1(t, s)\, ds \leqq c_1 = const.$$

3. *Для всех* $(x, u, z, \lambda) \in G^*(x, u, z) \times \Lambda$ *и* $(t, x, u, \lambda) \in G^*(t, x, u) \times \Lambda$ *существуют независящие от параметра* λ *пределы*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X\left(t, x, \psi(t, x, u, \lambda, T), u, \frac{\partial}{\partial t}\psi(t, x, u, \lambda, T), z\right) dt = X(x, u, z); \tag{29}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi\left(t, s, x, \psi(s, x, u, \lambda, T), u, \frac{\partial}{\partial s}\psi(s, x, u, \lambda, T)\right) ds = \bar{\varphi}(t, x, u), \tag{30}$$

причем предельные переходы в (29) *и* (30) *происходят равномерно относительно совокупности* $(x, u, z, \lambda) \in G^*(x, u, z) \times \Lambda$ *и* $(t, x, u, \lambda) \in G^*(t, x, u) \times \Lambda$. *Средние значения* $\bar{X}(x, u, z)$ *и* $\bar{\varphi}(t, x, u)$ *определены, непрерывны и равномерно ограничены для всех* $(x, u, z) \in G^*(x, u, z)$ *и* $(t, x, u) \in G^*(t, x, u)$ *и удовлетворяют условиям*

$$\|\bar{X}(x, u, z) - \bar{X}(x', u', z')\| \leq c(\|x - x'\| + \|u - u'\| + \|z - z'\|),$$

$$\|\bar{\varphi}(t, x, u)\| \leq \sigma_2(t),$$

$$\|\varphi(t, s, x, y, u, v) - \bar{\varphi}(t, x, u)\| \leq \sigma_3(t, s),$$

где c*-положительная постоянная, а функции* $\sigma_2(t)$ *и* $\sigma_3(t)$ *удовлетворяют условиям*

$$\frac{1}{t}\int_0^t \sigma_2(\tau)\, d\tau \to 0 \qquad \text{при } t \to \infty,$$

$$\frac{1}{t}\int_0^t d\tau \int_0^1 \sigma_3(\tau, s)\, ds \to 0 \qquad \text{при } t \to \infty.$$

4. *Решение усредненного уравнения*

$$\dot{\xi}(t) = \varepsilon \overline{X}\left(\xi(t), \dot{\xi}(t), \int_0^1 \overline{\varphi}(t, \xi(s), \dot{\xi}(s))\, ds\right) \tag{31}$$

удовлетворяющее начальному условию $\xi(0) = x^0$, *существует и однозначно определено при* $t \geq 0$ *для всех значениях* $\varepsilon \in (0, \varepsilon^*]$ *и кривая* $\{\xi(t), \dot{\xi}(t)\}$ *соответствующая этому решению, на этом интервале не выходит из области* $G^*(x, u)$.

5. *Уравнение в частных производных*

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial y} Y + \frac{\partial W}{\partial v}\left(I_2 - \frac{\partial Y}{\partial v}\right)^{-1}\left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial y} Y + \frac{\partial Y}{\partial w}\int_0^1 \frac{\partial \varphi_1}{\partial t}\, ds\right) =$$

$$= X(t, x, y, u, v, z) - \overline{X}(x, u, z), \tag{32}$$

$$W|_{t=0} = 0, \quad x = const, \quad u = const,$$

обладает некоторым решением $W = W(t, x, y, u, v)$, *которое определено и равномерно ограничено вместе со своими частными производными первого порядка во всей области* G^*.

Тогда, если $\{x(t), y(t)\}$-*решение краевой задачи* (1), (2), *а* $\xi(t)$-*решение усредненного уравнения* (31) *с начальным условием* $\xi(0) = x^0$, *то для любого сколь угодно малого* δ *можно указать такое* $\varepsilon^0 > 0$, *что при* $0 < \varepsilon \leq \varepsilon^0$ *на отрезке* $0 \leq t \leq T$ *будет выполняться неравенство*

$$\|x(t) - \xi(t)\| < \delta.$$

ДОКАЗАТЕЛЬСТВО. Оценим разность $\dot{\eta}(t) - \dot{\xi}(t)$, где $\dot{\xi}(t)$-решение усредненного уравнения (31) с начальным условием $\xi(0) = x^0$, а $\eta(t)$-выражение (12).

Имеем

$$\dot{\eta}(t) - \dot{\xi}(t) = \dot{x}(t) - \varepsilon\left\{\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}\dot{x}(t) + \frac{\partial W}{\partial y}\dot{y}(t) + \frac{\partial W}{\partial \dot{x}}\ddot{x}(t) + \right.$$

$$\left. + \frac{\partial W}{\partial \dot{y}}\ddot{y}(t) + \overline{X}(\xi(t), \dot{\xi}(t), \zeta(t))\right\}. \tag{33}$$

где

$$\zeta(t) = \int_0^1 \overline{\varphi}(t, \xi(s), \dot{\xi}(s))\, ds.$$

Продифференцировав систему (1) и подставляя (15) в (33) получаем

$$\dot{\eta}(t)-\dot{\xi}(t) = -\varepsilon\left\{\frac{\partial W}{\partial t}+\frac{\partial W}{\partial y}Y+\frac{\partial W}{\partial \dot{y}}\left(I_2-\frac{\partial Y}{\partial \dot{y}}\right)^{-1}\cdot\right.$$

$$\cdot\left(\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial y}Y+\frac{\partial Y}{\partial w}\int_0^1\frac{\partial \varphi_1}{\partial t}ds\right)-X(t,x,y,\dot{x},\dot{y},z)+\overline{X}(x,\dot{x},z)\Big\}+$$

$$+\varepsilon[\overline{X}(\eta,\dot{\eta},\zeta)-\overline{X}(\xi,\dot{\xi},\zeta)]+\varepsilon[\overline{X}(x,\dot{x},z)-\overline{X}(\eta,\dot{\eta},\zeta)]-$$

$$-\varepsilon^2\left(\frac{\partial W}{\partial x}X+\frac{\partial W}{\partial \dot{x}}P+\frac{\partial W}{\partial \dot{y}}Q\right).$$

В силу условий теоремы 2 и неравенств (18), (19) и (20) находим

$$\|\dot{\eta}(t)-\dot{\xi}(t)\| \leq \frac{\varepsilon c}{1-\varepsilon c}\|\eta(t)-\xi(t)\|+\frac{\varepsilon^2}{1-\varepsilon c}[c(M_1+M_2)(1+c_1)+M]+$$

$$+\frac{\varepsilon c}{1-\varepsilon c}[2\sigma_2(t)+\sigma_{30}(t)],$$

где

$$\sigma_{30}(t) = \int_0^1 \sigma_3(t,s)\,ds.$$

Отсюда, так как $\eta(0)=\xi(0)$, следует

$$\|\eta(t)-\xi(t)\| \leq \frac{1}{1-\varepsilon c}\int_0^t\{\varepsilon^2[c(M_1+M_2)(1+c_1)+M]+$$

$$+\varepsilon c[2\sigma_2(\tau)+\sigma_{30}(\tau)]\}\exp\{\varepsilon c(1-\varepsilon c)^{-1}(t-\tau)\}\,d\tau.$$

Введя функции

$$\gamma_1(t) = \frac{1}{t}\int_0^t \sigma_2(\tau)\,d\tau \to 0 \qquad \text{при} \quad t\to\infty,$$

$$\gamma_2(t) = \frac{1}{t}\int_0^t \sigma_{30}(\tau)\,d\tau \to 0 \qquad \text{при} \quad t\to\infty,$$

находим, что на отрезке $0\leq t\leq T$

$$\|\eta(t)-\xi(t)\| \leq \frac{1}{1-\varepsilon c}\exp\left\{\frac{cL}{1-\varepsilon c}\right\}\{\varepsilon[c(M_1+M_2)(1+c_1)+M]L+$$

$$+cL[2\gamma_1(L\varepsilon^{-1})+\gamma_2(L\varepsilon^{-1})]\}.$$

Из этого неравенства и из выражения (12) для $\eta(t)$ видно, что всегда можно выбрать $\varepsilon(0<\varepsilon\leq\varepsilon^0, \varepsilon^0\leq\varepsilon^*)$ таким образом, чтобы на отрезке $0\leq t\leq T$ было справедливо неравенство

$$\|x(t)-\xi(t)\|<\delta.$$

Этим теорема 2 доказана.

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ON SEMISIMPLE CLASSES OF TOPOLOGICAL RINGS

By

PHAM NGOC ANH

Eötvös Loránd University, Budapest

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§ 1. Introduction

The structure of topological rings has been extensively studied by several authors. Less attention has been paid to the study of subclasses of topological rings. In this paper we make an attempt to develop the theory of semisimple classes in certain classes of topological rings. A class of topological rings, designated for a semisimple theory, should be closed under continuous homomorphisms and under taking closed ideals. Moreover, it should consist of complete topological rings (this requirement discards inconveniences by forming inverse limits). The class **K** of all compact topological rings satisfies all the above requirements and so does the class **W** of all linearly compact rings in the narrow sense whose Jacobson radicals are linearly compact rings in the narrow sense. In this paper we shall define semisimple classes (in the sense of Kurosh and Amitsur) for topological rings and we shall give several characterizations of semisimple classes in **K** and **W**. Our main result states that the semisimple classes of these categories are exactly those which are hereditary, closed under extensions and inverse limits. This result can be regarded as a counterpart of the characterization of semisimple classes of groups given by TRAN VAN HAO [10] and of the recent characterization of semisimple classes of associative and alternative rings given by L. C. A. van LEEUWEN, C. ROOS and R. WIEGANDT [4]. In the proof we use the method developed in [4] as well as several statements and consideration concerning topological rings due to ZELINSKY [13], [14], H. LEPTIN [6], [7] and A. WIDIGER [11]. In § 3 we shall give two further characterizations of semisimple classes, one of them corresponds to a theorem of N. DIVINSKY [3]. Concerning the Kurosh – Amitsur radical and semisimple theory we hint to [12]. For details of topological algebra we refer to [8] and [13].

On end, I should like to say thank to my teacher, Dr. R. WIEGANDT for numerous valuable advices and the permission of using his manuscripts which meant a great help in my work.

§ 1. Preliminaries

In this section we recall some definitions and theorems we shall use later on.

All topological spaces considered, will be Hausdorff spaces.

The class of all compact topological rings will be denoted by **K**.

A topological A-module (M, τ) is called *linearly compact in the narrow sense* if M is the inverse limit of discrete A-modules satisfying the mininum-condition on submodules and the topology τ is that which is induced by the inverse limit. A ring A is *linearly compact in the narrow sense* if A as an A-module is linearly compact in the narrow sense and the open ideals form a fundamental system for neighborhoods of zero. This definition slightly differs from that of H. LEPTIN [6].

We denote by **L** the class of linearly compact rings in the narrow sense. Further, **W** will denote the class of the **L**-rings the Jacobson radicals of which are also **L**-rings.

DEFINITION: A subclass **S** of topological rings is said to be a *semisimple class* if **S** satisfies the following two conditions:

(A) If A is an **S**-ring, then every non-zero closed ideal of A has a non-zero continuous homomorphic image in **S**.

(B) If every non-zero closed ideal of a topological ring A has a non-zero continuous homomorphic image in **S**, then **S** contains A.

For discrete rings the above definition gives the usual definition of semisimple classes in the sense of Kurosh and Amitsur.

A subclass **S** of topological rings is called *hereditary* if **S** contains every closed ideal of each **S**-ring.

The fact that B is a closed ideal of a topological ring A, will be denoted by $B \lhd A$.

A subclass **S** is said to be *closed under extensions* if the conditions $B \lhd A$, $B \in \mathbf{S}$ and $A/B \in \mathbf{S}$ imply $A \in \mathbf{S}$.

We say that **S** is *closed under inverse limits* if for every inverse system $\{A_\alpha | \pi^\alpha_\beta : A_\alpha \to A_\beta\}$ where each π^α_β is a continuous surjective homomorphism and $A_\alpha \in \mathbf{S}$ for every α it follows that $\varprojlim \{A_\alpha\} \in \mathbf{S}$.

A linearly topologized ring A is said to be *strictly linearly compact* if A as a module over the integers, is linearly compact.

In [11] A. WIDIGER has proved

(1) ([11] Satz 5) *A ring A is a* **W**-*ring iff*

$$A = \sum_{\mu \in I} S^{(\mu)}_{n_\mu} \oplus R$$

where each $S^{(\mu)}_{n_\mu}$ is an $n_\mu \times n_\mu$ full matrix ring over an infinite (discrete) skew field $S^{(\mu)}$ and R is a strictly linearly compact ring.

(2) ([11], Satz 6) *A ring A is a* **W**-*ring iff every closed ideal of A is an* **L**-*ring. Hence the class* **W** *is hereditary.*

Let A be a **W**-ring and I an arbitrary closed ideal of A. By a theorem of ANDERSON – DIVINSKY – SULINSKI ([2], c.f. [12] Theorem 5.2) the Jacobson radical $J(I)$ of I is a closed ideal of A, hence $J(I)$ is linearly compact in the narrow sense, i. e. I belongs to **W**.

In view of TYCHONOFF's theorem and (1) we have

(3) *The classes* **K** *and* **W** *are closed unter taking complete direct sums.*

(4) ([5], Proposition (I. 38. 3)) *If* $\{A_\lambda\}$ *is an inverse system of topological spaces, then* $\varprojlim A_\lambda$ *is a closed set of the complete direct product* $\prod_\lambda A_\lambda$.

As an obvious consequense of the above assertions we have

(5) **K** *is closed under taking inverse limits.*

We say that the inverse system $\Omega = \{A_\alpha, \pi_\beta^\alpha\}$ of the rings $A_\alpha (\alpha \in \Gamma)$ is a direct sum of the inverse systems $\Omega_\lambda = \{A_{\alpha\lambda}, \varrho_{\beta\lambda}^\alpha\}$ if $A_\alpha = \sum_\lambda A_{\alpha\lambda}$ $(\alpha \in \Gamma)$ and $\varrho_{\beta\lambda}^\alpha$ is the restriction of the mapping π_β^α onto $A_{\alpha\lambda}$. The fact that Ω is a direct sum of the inverse systems Ω_λ, will be denoted by $\Omega = \sum_\lambda \Omega_\lambda$.

(6) (ZELINSKY [14], Theorem 1)

$$\varprojlim \sum_\lambda \Omega_\lambda = \sum_\lambda \varprojlim \Omega_\lambda$$

holds in algebraic and topological sense.

Now we are in the position to prove a statement analogous to (5)

(7) **W** *is closed under taking inverse limits.*

PROOF. Let $\{A_\alpha, \pi_\beta^\alpha\}$ be an arbitrary inverse system of a **W**-rings A_α. By (1) A_α has the following decomposition

$$A_\alpha = \sum_{\mu \in I}^{\alpha} S_{n_\mu}^{(\mu)} \oplus R_\alpha \cong S_\alpha \oplus R_\alpha$$

where

$$S_\alpha = \sum_{\mu \in I}^{\alpha} S_{n_\mu}^{(\mu)}.$$

One can see that $\{S_\alpha, \varrho_\beta^\alpha\}$ and $\{R_\alpha, \varrho_\beta^{*\alpha}\}$ form inverse systems where ϱ_β^α, $\varrho_\beta^{*\alpha}$ are the restrictions of π_β^α to S_α and R_α, respectively. By (6) we have

$$\varprojlim \{A_\alpha\} = \varprojlim \{S_\alpha\} \oplus \varprojlim \{R_\alpha\}.$$

It is evident that $\varprojlim \{S_\alpha\}$ is Jacobson semisimple and

$$\varprojlim \{S_\alpha\} = \sum_{\mu \in I^*} S_{n_\mu}^{*(\mu)}$$

where $S_{n_\mu}^{*(\mu)}$-s are $n_\mu \times n_\mu$ full matrix rings over infinite (discrete) division rings $S^{*(\mu)}$.

As modules over the integers, the rings R_α-s are linearly compact, so $\varprojlim\{R_\alpha\}$ is a linearly compact module over the integers i. e. $\varprojlim\{R_\alpha\}$ is a strictly linearly compact ring. so by (1) $\varprojlim\{A_\alpha\}\in\mathbf{W}$ holds.

(8) *If f is a continuous homomorphism from a* **W**-*ring A onto a linearly topologized ring B, then B is a* **W**-*ring too.*

PROOF. By (2) it is sufficient to show that every closed ideal I of B is a linearly compact ring in the narrow sense. Indeed, $I^* = f^{-1}(I)$ is a closed ideal of A, hence I^* is linearly compact in the narrow sense. We define a scalar multiplication for all $a\in I^*$, and $x\in I$ by $ax = f(a)\,x$, so f is the continuous I^*-homomorphism from the I^*-module I^* onto the I^*-module I. Hence the I^*-module I is linearly compact in the narrow sense. This implies that the ring I is linearly compact in the narrow sense, because the I^*-submodules of I are exactly the left ideals of I.

In [8] L. S. PONTRJAGIN has proved the following statement

(9) *Let G be an arbitrary topological group, H a compact normal subgroup of G and f the natural mapping of G onto G/H. If a set $Q\subseteq G/H$ is compact, so is the set $P = f^{-1}(Q)$.*

As an evident corollary of (9) we have

(10) *Let A be an arbitrary topological ring. I a compact ideal of A, and f the natural mapping of A onto A/I. If a set $Q\subseteq A/I$ is compact, so is the set $P = f^{-1}(Q)$.*

The first isomorphism theorem: Let B and C be a closed subring and a closed ideal of a topological ring A, respectively. Provided that $B+C$ is also closed in A,

$$B/(B\cap C)\cong(B+C)/C$$

holds in both algebraic and topological sense.

The second isomorphism theorem: Let B and C be two closed ideals of a topological ring A such that $B\subseteq C$, then:

$$\frac{A/B}{C/B}\cong A/C$$

holds in the algebraic and topological sense.

(11) *If $A\in\mathbf{K}$ or* **W** *and I_1, I_2 are two closed ideals of A, then I_1+I_2 is also closed.*

PROOF. For the **W**-ring our assertion is the special case of Satz 4 in [6]. For a compact ring by the first isomorphism theorem we have

$$(I_1+I_2)/I_1\cong I_2/(I_1\cap I_2)\,.$$

I_1 is compact and $I_2/(I_1\cap I_2)$ as a continuous image of the compact space I_2 and so $(I_1+I_2)/I_1$ are also compact. By (10) I_1+I_2 is compact and hence it is closed.

As a trivial consequense of (1.1) of [7] we have:

(12) *A linearly topologized ring A is linearly compact in the narrow sense, provided I and A/I are linearly compact in the narrow sense where $I \lhd A$.*

Let now A be a linearly compact ring in the narrow sense or a compact ring and $\{I_\alpha\}$ a filter base consisting of closed ideals of A for which $\bigcap_\alpha I_\alpha = 0$. Obviously, the rings $A_\alpha = A/I_\alpha$-s from an inverse system together with the natural surjective homomorphisms

$$\pi_\beta^\alpha : A/I_\alpha \to A/I_\beta \cong \frac{A/I_\alpha}{I_\beta/I_\alpha} \quad \text{for all} \quad I_\alpha \subseteq I_\beta .$$

Similarly to ZELINSKY's Theorem 1 in [13] we can prove

(13) *A is isomorphic and homeomorphic to* $\varprojlim \{A_\alpha\}$

PROOF. Taking a mapping φ where $\varphi(x) = (\ldots, x_\alpha, \ldots)$ with $x_\alpha = x + I_\alpha \in A_\alpha$ for every $x \in A$, we have a homomorphism from A into $\varprojlim \{A_\alpha\}$. Further $\varphi(x) = 0$ implies $x_\alpha = 0$ for each I_α i. e. $x \in \bigcap I_\alpha = 0$, hence $x = 0$, so φ is an injective homomorphism.

Conversely, consider an arbitrary element $y = (\ldots, y_\alpha, \ldots) \in \varprojlim \{A_\alpha\}$ For each I_α denote S_α the set $S_\alpha = \{x | x \in A \text{ and } x + I_\alpha = y_\alpha\}$ It is obvious that S_α is a closed set of A for every I_α. For each index α and β there exists an index γ such that $I_\gamma \subseteq I_\alpha \cap I_\beta$ which implies $S_\gamma \subseteq S_\alpha \cap S_\beta$, so $\{S_\alpha\}$ is a filterbase consisting of cosets of closed ideals of A which has an adherent point x^*, since A is linearly compact or compact, respectively. It is trivial that $\varphi(x^*) = y$, hence φ is an ismorphism.

By the continuity of φ it is sufficient to show that every open subbasic set U of $\prod_\alpha A_\alpha$ has an open counterimage in A by φ. By definition U has the form $U = \prod_\alpha{}^* A_\alpha \times U'$ where the symbol $\prod_\alpha{}^* A_\alpha$ denotes the Cartesian product of all A_α without only one index for example A_β, and U' is an open set of A_β. Obviously $\varphi^{-1}(U) = \pi_\beta^{-1}(U')$ holds where π_β is the natural mapping of A onto A_β. By the definition of the quotient topology $\pi_\beta^{-1}(U')$ is always open in A, hence φ is continuous.

If $A \in \mathbf{W}$, then φ is an open mapping; if $A \in \mathbf{K}$, then φ is a closed mapping, so in both cases this implies that φ is a homeomorphism between A and $\varprojlim \{A_\alpha\}$.

§ 2. Characterization of semisimple classes

In this section we shall characterize the semisimple classes of the categories **K** and **W** and we shall give examples for the semisimple classes of **K** and **W**.

THEOREM 1. *A subclass* **S** *of* **K** *or* **W** *is a semisimple class iff* **S** *satisfies the following three conditions:*

(1) **S** *is hereditary,*
(2) **S** *is closed unter extensions,*
(3) **S** *is closed under inverse limits.*

PROOF. *Necessity:* Let B be a closed ideal of a ring A of **K** or **W**, respectively, such that B and A/B are contained in **S**. Consider an arbitrary closed ideal I of A. If $0 \neq I \subseteq B$, then by $B \in \mathbf{S}$ condion (A) implies that I has a non-zero continuous homomorphic image in **S**. If $I \nsubseteq B$, then by the first isomorphism theorem, (6) and (9) we have:

$$0 \neq I/(B \cap I) \cong (I+B)/B \lhd A/B .$$

Hence by (A) $I/(B \cap I)$ as well as I has a non-zero continuous homomorphic image in **S**, so condition (B) implies $A \in \mathbf{S}$. Thus (2) has been verified.

Now we turn to exhibit the validity of condition (3). Let $\Omega = \{A_\alpha, \pi_\beta^\alpha\}$ be an inverse system of **S**-rings such that each π_β^α is surjective. By (5) $\overleftarrow{\lim}\, A_\alpha$ is contained in the class **K** or **W**, respectively. Let I be an arbitrary non-zero closed ideal of $\overleftarrow{\lim}\, A_\alpha$. Then there exists an index α such that $\pi_\alpha(I) \neq 0$, where π_α is the projection of $\overleftarrow{\lim}\, A_\alpha$ onto A_α. Since π_α is continuous and A_α are in **K** and **W**, respectively, so $\pi_\alpha(I)$ is closed in A_α. Hence (A) implies that both $\pi_\alpha(I)$ and I have a non-zero continuous homomorphic image in **S**. Thus (B) implies that $\overleftarrow{\lim}\, A_\alpha \in \mathbf{S}$.

The necessity part of the proof will be finished, if we proceed to show that **S** is hereditary. For this purpose for every ring A (of **K** and **W**) we define the ideal $R(A)$ as the smallest closed ideal of A which contains all closed ideal of A having only trivial continuous homomorphic image in **S**. One can easily see that:

$$R(A) = Cl \sum (I_\alpha \mid I_\alpha \lhd A \text{ and } I_\alpha \text{ has only trivial image in } \mathbf{S}).$$

holds where Cl means the topological closure. It is obvious that **S** contains a ring A iff $R(A) = 0$.

To prove the validity of condition (1) we need the following

LEMMA 1. *$R(I) \lhd A$ for every closed ideal I of A.*

PROOF. $R(I)$ is trivially closed in A. We can suppose that $R(I) \neq 0$ otherwise the assertion is trivial. We assume indirectly that $R(I)$ is not an ideal of A, i.e. there exists at least one element a or b of A such that $aR(I)$ or $R(I)b$ is not contained in $R(I)$, for instance $aR(I) \nsubseteq R(I)$, $aR(I) + R(I)$ is an ideal of I, namely for any element y of I, we have

$$y(aR(I)+R(I)) = (ya)R(I) + yR(I) \subseteq I\,R(I) + R(I) = R(I)$$

and

$$(aR(I)+R(I))\,y = a(R(I)\,y) + R(I)\,y \subseteq aR(I) + R(I) .$$

We define the following mapping

$$\varphi : R(I) \to (aR(I)+R(I))/R(I)$$

by taking $\varphi(x) = ax+R(I)$ for any $x \in R(I)$.

φ is a homomorphism, for

$$\varphi(x+y) = a(x+y)+R(I) = ax+R(I)+ay+R(I) = \varphi(x)+\varphi(y).$$

$$\varphi(xy) = axy+R(I) \subseteq I \cdot R(I)+R(I) \subseteq R(I)+R(I) = R(I) = \bar{0}$$

and

$$\varphi(x)\,\varphi(y) = ax \cdot ay+R(I) = (axa)\,y+R(I) \subseteq I \cdot R(I)+R(I) = \bar{0}.$$

Furthermore, let U be an arbitrary open set of $(aR(I)+R(I))/R(I)$. Then $\pi^{-1}(U)$ is open in $aR(I)+R(I)$ where π is the natural mapping of $aR(I)+R(I)$ onto $(aR(I)+R(I))/R(I)$. This shows that $V = \pi^{-1}(U) \cap R(I)$ is open in $R(I)$. The continuity of φ follows from the equality $\varphi(V) = U$. $(aR(I)+R(I))/R(I)$ as a continuous homomorphic image of $R(I)$ is again in **K** or **W**, respectively. This implies that $aR(I)+R(I)$ is also in **K** or **W**, respectively; hence $aR(I)+R(I)$ is closed in I, therefore in A.

We say that A *has property* (R), *if*

(R) A has not non-trivial continuous homomorphic image in **S**.

We prove that $R(I)$ has property (R). Otherwise there exists a non-zero continuous homomorphism f such that $0 \neq f(R(I)) \in \mathbf{S}$. Thus by (A) every non-zero closed ideal of $f(R(I))$ has a non-zero continuous homomorphic image in **S**. But the closed ideals of $f(R(I))$ are exactly images of those of $R(I)$ by f, contradicting the definition of $R(I)$. Moreover,

$$(aR(I)+R(I))/R(I)$$

as image of $R(I)$ by φ has also property (R).

Let f be now an arbitrary continuous surjective homomorphism of $aR(I)+R(I)$ for which $0 \neq f(aR(I)+R(I)) \in \mathbf{S}$.

If $R(I) \subseteq \ker f$, then the second isomorphism theorem yields

$$f(aR(I)+R(I)) \cong \frac{(aR(I)+R(I))}{\ker f} \cong \frac{\dfrac{(aR(I)+R(I))}{R(I)}}{\dfrac{\ker f}{R(I)}}$$

in both algebraic and topological sense.

If $R(I) \nsubseteq \ker f$, then by the first isomorphism theorem we obtain

$$\frac{R(I)}{(\ker f \cap R(I))} \cong \frac{(\ker f+R(I))}{\ker f} \lhd \frac{(aR(I)+R(I))}{\ker f} \cong f(aR(I)+R(I))$$

in both algebraic and topological sense, since any continuous injective homomorphism in both **K** and **W**, is a homeomorphism.

This implies that neither $(aR(I)+R(I))/R(I)$ nor $R(I)$ has property (R) which is a contradiction. Hence $aR(I)+R(I)$ has property (R) and consequently $aR(I)+R(I)\subseteq R(I)$ contradicting the assumption $aR(I)\subsetneqq R(I)$, so $R(I)$ is a left ideal.

Analogously we get that $R(I)$ is a right ideal too. Thus $R(I)\lhd A$ is proved.

Let A be now an arbitrary **S**-ring and I a closed ideal of A then $R(I)\subseteq R(A)$, hence $R(I) = R(A) = 0$ i. e. $I\in\mathbf{S}$.

SUFFICIENCY. It is clear that property (A) follows from property (1) We assume that the property (B) is not true, i. e. there exists a ring A of **K** or **W** such that every non-zero closed ideal of A has a nontrivial image in **S**, but $A\notin\mathbf{S}$. Consider the following operator T which assigns to each ring A in **K** or **W** the ideal

$$T(A) = \cap(I_\alpha | I_\alpha \lhd A \text{ and } A/I_\alpha\in\mathbf{S}).$$

We show that the system $\{I_\alpha\}$ forms a directed set by the set theoretic inclusion. Indeed, let I_1 and I_2 be two closed ideals of A for which A/I_1 and A/I_2 are contained in **S**.

Taking into consideration (9) by the first isomorphism theorem and the second isomorphism theorem we have

$$\frac{A/(I_1\cap I_2)}{I_1/(I_1\cap I_2)} \cong A/I_1$$

and

$$\frac{I_1}{(I_1\cap I_2)} \cong \frac{(I_1+I_2)}{I_2} \lhd \frac{A}{I_2}$$

algebraically and topologically where A/I_1, and A/I_2 are in **S**. The hereditariness of **S** implies that $(I_1+I_2)/I_1$ and so $I_1/(I_1\cap I_2)$ are in **S**. By condition (2) $A/(I_1\cap I_2)$ is in **S**, i. e. $I_1\cap I_2\in\{I_\alpha\}$. Thus the system $\{A/I_\alpha, \pi^\alpha_\beta\}$ forms an inverse system with the continuous homomorphic projection

$$\pi^\alpha_\beta : A/I_\alpha \to A/I_\beta \cong \frac{A/I_\alpha}{I_\beta/I_\alpha} \quad \text{for all} \quad I_\alpha\subseteq I_\beta .$$

Therefore by condition (3) and (11) we have $A/T(A) \cong \varprojlim A/I_\alpha\in\mathbf{S}$. Since $A\notin\mathbf{S}$ it follows $T(A)\neq 0$. Similarly one can easily see that

$$T(A)/T(T(A))$$

belongs also to **S**. If we proved that $T(T(A))\lhd A$ then we could complete the proof as follows. By the second ismorphism theorem we have

$$\frac{A/T(T(A))}{T(A)/T(T(A))} \cong \frac{A}{T(A)} \in\mathbf{S} .$$

Hence by $T(A)/T(T(A))\in\mathbf{S}$ condition (2) ensures that $A/T(T(A))$ belongs to $\mathbf{S}$. Thus by the definition of $T(A)$ we have $T(A)\subseteq T(T(A))$, but evidently $T(T(A))\subseteq T(A)$ holds, i.e. $T(A) = T(T(A))$. By our assumption $T(A)$ is not zero and the equality $T(A) = T(T(A))$ implies that $T(A)$ has property (R). But $T(A)$, as a closed ideal of A, cannot have property (R). This contradiction proves the validity of property (B).

To come to end of the proof of Theorem 1 we still have to prove

LEMMA 2. $T(T(A)) \dot{\lhd} A$.

PROOF. $T(T(A))$ is obviously closed in A. For an arbitrary element a of A one can see that $aT(T(A))+T(T(A))$ is a closed ideal of $T(A)$ as we have done it at the proof of the heredutariness of the semisimple classes. We define a mapping

$$\varphi : T(T(A)) \to \frac{\big(aT(T(A))+T(T(A))\big)}{T(T(A))}$$

by taking $\varphi(x) = ax+T(T(A))$ for every $x\in T(T(A))$.

One can easily check that φ is a continuous surjective homomorphism. By the hereditariness of $\mathbf{S}$ the closed ideal $\big(aT(T(A))+T(T(A))\big)/T(T(A))$ of the $\mathbf{S}$-ring $T(A)/T(T(A))$ is in $\mathbf{S}$.

On other hand we have

$$\frac{T(T(A))}{\ker\varphi} \cong \frac{\big(aT(T(A))+T(T(A))\big)}{T(T(A))\in\mathbf{S}}.$$

algebraically and topologically.

For every $x\in T(A)$, $y\in\ker\varphi$ it is true that

$$a(xy) = (ax)y\in T(T(A))$$

and

$$a(yx) = (ay)x\in T(T(A)),$$

i.e. $\{xy, yx\}\subseteq\ker\varphi$, hence $\ker\varphi$ is an ideal of $T(A)$.

By the second isomorphism theorem we have

$$\frac{T(A)/\ker\varphi}{T(T(A))/\ker\varphi} \cong \frac{T(A)}{T(T(A))\in\mathbf{S}}.$$

Using condition (2) we have $T(A)/\ker\varphi\in\mathbf{S}$ which implies by definition of $T(T(A))$ that $T(T(A))\subseteq\ker\varphi$. But clearly $\ker\varphi\subseteq T(T(A))$ holds, so $T(T(A)) = \ker\varphi$; hence $aT(T(A))\subseteq T(T(A))$. Similarly, one can prove that $T(T(A))\,a\subseteq T(T(A))$ which implies that $T(T(A))$ is an ideal of A.

REMARKS. In the proof we have followed the ideas of L. C. A. van LEEUWEN C. ROOS, R. WIEGANDT [4]. Nevertheless the statement of Theorem 1 differs essentially from the characterization of semisimple claases of associative (or alternative) rings. In [4] it has been proved that a

class **S** is a semisimple class of rings iff **S** is hereditary, closed under extensions as well as subdirect sums. As it has been mentioned in [4], the class of all Jacobson radical rings is hereditary, closed under extensions and under taking inverse limits, but it is not a semisimple class in the category of all associative rings.

As examples for semisimple classes of **K** and **W**, let us consider the classes **J** and S**J** of all Jacobson rings in **K** or **W** and of all Jacobson semisimple rings in **K** or **W**, respectively.

Clearly both **J** and S**J** are semisimple classes of **K** and **W**, respectively (cf. also Theorem 1).

An assertion analogous to Theorem 1, is not valid in the category **L**. Since **L** itself is not hereditary, a subclass of **L** satisfying conditions (A) and (B) is not necessarily hereditary either.

§ 3. Further characterization of semisimple classes

Before we give other characterizations for semisimple classes, let us draw up an easy proposition. Let **S*** be an arbitrary semisimple class of associative rings, i. e. **S*** satisfies:

($\overline{A}$) If $A \in$ **S***, then every non-zero ideal of A has non-zero homomorphic image in **S***.

($\overline{B}$) If every non-zero ideal of a ring A has non-zero homomorphic image in **S***, then $A \in$ **S***.

Let **S** denote the class of all topological rings (A, τ) where A is an element of **S***. Evidently there holds the following

PROPOSITION. **S** $\cap$ **K** and **S** $\cap$ **W** *is a semisimple in* **K** *and* **W**, *respectively.*

THEOREM 2. *In Theorem* 1 *condition* (3) *can be replaced by the coinductive property:*

(4) *If* $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_\alpha \supseteq \ldots$ *is a descending chain of closed ideals of any ring* A *in* **K** *or* **W** *such that* $A/B_\alpha \in$ **S** *for all* α *and* $B = \bigcap_\alpha B_\alpha$ *then also* $A/B \in$ **S**.

The satement corresponds to that of Theorem 1 in [9] and Theorem 3 in [4] and the its proof is a slight modification of Theorem 3 in [4], therefore we omit it.

To be able to draw up our next result we begin with the following definition. A subring B of a ring A in **K** or **W** is said to be *accesible* if there exist finitely many subrings $B_0, B_1, B_2, \ldots, B_n$ of A such that $B = B_0$, $A = B_n$ and each B_i is a closed ideal of B_{i+1}, $i = 0, 1, \ldots, n-1$. It follows clearly from this definition that B is again in **K** or **W**, respectively and B is closed in A. Now, we are in the position to draw up following

THEOREM 3. *A subclass* **S** *of* **K** *or* **W** *is semisimple if and only if* **S** *satisfies*

(A*) *Every non-zero accesible subring of* **S**-*ring has a non-zero continuous homomorphic image in* **S**.

(B*) *If every non-zero accesible subring of a ring A in* **K** *(or in* **W***) has non-zero continuous homomorphic image in* **S**, *then the class* **S** *contains the ring A.*

PROOF. Condition (A*) follows immidiately from the hereditariness of the semisimple classes.

We suppose indirectly that (B*) is false, i.e. there exists a ring A of **K** or **W** such that every non-zero accesible subring of which has a non-zero image in **S**, but $A \notin$ **S**. Since **S** does not contain A the closed ideal $R(A)$ is not zero, so $R(A)$ has a non-zero imageo in **S** which contradicts the definition of $R(A)$. This contradiction ends the proof for the necessity part of the Theorem.

Conversely, let **S** satisfies conditions (A^*) and (B^*). **S** is hereditary, namely let I be a closed ideal of an arbitrary **S**-ring A. It is clearly that every non-zero accesible subring C of I is also an accesible subring of A, i. e. C has a non-zero image in **S**, hence by (B^*) **S** contains obviously I.

Let A be an arbitrary ring in **K** or **W** such that a closed ideal I of which and the factor ring A/I are in **S**. Let $B = B_0$ be an arbitrary non-zero accesible subring of A, then there exist $B_1, \ldots, B_n = A$ subrings of A such that each B_i is a closed ideal of B_{i+1} for $i = 0, 1, \ldots, n-1$.

If $B \subseteq I$, then $B = B_0 \dot{\lhd} B_1 \cap I \dot{\lhd} \ldots \dot{\lhd} B_n \cap I = I$, i.e. B is also an accesible subring of I, so B has a non-zero image in **S**.

If $B \nsubseteq I$, then

$$\frac{B}{(B \cap I)} \cong \frac{(B+I)}{I} \dot{\lhd} \frac{(B_1+I)}{I} \dot{\lhd} \ldots \dot{\lhd} \frac{(B_n+I)}{I} = \frac{A}{I}$$

which implies that $B/(B \cap I)$ and so B has a non-trivial image in **S** and by (B^*) we get $A \in$ **S**.

Let now $\{A_\alpha, \pi_\beta^\alpha\}$ be an inverse system of **S**-rings. For any non-zero accesible subring $B = B_0$ of $\varprojlim A_\alpha$ there exists an index α such that $\pi_\alpha(B) \neq 0$ where π_α is the projection of $\varprojlim A_\alpha$ onto A_α. One can easily see that $\pi_\alpha(B)$ is a non-zero accesible subring of A_α. By $(A^*)\pi_\alpha(B)$ and so B have a non-zero image in **S**. Hence (B^*) implies $\varprojlim A_\alpha \in$ **S**. Thus **S** satisfies conditions (1), (2) and (3) in Theorem 1, so **S** is a semisimple class.

Let us remark that Theorem 2 corresponds to a characterization of semisimple classes given by N. DIVINSKY in [3].

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GROUP REPRESENTATIONS IN MECHANICS

By

T. MATOLCSI

II. Department of Analysis of the Eötvös Loránd University, Budapest

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1. Introduction

Group representations play an important role in modern physics. It seems, however, that the usual definition of group representations hinders this theory from being a more general, all-embracing branch of mathematics and from being a more powerful tool in physics.

Elementary particles are classified in quantum mechnanics by representations of the space-time symmetry groups. Recently a classification of elementary particles appeared also in classical mechanics which seems similar to the quantal one ([1], [2], [3], [4]). However, the analogy is not perfectly clear and "it would be desirable to formulate a correspondence principle to relate the classifications" ([4]). To establish the desired correspondence, one must first recognize that some sort of group representations should be used also in classical mechanics.

I defined a general notion of group representations ([6]) which includes the usual representations, that is, the linear and unitary representations, ray representations and actions of transformation groups as well. By the aid of this general notion of group representations we can state a perfect parallelism between classical and quantal elementary particles.

After recalling the general notion of group representations, here I shall examine the special feature of representations needed in quantum mechanics and in classical mechanics. The investigations are based on the formulation of mechanics using the algebraic structure of events, the so called logic of a physical system ([7], [8]).

2. Representations of groups

A category consists of objects and morphisms satisfying some axioms. We are concerned, in our applications, with so called concrete categories, the objects of which are sets with a certain structure and the morphisms are maps related to the structure (e.g. preserving it). An isomorphism is a one-to-one and onto morphism whose inverse is also a morphism;

an automorphism of an object is an isomorphism of the object onto itself. The automorphisms of an object X form a group under the composition of maps; we denote this group by $\mathrm{Aut}(X)$. One generally defines the notion of subobjects in concrete categories in a natural way and it has an intrinsic meaning that a subobject is invariant for an automorphism; hence, we do not give here the precise and convenient definition of subobjects and invariant subobjects in category theory. These definitions are given for an arbitrary (not necessarily concrete) category in [6]. For further details on categories we refer to [9]

Definition 1. Let G be a group and $\mathcal{C}$ a concrete category. A *representation* A of G on the object X of $\mathcal{C}$ is a group homomorphism

$$A:G \to \mathrm{Aut}(X),\ g \mapsto A_g .$$

The representation is *faithful* if it is one-to-one.

The representation is *irreducible* if there is no subobject of X invariant for all A_g and not invariant for all automorphisms of X.

The representation is *weakly irreducible* if there is no element of X invariant for all A_g and not invariant for all automorphisms of X.

Two representations of the same group G, A on X and B on Y, X and Y being objects of $\mathcal{C}$, are *equivalent* if there is an isomorphism $i:X \to Y$ such that $i \circ A_g = B_g \circ i$ for all $g \in G$.

If G is a topological group and the structure of X contains a topology too, we require that the map $G \times X \to X$, $(g, x) \mapsto A_g(x)$ be continuous. Such representations are called *continuous*.

If G is a Lie group and X has a differentiable structure, we require that the map $G \times X \to X$, $(g, x) \mapsto A_g(x)$ be differentiable. Such a representation is called *differentiable*.

As an example, take the category whose objects are Hilbert spaces and morphisms are linear contractions. Then continuous representations on such objects are exactly the strongly continuous unitary representations.

As a second example, let us consider the category of topological spaces and continuous maps. Then we obtain topological transformation groups in the customary sense ([10], p. 110). Effective actions are nothing else than faithful representations. Transitive actions are irreducible representations; the converse is only true, in general, on those spaces whose homeomorphisms form a transitive transformation group; such spaces are called *homogeneous*. In the theory of topological transformation groups one examines a given single action. In representation theory one is looking for different, that is, inequivalent representations. The two theories meet in the theorem that every continuous irreducible representation A of a locally compact second countable group G on a locally compact homogeneous Hausdorff space X is equivalent to the representation L on G/H where H is a closed subgroup of G and L is the canonical or standard action (left translation) of G ([10], p. 111). Recall that equivalence means here that there is a homeomorphism $f:G/H \to X$ such that $f \circ L_g = A_g \circ f$

for all $g \in G$. Quite the same is true for differentiable representations of a Lie group on a differentiable manifold ([10], p. 114). However, if X has a further additional structure then representations and equivalences are given by maps related to this structure too and there can be much more inequivalent representations of G on X.

3. Representations in quantum mechanics

The logic of a physical system in quantum mechanics is the orthocomplemented σ-lattice $\mathcal{P}(\mathcal{H})$ of projections on a separable complex Hilbert space $\mathcal{H}$, called the structure space (of the physical system). The lattice structure, that is, the ordering of projections is defined by the ordering of closed linear subspaces. Now we introduce a category in which objects are such lattices and morphisms are orthocomplementation preserving lattice σ-homomorphisms.

We assume from now on that the dimension of $\mathcal{H}$ is greater than two. By a theorem of Wigner ([7], vol I, p. 199), every isomorphism $S:\mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}')$ is induced by a unitary or antiunitary map $U:\mathcal{H} \to \mathcal{H}'$, determined uniquely up to a unit factor, such that $S(E) = UEU^{-1}$ for all $E \in \mathcal{P}(\mathcal{H})$. Consequently, as it is known, a representation of a group on $\mathcal{P}(\mathcal{H})$ gives rise to a unitary-antiunitary *ray representation* on $\mathcal{H}$ ([11]). Conversely, a ray representation on $\mathcal{H}$ determines a representation on $\mathcal{P}(\mathcal{H})$. Representations on projection lattices will be called *projective representations*. Let us see more closely the connection between ray representations and projective representations.

Let A be a projective representation of a group G on $\mathcal{P}(\mathcal{H})$ and (U, τ) its realisation as a ray representation on $\mathcal{H}$. It means that we have a map $g \mapsto U_g$, U_g being a unitary or antiunitary operator $\mathcal{H}$, and we have a function $\tau:G \times G \to \mathbf{T}$ ($\mathbf{T}$ is the complex unit circle) such that

$$A_g(E) = U_g E U_g^{-1}, \tag{1}$$

$$U_g U_h = \tau(g, h)\, U_{gh}, \tag{2}$$

$$\tau(e, e) = 1 \quad (e \text{ is the unit element of } G) \tag{3}$$

$$\tau(g, h)\, \tau(gh, f) = \tau(h, f)\, \tau(g, hf) \tag{4}$$

for all $E \in \mathcal{P}(\mathcal{H})$ and for all $g, h, f \in G$. In the sequel, a function $\tau:G \times G \to \mathbf{T}$ satisfying (3) and (4) will be called a *unitary cocycle* of G. Two unitary cocycles τ and τ' will be said *cohomologous* if there is a function $\varrho:G \to \mathbf{T}$ such that

$$\tau'(g, h) = \tau(g, h) \frac{\varrho(g)\, \varrho(h)}{\varrho(gh)} \quad \text{for all } g, h \in G.$$

τ and τ' are *weakly cohomologous* if either τ and τ' or $\bar{\tau}$ (the complex conjugate of the function τ) and τ' are cohomologous.

Now, first we mention the known fact that if G is a connected topological group then U_g is unitary for all $g \in G$ ([11]). Then we state

PROPOSITION 1. *The projective representation A of a connected topological group G is weakly irreducible if and only if (U, τ) is irreducible in the usual sense.*

PROOF. We have $U_{g^{-1}} = \tau(g, g^{-1})\, U_g^{-1}$ from (2) and thus we can repeat the familiar arguments for unitary representations to get that a closed linear subspace is invariant for all U_g if and only if the corresponding projection commutes with all U_g. Consequently, we conclude from (1) that a projection is invariant for all A_g if and only if the corresponding subspace is invariant for all U_g. Recall that A is weakly irreducible if the only elements of the projection lattice, which are invariant for all A_g, are the identity and the zero.

Recall now that it follows from our general definition concerning group representations that two projective representations of the same group, A on $\mathcal{P}(\mathcal{H})$ and A' on $\mathcal{P}(\mathcal{H}')$, are equivalent if and only if there is an isomorphism $S:\mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}')$ such that

$$S \circ A_g = A'_g \circ S \quad \text{for all} \quad g \in G. \tag{5}$$

PROPOSITION 2. *Let A and A' be two projective representations of the connected group G and let (U, τ) and (U', τ') be their realizations as ray representations. A and A' are equivalent if and only if there is a unitary or antiunitary map V and a function $\varrho: G \to \mathbf{T}$ such that*

$$V U_g = \varrho(g)\, U'_g\, V \quad \textit{for all} \quad g \in G. \tag{6}$$

Consequently, τ and τ' or $\bar{\tau}$ and τ', depending on whether V is unitary or antiunitary, are cohomologous by ϱ.

PROOF. (5) and (6) are simple consequences of each other. Furthermore, multiplying the equality (2) from the left and from the right by V and by V^{-1} respectively and using (6) we get that, according to the nature of V, τ and τ' or $\bar{\tau}$ and τ' are cohomologous by ϱ.

As a consequence of Proposition 2, if (U, τ) and (U', τ') are ray representations such that τ and τ' are not weakly cohomologous, the projective representations determined by the ray representations are not equivalent. In other words, the *weak cohomology classes* of unitary cocycles of a connected group can be used for certain *labelling* of the equivalence classes of projective representations of the group.

Now we want to consider projective representations continuous in the sense of our definition. For this purpose recall that there is a well working notion of continuity of ray representations. A ray representation (U, τ) of a topological group G is continuous in the sence of Bargmann if there exists a neighbourhood of the identity of G so that U is strongly continuous and τ is continuous in this neighbourhood. The continuity of ray representations can be related to projective representations as follows. A distance d is defined on the set of one-dimensional projections (on the rays) by

$$d(E, F) = \inf\{\|x-y\|,\ x \in E\mathcal{H},\ y \in F\mathcal{H},\ \|x\| = \|y\| = 1\}.$$

Let A be a projective representation induced by a continuous ray representation of G. Then the map $g \mapsto A_g(E)$ is continuous in the distance d for every one-dimensional projection E. Conversely, if A is a projective representation for which the map $g \mapsto A_g(E)$ is continuous for all one-dimensional E, then one can find a continuous ray representation inducing A.

We would like to define a topology on a projection lattice $\mathcal{P}(\mathcal{H})$ with the following conditions:

1. The topology is compatible with the structure of $\mathcal{P}(\mathcal{H})$ that is the orthocomplementation and intersection are continuous operations.
2. The automorphisms of $\mathcal{P}(\mathcal{H})$ are continuous.
3. The continuous projective representations can be given by continuous ray representations and continuous ray representations induce continuous projective representations.

PROPOSITION 3. *There is no Hausdorff topology on the projection lattice $\mathcal{P}(\mathcal{H})$ satisfying conditions* 1 *and* 3.

PROOF. Let e_n ($n = 1, 2, \ldots, \dim \mathcal{H}$) be an orthonormal basis of $\mathcal{H}$. The formula

$$U_t e_n = \begin{cases} \exp(it)\, e_n & \text{if } n = 1 \\ e_n & \text{if } n \neq 1 \end{cases} \quad (t \in \mathbf{R})$$

determines a ray representation of the additive group of real numbers. The ray representation is continuous in Bargmann's sense. Let $\mathcal{M}$ be the subspace generated by the two vectors $e_1 + e_2$ and e_3, and let $\mathcal{N}$ be the subspace generated by e_3. If there is a topology T with the properties listed in conditions 1 and 3, then

$$\mathcal{N} = T - \lim_{\substack{t \to 0 \\ t \neq 0}} \left(U_t(\mathcal{M}) \cap \mathcal{M}\right) = \left(T - \lim_{\substack{t \to 0 \\ t \neq 0}} U_t(\mathcal{M})\right) \cap \mathcal{M} = \mathcal{M},$$

which is a contradiction.

Therefore we are forced to change Condition 1 into the following weaker one:

1′. The orthocomplementation and the intersection are continuous operations in every distributive sublattice of $\mathcal{P}(\mathcal{H})$.

We say in this case that the topology is *partially compatible* with the lattice structure.

PROPOSITION 4. *The topology on $\mathcal{P}(\mathcal{H})$, induced by the strong operator topology, satisfies conditions* 1′, 2 *and* 3.

PROOF. Condition 2 is trivially satisfied. It is also simple to see that Condition 1′ is fulfilled. The orthocomplementation $E \mapsto I - E (E \in \mathcal{P}(\mathcal{H}))$ is evidently continuous. A sublattice of $\mathcal{P}(\mathcal{H})$ is distributive if and only if it is commutative. Furthermore, if E and F are commuting projections then their intersection $E \wedge F$ is their product EF. The multiplication

in a set of operators bounded in norm is jointly continuous in the two variables with respect to the strong topology, hence the intersection is continuous operation in a distributive sublattice.

Regarding Condition 3, standard arguments show that the norm topology, the strong topology and the topology of the distance d on the set of one dimensional projections are equivalent. As a consequence, the continuity of a projective representation implies the continuity of ray representations.

Conversely, let the ray representation (U, τ) be continuous. Then we get the continuity of the projective representation A induced by (U, τ) from the equality

$$A_g(F)-E = U_g(F-E)U_g^{-1}+(U_g E U_g^{-1}-E)$$

and from the fact that the multiplication of operators in a set bounded in norm is continuous in the two variables.

REMARK 1. One can see by a simple example that the topology on $\mathcal{P}(\mathcal{H})$, given by the distance

$$d(E, F) = \inf\{\|x-y\|,\ x\in(E-E\wedge F)\mathcal{H},\ y(F-E\wedge F)\mathcal{H},\ \|x\| = \|y\| = 1\}$$

satisfies Conditions 1′ and 2 but it does not satisfy Condition 3.

From now on we equip the projection lattices with the strong operator topology.

Let us now consider continuous projective representations of connected Lie groups. To every unitary cocycle continuous in a neighbourhood of the identity of the Lie group, one can assign a closed bilinear antisymmetric real valued function on the Lie algebra of the group. Closedness means that if β is a bilinear antisymmetric function on a Lie algebra $\mathcal{G}$ then

$$\beta([a, b], c)+\beta([b, c], a)+\beta([c, a], b) = 0$$

for all $a, b, c\in\mathcal{G}$. A closed bilinear antisymmetric function on a Lie algebra will be called a *commutator cocycle*. Two commutator cocycles β and β' are said *cohomologous* if there is a linear function α such that $\beta' = \beta+\alpha\circ[,]$ where $[,]$ denotes the commutator on the Lie algebra. β and β' are *weakly cohomologous* if either β and β' or $-\beta$ and β' are cohomologous.

Collecting the results of [11] we get

COROLLARY TO PROPOSITION 2. *Let A and A' be continuous projective representations of the connected Lie group G. Assume (U, τ) and (U', τ') are realizations of A and A' as continuous ray representations. Let β and β' be the commutator cocycles on the Lie algebra $\mathcal{G}$ of G, corresponding to τ and to τ' respectively. Let H_a and H'_a denote the self-adjoint infinitesimal generators corresponding to $a\in\mathcal{G}$ in the ray representation (U, τ) and (U', τ') respectively. A and A' are equivalent if and only if there is a unitary or antiunitary map V and a linear function $\alpha: G\to\mathbf{R}$, $a\mapsto\alpha_a$ such that if V is uniary then β and β' are cohomologous by α and*

$$H'_a = VH_a V^{-1} + \alpha_a \quad \textit{for all} \quad a \in \mathcal{G}.$$

If V is antiunitary then $-\beta$ and β' are cohomologous by α and

$$-H'_a = VH_a V^{-1} + \alpha_a \quad \textit{for all} \quad a \in \mathcal{G}.$$

4. Representations in classical mechanics

Let us turn to classical mechanics. The question is what objects are logics of physical systems in this theory. One knows a partial answer: the logic of a physical system is the σ-algebra $\mathcal{B}(M)$ of Borel subsets of a differentiable manifold M called the phase space (of the physical system) ([7]). Morphisms between such objects are defined as differentiable Boolean σ-homomorphisms. A Boolean σ-homomorphism $u:\mathcal{B}(M) \to \mathcal{B}(M')$ is called differentiable if it can be given by a (necesrarily uniquely determined) differentiable map $f_u:M' \to M$ such that $u(E) = f_u^{-1}(E)$ for all $E \in \mathcal{B}(M)$. This choice seems, however, too large, because of the previously mentioned fact that all good representations on such objects are equivalent to a few ones.

We assume that on the phase spaces there is given some structure inducing a Borel measure in a canonical way. Then we can require that morphisms, besides being differentiable Boolean σ-homomorphisms, be related to this structure or only to the measure induced, but in any way in such a manner that isomorphisms, which are given by diffeomorphisms of phase spaces, preserve the measure. The most simple examples: the phase spaces can be Riemannian manifolds, contact manifolds or symplectic manifolds. Here only one possibility will be investigated.

Some preliminary notions are necessary for our work. A closed two-form with constant rank on a differentiable manifold will be called a *presymplectic* form. If the rank of a presymplectic form is maximal then the form is called *symplectic*. A *symplectic manifold* is a pair $(M; \omega)$ where M is a differentiable manifold and ω is a symplectic form on M.

DEFINITION 2. Let ω and ω' be two presymplectic forms with rank r and r' on the same manifold. We write $\omega < \omega'$ if $r < r'$ and if there is a presymplectic form ω_0 with rank $r_0 = r' - r$ such that $\omega + \omega_0 = \omega'$. We write $|\omega| < |\omega'|$ if either $\omega < \omega'$ or $-\omega < \omega'$.

Now we introduce a category whose objects are σ-algebras of Borel subsets in symplectic manifolds and morphisms are given as follows. Let (M, ω) and (M', ω') be two symplectic manifolds. $\mathcal{B}(M, \omega)$ and $\mathcal{B}(M', \omega')$ will denote the corresponding Borel σ-algebras. A morphism $\mathcal{B}(M, \omega) \to \mathcal{B}(M', \omega')$ is a differentiable map $f:M' \to M$ such that $|f^*\omega| \leq |\omega'|$. The morphisms will be called *semi-symplectic* maps. Note that if $f:M' \to M$ represents an isomorphism then f is a diffeomorphism and $|f^*\omega| = |\omega'|$, that is either $f^*\omega = \omega'$ (f is *symplectic*) or $f^*\omega = -\omega'$ (f is *antisymplectic*).

We define that logics of physical systems in classical mechanics are objects of this category.

Let us consider group representations on the objects of the above category. Let (M, ω) be a symplectic manifold; a representation A of a group G on $\mathcal{B}(M, \omega)$ is a group homomorphism $g \mapsto A_g$ from G into the group of semi-symplectic diffeomorphisms of M. That is why we shall call such representations *semi-symplectic.*

In other words, a semi-symplectic representation is nothing else than a special "action" of G on M, an "action" which preserves in some sense the additional structure of M. The "action" will be identified with the representation and will be denoted by the same symbol A. For topological resp. Lie groups we consider exclusively continuous resp. differentiable semi-symplectic representations, which means that the corresponding "action" is a topological resp. Lie transformation group.

We shall say for simplicity that a semi-symplectic representation is given on a symplectic manifold. One must not forget, however, about the Borel structure because the meaning of irreducibility and weak irreducibility is related to it.

If the representation A on $\mathcal{B}(M, \omega)$ is not weakly irreducible then there exists a non-trivial Borel subset invariant for all A_g. The converse is true only if there is no non-trivial Borel subset invariant for all automorphisms of $\mathcal{B}(M, \omega)$. It is the case if $\mathcal{B}(M, \omega)$ is *homogenous* in the sense that Aut $(\mathcal{B}(M, \omega))$ constitutes a transitive transformation group of M. Every orbit of a locally compact second countable topological transformation group is a Borel set, hence we have.

PROPOSITION 5. *If the "action" on M corresponding to a representation on $\mathcal{B}(M, \omega)$ is transitive then the representation is weakly irreducible. If a continuous representation of a locally compact second countable group on a homogeneous $\mathcal{B}(M, \omega)$ is weakly irreducible then the corresponding "action" is transitive.*

We shall call a semi-symplectic representation *transitive* if the corresponding "action" is transitive. We used the term action only for pointing out how we arrived at this familiar notion; quotation marks were applied to underline that, as it was mentioned earlier, an action is a special sort of representations. Later we retain the term action only for standard actions (left translations) on coset spaces.

We shall now restrict our investigations to differentiable semi-symplectic representations of Lie groups. If G is connected we have the following simple but important facts. First: A_g is symplectic for all $g \in G$. Secondly: if the representation is transitive then also M is connected.

Let us denote the Lie algebra of G by $\mathcal{G}$ and that of the infinitesimal generators of the representation by $\mathcal{L}_A$. Recall that the infinitesimal generator of a one-parameter group $g(\bullet)$ is the vector field defined by $x \mapsto (d/dt)_{t=0} A_{g(t)}(x)$ $(x \in M)$. There is a surjective algebra anti-homomorphism $Z: \mathcal{G} \to \mathcal{L}_A$, $a \mapsto Z_a$ ([3] Chap. I. 4.). Let $\mathcal{D}(M)$ denote the algebra of differentiable functions on M, and assume the usual notation i_X for the interior product of differential forms by a vector field X.

PROPOSITION 6. *Consider a differentiable semi-symplectic representation of a Lie group on a simply connected symplectic manifold* (M, ω). *Then there exists a generating function* $H_a \in \mathcal{D}(M)$ *for all* Z_a *such that* $i_{Z_a}\omega = dH_a$. *Evidently,* H_a *is determined only up to an additive constant.*

PROOF. Z_a is the infinitesimal generator of a one-parameter group of symplectic diffeomorphisms, hence $L_{Z_a}\omega = 0$ ([12], Chap. V. 2). Now, $L_{Z_a} = i_{Z_a} \circ d + d \circ i_{Z_a}$ and thus $i_{Z_a}\omega$ is closed. We conclude from the simply connectedness of M and from Poincaré's lemma ([12], Chap. IV.) that $i_{Z_a}\omega$ is exact.

REMARK 2. The simply connectedness of M is not necessary at all for the existence of generating functions. We suppose in the sequel that M is connected and there is a generating function for every infinitesimal generator.

We start our further investigations recalling that the symplectic form ω induces an isomorphism j_ω from the module of vector fields onto the module of one-forms on M, $X \mapsto j_\omega(X) = i_X\omega$ ([12], Chap. VII. 1). The Lie structure is transferred by this isomorphism from the vector fields to the one-forms:

$$[j_\omega(X), j_\omega(Y)]_1 := j_\omega[X, Y]. \tag{7}$$

Furthermore one defines a Lie structure on $\mathcal{D}(M)$ by the Poisson bracket:

$$\{F, H\} = -\omega(j_\omega^{-1}(dF), j_\omega^{-1}(dH)) \qquad (F, H \in \mathcal{D}(M)). \tag{8}$$

One has the identity

$$d\{F, H\} = [dF, dH]_1. \tag{9}$$

PROPOSITION 7. *The map* $H: \mathcal{G} \to \mathcal{G}(M)$, $a \mapsto H_a$ (cf. Proposition 6 and Remark 2) *can be chosen to be linear. Furthermore, by such a choice, there is a commutator cocycle* β *on* $\mathcal{G}$ *so that*

$$\{H_a, H_b\} = -H_{[a,b]} + \beta_{a,b} \quad \textit{for all} \quad a, b \in \mathcal{G}. \tag{10}$$

(We find it convenient to write $\beta_{a,b}$ instead of $\beta(a, b)$.)

PROOF. The kernel in $\mathcal{D}(M)$ of the $\mathbf{R}$-linear map d is $\mathbf{R}$, viewed as the constant functions on M. Let $\Phi: \mathcal{D}(M) \to \mathcal{D}(M)/\mathbf{R}$ be the natural surjection. $\mathbf{R}$ is in the center of $\mathcal{D}(M)$ with respect to the Poisson bracket, thus one can define a Lie structure on $\mathcal{D}(M)/\mathbf{R}$ by

$$\{\Phi(F), \Phi(H)\} := \Phi\{F, H\}. \tag{11}$$

Now, d can be factored, $d = D \circ \Phi$. The image of d is the submodule consisting of exact one-forms. Then D^{-1} is an $\mathbf{R}$-linear isomorphism from the module of exact one-forms onto $\mathcal{D}(M)/\mathbf{R}$. Let us transfer the Lie structure defined by (7) from the one-forms to $\mathcal{D}(M)/\mathbf{R}$. Then we obtain by (8) and (11):

$$[D^{-1}(dF), D^{-1}(dH)] := D^{-1}[dF, dH]_1 =$$
$$(12) \qquad = D^{-1}d\{F, H\} = \{\Phi(F), \Phi(H)\}.$$

That is, the Lie structures on $\mathcal{D}(M)/\mathbf{R}$, induced by the Poisson bracket and by the Lie structure of the vector fields, coincide.

$(\mathcal{D}(M)/\mathbf{R})\times\mathbf{R}$ can be linearly imbedded in – and onto – $\mathcal{D}(M)$. It can be done in such a manner that the imbedding followed by the natural surjection Φ gives the projection from $(\mathcal{D}(M)/\mathbf{R})\times\mathbf{R}$ onto $\mathcal{D}(M)/\mathbf{R}$.

Now, by Proposition 6, the elements of $j_\omega(\mathcal{L}_A)$ are exact one-forms, that is, we can consider $\mathcal{D}_A := D^{-1}(j_\omega(\mathcal{L}_A))$. It is a subalgebra of $\mathcal{D}(M)/\mathbf{R}$. Thus there is a linear imbedding $i: \mathcal{D}_A\times\mathbf{R}\to\mathcal{D}(M)$ so that $\Phi(i(\overline{X}, \lambda)) = \overline{X}$ for all $\overline{X}\in\mathcal{D}_A$ and for all $\lambda\in\mathbf{R}$. Furthermore, $\Phi(F)\in\mathcal{D}_A$ if and only if $F\in i(\mathcal{D}_A\times\mathbf{R})$. Thus we conclude from (11) that the Poisson bracket of two elements of $i(\mathcal{D}_A\times\mathbf{R})$ is again in $i(\mathcal{D}_A\times\mathbf{R})$. In other words, $i(\mathcal{D}_A\times\mathbf{R})$ with the Poisson bracket is a central extension of $\mathcal{D}_A$ by $\mathbf{R}$.

Equivalence classes of such extensions are in one-to-one correspondence with cohomology classes of commutator cocycles of $\mathcal{D}_A$ ([14], Chap. 3.8). Let $\overline{\beta}$ be a commutator cocycle of $\mathcal{D}_A$; the extension defined by $\overline{\beta}$ is given by the commutator

$$(13) \qquad \{i(\overline{X}, \lambda), i(\overline{Y}, \mu)\} = i([\overline{X}, \overline{Y}], \overline{\beta}(\overline{X}, \overline{Y}) \quad (\overline{X}, \overline{Y}\in\mathcal{D}_A;\ \lambda, \mu\in\mathbf{R})$$

Put now $\overline{Z}_a := D^{-1}(j_\omega(Z_a))$ and define the linear map $H: \mathcal{G}\to\mathcal{D}(M)$ by $H_a = i(\overline{Z}_a, 0)$. Then $\Phi(H_a) = \overline{Z}_a$ and by $d = D\circ\Phi$ we get $dH_a = j_\omega(Z_a)$, hence H_a is a generating function for Z_a. Furthermore, it is obvious from (12) that the mapping $(a, b)\mapsto\beta_{a,b} := \overline{\beta}(\overline{Z}_a, \overline{Z}_b)$ defines a commutator cocycle β on $\mathcal{G}$. Then equality (10) follows from (13) and from the fact that Z is a Lie algebra antihomomorphism.

Later we need the following facts. Let X be a vector field on a differentiable manifold M. Consider a manifold M' and a diffeomorphism $f: M\to M'$. Then $f_*(X) := df\circ X\circ f^{-1}$ is a vector field on M'. If φ' is a p-form on M' then

$$(14) \qquad (f^*\varphi')(X_1, X_2, \ldots, X_p) = \varphi'(f_*(X_1), f_*(X_2), \ldots, f_*(X_p))\circ f$$

for all vector fields $X_1, X_2, \ldots, X_p$ on M ([10], p. 24). Making use of this equality one easily proves the following

LEMMA. *Let (M, ω) and (M', ω') be symplectic manifolds, $f: M\to M'$ a symplectic (resp. antisymplectic) diffeomorphism. Suppose X and X' are vector fields on M and on M' respectively and put $\alpha := j_\omega(X)$, $\alpha' := j_{\omega'}(X')$. Then $X' = f_*(X)$ if and only if $\alpha = f^*\alpha'$ (resp. $-\alpha = f^*\alpha'$).*

We can now state

PROPOSITION 8. *Let A and A' be differentiable semi-symplectic representations of the connected Lie group G on the connected symplectic manifolds (M, ω) and (M', ω') respectively. Let $\mathcal{G}$ denote the Lie algebra of G.*

Assume that H, β and H', β' corresponding to A and A' respectively, are the same as in Proposition 7. A and A' are equivalent if and only if there is a symplectic or antisymplectic diffeomorphism $f:M\to M'$ and a linear function $\alpha:\mathcal{G}\to\mathbf{R}$, $a\mapsto\alpha_a$, such that if f is symplectic then β and β' are cohomologous by α and

$$H'_a = H_a\circ f^{-1}+\alpha_a \quad \text{for all} \quad a\in\mathcal{G}. \tag{15}$$

If f is antisymplectic then $-\beta$ and β' are cohomologous by α and

$$-H_a = H_a\circ f^{-1}+\alpha_a \quad \text{for all} \quad a\in\mathcal{G}.$$

PROOF. Suppose first that A and A' are equivalent. Then there exists a symplectic or antisymplectic diffeomorphism $f:M\to M'$ such that $f\circ A_g = A'_g\circ f$ for all $g\in G$. Let f be symplectic; the other case can be treated similarly. The equivalence of the representations implies that the corresponding infinitesimal generators satisfy the equality $df\circ Z_a = Z'_a\circ f$ that is

$$Z_a = f_*(Z_a) \quad \text{for all} \quad a\in\mathcal{G}. \tag{16}$$

Consequently, $dH_a = f^*(dH'_a)$. Now, f^* and d commute ([12], Chap. IV. 2) thus there exist real numbers α_a such that $H_a = f^*(H'_a)-\alpha_a$ for all $a\in\mathcal{G}$. It can be easily seen from the linearity of H and H' and from the identity $f^*(H'_a) = H'_a\circ f$ that the correspondence $a\mapsto\alpha_a$ defines a linear map on $\mathcal{G}$, whence we obtain equality (15). Furthermore, because of the identity $\{H_a, H_b\} = -\omega(Z_a, Z_b)$, we get from (10), (15), (16) that

$$\beta_{a,b} = -\omega(Z_a, Z_b)+H_{[a,b]}$$

$$\beta'_{a,b} = -\omega'(f_*(Z_a), f_*(Z_b))+H_{[a,b]}\circ f^{-1}+\alpha_{[a,b]}$$

for all $a, b\in\mathcal{G}$. Using (14) and the fact that $\beta_{a,b}\circ f^{-1} = \beta_{a,b}$, we get the desired result concerning the cohomology of β and β'.

Suppose now that we have an f and an α with the properties listed in the proposition. Then $dH_a = f^*(dH'_a)$ or $-dH_a = f^*(dH'_a)$ which implies (16). Let s_a be the one-parameter subgroup of G generated by $a\in\mathcal{G}$. Then we have $f\circ A_{s_a} = A'_{s_a}\circ f$. Since every element of G can be written as a product of elements taken from one-parameter subgroups, we obtain the equality $f\circ A_g = A'_g\circ f$ for all $g\in G$ which proves our assertion.

5. Discussion

There is a well known analogon of Proposition 7 for ray representations which, together with Proposition 8 and Corollary of Proposition 2, shows how a complete parallelism can be established between group representations in quantum and in classical mechanics. Unitary resp. antiunitary maps in quantum mechanics correspond to symplectic resp. antisymplectic diffeomorphisms in classical mechanics. As an essential consequence of our definition of representations and their equivalence,

weak cohomology classes of commutator cocycles appear in labelling equivalence classes of representations of connected Lie groups, both in quantum and in classical mechanics.

Souriau's treatment of dynamic systems, translated to our language, is similar to our treatment of representations in classical mechanics, but he considers only symplectic representations and follows a somewhat different way. Our line was worked out to make the parallelism between quantum and classical mechanics apparent.

On the other hand, one does not see the connection between Arens' method and ours at first sight. Arens considers cononical left actions of groups on their coset spaces modulo closed subgroups and he looks for symplectic structures which make the actions symplectic. The following method would seem corresponding in quantum mechanics: one constructs some canonical linear representations of groups and then one looks for inner products which make the representations unitary. However, Arens' method can be arrived at from ours by the reasoning below.

Suppose we have to find transitive differentiable semi-symplectic representations of a connected Lie group G. Let A be such a representation of G on the symplectic manifold (M, ω). Then, as we know, there is a closed subgroup H of G and a diffeomorphism $f: G/H \to M$ such that $f \circ L_g = A_g \circ f$ for all $g \in G$, where L is the canonical action of G on G/H. Then $(G/H, f^*\omega)$ is a symplectic manifold and $L =: (L, f^*\omega)$ will be a semi-symplectic representation of G. One easily proves the following.

PROPOSITION 9. *Let A and A' be two transitive differentiable semi-symplectic representation of the connected Lie group G on (M, ω) and on (M', ω') respectively, and let f and f' be the diffeomorphisms sending A and A' to canonical actions on the coset spaces G/H and G/H' respectively. Then $(L; f^*\omega)$ and $(L', f'^* \omega')$ are equivalent if and only if A and A' are equivalent.*

In this way we obtained, for representations in classical mechanics, a method of "symplectification" of canonical actions on coset spaces, and thus there is given directy a correspondence desired in [4].

6. Elementary particles

We define elementary particles both in quantum and in classical mechanics as follows. The Galilean group in non-relativistic theories and the Poincaré group in relativistic theories will be called the space-time symmetry group.

DEFINITION 3. An elementary particle is a physical system on whose logic there is given a weakly irreducible faithful representation of the identity component of the space-time symmetry group. The properties of an elementary particle are characterized by the corresponding representation so that equivalent representations, and only those, describe the same particle.

SOURIAU ([1]) stated a general method of finding the representations we need for elementary particles in classical mechanics and he gave ex-

plicitly the representations both of the Galilean and of the Poincaré group. ARENS ([3], [4]) obtained a similar result for the Poincaré group. There are two aspects in which the given form of representations is not sufficient. First: the equivalence classes are not the same as ours because equivalence made by an antisymplectic map is not considered. Secondly: we would like to get a member of an equivalence class of representations which reflects the dynamics of the particle in a customary way. This means the following.

States of a physical system are defined to be probability measures on the logic of the system. Let ϱ be a probability measure on the logic $\mathcal{L}$ of a classical or a quantal system and let $t \mapsto A_t$ be a "convenient" representation on $\mathcal{L}$ of the time translation group. Then the map

$$t \mapsto \varrho_t := \varrho \circ A_t^{-1}$$

describes the dynamics of the system. We mean by "convenient" that some condition must be fulfilled; for instance, for an elementary particle, the representation of the time translation group is obtained by restricting the representation of the space-time symmetry group.

Here we mention that pure states can be identified, both in classical and in quantum mechanincs, with the atoms of the logics ([7] vol. I. p. 116, p. 160). Consequently, the points of the phase space in classical mechanics and the vector rays of the structure space in quantum mechanics can be considered the pure states of a physical system and thus the representations on the logics can be transferred directly to the pure states.

Now I give a form of transitive semi-symplectic representations of the Galilean group and of the Poincaré group which is in perfect accordance with the results in quantum mechanics and with the canonical formulation of classical mechanics.

Let $\mathbf{S}^2_\sigma$ denote the two dimensional sphere in $\mathbf{R}^3$ with radius $\sigma \geq 0$. Let $x = (q, p, s)$ be an element of $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma$ with an obvious meaning in the case $\sigma = 0$. A vector field V on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma$ is given by three components (X, Y, Z) such that $X(q, p, s)$, $Y(q, p, s)$ and $Z(q, p, s)$ are in the tangent space of the points q, p and s respectively. Let us define a symplectic form ω_σ on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma$ by

$$\omega_\sigma(V, V')(x) := \langle X(x), Y'(x)\rangle - \langle Y(x), X'(x)\rangle + \langle s, Z(x) \times Z'(x)\rangle$$

where $\langle , \rangle$ resp. $\times$ denotes the inner resp. exterior product in $\mathbf{R}^3$. We shall write p^2 for $\langle p, p\rangle$, and q_i, p_i, s_i $(i = 1, 2, 3)$ for the Cartesian coordinates in $\mathbf{R}^3$.

We state our theorems based on the results in [1].

THEOREM 1. *There is a family of inequivalent differentiable transitive semi-symplectic representations of the connected Galilean group labelled by two real numbers $\mu > 0$ and $\sigma \geq 0$. The representation $A^{(\mu,\sigma)}$ can be given, up to equivalence, on $(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma, \omega_\sigma)$ by the following generating functions for the canonical base of the Lie algebra of the Galilean group:*

$$K_i(q, p, s) = \mu q_i \qquad \text{(pure Galilean subgroup)},$$
$$P_i(q, p, s) = p_i \qquad \text{(space translation subgroup)},$$
$$J_i(q, p, s) = \epsilon_{ijk} q_j p_k + s_i \qquad \text{(space rotation subgroup)},$$
$$H(q, p, s) = \frac{p^2}{2\mu} \qquad \text{(time translation subgroup)};$$
$$(i, j, k = 1, 2, 3; \quad (q, p, s) \in \mathbf{R}^3 \times \mathbf{R}^3 \times S^2_\sigma).$$

μ enumerates the weak cohomology classes of commutator cocycles and appears in the commutator (Poisson bracket) $\{K_i, P_j\} = \mu \delta_{ij}$.

One can write down the representations explicitly:

$$A^{(\mu,\sigma)}_{(t,a,v,R)}(q, p, s) = \left(Rq + a + \frac{Rp - \mu v}{\mu} t, Rp - \mu v, Rs\right).$$

(t is a time translation, a is a space translation, v is a pure Galilean transformation and R is a rotation in space.)

The representation given in Theorem 1 can be extended, in a unique way, to semi-symplectic representations of the Galilean group with space and time inversion. It is an easy task to check that the representation I_S of the space inversion S and the representation I_T of the time inversion T are obtained as follows:

$$I_S(q, p, s) = (-q, -p, s), \quad I_T(q, p, s) = (q, -p, -s).$$

Note that the time inversion is represented by an anti-symplectic diffeomorphism; in quantum mechanics, as it is known, the time inversion is represented by an antiunitary operator.

THEOREM 2. *There is a family of inequivalent differentiable transitive semi-symplectic representations of the connected Poincaré group labelled by two real numbers $\mu > 0$, $\sigma \geq 0$. The representations can be given, up to equivalence, on $(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma, \omega_\sigma)$ by the following generating functions for the canonical base of the Lie algebra of the Poincaré group:*

$$P_i(q, p, s) = p_i \qquad \text{(space translation subgroup)}$$
$$H(q, p, s) = p_0 = \sqrt{p^2 + \mu^2} \qquad \text{(time translation subgroup)}$$
$$J_i(q, p, s) = \epsilon_{ijk} q_j p_k + s_i \qquad \text{(space rotation subgroup)}$$
$$K_i(q, p, s) = p_0 q_i + \epsilon_{ijk} p_j s_k / (p_0 + \mu) \qquad \text{(pure Lorentz one-parameter subgroups)}$$
$$(i, j, k = 1, 2, 3, \quad (q, p, s) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2_\sigma).$$

The interpretation of Theorem 1 and Theorem 2 is straightforward: the representations describe free particles with mass μ and spin σ.

There is another family of representations which correspond to particles with zero mass. These representations, however, lie outside of the framework of mechanics, and do not describe particles in mechanical sense ([1] p. 220, [7] vol. II. p. 217).

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ON SOME SPECTRAL PROPERTIES OF OPERATOR FUNCTIONS

By

B. NAGY

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It is well-known that the spectral classification of the point 0 with respect to the classical parts of the spectrum is identical for every $t>0$ in the case of a semigroup $T(t)$ of linear operators acting in a Banach space X. In this paper we show that this property extends with one exception to the various essential spectra surveyed recently by B. GRAMSCH and D. LAY [1]. It is also shown that cosine operator functions behave quite differently in this respect.

1. Preliminaries

Let X be a complex Banach space and $B(X)$ the space of bounded linear operators from X to X. R will denote the real line, R^+ its positive part, and we shall assume the semigroup $\{T(t);\ t\in R^+\}\subset B(X)$ to be of class (A) [2, p. 342].

If $V\in B(X)$, let $R(V)$ denote its range and $\varrho(V)$ its resolvent set. The nullity of V, $n(V)$ is the dimension of the null space $N(V)$. The defect of V, $d(V)$ is the dimension of the null space $N(V)$. The defect of V, $d(V)$ is the dimension of the quotient space $X/R(V)$. The index of V, ind (V) is $n(V)-d(V)$, where $\infty-\infty$ remains undefined. The ascent of V, $a(V)$ is the smallest nonnegative integer p such that $N(V^p)=N(V^{p+1})$. The descent of V, $e(V)$ is the smallest nonnegative integer q such that $R(V^q)=R(V^{q+1})$. If no such p or q exist, set $a(V)=\infty$ or $e(V)=\infty$, respectively.

A survey of essential spectra was given in [1]. We shall define them by means of the regularity sets G_i, where $V\in G_i$ $(i=1, 2, \ldots, 11)$ means

G_1 : $V^{-1}\in B(X)$,
G_2 : ind $(V)=0$ and $a(V)=e(V)<\infty$,
G_3 : ind $(V)=0$,
G_4 : ind (V) is finite,
G_5 : $n(V)<\infty$ and $R(V)=R(P)$ for some $P=P^2\in B(X)$,

G_6 : $d(V)<\infty$ and $N(V)=R(P)$ for some $P=P^2\in B(X)$,
G_7 : $n(V)<\infty$ and $R(V)$ is closed,
G_8 : $d(V)<\infty$,
G_9 : $G_7\cup G_8$,
G_{10} : $R(V)$ is closed,
G_{11} : $a(V)<\infty$ and $e(V)<\infty$.

Clearly, the following inclusions hold:

$$\begin{tikzcd}[column sep=small] & & & \subset G_6\subset G_8\subset \arrow[dr] & \\ G_1\subset G_2\subset G_3\subset G_4 \arrow[urr] \arrow[drr] & & & & G_9\subset G_{10}. \\ \cap & & & \subset G_5\subset G_7\subset \arrow[ur] & \\ G_{11} & & & & \end{tikzcd}$$

However, the relation $G_{11}\subset G_{10}$ is not true (see [4]).

The essential spectrum $s_i(V)$ is defined as the set of all complex numbers c such that $V-c=V-cI\notin G_i$ ($i=2, 3, \ldots, 11$; for $i=1$ we obtain the spectrum of V). In Section 2 we shall study whether $0\in s_i(T(p))$ for some $p>0$ implies $0\in s_i(T(t))$ for every $t>0$ or, equivalently, whether $T(p)\in G_i$ implies $T(t)\in G_i$ ($i\neq 1$ fixed). The affirmative answers for $i=2, 3, \ldots, 9$ and 11 define regularity classes of semigroups by means of the sets G_i. Section 3 investigates the similar problem for cosine operator functions; here Ps and Cs will denote the point and continuous spectra, respectively.

2. Regularity classes of semigroups of class (A)

In this section $T(t)$ ($t>0$) denotes a semigroup of class (A). The following lemma will be fundamental.

LEMMA 1. If there is a positive p for which $n(T(p))<\infty$, then for every $t>0$ $n(T(t))=0$. Similar statement holds with defect replacing nullity throughout.

PROOF. Clearly, $N(T(t))$ is monotonically nondecreasing and $R(T(t))$ is nonincreasing, if t increases. Define for $k=1, 2, \ldots$ $n(k)=\inf\{t>0;\ n(T(t))\geq k\}$, $d(k)=\inf\{t>0;\ d(T(t))\geq k\}$, $n(k)=\infty$ or $d(k)=\infty$ if $n(T(t))<k$ or $d(T(t))<k$ for every $t>0$, then both extended real-valued sequences are nondecreasing. If $n(1)>0$, then there is an $s>0$ with $n(T(s))=0$, and [2, Theorem 16. 7. 5] implies that the nullity is identically 0. If $n(1)=0$ and $n(T(p))<\infty$, then for some positive integer r $0=n(r)$ and $0<n(r+1)\leq\infty$. Hence $N(T(s))=X_r$ is a fixed r-dimensional subspace of X for $0<s<n(r+1)$, though [2, p. 346 Corollary 1] implies that $X_r=0$, a contradiction.

Similarly, if $d(1)>0$, then there is an $s>0$ with $T(s)X=X$, hence for every positive integer k $T(ks)X=T(s)^kX=X$. Thus the monotonicity of $R(T(t))$ implies $d(T(t))=0$ for every $t>0$. If $d(1)=0$ and $d(T(p))<\infty$, then there exists a positive integer r with $0=d(r)$ and $0<d(r+1)\leq$

$\leq \infty$. Hence $R(T(s)) = X^r$ is a fixed subspace of X with codimension r for $0<s<d(r+1)$. Then

$$\bigcup_{t>0} R(T(t)) = X^r$$

and, X^r being closed, this would imply $X^r = X$ [2, p. 321]. Contradiction.

THEOREM 1. *If there is a positive p for which $T(p)\in G_i$, then for every $t>0$ $T(t)\in G_i$ ($i = 7, 8$).*

PROOF. For $i = 8$ this follows from Lemma 1. Further, if $T(p)\in G_7$, then for $t>0$ we have $n(T(t)) = 0$. Hence the closedness of $R(T(t))$ is equivalent to the existence of a continuous inverse of $T(t)$ defined on $R(T(t))$. This, on the other hand, is equivalent to $T(t)$ not having the spectral property P_3 [2, p. 54]. However, if $T(p)$ dose not have P_3, then no $T(t)$ has it (see the proof of [2, Theorem 16, 7. 5]), thus the proof is complete.

COROLLARY 1. If there is a positive p for which $T(p)\in G_i$ ($i = 2, 3, 4$), then for $t>0$ $T(t)\in G_1$ or, equivalently, $\{T(t)\}$ can be embedded in a strongly continuous group of operators. Further, $T(p)\in G_9$ implies $T(t)\in G_9$ for every $t>0$.

PROOF. This follows from the preceding results and [2, Theorem 16, 7. 5].

The question naturally arises whether milder regularity conditions on $T(p)$ still imply that it can be embedded in a strongly continuous group of operators. However, we have

THEOREM 2. *There exist Banach spaces X_i and semigroups $T_i(t)$ of class (C_0) in them such that for $t>0$ $T_i(t)\in G_i$, but $T_i(t)\notin G_1$ ($i = 5, 6, 11$).*

PROOF. We exhibit the following examples.

EXAMPLE 1. Let $X = L_p(0, \infty)$, $1\leq p<\infty$, and for $x\in X$ put

$$[T(t)x](s) = x(s+t) \quad (t>0).$$

Then $R(T(t)) = X$, hence $d(T(t)) = 0$, further

$$N(T(t)) = \{x\in X;\ x(s) = 0 \text{ a.e. for } s\geq t\}.$$

Define

$$[P_t x](s) = \begin{cases} x(s) & \text{if } 0<s<t \\ 0 & s\geq t \end{cases}.$$

Then P_t is a continuous projection of X onto $N(T(t))$, thus $T(t)\in G_6$, though clearly $T(t)\notin G_1$.

EXAMPLE 2. Let $X = L_p(0, \infty)$, $1<p<\infty$, and for $x\in X$ put

$$[T(t)x](s) = \begin{cases} x(s-t) & \text{if } s>t \\ 0 & 0<s<t \end{cases} \quad (t>0).$$

It is easily seen that $T(t)$ is of class (C_0), $n(T(t)) = 0$ and

$$R(T(t)) = \{x \in X;\ x(s) = 0 \text{ a.e. for } 0 < s < t\}.$$

The projection

$$[P_t x](s) = \begin{cases} x(s) & \text{if } s \geq t \\ 0 & 0 < s < t \end{cases}$$

maps X onto $R(T(t))$, hence $T(t) \in G_5$, though $T(t) \notin G_1$.

EXAMPLE 3. The nilpotent semigroup of [2, p. 537]. Obviously, here $a(T(t)) < \infty$, $e(T(t)) < \infty$, thus $T(t) \in G_{11}$, but $T(t) \notin G_1$.

THEOREM 3. *If there is a positive p with $T(p) \in G_i$, then for every $t > 0$ $T(t) \in G_i$ $(i = 5, 6)$.*

PROOF. If the assumption holds for a given i, then for $r = np$ (n being ary positive integer) we have

$$T(r) = (T(p))^n \in G_i \quad (i = 5 \text{ or } 6); \text{ see } [5, \text{Satz } 2.1., 1°].$$

Let $0 < t < r$. If $T(r) \in G_5$, then $T(r) = T(r-t)\,T(t)$ implies $T(t) \in G_5$, while if $T(r) \in G_6$, then $T(r) = T(t)\,T(r-t)$ implies $T(t) \in G_6$; see [5, Satz 2.1., 2°]. Since r can be arbitrarily large, the proof is complete.

LEMMA 2. The following three conditions are equivalent:

A1 : There is a $p > 0$ with $a(T(p)) = \infty$,
A2 : For every $t > 0$ $a(T(t)) = \infty$,
A3 : $N(T(t))$ is strictly monotonically increasing in $(0, \infty)$.

Similarly, conditions E1 – E3 below are equivalent:

E1 : There is a $p > 0$ with $e(T(p)) = \infty$,
E2 : For every $t > 0$ $e(T(t)) = \infty$,
E3 : $R(T(t))$ is strictly monotonically decreasing in $(0, \infty)$.

PROOF. For every positive integer k $T(t)^k = T(kt)$, hence A3 implies A2, while A2 clearly implies A1. Suppose now that A3 dose not hold, i.e. there are real numbers $0 < r < s$ such that $N(T(r)) = N(T(s))$. Then for $t = s - r$ and every k the relation $T(r + kt)\,x = 0$ implies

$$\begin{aligned} 0 &= T(s + (k-1)t)\,x = T(s)\,T((k-1)t)\,x = \\ &= T(r)\,T((k-1)t)\,x = T(r + (k-1)t)\,x, \end{aligned}$$

thus $T(r)\,x = 0$. Hence for every k $N(T(r + kt)) = N(T(r))$, consequently $N(T(u)) = N)T(r))$ for every $u > r$. But then A1 cannot hold, and the proof for conditions A1 – A3 terminates.

Similarly, it is obvious that E3 implies E2 and E2 implies E1. If E3 does not hold, then there are real numbers $0 < r < s$ such that $R(T(r) = R(T(s))$. With $t = s - r$ we get for every k

$$T(r+kt)X = T((k-1)t)\,T(s)X = T((k-1)t)\,T(r)X = \\ = T(r+(k-1)t)X,$$

thus $R(T(r+kt)) = R(T(r))$, hence $R(T(u)) = R(T(r))$ for every $u>r$. Therefore E1 can not hold, and the proof is complete.

THEOREM 4. *If there is a $p>0$ with $T(p)\in G_{11}$, then for every $t>0$ $T(t)\in G_{11}$.*

PROOF. This follows from Lemma 2.

REMARK. It can happen that $N(T(t))$ is strictly increasing and $R(T(t))$ is strictly decreasing for $0<t<b$, then both become constant. Take e.g. the nilpotent semigroup of [2, p. 537].

THEOREM 5. *There exist a Banach space X and a semigroup $\{T(t)\}\subset \subset B(X)$ (C_0) of class such that $T(p)\in G_{10}$ and $T(r)\notin G_{10}$ for some $p, r\in R^+$.*

PROOF. Let $b>0$, $f(t) = (b-t)^{-1}$ for $0\leq t<b$, and $S = [0, b]$. Let L denote the class of Lebesgue measurable subsets of S, m the Lebesgue measure on L, and $F(E) = \int_E f(t)\,dt$ for $E\in L$. Put $X = L_1(S, L, F)$, and for $x\in X$, $t>0$

$$[T(t)x](s) = \begin{cases} x(s+t) & \text{if } 0\leq s\leq \max(0, b-t) \\ 0 & \max(0, b-t)<s\leq b \end{cases}.$$

Then $T(t)$ is a contraction semigroup, and for $t>b$ $R(T(t)) = \{0\}$, hence closed. We show that $T(t)$ is of class (C_0).

For $x\in X$, $0<t<b$

$$\|T(t)x-x\| = \int_0^{b-t} |x(s+t)-x(s)|\,dF(s) + \int_{b-t}^{b} |x(s)|\,dF(s).$$

The second term on the right side converges to 0 as $t\to 0+$. As for the other term, we first prove that the set H consisting of all functions h which are continuous on S and have their support in $[0, b-q]$ for some $q>0$ depending on h is dense in $L_1(S, L, F)$. It suffices to show that the characteristic function k_E of an $E\in L$ with $F(E)<\infty$ can be approximated by elements of H.

Given $\varepsilon>0$ there is a $q = q(\varepsilon)>0$ such that with the notation $Q = = E\cap[q, b-2q]$ we have $F(E\setminus Q)<\varepsilon$. Moreover, we can find a $\delta = \delta(\varepsilon)>0$ such that the relations $D\in L$, $D\subset[0, b-q]$, $m(D)<\delta$ imply $F(D)<\varepsilon$. Choose an open set U and a closed set V such that $V\subset Q\subset U\subset[0, b-q]$, $m(U\setminus Q)<\delta$ and $m(Q\setminus V)<\delta$. By Urysohn's theorem there exists a $g\in H$ such that $0\leq g(s)\leq 1$, $g(V) = 1$, $g(S\setminus U) = 0$. Hence

$$\|k_E-g\|\leq\|k_E-k_Q\|+\|k_Q-g\|<3\varepsilon.$$

Thus H is dense in X.

Given $x \in X$, $\varepsilon > 0$ we choose a $h \in H$ with $\|x-h\| < \varepsilon$. Then

$$\int_0^{b-t} |x(s+t)-h(s+t)|\, dF(s) = \int_t^b |x(s)-h(s)|\, f(s-t)\, ds \leq \|x-h\| < \varepsilon,$$

hence

$$\int_0^{b-t} |x(s+t)-x(s)|\, dF(s) < 2\varepsilon + \int_0^b |h(s+t)-h(s)|\, f(s)\, ds.$$

Since $h \in H$ and h is uniformly continuous, the last term converges to 0 if $t \to 0+$. Thus $T(t)$ is of class (C_0).

Now we prove that $R(T(t))$ is not closed for $0 < t < b$. Clearly

$$N_t = N(T(t)) = \{x \in X,\ x(s) = 0 \ \text{ for } \ t \leq s \leq b\}.$$

If $R(T(t))$ were closed, we should have with some $p > 0$

$$\|T(t)\, x\| > p.\ \text{dist}\ (x.N_t),$$

that is

$$\int_0^{b-t} |x(s+t)|\, f(s)\, ds \geq p.\ \inf_{y \in N_t} \int_0^b |x(s)-y(s)|\, f(s)\, ds,$$

or

$$\int_t^b |x(s)|\, f(s-t)\, ds \geq p \int_t^b |x(s)|\, f(s)\, ds \quad \text{for} \quad x \in X.$$

Put

$$t < a_n < b, \quad b_n = b-(b-a_n)\, e^{-1}, \quad x_n = k_{(a_n,\, b_n)} \in X.$$

Then

$$\int_t^b |x_n(s)|\, f(s)\, ds = 1,$$

whereas

$$\int_t^b |x_n(s)|\, f(s-t)\, ds \leq f(b-t) \cdot (b_n - a_n) \to 0 \quad \text{if} \quad a_n \to b.$$

Hence $R(T(t))$ is not closed.

3. Cosine operator functions

A cosine operator function is a strongly continuous mapping C from R into $B(X)$, satisfying $C(0) = I$ and

$$C(s+t)+C(s-t) = 2C(s)\, C(t) \quad (s, t \in R),$$

see e.g. [3], [6]. With respect to the spectral character of 0, cosine operator functions show a behavior completely different from that of semigroups. We have

THEOREM 6. *There exist a Banach space X and a cosine operator function $\{C(t); t\in R\}\subset B(X)$ such that the following sets are all dense in R:*

$$E = \{t;\ 0\in\varrho(C(t))\},$$
$$F = \{t;\ 0\in Ps(C(t))\},$$
$$H = \{t;\ 0\in Cs(C(t))\},$$
$$K = \{t;\ 0\in s_i(C(t))\} \qquad (i = 1, 2, \ldots, 11).$$

PROOF. Put $X = l_2$ and for $x = (x_1, x_2, \ldots)\in l_2$, $t\in R$ define $C(t)x = \{(\cos nt)x_n;\ n = 1, 2, \ldots\}$. If t and π are rationally independent, then by Kronecker's approximation theorem (see [2, p. 146]) the set

$$Ps(C(t)) = \{\cos nt;\ n = 1, 2, \ldots\}$$

is dense in the interval $[-1, 1]$ and, clearly, $0\notin Ps(C(t))$. Applying the reasoning of the proof of [3, Theorem 6] we see that $0\in Cs(C(t))$. Hence $R(C(t))$ cannot be closed, so that also $0\in s_i(C(t))$ $(i = 1, 2, \ldots, 10)$. Moreover, $C(t)$ is a one-to-one mapping, therefore $a(C(t)) = 0$, whereas $R(C(t))\neq X$, hence by [7, Theorem 5.41 – E], $e(C(t)) = \infty$. Thus $0\in s_{11}(C(t))$ and, since the numbers $t = q\pi$ with q irrational are dense in R, the statement of the theorem holds for the sets H and K_i $(i = 1, 2, \ldots, 11)$.

Suppose now that $t = \frac{p}{r}\pi$, where $r>0$ and p are relative primes. Then $\cos(n+2r)t = \cos nt$, so that the set $\{\cos nt;\ n = 1, 2, \ldots\}$ contains at most $2r$ different numbers. If none of them vanishes, then for every $x = (x_1, x_2, \ldots)\in X$ we have $C(t)^{-1}x = \{(\cos nt)^{-1}x_n;\ n = 1, 2, \ldots\}$, hence $0\in\varrho(C(t))$. If, on the other hand, one of the values $\cos nt$ is zero, then clearly $0\in Ps(C(t))$. But the relation $\cos npr^{-1}\pi = 0$ holds for at least one positive integer n if and only if p is odd and r is even; hence

$$E = \left\{t;\ t = \pi\frac{\text{integer}}{\text{odd}}\right\},\quad F = \left\{t;\ t = \pi\ \begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}.$$

Thus the statements of the theorem follow. Moreover, $R = E\cup F\cup H$.

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ON SOME CLASSES OF SEMIGROUPS, II

By

S. LAJOS

Department of Math. of the K. Marx University, Budapest

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This note is a sequel to [3]. For the terminology not defined here we refer to A. H. CLIFFORD and G. B. PRESTON [2]. A semigroup S is said to be *left quasi-regular* if every left ideal L of S is globally idempotent, i.e. $L^2 = L$ (cf. CALAIS [1]). Three classes of semigroups will be characterized here in terms of bi-ideals.

THEOREM 1. *A semigroup S is regular if and only if the relation*

$$B \cap L \cap R \subseteq RBL \tag{1}$$

holds for every bi-ideal B, left ideal L, and right ideal R of S.

PROOF. Let S be a regular semigroup. Then, for any bi-ideal B of S the representation

$$B = R' L' = R' \cap L' \tag{2}$$

holds, where L' is a left ideal, R' is a right ideal of S (see [4], Theorem 9). On the other hand, we have

$$L \cap L' \subseteq L' L \tag{3}$$

for any couple L, L' of left ideals of S because a regular semigroup is left (and right) quasi-regular. By making use (2), (3), and the well-known ISÉKI-KOVÁCS regularity criterion we get

$$\begin{aligned} B \cap L \cap R &= R' \cap L' \cap L \cap R \subseteq \\ &\subseteq RR' \cap L' L \subseteq \\ &\subseteq RR' L' L = \\ &= RBL \end{aligned}$$

for any bi-ideal B, left ideal L, and right ideal R of S.

The sufficiency of condition (1) follows easily because

$$L \cap R \subseteq RSL \subseteq RL \subseteq L \cap R \tag{4}$$

holds for every left ideal L and every right ideal R of S. Hence $L \cap R = RL$, i.e. S is a regular semigroup.

THEOREM 2. *A semigroup S is both regular and intra-regular if and only if the condition*

$$B \cap L \cap R \subseteq BRL \tag{5}$$

holds for every bi-ideal B, left ideal L, and right ideal R of S.

PROOF. It is known that a semigroup S is both regular and intra-regular if and only if every bi-ideal B of S is globally idempotent (see [5], Theorem 39). Hence it follows that the condition (5) is necessary.

Conversely, let S be a semigroup with property (5) for every bi-ideal B, left ideal L, and right ideal R of S. Then, for any left ideal L and right ideal R of S, (5) implies

$$L \cap R \subseteq R^2 L \subseteq RL, \tag{6}$$

whence $L \cap R = RL$, i.e. S is regular. On the other hand, (5) implies

$$L \cap R \subseteq LRL \subseteq LR \tag{7}$$

for every left ideal L, and every right ideal R of S. Applying (7) for principal left and principal right ideals of S generated by an arbitrary element a of S we get

$$a \in L(a) \cap R(a) \subseteq L(a)\, R(a) \subseteq S^1 a^2 S^1, \tag{8}$$

whence S is intra-regular.

THEOREM 3. *A semigroup S is intra-regular and left quasi-regular if and only if the condition*

$$B \cap L \cap R \subseteq LRB \tag{9}$$

holds for every bi-ideal B, left ideal L, and right ideal R of S.

PROOF. It is known (see [7]) that for any bi-ideal B and left ideal L of an intra-regular and left quasi-regular semigroup S the following condition holds:

$$B \cap L \subseteq LB. \tag{10}$$

This implies (9) for any bi-ideal B, left ideal L, and right ideal R of S.

Conversely, suppose that S is a semigroup with property (9). Then (7) follows for every left ideal L and every right ideal R of S, whence S is intra-regular. In case of $B = L$, $R = S$, (9) implies

$$L \subseteq LSL \subseteq L^2 \tag{11}$$

for any left ideal L of S. Hence $L = L^2$ for every left ideal L of S, that is, S is left quasi-regular indeed.

COROLLARY. *For a commutative [duo] semigroup S the following are equivalent conditions:*

(i) *S is regular.*
(ii) *S is both regular and intra-regular.*
(iii) *S is left quasi-regular and intra-regular.*

Finally we have the following criterion.[1]

THEOREM 4. *A semigroup S is intra-regular if and only if the condition*

$$B \cap L \cap R \subseteq LBR$$

holds for every bi-ideal B, left ideal L, and right ideal R of S.

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ОДИН КРИТЕРИЙ p-НИЛЬПОТЕНТНОСТИ КОНЕЧНЫХ ГРУПП

Ф. ДЬАПЙАШ

Кафедра вычислительной математики и кибернетики,
Университета им. Л. Этвеша, Будапешт

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Целью настоящей работы является доказатеьство следующих утверждений.

ТЕОРЕМА 1. *Пусть P — группа порядка p^{n+1}. Если*

а) подгруппа $\mathbf{Z}(P)$ циклическая,
б) $P' \leq \Omega_1(P)$,
в) $|\Omega_1(P)| = p^2$,

тогда группа P может быть только следующих типов:

I. $p = 2,\ n > 3$, *либо* $p > 2,\ n \geq 3$,
$P \cong \langle a, b \mid a^{p^n} = b^p = e,\ a^b = a^{1+p^{n-1}} \rangle$.

II. $p = 2,\ n = 4$,
$P \cong \langle a, b, c \mid a^8 = b^2 = e,\ a^b = a^5,\ a^c = ab,\ b^c = a^4 b,\ c^2 = a^6 b \rangle$

III. $p = 3,\ n = 3$,
$P \cong \langle a, b, c \mid a^9 = a^3 = e,\ a^b = a^4,\ a^c = a^3 b,\ c^3 = a^6 \rangle$.

ТЕОРЕМА 2. *Пусть G — конечная группа, p наименьшее простое число, делящее порядок группы G. Пусть P — силовская p-подгруппа группы G. Если*

а) подгруппа $\mathbf{Z}(P)$ — циклическая,
б) $P' \leq \Omega_1(P)$,
в) $|\Omega_1(P)| = p^2$,

то G — p-нильпотентна.

В работе будут использованы стандартные обозначения.

Для доказательства теоремы 1 нам понадобится следующее утверждение.

ЛЕММА. (см. [2], стр. 349. задача 30.) Если P — 2-группа, $P' \neq \langle e \rangle$, $|P| = 16$, $\operatorname{Exp} P = 4$, то группа P изоморфна группе, одному из следующих типов:

1. $D \curlyvee Z = \langle a, b, z | a^4 = b^2 = e,\ a^b = a^{-1},\ z^2 = a^2 \rangle$.
2. $H \times D$, где $|H| = 2$, D-группа диэдра порядок 8.
3. $H \times D$, где $|H| = 2$, Q-группа кватернионов.
4. метациклическая 2-группа.
5. $\langle a, b, c | a^4 = b^2 = c^2 = e,\ [a, b] = [b, c] = e,\ [a. c] = b \rangle$.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 1.

1. Если подгруппа A группы P является обобщенной группой кватернионов, то $|A| = 8$:

Если

$$A = \langle a, b | a^{2^k} = e,\ b^2 = a^{2^{k-2}},\ a^b = a^{-1} \rangle,$$

то

$$A' = \langle a^2 \rangle,\ A' \leq P' \leq \Omega_1(P).$$

Из этого следует, что A — группа кватернионов.

2. Если $\Omega_1(P) \leq \Phi(P)$, то

$$P \cong \langle a, b | a^{p^n} = b^p = e,\ a^b = a^{1+p^{n-1}} \rangle: \tag{1}$$

Из $\Omega_1(P) \not\leq \Phi(P)$ следуеет, что группа P обладает максимальной подгруппой A, содержащей одну подгруппу порядка p. Подгруппа является или циклической, или обобщенной группой кватернионов. Предположим, что A-обобщенная группа кватернионов. Покажем, что это предположение приводит к противоречию. Согласно пункт 1, из $A' \leq P' \leq \Omega_1(P)$ следует, что

$$o(a^2) = 2,\ n = 3,\ |P| = 16,\ \operatorname{Exp} P = 4,\ P' \neq \langle e \rangle.$$

Отсюда следует, что P — группа одного из типов, перечисленных в лемме. Центры групп типов 2.—5. содержат элементарны абелевы подгруппы порядок 4. Но это противоречит условию *а)*. Группа P содержит 3 элемента порядка 2, и так $P \not\cong D \curlyvee Z$. Поэтому, подгруппа A циклическая.

По доказанному, группа P обладает циклической максимальной подгруппой. В силу этого (см. [2], стр. 90. теорема 14.9), если не выпольняется (1), то $p = 2$, и группа P может быть только следующих типов: лгруппа диэдра, обобщенная группа кватернионов, группа семидиэдрическая. В последних случаях условии *а)*, *б)*, *в)* не выпольняются. Из этого следует, что (1) действителен.

В дальнейшем мы предположим, что

$$\Omega_1(P) \leq \Phi(P). \tag{2}$$

Мы пользуемся обозначениями:

$$B := \mathbf{C}_P(b) \quad (b \in \Omega_1(P) - Z(P)),$$
$$A := \mathbf{C}_P(a) \quad (a \in P - B),$$
$$D := \langle b, A \rangle.$$

3. $|B| = p^n$:

Для любых элементов u, v группы P

$$w := v^{-1} u v = u[u, v] \in uP'. \tag{3}$$

Согласно (3) из условий *б)* и *в)* следует, что $|P : \mathbf{C}_P(u)| \leq p^2$.

Подгруппа $\Omega_1(P) \cap \mathbf{Z}(P) \neq \langle e \rangle$ является характеристической подгруппой. Поэтому $|P : \mathbf{C}_P(b)| < p^2$. Следовательно, $|P : \mathbf{C}_P(b)| = p^2$.

4. $|A| = p^{n-1}$, $P' = \Omega_1(P)$, $|D| = p^n$:

Из соотношений $\mathbf{C}_P(a) < P$, (3) и условий *б)*, *в)* следует, что $p^{n-1} \leq |A| \leq p^n$. В силу нашего выбора, $b \notin A$. Если $|\mathbf{C}_P(a)| = p^n$, то

$$b \in \Omega_1(P) \leq \Phi(P) \leq A,$$

что противоречит соотношению $b \notin A$. Согласно (3), из $|A| = p^{n-1}$ следует, что $P' = \Omega_1(P)$.

Если $D = P$, то из $b \notin \Omega_1(P) \leq \Phi(P)$ следует, что $P = \langle b, A \rangle = A$. Это противоречит соотношению $A < P$.

5. $n \geq 3$, даже, если $p = 2$, то $n \geq 3$:

В силу условий *а)*, *б)*, *в)* и равентства $P' = \Omega_1(P)$, следует, что $c(P) = 3$. Следовательно, $n \geq 3$. При $p = 2$ и $n = 3$ (см. [2], стр. 339. теорема 11.9.), группа может быть только следующих типов: группа диэдра, обобщенная группа кватнрнионов, группа семидиэдрическая, что противоречит условию *в)*. Поэтому, если $p = 2$, то $n > 3$.

6. $A = \langle a \rangle$ и для подходящего элемент b

$$D = \langle a, b \mid a^{p^{n-1}} = b^n = e, \ a^b = a^{1+p^{n-2}} \rangle:$$

Подгруппа A содержит только одну подгруппу порядка p. Подгруппа A является или циклической, или обобщенной группой кватернионов. Предположим, что A — обобщенная группа кватернионов. Покажем, что это предположение приводит к противоречию. Согласно пункт 1, A является группой кватерионов. Пусть

$$A = \langle v, w \mid v^4 = e, \ w^2 = v^2, \ v^w = v^{-1} \rangle.$$

$$b, v^2 \in \Omega_1(P), \ b \notin A$$

поэтому

$$\Omega_1(P) = \langle v^2, b \rangle.$$

Из $\mathbf{Z}(P) \leq A$ следует, что $\mathbf{Z}(P) \leq \mathbf{Z}(A) = \langle v^2 \rangle$. Следовательно, из $a \in \mathbf{Z}(P)$ следует, что $a = v^2$, $b^{-1} ab = a$, что противоречит к соотношению $a \notin B$. Поетому подгруппа A циклична. Если $A = \langle u \rangle$ то $u \in P - B$, $o(u) = p^{n-1}$. Пусть $a = u^t$. Если $t \equiv 0 \pmod p$, то $a \in \Phi(P)$, $P = \langle a, B \rangle$, что противо-

речит к соотношению $|B| = p^n$. Из этого следует, что $A = \langle a \rangle$. $A = \langle a \rangle$ является максимальной подгруппой группы $D = \langle a, b \rangle$. Поэтому из соотношений $[a, b] \neq e$, $[a, b]^p = e$, следует, что

$$b^{-1} ab = a^{1+k \cdot p^{n-2}}, \quad k \equiv 1 \pmod{p^{n-2}}. \tag{4}$$

Если $mk \equiv 1 \pmod{p}$, то из (4) следует, что

$$b^{-m} ab^m = a^{1+kmp^{n-2}} = a^{1+p^{n-2}}.$$

Если мы положим $b_1 = b^m$, то

$$\langle b_1 \rangle = \langle b \rangle, \quad b_1^{-1} ab_1 = a^{1+p^{n-2}}.$$

Поэтому, не ограничивая общности, можно считать, что

$$D = \langle a, b \mid a^{p^{n-1}} = b^p = e, \ a^b = a^{1+p^{n-2}} \rangle. \tag{5}$$

7. Группа P содержит такой элемент c, что $c \notin B$, $c^{-1} ac = ab$, $P = \langle a, b, c \rangle$:

$|P'| = |P : A| = p^2$, поэтому существует такой $u \in P$, что $u^{-1} au = ab$. $Au \nsubseteq B$, поэтому существует такой $c \in P - C$, что

$$c^{-1} ac = ab. \tag{6}$$

Из $A \lhd D$ следует, что $c \in D$ и так $P = \langle a, b, c \rangle$. Из соотношений $a^b = a^{1+p^{n-2}}$ и (6) следует, что

$$c^{-1} a^k c = a^{k-\binom{k}{2} p^{n-2}} b^k. \tag{7}$$

8. Если $p = 2$, то

$$P \cong \langle a, b, c \mid a^{2^{n-1}} = b^2 = e, \ a^b = {}^{1+2^{n-2}}, \ a^c = ab, \ b^c = a^{2^{n-2}} b, \ c^2 = a^{2m} b \rangle,$$

где m нечетное.

Из (7) следует, что

$$c^{-1} a^2 c = a^{2+2^{n-2}}, \quad c^{-1} a^4 c = a^4. \tag{8}$$

Из (8) и

$$a^b = a^{1+2^{n-2}} \tag{9}$$

следует, что

$$a^4 \in \mathbf{Z}(P), \quad a^{2^{n-2}} \in \mathbf{Z}(P), \quad (n \geq 4) \tag{10}$$

$[b, c] \neq b$, $a^{2^{n-2}} b$, поэтому $c^{-1} bc = a^{2^{n-2}} b$.

Из $c^{-2} ac^2 = a^{1+2^{n-2}} = b^{-1} ab$ следует, что

$$(c^2 b^{-1})^{-1} a (c^2 b^{-1}) = a, \quad c^2 \in Ab. \tag{11}$$

Если $c^2 = a^k b$, то $c^4 = a^{2k+k \cdot 2^{n-2}}$. $a(c) = 2^{n-1}$, $n \geq 4$, поэтому $e = (c^4)^{2n-3} = a^{k \cdot 2^{n-2}}$, $k = 2m$, $(m, 2) = 1$.

9. Если $p = 2$, то

$$P \cong \langle a, b, c \mid a^8 = b^2 = e,\ a^b = a^5,\ a^c = ab,\ b^c = a^4 b,\ c^2 = a^6 b\rangle:$$

Согласно пункт 8.

$$P \cong \langle a, b, c \mid a^{2^{n-2}} = b^2 = e,\ a^b = a^{1+2^{n-2}},\ a^c = ab,\ b^c = a^{2^{n-2}} b,\ c^2 = a^{2m} b\rangle.$$

Если, мы положим $c_1 = ca^{-4t}$, то из $a^4 \in \mathbf{Z}(P)$ следует, что

$$P \cong \langle a, b, c_1 \mid a^{2^{n-1}} = b^2 = e,\ a^b = a^{1+2^{n-2}},$$
$$a^{c_1} = ab,\ b^{c_2} = a^{2^{n-2}} b\ \ c_1^2 = a^{2m-4} b\rangle.$$

Поэтому, не ограничивая общности, можно считать, что $m = 1$, или $m = 3$.

Посмотрим случай $n > 4$. Если мы положим

$$a_1 = a^{1+2^{n-3}},\ b_1 = b,\ c_1 = ca^{-2}$$

то из $a^{2^{n-3}} \in \mathbf{Z}(P)$ следует, что

$$P \cong \langle a_1, b_1, c_1 \mid a_1^{2^{n-1}} = b_1^2 = e,\ a_1^{b_1} = a_1^{1+2^{n-2}},$$
$$a_1^{c_1} = a_2 b_1,\ b_1^{c_2} = a_1^{2^{n-2}} b_1,\ c_1^2 = a_1^{2m-4} b\rangle.$$

Поэтому, не ограничивая общности, в случае $n > 4$, можно считать, что $m = 1$.

Из определяющих отношений следует, что

$$(c\, a^k b^l)^2 = a^{2m+2k+\binom{k}{2}\cdot 2^{n-2}+k\cdot 2^{n-2}} b^{k+1}. \tag{12}$$

Из (12) следует, что $|\Omega_1(P)| > 4$ при $m = 1$. Случаи $n > 4$ и $n = 4$, $m = 1$ отпадают. Если $n = 4$, $m = 3$, то

$$\mathbf{Z}(P) = \langle a^4\rangle,\ P' = \Omega_r(P) = \langle a^4, b\rangle.$$

Пусть

$$(a^k b^l)^\alpha := a^{k+\binom{k}{2}\cdot 2^{n-2}+l\cdot 2^{n-2}} b^{k+l}.$$

Легко проверяется, что α — автоморфизм группы D, кроме того

$$(a^6 b)^\alpha = a^6 b,\ (a^k b^l)^{\alpha^2} = (a^6 b)^{-1}(a^k b^l)(a^6 b).$$

Из этого следует, что группа P изоморфна группе циклического расширения группы D при помощи группы $\langle \alpha \mid \alpha^2 = e\rangle$, и так $|P| = 32$.

10. Если $p > 2$, то $p = 3$ и

$$P \cong \langle a, b, c \mid a^9 = b^3 = e,\ a^b = a^4,\ a^c = ab,\ b^c = a^3 b,\ c^3 = a^6\rangle:$$

Из $|\Omega_1(P)| = p^2$ следует, что группа P имеет $p+1$ подгрупп порядка p. Если $p > 3$, то P — метациклическая группа (см. [2], стр. 338.

теорема 11.6). Поэтому группа P' — циклична, что противоречит к соотношениям.

Таким образом, случай $p>3$ отпадает. Пусть $p=3$. Из (6) и (8) следует, что

$$c^{-1}a^3c = a^3,\ \ a^3\in Z(P),\ \ P' = \langle a^{3^{n-2}}, b\rangle .$$

Отсюда

$$c^{-1}bc = a^{s\cdot 3^{n-2}}b^t \quad (0\leq s<3,\ 0<t<3) \tag{13}$$

$$c^{-2}bc^2 = a^{s(1+t)\cdot 3^{n-2}}b^{t^2} \tag{14}$$

$c\notin B$, поэтому, из (13) и (14) следует, что $s\neq 0$. Если $s=2$, то

$$(ac)^{-1}b(ac) = a^{3^{n-2}}b,\ \ (ac)^{-1}a(ac) = ab .$$

Поэтому, можно считать, что

$$c^{-1}bc = {}^{3^{n-2}}b . \tag{15}$$

Из (7) и (15) следует, что $c^3\in A = \langle a\rangle$, $a(c) = 3^{n-1}$, поэтому $c^3 = a^{3k}$, где $(k,1) = 1$. Пусть $k = l+3t$, $0<l<3$. Пусть (x, y) решение в целых числах уравнения

$$k+3^{p-2}x = l(1+3y) .$$

Если, мы положим, $a_1 = a^{1+3y}$, то

$$A = \langle a_1\rangle,\ \ b^{-1}a_1b = a_1^{1+3^{n-2}},\ \ c^{-1}a_1c = a_1b .$$

Поэтому, не ограничивая общности, можно считать, что $0<k<3$. Легко проверяются, что

$$\begin{array}{ll} \langle a^3, b, ca^{-1}\rangle, & \text{если}\ \ n=3,\ k=1, \\ \langle a^3, b\rangle, & \text{если}\ \ n=3,\ k=2, \\ \langle a^{3^{n-2}}, b, ca^{-1}\rangle, & \text{если}\ \ n>3,\ k=1, \\ \langle a^{3^{n-2}}, b, ca^{3^{n-3}+1}\rangle, & \text{если}\ \ n>3,\ k=2. \end{array}$$

Поэтому

$$P\cong\langle a,b,c\,|\,a^9 = b^3 = e,\ \ a^b = a^4,\ \ a^c = ab,\ \ b^c = a^3b,\ \ c^3 = a^6\rangle .$$

Аналогично, как при $p=2$, можно доказать, что группа P изоморфна группе циклического расширения группы D, кроме того $|P| = 81$.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 2.

Пусть $|P| = p^{n+1}$. Если $p>2$, то теорема является специальным случаем следующей теоремы:

Пусть G — конечная группа нечотном порядком. Пусть p — наименьшее простое число, делящее порядок группы G. Если группа G не содержит элементарную абелеву подгруппу, порядок которой p^3, то G p-нильпотент. (см. [1], лемма 8.5. или [3], теорема B′).

Если $\Omega_1(P) \not\leq \Phi(P)$, то согласно теореме 1,

$$P \cong \langle a, b \mid a^{p^n} = b^p = e,\ a^b = e \rangle .$$

Поэтому, если $p = 2$, $\Omega_1(P) \leq \Phi(P)$, то теорема 2 следует из теоремы Вонга (см. [4], или [2], стр. 424. теорема 3.5). Поэтому, мы можем ограничиваться рассмотрением случаев, когда $\Omega_1(P) \leq \Phi(P)$ и $p = 2$. Мы отмечаем, что данное доказательство правильно при $p = 3$. Доказательство приведем польной индукцией. Пусть группа G наименьший контрапример для теоремы. Посмотрим случай $P \not\trianglelefteq G$. Тогда $P \leq \mathbf{N}_G(P) < P$, следовательно, группа $\mathbf{N}_G(P)$ p-нильпотентна. Поэтому

$$\mathbf{N}_G(P)' \cap P = P' .$$

Согласно теореме Грюна

$$G' \cap P = \langle \mathbf{N}_G(P)' \cap P,\ P^x \cap P \mid x \in G \rangle ,$$

$P' = \Omega_1(P) \leq \Phi(P)$, поэтому $(P^x)' \cap P \leq \Omega_1(P) \leq \Phi(P)$ для любых $x \in G$. И так

$$G' \cap P \leq \Phi(P) .$$

Согласно теореме Тате (см. [2], стр. 432. теорема 4.8), группа G p-нильпотентна, что противоречит условию.

Посмотрим случай $P \lhd G$. Тогда $P' \lhd G$, $\mathbf{Z}(P) \lhd G$, $\Omega_1(\mathbf{Z}(P)) \lhd G$. p наименьшее простое число, делящее порядок группы G, поэтому из $|\Omega_1(\mathbf{Z}(P))| = p$ следует, что $\Omega_1(\mathbf{Z}(P)) \leq \mathbf{Z}(G)$. Если $b \in P' - \mathbf{Z}(G)$, то из соотношений $|P'| = p^2$, $P' \lhd G$, $\Omega_1(\mathbf{Z}(P)) \leq \mathbf{Z}(G)$ следует, что

$$|G : \mathbf{C}_P(b)| = p, \quad \text{и так} \quad \mathbf{C}_G(b) \lhd G .$$

Поэтому подгруппа $\mathbf{C}_P(b) = P \cap \mathbf{C}_G(b)$ является силовской p-подгруппой группы $\mathbf{C}_G(b)$, кроме того $B = \mathbf{C}_P(b) \lhd G$.

Согласно теореме Цассенхауза, из соотношений

$$P \lhd G \quad \text{и} \quad (G : P|, |P|) = 1$$

следует, что существует собственная подгруппа H группы G, для которого $|H| = |G : P|$. Как известно, из этих условий следует, что $|H| = q^m$, , где q простое число, делящее порядок группы G. Напомним кратко на доказательсво этого факта. Пусть $|H| = q_s^{m_1} \ldots q_s^{m_s}$ $(s \geq 2)$, где q_i-простые числа, делящие порядок группы G. Если Q_i-силовская q_i-подгруппа H и $R_i = PQ_i$, то легко получается, что $R_i = P \times Q_i$, и так $H \leq \mathbf{C}_G(P)$. Отсюда следует, что $H \lhd G$. Таким образом, G p-нильпотентна, что противоречит индуктивному предположению. Поэтому, можно считать, что $|H| = q^m$.

Согласно теореме 1, $|P/\Phi(P)| = p^2$, поэтому группа P обладает $p + 1$ разных максимальных подгрупп, скажем $B, D_1, \ldots, D_p$. $B \lhd G$, поэтому для любых $x \in H$

$$\pi := \begin{pmatrix} D_i^x \\ D_i \end{pmatrix} \in \mathbf{S}_p$$

где $\mathbf{S}_p$ — группа подстановок множества $D_1, \ldots, D_p$. Порядок π делит $|H| = q^m$ и $p!$, Поэтому, из $(q^m, p!) = 1$ следует, что $D_i^x = D_i$ $(1 \leq i \leq s)$ для любых $x \in H$. Пусть D-максимальная подгруппа группы P, определенная в теореме 1, кроме того $L := DH$. Очевидно, что D-силовская p-подгруппа группы L. Кроме того выпольняются условия теоремы 2. Поэтому группа L — p-нильпотентна. Таким образом, $H \lhd L$, т.е. $L \leq \mathbf{N}_G(H)$.

H — силовская q-подгруппа группы L. Таким образом, согласно теорема Силова, из $L = \mathbf{N}_G(H)$ следует, что

$$|G : L| = p \equiv 1 \pmod q$$

что противоречит $p < q$. Из этого следует, что $H \lhd G$. что противоречит индуктивному предположению. Этим завершается доказательство теоремы 2.

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INVERSE FUNCTION THEOREM — WITHOUT ASSUMING CONTINUOUS DIFFERENTIABILITY

By

T. SZILÁGYI

II. Department of Analysis of the Eötvös Loránd University, Budapest

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In our paper we give another proof for a theorem of H. HALKIN (see [1], Theorem G), which shows that this theorem is very close to certain results of M. R. HESTENES obtained at least ten years ago. The proof is based on the following

LEMMA. Let V be an arbitrary set, X — an Euclidean space of finite dimension, ε_0 a positive number, G — the unit ball in X with centre 0,

$$F: G\times\,]0,\ \varepsilon_0[\,\times V\to X,$$

A a linear map from X onto X, and let us denote

$$N(t):=\sup_{(z,\,v)\in G\times V}\|Az-F(z,t,v)\|.$$

If

(i) $\lim\limits_{+0} N=0$ and

(ii) for every $(t,v)\in\,]0,\ \varepsilon_0[\,\times V$ the function $F(\circ\,,t,v):G\to X$ is continuous, then there exists (for some positive ω) a function $z:\,]0,\ \omega[\,\times V\to G$ such that

a) $\lim\limits_{+0} z=0$ uniformly in $v\in V$,

b) $F(z(t,v),t,v)=0\quad\forall\,(t,v)\in\,]0,\ \omega[\,\times V.$

This lemma is a consequence of Brouwer's fixed point theorem, its proof can be found in Chapter IV of [2] (see the proof of Theorem 7.2)

THEOREM. *Let $x_0\in\mathbf{R}^n$, $U\subset\mathbf{R}^n$ be a neighborhood of x_0, $f:U\to\mathbf{R}^n$ a continuous function which is differentiable at x_0, and suppose that $Df(x_0)$ is invertible. Then there exist a neighborhood W of $f(x_0)$ and a function $h:W\to U$ such that*

I $h(f(x_0)) = x_0$,
II $f(h(y)) = y \quad \forall y \in W$,
III *h is differentiable at $f(x_0)$ and* $Dh(f(x_0)) = [Df(x_0)]^{-1}$.

PROOF. Let $A := Df(x_0)$, G be the unit ball of $\mathbf{R}^n$ with centre O again, V its boundary, and ε_0 a positive number such that $\|z\| < 2\varepsilon_0$ implies $x_0 + z \in U$. Then we have for every $z \in G$, $v \in V$, and $t \in]0, \varepsilon_0[$

$$x_0 + t(z+v) \in U. \tag{1}$$

Finally the function $F: G \times]0, \varepsilon_0[\times V \to \mathbf{R}^n$ will be defined by

$$F(z,t,v) := \frac{1}{t}[f(x_0 + t(z+v)) - f(x_0)] - Av.$$

We shall apply the Lemma. The condition (ii) is satisfied because of (1) and the continuity of f. As for the condition (i) let ε be a positive number; f is differentiable at x_0, so there exists a positive number δ such that

$$\frac{\|f(x) - f(x_0) - A[x - x_0]\|}{\|x - x_0\|} < \frac{\varepsilon}{2}.$$

whenever $\|x - x_0| < \delta$. In particular, if $0 < t < \frac{\delta}{2}$, $z \in G$, and $v \in V$, then

$$\frac{\|f(x_0 + t(z+v)) - f(x_0) - A[t(z+v)]\|}{\|t(z+v)\|} < \frac{\varepsilon}{2},$$

thus

$$\left\| \frac{1}{t}[f(x_0 + t(z+v)) - f(x_0)] - Av - Az \right\| < \frac{\varepsilon \|z+v\|}{2} \leq \varepsilon,$$

consequently if $0 < t < \frac{\delta}{2}$, then $0 \leq N(t) \leq \varepsilon$.

By the Lemma we have a function $z:]0, \omega[\times V \to G$ which satisfies the identity *b)*, that is

$$f\big(x_0 + t(z(t,v) + v)\big) = f(x_0) + tAv \quad \text{for all} \quad (t,v) \in]0, \omega[\times V. \tag{2}$$

Let

$$W := \{y \in \mathbf{R}^n \mid \exists\, 0 \leq t < \omega \quad \text{and} \quad v \in V \quad \text{such that} \quad y = f(x_0) + tAv\};$$

since A^{-1} exists, W is a neighborhood of $f(x_0)$. The function h will be defined on W by

$$h(f(x_0) + tAv) := x_0 + t(z(t,v) + v).$$

First of all we must verify that this definition is correct, that is if

(3) $\quad t_1, t_2 \in [0, \omega[, \; v_1, v_2 \in V; \quad \text{and} \quad f(x_0)+t_1 Av_1 = f(x_0)+t_2 Av_2,$

then

(4) $$x_0+t_1(z(t_1, v_1)+v_1) = x_0+t_2(z(t_2, v_2)+v_2).$$

Really, if $t_1 \cdot t_2 = 0$, then $t_1 = t_2 = 0$ because A^{-1} exists, thus in this case

$$x_0+t_i(z(t_i, v_i)+v_i) = x_0 \quad \text{for} \quad i = 1, 2.$$

If $t_1 > 0$ and $t_2 > 0$, then from (3) and the invertibility of A we get

$$t_1 v_1 = t_2 v_2, \qquad v_1 = \frac{t_2}{t_1} v_2$$

and herefrom – since $\|v_1\| = \|v_2\| = 1$ – that $t_1 = t_2$ and $v_1 = v_2$, so (4) holds. Now from the definition of h and from the identity (2) one can see that h has the properties I and II. We must prove III:

$$h(f(x_0)+tAv)-h(f(x_0)) = t(z(t, v)+v) = A^{-1}(tAv)+tz(t, v),$$

therefore the proof will be complete, if we show that

(5) $$tz(t, v) = o(tAv).$$

To this end first we observe that there exists a positive number m such that

(6) $$\|Av\| \geq m \quad \text{for all} \quad v \in V.$$

Let ε be a positive number. From statement *a)* of the Lemma we get a $\delta > 0$ such that $0 < t < \delta$ and $v \in V$ imply

$$\|z(t, v)\| < \varepsilon \cdot m.$$

To obtain (5) we show that if $\|tAv\| < \delta m$, then

$$\frac{\|tz(t, v)\|}{\|tAv\|} < \varepsilon.$$

From (6) we know that $\|tAv\| < \delta \cdot m$ implies $t < \delta$, thus

$$\frac{\|tz(t, v)\|}{\|tAv\|} = \frac{\|z(t, v)\|}{\|Av\|} \leq \frac{\|z(t, v)\|}{m} < \varepsilon.$$

REMARK 1. The Theorem could be called "right-inverse function theorem".

REMARK 2. Under the assumptions of the Theorem f has no inverse in general. This fact can be shown by the following simple example:

$$n := 1, \quad x_0 := 0, \quad f(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ x^2 \sin \dfrac{1}{x} + x & \text{if} \quad x \neq 0 \end{cases}.$$

REMARK 3. The function F used in this proof appears in Chapter IV of [2], too, in the proof of the Theorem 7.3, that can be regarded as a special case of the present Theorem.

REMARK 4. It would be interesting to know whether the Theorem remains valid (my conjecture is that it does not) if $\mathbf{R}^n$ is substituted by an infinite dimensional normed space (assuming that $[Df(x_0)]^{-1}$ is defined on the whole space and is continuous).

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SOME METRIC PROBLEMS IN THE ADDITIVE NUMBER THEORY, II.

By

A. SÁRKÖZY

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

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1. Notations and definitions

We shall use mostly the same notations and definitions as in Part I of this paper (see [3]). However, we shall use also some new concepts and some notations will be modified. Thus we are going to recapitulate the notations and definitions needed also here.

Sequence will mean a strictly increasing (finite or infinite) sequence of non-negative integers. The sequences will be denoted by upper case Gothic letters, their elements by the corresponding lower case letters, while their accounting functions (the number of the elements not exceeding x) by the corresponding capitals:

$$\mathfrak{A} = \{a_1, a_2, \ldots\} \quad \text{where} \quad a_1 < a_2 < \ldots,$$

$$A(x) = \sum_{a_i \leq x} 1.$$

For $j = 0, 1, 2, \ldots$, we denote the set of those elements of $\mathfrak{A}$ which do not exceed j, by $\mathfrak{A}_j$:

$$\mathfrak{A}_j = \mathfrak{A} \cap \{0, 1, \ldots, j\}.$$

Sets of sequences will be denoted by upper case Greek letters. In particular, let $\sum$, $\sum^*$ and $\sum^\infty$ denote the set of all sequences, finite sequences and infinite sequences, respectively. For some set $\Pi \subset \sum$ and for $j = 0, 1, 2, \ldots$, let Π_j denote the set of those sequences

$$\mathfrak{A} \subset \{0, 1, 2, \ldots, j\}$$

for which there exists a sequence $\mathfrak{B} \in \Pi$ such that $\mathfrak{B}_j = \mathfrak{A}$.

The *sum* of the sequences $\mathfrak{A}$, $\mathfrak{B}$, i.e. $\mathfrak{A} + \mathfrak{B}$ is defined as the sequence consisting of those integers which can be written in form $a_i + b_j$ (where $a_i \in \mathfrak{A}$, $b_j \in \mathfrak{B}$). For $n = 2, 3, \ldots$, we define $n\mathfrak{A}$ by $2\mathfrak{A} = \mathfrak{A} + \mathfrak{A}$ and $k\mathfrak{A} = (k-1)\mathfrak{A} + \mathfrak{A}$ for $k = 3, 4, \ldots$.

For $k = 2, 3 \ldots$, the sequence $\mathfrak{A}$ is said to be an *asymptotic basis of order* k if there exists a non-negative integer n_0 such that

$$\{n_0, n_0+1, n_0+2, \ldots\} \subset k\mathfrak{A}$$

(i.e. any integer $n \geqq n_0$ can be written in form $n = a_{i_1} + a_{i_2} + \ldots + a_{i_k}$). For $k = 2, 3, \ldots$, let $\Phi^{(k)}$ denote the set of those sequences which are not asymptotic bases of order k.

To investigate subsets of $\sum$ by means of measure theory, we use the usual mapping of $\sum$ into the interval $[0, 1]$: for some $\mathfrak{A} \in \sum$, let

$$\varrho(\mathfrak{A}) = \sum_{a_i \in \mathfrak{A}} \frac{1}{2^{a_i+1}}. \tag{1}$$

(Obviously, (1) defines a one-to-one correspondence between the infinite sequences and the points in the interval $(0, 1]$.) For some $\Gamma \subset \sum$, let $\varrho(\Gamma)$ denote the set consisting of the points $\varrho(\mathfrak{A})$ where $\mathfrak{A} \in \Gamma$.

We need also the concepts of the Hausdorff-measure and Hausdorff-dimension. The definitions and notations concerning these concepts can be found in Part I of this paper. In particular, we denote the Hausdorff-dimension of a set M by $\dim M$.

2. The problem and some lemmas

In this paper, *our aim is to investigate the sets* $\Phi^{(k)}$ $(k = 2, 3, \ldots)$ *by means of the measure theory;* more exactly, *we are going to estimate* $\dim \varrho(\Phi^{(k)})$. In Section 3, we shall determine $\dim \varrho(\Phi^{(2)})$. For $k \geqq 3$, we shall not be able to determine $\dim \varrho(\Phi^{(k)})$; instead, it will be shown in Section 4 that

$$\lim_{k \to +\infty} \dim \varrho(\Phi^{(k)}) = \frac{1}{2}.$$

We shall need the following lemmas:

LEMMA 1. (B. Volkmann, [4]):

For some sequence $\mathfrak{A}$, *let* $\langle \mathfrak{A} \rangle$ *denote the set of those sequences* $\mathfrak{B}$ *for which* $\mathfrak{B} \subset \mathfrak{A}$ *holds. Then*

$$\dim \varrho(\langle \mathfrak{A} \rangle) = \liminf_{x \to +\infty} \frac{A(x)}{x}.$$

For $0 \leqq x \leqq 1$, let us define the function $d(x)$ in the following way:

$$d(x) = \begin{cases} -\dfrac{1}{\log 2}\left(x \log x + (1-x)\log(1-x)\right) & \text{for } 0 < x < 1 \text{ and} \\ 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = 1 \end{cases}.$$

This function $d(x)$ is continuous in $[0, 1]$, increasing in $[0, 1/2]$, $d(1/2) = 1$ and $d(1-x) = d(x)$ for $0 \leqq x \leqq 1$.

LEMMA 2. (see [3]):

Let $0 \leqq a < b$ and $\varepsilon > 0$. There exist a positive number $\delta = \delta(a, b, \varepsilon)$ and a positive integer $m_0(a, b, \varepsilon)$ such that

$$m \geqq m_0(a, b, \varepsilon),$$

$$|u - bm| < \delta m$$

and

$$|v - am| < \delta m$$

imply that

$$\binom{u}{v} < 2^{(bd(a/b)+\varepsilon)m} \tag{2}$$

and (provided that $0 < a$)

$$\binom{u}{v} > 2^{(bd(a/b)-\varepsilon)m}. \tag{3}$$

LEMMA 3. (see [3]):

Let $\Theta^{(j)} \subset \sum$ for $j = 0, 1, 2, \ldots$ and let

$$\Theta \subset \bigcap_{i=0}^{+\infty} \left(\bigcup_{j=0}^{+\infty} \Theta^{(j)} \right).$$

For fixed j, let $\varphi(\Theta^{(j)})$ denote the cardinality of $\Theta_j^{(j)}$ (i.e. the number of those sequences $\mathfrak{B} \subset \{0, 1, 2, \ldots, j\}$ for which there exists a sequence $\mathfrak{A}$ such that $\mathfrak{A}_j = \mathfrak{B}$ and $\mathfrak{A} \in \Theta^{(j)}$ hold). Let $0 \leqq \mu < 1$ and let us assume that for $\varepsilon > 0$ arbitrary small, $j > j_0(\varepsilon)$ implies that

$$\varphi(\Theta^{(j)}) < 2^{(\mu+\varepsilon)j} \quad (j > j_0(\varepsilon)). \tag{4}$$

Then

$$\dim \varrho(\Theta) \leqq \mu.$$

LEMMA 4. (H. G. Eggleston, [1]; see also P. ERDŐS and S. J. TAYLOR, [2]):

Suppose I_k $(k = 1, 2, \ldots)$ is a linear set consisting of N_k closed intervals each of length δ_k. Let each interval of I_k contain

$$n_{k+1} \geqq 2 \tag{5}$$

closed intervals of I_{k+1} so distributed that their minimum distance apart is

$$\varrho_{k+1} > \delta_{k+1}. \tag{6}$$

Let

$$P = \bigcap_{k=1}^{+\infty} I_k.$$

Then

$$\liminf_{k\to+\infty} N_{k+1}\,\varrho_{k+1}\,\delta_k^{s-1}>0 \tag{7}$$

implies that

$$\dim P\geqq s\,. \tag{8}$$

Lemmas 1, 2 and 3 can be proved relatively easily. Lemma 2 is an extension of Lemma 3 in [3]. Namely, Lemma 3 in [3] contained only the upper estimate (2) while here Lemma 2 includes also the lower estimate (3). Using Stirling's formula, the lower estimate (3) can be deduced in the same way as (2). Lemma 3 is identical to Lemma 2 in [3].

Lemma 4 is deeper than the first three lemmas. It is a consequence of Theorem 6 in [1]. In fact, Lemma 4 is identical to Theorem A in [2] (which is stated there as a special case of Theorem 6 in [1]), except that (8) is replaced there by the slightly stronger statement that the set P has positive s-dimensional Hausdorff-measure.

3. In this section, we will prove the following theorem:

THEOREM 1.

$$\dim \varrho(\Phi^{(2)}) = \frac{\log 3}{\log 4}\,.$$

PROOF:

It is sufficient to show that both

$$\dim \varrho(\Phi^{(2)})\leqq \frac{\log 3}{\log 4} \tag{9}$$

and

$$\dim \varrho(\Phi^{(2)})\geqq \frac{\log 3}{\log 4} \tag{10}$$

hold. The upper estimate (9) can be deduced relatively easily (by using Lemmas 2 and 3). The lower estimate of $\dim \varrho(\Phi^{(2)})$ will be much more difficult; it will be based on Lemma 4.

UPPER ESTIMATE OF $\dim \varrho(\Phi^{(2)})$

For $j = 0, 1, 2, \ldots$, let $\Theta^{(j)}$ denote the set of those sequences $\mathfrak{A}$ for which $j\notin 2\mathfrak{A}$ holds. Obviously,

$$\Phi^{(2)} = \bigcap_{i=0}^{+\infty}\left(\bigcup_{j=i}^{+\infty}\Theta^{(j)}\right). \tag{11}$$

In order to apply Lemma 3, we have to estimate $\varphi(\Theta^{(j)})$.

Let us take some sequence $\mathfrak{A} \in \Theta^{(j)}$. Then the elements of

$$\mathfrak{A}_j \cap \left\{0, 1, \ldots, \left[\frac{j}{2}\right]\right\} (= \mathfrak{A}_{[j/2]}) \tag{12}$$

can be chosen from the $\left[\frac{j}{2}\right]+1$ integers $0, 1, \ldots, \left[\frac{j}{2}\right]$. Thus if we fix $A\left(\left[\frac{j}{2}\right]\right) = k$, then the elements of the sequence (12) can be chosen in $\binom{[j/2]+1}{k}$ ways.

Let us suppose now that k and the sequence (12) have been fixed. Then

$$j - \left[\frac{j}{2}\right] \leqq j - a_i \leqq j \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Furthermore, $\mathfrak{A} \in \Theta^{(j)}$ implies that $j \notin 2\mathfrak{A}$, thus

$$\mathfrak{A}_j \cap \left\{j - \left[\frac{j}{2}\right],\ j - \left[\frac{j}{2}\right] + 1,\ \ldots,\ j\right\} \tag{13}$$

must not contain the k integers $j-a_1, j-a_2, \ldots, j-a_k$. Hence, for fixed k, the elements of the sequence (13) can be chosen from at most

$$j - \left(j - \left[\frac{j}{2}\right]\right) + 1 - k = \left[\frac{j}{2}\right] + 1 - k$$

integers, i.e. in at most $2^{[j/2]+1-k}$ ways. Summarizing, we obtain that $\mathfrak{A}_j$ can be chosen in

$$\varphi(\Theta^{(j)}) \leqq \sum_{k=0}^{[j/2]+1} \binom{[j/2]+1}{k} 2^{[j/2]+1-k} =$$

$$= (2+1)^{[j/2]+1} = 3^{[j/2]+1} \leqq 3^{j/2+1} = 2^{\frac{\log 3}{\log 4} j + \frac{\log 3}{\log 2}} \tag{14}$$

ways. Applying Lemma 3, we obtain (9) from (11) and (14).

Lower Estimate of $\dim \varrho(\Phi^{(2)})$

In order to prove (10), it is sufficient to show that there exists a set Ψ for which

$$\Psi \subset \Phi^{(2)} \tag{15}$$

and

$$\dim \varrho(\Psi) \geqq \frac{\log 3}{\log 4} \tag{16}$$

hold.

Let us define the sequence $\mathfrak{B}$ in the following way: let

$$b_0 = 30, \quad b_{i+1} = b_i^2 \quad \text{for} \quad i = 0, 1, 2, \ldots$$

and

$$\mathfrak{B} = \{b_0, b_1, b_2, \ldots\}.$$

Furthermore, let

$$\mathfrak{C}^{(0)} = \{0, 30, 60, \ldots, 870\} = \{0, b_0, 2b_0, \ldots, b_1 - b_0\},$$

$$\mathfrak{C}^{(i)} = \{b_i, b_i + b_{i-1}, b_i + 2b_{i-1}, \ldots, b_{i+1} - b_{i-1}\} \quad \text{for} \quad i = 1, 2, \ldots$$

(obviously, b_{i-1}/b_i and b_{i-1}/b_{i+1}, thus also $b_{i-1}/b_{i+1} - b_i$) and

$$\mathfrak{C} = \{c_0, c_1, c_2, \ldots\} = \bigcup_{i=0}^{+\infty} \mathfrak{C}^{(i)}.$$

Finally, let Ψ denote the set of those sequences $\mathfrak{A} \in \sum$ for which the following conditions hold:

$$b_i \notin 2\mathfrak{A} \quad \text{for} \quad i = 0, 1, 2, \ldots, \tag{17}$$

$$c_i - 1 \notin \mathfrak{A} \quad \text{for} \quad i = 1, 2, \ldots, \tag{18}$$

$$c_i \notin \mathfrak{A} \quad \text{for} \quad i = 0, 1, 2, \ldots, \tag{19}$$

$$c_i + 1 \notin \mathfrak{A} \quad \text{for} \quad i = 0, 1, 2, \ldots \tag{20}$$

and

$$A(c_{i+1}) - A(c_i) = \frac{c_{i+1} - c_i}{3} \quad \text{for} \quad i = 0, 1, 2, \ldots \tag{21}$$

(obviously, $c_{i+1} - c_i$ is integer for $i = 0, 1, 2, \ldots$).

For this set Ψ, (15) holds trivially. In order to prove that also (16) holds, we are going to apply Lemma 4. We have to define the sets I_k at first.

For $k = 1, 2, \ldots$, let

$$I_k = \bigcup_{\mathfrak{A} \in \Psi_{c_k}} \left[\varrho(\mathfrak{A}), \varrho(\mathfrak{A}) + \frac{1}{2^{c_k+1}}\right]$$

and

$$J_k = \bigcup_{\mathfrak{A} \in \Psi_{c_k}} \left[\varrho(\mathfrak{A}), \varrho(\mathfrak{A}) + \frac{1}{2^{c_k+1}}\right).$$

Obviously,

$$\varrho(\Psi) = \bigcap_{k=1}^{+\infty} J_k. \tag{22}$$

Furthermore, if $\mathfrak{A} \in \Psi_{c_k}$ for some positive integer k then (19) and (20) (with $i = k$) imply that

$$\varrho(\mathfrak{A}) + \frac{1}{2^{c_{k+1}}} \notin I_{k+1}.$$

Thus

$$\bigcap_{k=1}^{+\infty} I_k = \bigcap_{k=1}^{+\infty} J_k. \tag{23}$$

By (22) and (23),

$$\varrho(\Psi) = \bigcap_{k=1}^{+\infty} I_k. \tag{24}$$

We have to show that the sets I_k satisfy (5), (6) and (7) $\left(\text{with } s = \frac{\log 3}{\log 4} - \varepsilon\right)$.

To prove that (6) holds, we have to estimate ϱ_{k+1}

$$\left(\text{obviously, } \delta_{k+1} = \frac{1}{2^{c_{k+1}}+1}\right).$$

If $\mathfrak{A} \in \Psi_{c_{k+1}}$, $\mathfrak{A}^* \in \Psi_{c_{k+1}}$ and $\varrho(\mathfrak{A}) > \varrho(\mathfrak{A}^*)$, then (18) and (19) (with $i = k+1$) imply that

$$\varrho(\mathfrak{A}) - (\varrho(\mathfrak{A}^*) + \delta_{k+1}) = (\varrho(\mathfrak{A}) - \varrho(\mathfrak{A}^*)) - \delta_{k+1} \geqq$$
$$\geqq \frac{1}{2^{(c_{k+1}-2)+1}} - \delta_{k+1} = 4 \cdot \frac{1}{2^{c_{k+1}+1}} - \delta_{k+1} = 4\delta_{k+1} - \delta_{k+1} = 3\delta_{k+1}.$$

Hence

$$\varrho_{k+1} \geqq 3\delta_{k+1}, \tag{25}$$

which proves (6).

To show that also (5) holds, let us introduce the following notation: for some non-negative integer k and for a sequence

$$\mathfrak{A} \in \Psi_{c_k}, \tag{26}$$

let $f(\mathfrak{A}, k)$ denote the number of those sequences $\mathfrak{B}$ for which

$$\mathfrak{B}_{c_k} = \mathfrak{A} \tag{27}$$

and

$$\mathfrak{B} \in \Psi_{c_{k+1}} \tag{28}$$

hold (i.e. $\mathfrak{A}$ can be continued in $f(\mathfrak{A}, k)$ ways to obtain a sequence belonging to $\Psi_{c_{k+1}}$). With respect to the construction of the sequence $\mathfrak{C}$, $b_i \in \mathfrak{C}$ and $\frac{b_i}{2} \in \mathfrak{C}$ hold for any non-negative integer i. Thus for fixed k, there exists a uniquely determined non-negative integer j such that either

(29) $$b_j \leqq c_k < c_{k+1} \leqq \frac{b_{j+1}}{2}$$

or

(30) $$\frac{b_{j+1}}{2} \leqq c_k < c_{k+1} \leqq b_{j+1}$$

holds.

If (29) holds and the sequence $\mathfrak{A}$ satisfies (26), then (27) and (28) imply that the sequence

(31) $$\mathfrak{B} \cap \{c_k+1, c_k+2, \ldots, c_{k+1}\}$$

must not contain the integers c_k+1, $c_{k+1}-1$, c_{k+1} (by (18), (19), (20)) and the number of its elements is

$$B(c_{k+1})-B(c_k) = \frac{c_{k+1}-c_k}{3}$$

(by (21)), and these are the only restrictions concerning the sequence (31). In other words, to obtain the sequence (31), we have to choose $\frac{c_{k+1}-c_k}{3}$ of the $c_{k+1}-c_k-3$ integers

(32) $$c_k+2, c_k+3, \ldots, c_{k+1}-2.$$

Thus

(33) $$f(\mathfrak{A}, k) = \binom{c_{k+1}-c_k-3}{\frac{c_{k+1}-c_k}{3}} \quad \text{(in case (29))}.$$

If (30) holds then the sequence (31) must satisfy one more condition. Namely, by (17),

$$b_{j+1} \notin 2\mathfrak{A}.$$

Thus if

(34) $$n \in \{c_k+2, c_k+3, \ldots, c_{k+1}-2\}$$

and

(35) $$b_{j+1}-n \in \mathfrak{A}$$

hold then n must not belong to the sequence (31). Let us put $m = b_{j+1}-n$. Then the conditions (34) and (35) can be rewritten in the form

(36) $$m \in \{b_{j+1}-c_{k+1}+2, b_{j+1}-c_{k+1}+3, \ldots, b_{j+1}-c_k-2\},$$

(37) $$m \in \mathfrak{A}.$$

The construction of the sequence $\mathfrak{C}$ implies that $b_{j+1}-c_{k+1}\in\mathfrak{C}$ and $b_{j+1}-c_k\in\mathfrak{C}$. Let us write

$$b_{j+1}-c_{k+1}=c_r,\quad b_{j+1}-c_k=c_s.$$

Then (18), (19), (20) and (21) imply that (36) and (37) hold for

$$A(b_{j+1}-c_k-2)-A(b_{j+1}-c_{k+1}+1)=$$

$$=A(c_s-2)-A(c_r+1)=A(c_s)-A(c_r)=\frac{c_s-c_r}{3}=$$

$$=\frac{(b_{j+1}-c_k)-(b_{j+1}-c_{k+1})}{3}=\frac{c_{k+1}-c_k}{3}$$

integers m. In other words, when we construct the sequence (31), $\frac{c_{k+1}-c_k}{3}$ of the $c_{k+1}-c_k-3$ integers in (32) must be excluded. Thus we have to choose $\frac{c_{k+1}-c_k}{3}$ of

$$(c_{k+1}-c_k-3)-\frac{c_{k+1}-c_k}{3}=\frac{2(c_{k+1}-c_k)}{3}-3$$

integers:

$$f(\mathfrak{A},k)=\begin{pmatrix}\dfrac{2(c_{k+1}-c_k)}{3}-3\\[2ex] \dfrac{c_{k+1}-c_k}{3}\end{pmatrix}\quad\text{(in case (30))}. \tag{38}$$

(33) and (38) show that in fact, $f(\mathfrak{A},k)$ is independent of $\mathfrak{A}$ thus we may put $n_{k+1}=f(\mathfrak{A},k)$ and

$$n_{k+1}=\begin{cases}\begin{pmatrix}c_{k+1}-c_k-3\\ \dfrac{c_{k+1}-c_k}{3}\end{pmatrix} & \text{for}\quad b_j\leqq c_k<c_{k+1}\leqq\dfrac{b_{j+1}}{2},\\[2ex] \begin{pmatrix}\dfrac{2(c_{k+1}-c_k)}{3}-3\\[2ex] \dfrac{c_{k+1}-c_k}{3}\end{pmatrix} & \text{for}\quad \dfrac{b_{j+1}}{2}\leqq c_k<c_{k+1}\leqq b_{j+1}.\end{cases} \tag{39}$$

Since $c_{k+1}-c_k\geqq 30$ for any non-negative integer k, (39) implies (5).

Finally, in order to prove that also (7) holds $\left(\text{for } s=\dfrac{\log 3}{\log 4}-\varepsilon\right)$, we have to give a lower estimate for N_{k+1} (for large k). Since

$$N_{k+1} = N_1 \cdot \frac{N_2}{N_1} \cdot \frac{N_3}{N_2} \cdot \ldots \cdot \frac{N_k}{N_{k-1}} \cdot \frac{N_{k+1}}{N_k} =$$

$$= N_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k \cdot n_{k+1}, \tag{40}$$

it suffices to estimate n_i for large i.

Let us fix some $\varepsilon > 0$. Applying Lemma 2, we get from (39) that

$$n_{i+1} > 2^{(d(1/3)-\varepsilon)(c_{i+1}-c_i)} \quad \text{for} \quad j \geqq j_0(\varepsilon), \tag{41}$$

$$b_j \leqq c_j < c_{i+1} \leqq \frac{b_{j+1}}{2}$$

and

$$n_{i+1} > 2^{((2/3)\cdot d(1/2)-\varepsilon)(c_{i+1}-c_i)} = \tag{42}$$

$$= 2^{(2/3-\varepsilon)(c_{i+1}-c_i)} \text{ for } j \geqq j_0(\varepsilon),\ \frac{b_{j+1}}{2} \leqq c_i < c_{i+1} \leqq b_{j+1}.$$

The constructions of the sequences $\mathfrak{B}$ and $\mathfrak{C}$ imply that $b_j \in \mathfrak{C}$ and $\frac{b_{j+1}}{2} \in \mathfrak{C}$ for $j = 0, 1, 2, \ldots$; let $b_j = c_{u_j}$ and $\frac{b_{j+1}}{2} = c_{v_j}$. We are going to show that if

$$j \geqq j_0(\varepsilon), \quad b_j = c_{u_j} < c_t \leqq b_{j+1} = c_{u_{j+1}} \tag{43}$$

then

$$n_{u_j+1}\, n_{u_j+2} \ldots n_t > 2^{\left(\frac{\log 3}{\log 4}-2\varepsilon\right)(c_t-c_{u_j})}. \tag{44}$$

Let us assume at first that $c_t \leqq \frac{b_{j+1}}{2} = c_{v_j}$. Then by (41),

$$n_{u_j+1}\, n_{u_j+2} \ldots n_t > \prod_{i=u_j}^{t-1} 2^{(d(1/3)-\varepsilon)(c_{i+1}-c_i)} = \tag{45}$$

$$= 2^{(d(1/3)-\varepsilon)(c_t-c_{u_j})} \left(\text{for } c_t \leqq \frac{b_{j+1}}{2}\right).$$

Since

$$d(1/3) = -\frac{1}{\log 2}\left(\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3}\right) = \tag{46}$$

$$= -\frac{2}{3} + \frac{\log 3}{\log 2} > -\frac{\log 3}{2\log 2} + \frac{\log 3}{\log 2} = \frac{\log 3}{\log 4},$$

(45) implies (44) in this case.

Let us assume now that $\frac{b_{j+1}}{2} = c_{v_j} < c_t$.

Then by (42) and (45):

$$(47)\qquad n_{u_j+1}\, n_{u_j+2} \ldots n_t = (n_{u_j+1}\, n_{u_j+2} \ldots n_{v_j}) \cdot (n_{v_j+1}\, n_{v_j+2} \ldots n_t) >$$

$$> 2^{(d(1/3)-\varepsilon)(c_{v_j}-c_{u_j})} \prod_{i=v_j}^{t-1} 2^{(2/3-\varepsilon)(c_{i+1}-c_i)} =$$

$$= 2^{(d(1/3)-\varepsilon)(c_{v_j}-c_{u_j})} 2^{(2/3-\varepsilon)(c_t-c_{v_j})} =$$

$$= 2^{d(1/3)(c_{v_j}-c_{u_j})+(2/3)(c_t-c_{v_j})} \cdot 2^{-\varepsilon(c_t-c_{u_j})}.$$

In view of (46), the exponent of the first factor can be estimated in the following way:

$$(48)\qquad d\left(\frac{1}{3}\right)(c_{v_j}-c_{u_j})+\frac{2}{3}(c_t-c_{v_j}) = \frac{\log 3}{\log 4}(c_{v_j}-c_{u_j})+$$

$$+\left(d\left(\frac{1}{3}\right)-\frac{\log 3}{\log 4}\right)(c_{v_j}-c_{u_j})+\frac{\log 3}{\log 4}(c_t-c_{v_j})+\left(\frac{2}{3}-\frac{\log 3}{\log 4}\right)(c_t-c_{v_j}) =$$

$$= \frac{\log 3}{\log 4}(c_t-c_{u_j})+\left(\frac{\log 3}{\log 4}-\frac{2}{3}\right)(c_{v_j}-c_{u_j})+\left(\frac{2}{3}-\frac{\log 3}{\log 4}\right)(c_t-c_{v_j}) =$$

$$= \frac{\log 3}{\log 4}(c_t-c_{u_j})+\left(\frac{\log 3}{\log 4}-\frac{2}{3}\right)(2c_{v_j}-c_t-c_{u_j}) =$$

$$= \frac{\log 3}{\log 4}(c_t-c_{u_j})+\left(\frac{\log 3}{\log 4}-\frac{2}{3}\right)(b_{j+1}-c_t)-b_j\left(\frac{\log 3}{\log 4}-\frac{2}{3}\right).$$

Here

$$(49)\qquad b_{j+1}-c_t \geqq 0,$$

$$(50)\qquad 0 < \frac{\log 3}{\log 4}-\frac{2}{3} < 1$$

and

$$b_j = \sqrt{b_{j+1}} < \varepsilon\left(\frac{b_{j+1}}{2}-\sqrt{b_{j+1}}\right) = \varepsilon\left(\frac{b_{j+1}}{2}-b_j\right) \leqq$$

$$(51)\qquad \leqq \varepsilon(c_t-b_j) = \varepsilon(c_t-c_{u_j}).$$

We get from (47), (48), (49), (50) and (51) that

$$n_{u_j+1}\, n_{u_j+2} \ldots n_t > 2^{\frac{\log 3}{\log 4}(c_t-c_{u_j})-\varepsilon(c_t-c_{u_j})}\, 2^{-\varepsilon(c_t-c_{u_j})} =$$

$$= 2^{\left(\frac{\log 3}{\log 4}-2\varepsilon\right)(c_t-c_{u_j})}$$

which completes the proof of (44) (provided that (43) holds).

Starting out from (40) and using (44), N_{k+1} can be estimated easily. Let us define the non-negative integer q by

$$u_q < k+1 \leqq u_{q+1}$$

(where u_j is defined by $c_{u_j} = b_j$). Then we get from (40) and (44) that for $k \geqq k_0(\varepsilon)$,

$$N_{k+1} = N_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k \cdot n_{k+1} = \tag{52}$$

$$= N_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_{u_{j_0}} \cdot \prod_{j=j_0}^{q-1} \left(\prod_{i=u_j+1}^{u_{j+1}} n_i \right) \cdot \prod_{i=u_q+1}^{k+1} n_i >$$

$$> 1 \cdot \prod_{j=j_0}^{q-1} 2^{\left(\frac{\log 3}{\log 4} - 2\varepsilon\right)(c_{u_{j+1}} - c_{u_j})} \cdot 2^{\left(\frac{\log 3}{\log 4} - 2\varepsilon\right)(c_{k+1} - c_{u_q})} =$$

$$= 2^{\left(\frac{\log 3}{\log 4} - 2\varepsilon\right)(c_{k+1} - c_{u_{j_0}})} > 2^{\left(\frac{\log 3}{\log 4} - 2\varepsilon\right) c_{k+1} - c_{u_{j_0}}} >$$

$$> 2^{\left(\frac{\log 3}{\log 4} - 2\varepsilon\right) c_{k+1} - \varepsilon c_{k+1}} = 2^{\left(\frac{\log 3}{\log 4} - 3\varepsilon\right) c_{k+1}}.$$

Let us put $s = \dfrac{\log 3}{\log 4} - 4\varepsilon$. Then we get from (25) and (52) that for $k \geqq k_0(\varepsilon)$,

$$N_{k+1} \varrho_{k+1} \delta_k^{s-1} > 2^{\left(\frac{\log 3}{\log 4} - 3\varepsilon\right) c_{k+1}} \cdot 3\delta_{k+1} \delta_k^{s-1} =$$

$$= 2^{(s+\varepsilon) c_{k+1}} \cdot 3 \cdot \frac{1}{2^{c_{k+1}+1}} \cdot \frac{1}{2^{(c_k+1)(s-1)}} = 3 \cdot 2^{\varepsilon c_{k+1} + (s-1)(c_{k+1} - c_k) - s}.$$

By the construction of the sequence $\mathfrak{C}$, $c_{k+1} - c_k = o(c_{k+1})$. Thus for $k \geqq k_1(\varepsilon)$,

$$N_{k+1} \varrho_{k+1} \delta_k^{s-1} > 3 \cdot 2^{\varepsilon c_{k+1} - (\varepsilon/2) c_{k+1}} = 3 \cdot 2^{(\varepsilon/2) c_{k+1}}.$$

Hence

$$\liminf_{k \to +\infty} N_{k+1} \varrho_{k+1} \delta_k^{s-1} = +\infty$$

which proves (7).

Summarizing: we have proved that all the conditions in Lemma 4 hold with $P = \varrho(\Psi)$, $s = \dfrac{\log 3}{\log 4} - 4\varepsilon$. Thus applying Lemma 4, we get that

$$\dim \varrho(\Psi) \geqq \frac{\log 3}{\log 4} - 4\varepsilon.$$

This inequality holds for any $\varepsilon > 0$ which implies (16) and thus also (10), and the proof of Theorem 1 is completed.

4. It seems to be hopeless to determine $\dim \varrho(\Phi^{(k)})$ for $k \geqq 3$. Instead, we shall prove the following theorem:

THEOREM 2.

$$\lim_{k \to +\infty} \dim \varrho(\Phi^{(k)}) = \frac{1}{2}. \tag{53}$$

PROOF.

The lower estimate of $\dim \varrho(\Phi^{(k)})$ is near trivial. Namely, let $\mathfrak{M} = \{0, 2, 4, \ldots, 2i, \ldots\}$. Obviously, $\mathfrak{A} \subset \mathfrak{M}$ implies that $\mathfrak{A} \subset \bigcap_{k=2}^{+\infty} \Phi^{(k)}$. In other words, $\langle \mathfrak{M} \rangle \subset \bigcap_{k=2}^{+\infty} \Phi^{(k)}$, thus by Lemma 1,

$$\dim \varrho(\Phi^{(k)}) \geqq \dim \varrho\left(\bigcap_{k=2}^{+\infty} \Phi^{(k)}\right) \geqq \dim \varrho(\langle \mathfrak{M} \rangle) = \tag{54}$$

$$= \liminf_{x \to +\infty} \frac{M(x)}{x} = \frac{1}{2} \quad \text{for} \quad k = 2, 3, \ldots .$$

On the other hand, we are going to show that

$$\limsup_{k \to +\infty} \dim \varrho(\Phi^{(k)}) \leqq \frac{1}{2}. \tag{55}$$

It suffices to show that

$$\limsup_{k \to +\infty} \dim \varrho(\textstyle\sum^{\infty} \cap \Phi^{(k)}) \leqq \frac{1}{2} \tag{56}$$

(since $\dim \varrho(\sum^*) = 0$). In fact, we are going to prove that for fixed $\varepsilon > 0$, $k > k_0(\varepsilon)$ implies that

$$\dim \varrho(\textstyle\sum^{\infty} \cap \Phi^{(k+2)}) \leqq \frac{1}{2} + \varepsilon \quad (k > k_0(\varepsilon)). \tag{57}$$

Let

$$\mathfrak{A} = \{a_1, a_2, \ldots\} \in (\textstyle\sum^{\infty} \cap \Phi^{(k+2)}) \tag{58}$$

and let us put $a = a_2 - a_1$. Then (58) and

$$\begin{aligned}(k+2)\mathfrak{A} &= k\mathfrak{A} + 2\mathfrak{A} \supset k\{a_1, a_2\} + 2\mathfrak{A} = \\ &= k\{a_1\} + k\{0, a_2 - a_1\} + k\mathfrak{A} = k\{a_1\} + k\{0, a\} + k\mathfrak{A} = \\ &= \{ka_1\} + \{0, a, 2a, \ldots, ka\} + 2\mathfrak{A}\end{aligned}$$

imply that there exist infinitely many positive integers j_i $(i = 1, 2, \ldots)$ for which

$$j_i \notin \{0, a, 2a, \ldots, ka\} + 2\mathfrak{A} \quad (i = 1, 2, \ldots). \tag{59}$$

For $a = 1, 2, \ldots$, let $\Theta^{(a)}$ denote the set of those sequences $\mathfrak{A} \in \sum^{\infty}$ for which $a_2 - a_1 = a$ holds and there exist infinitely many positive integers j_i satisfying (59). As we showed,

$$\bigcup_{b=1}^{+\infty} \Theta^{(a)} \supset \sum^{\infty} \cap \Phi^{(k+2)}. \tag{60}$$

Furthermore, for fixed a, j $(a = 1, 2, \ldots;\ j = 1, 2, \ldots)$, let $\Theta^{(a\ j)}$ denote the set of those sequences $\mathfrak{A}$ for which

$$j \notin \{0, a, 2a, \ldots, ka\} + 2\mathfrak{A} \tag{61}$$

holds. Obviously,

$$\Theta^{(a)} \subset \bigcap_{i=0}^{+\infty} \left(\bigcup_{j=i}^{+\infty} \Theta^{(a,\, j)} \right). \tag{62}$$

In order to apply Lemma 3 with $\Theta = \Theta^{(a)}$, $\Theta^{(j)} = \Theta^{(a,\, j)}$, we have to estimate $\varphi(\Theta^{(a\ j)})$ for fixed a and large j. We may suppose that

$$a < \left[\frac{j}{2}\right] \tag{63}$$

and let $\mathfrak{A} \in \Theta^{(a,\, j)}$.

For $r = 0, 1, \ldots, a-1$, let us define the sequence $\mathfrak{A}^{(r)}$ in the following way: $n \in \mathfrak{A}^{(r)}$ should hold if and only if

$$0 \leqq n \leqq \left[\frac{j}{2}\right], \quad n \equiv r \pmod a \quad \text{and} \quad n \in \mathfrak{A}$$

or

$$\left[\frac{j}{2}\right] < n \leqq j, \quad n \equiv j - r \pmod a \quad \text{and} \quad n \in \mathfrak{A}$$

hold. Obviously,

$$\mathfrak{A}^{(p)} \cap \mathfrak{A}^{(q)} = \emptyset \quad \text{for} \quad 0 \leqq p < q \leqq a-1$$

and

$$\bigcup_{r=0}^{a-1} \mathfrak{A}^{(r)} = (\mathfrak{A} \cap \{0, 1, \ldots, j\} =)\, \mathfrak{A}_j. \tag{64}$$

For fixed $\mathfrak{A}^{(r)}$, let us define the sequences

$$\mathfrak{B}^{(r)} = \{b_0^{(r)}, b_1^{(r)}, \ldots, b_s^{(r)}\}, \quad \mathfrak{C}^{(r)} = \{c_0^{(r)}, c_1^{(r)}, \ldots, c_t^{(r)}\}$$

by the following recursion:

Let $b_0^{(r)} = r$ (by (63), $b_0^{(r)} < [j/2]$). Furthermore, let $c_0^{(r)}$ be the smallest positive integer c for which $c > b_0^{(r)}$, $c \in \mathfrak{A}^{(r)}$ and $c \leqq [j/2]$ hold. If such number c does not exist then we end the construction, i.e. let $\mathfrak{B}^{(r)} = \{b_0^{(r)}\}$, $\mathfrak{C}^{(r)} = \emptyset$.

Let us suppose now that

$$b_0^{(r)} < c_0^{(r)} \leqq b_1^{(r)} < c_1^{(r)} \leqq \ldots \leqq b_i^{(r)} < c_i^{(r)} \quad (i \geqq 0)$$

have been defined. Then let $b_{i+1}^{(r)}$ be the smallest positive integer b for which $b \in \mathfrak{A}^{(r)}$, $c_i^{(r)} \leqq b \leqq [j/2] - ka$ and $\mathfrak{A}^{(r)} \cap \{b+a, b+2a, \ldots, b+ka\} = \emptyset$ hold. If such number b does not exist then we end the construction, i.e. let $s = t = i$. If $b_{i+1}^{(r)}$ exists then let $c_{i+1}^{(r)}$ be the smallest positive integer for which $c \in \mathfrak{A}^{(r)}$ and $b_{i+1}^{(r)} < c \leqq [j/2]$ hold. If such number c does not exist then we end the construction, i.e. let $s = i+1$, $t = i$.

Let

$$c_0^{(r)} - b_0^{(r)} = x_0, \quad c_1^{(r)} - b_1^{(r)} = x_1, \ldots, c_t^{(r)} - b_t^{(r)} = x_r;$$

in case $s = t+1$ (including the case $\mathfrak{B}^{(r)} = \{b_0^{(r)}\}$, $\mathfrak{C}^{(r)} = \emptyset$), let also

$$a\left(\left[\frac{[j/2] - b_s^{(r)}}{a}\right] + 1\right) = x_s.$$

Obviously,

$$(65) \qquad x_0 + x_1 + \ldots + x_s \leqq \frac{j}{2} + a, \quad a/x_i \quad \text{for} \quad i = 0, 1, 2, \ldots, s,$$

$$x_0 \geqq a \quad \text{and} \quad x_i \geqq (k+1)a \quad \text{for} \quad i = 0, 1, 2, \ldots, s.$$

Let us put

$$(66) \qquad x_0 + x_1 + \ldots + x_s = aq,$$
$$x_0 = ay_0$$

and

$$x_i - ak = ay_i \quad \text{for} \quad i = 1, 2, \ldots, s.$$

Then

$$(67) \quad y_0 + y_1 + \ldots + y_s = q - sk \quad \text{and} \quad y_i \geqq 1 \quad \text{for} \quad i = 0, 1, 2, \ldots, s.$$

We are going to estimate the number of those sequences $\mathfrak{D} \subset \{0, 1, 2, \ldots, [j/2]\}$ for which $\mathfrak{D} = \mathfrak{A}_{[j/2]}^{(r)}$ holds for some $\mathfrak{A} \in \Theta^{(a,j)}$ and fixed r. To get a such sequence $\mathfrak{D}$, let us fix the numbers $y_0, y_1, \ldots, y_s$ at first. These numbers satisfy the diophantine equation (67), thus for fixed s, the numbers y_i can be chosen in $\binom{q-sk-1}{s}$ ways. The y_i determine also the numbers x_i. In the second step, we fix the sequence $\mathfrak{B}^{(r)}$. $b_0^{(r)} = r$ is fixed. The integers $b \in \mathfrak{B}^{(r)}$, $b \neq b_0^{(r)}$ satisfy

$$b \equiv r \pmod{a}$$

and $r < b < [j/2]$, thus these numbers can be chosen from at most $[j/2a]$ integers, and their number is s. Hence, for fixed s, the sequence $\mathfrak{B}^{(r)}$ can be chosen in at most $\binom{[j/2a]}{s}$ ways. $\mathfrak{B}^{(r)}$ and the numbers x_i determine

also the sequence $\mathfrak{C}^{(r)}$. If $\mathfrak{B}^{(r)}$, $\mathfrak{C}^{(r)}$ and the x_i have been fixed then also the sequences

$$\{b_0^{(r)}+a,\ b_0^{(r)}+2a,\ \ldots,\ c_0^{(r)}\}\cap\mathfrak{A}^{(r)}=\{c_0^{r}\}.$$

$$\{b_i^{(r)},\ b_i^{(r)}+a,\ b_i^{(r)}+2a,\ \ldots,\ c_i^{(r)}\}\cap\mathfrak{A}^{(r)}=\{b_i^{(r)},\ c_i^{(r)}\}\quad(\text{for } i=1,2,\ldots,s-1)$$

and

$$\{b_s^{(r)},\ b_s^{(r)}+a,\ b_s^{(r)}+2a,\ \ldots,\ b_s^{(r)}+(x_s-a)\}\cap\mathfrak{A}^{(r)}=\{b_s^{(r)}\}$$

are fixed. Thus to get $\mathfrak{A}^{(r)}$, we have to choose only from those integers n which satisfy the following conditions:

(i) $n\equiv r \pmod a$;
(ii) $0\leqq n\leqq[j/2]$;
(iii) n is different from each of

$$b_0^{(r)}+a,\ b_0^{(r)}+2a,\ \ldots,\ c_0^{(r)}=b_0^{(r)}+x_0;$$

$$b_i^{(r)},\ b_i^{(r)}+a,\ b_i^{(r)}+2a,\ \ldots,\ c_i^{(r)}=b_i^{(r)}+x_i\ (\text{where } i=1,2,\ldots,s-1);$$

$$b_s^{(r)},\ b_s^{(r)}+a,\ b_s^{(r)}+2a,\ \ldots,\ b_s^{(r)}+(x_s-a).$$

Obviously, (i) and (ii) hold for at most $[j/2a]+1$ integers n, but by (iii),

$$\frac{x_0}{a}+\sum_{i=1}^{s-1}\left(\frac{x_i}{a}+1\right)+\frac{x_s}{a}$$

of these integers must be excluded. Thus for fixed $\mathfrak{B}^{(r)}$, $\mathfrak{C}^{(r)}$, $x_0, x_1, \ldots, x_s$, we may choose only from at most

$$\left[\frac{j}{2a}\right]+1-\frac{x_0}{a}-\sum_{i=1}^{s-1}\left(\frac{x_i}{a}+1\right)-\frac{x_s}{a}\leqq\left[\frac{j}{2a}\right]+1-\frac{1}{a}\sum_{i=0}^{s}x_i=\left[\frac{j}{2a}\right]+1-q$$

integers (with respect to (66)), i.e. in at most $2^{[j/2a]+1-q}$ ways.

Summarizing: for fixed q, s, $\mathfrak{A}^{(r)}_{[j/2]}$ can be chosen in at most

$$\binom{q-sk-1}{s}\binom{[j/2a]}{s}2^{[j/2a]+1-q}$$

ways.

Now we are going to investigate

$$\mathfrak{A}^{(r)}\cap\{[j/2]+1,\ [j/2]+2,\ \ldots,\ j\} \tag{68}$$

(for $\mathfrak{A}\in\Theta^{(a,j)}$). We will show that

$$j-b_{i+1}^{(r)}\leqq n\leqq j-c_i^{(r)}\quad(\text{where}\quad i=0,1,\ldots,s-1) \tag{69}$$

and case $s=t$,

$$j-\left[\frac{j}{2}\right]<n\leqq j-c_t^{(r)} \tag{70}$$

imply that $n\notin\mathfrak{A}^{(r)}$.

By the construction of the sequence $\mathfrak{A}^{(r)}$, it suffices to investigate those integers n which satisfy

$$n \equiv j-r \pmod{a}. \tag{71}$$

(69) can be rewritten in form

$$c_i^{(r)} \leqq j-n \leqq b_{i+1}^{(r)}.$$

Thus the constructions of the sequences $\mathfrak{B}^{(r)}$ and $\mathfrak{C}^{(r)}$ imply that there exists an integer a_x for which $a_x \in \mathfrak{A}^{(r)}$,

$$a_x \equiv r \pmod{a} \tag{72}$$

and

$$j-n-ka \leqq a_x \leqq j-n$$

hold. The last inequality can be rewritten in the form

$$0 \leqq j-n-a_x \leqq ka. \tag{73}$$

Furthermore, we get from (71) and (72) that

$$j-n-a_x \equiv j-(j-r)-r \equiv 0 \pmod{a}. \tag{74}$$

By (73) and (74), $n \in \mathfrak{A}^{(r)}$ would imply that

$$j = (j-n-a_x)+n+a_x \in \{0, a, 2a, \ldots, ka\}+2\mathfrak{A}$$

which would contradict (60).

It can be shown in the same way that also (70) implies that $n \notin \mathfrak{A}^{(r)}$.

Thus to get the sequence (68), we must not choose integers from the intervals (69) and (70). Also we have to drop those integers which do not satisfy (71). The remaining integers are:

$$j-r = j-b_0^{(r)},\ j-b_0^{(r)}-a,\ j-b_0^{(r)}-2a, \ldots, j-c_0^{(r)}+a = j-b_0^{(r)}-(x_0-a),$$

$$j-b_i^{(r)}-a,\ j-b_i^{(r)}-2a, \ldots, j-c_i^{(r)}+a = j-b_i^{(r)}-(x_i-a)$$

$$\text{where } i = 1, 2, \ldots, s-1;$$

$$j-b_s^{(r)}-a,\ j-b_s^{(r)}-2a, \ldots, j-b_s^{(r)}-x_s.$$

The number of these integers is at most

$$\frac{x_0}{a}+\sum_{i=1}^{s-1}\frac{x_i-a}{a}+\frac{x_s}{a} \leqq \frac{1}{a}\sum_{i=0}^{s} x_i = q.$$

thus the sequence (68) can be chosen in at most 2^q ways (for fixed $\mathfrak{A}_{[j/2]}^{(r)}$).

Summarizing the estimates above, we get that for fixed r, s, q, the sequence $\mathfrak{A}^{(r)}$ can be chosen in at most

$$\binom{q-sk-1}{s}\binom{[j/2a]}{s}2^{[j/2a]+1-q}\cdot 2^q = \binom{q-sk-1}{s}\binom{[j/2a]}{s}2^{[j/2a]+1}$$

ways. By (65), (66) and (67), s and q must satisfy

$$q \leqq \left[\frac{j}{2x}+1\right] = \left[\frac{j}{2x}\right]+1$$

and

$$s \leqq \left[\frac{q}{k}\right] \leqq \left[\frac{[j/2a]+1}{k}\right] \leqq \left[\frac{j}{2ak}\right]+1 .$$

Thus for $k \geqq 3$, large j and fixed r, $\mathfrak{A}^{(r)}$ can be chosen in at most

$$\sum_{q=0}^{[j/2a]+1} \sum_{s=0}^{[j/2ak]+1} \binom{q-sk-1}{s} \binom{[j/2a]}{s} 2^{[j/2a]+1} <$$

$$< \sum_{s=0}^{[j/2a]+1} \sum_{s=0}^{[j/2ak]+1} \binom{([j/2a]+1)-1}{[j/2ak]+1} \binom{[j/2a]}{[j/2ak]+1} 2^{\frac{j}{2a}+1} \leqq$$

$$\leqq j^2 \binom{[j/2a]}{[j/2ak]+1}^2 2^{\frac{j}{2a}+1} \quad (k \geqq 3, \quad j > j_0(a))$$

ways.

Thus by (64), the sequence $\mathfrak{A}_j$ can be chosen in

$$\varphi(\Theta^{(a,j)}) < \left\{ j^2 \binom{[j/2a]}{[j/2ak]+1}^2 2^{\frac{j}{2a}+1} \right\}^a = j^{2a} \binom{[j/2a]}{[j/2ak]+1}^{2a} 2^{\frac{j}{2}+a} \tag{75}$$

ways (for $k \geqq 3$, $j > j_0(a)$). Fixing some $\varepsilon > 0$ and applying Lemma 2, we get for $k \geqq k_0(\varepsilon)$, $j > j_1(a, \varepsilon)$ that

$$\varphi(\Theta^{(a,j)}) < 2^{\frac{\varepsilon}{3} j} \left(2^{\frac{\varepsilon}{3} \cdot \frac{j}{2a}}\right)^{2a} 2^{\frac{j}{2}+\frac{\varepsilon}{3} j} = 2^{\left(\frac{1}{2}+\varepsilon\right)} \tag{76}$$

(for $k \geqq k_0(\varepsilon)$, $j > j_1(a, \varepsilon)$) since $\lim\limits_{x \to 0+} d(x) = 0$.

Applying Lemma 3, we get from (62) and (76) that for $k \geqq k_0(\varepsilon)$ and $a = 1, 2, \ldots,$

$$\dim \varrho(\Theta^{(a)}) \leqq \frac{1}{2} + \varepsilon \quad (k \geqq k_0(\varepsilon), \quad a = 1, 2, \ldots).$$

By (60), this implies (57) and thus also (56) and (55) which completes the proof of Theorem 2.

We remark that for $k \geqq 3$, also the estimate

$$\dim \varrho(\Phi^{(k+2)}) \leqq \frac{1}{2} + d\left(\frac{1}{k}\right)$$

can be deduced from (75).

Furthermore, our proof gives also the following theorem:

THEOREM 3.

$$\dim \varrho \left(\bigcap_{k=2}^{+\infty} \Phi^{(k)} \right) = \frac{1}{2}.$$

PROOF.

Theorem 2 implies that

$$\dim \varrho \left(\bigcap_{k=2}^{+\infty} \Phi^{(k)} \right) \leqq \frac{1}{2},$$

while

$$\dim \varrho \left(\bigcap_{k=2}^{+\infty} \Phi^{(k)} \right) \geqq \frac{1}{2}$$

holds by (54).

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ÜBER DAS MINIMUM DER ν-POTENZSUMME DER SEITEN SOLCHER n-ECKE, DIE IN EINEN GEGEBENEN KREIS EINGESCHRIEBEN SIND, II.

Von

ÁGOTA TEMESVÁRI

Lehrstuhl für Darstellende Geometrie der Eötvös Loránd Universität, Budapest

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Schreiben wir in einen gegebenen Kreis ein solches konvexes n-Eck ein, für das der Mittelpunkt des Kreises ein innerer Punkt oder ein Grenzpunkt des n-Eckes ist. Betrachten wir die Quadratsumme der Seiten dieses n-Eckes. P. SZÁSZ bewies [1], daß das Minimum dieser Quadratsumme

im Falle $n \geq 7$ auftritt, wenn das n-Eck regulär ist;
im Falle $6 \geq n \geq 4$ auftritt, wenn eine Seite des n-Eckes der Durchmesser des Kreises ist und die anderen Seiten miteinander gleich sind. (Dieses n-Eck nennen wir Typ F_n.)

Wenn $n = 3$ ist, dann geben die rechtwinkligen Dreiecke das Minimum.

Betrachten wir statt der Quadratsumme die ν-Potenzsumme der Seiten, wo $\nu > 1$ eine reelle Zahl ist. Da gilt der folgende Satz:

SATZ. *Es seien $n \geq 3$ eine ganze, $\nu > 1$ eine reelle Zahl, sowie $\nu_1(n) > 1$ und $\nu_0(n) > 1$ solche reelle Zahlen, für die die folgenden Gleichheiten*

$$n \cdot \sin^{\nu_1(n)} \frac{\pi}{n} = 1 + (n-1) \sin^{\nu_1(n)} \frac{\pi}{2(n-1)} \tag{1}$$

und

$$1 = (n-1) \sin^{\nu_0(n)} \frac{\pi}{2(n-1)} \tag{2}$$

gelten.

Die ν-Potenzsumme $S_\nu(n)$ der Seiten solcher konvexen n-Ecke, die in einen gegebenen Kreis so eingeschrieben sind, daß der Mittelpunkt des Kreises ein innerer Punkt oder ein Grenzpunkt der n-Ecke ist, wird minimal

im Falle $\nu \geq \nu_1(n)$, wenn das n-Eck regulär ist;
im Falle $\nu_1(n) \geq \nu \geq \nu_0(n)$, wenn das n-Eck Typ F_n ist.
Wenn $\nu_0(n) > \nu > 1$ ist, dann existiert $\min S_\nu(n)$ *nicht und* $\inf S_\nu(n) = 2^{\nu+1}$ *ist, wenn der Radius des gegebenen Kreises eine Einheit ist.*

Im [2] wurde der Satz im Falle $\nu > 2$ bewiesen und dort kann man zwei Hilfssätze finden, die wir später verwenden werden. Man kann leicht einsehen, daß der erste Hilfssatz auch im Falle $\nu > 1$ gültig ist. Im [2] führten wir die folgenden Bezeichnungen ein. A und B seien zwei Punkte eines Kreises mit dem Mittelpunkt O, dessen Radius eine Einheit ist, sowie F der Halbierungspunkt des Bogens $\widehat{AB}$, 2α der Winkel AOF und C solcher Punkt des Bogens $\widehat{AFB}$, für den die folgenden Ungleichungen gelten:

$$\overline{AC} \geq \overline{CB}, \quad \sphericalangle AOC \leq \pi \quad \text{und} \quad \sphericalangle AOC \leq 4\alpha .$$

Wir bezeichnen $\sphericalangle FOC$ mit 2δ. Da wird die ν-Potenzsumme der Strecken $\overline{AC}$ und $\overline{CB}$

$$H_\alpha(\delta) = 2^\nu[\sin^\nu(\alpha+\delta) + \sin^\nu(\alpha-\delta)] ,$$

wo

$$0 \leq \delta \leq \alpha \quad \text{und} \quad \alpha \leq \alpha + \delta \leq \frac{\pi}{2}$$

sind.

Wir bezeichnen denjenigen Winkel mit $i(\nu)$, wo die Funktion $y = \sin^\nu x$ einen Wendepunkt hat, wo $x \in \left[0; \frac{\pi}{2}\right]$ ist.

Zum Beweis des Satzes sind noch weitere Hilfssätze notwendig.

HILFSSATZ III. Es sei $1 < \nu < 2$ eine reelle Zahl.
Wenn

1. $\frac{\pi}{4} \leq \alpha < \frac{\pi}{2}$ ist, dann ist $H_\alpha(\delta)$ abnehmend;
2. $i(\nu) < \alpha < \frac{\pi}{4}$ ist, dann hat $H_\alpha(\delta)$ ein Minimum im Inneren des untersuchten Intervalls;
3. $0 < \alpha \leq i(\nu)$ ist, dann ist $H_\alpha(\delta)$ zunehmend.

BEWEIS. Weil $1 < \nu < 2$ eine fixe reelle Zahl ist, ist es genug die Funktion

$$h_\alpha(\delta) = \sin^\nu(\alpha+\delta) + \sin^\nu(\alpha-\delta)$$

statt der Funktion $H_\alpha(\delta)$ zu untersuchen.

1. Es sei $\frac{\pi}{4} \leq \alpha < \frac{\pi}{2}$. Wenn $\alpha = \frac{\pi}{4}$ ist, dann ist

$$h_{\frac{\pi}{4}}(\delta) = \sin^\nu\left(\frac{\pi}{4} + \delta\right) + \cos^\nu\left(\frac{\pi}{4} + \delta\right),$$

deren erste Ableitung

$$h'_{\frac{\pi}{4}}(\delta) = \nu\left[\sin^{\nu-1}\left(\frac{\pi}{4}+\delta\right)\cdot\cos\left(\frac{\pi}{4}+\delta\right)-\cos^{\nu-1}\left(\frac{\pi}{4}+\delta\right)\sin\left(\frac{\pi}{4}+\delta\right)\right]$$

ist. Die Funktion $h'_{\frac{\pi}{4}}(\delta)$ hat dasselbe Vorzeichen im Falle $\delta \neq \frac{\pi}{4}$ wie die Funktion

$$y_{\frac{\pi}{4}}(\delta) = \operatorname{tg}^{\nu-1}\left(\frac{\pi}{4}+\delta\right)-\operatorname{tg}\left(\frac{\pi}{4}+\delta\right).$$

Weil $1<\nu<2$ und $\delta\geq 0$ sind, ist $y'_{\frac{\pi}{4}}(\delta)<0$, also ist $h_{\frac{\pi}{4}}(\delta)$ abnehmend.

Aus den vorangehenden und aus dem ersten Hilfssatz folgt die erste Behauptung.

2. Es seien $i(\nu)<\alpha<\frac{\pi}{4}$, sowie

$$y_1(\delta) = \sin^{\nu-1}(\alpha+\delta)\cdot\cos(\alpha+\delta)$$

und

$$y_2(\delta) = \sin^{\nu-1}(\alpha-\delta)\cdot\cos(\alpha-\delta).$$

Im folgenden erweisen sich die Eigenschaften der Funktionen $y_1(\delta)$ und $y_2(\delta)$ als notwendig. Die erste Ableitung der Funktion $y_1(\delta)$ ist

$$y'_1(\delta) = \sin^{\nu-2}(\alpha+\delta)\,[\nu\cdot\cos^2(\alpha+\delta)-1].$$

Weil $\alpha+\delta>i(\nu)$ ist, ist $\cos^2(\alpha+\delta)<\frac{1}{\nu}$, $y_1(\delta)$ ist also abnehmend. Die erste Ableitung der Funktion $y_2(\delta)$ ist

$$y'_2(\delta) = \sin^{\nu-2}(\alpha-\delta)\,[-\nu\cdot\cos^2(\alpha-\delta)+1].$$

δ_s sei ein solcher Winkel, für den die Gleichung $\cos^2(\alpha-\delta_s) = \frac{1}{\nu}$ gilt. Im Falle $\delta = \delta_s$ hat die Funktion $y_2(\delta)$ ein Maximum. Wenn $0\leq\delta\leq\delta_s$ ist, dann ist $y_2(\delta)$ zunehmend. Weil $y_1(0) = y_2(0)$ ist, bekommen wir auf Grund der vorangehenden, daß

$$h'_\alpha(\delta)<0 \quad \text{im Falle} \quad 0\leq\delta\leq\delta_s \tag{3}$$

ist.

Mit der Anwendung der folgenden Identität

$$\sin(\alpha+\delta) = \sin 2\alpha\cdot\cos(\alpha-\delta)-\cos 2\alpha\cdot\sin(\alpha-\delta)$$

und

$$\cos(\alpha+\delta) = \cos 2\alpha\cdot\cos(\alpha-\delta)+\sin 2\alpha\cdot\sin(\alpha-\delta)$$

nimmt die erste Ableitung der Funktion $h_\alpha(\delta)$ folgende Form auf:

$$h'_\alpha(\delta) = \nu\cdot\sin^{\nu-1}(\alpha-\delta)\cdot\cos(\alpha-\delta)\,[f(\delta)-1], \tag{4}$$

wo

(5) $f(\delta) = [\sin 2\alpha \cdot \operatorname{ctg}(\alpha-\delta) - \cos 2\alpha]^{\nu-1} [\cos 2\alpha + \sin 2\alpha \cdot \operatorname{tg}(\alpha-\delta)]$

ist. Weil $\nu \cdot \sin^{\nu-1}(\alpha-\delta) \cdot \cos(\alpha-\delta) > 0$ ist, so werden wir das Vorzeichen der Funktion $f(\delta)$ statt des Vorzeichens der Funktion $h'_\alpha(\delta)$ untersuchen. Wenn wir die Bezeichnungen

$$\sin 2\alpha = a, \quad \cos 2\alpha = b \quad \text{und} \quad \operatorname{ctg}(\alpha-\delta) = x$$

einführen, dann wird die erste Ableitung der Funktion $f(\delta)$:

$$f'(\delta) = (\nu-1)(ax-b)^{\nu-2} \cdot a \cdot x' \left(b + a \cdot \frac{1}{x}\right) - (ax-b)^{\nu-1} \cdot a \cdot \frac{1}{x^2} \cdot x',$$

wo die Ungleichungen

$$x' = \frac{1}{\sin^2(\alpha-\delta)} > 0, \quad a > 0, \quad b > 0, \quad x > 0$$

und

$$ax - b = \sin 2\alpha \cdot \operatorname{ctg}(\alpha-\delta) - \cos 2\alpha = \sin 2\alpha [\operatorname{ctg}(\alpha-\delta) - \operatorname{ctg} 2\alpha] > 0$$

gelten. So zeigt die folgende quadratische Funktion das Vorzeichen der Funktion $f'(\delta)$:

(6) $$f_1(\delta) = (\nu-1) \cdot b \cdot x^2 + (\nu-2) \cdot a \cdot x + b.$$

So hat die Funktion $f(\delta)$ höchstens zwei Extremwerte. Aus (3) und daraus, daß $f(0) = 1$ und $\lim\limits_{\delta \to \alpha} f(\delta) = +\infty$ sind, folgt, daß die Funktion $h'_\alpha(\delta)$ eine einzige Nullstelle hat.

Auf Grund der oben Erwähnten gilt auch die Behauptung 2.

3. Es sei $0 < \alpha \leq i(\nu)$.

Wenn $\alpha = i(\nu)$ ist, so kann man ähnlich der Behauptung 2. beweisen, daß $\delta_0 = 0$ eine einzige Nullstelle der Funktion (6) ist und wenn $\delta_0 < \delta$ ist, dann ist die Funktion (6) zunehmend, also $h'_\alpha(\delta) \geq 0$, so ist $h_\alpha(\delta)$ zunehmend. Aus diesem und aus dem Hilfsatz I. folgt die Behauptung 3.

Hilfssatz IV. Betrachten wir ein solches konvexes n-Eck ($n \geq 4$), das in einen gegebenen Kreis mit Radius eins eingeschrieben ist und dessen eine Seite der Durchmesser des gegebenen Kreises ist und aus den weiteren $(n-1)$ Seiten die $(n-2)$ Seiten einander gleich sind. Wir bezeichnen jene Zentriwinkel, die zu den gleichen Seiten gehören, mit 2β, und jenen Zentriwinkel, der zu der Seite gehört, die dem Durchmesser nicht gleich ist, bezeichnen wir mit 2α.

Wenn $\alpha \geq i(\nu)$, $\beta \leq i(\nu)$, $1 < \nu < 2$ eine reelle Zahl und

$$i(\nu) \leq \frac{\alpha+\beta}{2} < \frac{\pi}{4}$$

sind, dann gibt das n-Eck Typ F_n das Minimum der Funktion $S_\nu(n)$, oder existiert das Minimum nicht und in diesem Fall ist $\inf S_\nu(n) = 2^{\nu+1}$.

BEWEIS. Für das obige n-Eck ist

$$S_\nu(n) = 2^\nu\,[1+\sin^\nu\alpha+(n-2)\sin^\nu\beta],$$

wo

$$(n-2)\beta+\alpha = \frac{\pi}{2}$$

ist. Aus den Bedingungen $i(\nu)\leq\frac{\alpha+\beta}{2}$ bzw. $\alpha\geq\beta$ geht hervor, daß

$$\beta\leq\frac{\frac{\pi}{2}-2\cdot i(\nu)}{n-3}\quad\text{bzw.}\quad\beta\leq\frac{\pi}{2(n-1)}$$

sind.

Es ist genug, statt des Minimums der Funktion $S_\nu(n)$ das Minimum der Funktion

$$g(\beta) = 1+\sin^\nu\alpha+(n-2)\sin^\nu\beta$$

zu bestimmen, deren erste Ableitung

$$g'(\beta) = \nu(n-2)\,[\sin^{\nu-1}\beta\cdot\cos\beta-\cos^{\nu-1}(n-2)\beta\cdot\sin(n-2)\beta]$$

ist. Das Vorzeichen der Funktion $g'(\beta)$ ist dasselbe, wie das Vorzeichen der Funktion

$$g_1(\beta) = 1-F(\beta),$$

wo

$$F(\beta) = \frac{\cos^{\nu-1}(n-2)\beta\cdot\sin(n-2)\beta}{\sin^{\nu-1}\beta\cdot\cos\beta}$$

und $\beta\neq 0$ sind. Das Vorzeichen der ersten Ableitung von $F(\beta)$ ist dasselbe, wie das Vorzeichen der Funktion

$$F_1(\beta) = (n-2)\sin 2\beta\,[-\nu\cdot\sin^2(n\quad 2)\beta+1]-$$
$$-\sin 2(n-2)\beta[\nu\cdot\cos^2\beta-1].$$

Es ist offenbar, daß die Ungleichung

$$(n-2)\sin 2\beta-\sin 2(n-2)\beta>0 \tag{7}$$

gilt. Die erste Ableitung der Funktion

$$k(\beta) = -\nu\cdot\sin^2(n-2)\beta+1-[\nu\cdot\cos^2\beta-1] \tag{8}$$

ist

$$k'(\beta) = -(n-2)\cdot\nu\cdot\sin 2(n-2)\beta+\nu\cdot\sin 2\beta.$$

Wenn

$$0\leq\beta\leq\frac{\pi}{2(n-1)}$$

ist, dann gilt die Ungleichung

$$\sin 2\beta \leq \sin 2(n-2)\beta$$

und die Gleichung gilt nur in den Fällen $\beta = 0$ und $\beta = \frac{\pi}{2(n-1)}$. So ist $k'(\beta) < 0$, wenn $\beta \neq 0$ ist, die Funktion $k(\beta)$ ist also abnehmend. Betrachten wir das Vorzeichen der Funktion $k(\beta)$ im Endpunkt. Wenn

$$i(\nu) \leq \frac{\pi}{2(n-1)}$$

ist, dann ist leicht einzusehen, daß

$$k\left(\frac{\pi}{2(n-1)}\right) \geq 0$$

ist. Wenn

$$i(\nu) > \frac{\pi}{2(n-1)}$$

ist, dann gilt die Ungleichung

$$\frac{\pi}{2(n-1)} > \frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}$$

so ist der Funktionswert im Endpunkt

$$k\left(\frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}\right) = -\nu \cos^2\left(2 \cdot i(\nu) - \frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}\right) +$$

$$+ 1 - \nu \cdot \cos^2\left(\frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}\right) + 1 .$$

Weil $i(\nu) < \frac{\pi}{4}$ ist, folgt aus den Eigenschaften der Funktion $\cos^2 x$ $\left(x \in \left[0; \frac{\pi}{2}\right]\right)$, daß

$$\frac{\cos^2\left(2 \cdot i(\nu) - \frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}\right) + \cos^2\left(\frac{\frac{\pi}{2} - 2 \cdot i(\nu)}{n-3}\right)}{2} < \cos^2 i(\nu) = \frac{1}{\nu}$$

ist, also

$$k\left(\frac{\frac{\pi}{2}-2\,i(\nu)}{n-3}\right)>0.$$

Weil $k(\beta)$ abnehmend ist, ist $k(\beta)\geq 0$, also $F_1(\beta)>0$ auf Grund (7), so ist $F(\beta)$ zunehmend. Es gilt auch, daß $\lim\limits_{\beta\to 0} F(\beta)=0$ ist. Aus den obigen kommt, daß die Funktion $g(\beta)$ eine einzige Maximumstelle hat, $g(\beta)$ hat also ein Minimum entweder im Falle $\beta=0$ oder im Falle

$$\beta=\frac{\frac{\pi}{2}-2\cdot i(\nu)}{n-3}.$$

Wenn

$$i(\nu)<\frac{\pi}{2(n-1)}$$

ist, hat $g(\beta)$ ein Minimum im Falle $\beta=0$.

Wenn $\beta=0$ ist, dann gibt es kein Minimum, wenn

$$\beta=\frac{\frac{\pi}{2}-2i(\nu)}{n-3}$$

ist, dann ist

$$\frac{\alpha+\beta}{2}=i(\nu),$$

also man kann den Hilfssatz III./3. anwenden und so bekommt man das n-Eck Typ F_n.

Hilfssatz V. Wir schreiben in einen gegebenen Kreis mit Radius 1 ein solches konvexes n-Eck ($n\geq 4$), das den Mittelpunkt des Kreises enthält und dessen $(n-1)$ Seiten einander gleich sind. Wir bezeichnen jene Zentriwinkel, die zu den gleichen Seiten gehören mit 2β und jenen Zentriwinkel mit 2α, der zu der n-ten Seite gehört. Wenn $\alpha\geq i(\nu)$, $\beta\leq i(\nu)$, $1<\nu<2$ eine reelle Zahl und $i(\nu)\leq\frac{\alpha+\beta}{2}\leq\frac{\pi}{4}$ sind, dann gibt das reguläre n-Eck das Minimum, oder tritt $\inf S_\nu(n)$ beim $(n-1)$-Eck Typ F_{n-1} auf.

Den fünften Hilfssatz kann man ebenso beweisen wie den vierten Hilfssatz, deshalb gehen wir darauf nicht näher ein.

Hilfssatz VI. Wenn die Potenzsumme $S_\nu(n)$ bei einem Exponenten $1<\nu(n)$ für das reguläre n-Eck minimal ist, dann

1. gibt das reguläre n-Eck das Minimum auch bei jedem $\nu(n)<\nu$, wenn n eine fixe natürliche Zahl ist;

2. ist die Summe $S_\nu(N)$ für das reguläre N-Eck kleiner als die für das N-Eck Typ F_N, wenn $n \leq N$ ist und der Exponent $\nu(n) \leq \nu$ eine fixe reelle Zahl ist.

BEWEIS. 1. Es seien die Seiten des n-Eckes $a_1, \ldots, a_n$. Die ν-te Potenzmitte der Seiten ist

$$\left(\frac{\sum_1^n a_i^\nu}{n} \right)^{\frac{1}{\nu}}.$$

Auf Grund der Bedingung ist

$$b \leq \left(\frac{\sum_1^n a_i^{\nu(n)}}{n} \right)^{\frac{1}{\nu(n)}},$$

wo b die Seite des in den Kreis eingeschriebenen regulären n-Eck ist. Die Gleichheit tritt nur im Falle $a_1 = \ldots = a_n = b$ auf. Weil die Potenzmitte zunehmend ist, wenn ν zunimmt, ist

$$\left(\frac{\sum_1^n a_i^{\nu(n)}}{n} \right)^{\frac{1}{\nu(n)}} \leq \left(\frac{\sum_1^n a_i^\nu}{n} \right)^{\frac{1}{\nu}},$$

wo $\nu(n) < \nu$ ist und die Gleichheit ist dann und nur dann, wenn $a_1 = \ldots = a_n$ ist. So gilt die Behauptung 1.

2. Wenn $i(\nu(n)) < \frac{\pi}{n}$ ist, dann kann das reguläre n-Eck das Minumum offenbar nicht geben, also ist $i(\nu(n)) \geq \frac{\pi}{n}$. Die erste Ableitung der Funktion $l(z) = z \cdot \sin \frac{\pi}{z}$ $(\nu > 1,\ z \geq 4)$ ist

$$l'(z) = \sin^{\nu-1} \frac{\pi}{z} \cdot \cos \frac{\pi}{z} \left(\operatorname{tg} \frac{\pi}{z} - \frac{\nu\pi}{z} \right). \tag{9}$$

Weil

$$i(\nu) \geq \frac{\pi}{z},$$

also

$$\nu \geq \frac{1}{\cos^2 \frac{\pi}{z}}$$

ist, so ist

$$l'(z) \leq \frac{1}{2} \sin^{\nu-1} \frac{\pi}{z} \cdot \frac{1}{\cos \frac{\pi}{z}} \left(\sin \frac{2\pi}{z} - \frac{2\pi}{z} \right).$$

Die Ungleichung

$$\sin \frac{2\pi}{z} - \frac{2\pi}{z} < 0$$

gilt, das heißt $l'(z) < 0$, so ist $l(z)$ abnehmend. Die zweite Ableitung von $l(z)$ ist

$$l''(z) = \frac{\nu\pi^2}{z^3} \sin^{\nu-2} \frac{\pi}{z} \left(\nu \cos^2 \frac{\pi}{z} - 1 \right),$$

die wegen $i(\nu) \geq \frac{\pi}{z}$ positiv ist, also $l(z)$ ist eine konvexe Funktion. Zum Beweis der zweiten Behauptung werden wir zeigen, daß die Funktion

$$m(x) = x \cdot \sin^{\nu(n)} \frac{\pi}{x} = \left[(x-1) \sin^{\nu(n)} \frac{\pi}{2(x-1)} + 1 \right]$$

im Falle $x \geq n$ abnehmend ist. Die erste Ableitung von $m(x)$ ist

$$m'(x) = l'(x) - l'(2)(x-1).$$

Weil $l(z)$ abnehmend und konvex ist, ist $m'(x) < 0$.

In den Fällen $\nu(n) < \nu$ ist die zweite Behauptung die Konsequenz der ersten.

Hilfssatz VII. Wenn die Potenzsumme $S_\nu(n)$ bei einem Exponenten $1 < \nu(n)$ für das n-Eck Typ F_n kleiner als $2^{\nu(n)+1}$ ist (der Radius des gegebenen Kreises ist die Einheit), dann ist

1. $S_\nu(n) < 2^{\nu+1}$ bei jedem Exponenten $\nu(n) < \nu$ und bei jedem n-Eck;
2. $S_\nu(n) < 2^{\nu(n)+1}$ bei jedem N-Eck, wo $n \leq N$ ist, und bei jedem Exponenten $\nu(n) \leq \nu$.

Beweis. Die Richtigkeit von 1. folgt auch daraus, daß die Potenzmitte zunehmend ist.

2. Aus (9) geht hervor, daß die Funktion $l(z) = z \cdot \sin^\nu \frac{\pi}{z}$ eine einzige Maximumstelle im Falle $z \geq 4$ hat. Wenn $\nu \geq 2$ ist, dann ist $l(z)$ abnehmend, wenn $1 < \nu < 2$ gilt, dann ist $l(4) > 2$. Daraus folgt: wenn $S_\nu(n)$ bei dem n-Eck Typ F_n kleiner als $2^{\nu(n)+1}$ ist, das heißt

$$2 > 2(n-1) \sin^{\nu(n)} \frac{\pi}{2(n-1)}$$

ist, dann ist die Funktion $l(z)$ an der Stelle $z = 2(n-1)$ schon sicher abnehmend, also gilt die Behauptung 2.

DER BEWEIS DES SATZES. Bezeichnen wir die Eckpunkte des n-Eckes mit $A_1, \ldots, A_n$ und mit $2\alpha_i$ jene Zentriwinkel, der von der Seite $A_i A_{i+1}$ bestimmt wird. Man kann voraussetzen, daß $\alpha_i \leq \alpha_{i+1}$ ist und der Radius des Kreises die Einheit ist. Da ist

$$S_\nu(n) = \sum_1^n \overline{A_i A_{i+1}}^\nu = 2^\nu \sum_1^n \sin^\nu \alpha_i \quad (A_{n+1} \equiv A_n), \tag{10}$$

wo $\sum_1^n \alpha_i = \pi$ und $0 < \alpha_i \leq \frac{\pi}{2}$ sind.

Man kann die Summe $S_\nu(n)$ folgenderweise vermindern. Wenn es im Intervall $\left(i(\nu); \frac{\pi}{2}\right)$ mindestens zwei Winkel, α_i und α_j gibt, dann werden diese zwei Winkel so verändert, daß ihre Summe konstant und eine der Winkel entweder $i(\nu)$ oder $\frac{\pi}{2}$ sei. Wenn wir diese Methode wiederholen, bleibt höchstens ein Winkel α im Intervall $\left(i(\nu); \frac{\pi}{2}\right)$. Auf Grund der Jensenschen Ungleichung kann die Potenzsumme weiter vermindert werden, wenn die Winkel im Intervall $(0; i(\nu)]$ durch ihre aritmetische Mitte (β) ersetzt werden. So bekommt man folgende Fälle:

1. Wenn im Intervall $\left(i(\nu); \frac{\pi}{2}\right)$ kein Winkel ist, dann bekommt man entweder das reguläre n-Eck oder das n-Eck Typ F_n.

2. Ein Winkel ist im Intervall $\left(i(\nu); \frac{\pi}{2}\right)$, ein Winkel ist gleich $\frac{\pi}{2}$, die anderen Winkel sind einander gleich, das heißt

$$S_\nu(n) = 2^\nu [1 + \sin^\nu \alpha + (n-2) \sin^\nu \beta].$$

3. Ein Winkel ist im Intervall $\left(i(\nu); \frac{\pi}{2}\right)$ und die anderen Winkel sind einander gleich, so ist

$$S_\nu(n) = 2^\nu [\sin^\nu \alpha + (n-1) \sin^\nu \beta].$$

Im Falle $\nu \geq 2$ gibt das Minimum entweder das reguläre n-Eck oder das n-Eck Typ F_n nach [2].

Es sei $1 < \nu < 2$.

Im Falle $n = 3$ kann man auf Grund des Hilfssatzes III. leicht beweisen, daß $S_\nu(3)$ kein Minimum hat. Es sei im folgenden $n \geq 4$.

Im Fälle 2., wenn $\frac{\alpha+\beta}{2} < i(\nu)$ ist, dann kann man den Hilfssatz III./3. anwenden und man bekommt den Typ F_n, wenn

$$i(\nu) \leq \frac{\alpha+\beta}{2} < \frac{\pi}{4}$$

ist, dann gibt sich entweder der Typ F_n, oder daß $\inf S_\nu(n) = 2^{\nu+1}$ ist, auf Grund des Hilfssatzes IV.

3. Wenn $\frac{\alpha+\beta}{2} < i(\nu)$ ist, dann kann man den Hilfssatz III./3. anwenden und man bekommt das reguläre n-Eck. Wenn $i(\nu) \leq \frac{\alpha+\beta}{2} \leq \frac{\pi}{4}$ ist, dann gibt sich entweder das reguläre n-Eck oder, daß das $(n-1)$-Eck Typ F_{n-1} das Infinum von $S_\nu(n)$ gibt, auf Grund des Hilfssatzes V.

Aus dem Hilfssatz VI. folgt: wenn die Gleichheit (1) besteht, dann wird das reguläre n-Eck das minimale n-Eck im Falle $\nu \geq \nu_1(n)$. Wenn $\nu = \nu_0(n)$ ist, das heißt die Gleichheit (2) gilt, dann kann man aus dem Beweis des Hilfssatzes VII. auch das sehen, daß $\inf S_{\nu_0}(n) = 2^{\nu_0(n)+1}$ bei jedem $n_1 < n$ und bei jedem fixen Exponenten $\nu_0(n)$ kleiner als die Potenzsumme für den Typ F_n ist, also der Typ F_{n-1} kann bei dem Exponenten $\nu_0(n)$ nicht minimal sein. Wenn $1 < \nu < \nu_0(n)$ ist, dann werden die Werte $2(n-1) \cdot \sin^\nu \frac{\pi}{2(n-1)}$ größer als im Falle $\nu = \nu_0(n)$, also kann weder der Typ F_n noch der Typ F_{n-1} das Minimum geben. In diesem Fall kann das reguläre n-Eck das Minimum nicht geben $(n < 2(n-1))$. Also das Minimum von $S_\nu(n)$ existiert im Falle $1 < \nu < \nu_0(n)$ nicht und ist $\inf S_\nu(n) = 2^{\nu+1}$.

Aus dem Hilfssatz VII. folgt auch, daß das reguläre n-Eck und das vom Typ F_n verglichen werden muß, wenn $\nu_0(n) \leq \nu \leq \nu_1(n)$ ist. Wenn bei jedem solchen Exponenten ν $i(\nu) < \frac{\pi}{n}$ ist, wird das Minimum nur der Typ F_n offenbar geben. Es sei $\nu_0(n) < \nu_k < \nu_1(n)$, wo auf ν_k gilt, daß $i(\nu) \geq \frac{\pi}{n}$ und $i(\nu_k) = \frac{\pi}{n}$ im Falle $\nu_k \leq \nu$ sind. Wenn $\nu_0(n) \leq \nu < \nu_k$ gilt, dann ist $i(\nu) < \frac{\pi}{n}$, also die Funktion

$$m(\nu) = n \cdot \sin^\nu \frac{\pi}{n} - \left[(n-1) \sin^\nu \frac{\pi}{2(n-1)} + 1\right]$$

ist positiv. Es gilt auch, daß $m(\nu_1(n)) = 0$ ist. Die erste Ableitung der Funktion $m(\nu)$ ist

$$m'(\nu) = n \cdot \sin^\nu \frac{\pi}{n} \cdot \ln \sin \frac{\pi}{n} - (n-1) \sin^\nu \frac{\pi}{2(n-1)} \cdot \ln \sin \frac{\pi}{2(n-1)}$$

$$(\nu_0(n) \leq \nu \leq \nu_1(n)).$$

Das Vorzeichen der Funktion $m'(\nu)$ ist dasselbe wie das Vorzeichen der Funktion

$$p(\nu) = \left(\frac{\sin \dfrac{\pi}{2(n-1)}}{\sin \dfrac{\pi}{n}} \right)^{\nu} - \frac{\ln \sin \dfrac{\pi}{n}}{\ln \sin \dfrac{\pi}{2(n-1)}}.$$

Weil die Funktion

$$\left(\frac{\sin \dfrac{\pi}{2(n-1)}}{\sin \dfrac{\pi}{n}} \right)^{\nu}$$

abnehmend ist, hat $m(\nu)$ eine einzige Maximumstelle und auf Grund des vorangehenden ist $m(\nu) > 0$, wenn $\nu \neq \nu_1(n)$ ist, so wird das Minimum im Falle $\nu_0(n) \leq \nu \leq \nu_1(n)$ vom Typ F_n gegeben.

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RENORMING A BANACH SPACE FOR CONTINUITY OF A METRIC PROJECTION

By

A. BOSZNAY, JR.

Department of Probability Theory of the L. Eötvös University Budapest

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Let $(X, \|\cdot\|)$ be a Banach space, and L be a subspace of X. Let us define for $\forall x \in X$

$$P_L(x) \equiv P_{L,\|\cdot\|}(x) = \{y; y \in L, \|x-y\| = \inf_{l \in L} \|x-l\|\}.$$

P_L is called the metric projection to L. In the case where $\forall x \in X\ P_L(x) \neq \emptyset$, and P_L is single valued, we call L a Chebyshev subspace of X (we use the terminology of [2]).

It is well known that the metric projection to a Chebyshev subspace is not always continuous.

Oshman [4] gave several necessary and sufficient conditions for the continuity of all metric projections, when all subspaces in X are Chebyshev subspaces. Here we only mention that reflexivity is a necessary condition (in the case of strict convexity).

In this paper we give an arbitrarily small renorming of a separable Banach space X so that a given complemented subspace L becomes a Chebyshev subspace and the corresponding metric projection is continuous. The construction is such that the points of L nearest to the points of a certain dense subset of X are contained in a given dense linear subspace of L.

We first fix some notations.

Let $(X, \|\cdot\|)$ be a Banach space. The topological concepts to be used will be referring to the norm topology. If $H_1\ H_2 \subset X$, denote coH_1 the convex hull of H_1, and by $H_1 \oplus H_2$ the direct sum of H_1 and H_2 (if it exists).

If $L \subset X$ is a closed linear subspace, define

$$\|x+L\| = \inf_{l \in L} \|x+l\|.$$

For $x, y \in X$, $r \in R$, $r \geq 0$, set

$$S_{\|\cdot\|}(x, r) = \{y \in X; \|x-y\| \leq r\},$$
$$F_{\|\cdot\|}(x, r) = \{y \in X; \|x-y\| = r\},$$
$$[x, y] = \{\lambda x + (1-\lambda) y; 0 \leq \lambda \leq 1\},$$
$$(x, y) = \{\lambda x + (1-\lambda) y; 0 < \lambda < 1\}.$$

DEFINITION. Let X be a vector space, $\varepsilon > 0$ fixed, $\|\cdot\|$ and $|\cdot|$ norms on X. $\|\cdot\|$ and $|\cdot|$ are said to be ε-equivalent, if either

$$(1-\varepsilon)\,|x| \leq \|x\| \leq (1+\varepsilon)\,|x|$$

for $\forall\, x \in X$, or

$$(1-\varepsilon)\,\|x\| \leq |x| \leq (1+\varepsilon)\,\|x\|$$

for $\forall\, x \in X$.

We shall often use the following simple properties of a Chebyshev subspace L (see [3]):

a) $P_L(tx) = tP_L(x)$ for $\forall\, t \in R$,
b) $P_L(x+l) = P_L(x) + l$ for $\forall\, l \in L$,
c) $\|P_L(x)\| \leq 2\,\|x\|$,
d) $L \oplus P_L^{-1}(o) = X$.

THEOREM. *Let $(X, \|\cdot\|)$ be a real, separable Banach space, L_1 a* (not necessarily closed) *linear subspace with complemented closure L. Then for every $n \in \mathbf{N}$ there exists a norm $|||\cdot|||_n$ on X, $\frac{1}{n}$-equivalent to $\|\cdot\|$, such that L is a Chebyshev subspace in $(X, |||\cdot|||_n)$, the metric projection $P_{L,|||\cdot|||_n} = P_n$ is continuous, and a suitable dense set V satisfies $P_n(v) \in L_1$ for all $v \in V$ and $n \in \mathbf{N}$.*

PROOF. Before we go to the technical details of the proof, it will be useful to see the idea of it. First we construct a part of the surface of the new unit ball: the set whose image according to P_L is 0. After this, the new unit ball will be the convex closed hull of these points and the ball in the original norm (with radius a little less that 1). To construct this, we shall apply the Michael selection theorem (to the "almost nearest" elements to x in L to obtain a continuous P_L), and lemma 2 (to obtain a set V with the property mentioned in the theorem). Lemma 1 is necessary to assure the Chebyshev property of L (we must construct the surface of the unit ball so that no set $y+L$ disjoint from the interior of the new unit ball can contain more than one element of the new unit ball).

Let

$$(1) \qquad F_n : (X \setminus L) \to 2^L,\ F_n(x) = \left\{y; y \in L, \|x-y\| \leq \|x+L\| \left(1 + \frac{1}{10n}\right)\right\}.$$

We shall demonstrate that F_n is lower semi-continuous, i.e. $\{x; F_n(x) \cap \cap U \neq \emptyset\}$ is open in the relative topology of $X \setminus L$, if U is open in the relative topology of L. If $F_n(y) \cap U \neq \emptyset$, then for some $z \in F_n(y)$ and $\delta > 0$ we have $\{v; v \in L, \|z-v\| < \delta\} \subset U$. We distinguish between two cases.

First let $\|y-z\| = \|y+L\|$. Consider a point $w \in X$ with

$$\|w-y\| \leq \|y+L\| \frac{1}{40n}.$$

Obviously,

$$\|w-z\| \leq \|y+L\| \left(1+\frac{1}{40n}\right), \|w+L\| \geq \|y+L\| \left(1-\frac{1}{40n}\right).$$

Therefore

$$\left(1+\frac{1}{10n}\right) \|w+L\| \geq \|w-z\|.$$

Hence, in view of (1), $Z \in F_n(w)$, i.e. $U \cap F_n(w) \neq \emptyset$.

Next consider the case

$$\|y+L\| < \|y-z\| \leq \|y+L\| \left(1+\frac{1}{10n}\right).$$

Then there exists $z' \in L$ with $\|y-z'\| < \|y-z\|$. For $z'' \in (z', z)$, the triangle-inequality yields $\|y-z''\| < \|y-z\|$, Let $z'' \in (z', z)$ satisfy also

(2) $$\|z-z''\| < \delta, \quad \text{i.e.} \quad z'' \in U.$$

If

(3) $$\|y-w\| < \frac{\left(1+\frac{1}{10n}\right)\|y+L\| - \|y-z''\|}{2+\frac{1}{10n}},$$

then

$$\|w-z''\| < \|y-z''\| + \frac{\left(1+\frac{1}{10n}\right)\|y+L\| - \|y-z''\|}{2+\frac{1}{10n}}$$

and

$$\|w+L\| \geq \|y+L\| - \frac{\left(1+\frac{1}{10n}\right)\|y+L\| - \|y-z''\|}{2+\frac{1}{10n}}.$$

From the last two formulas we obtain

$$\left(1+\frac{1}{10n}\right) \|w+L\| \geq \|w-z''\|.$$

So, $z'' \in F_n(w)$, and by (2), $F_n(w) \cap U \neq \emptyset$.

We have proved that F_n is lower semi-continuous. Clearly, the elements of its range are convex, closed, non-empty subsets of the Banach space L. Therefore Michael's theorem ([1], p. 85) ensures the existence of a continuous selection f_n of F_n. We shall also need the following obvious property of F_n:

$$F_n(tx) = tF_n(x) \quad \text{for} \quad \forall\, t \in R\,. \tag{4}$$

Now consider a decomposition

$$X = L \oplus M\,, \tag{5}$$

where M is a closed subspace, and the corresponding projection

$$m : X \to M;\ x - m(x) \in L \quad \text{for all} \quad x \in X\,. \tag{6}$$

m is linear and closed, thus continuous. Let

$$g_n : \{x;\, x \in X;\, \|x + L\| = 1\} = Dg_n \to X\,, \tag{7}$$

$$g_n(u) = \frac{f_n(m(u)) - f_n(m(-u))}{2} - m(u)\,. \tag{8}$$

Then, in particular

$$g_n(-u) = -g_n(u)\,, \tag{9}$$

and g_n is continuous. Using (8), (4) and the convexity of $F_n(m(u))$ we find

$$g_n(u) \in F_n(m(u)) - m(u)\,. \tag{10}$$

Before proceeding with the proof of the theorem, we need two lemmas.

LEMMA 1. Let $(X, |\cdot|)$ be a real separable Banach space. Then for any $\varepsilon > 0$ there exists a locally uniformly convex norm $|||\cdot|||$ on X which is ε-equivalent to $|\cdot|$.

PROOF OF LEMMA 1. We replace relation (3) of [1], p. 177 (proof of theorem 3.1) by

$$\|x\| = w_0(x) + \sum_{K=1}^{\infty} 2^{-K} \frac{\varepsilon}{8} w_K(x)\,.$$

Then

$$|x| \leq \|x\| \leq \left(1 + \frac{\varepsilon}{4}\right) |x|\,,$$

while the other parts of the proof remain unchanged. So, instead of the conclusion of theorem 3.1, we obtain an $\frac{\varepsilon}{4}$-equivalent Kadec norm. In the proof of proposition 3.1. of [1], p. 178, let $a_i > 0\,(i = 1, \ldots)$ be such that

$$\sum_{i=1}^{\infty} a_i \|x_i^*(x)|^2 \leq \frac{\varepsilon}{2} \|x\|^2 .$$

Using these as well as relation (8) of the same p. 178, we find

$$|x| \leq |||x||| \leq \left[\left(1+\frac{\varepsilon}{4}\right)\left(1+\frac{\varepsilon}{2}\right)\right]^{1/2} |x| ,$$

and also $|||\cdot|||$ is locally uniformly convex. Lemma 1 is proved.

LEMMA 2. Let $(X, |\cdot|)$ be a separable real Banach space, U a dense (linear) subspace, $K>0$, Then for any $\varepsilon>0$ there exists a continuous mapping $l_\varepsilon : S_{|\cdot|}(O, K) \to U$ such that

$$|l_\varepsilon(u) - u| \leq \varepsilon \tag{11}$$

and

$$l_\varepsilon(-u) = -l_\varepsilon(u) \tag{12}$$

for $\forall\, u \in S_{|\cdot|}(O, K)$.

PROOF OF LEMMA 2. Let $\{y_i'\}_{i\in\mathbf{N}}$ be an $\frac{\varepsilon}{8}$-net in $U \cap S_{|\cdot|}(O, K)$, $|y_i'| \leq K$. Then (because of the density of U) there exists an $\bar{y} \in U$, with

$$\left|\sum_{i=1}^{\infty} \frac{1}{2^i} y_i' - \bar{y}\right| < \frac{\varepsilon}{16} ,$$

and $\exists \{z_i\}_{i\in\mathbf{N}} \subset U$ with

$$|z_i - z_{i-1}| < \frac{\varepsilon}{16} \cdot \frac{1}{2^i} ,$$

$$\lim_i z_i = \bar{y} - \sum_{i=1}^{\infty} \frac{1}{2^i} y_i' .$$

So, if

$$y_i = y_i' + 2^i (z_i - z_{i-1}) \quad (i>1), \quad y_1 = y_1' + z ,$$

then

$$\sum_{i=1}^{\infty} \frac{1}{2^i} y_i = \sum_{i=1}^{\infty} \frac{1}{2^i} y_i' + \lim_i z_i = \bar{y} \in U . \tag{13}$$

The sequence $\{y_i\}_{i\in\mathbf{N}}$ is an $\frac{\varepsilon}{4}$-net in $U \cap S_{|\cdot|}(O, K)$, and

$$|y_i| \leq K + \frac{\varepsilon}{16} = K' .$$

Set

$$l_i(y) = \max\left\{0, \frac{\varepsilon}{2} - |y - y_i|\right\}, \quad n(y) = \sum_{i=1}^{\infty} \frac{1}{3^i} l_i(y) . \tag{14}$$

Obviously,

$$0 \leq l_i(y) \leq \frac{\varepsilon}{2}, \quad 0 < n(y). \tag{15}$$

Moreover, for some $i_0 = i_0(y)$

$$\max\left\{\frac{1}{3^i} l_i(y), \frac{1}{2^i} n(y)\right\} = \frac{1}{2^i} n(y) \quad \text{if} \quad i \geq i_0, \tag{16}$$

since otherwise for infinitely many i

$$\left(\frac{3}{2}\right)^i n(y) < l_i(y),$$

which is impossible by (15). Set

$$l'_\varepsilon(y) = \frac{\sum_{i=s+1}^{\infty} \max\left\{\frac{1}{3^i} l_i(y), \frac{1}{2^i} n(y)\right\} y_i + \sum_{i=1}^{s} \frac{1}{3^i} l_i(y) y_i}{n(y)},$$

where S is chosen so large that

$$\sum_{i=s+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2K'}. \tag{17}$$

The mapping l'_ε is continuous, because

$$\sum_{i=s+1}^{\infty} \left| \frac{\max\left\{\frac{1}{3^i} l_i(y), \frac{1}{2^i} n(y)\right\}}{n(y)} y_i \right| \leq$$

$$\leq \sum_{i=s+1}^{\infty} \frac{\frac{1}{3^i} l_i(y)}{n(y)} |y_i| + \sum_{i=s+1}^{\infty} \frac{1}{2^i} |y_i| \leq 2K'.$$

Also, by (13) and (16), for some index m we have

$$l'_\varepsilon(y) = \bar{y} + \sum_{i=1}^{m} c_i y_i.$$

Therefore $l'_\varepsilon(y) \in U$. Further

$$|l'_\varepsilon(y) - y| = \left| \frac{\sum_{i=1}^{\infty} \frac{1}{3^i} l_i(y) y_i}{n(y)} - \frac{\sum_{i=1}^{\infty} \frac{1}{3^i} l_i(y) y}{n(y)} + \right.$$

$$\left. + \frac{\sum_{i=s+1}^{\infty} \left[\max\left\{\frac{1}{3^i} l_i(y), \frac{1}{2^i} n(y)\right\} - \frac{1}{3^i} l_i(y)\right] y_i}{n(y)} \right| \leq$$

$$\leq \left| \frac{\sum_{i=1}^{\infty} \frac{1}{3^i} l_i(y)(y-y_i)}{n(y)} \right| +$$

$$+K' \sum_{i=s+1}^{\infty} \frac{\max\left\{\frac{1}{3^i} l_i(y), \frac{1}{2^i} n(y)\right\} - \frac{1}{3^i} l_i(y)}{n(y)} \leq \frac{\varepsilon}{2} + K' \frac{\varepsilon}{2K'}$$

(at the last step we used (14) and (17)).

Consequently, the mapping

$$l_\varepsilon(y) = \frac{l'_\varepsilon(y) - l'_\varepsilon(-y)}{2}$$

satisfies (11) and (12). Lemma 2 is proved.

Returning to the proof of the theorem, let $\|\cdot\|'_n$ be a locally uniformly convex norm on X/L such that

(18) $$\|y+L\|\left(1-\frac{1}{10n}\right) \leq \|y+L\|'_n \leq \|y+L\|.$$

By (18)

$$\sup\left\{\left\|\frac{g_n(u)+m(u)}{\|g_n(u)+L\|'_n}\right\|;\ u \in Dg_n\right\} < T < +\infty$$

is valid, because from (1), (5), (6), (7) and (10) we obtain

(19) $$\|g_n(u)\| \leq \left(1+\frac{1}{10n}\right)\|(m(u)\|,$$

and by the definition and continuity of m, for some $H>0$,

(20) $$\|m(u)\| \leq H\|u+L\|.$$

Let $D_{tn} = Dg_n$,

(21) $$t_n = l_{1/10n}\left[\frac{g_n(u)+m(u)}{\|g_n(u)+L\|'_n}\right] - \frac{m(u)}{\|g_n(u)+L\|'_n},$$

where

$$l_{1/10n}: (L \cap S_{\|\cdot\|}(O,T)) \to L_1$$

and $l_{1/10n}$ satisfies (11) and (12) with $\varepsilon = \frac{1}{10n}$. The mapping t_n is continuous. Moreover, using (9) and (12)

(22) $$t_n(-u) = -t_n(u).$$

Let H_n be the closure of

$$co\left[S_{\|\cdot\|}\left(0,1-\frac{1}{n}\right)\cup\{t_n(v);\ v\in D_{t_n}\}\right].$$

Obviously, $H_n\supset S_{\|\cdot\|}\left(0,1-\frac{1}{n}\right)$. On the other hand, by (11) and (21)

$$\left\|t_n(u)-\frac{g_n(u)}{\|g_n(u)+L\|'_n}\right\|\leq\frac{1}{10n}.$$

Using (1), (10) and (18) we find

$$\|t_n(u)\|\leq\frac{1}{10n}+\frac{1}{1-\frac{1}{10n}}\left(1+\frac{1}{10n}\right),$$

so that

$$t_n(u)\in S_{\|\cdot\|}\left(0,1+\frac{1}{2n}\right). \tag{23}$$

By (22) H_n is symmetrical to 0, therefore its Minkowski functional is a norm $\frac{1}{n}$-equivalent to $\|\cdot\|$. Let us denote this norm by $|||\cdot|||_n$. Now, we shall prove that the norm $|||\cdot|||_n$ satisfies the requirements of the theorem.

Clearly,

$$S_{|||\cdot|||_n}(0,1)=H_n.$$

We are going to prove that for all $u\in D_{t_n}$

$$(t_n(u)+L)\cap H_n=\{t_n(u)\} \tag{24}$$

and

$$\lambda t_n(u)\notin H_n \quad \text{if} \quad \lambda>1. \tag{25}$$

Suppose that either (24) or (25) is not valid. Then, for each n, there exists a sequence $\{w_K^n\}_{K\in\mathbf{N}}$,

$$w_K^n\in co\left[S_{\|\cdot\|}\left(0,1-\frac{1}{n}\right)\cup\{t_n(v);\ v\in D_{t_n}\}\right],$$

with

$$\lim_K w_K^n=t_n(u)+r_n \quad (r_n\in L,\ r_n\neq 0) \tag{26}$$

or with

$$\lim_K w_K^n=\lambda t_n(u). \tag{27}$$

We have

$$w_K^n = \sum_{i=1}^{P(K)} \lambda_i^K t_n(u_i^K) + \sum_{i=1}^{P(K)} \mu_i^K v_i^K, \tag{27'}$$

where

$$\sum_{i=1}^{P(K)} (\lambda_i^K + \mu_i^K) = 1; \ \lambda_i^K, \ \mu_i^K \geq 0, \tag{28}$$

$$u_i^K \in D_{t_n}, \quad v_i^K \in S_{\|\cdot\|}\left(0, 1 - \frac{1}{n}\right), \tag{29}$$

and λ_i^K, μ_i^K, u_i^K, v_i^K, $P(K)$ depend on n. By (26)

$$t_n(u) + L = \lim_K \left[\sum_{i=1}^{P(K)} \lambda_i^K (t_n(u_i^K) + L) + \sum_{i=1}^{P(K)} \mu_i^K (v_i^K + L)\right]. \tag{30}$$

Using (10) and (21)

$$\|t_n(u) + L\|_n' = 1, \tag{31}$$

$$\|t_n(u_i^K) + L\|_n' = 1. \tag{32}$$

Further, by (18) and (29)

$$\|v_i^K + L\|_n' \leq \|v_i^K + L\| \leq 1 - \frac{1}{n}. \tag{33}$$

From (27), (27′), (28), (31), (32) and (33) we obtain a contradiction, which proves (25).

Going on with the proof of (24), from (26) and (31) we find

$$\lim_K \|w_K^n + L\|_n' = 1. \tag{34}$$

Let

$$\varepsilon > \varepsilon_1 > 0. \tag{35}$$

If $K > K_0(\varepsilon_1, n)$, then by (34) and (26)

$$\|w_K^n + L\|_n' \geq 1 - \frac{\varepsilon_1}{n} \tag{35'}$$

and

$$\|w_K^n - t_n(u) - r_n\| < \varepsilon_1. \tag{36}$$

Relations (27′), (28), (32), (33) and (35′) yield

$$\sum_{i=1}^{P(K)} \mu_i^K \leq \varepsilon_1. \tag{37}$$

Setting

$$x_K^n = \sum_{i=1}^{P(K)} \lambda_i^K t_n(u_i^K) \tag{38}$$

and making use of (27′), (33), (36) and (37) we obtain

$$\|x_K^n - t_n(u) - r_n\| < 2\varepsilon_1 . \tag{39}$$

Therefore, according to (18) and (26),

$$\|x_K^n - t_n(u) + L\|_n' < 2\varepsilon_1 .$$

Hence by (28), (31) and (37)

$$2 - 2\varepsilon_1 \leq \|x_K^n + t_n(u) + L\|_n' =$$

$$= \left\| \sum_{i=1}^{P(K)} \lambda_i^K \left(t_n(u_i^K) + t_n(u) + L\right) + \left(1 - \sum_{i=1}^{P(K)} \lambda_i^K\right)\left(t_n(u) + L\right) \right\|_n' \leq$$

$$\leq \sum_{i=1}^{P(K)} \lambda_i^K \|t_n(u_i^K) + t_n(u) + L\|_n' + \varepsilon_1 .$$

Thus, in view of (28), (31) and (32),

$$\sum_{i \in M_K^n} \lambda_i^K \leq \sqrt{\varepsilon_1} \tag{40}$$

where

$$M_K^n = \{j,\ 1 \leq j \leq P(K),\ \|t_n(u_j^K) + t_n(u) + L\|_n' \leq 2 - 3\sqrt{\varepsilon_1}\} .$$

By the definition of M_K^n

$$\|t_n(u_i^K) + t_n(u) + L\|_n' > 2 - 3\sqrt{\varepsilon_1} \quad \text{for} \quad i \notin M_K^n . \tag{41}$$

Let

$$\varepsilon > \varepsilon_2 > 0 . \tag{42}$$

Choose ε_1 so small (i.e., K so large) that from (41)

$$\|t_n(u_i^K) - t_n(u) + L\|_n' < \varepsilon_2 \quad \text{for} \quad i \notin M_K^n \tag{43}$$

(this is possible on account of (31), (32) and the local uniform convexity of $\|\cdot\|_n'$.)

By (18), (21) and (43), for $i \notin M_K^n$,

$$\left\| \frac{m(u)}{\|g_n(u) + L\|_n'} - \frac{m(u_i^K)}{\|g_n(u_i^K) + L\|_n'} + L \right\| < \varepsilon_2 \frac{1}{1 - \frac{1}{10n}} < 2\varepsilon_2 . \tag{44}$$

As u, $u_i^K \in D_{t_n} = Dg_n$, relation (7) and the definition of m yield

$$\|m(u)+L\| = \|m(u_i^K)+L\| = 1 . \tag{45}$$

From (44) we obtain

$$-2\varepsilon_2 \|g_n(u)+L\|_n' + \frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} \|m(u_i^K)+L\| \leq \|m(u)+L\| \leq$$

$$\leq \|m(u_i^K)+L\| \frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} + 2\varepsilon_2 \|g_n(u)+L\|_n' ,$$

so that by (45)

$$-2\varepsilon_2 \|g_n(u)+L\|_n' + \frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} \leq 1 \leq$$

$$\leq \frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} + 2\varepsilon_2 \|g_n(u)+L\|_n' . \tag{46}$$

According to relations (1), (6), (7), (10) and (18)

$$\|g_n(u)+L\|_n' \leq 1 .$$

Thus from (46)

$$1-2\varepsilon_2 \leq \frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} \leq 1+2\varepsilon_2 .$$

Hence, making use of (10), (18), (44) and (45),

$$\|m(u)-m(u_i^K)+L\| \leq \left\| m(u)-m(u_i^K)\frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} + L \right\| +$$

$$+ \left\| m(u_i^K)\left(\frac{\|g_n(u)+L\|_n'}{\|g_n(u_i^K)+L\|_n'} - 1\right) + L \right\| \leq 2\varepsilon_2 + 2\varepsilon_2 = 4\varepsilon_2 ,$$

and in view of (20)

$$\|m(u)-m(u_i^K)\| = \|m[m(u)-m(u_i^K)]\| \leq 4H\varepsilon_2 .$$

So, if ε_2 is sufficiently small (to this end, also ε_1 must be sufficiently small), then with the aid of (8), (21) and the continuity of t_n we obtain

$$\|t_n(u_i^K)-t_n(u)\| < \varepsilon$$

(K sufficiently large, $1 \leq i \leq P(K)$, $i \notin M_K^n$). Consequently, by (23), (28), (35), (37), (38) (39), (40) and (42),

$$\|r_n\| \leq \|x_K^n - t_n(u) - r_n\| + \|t_n(u) - x_K^n\| \leq$$

$$\leq 2\varepsilon + \varepsilon + 2\sqrt{\varepsilon}\left(1+\frac{1}{2n}\right) + \varepsilon\left(1+\frac{1}{2n}\right)$$

for every $\varepsilon > 0$. It follows that $r_n = 0$, which contradicts (26). Thus (24) is valid.

We are going to prove that L is a Chebyshev subspace in $(X, |||\cdot|||_n)$ and P_n is continuous.

At points of L the existence and uniqueness of the metric projection are obvious. If we already know that L is a Chebyshev subspace, then also the continuity of P_n is clear at these points, since *b)* and *c)* yield

$$|||P_n(x) - P_n(l)|||_n = |||P_n(x-l)|||_n \leq 2\,|||x-l|||_n \quad \text{for} \quad l \in L.$$

So let $x \in (X \setminus L)$. We define

$$P_n'(x) = \|x+L\|_n' \left[t_n\left(\frac{x}{\|x+L\|}\right) + \frac{x}{\|x+L\|_n'} \right],$$

and show that $P_n'(x)$ is the only nearest point to x in L according to $|||\cdot|||_n$. By (8) and (21) $P_n'(x) \in L$. Further, as

$$P_n'(x) - x = \|x+L\|_n' \, t_n\left(\frac{x}{\|x+L\|}\right), \tag{47}$$

making use of (25) we obtain

$$P_n'(x) \in F_{|||\cdot|||_n}(x, \|x+L\|_n'),$$

and by (24)

$$S_{|||\cdot|||_n}(x, \|x+L\|_n') \cap L = \{P_n'(x)\}.$$

Thus L is a Chebyshev subspace, $P_n'(x) = P_n(x)$, and P_n is continuous.

Finally, we must prove the existence of a dense subset $V \subset X$ such that $P_n(v) \in L_1$ for all $v \in V$ and $n \in \mathbf{N}$. In other words, we must prove that the set

$$X \setminus \bigcup_{n=1}^{\infty} \bigcup_{y \in L \setminus L_1} (y + P_n^{-1}(0)) \tag{48}$$

is dense in X. To this end it is in turn sufficient to show that, for fixed $m_0 \in M$, the set

$$(L + m_0) \setminus \bigcup_{n=1}^{\infty} \bigcup_{y \in L \setminus L_1} (y + P_n^{-1}(0)) \tag{49}$$

is dense in $L + m_0$.

Note that by (47) and property *a)*

$$P_n^{-1}(0) = \{\lambda\, t_n(u);\ \lambda \in R,\ u \in D_{t_n}\}. \tag{50}$$

Now, if $y + p_n \in L + m_0$, $y + p_m \in L + m_0$ $(p_n \in P_n^{-1}(0),\ p_m \in P_m^{-1}(0))$, then $p_n - p_m \in L$, so that in view of (5), (21) and (50)

$$p_n - p_m \in L_1. \tag{51}$$

According to property *d)*, $L \oplus P_n^{-1}(0) = X$. So, if

$$y + p_n \in L + m_0 \, (y \in L, \; p_n \in P_n^{-1}(0)) \, ,$$

then there is no $p'_n \neq p_n$ with $y' + p'_n \in L + m_0 \, (y' \in L, \; p'_n \in P_n^{-1}(0))$. Thus, assuming $L \neq L_1$, we can find elements $p_n \in P_n^{-1}(0)$ for all $n \in \mathbf{N}$, such that

$$(L + m_0) \cap \left[\bigcup_{n=1}^{\infty} \bigcup_{y \in L \setminus L_1} (y + P_n^{-1}(0)) \right] = \bigcup_{n=1}^{\infty} (p_n + (L \setminus L_1)) \, . \tag{52}$$

Let $Q = \{p_n - p_1; \; n \in \mathbf{N}; \; n > 1\}$. By (51), $Q \subset L_1$. Further, $p_1 + L_1$ is dense in $L + m_0$, and $(p_1 + L_1) \cap (p_1 + (L \setminus L_1)) = \emptyset$. Moreover, $(p_1 + L_1) \cap \cap (p_n + (L \setminus L_1)) = \emptyset$ $(n > 1)$. In fact, if there is an element s in the set $(p_1 + L_1) \cap (p_n + (L \setminus L_1))$, $s = p_1 + l_1 = p_n + l$ $(l \in L, \; l_1 \in L \setminus L_1)$, then $p_n - p_1 = = l_1 - l \in L \setminus L_1$, contrary to the relation $Q \subset L$. Thus the set (49), the complement of the set (52) relative to $L + m_0$, is dense in $L + m_0$.

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ON THE GENERALIZED BUTZER-WAGNER TYPE a.e. DIFFERENTIABILITY OF INTEGRAL FUNCTION

By

J. PÁL and P. SIMON

Department II. for Analysis and Department for Computer Science of the L. Eötvös University, Budapest

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1. §

In this section we introduce some notations and definitions. Let

(1) $$m = (m_0, m_1, \ldots, m_k, \ldots) \quad (2 \leq m_k, m_k \in \mathbf{N}, k \in \mathbf{N} := \{0, 1, \ldots\})$$

be a sequence of natural numbers and denote by Z_{m_k} the m_k the discret cyclic group, i.e.

(2) $$Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \quad (k \in \mathbf{N}).$$

Furthermore, if we define the group G_m as the direct product of groups Z_{m_k}, then G_m is a compact Abelian group. Thus the elements of G_m are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $0 \leq x_k < m_k$ $(k \in \mathbf{N})$ and for x, y in G_m their sum $x \dot{+} y$ is obtained by adding the nth coordinate of x and y modulo m_k $(k \in \mathbf{N})$. We define the subgroups $I_n(x)$ of G_m as follows:

(3) $$I_n(x) := \{y \in G_m : y = (x_0, \ldots, x_{n-1}, y_n, \ldots)\} \quad (x \in G_m, n \in \mathbf{N}).$$

Then the $I_n(0)$'s $(0 \in G_m, n \in \mathbf{N})$ form a basis for the neighborhoods of $0 \in G_m$ in G_m and these subgroups completely determine the topology of G_m.

Let $\Gamma(m) := \{\Psi_n : n \in \mathbf{N}\}$ denote the character group of G_m. We enumerate the elements of $\Gamma(m)$ as follows. For $k \in \mathbf{N}$ and $x \in G_m$ let r_k be the function defined by

(4) $$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (x \in G_m, i := \sqrt{-1}, k \in \mathbf{N}).$$

If we define the sequence $(M_k, k \in \mathbf{N})$ by $M_0 := 1$ and $M_k := m_0 m_1 \cdot \ldots \cdot m_{k-1}$ $(k \in \mathbf{P} := \{1, 2, \ldots\})$ then each $n \in \mathbf{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $0 \leq n_k < m_k$ $(k \in \mathbf{N})$. For such $n \in \mathbf{N}$ we define the function Ψ_n by

$$\Psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k}. \tag{5}$$

We remark that $\Gamma(m)$ is a complete orthonormal system relate to the normalized Haar measure dx on G_m [10]. Furthermore, if $m_n = 2$ $(n \in \mathbf{N})$ then G_m is the so-called dyadic group and the elements of the character group $\Gamma(m)$ are the Walsh-Paley functions.

For $f \in L^1(G_m)$ we define its partial sums and $(C, 1)$ means by

$$\hat{f}(n) := \int_{G_m} f(t)\overline{\Psi_n(t)}\,dt, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\Psi_k \quad (n \in \mathbf{P},\ S_0 \equiv 0), \tag{6}$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f \quad (n \in \mathbf{P}).$$

Then we have the formula

$$\begin{aligned} S_n f(x) &= \int_{G_m} f(t)\, D_n(x \dot{-} t)\, dt = (f * D_n)(x), \\ \sigma_n f(x) &= \int_{G_m} f(t)\, K_n(x \dot{-} t)\, dt = (f * K_n)(x), \end{aligned} \tag{7}$$

$x \in G_m$, $n \in \mathbf{N}$, where D_n, K_n are the so-called Dirichlet and Fejér kernel, resp. In [9] was proved the following formula:

$$\begin{aligned} D_n &= \Psi_n \sum_{k=0}^{\infty} \sum_{j=m_k-n_k}^{m_k-1} (r_k)^j D_{M_k} \\ D_{M_n} &= \prod_{k=0}^{n-1} \left(1 + r_k + \ldots + (r_k)^{m_k-1}\right). \end{aligned} \tag{8} \qquad (n \in \mathbf{N})$$

C. W. ONNEWEER has given the following generalization of the concept of a derivative of P L. BUTZER and H. J. WAGNER:

DEFINITION ([4]): Let

$$d_n f(x) := \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} \frac{k}{m_j} \sum_{s=0}^{m_j-1} \overline{r_j(se_j)}^k f(x \mathbin{\mathring{+}} se_j)$$

$\Big(x \in G_m, n \in \mathbf{P}, e_j := \big(0, 0, \ldots, 0, \overset{j}{\smile{1}}, 0, \ldots\big) \in G_m, se_j := \overset{1}{\smile{e_j}} \mathbin{\mathring{+}} \ldots \mathbin{\mathring{+}} \overset{s}{\smile{e_j}}, f$ is an arbitrary function on $G_m\Big)$.

If for the function f there exists $\lim_n d_n f(x)$ at some point $x \in G_m$ then this limes is called the pointwise derivative of f at x and is denoted by $f^{[1]}(x)$.

The following relation is easily verified:

$$\Psi_n^{[1]}(x) = n\Psi_n(x) \quad (n \in \mathbf{N}, x \in G_m).$$

Further we suppose that the sequence (1) is bounded.

2. §

Let $W \in L^2(G_m)$ denote the function for which

$$\widehat{W}(k) := \begin{cases} 1 & (k = 0) \\ 1/k & (k \in \mathbf{P}) \end{cases}$$

holds and let

$$If := W * f \quad (f \in L^1(G_m)).$$

The system $\Gamma(m)$ is complete, therefore the function W is uniquely determined.

Let T denote the maximal operator of the operator sequence

$$d_n : L^1(G_m) \to L^1(G_m) \quad (n \in \mathbf{P}), \quad \text{i.e.}$$
$$Tf(x) := \sup_n |d_n(If)(x)| \quad (f \in L^1(G_m),\ x \in G_m).$$

Then the following maximal theorem holds.

THEOREM. *The operator $T : L^1(G_m) \to L^1(G_m)$ has weak type* (1,1) *and has type* (∞, ∞), *i.e. the following inequalities hold:*

(i) $\|Tf\|_\infty \leq A\|f\|_\infty \quad (f \in L^\infty(G_m))$,
(ii) $\operatorname{mes}\{x \in G_m : Tf(x) > y\} \leq B\|f\|_1/y \quad (f \in L^1(G_m),\ y > 0)$,

where the A and B are absolute constants.

Since $I\Psi_n = 1/n\Psi_n$ $(n \in \mathbf{P})$, for an arbitrary "polynomial" of the form $P = \sum_{j=1}^{n} a_j \Psi_j$ $(n \in \mathbf{P})$ we have $\lim_k d_k(IP)(x) = P(x)$ $(x \in G_m)$. From this, applying that the set of the polynomials is closed in $L^1(G_m)$, follows the

COROLLARY 1. For an arbitrary function $f \in L^1(G_m)$, $\hat{f}(0) = 0$ the following statement holds:

$$\lim_n d_n(If)(x) = f(x) \quad (\text{a.e. } x \in G_m).$$

The above COROLLARY 1. is similar to the well-known theorem of the classical analysis on the a.e. differentiability of the integral function.

An immediate consequence of the THEOREM and of the interpolation theorem of Marcinkiewicz [12] is the following

COROLLARY 2. The operator $T : L^p(G_m) \to L^p(G_m)$ has type (p, p) for all $1 < p \leq \infty$.

We remark that the above statements in the case $m_n = 2$ $(n \in \mathbf{N})$ was proved and generalized for measures by F. SCHIPP [6], [7]. We follow the method of these papers.

3. §

PROOF OF THEOREM. To the proof we shall need also some lemmas.

LEMMA 1. Let x denote an arbitrary element of $G_m\backslash\{0\}$. Then the series $\sum_{k=1}^{\infty} \Psi_k(x)/k$ is convergent. Furthermore, for the function W, defined by

$$W(x) := \sum_{k=1}^{\infty} \Psi_k(x)/k \quad (x \in G_m\backslash\{0\})$$

the following estimation holds:

$$|W| \leq 0(1) \sum_{k=1}^{\infty} \frac{1}{M_{k-1}} D_{M_k}.$$

PROOF OF LEMMA 1. Let $x \in G_m\backslash\{0\}$ and $n \in \mathbf{N}$ be fixed. Then we have on the basis of (8) that

$$|D_n(x)| \leq 0(1) \sum_{k=0}^{\infty} D_{M_k}(x),$$

i.e. the series $\sum_{k=1}^{\infty} \Psi_k(x)/k$ is convergent.

Applying Abel transformation we obtain that

$$W(x) = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} D_k(x) - 1,$$

from which

$$W(x)+1 = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} D_k(x) =$$

$$= \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \Psi_k(x) \sum_{i=0}^{\infty} \sum_{j=m_i-k_i}^{m_i-1} (r_i(x))^j D_{M_i}(x) =$$

$$= \sum_{i=0}^{\infty} \sum_{j=m_i-k_i}^{m_i-1} (r_i(x))^j D_{M_i}(x) \sum_{\substack{k \geq 2 \\ m_i > k_i \geq 1}} \frac{1}{k(k-1)} \Psi_k(x)$$

follows. Since

$$\sum_{\substack{k \geq 2 \\ k_i \geq 1}} \left| \frac{1}{k(k-1)} \Psi_k(x) \right| \leq \sum_{k=M_i}^{\infty} \frac{1}{k(k-1)} = \frac{1}{M_i - 1} \leq \frac{1}{M_{i-1}},$$

the last estimation implies that

$$|W(x)| \leq 0(1) \sum_{i=1}^{\infty} \frac{1}{M_{i-1}} D_{M_i}(x).$$

Lemma 1. is proved.

LEMMA 2. For an arbitrary $n \in \mathbf{P}$ we have the following decomposition:

$$d_n W = D_{M_n} - 1 + F_n + H_n,$$

where

(i) $$|F_n(x)| \leq 0(1) \sum_{i=1}^{\infty} \frac{1}{M_{n+i-1}} \left(\sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} D_{M_{n+i}}(x \dot{+} q e_j) \right),$$

(ii) $$|H_n(x)| \leq 0(1) \left(D_{M_n}(x) + |K_{M_{n-1}}(x)| + \frac{1}{M_n} \sum_{i=1}^{M_n-1} |K_i(x)| \right) \quad (x \in G_m).$$

PROOF OF LEMMA 2. From the definition of d_n follows that

$$d_n W = \sum_{s=1}^{\infty} 1/s \Psi_s \left(\sum_{i=0}^{n-1} s_i M_i \right) = \sum_{s=1}^{M_n-1} \Psi_s + \sum_{i=1}^{\infty} \sum_{k=0}^{M_n-1} \frac{k}{iM_n + k} \Psi_{iM_{n+k}}.$$

Let

$$a_k(i, n) := i \left(\frac{k}{M_n} - \frac{ik}{iM_n + k} \right) \quad (i \in \mathbf{P},\ n,\ k \in \mathbf{N}).$$

Then

$$d_n W(x) = D_{M_n}(x) - 1 + \sum_{i=1}^{\infty} 1/i \Psi_{iM_n}(x) \sum_{k=0}^{M_n-1} \frac{ik}{iM_n + k} \Psi_k(x) =$$

$$= D_{M_n}(x) - 1 + \sum_{i=1}^{\infty} 1/i \Psi_{iM_n}(x) \sum_{k=0}^{M_n-1} k/M_n \Psi_k(x) -$$

$$- \sum_{i=1}^{\infty} 1/i^2 \Psi_{iM_n}(x) \sum_{k=0}^{M_n-1} a_k(i, n) \Psi_k(x) \quad (x \in G_m).$$

Let $G_m^{(n)}$ $(n \in \mathbf{N})$ denote the direct product of groups Z_{m_k} $(k = n, n+1, \ldots)$ and let $\Psi_i^{(n)}$ and $r_i^{(n)}$ $(i \in \mathbf{N})$ be the characters of $G_m^{(n)}$ defined in the same way as Ψ_i and r_i are defined in (5) and (4), respectively. We can easily verified that

$$\Psi_{iM_n}(x) = \Psi_i^{(n)}(x^{(n)}),$$

where

$$x = (x_0, x_1, \ldots) \in G_m,\ x^{(n)} := (x_n, x_{n+1}, \ldots) \in G_m^{(n)},\ n \in \mathbf{N},\ i \in \mathbf{P}.$$

Applying this we obtain the following decomposition for $d_n W$:

$$d_n W(x) = D_{M_n}(x) - 1 + \sum_{i=1}^{\infty} 1/i \Psi_i^{(n)}(x^{(n)}) \sum_{k=0}^{M_n-1} k/M_n \Psi_k(x) -$$

$$- \sum_{i=1}^{\infty} 1/i^2 \Psi_{iM_n}(x) \sum_{k=0}^{M_n-1} a_k(i, n) \Psi_k(x) =: D_{M_n}(x) - 1 + F_n(x) + H_n(x).$$

Let

$$W^{(n)} := \sum_{i=1}^{\infty} 1/i\, \Psi_i^{(n)}.$$

Since

$$1/M_n \sum_{k=0}^{M_n-1} k\, \Psi_k(x) = 1/M_n (d_n D_{M_n})(x),$$

we have that

$$F_n(x) = W^{(n)}(x^{(n)})\, 1/M_n \sum_{j=0}^{n-1} M_j \sum_{p=0}^{m_j-1} p/m_j \sum_{q=0}^{m_j-1} \overline{r_j(qe_j)}^p D_{M_n}(x \dot{+} qe_j).$$

Applying the Lemma 1. for the function $W^{(n)}$ we obtain that for $x^{(n)} \neq 0$

$$|F_n(x)| \leq 0(1) \sum_{i=1}^{\infty} \frac{1}{M_{i-1}^{(n)}} D_{M_i^{(n)}}^{(n)}(x^{(n)}) \left[\frac{1}{M_n} \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} D_{M_n}(x \dot{+} qe_j)\right],$$

where

$$M_i^{(n)} := m_n m_{n+1} \cdot \ldots \cdot m_{n+i-1},$$

$$D_{M_i^{(n)}}^{(n)}(x^{(n)}) := \sum_{k=0}^{M_i^{(n)}-1} \Psi_k^{(n)}(x^{(n)}) \quad (n \in \mathbf{N},\ i \in \mathbf{P},\ x^{(n)} \in G_m^{(n)}).$$

On the basis of (8) the following equation is valid for all $j = 0, \ldots$ $\ldots, n-1, q = 0, \ldots, m_j - 1$:

$$D_{M_i^{(n)}}^{(n)}(x^{(n)})\, D_{M_n}(x \dot{+} qe_j) =$$

$$= \prod_{s=0}^{i-1} [1 + \ldots + (r_s^{(n)}(x^{(n)}))^{m_{n+s}-1}] \prod_{t=0}^{n-1} [1 + \ldots + (r_t(x \dot{+} qe))^{m_t-1}] =$$

$$= \prod_{s=0}^{i-1} [1 + r_{n+s}(x \dot{+} qe_j) + \ldots +$$

$$+ (r_{n+s}(x \dot{+} qe_j))^{m_{n+s}-1}] \prod_{t=0}^{n-1} [1 + \ldots + (r_t(x \dot{+} qe))^{m_t-1}] = D_{M_{n+i}}(x \dot{+} qe_j).$$

From this follows that

$$|F_n(x)| \leq 0(1) \sum_{i=1}^{\infty} \frac{1}{M_{n+i-1}} \left(\sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} D_{M_{n+i}}(x \dot{+} qe_j)\right).$$

To the estimation for H_n let

$$B_n := \sup_i \left| \sum_{k=0}^{M_n-1} a_k(i, n)\, \Psi_k \right|.$$

We prove that

$$B_n \leq 12 \left(D_{M_n} + |K_{M_n-1}| + 1/M_n \sum_{k=0}^{M_n-1} |K_i|\right).$$

Then

$$|H_n| \leq B_n \sum_{k=1}^{\infty} 1/k^2 \leq 2B_n \quad (n \in \mathbf{N})$$

and from this follows the statement concerning with H_n.

Let g_i denote the function defined by

$$g_i(t) := \frac{t^2}{1+t/i} \quad (i \geq 1,\ 0 \leq t \leq 1).$$

Then $a_k(i, n) = g_i(k/M_n)$. Applying Abel transformation we obtain that

$$\sum_{k=0}^{M_n-1} a_k(i,n)\Psi_k = \sum_{k=1}^{M_n-1} \left(g_i\left(\frac{k-1}{M_n}\right) - 2g_i(k/M_n) + g_i\left(\frac{k+1}{M_n}\right)\right) kK_k +$$

$$+(M_n-1)\left(g_i\left(\frac{M_n-2}{M_n}\right) - g_i\left(\frac{M_n-1}{M_n}\right) K_{M_n-1} + g_i\left(\frac{M_n-1}{M_n}\right)\right) D_{M_n}.$$

Since

$$\max_{0 \leq j \leq 2} \max_{0 \leq t \leq 1} |g_i^{(j)}(t)| \leq 6 \quad (i \in \mathbf{P}),$$

the following estimations are valid:

$$\left|g_i\left(\frac{M_n-1}{M_n}\right)\right| \leq 6, \quad \left|g_i\left(\frac{M_n-2}{M_n}\right) - g_i\left(\frac{M_n-1}{M_n}\right)\right| \leq 6/M_n,$$

$$\left|g_i\left(\frac{k-1}{M_n}\right) - 2g_i(k/M) + g_i\left(\frac{k+1}{M_n}\right)\right| \leq 12/M_n^2.$$

From this it is obvious that

$$\left|\sum_{k=0}^{M_n-1} a_k(i,n)\Psi_k\right| \leq 12\left(D_{M_n} + |K_{M_n-1}| + 1/M_n \sum_{i=1}^{M_n-1} |K_i|\right).$$

Lemma 2. is proved.

On the basis of Theorem 3. of [5] we obtain the following

LEMMA 3. Let L_n denote the function defined by

$$L_n(x) := 1/M_n \sum_{j=0}^{n-1} M_j \sum_{i=j}^{n-1} \left[D_{M_i}(x) + \sum_{q=0}^{m_j-1} D_{M_i}(x \dotplus qe_j)\right] \quad (x \in G_m,\ n \in \mathbf{P}).$$

Then

$$|K_n| \leq 0(1)\, L_n \quad (n \in \mathbf{P}).$$

A modified version of a theorem of F. SCHIPP will be used, which has the following form ([8], Satz):

LEMMA 4. Let $A = (a_{ij}^{nq})$ $(i, j, q, n \in \mathbf{N})$ be a matrix having the following properties:

(i) $a_{ij}^{nq} \geq 0 \quad (i, j, q, n \in \mathbf{N})$,

(ii) $\sup_q \sup_n \sum_{i,j=0}^{\infty} a_{ij}^{qn} < \infty$,

(iii) $\sup_q \sup_s \sum_{i=0}^{s} \sum_{j=s}^{\infty} \left(\sup_{n<j} a_{ij}^{nq} \right) < \infty$,

(iv) $\sup_q \sup_k \sum_{i=0}^{k} \sum_{j=k}^{\infty} 1/M_j \sup_{n \geq j} (M_n a_{ij}^{nq}) < \infty$.

Furthermore let

$$f_n(x) := \sum_{i,j=0}^{\infty} \sum_{q=0}^{m_i-1} a_{ij}^{nq} D_{M_j}(x \mathbin{\dot{+}} qe_i) \quad (n \in \mathbf{N},\ x \in G_m).$$

Then the maximal operator $Tf := \sup_n |(f_n * f)|$ has type (∞, ∞) and weak type (1,1).

PROOF OF THEOREM. Let $f \in L^1(G_m)$, $\hat{f}(0) = 0$. By Lemma 2. we have

$$d_n(If) = d_n(W * f) = (d_n W) * f = D_{M_n} * f + F_n * f + H_n * f.$$

We introduce the following notations:

$$f_n^1 := D_{M_n}, \quad f_n^2 := 1/M_n \sum_{j=0}^{n-1} M_j \sum_{i=j}^{n-1} D_{M_i},$$

$$f_n^3 := 1/M_n \sum_{j=0}^{n-1} M_j \sum_{i=j}^{n-1} \sum_{q=0}^{m_j-1} D_{M_i}(\cdot \mathbin{\dot{+}} qe_j),$$

$$f_n^4 := \sum_{i=1}^{\infty} \frac{1}{M_{n+i-1}} \sum_{j=0}^{n-1} M_j \sum_{q=0}^{m_j-1} D_{M_{n+i}}(\cdot \mathbin{\dot{+}} qe_j) \quad (n \in \mathbf{P}).$$

It is easily verified that the functions f_n^i $(i = 2, 3, 4)$ satisfy the conditions of Lemma 4. From this follows that the operators $T^i f := \sup_n |(f_n^i * f)|$ have type (∞, ∞) and weak type (1,1). It is obvious that the operator $T^1 f := \sup_n |(f_n^1 * f)|$ has the same property. Since by Lemma 3. $|K_n| \leq \leq 0(1)(f_n^2 + f_n^3)$, the operator $T^5 f := \sup_n |(|K_n| * f)|$ has also type (∞, ∞) and weak type (1,1) ([5]). From this by Lemma 2. we obtain that $\sup_n |(H_n * f)| \leq 0(1)(T^1 f + T^5 f)$, i.e. the operator $H^* := \sup_n |(H_n * f)|$ has the above type. Applying Lemma 2. we have that

$$Tf \leq 0(1)(T^1 f + T^4 f + H^* f),$$

hence the operator T has type (∞, ∞) and weak type (1,1).

This completes the proof of THEOREM.

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SU UNA CONGETTURA DI SEPPO ILKKA

F. KÁRTESZI
Istituto di Geometria, Università L. Eötvös, Budapest

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1. Ci occuperemo nella presente Nota della nozione di *semiordine* in uno affine di GALOIS di ordine dispari. P. KUSTAANHEIMO [1] è stato fra i primi chi studiava le geometrie di GALOIS in relazione all'astronomia ed alla fisica e tra alcuni anni — nel 1957 — ha introdotto una certa nozione di *ordine* (o meglio di semiordine), Fra questa nozione e quella classica si presentano analogia, ma nello stesso tempo alcuni diversioni sorprendenti. Un vasto ed interessante lavoro di SEGRE [2] contiene tutti i risultati che sono finora ottenuti, segnalando le questioni apperte. Tra l'altro dimostrava una congettura di S. ILKKA (rispetto alle nozioni «*interno*» ed «*esterno*» ad una conica e ad un segmento).

Per svolgere il presente studio è opportuno premettere alcuni proposizioni, in parte ben note.

Nei seguenti — per la brevità — *piano* vuol dire: «piano affine di GALOIS di ordine dispari». Per ogni $k \in GF(q)$ (dove $GF(q)$ è un campo di GALOIS di q elementi), se $k \neq 0$, scriveremo che

$$k = \square \quad \text{oppure} \quad k = \triangle \tag{1.1}$$

a seconda che in $GF(q)$ esista o non esista un elemento $x(\neq 0)$ tale che $x^2 = k$. Il prodotto o quoziente di un *quadrato* ed un *non-quadrato* è sempre un non-quadrato, mentre il prodotto o quoziente di due quadrati o di due non-quadrati è sempre un quadrato. Il numero di elementi quadrati ed anche di non-quadrati: $(q-1)/2$. Gli elementi non nulli di $GF(q)$ che sono quadrati formano un sottogruppo moltiplicativo di $GF(q)$.

Ora la definizione del punto interno rispetto al segmento P_1P_2 può regolarsi al modello classico. Considerando quest' ultimo, siano P_1, P_2, P tre punti allineati che hanno il rapporto semplice $(P_1P_2P) = k$; il punto P è un punto interno al segmento P_1P_2, se è soltanto se $k<0$. Se (x_1, y_1), (x_2, y_2), (x, y) sono le coordinate di tre punti, si ha

$$k = \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2}. \tag{1.2}$$

Sul piano di GALOIS è lecito adottare per definizione del rapporto semplice la formula (1.2), se le lettere denotano elementi di $GF(q)$, ma $k \neq 0, 1$. *Se e soltanto se* $k = \triangle$ *– cioè non-quadrato – diremo che* P *è un punto interno rispetto al segmento* $P_1 P_2$. (Punto *esterno* se $k = \square$.)

Come sul piano euclideo anche sul piano di GALOIS il punto P non situato su un'ellisse $\mathcal{E}$ è un punto esterno od interno secondocche per esso passano o non passano tangenti di $\mathcal{E}$.

Nella geometria euclidea un punto esterno al segmento $P_1 P_2$ è anche esterno rispetto ad ogni ellisse che passa per i due punti P_1 e P_2. Nel caso del piano di GALOIS non è così. Recentemente S. ILKKA [3] ha studiato questo fatto utilizzando un elaboratore elettronico e ha formulato una congettura relativo al punto *pseudoregolare* (rispetto ad una conica).

Ma prima di tutto daremo la definizione di punto pseudoregolare. Diremo che un punto P non giacente sull'ellisse $\mathcal{E}$ risulta pseudoregolare rispetto a $\mathcal{E}$, quando P sia esterno rispetto a $\mathcal{E}$ ma interno rispetto ad ognuno dei segmenti intercetti da $\mathcal{E}$ sulle singole rette per P secanti la $\mathcal{E}$.

Ora possiamo formulare brevemente la congettura di ILKKA: *Punti pseudoregolari rispetto ad un'ellisse possano esistere soltanto per piccoli valori di* q.

Nella presente nota non intendiamo parlare delle iperbole e parabola. Ma rileviamo ancora due lemmi utili e necessarii.

È noto che

$$-1 = \square \quad \text{oppure} \quad -1 = \triangle$$

secondo che

$$q \equiv 1 \pmod 4 \quad \text{oppure} \quad q \equiv 3 \pmod 4 . \tag{1.3}$$

L'insieme dei punti di un'ellisse $\mathcal{E}$ può essere rappresentato con un $(q+1)$-agono regolare del piano euclideo che rifletta tutti i parallelismi che si presentano fra i secanti (e tangenti) di $\mathcal{E}$. La spiegazione di questo fatto segue da un noto lemma: Considerando i punti di $\mathcal{E}$ esiste una tale permutazione ciclica di essi, $P_0 P_1 \ldots P_q$, che le rette $P_j P_h$ e $P_k P_l$ siano parellele se e soltanto se

$$j + h \equiv k + l \pmod q \tag{1.4}$$

($P_j P_j$ denota le tangente di $\mathcal{E}$ nel punto P_j). Cfr. [4], [5].

Nella presente modesta nota daremo (per contributo ad [2]) due modelli euclidei: la configurazione dei punti di un'ellisse se $q = 5$ oppure $q = 7$.

2. Se $q = 5$, per determinare un'ellisse sarà opportuna la forma

$$\mathcal{E} : y^2 \equiv 3x^2 + 1 \pmod 5 . \tag{2.1}$$

Questa forma mediante una traszformazione delle coordinate affine è sempre ottenibile.

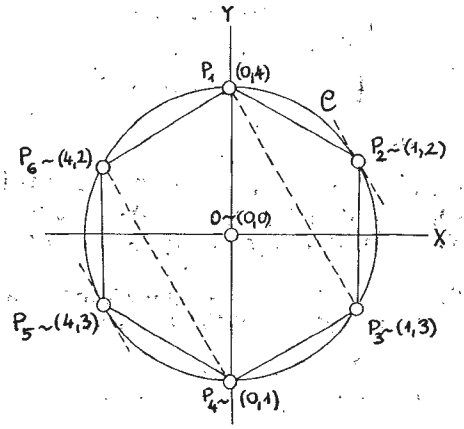

Fig. 1.

Ne segue che i punti di $\mathcal{C}$ sono i seguenti:

$$P_1 \sim (0,4),\ P_2 \sim (1,2),\ P_3 \sim (1,3),\ P_4 \sim (0,1),\ P_5 \sim (4,3),\ P_6 \sim (4,2). \tag{2.2}$$

L'equazione della tangente nel punto P_k della $\mathcal{C}$:

$$y_k y \equiv 3x_k x + 1 \pmod 5.$$

Ora consideriamo i punti tangenziali P_j e P_k e le tangenti in essi. Prescindendo dai casi se $|j-k| \equiv 3 \pmod 5$, le due tangenti s'incontrano in un punto, che sia denotato con P_{jk} $(j \neq k)$. In tal modo otteniamo tutti i punti esterni a $\mathcal{C}$:

$$\begin{cases} P_{12} \sim (4,4),\ P_{13} \sim (2,4),\ P_{15} \sim (3,4),\ P_{16} \sim (1,4); \\ P_{23} \sim (3,0),\ P_{24} \sim (2,1),\ P_{26} \sim (0,3);\ P_{34} \sim (4,1),\ P_{35} \sim (0,2); \\ P_{45} \sim (1,1),\ P_{46} \sim (3,1),\ P_{56} \sim (2,0). \end{cases} \tag{2.3}$$

I punti diametralmente opposti — con tangenti in essi coppiamente parallele — formano le coppie:

$$P_1,\ P_4;\ P_2,\ P_5;\ P_3,\ P_6.$$

Le rette P_1P_4, P_2P_5, P_3P_6 s'incontrano nel punto $O \sim (0,0)$, cioè nel centro di $\mathcal{C}$.

I punti rimanenti del piano:

$$(1,0),\ (2,2),\ (2,3),\ (3,2),\ (3,3),\ (4,0).$$

Questi punti e il centro sono i punti interni a $\mathcal{C}$.

I 25 punti prescindendo dal centro O e dai punti di $\mathcal{C}$ formano ancora tre ellissi ulteriori determinate per le equazioni:

$$(2.4)\qquad \begin{cases} \mathcal{C}_1 : y^2 \equiv 3x^2+2 \pmod 5, \\ \mathcal{C}_2 : y^2 \equiv 3x^2+3 \pmod 5, \\ \mathcal{C}_3 : y^2 \equiv 3x^2+4 \pmod 5. \end{cases}$$

Particolalareggiamente:

$\mathcal{C}_1 : \{(2,2), (4,0), (2,3), (3,3), (1,0), (3,2)\}$ – punti interi a $\mathcal{C}$ –,
$\mathcal{C}_2 : \{(1,1), (2,0), (1,4), (4,4), (3,0), (4,1)\}$ – punti esterni a $\mathcal{C}$ –,
$\mathcal{C}_3 : \{(0,2), (3,1), (3,4), (0,3), (2,4), (2,1)\}$ – punti esterni a $\mathcal{C}$ –.

Per mezzo dei risultati finora ottenuti potremo disegnare i modelli euclidei che fanno vedere eleganti configurazioni.

3. *La relazone fra* $\mathcal{C}$ *e* $\mathcal{C}_1$ (fig. 2).
Ogni secondo lato dell'essagono costituiti dai punti di $\mathcal{C}_1$ è lato di un triangolo iscritto in $\mathcal{C}$.

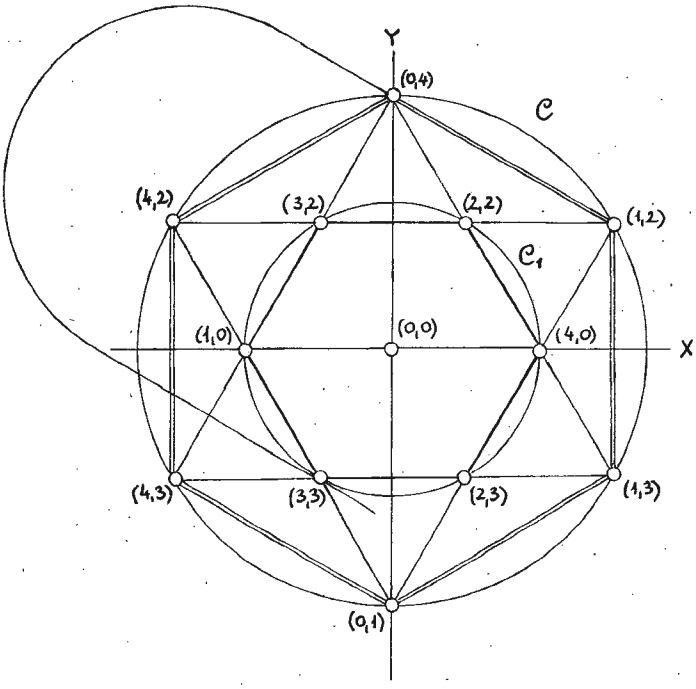

Fig. 2.

L'esagono costituito dai punti di $\mathcal{C}$ è un poligono circoscritto a $\mathcal{C}_1$. – Ad esempio la retta [(0,4), (1,2)] è la tangente di $\mathcal{C}_1$ nel punto (3,3). L'equazione di questa tangente:

$$3y \equiv 3 \cdot 3x+2 \pmod 5 .$$

Le coordinate 0,4 e 1,2 soddisfano all'equazione. — Particolareggiamente: le rette P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6, P_6P_1 toccano la $\mathcal{C}_1$ in ordine nei punti (3,3), (1,0), (3,2), (2,2), (4,0), (2,3).

La relazione fra $\mathcal{C}$ e $\mathcal{C}_2$ (fig. 3)

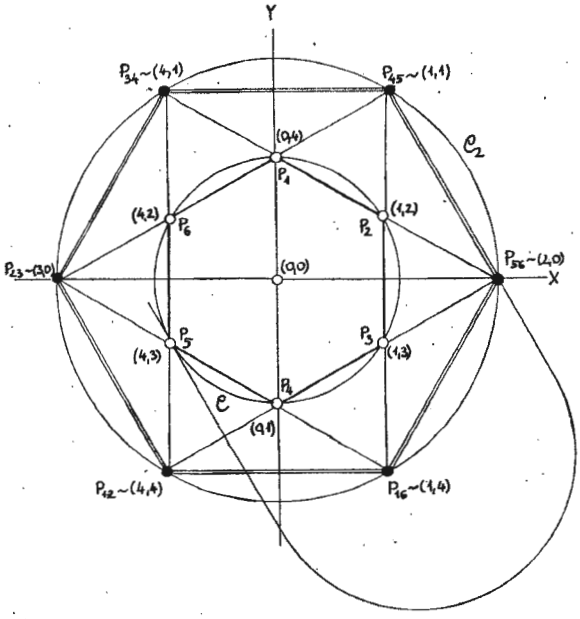

Fig. 3.

Similmente alla precedenti è facile verificare che: Ogni secondo lati dell'esagono $P_1P_2P_3P_4P_5P_6$ risultano — come lati di due triangoli iscriti nella $\mathcal{C}_2$:

$$P_{56}P_{12}P_{34} \text{ e } P_{16}P_{23}P_{45}\,.$$

I lati dell'esagono $P_{45}P_{56}P_{16}P_{12}P_{23}P_{34}$ sona le tangenti di $\mathcal{C}$ nei punti P_5, P_6, P_1, P_2, P_3, P_4. — Ad esempio le equazioni delle tangenti nel punto (0,1) e nel punto (4,3):

$$y \equiv 3\cdot 0\cdot x+1 \pmod 5 \text{ e } 3y \equiv 3\cdot 4\cdot x+1 \pmod 5\,.$$

Ne segue che le rette s'incontrano nel punto (1.1). —

Dunque la relazione fra $\mathcal{C}_2$ e $\mathcal{C}$ è confórme alla relazione fra $\mathcal{C}$ e $\mathcal{C}_1$: *un esagono iscritto nella prima e circoscritto alla seconda.*

La relazione fra $\mathcal{C}$ e $\mathcal{C}_3$ (fig. 4).

Ci sono ancora sei punti del piano, sei punti esterni a $\mathcal{C}$: P_{35}, P_{46}, P_{15}, P_{26}, P_{13}, P_{24}. — Considerando il modello euclideo dobbiano scorgere che i rappresentanti di questi punti sono (necessariamente) interni a $\mathcal{C}$. —

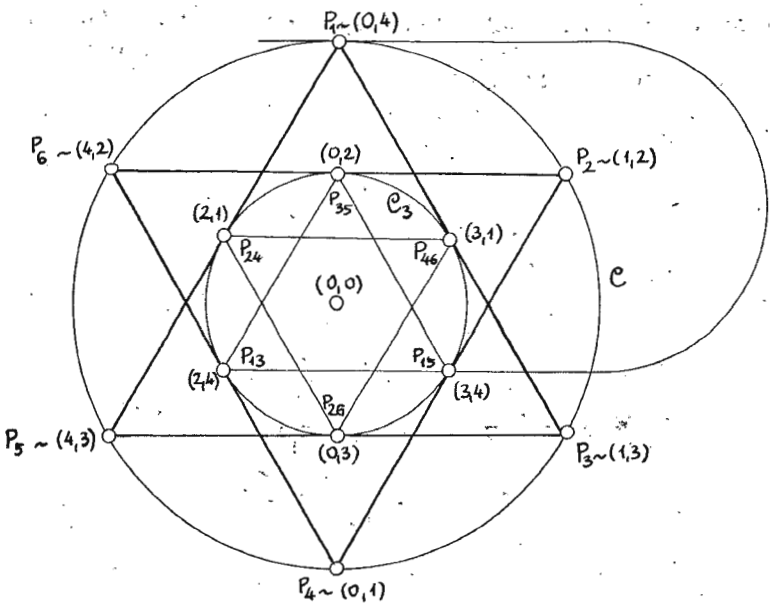

Fig. 4.

Come precedentemente si può verificare la seguente proprietà: *Si può rilevare due triangoli iscritti in una qualunque delle* $\mathcal{C}$ *e* $\mathcal{C}_3$ *che siano circoscritti all' altra.* Una delle coppie di triangoli: $P_1 P_3 P_5$ e $P_2 P_4 P_6$; l'altra $P_{35} P_{15} P_{13}$ e $P_{46} P_{26} P_{24}$. — Sul piano euclideo tale relazione simmetrica è ben conosciuta, ma con numero infinito di tali triangoli, come dimostrà PONCELET. In tal caso le due coniche s'incontrano in alcuni punti. —

Per ultima osservazione rileviamo che *i punti di* $\mathcal{C}_2$, *ma soltanto essi sono pseudoregolari rispetto a* $\mathcal{C}$. — Cfr. [2] —

Per ogni punto di $\mathcal{C}_2$ passano due secanti alla $\mathcal{C}$. Consideriamo — ad esempio — la secante $P_1 P_6$ che passa per P_{45}. Essendo $P_1 \sim (0,4)$, $P_6 \sim (4,2)$ e $P_{45} \sim (1,1)$, in virtù della (1.2)

$$(P_1 P_6 P_{45}) \equiv \frac{1-0}{1-4} \equiv \frac{1-4}{1-2} \equiv 3 \pmod 5$$

e $3 = \triangle$, ne discende che il punto esterno P_{45} è un punto interno rispetto al segmento $P_1 P_6$. Calcolando in tal modo, si può verificare la proposizione di sopra.

4. Se $q = 7$, l'equazione canonica di un'ellisse è

$$\mathcal{C} : y^2 \equiv 3x^2 + 1 \pmod 7 . \tag{4.1}$$

Se consideriamo lo scopo — cioè costruire un modello euclideo che rappresenta tutti i parallelismi delle corde e tangenti appartenenti a $\mathcal{C}$ — sarà utile ordimare i punti di $\mathcal{C}$ in una ordine ciclica:

$$(4.2)\qquad \begin{cases} P_1 \sim (0,1),\ P_2 \sim (1,2),\ P_3 \sim (4,0),\ P_4 \sim (1,5), \\ P_5 \sim (0,6),\ P_6 \sim (6,5),\ P_7 \sim (3,0),\ P_8 \sim (6,2). \end{cases}$$

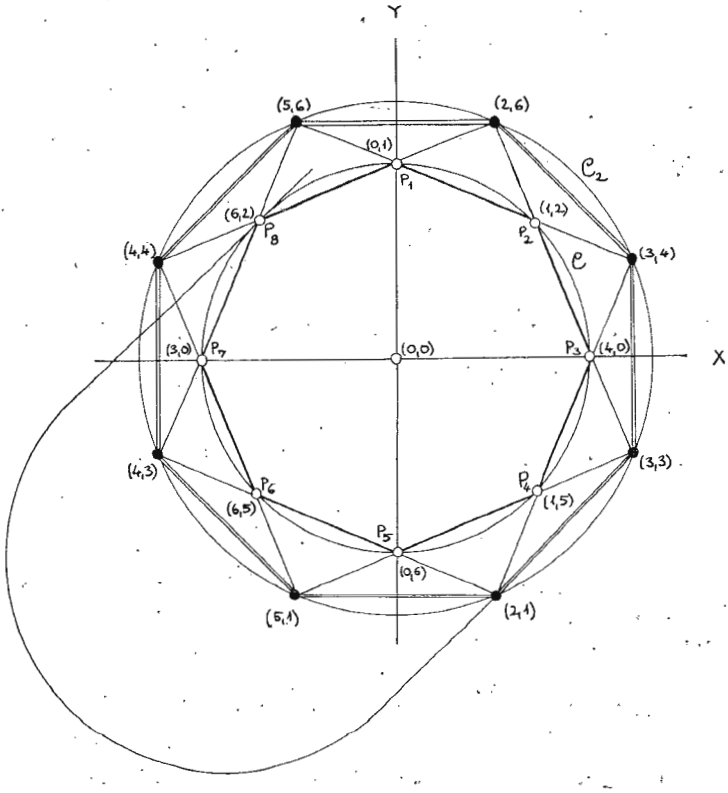

Fig. 5.

Con un calcolo sempiicissimo si ottiene l'insieme dei punti pseudoregolari rispetto a $\mathcal{C}$. Sia denotata con l_{jk} la retta P_jP_k. Ora i punti interni rispetto al segmento P_jP_k sono i seguenti:

$$(4.3)\qquad \left(\frac{x_j+x_k}{2},\ \frac{y_j+y_k}{2}\right),\ \left(\frac{x_j+2x_k}{3},\ \frac{y_j+2y_k}{3}\right),\ \left(\frac{x_j+4x_k}{5},\ \frac{y_j+4y_k}{5}\right),$$

essendo 3, 5, 6 i non-quadrati e

$$-3 \equiv 4 \pmod 7,\quad -5 \equiv 2 \pmod 7,\quad -6 \equiv 1 \pmod 7.$$

Ne segue — ad esempio-, che i punti interni rispetto al segmento P_1P_2 sono i seguenti:

$$(4,5),\ (3,4),\ (5,6)\,.$$

Ma si può controllare facilmente che essi sono punti esterni alla conica $\mathcal{C}$. Similmente i punti interni a P_3P_4 ed i punti interni a P_7P_8 sono i seguenti:

$$(6,6),\ (2,1),\ (3,4)\ \text{e}\ (3,4),\ (4,1),\ (2,0)\,,$$

ma nello stesso tempo essi sono punti esterni rispetto alla conica $\mathcal{C}$. Come si vede le secanti passano per punto (3,4), dunque (3,4) è un punto pseudoregolare rispetto a $\mathcal{C}$.

In tal modo otteniamo tutti i punti pseudoregolari:

$$(5,6),\ (2,6),\ (3,4),\ (3,3),\ (2,1),\ (5,1),\ (4,3),\ (4,4)\,. \tag{4.4}$$

Ora si vede questi punti formano un'ellisse di equazione

$$\mathcal{C}_2 : y^2 \equiv 3x^2 + 3 \pmod 7\,. \tag{4.5}$$

La tangente di $\mathcal{C}$ che tocca essa nel punto (x_0, y_0) ha l'equazione:

$$y_0\, y \equiv 3x_0\, x + 3 \pmod 7\,. \tag{4.6}$$

Consideriamo — ad esempio — la retta [(3,3), (2,1)] e la tangente di $\mathcal{C}$ nel punto (6,2) si vede subito che esse coincidono. In tal modo discende la lista delle rette

$$\begin{gathered}[(2,1),(5,1)],\ [(5,1),(4,3)],\ [(4,3),(4,4)],\ [(4,4),(5,6)]\,,\\ [(5,6),(2,6)],\ [(2,6),(3,4)],\ [(3,4),(3,3)],\ [(3,3),(2,1)]\,.\end{gathered} \tag{4.7}$$

toccano la $\mathcal{C}$ in ordine nei punti di essa

$$(0,1),\ (1,2),\ (4,0),\ (1,5),\ (0,6),\ (6,5),\ (3,0),\ (6,2)\,. \tag{4,8}$$

Ne segue che l'ottagono iscritto in $\mathcal{C}_2$ e circoscritto a $\mathcal{C}$.

Dunque la relazione fra $\mathcal{C}_2$ e $\mathcal{C}$ nel caso $q = 7$ è la stessa che si presenta nel case se $q = 5$.

Osservazione. — L'autore non ha intenzione continuare di questo studio perchè alcuni dei suoi allievi desiderebbero studiare ed analizzare le questioni generali che rimangono ancora apperti in [2] appoggiandosi sui modelli iniziali della presente Nota.

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CONVERGENT FILTERS IN THE FINAL TOPOLOGY AND A CHARACTERIZATION OF REGULARITY

By

Cs. NÉMETHI

Babeş–Bolyai University, Cluj–Napoca (Romania)

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1. Filter-Convergence with Respect to the Final Topology

The purpose of this section is to examine some problems concerning convergent filters, similar to those considered for nets in the author's paper [4]. We shall use when possible the same terminology and notations as in the case of nets. We omit those proofs which are simple modifications of the corresponding ones for nets.

Let X be a set. We denote by $\Phi(X)$ the set of all (proper) filters on X. If $\mathcal{F}\in\Phi(X)$, we define the set $\mathcal{G}(\mathcal{F})$ by the equality:

$$\mathcal{G}(\mathcal{F})=\{D\subseteq X\,|\,(\forall\, F\in\mathcal{F})\, D\cap F\neq\emptyset\}.$$

For every $x\in X$ the principal filter on X generated by the set $\{x\}$ will be denoted by $\dot{x}$. If Y is a set and $\mathcal{F}\in\Phi(X)$, then for any mapping $f:X\to Y$ we denote by $f(\mathcal{F})$ the image of $\mathcal{F}$ by f, considered as a filter on Y. If $(\mathcal{F}_i)_{i\in I}$ is a set-indexed family of filters on X and $\mathcal{J}\in\Phi(I)$, we shall denote by $\underset{i\in I,\,\mathcal{J}}{K}\,\mathcal{F}_i$, or simply by $K\mathcal{F}_i$ the filter on X for which a base is the collection of all unions of the form $\bigcup\limits_{i'\in I'} F_{i'}$, where $I'\in\mathcal{J}$ and $F_{i'}\in\mathcal{F}_{i'}$ for each $i'\in I'$.

The following result will be useful later:

LEMMA 1. *Let $\mathcal{F}\in\Phi(X)$ and $A\subseteq X$. Then $A\in\mathcal{F}$ iff $A\cap D\neq\emptyset$ for every $D\in\mathcal{G}(\mathcal{F})$.*

By a *generalized convergence stucture* (GCS) on X we shall mean a set $\mathcal{C}\subseteq\Phi(X)\times X$. A GCS on X may have any or all of the following properties:

(C1) If $(\mathcal{F}, x)\in\mathcal{C}$ and $\mathcal{F}'\in\Phi(X)$ with $\mathcal{F}\subseteq\mathcal{F}'$, then $(\mathcal{F}', x)\in\mathcal{C}$;

(C2) If the filter $\mathcal{F}\in\Phi(X)$ possesses the property:

$$(1)\qquad (\forall\, D\in\mathcal{G}(\mathcal{F}))\,(\exists\,\mathcal{F}_1\in\Phi(X))\,[D\in\mathcal{F}_1\wedge(\mathcal{F}_1, x)\in\mathcal{C}],$$

then $(\mathcal{F}, x)\in\mathcal{C}$;

(C2′) If the filter $\mathcal{F}\in\Phi(X)$ possesses the property:

$$(2)\qquad (\forall\mathcal{F}'\in\Phi(X) \text{ with } \mathcal{F}\subseteq\mathcal{F}')(\exists\mathcal{F}''\in\Phi(X))\ [\mathcal{F}'\subseteq\mathcal{F}''\wedge(\mathcal{F}'',x)\in\mathcal{C}],$$

then $(\mathcal{F},x)\in\mathcal{C}$;

(C3) If $x\in X$, $(\mathcal{F}_i)_{i\in I}$ is a set-indexed family of filters on X, $\mathcal{J}\in\Phi(I)$ and $\varphi:I\to X$ is a mapping such that $(\varphi(\mathcal{J}),x)\in\mathcal{C}$ and $(\mathcal{F}_i,\varphi(i))\in\mathcal{C}$ for each $i\in I$, then there exists an $\mathcal{F}\in\Phi(X)$ with $\bigcap_{i\in I}\mathcal{F}_i\subseteq\mathcal{F}$ and $(\mathcal{F},x)\in\mathcal{C}$;

(C3′) Under the hypotheses of (C3) $(K\mathcal{F}_i,x)\in\mathcal{C}$;

(C4) $(\forall x\in X)\ (\dot{x},x)\in\mathcal{C}$.

Obviously (C2)⇒(C2′) and (C3′)⇒(C3).

LEMMA 2. [1]. *Suppose that $\mathcal{C}$ satisfies Axioms* (C3′), (C4) *and let* $x\in X$, $\mathcal{F}\in\Phi(X)$, *If*

$$(\forall F\in\mathcal{F})(\exists\mathcal{F}_1\in\Phi(X))\,[F\in\mathcal{F}_1\wedge(\mathcal{F}_1,x)\in\mathcal{C}],$$

then there exists an $\mathcal{F}'\in\Phi(X)$ with $\mathcal{F}\subseteq\mathcal{F}'$ and $(\mathcal{F}',x)\in\mathcal{C}$.

With the aid of this lemma one can prove:

THEOREM 1. *If $\mathcal{C}$ satisfies Axioms* (C3′), (C4), *then for every $x\in X$ and every $\mathcal{F}\in\Phi(X)$ the statements* (1) *and* (2) *are equivalent.*

COROLLARY. *If $\mathcal{C}$ satisfies Axioms* (C3′), (C4), *then the properties* (C2) *and* (C2′) *are equivalent.*

In the paper [4] a set $\mathcal{T}\subseteq\mathcal{P}(X)$ was called a *generalized topology* (GT) on X. We recall the possible axioms for $\mathcal{T}$:

(01) If $\mathcal{A}\subseteq\mathcal{T}$, then $\bigcup\mathcal{A}\in\mathcal{T}$;

(02) $X\in\mathcal{T}$ and if $G_1, G_2\in\mathcal{T}$, then $G_1\cap G_2\in\mathcal{T}$.

Given a GCS $\mathcal{C}$ on the set X, we can associate two GTs $\mathcal{T}_1(\mathcal{C})$ and $\mathcal{T}_2(\mathcal{C})$ with it, defined as follows:

$$\mathcal{T}_1(\mathcal{C})=\{G\subseteq X\,|\,[(\mathcal{F},x)\in\mathcal{C}\wedge x\in G]\Rightarrow G\in\mathcal{G}(\mathcal{F})\};$$

$$\mathcal{T}_2(\mathcal{C})=\{G\subseteq X\,|\,[(\mathcal{F},x)\in\mathcal{C}\wedge x\in\mathcal{G}]\Rightarrow G\in\mathcal{F}\}.$$

Obviously $\mathcal{T}_2(\mathcal{C})\subseteq\mathcal{T}_1(\mathcal{C})$.

THEOREM 2. *If $\mathcal{C}$ satisfies Axiom* (C1), *then $\mathcal{T}_1(\mathcal{C})=\mathcal{T}_2(\mathcal{C})$.*

If $\mathcal{T}_1(\mathcal{C})$ and $\mathcal{T}_2(\mathcal{C})$ are equal, we denote their common value by $\mathcal{T}(\mathcal{C})$. For any GT $\mathcal{T}$ *on the set X we define the* GCS $\mathcal{C}(\mathcal{T})$ *on X as follows:*

$$\mathcal{C}(\mathcal{T})=\{(\mathcal{F},x)\in\Phi(X)\times X\,|\,(G\in\mathcal{T}\wedge x\in G)\Rightarrow G\in\mathcal{F}\}.$$

LEMMA 3. *$\mathcal{T}_1(\mathcal{C})$ satisfies Axiom* (01) *and $\mathcal{T}_2(\mathcal{C})$ satisfies Axioms* (01), (02).

LEMMA 4. *$\mathcal{C}(\mathcal{T})$ satisfies Axioms* (C1), (C2), (C2′), (C3), (C3′), (C4).

Let $\mathcal{C}$ be a GCS on X and $M \subseteq X$. We define the closure $C_1(M) = C_1^{\mathcal{C}}(M)$ (resp. $C_2(M) = C_2^{\mathcal{C}}(M)$) of M as the least of all sets $H \subseteq X$ with $M \subseteq H$ and $X \setminus H \in \mathcal{T}_1(\mathcal{C})$ (resp. $X \setminus H \in \mathcal{T}_2(\mathcal{C})$). Taking into account the relation between $\mathcal{T}_1(\mathcal{C})$ and $\mathcal{T}_2(\mathcal{C})$ we deduce that $C_1(M) \subseteq C_2(M)$ and if $\mathcal{C}$ satisfies Axiom (C1), then the equality holds.

In order to find an expression of these two closures let us consider for any $M \subseteq X$ the sets:

$$\mathrm{Lim}_1(M) = \mathrm{Lim}_1^{\mathcal{C}}(M) = \{x \in X \mid (\exists \mathcal{F} \in \Phi(X))\, [M \in \mathcal{F} \wedge (\mathcal{F}, x) \in \mathcal{C}]\};$$

$$\mathrm{Lim}_2(M) = \mathrm{Lim}_2^{\mathcal{C}}(M) = \{x \in X \mid (\exists \mathcal{F} \in \Phi(X))\, [M \in \mathcal{G}(\mathcal{F}) \wedge (\mathcal{F}; x) \in \mathcal{C}]\}.$$

Again we have $\mathrm{Lim}_1(M) \subseteq \mathrm{Lim}_2(M)$ and if $\mathcal{C}$ satisfies Axiom (C1), then the equality holds.

THEOREM 3. a) *If $\mathcal{C}$ satisfies Axioms* (C3), (C4), *then* $C_1(M) = \mathrm{Lim}_1(M)$.

b) *For each $\mathcal{F} \in \mathrm{pr}_1(\mathcal{C}) = \{\mathcal{F} \in \Phi(X) \mid (\exists x \in X)\ (\mathcal{F}, x) \in \mathcal{C}\}$ let $\mathcal{B}_{\mathcal{F}}$ be a base of $\mathcal{F}$. Let δ be a regular ordinal with $\delta > 1$ (i.e., $\delta \geq \omega$), having a cardinality strictly greater than that of each $\mathcal{B}_{\mathcal{F}}$, and let α_0 be a fixed ordinal with $0 < \alpha_0 < \delta$. Then*

$$C_2(M) = \bigcup_{\alpha < \delta} M_\alpha, \text{ where } M_0 = M,\ M_\alpha \subseteq \mathrm{Lim}_2\Big(\bigcup_{\beta < \alpha} M_\beta\Big)$$

is arbitrary for $0 < \alpha < \alpha_0$ and

$$M_\alpha = \mathrm{Lim}_2\Big(\bigcup_{\alpha < \beta} M_\beta\Big) \text{ for } \alpha_0 \leq \alpha < \delta.$$

PROOF. We show only that $X \setminus \bigcup_{\alpha < \delta} M_\alpha \in \mathcal{T}_2(\mathcal{C})$. For this purpose let $(\mathcal{F}, x) \in \mathcal{C}$ such that $\bigcup_{\alpha < \delta} M_\alpha \in \mathcal{G}(\mathcal{F})$. Then for each $B \in \mathcal{B}_{\mathcal{F}}$ we have $B \cap \bigcap\Big(\bigcup_{\alpha < \delta} M_\alpha\Big) \neq \emptyset$, thus there exists an $\alpha_B < \delta$ with $B \cap M_{\alpha_B} \neq \emptyset$. The set $\{\alpha_B \mid B \in \mathcal{B}_{\mathcal{F}}\}$ is contained in the segment $[0, \delta)$ and has a cardinality strictly less than that of δ, thus its ordinal number is strictly less than δ. Using the regularity of δ we can find an ordinal $\gamma < \delta$ such that $\alpha_0 \leq \gamma$ and $\alpha_B < \gamma$ for each $B \in \mathcal{B}_{\mathcal{F}}$. Then $B \cap \Big(\bigcup_{\beta < \gamma} M_\beta\Big) \neq \emptyset$ for every $B \in \mathcal{B}$, i.e., $\bigcup_{\beta < \gamma} M_\beta \in \mathcal{G}(\mathcal{F})$. It follows that

$$x \in \mathrm{Lim}_2\Big(\bigcup_{\beta < \gamma} M_\beta\Big) = M_\gamma \subseteq \bigcup_{\alpha < \delta} M_\alpha.$$

REMARK. In the case of nets [4] the author has introduced two GTs: $\mathcal{T}_1(\mathcal{C})$, $\mathcal{T}_2(\mathcal{C})$ and a single notion of closure, corresponding to $\mathcal{T}_1(\mathcal{C})$. In the reality one can define for nets one more GT, which lies between $\mathcal{T}_2(\mathcal{C})$ and $\mathcal{T}_1(\mathcal{C})$ (replacing the word "eventually" by "frequently" in the definition of $\mathcal{T}_2(\mathcal{C})$) and two more notions of closure, corresponding to $\mathcal{T}_2(\mathcal{C})$ and to this new GT. Thus one can consider everywhere besides $\mathcal{T}_1(\mathcal{C})$ and $\mathcal{T}_2(\mathcal{C})$ this intermediary GT and Theorem 3 in [4] can be completed by two more statements concerning the two new closures.

Having a GCS $\mathcal{C}$ on X we define the GCS $\hat{\mathcal{C}}$ on X as follows:

$$\hat{\mathcal{C}} = \{(\mathcal{F}, x) \in \Phi(X) \times X \mid (\forall D \in \mathcal{G}(\mathcal{F}))\, x \in C_1(D)\}.$$

THEOREM 4. *We have the equality:* $\hat{\mathcal{C}} = \mathcal{C}(\mathcal{T}_1(\mathcal{C}))$.

Like in the case of nets, the proof is facilited by using Lemma 1.

One can easily translate the remaining results and definitions from [4] for the case of filters. We recall the last theorem in terms of filters:

THEOREM 5. *Let* $((X_p, \mathcal{T}_p))_{p \in P}$ *be a family of topological spaces (P is a class), Y a set and, for each* $p \in P$, φ_p *a mapping of* X_p *into Y. Denoting by* τ *the final topology of the topologies* $\mathcal{T}_p$, $p \in P$, *with respect to the mappings* φ_p, $p \in P$, *we have the equalities:*

$$\mathcal{C}(\tau) = \mathcal{C}(\mathcal{T}(\mathcal{C}^*)) = \hat{\mathcal{C}}^*,$$

where

$$\mathcal{C}^* = \{(\mathcal{F}, y) \in \Phi(Y) \times Y \mid (\exists p \in P)\, (\exists (\mathcal{F}_p, x_p) \in \mathcal{C}(\mathcal{T}_p))$$
$$[\varphi_p(\mathcal{F}_p) \subseteq \mathcal{F} \wedge \varphi_p(x_p) = y]\}.$$

2. The Equivalence of Two Regularity Axioms

In their paper [1] C. H. COOK and H. R. FISCHER have given a new characterization for the regularity of a topological space in terms of filters and have shown that if this characterization, as well as J. DIEUDONNÉ's one [2] are translated for FISCHER's convergence spaces [3], the second implies the first. We shall prove that they are in fact equivalent.

For any two sets A, B and any $\mathcal{U} \in \Phi(A)$, $\mathcal{V} \in \Phi(B)$ let us denote by $\mathcal{U} \otimes \mathcal{V}$ the product filter on $A \times B$, for which a base is formed of the sets $U \times V$, with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

THEOREM 6. *Let* $\mathcal{C}$ *be a GCS on a set X, satisfying Axiom* (C1). *Then the following two statements are equivalent:*

(i) (COOK-FISCHER) *If* $x \in X$, $(\mathcal{F}_i)_{i \in I}$ *is a set-indexed family of filters on X,* $\mathcal{I} \in \Phi(I)$ *and* $\varphi: I \to X$ *is a mapping such that* $(K\, \mathcal{F}_i, x) \in \mathcal{C}$ *and* $(\mathcal{F}_i, \varphi(i)) \in \mathcal{C}$ *for each* $i \in I$, *then* $(\varphi(\mathcal{I}), x) \in \mathcal{C}$;

(ii) (DIEUDONNÉ) *If* $x \in X$, A, B *are sets,* $\mathcal{U} \in \Phi(A)$, $\mathcal{V} \in \Phi(B)$ *and* $f: A \times B \to X$, $g: A \to X$ *are mappings such that* $(f(\mathcal{U} \otimes \mathcal{V}), x) \in \mathcal{C}$ *and* $(f(\dot{a} \otimes \mathcal{V}), g(a)) \in \mathcal{C}$ *for each* $a \in A$, *then* $(g(\mathcal{U}), x) \in \mathcal{C}$.

PROOF. The implication (ii)$\Rightarrow$(i) was proved by COOK and FISCHER and we remark that their proof does not require Axiom (C1). Conversely, supposing that (i) holds, with the notations of (ii) let $\mathcal{F}_a = f(\dot{a} \otimes \mathcal{V})$ for each $a \in A$. As the sets $f(\{a\} \times V)$, with $V \in \mathcal{V}$, form a base for $\mathcal{F}_a$, it follows that the collection of all unions $\bigcup_{u \in U} f(\{u\} \times V_u)$, where $U \in \mathcal{U}$ and $V_u \in \mathcal{U}$ for each $u \in U$, is a base for $\underset{a \in A, \mathcal{U}}{K}\, \mathcal{F}_a$. At the same time, taking

the sets $f(U\times V) = \bigcup_{u\in U} f(\{u\}\times V)$, with $U\in\mathcal{U}$ and $V\in\mathcal{V}$, we obtain a base for $f(\mathcal{U}\otimes\mathcal{V})$. Thus we can see that $K\mathcal{F}_a\supseteq f(\mathcal{U}\otimes\mathcal{V})$ and hence, by Axiom (C1), $(K\mathcal{F}_a, x)\in\mathcal{C}$. In addition, by the construction of $\mathcal{F}_a$, we have $(\mathcal{F}_a, g(a))\in\mathcal{C}$ for each $a\in A$. Thus, using (i), it follows that $(g(\mathcal{U}), x)\in\mathcal{C}$.

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DISTRIBUTION OF DIGITS IN CANONICAL REPRESENTATION OF GAUSSIAN INTEGERS

By

I. KÁTAI

Department of Numerical Methods and Computer Scince of the
L. Eötvös University, Budapest

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1. In a recent paper [1] we have considered the following problem. We say that a Gaussian integer ϑ is a canonical base, if every Gaussian integer α can be represented and uniquely in the form

$$\alpha = a_0 + a_1 \vartheta + \ldots + a_k \vartheta^k, \tag{1.1}$$

$$a_j \in \mathfrak{A} = \{0, 1, \ldots, N(\vartheta) - 1\}, \tag{1.2}$$

$$N(\vartheta) = |\vartheta|^2.$$

We have proved that ϑ is a canonical base, if and only if $Re\,\vartheta < 0$ and $Im\,\vartheta = \pm 1$.

In what follows ϑ denotes a canonical base, α denotes a general Gaussian integer. Let $f(a)$ be a function defined on $\mathfrak{A}$,

$$A(\alpha) = \sum_{j=0}^{k} f(a_j). \tag{1.3}$$

Our purpose is to consider the distribution of $A(\alpha)$. Let $C(r, b)$ denote the disc with radius r and centre b. Let B be a positive number satisfying the inequality

$$B \geq \frac{N(\vartheta)}{|\vartheta| - 1}. \tag{1.4}$$

For an α stated in (1.1) we write

$$\alpha = \alpha_1 \vartheta + a_0, \quad \alpha_1 = \alpha_2 \vartheta + a_1, \quad \ldots.$$

First we observe that if α runs over the integers of $C(r, b)$, then $\alpha_1, \alpha_2, \ldots$ run over the integers of

$$C\left(\frac{r}{|\vartheta|}, b_1\right), \; C\left(\frac{r}{|\vartheta|^2}, b_2\right), \; \ldots,$$

respectively, where

$$b_1 = \frac{b-a_0}{\vartheta}, \quad b_2 = \frac{b-a_0}{\vartheta^2} - \frac{a_1}{\vartheta}, \quad \ldots,$$

$$b_\nu = \frac{b-a_0}{\vartheta^\nu} - \frac{a_1}{\vartheta^{\nu-1}} - \ldots - \frac{a_{\nu-1}}{\vartheta}.$$

Furthermore, if $|b| \leq B$, then $|b_\nu| \leq B$ for every ν. Let

$$g = \sum_{a=0}^{N(\vartheta)-1} f(a), \tag{1.5}$$

and

$$T_l(r,b) = \sum_{\alpha \in C(r,b)} \left(A(\alpha) - \frac{g}{N(\vartheta)} \frac{\log r}{\log |\vartheta|} \right)^l. \tag{1.6}$$

Let

$$U = \max_{a \in \mathfrak{A}} |f(a)|. \tag{1.7}$$

From $A(\alpha) = f(a_0) + A(\alpha_1)$ we deduce that

$$\max_{|b| \leq B} \max_{\alpha \in C(r,b)} |A(\alpha)| \leq U + \max_{|b| \leq B} \max_{\alpha_1 \in C\left(\frac{r}{|\vartheta|}, b\right)} |A(\alpha_1)|,$$

whence we get

$$\max_{|b| \leq B} \max_{\alpha \in C(r,b)} |A(\alpha)| = O(\log r). \tag{1.8}$$

Since the number of integers in the sets $C(r,b_1) \backslash C(r,b_2)$ for $|b_1|, |b_2| \leq B$ is $O(r)$, therefore

$$T_l(r,b) - T_l(r,o) = O(r(\log r)^l), \quad |b| \leq B. \tag{1.9}$$

Let

$$c_h = \frac{1}{N(\vartheta)} \sum_{a=0}^{N(\vartheta)-1} \left(f(a) - \frac{g}{N(\vartheta)} \right)^h.$$

It is obvious that $c_0 = 1$, $c_1 = 0$.

Since

$$A(\alpha) - \frac{g}{N(\vartheta)} \cdot \frac{\log r}{\log |\vartheta|} = \left(f(a_0) - \frac{g}{N(\vartheta)} \right) +$$

$$+ \left(A(\alpha_1) - \frac{g}{N(\vartheta)} \frac{\log(|r|/|\vartheta|)}{\log |\vartheta|} \right),$$

therefore

$$T_l(r,b) = \sum_{h=0}^{l} \binom{l}{h} \sum_{a_0=0}^{N(\vartheta)-1} \left(f(a_0) - \frac{g}{N(\vartheta)}\right)^h T_{l-h}\left(\frac{r}{|\vartheta|}, \frac{b-a_0}{\vartheta}\right).$$

Consequenty by (1.9) we get

$$T_l(r,o) = \sum_{h=0}^{l} \binom{l}{h} N(\vartheta)\, c_h T_{l-h}\left(\frac{r}{|\vartheta|}, o\right) + O(r(\log r)^l), \tag{1.10}$$

and by using the notation

$$t_l(r) = \frac{T_l(r,o)}{\pi r^2}, \tag{1.11}$$

that

$$t_l(r) = \sum_{h=0}^{l} \binom{l}{h} c_h t_{l-h}\left(\frac{r}{|\vartheta|}\right) + O\left(\frac{(\log r)^l}{r}\right). \tag{1.12}$$

Now we give the asymptotic of $t_l(r)$ for even l, and the order of it for odd l.

Let μ^* be defined by $r = |\vartheta|^{\mu^*}$ and let $\mu = [\mu^*]$ be the integer part of μ. First we state without proof the following simple

LEMMA 1. *If*

$$H(x) = A(\log x)^c + O((\log x)^{c-1}), \tag{1.13}$$

where A, c are constants, $c>0$, then

$$\sum_{\nu=0}^{\mu} H\left(\frac{r}{|\vartheta|^\nu}\right) = \frac{A}{c+1} \frac{(\log r)^{c+1}}{\log|\vartheta|} + ((\log r)^c). \tag{1.14}$$

We prove the following

THEOREM 1. *We have*

$$T_{2j}(r,o) = A_{2j}\pi r^2 \left(\frac{\log r}{\log|\vartheta|}\right)^j + O(r^2(\log r)^{j-1}), \tag{1.15}$$

$$T_{2j+1}(r,o) = O(r^2(\log r)^j), \tag{1.16}$$

$(j = 0, 1, \ldots)$ *where*

$$A_{2j} = \frac{(2j)!}{2^j \cdot (j+1)!}.$$

PROOF. The relations (1.15), (1.16) are true for $j = 0$. Suppose that they are true for $j = 0, \ldots k-1$. We prove those for $j = k$. From (1.12) we get

$$t_{2k}(r) - t_{2k}\left(\frac{r}{|\vartheta|}\right) = t_{2k-2}\left(\frac{r}{|\vartheta|}\right) \cdot \binom{2k}{2} + O((\log r)^{k-2}).$$

Taking this for $\frac{r}{|\vartheta|^\nu}$ $(\nu = c, 1, \ldots, \mu-1)$, and adding, we get

$$t_{2k}(r) - t_{2k}\left(\frac{r}{|\vartheta|^\mu}\right) = \binom{2k}{2} \sum_{\nu=1}^{\mu} t_{2k-2}\left(\frac{r}{|\vartheta|^\nu}\right) + O((\log r)^{k-1}).$$

By using Lemma 1, and (1.15) for $j = k-1$, we deduce that

$$t_{2k}(r) = \frac{2k\cdot(2k-1)}{2(k+1)} \cdot A_{2k-2}\left(\frac{\log r}{\log|\vartheta|}\right)^k + O((\log r)^{k-1}),$$

which proves (1.15) for $j = k$. The proof of (1.16) is similar, and so we omit it.

2. Now we consider the distribution of

$$\frac{A(\alpha) - \frac{g}{N(\vartheta)} \frac{\log r}{\log|\vartheta|}}{\left(\frac{\log r}{\log|\vartheta|}\right)^{1/2}}$$

for $\alpha \in C(r, 0)$.

For this we need a well known result from the probability theory.

In the sequel $\delta, \delta_1, \delta_2, \ldots$ denote suitable positive constants.

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be completely independent random variables assuming the values $f(s)$ $(s = 0, 1, \ldots, N(\vartheta)-1)$ with probability

$$P(\xi_i = f(s)) = \frac{1}{N(\vartheta)} \quad (i = 1, 2, \ldots).$$

Let $M(\xi)$ and $D(\xi)$ denote the mean-value and the variance of a random variable ξ, respectively, further let for the sake of brevity

$$M = M(\xi_i) = \frac{g}{N(\vartheta)}, \quad D^2 = D^2(\xi_i) = c_2 = \frac{1}{N(\vartheta)} \sum_{s=0}^{N(\vartheta)-1} \left(f(s) - \frac{g}{N(\vartheta)}\right)^2$$

$$(i = 1, 2, \ldots).$$

Let $\eta_n = \xi_1 + \ldots + \xi_n$, so $M(\eta_n) = nM$, $D^2(\eta_n) = nD^2$, and $F_n(y)$ be the distribution function defined by

$$F_n(y) = P(\eta_n < nM + yD\sqrt{n}).$$

Then the following assertion holds.

LEMMA 2. *We have*

$$|F_n(y) - \varphi(y)| < \frac{\delta}{\sqrt{n}}$$

uniformly in $-\infty \leq y \leq \infty$, *where*

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt .$$

For the proof see the book of B. V. GNEDENKO – A. N. KOLMOGOROV [2] (Theorem 1, p. 201).

Let $N(r, b)$ denote the number of Gaussian integers in the disc with radius r and centre b. It is well-known that $N(r, b) = \pi r^2 + O(r)$.

Every α can be stated uniquely in the form

$$\alpha = \beta(\alpha) + \vartheta^{\nu} \alpha_{\nu} , \tag{2.1}$$

where

$$\beta(\alpha) = a_0 + a_1 \vartheta + \ldots + a_{\nu-1} \vartheta^{\nu-1} , \tag{2.2}$$

and

$$\alpha_{\nu} \in C\left(\frac{r}{|\vartheta|^{\nu}},\ -\frac{a_0}{\vartheta^{\nu}} - \frac{a_1}{\vartheta^{\nu-1}} - \ldots - \frac{a_{\nu-1}}{\vartheta}\right). \tag{2.3}$$

Let β denote a general integer of the form

$$\begin{cases} \beta = d_0 + d_1 \vartheta + \ldots + d_{\nu-1} \vartheta^{\nu-1} \\ d_j \in \mathfrak{A} \quad (j = 0, 1, \ldots, \nu-1) \end{cases} \tag{2.4}$$

Then the number of $\alpha \in C(r, 0)$ satisfying $\beta(\alpha) = \beta$ is equal to the number of integers in

$$C\left(\frac{r}{|\vartheta|^{\nu}},\ -\frac{d_0}{\vartheta^{\nu}} - \ldots - \frac{d_{\nu-1}}{\vartheta}\right),$$

and so

$$N\{\alpha \mid \alpha \in C(r, o),\ \beta(\alpha) = \beta\} = \pi \left(\frac{r}{|\vartheta|^{\nu}}\right)^2 + O\left(\frac{r}{|\vartheta|^{\nu}}\right). \tag{2.5}$$

Furthermore we have

$$A(\alpha) = A(\beta(\alpha)) + A(\alpha_{\nu}) . \tag{2.6}$$

Let $F_{\nu}(y)$ be the frequency of the number of β' s satisfying the condition

$$A(\beta) < \nu M + yD\sqrt{\nu} ,$$

i.e.

$$F_{\nu}(y) = \frac{1}{|\vartheta|^{2\nu}} \cdot N\{\beta \mid A(\beta) < \nu M + yD\sqrt{\nu}\} .$$

From Lemma 2 we have that

$$|F_{\nu}(y) - \varphi(y)| < \frac{\delta}{\sqrt{\nu}} .$$

Using Lemma 2 we have that for every pair of real numbers y_1, y_2 the inequality

$$|F_\nu(y_1)-F_\nu(y_2)| \leq |F_\nu(y_1)-\varphi(y_1)|+|\varphi(y_1)-\varphi(y_2)|+|\varphi(y_2)-F_\nu(y_2)| \leq$$

$$\leq \frac{2\delta}{\sqrt{\nu}}+|y_1-y_2| \tag{2.7}$$

is satisfied.

Let

$$K_r(y) = \frac{1}{N(r,0)} N\{\alpha \mid A(\alpha) < \mu^* M + yD\sqrt{\mu^*}\}. \tag{2.8}$$

Observing that

$$A(\beta(\alpha))-\delta_1(\mu^*-\nu) \leq A(\alpha) \leq A(\beta(\alpha))+\delta_1(\mu^*-\nu), \tag{2.9}$$

and by using the notation

$$\varrho = \mu^*-\nu \tag{2.10}$$

we get

$$K(r,y) \leq \frac{1}{N(r,0)} N\{\beta \mid A(\beta) < \nu M + \delta_1(y)\sqrt{\nu} D\}\left(\pi\left(\frac{r}{|\vartheta|^\nu}\right)^2+O\left(\frac{r}{|\vartheta|^\nu}\right)\right) =$$

$$= F_\nu(\delta_1(y))\frac{|\vartheta|^{2\nu}}{N(r,0)}\left(\pi\frac{r^2}{|\vartheta|^{2\nu}}+O\left(\frac{r}{|\vartheta|^\nu}\right)\right) = F_\nu(\delta_1(y))\,[1+O(|\vartheta|^{-\varrho})],$$

assuming that $\varrho < \frac{1}{2}\mu^*$. Here

$$\delta_1(y) = \frac{\varrho}{\sqrt{\nu}}\cdot\frac{M+\delta_1}{D}+y\sqrt{\frac{\mu^*}{\nu}} \approx y+\frac{\varrho}{\sqrt{\nu}}\,\frac{M+\delta_1}{D}+\frac{1}{2}\,\frac{\varrho}{\nu}\,y.$$

Similarly we have

$$K(r,y) \geq F_\nu(\delta_2(y))\,[1+O(|\vartheta|^{-\varrho})]$$

where

$$\delta_2(y) = \delta_1(y)-2\delta_1\frac{\varrho}{\nu}.$$

From (2.7) we have

$$|F_\nu(y)-F_\nu(\delta_i(y))| \leq \delta_2\cdot\frac{\varrho}{\sqrt{\nu}}\left[1+\frac{|y|}{\sqrt{\nu}}\right],$$

and so

$$|F_\nu(\delta_i(y))-\varphi(y)| \leq \delta_3\cdot\frac{\varrho}{\sqrt{\nu}}\left[1+\frac{|y|}{\sqrt{\nu}}\right],$$

if $\varrho \geq 1$.

Consequently we have

$$K_r(y) = \left[\varphi(y) + O\left(\frac{\varrho}{\sqrt{\nu}}\left(1 + \frac{|y|}{\sqrt{\nu}}\right)\right)\right][1 + O(|\vartheta|^{-\varrho})].$$

By choosing

$$\varrho = \left[\frac{1}{2}\,\frac{\log \mu^*}{\log |\vartheta|}\right],$$

we get

$$K_r(y) = \varphi(y) + O\left(\frac{\log \mu^*}{\sqrt{\mu^*}}\left(1 + \frac{|y|}{\sqrt{\mu^*}}\right)\right). \tag{2.11}$$

In the interval $|y| \leq \sqrt{\mu^*}$ we have

$$K_r(y) = \varphi(y) + O\left(\frac{\log \mu^*}{\sqrt{\mu^*}}\right). \tag{2.12}$$

But this relation holds for $|y| > \sqrt{\mu^*}$. To prove this, we need to apply the relation (1.15) for $j = 2$. Indeed, from this

$$N\left\{\alpha \,\middle|\, |\alpha| < r,\ \left|A(\alpha) - \frac{g}{N(\vartheta)}\,\frac{\log r}{\log |\vartheta|}\right| \geq \frac{\sqrt{c_2}}{\sqrt{\log |\vartheta|}} \log r\right\} \leq$$

$$\leq \delta_3 (\log r)^{-2} T_2(r, 0) \leq \delta_4 \frac{r^2}{\log r},$$

and hence (2.12) immediately follows.

So the following assertion holds.

THEOREM 2. *We have*

$$N\left\{\alpha \,\middle|\, |\alpha| < r,\ A(\alpha) < \frac{g}{N(\vartheta)}\,\frac{\log r}{\log |\vartheta|} + \sqrt{c_2}\left(\frac{\log r}{\log |\vartheta|}\right)^{1/2} y\right\} =$$

$$= \pi r^2 \left[\varphi(y) + O\left(\frac{\log \log r}{(\log r)^{1/2}}\right)\right],$$

uniformly in y as r tends to infinity.

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УЗКАЯ РЕШЕТЧАТАЯ УПАКОВКА ШАРОВ В РЕШЕТКАХ ПЕРВОГО ТИПА

Й. ХОРВАТ

Кафедра Начертательной и Проективной Геометрии
Университета им. Л. Этвеша

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К семидесятилетнюю со для рождения Ф. КАРТЕСИ.

Множество $\{K_i^n\}$ единичных шаров в пространстве E^n составляет упаковку шаров, если шары попарно не имеют общих внутренних точек. Рассмотрим такой шар K^n радиуса $\bar{r}$, не имеющий общих внутренних точек ни с одним из шаров $\{K_i^n\}$. — От Л. ФЕЙЕША ТОТА происходит вопрос, чему равен

$$\varrho(n) = \min(\sup \bar{r}) \tag{1}$$

и у которой упаковки единичных шаров достигается этот минимум. Такую упаковку шаров он назвал *узкой упаковкой шаров.*

Он занимался в [4] узкой упаковкой кругов на сфере. К. БЕРЕЦКИ в своей докторской диссертации решил проблему узкой упаковки шаров в 3-мерном пространстве.

В другом отношении занимался этим вопросом С. С. РЫШКОВ [3]. $\varepsilon \subset E^n$ называется дискретной системой или $(r:R)$-системой, если существуют такие числа $r < R$, что выполняются следующие два условия:

α) В открытом шаре радиуса r, описанном вокруг любой точки множества ε, нет больше точек этого множества.

β) В любом замкнутом шаре радиуса R обязательно лежит точка множества ε.

Пусть — $r^* = \sup r$ и $R^* = \inf R$. Отношение $\dfrac{R^*}{r^*}$ он назвал плотностью данной системы ε. Обозначим через

$$\varkappa(n) = \min \frac{R^*}{r^*}$$

по всем равномерно дискретным системам пространства E^n. РЫШКОВ в [3] доказал несколько теорем о плотности $(r; R)$-систем.

В этой работе мы решим проблему узкой решетчатой упаковки шаров первого типа.

В дальнейшем употребляются следующие обозначения: Γ^n — n-мерная решетка, определенная центрами шаров решетчатой упаковки. $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ — основной репер решетки Γ^n.

$З(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, \bar{a}_{n+1})$ — репер Зеллинга решетки Γ^n, т.е. произвольный основной репер $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ решетки Γ^n, дополенный вектором

$$\bar{a}_{n+1} = -\sum_{i=1}^{n} \bar{a}_i .$$

$$g_{ik} = \bar{a}_i \bar{a}_k. \text{ где } i \neq k \text{ и } i, k = 1, 2, \ldots, n+1 .$$

Легко видет, что

$$\bar{a}_i^2 = -\sum_{\substack{k=1 \\ i \neq k}}^{n+1} g_{ik} .$$

Решетка называется решеткой первого типа по Вороному, если существуют векторы репера Зеллинга $З(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1})$, которые удовлетворяют неравенствам

$$g_{ik} < 0 , \tag{2}$$

где $i \neq k$ и $i, k = 1, 2, \ldots, n+1$ [2].

Решетчатая упаковка единичных шаров называется *решетчатой упаковкой шаров первого типа*, если центры шаров составляют решетку первого типа по Вороному.

Шар (L), на котором лежит n-мерная совокупность точек решетки и не имеет точек решетки в его внутренности, будем называть пустым шаром (L) решетки Γ^n, а выпуклую оболочку всех точек решетки, лежащих на шаре (L), будем называть L-многогранником решетки. Известно, что L-многогранники решетки Γ^n образуют нормальное разбиение пространства E^n [2]. Это разбиение называется L-разбиением.

Звезда $[L_A^*]$ — совокупность всех L-многогранинков L-разбиения, имеющих общую вершину A. Решетка является общей, если элементы $[L_A^*]$ только симплексы, и специальной, если элементы $[L_A^*]$ не только симплексы.

Если решетка является решеткой первого типа, то вершины L-симплексов в $[L_A^*]$ задаются векторами

$$a_{i_1}, \bar{a}_{i_1} + \bar{a}_{i_2}, \ldots, \bar{a}_{i_1} + \bar{a}_{i_2} + \ldots + \bar{a}_{i_{n+1}}, \tag{3}$$

где $i_1, i_2, \ldots, i_{n+1}$ — некоторая перестановка чисел $1, 2, \ldots, n+1$ [2].

Пусть $R_1, R_2, \ldots, R_{(n+1)!}$ — радиус шара, описанного вокруг L-симплекса в $[L_A^*]$.

Доказывается следующая

ТЕОРЕМА. *Среди упаковок единичных шаров первого типа по Вороному узкую упаковку шаров составляет только главная решетка первого типа, т.е.*

$$(4)\qquad \begin{aligned} &|\bar{a}_i| = 2, \text{ где } i = 1, 2, \ldots, n+1 \text{ и} \\ &g_{ik} = -\frac{4}{n}, \text{ где } i \neq k \text{ и } i, k = 1, 2, \ldots, n+1. \end{aligned}$$

В этом случае минимум по формуле (1) *равно*

$$(5)\qquad \varrho_1(n) = \sqrt{\frac{n+2}{3}} - 1.$$

ДОКАЗАТЕЛЬСТВО: Рассмотрим решетчатую упаковку единичных шаров первого типа, заданную приведением репера Зеллинга по Вороному, т.е.

$$(6)\qquad \begin{aligned} &\bar{a}_i \bar{a}_k < 0 \quad i \neq k \text{ и } i, k = 1, 2, \ldots, n+1 \text{ и} \\ &|\bar{a}_i| \geq 2 \quad i = 1, 2, \ldots, n+1. \end{aligned}$$

Очевидно, если существует шар радиуса $\bar{r}$, не имеющих общих внутренних точек шаров упаковки, то в решетке существует пустой шар радиуса R_k, где $R_k \geq \bar{r} + 1$. Поэтому решение $\varrho_1(n)$ формулы (1) эквивалентна с формулой

$$(7)\qquad R = \min(\max R_k), \text{ где } k = 1, 2, \ldots, (n+1)!.$$

Известно [1], что

$$(8)\qquad \sum_{k=1}^{(n+1)!} R_k^2 = \frac{(n+2)!}{12(n+1)} \sum_{i=1}^{n+1} \bar{a}_i^2 + \sum_{k=1}^{(n+1)!} c_k^2,$$

где c_k — расстояние между центром тяжести и центром описанного шара L_k-симплекса.

Из (6) следует, что

$$(9)\qquad \sum_{k=1}^{(n+1)!} R_k^2 \geq \frac{(n+2)!}{3}$$

и равенство имеет место только тогда, если

$$(10)\qquad \begin{aligned} &|a_i| = 2, \text{ где } i = 1, 2, \ldots, n+1 \text{ и} \\ &C_k = 0, \text{ где } k = 1, 2, \ldots, (n+1)!. \end{aligned}$$

Легко видеть, что если существует решетка, удовлетворяющая условием (10) и равенствам

$$R_1 = R_2 = \ldots = R_{(n+1)!},$$

то

$$R = \min(\max R_k) = \sqrt{\frac{n+2}{3}},$$

т.е.

$$\varrho_1(n) = \sqrt{\frac{n+2}{2}} - 1 .$$

Мы докажем, что существует только одна такая решетка. Эта решетка задается формулами (4).

Рассмотрим два произвольных соседних L-симплекса S_1, и S_2. По (3)

вершины $S_1 : A, A_1, A_2, \ldots, A_{k-1}, A_k, A_{k+1}, \ldots, A_n$,
вершины $S_2 : A, A_1, A_2, \ldots, A_{k-1}, A'_k, A_{k+1}, \ldots, A_n$,

где $k = 1, 2, \ldots, n+1$,

$$\vec{AA}_j = \bar{a}_{i_1} + \bar{a}_{i_2} + \ldots + \bar{a}_{i_j} \quad j = 1, 2, \ldots, n ,$$

$$\vec{AA}'_k = \bar{a}_{i_1} + \bar{a}_{i_2} + \ldots + \bar{a}_{i_{k-1}} + \bar{a}_{i_{k+1}} \text{ и } \bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_n}$$

некоторая перестановка векторов $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1}$.

Пусть O_1 и T_1, O_2 и T_2 или O и T — центры описанного шара и центры тяжести симплексов S_1, S_2 или $(n-1)$-мерного симплекса A, $A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n$. Так как S_1 и S_2 — соседние симплексы, поэтому проекция точек O_1 и O_2 на общую грань будет O.

Поскольку $c_k = 0$ $(k = 1, 2, \ldots, (n+1)!)$, т.е. $T_1 = O_1$ и $T_2 = O_2$ и S_1, S_2 — основные симплексы, то следует, что точки T_1 и T_2 так симплексы S_1, и S_2 симметричны относительно гиперплоскости $A, A_1, A_2, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n$. Из этого следует, что

$$(\bar{a}_{i_j}; \bar{a}_{i_k})\sphericalangle = (\bar{a}_{i_j}; \bar{a}_{i_{k+1}})\sphericalangle \quad j = 1, 2, \ldots, k-1, k+2, \ldots, n$$

и

$$|\bar{a}_{i_k}| = |\bar{a}_{i_{k+1}}| .$$

Так как S_1 и S_2 — произвольные соседние симплексы звезды $]L^*_A]$, поэтому углы между любыми двумя векторами из $\bar{a}_1, \bar{a}_2, \ldots, a_{n+1}$ является равными и

$$|\bar{a}_i| = 2, \text{ где } i = 1, 2, \ldots, n+1 .$$

Таким образом в неравенстве (9) равенство имеет место тогда и только тогда, если решетка определена с формулами (4).

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ON THE ϱ-SYSTEM OF UNIT CIRCLES[1]

By

J. MOLNÁR

Department of Geometry of the L. Eötvös University, Budapest

Dedicated to F. KÁRTESZI on his seventieth birthday

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Let $\{C_i\}$ be a packing of circles in the euclidean plane. A circle is said to be a supporting circle of the circle system $\{C_i\}$ if has no common inner point with $\{C_i\}$ and touches at least three circles of $\{C_i\}$. If the radii of the supporting circles of $\{C_i\}$ are $\geq \varrho > 0$ and there exists a supporting circle whose radius is ϱ, then $\{C_i\}$ is called a ϱ-system of circles.

The density of a circle system $\{C_i\}$ with respect to the euclidean plane is defined by

$$\delta = \overline{\lim_{R\to\infty}} \frac{\sum C_i \cap C(R)}{C(R)},^2$$

where $C(R)$ is a circle of radius R and of center in a fixed point O of the plane.[3]

Our main result is the following

THEOREM: *The density of a ϱ-system of non-overlapping unit circles in the euclidean plane is*

$$\leq \frac{\pi(1+\varrho)^2}{8\sqrt{\varrho^2+2\varrho}}, \quad \text{if} \quad \varrho \leq \sqrt{2}-1,$$

resp.

$$\leq \frac{\pi}{2\left[\frac{\pi}{\alpha}\right]\sqrt{\varrho^2+2\varrho}+(1+\varrho)^2 \sin 2\left(\pi-\left[\frac{\pi}{\alpha}\right]\alpha\right)},$$

where

$$\alpha = \arcsin\frac{1}{1+\varrho}, \quad \text{if} \quad \varrho \geq \sqrt{2}-1.$$

Equality can be attained, for all values of ϱ.

[1] Communication made at the Colloquium of Discrete Geometry in May 1975 in Salzburg.

[2] We denote a domain and its area by the same symbol.

[3] It is easy to show that δ does not depend on the choice of O. See FEJES TÓTH [4].

Before proving the theorem first we shall give the intuitive meaning of the upper bounds of the density, i.e. of

$$d_1(\varrho) = \frac{\pi(1+\varrho)^2}{8\sqrt{\varrho^2+2\varrho}},$$

resp.

$$d_2(\varrho) = \frac{\pi}{2\left[\frac{\pi}{\alpha}\right]\sqrt{\varrho^2+2\varrho}+(1+\varrho)^2\sin 2\left(\pi-\left[\frac{\pi}{\alpha}\right]\alpha\right)}.$$

Let us consider a unit circle of centre O_1 touched by two other unit circles of centres O_2, O_3. If these circles have the same supporting circle

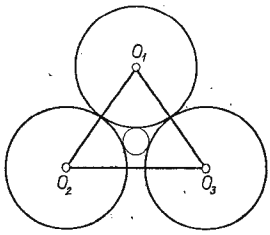


Fig. 1.

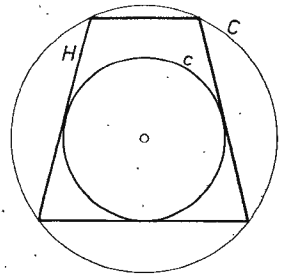


Fig. 2.

of radius $\varrho \leq \sqrt{2}-1$ (Fig. 1) then $d_1(\varrho)$ is the density of these three circles with respect to the triangle $O_1\,O_2\,O_3$, i.e.

$$d_1(\varrho) = \frac{\pi}{2\,O_1\,O_2\,O_3}.$$

Let c, resp. C be two concentric circles of radii 1, resp. $1+\varrho$ and H the polygon inscribed in C having its sides, except for at most one, tangent at c (Fig. 2). $d_2(\varrho)$ is the density of c with respect to H, i.e.

$$d_2(\varrho) = \frac{\pi}{H}.$$

In the proof of our theorem a prominent role plays the tessellation L^* which in general is a modified tessellation of the well known L-tessellation of VORONOÏ – DELAUNAY.[4]

We introduce the tessellation L^*.[5]

We consider a system $\{O_i\}$ of points in the plane having the following properties:

1. $O_i\,O_j \geqq a$ $(i, j = 1, 2, \ldots, i \neq j)$ and

[4] See DELAUNAY [2].

[5] This tessellation was introduced ten years ago by MOLNÁR (See MOLNÁR [5]).

2. no point of the plane at distance $b \geqq a$ can be added to the system $\{O_i\}$.

Such a system of $\{O_i\}$ points will be called briefly a saturated point system.

Associating with a point O_i the set D_i of all points P lying nearer to O_i than to any other point O_j more precisely, $d(P, O_i) \leqq d(P, O_j)$, $i \neq j$ where $d(P, O_i)$ denotes the distance of P from O_i, we obtain a convex polygon D_i (DIRICHLET-cell, VORONOÏ-polygon)[6]. It is known that the convex polygons $\{D_i\}$ constitute an L_D-tessellation (DIRICHLET-tessellation, DIRICHLET – VORONOÏ-tessellation).

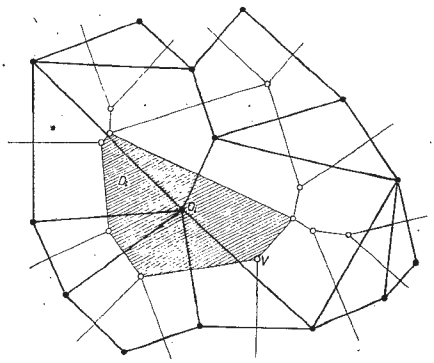


Fig. 3.

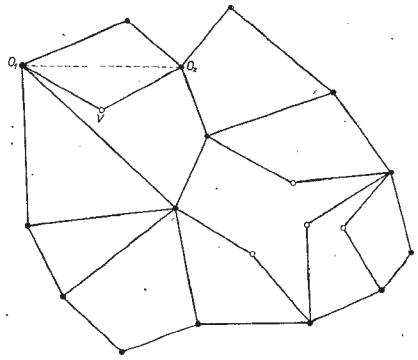


Fig. 4.

By replacing each edge of L separating the neighbouring faces D_i, D_j by the segment $O_i\ O_j$ we obtain the "dual" tessellation, the L-tessallation (VORONOÏ – DELAUNAY-tessellation)[7] (Fig. 3). The L-tessellation has the property, that the circumcentre V of a face is a vertex of a DIRICHLET-cell. Obviously the faces of L are also convex polygons.

Let us consider a polygon inscribed in a circle of centre V (Fig. 5). We call a side $O_1\ O_2$ of this polygon a *separating side* if the straight-line $O_1\ O_2$ separates the polygon from V. In such a case we call the broken line $O_1\ V\ O_2$ the *bridge of the polygon* and the segments $O_1\ V$ and $O_2\ V$ the *components of the* bridge.

It is easy to see that in an L-tessellation any separating side can be replaced by only one bridge.

LEMMA 1. *Let us consider an L-tessellation of a saturated system of points. If we replace each separating side of the face of L by its corresponding bridge, we obtain a tessellation.*

PROOF. Let $\{O_i\}$, resp. $\{V\}$ be the vertex system of the L-tessellation, resp. of its "dual" L-tessellation corresponding to the saturated point-system $\{O_1\}$. Replace every separating side of the faces of L by the corresponding bridge (Fig. 4).

[6] See DIRICHLET [3] and VORONOÏ [10].

[7] For a direct construction of the L-tessellation see for instance DELAUNAY [2].

To prove our lemma it is sufficient to prove, that

1. the components of the bridge do not intersect one another,
2. no bridge can intersect a non-replaced side of L.

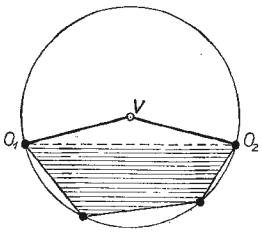


Fig. 5.

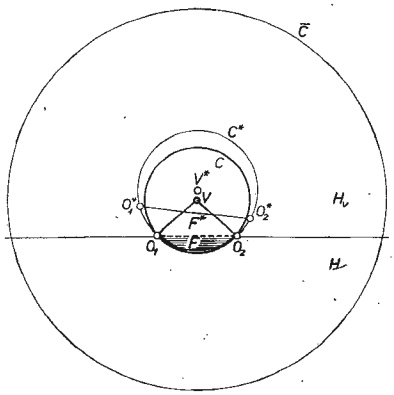


Fig. 6.

To prove 1. we start with the obvious remark, that the circle of centre V and of radius OV does not contain in its interior another vertex O_i of L. Let OV and $O^* V^*$ be components of two bridges. The perpendicular to OO^* at the midpoint of OO^* divides the plane into two halfplanes H, H^*. A simple consequence of our remark is that $V^* \notin H$ and $V \notin H^*$, hence OV and $O^* V^*$ do not intersect each other. OV and $O^* V^*$ have a common point iff $V = V^*$.

To prove 2.[8] we consider an arbitrary bridge $O_1 V O_2$ of the face F of L and we investigate the relation of this bridge with respect to the non-replaced sides of L. Since $O_1 V O_2$ exists, the straight-line $O_1 O_2$ divides the plane into two halfplanes H, resp. H_v containing F, resp. V (Fig. 6). Denote by C the circle of centre V and radius $O_1 V$. Let us consider now the neighbouring face F_1^* of F, having with F the common side $O_1 O_2$, i.e. the straight-line $O_1 O_2$ separates F_1^* from F. Since F_1^* is a face af L, F_1^* is convex and obviously all its vertices lie on a circle C^* having the radius $O_1 V^*$ greater than $O_1 V$. In the case when F_1^* contains the bridge $O_1 V O_2$, then the bridge does not intersect the sides of L. If however F_1^* does not contain the bridge $O_1 V O_2$, then it must have a side $O_1^* O_2^*$ which intersects the bridge (Fig. 6). In this case $O_1^* O_2^*$ is a separating side of F^*, i.e. $O_1^* V^* O_2^*$ is a bridge, that is the side $O_1^* O_2^*$ is a replaced side of L.

Since 1. $O_1 V = O_2 V \leq b$, 2. $O_i O_j \geq 2b$ and 3. in the circle C of centre V and radius $3b$ there are only a finite number of O_i, it follows obviously that the number of non-replaced $O_i O_j$ sides which can have common points with the bridge $O_1 V O_2$ is finite. Thus continuing the annexion of $\{F_i^*\}$ in a finite number of stepes the bridge $O_1 V O_2$ will be covered by the annexed faces $\{F_i^*\}$.

[8] The proof of this part deffers from our earlier proof (See MOLNÁR [5]).

The following two simple remarks will be valuable in the proof of our theorem. In an L^*-tessellation 1. $V V^*$ is perpendicular to $O_1 O_2$, 2. $V^* O_1 > V O_1$ and $V^* O_2 > V O_2$ (Fig. 6).

In the proof of our theorem we shall use also the following three known lemmas:[9]

LEMMA 2. (HAJÓS' lemma): *Let us consider two concentric circles c, C ($c \subset C$) and a polygon containing c and having its vertices not in the interior of C. The area of the poligon is minimal if the polygon is inscribed in C and its sides are tangent to c, except for at most one.*

LEMMA 3. *Denote by OTP_k a right-angled triangle $\left(\sphericalangle OTP = \frac{\pi}{2}\right)$ of hypotenuse OP_k and by c a circle of centre O. If $OP_1 < OP_2 < OP_3$, then*

$$\frac{s(P_1 OP_3)}{P_1 OP_3} < \frac{s(P_1 OP_2)}{P_1 OP_2},$$

where $s(P_i OP_j)$ denotes a sector of the circle c of $\sphericalangle P_i OP_j$.

LEMMA 4. *Let $OT_n P$ be a richt-angled triangle of hypotenuse OP and let c be a circle of centre O. If $\sphericalangle T_1 OP < \sphericalangle T_2 OP$ then*

$$\frac{s(T_1 OP)}{T_1 OP} < \frac{s(T_2 OP)}{T_2 OP}.$$

PROOF OF THE THEOREM. To prove our theorem we shall consider an L^*-tessellation if $\varrho \leq \sqrt{2}-1$, reps. the tessellation of DIRICHLET if $\varrho \geq \sqrt{2}-1$.

I. $\varrho \leq \sqrt{2}-1$. Of course we can restrect ourselves to saturated $\{C_i\}$ system of unit circles of centre system $\{O_i\}$. Let L^* be the corresponding tessellation. We distinguish two types of faces L^*: 1. face F, which is a polygon having only non-replaced sides, i.e. $F = O_1 O_2 \ldots O_k$, 2. face F^*, containing at least one bridge, i.e. $F = O_1 V_1 O_2 \ldots O_k$, resp. $F^* = O_1 V_1 O_2 \ldots V_k$.

Type 1. Let V be the corresponding centre radical of the face F.

a) If $V O_1 \leq \sqrt{2}$, then F is of course an acute triangle $\Delta = O_1 O_2 O_3$. It is easy to see that in this case Δ has minimal area if its sides are 2, 2, $4 \cos \frac{1}{1+\varrho}$ (Fig. 1). Thus the density of the corresponding unit circles with respect to Δ is $\leq d_1(\varrho)$.

b) If $V O_1 \geq \sqrt{2}$ we decompose the face $F = O_1 O_2 \ldots O_k$ in the triangles $\Delta_1 = O_1 V O_2$, $\Delta_2 = O_2 V O_3$, $\ldots$, $\Delta_k = O_k V O_1$ and we consider the density of the unit circles in these triangles. According to our Lemmas 3 and 4, the circle density in such a triangle Δ_i takes its maximal value $\frac{\pi}{4}$ if $O_i O_{i+1} = 2$ and $V O_1 = \sqrt{2}$. But $\frac{\pi}{4} \leq d_1(\varrho)$.

[9] See MOLNÁR [5], [6], [7], [8] [9].

Type 2. Let V^* be the centre radical corresponding to F^*. The existencs of a bridge implies $V^* O_1 \geq \sqrt{2}$. Indeed, if there is a bridge $O_1 V O_2$, then around V we may consider at least four circles at the same distance from V having theirs centre distances $O_i O_j \geq 2$. Thus $V O_1 > \sqrt{2}$, but $V^* O_1 > V O_1$, hence $V^* O_1 > \sqrt{2}$.

We decompose $F^* = O_1 V_1 O_2 \ldots O_k$, resp. $F^* = O_1 V_1 O_2 \ldots V_k$ into triangles $\Delta_1 = O_1 V_1 V^* = O_2 V_1 V^*, \ldots, \Delta_n = O_k V^* O_k$, resp. $\Delta_1 = O_1 V_1 \cdot V^* = O_2 V_1 V^*, \ldots, \Delta_k = O_k V^* V_k = O_1 V^* V_k$.

According to our Lemmas 3 and 4 the circle density in these triangles is $< \frac{\pi}{4}$. This density value is not greater as $d_1(\varrho)$. Thus the density of unit circles in a face of L^* of type 2 is not greater as $d_1(\varrho)$.

II. $\varrho \geq \sqrt{2} - 1$. Of course also in this case we can restrect ourselves to a saturated $\{C_i\}$ circle system. Let us consider the corresponding tessellation of DIRICHLET.[10] Let V be an arbitrary vertex of the face F corres-

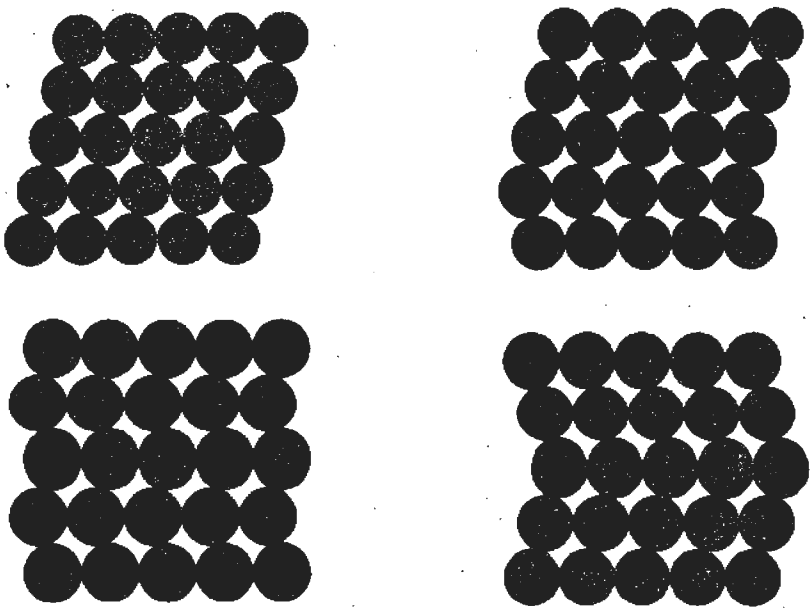
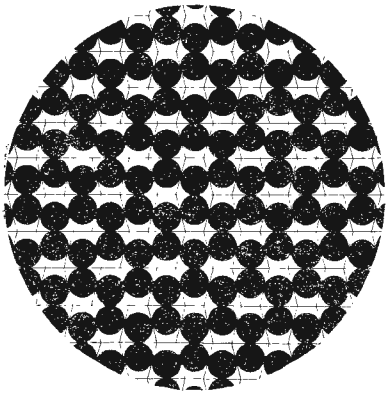


Fig. 7.

ponding to the circle C_i of centre O_i. Since $VO_i \geq 1 + \varrho$ according to HAJÓS' lemma (Lemma 2) the density of C_i with respect to the corresponding F is not greater as $d_2(\varrho)$.

[10] See MOLNÁR [9].



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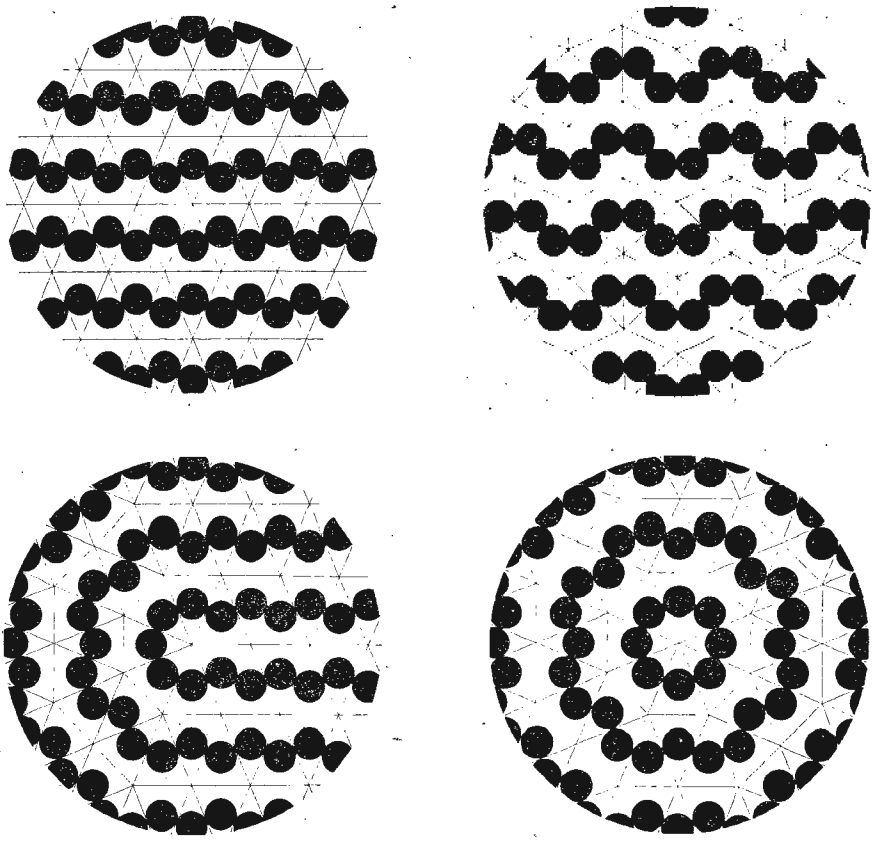
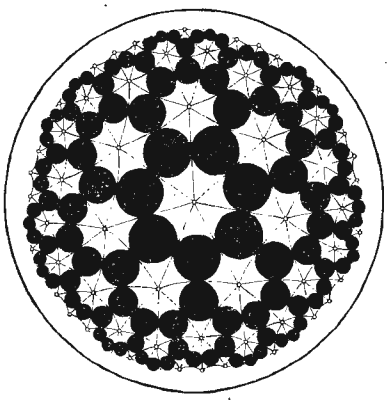


b)

Fig. 8.

We finish the proof of our theorem stressing the fact that the given upper bounds $d_1(\varrho)$, resp. $d_2(\varrho)$ obviously may be attained (Fig. 7, resp. 8).

REMARKS. 1 For every $\frac{2\sqrt{3}}{3}-1<\varrho<\sqrt{2}-1$ there are infinitely many packings of unit circles having the same density (Fig. 7).



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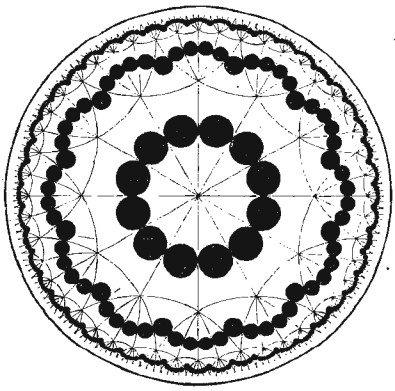

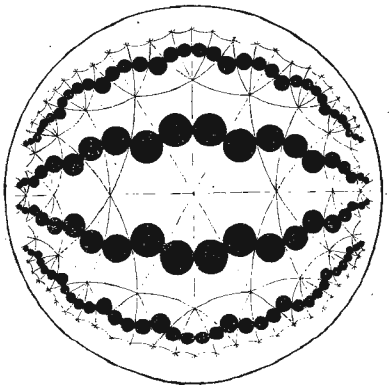

b)

Fig. 9.

2. The L^*-tessellation can be introduced essentially in the same way on the sphere and in the hyperbolic plane.

3. It is easy to see that our theorem can be extended to the case of the sphere and the hyperbolic plane.[11] The equality for the corresponding upper bounds $d_1(\varrho)$ and $d_2(\varrho)$ can be attained only for discrete values of ϱ (Fig. 9).

[11] In the hyperbolic plane the non-existence of a general notion of circle density (See BÖRÖCZKY [1]) forces us to use a density notion corresponding to a fixed type of tessellation (See MOLNÁR [8], [9]).

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