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SOME RESULTS ON ω_μ-METRIC SPACES

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1. Introduction

It is a well-known fact that the metrizable spaces are exactly those, which admit a uniformity with a countable base or, equivalently, one with a linearly ordered base of countable cofinality. Therefore — and also because of other reasons — it is just natural to investigate spaces having uniformities with linearly ordered bases of arbitrary cofinality. It appears that these spaces are exactly those, which can be "metrized" by a distance-function d on X with values in a totally ordered abelian group G, which has cofinality $\omega_\mu \geq \omega_0$. Usually such spaces (X, d) are called ω_μ-metric spaces. Good work has been done in this field, e.g. by F. HAUSDORFF, F. STEVENSON, W. THRON, A. HAYES, P. NYIKOS, I. JUHÁSZ, A. K. STEINER, E. F. STEINER, H. C. REICHEL and many others. See also the bibliographies. This paper deals with products of such spaces and investigates completeness and compactness properties of them. Let us now collect several prerequisites.

A totally ordered abelian group is an abelian group $(G, +)$ with an order $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ for all $a, b, c \in G$. The cofinality cof (G) of such a group G is the smallest ordinal ω_μ such that there is ω_μ-sequence $(x_\alpha)_{\alpha<\omega_\mu}$ in G with $x_\alpha \neq 0$ and which is converging to 0 in the order topology. Because of the definition cof $(G) = \omega_\mu$ has to be a regular ordinal. For notations concerning ordinals and cardinals see the book of JUHÁSZ [5]. All spaces are at least T_2. Let X be a set, G a totally ordered abelian group with cofinality ω_μ, then we call a function d from X^2 to G an ω_μ-metric on X iff it satisfies the usual axioms for a metric. (X, d) is then called an ω_μ-metric space. If additionally (i) holds, we call d a non-archimedean ω_μ-metric: (i) $d(x, y) \leq \max(d(x, z), d(z, y))$ for all $x, y, z \in X$. Clearly every ω_μ-metric space carries both a topology and a uniformity in a natural way. A base for this uniformity $\mathcal{U}_d$ is for example the set $\{U_\varepsilon : \varepsilon > 0, \varepsilon \in G\}$ with entourages $U_\varepsilon = \{(x, y) \in X^2 : d(x, y) < \varepsilon\}$. A base for the topology τ_d is $\{B(x, \varepsilon) : x \in X, \varepsilon \in G, \varepsilon > 0\}$ with $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. Clearly τ_d coincides with the topology $\tau_{\mathcal{U}_d}$ induced by $\mathcal{U}_d$. If we now order the entourages by $U \leq V$ iff $U \supseteq V$, we learn that $\mathcal{U}_d$ has a linearly ordered base of cofinality ω_μ.[1] Stevenson and Thron have shown the following: every ω_μ-metric space (X, d) carries a uniformity $\mathcal{U}_d$ with a linearly ordered (even more: well ordered) base of cofinality ω_μ.[2] If conversely $(X, \mathcal{U})$ is a uniform space with a linearly ordered base, which has cofinality ω_μ (then X has even an equivalent base of the same cofinality, which is well ordered), then a group G with $\operatorname{cof}(G) = \omega_\mu$ and an ω_μ-metric $d: X^2 \to G$ exist such that $\mathcal{U} = \mathcal{U}_d$. From this theorem and the footnote 2 it follows that a space can only be ω_μ-metrizable for one ω_μ, except if it is discrete.

Proposition 1.1.: Let $(X, \mathcal{U})$ be a uniform space with a linearly ordered base of cofinality $\omega_\mu > \omega_0$. Then for *every* group G with cof $(G) = \omega_\mu$ there is a non-archimedean ω_μ-metric d from X^2 to G with $\mathcal{U}_d = \mathcal{U}$. (As it is seen from the proof below, $d(X^2)$ is not the whole of G but only the set $\{s_\alpha : \alpha < \omega_\mu\}$ defined below. Compare [15]).

Proof: Take a well ordered base for $\mathcal{U}$, say $\{V_\alpha : \alpha < \omega_\mu\}$.[3] For all $\alpha < \omega_\mu$ there is a sequence $\beta_n < \omega_\mu$, such that $V_{\beta_n} \circ V_{\beta_n} \subseteq V_{\beta_m}$ for $n > m$ and $V_{\beta_n} \circ V_{\beta_n} \subseteq$ $\subseteq V_\alpha$ for all β_n. Put $W'_\alpha = \bigcap_{n=1}^{\infty} V_{\beta_n}$. Then we have $W'_\alpha \circ W'_\alpha \subseteq W'_\alpha \subseteq V_\alpha$. Now define W_α by transfinite induction: $W_1 = W'_1$ and if W_γ is defined for $\gamma < \alpha$, notice that $\bigcap_{\gamma<\alpha} W_\gamma$ is an entourage again since $\alpha < \omega_\mu$, $\mu > 0$. Therefore there is some $W'_\delta \subseteq \bigcap_{\gamma<\alpha} W_\gamma$ and we put $W_\alpha = W'_\delta \cdot \{W_\alpha : \alpha < \omega_\mu\}$ is a linearly ordered base with the property that $W_\beta \circ W_\beta \subseteq W_\alpha$ whenever $\alpha \leq \beta$. Now take a monotonically decreasing ω_μ-sequence $(s_\alpha)_{\alpha<\omega_\mu}$ in G, which converges to zero. Define $d(x, y) = s_\alpha$ iff $(x, y) \in W_\beta$ for $\beta < \alpha$ and $(x, y) \notin W_\alpha$, if such an α exists; if not let $d(x, y) = 0$. Then d is a non-archimedean (n.a.) ω_μ-metric generating $\mathcal{U}$.

There is no analogue of this proposition for $\mu = 0$, of course. Since every uniform space with a countable base is metrizable, clearly every ω_0-metrizable space is metrizable and vice versa. But not every metrizable space is ω_0-metrizable over *every* group of countable cofinality as the following example shows:

Example 1.0.: Take X as the set of real numbers R, equipped with the standard topology. Then X is metrizable but not ω_0-metrizable over the additive group Q of rationals, because a metric $d: R^2 \to Q$ would induce a continuous non-constant function $f(x) = d(x, 0)$ from R to Q.

For an interesting investigation in this field for groups with countable cofinality see [1]. Now we give an important example of an ω_μ-metric space:

[1] If we consider the uniformity as a Tukey-uniformity, the linearly ordered base consists of uniform covers $\mathcal{A}$ and the order is given by the relation "$\mathcal{A}$ refines $\mathcal{B}$".

[2] This part of the statement is only correct if X is not discrete. If X is discrete, its uniformity $\mathcal{U}_d$ has linearly ordered bases with arbitrary cofinality less than $|X|$. If X is not discrete, all compatible uniformities having linearly ordered bases have such with the same cofinality.

[3] Take symmetric V_α's.

EXAMPLE 1.1.: If A is an arbitrary set we denote by $A^{\omega_\mu} = \{(x_\alpha) : x_\alpha \in A, \alpha < \omega_\mu\}$ i.e. the set of all ω_μ-sequences in A. The "natural" topology ν on A^{ω_μ} has as a base the following sets: $x(\beta) = \{y \in A^{\omega_\mu} : x_\alpha = y_\alpha \text{ for all } \alpha < \beta\}$ whereby $x = (x_\alpha) \in A^{\omega_\mu}$ and $\beta < \omega_\mu$. The corresponding uniformity on A^{ω_μ} has as a linearly ordered base the set $\mathcal{B} = \{\{((x_\alpha), (y_\alpha)) : x_\alpha = y_\alpha \text{ for } \alpha < \beta\} : \beta < \omega_\mu\}$. From Proposition 1.1. it follows that there is a n.a. ω_μ-metric on A^{ω_μ}. Clearly, it can be chosen such that the balls $B(x, s_\alpha) = x(\alpha)$. (cf. [14]). The importance of A^{ω_μ} lies in the fact, that every ω_μ-metrizable space with $\mu > 0$ is embeddable in a suitable A^{ω_μ} (cf. [9]).

Every ω_μ-metrizable space is paracompact (cf. [3]). In the case $\mu > 0$ there can be proved much more: let $\mathcal{U}_1$ be an uniform cover of X. Then there exists another uniform cover $\mathcal{U}_2$ which starrefines $\mathcal{U}_1$, and so on. $\mathcal{U} = \bigwedge_{n=1}^{\infty} \mathcal{U}_n$ is then again an uniform cover since $\mu > 0$. But $\mathcal{U}$ starrefines itself, and therefore must be a partition of X. From that it is easily seen that every ω_μ-metrizable space ($\mu > 0$) has a well ordered base consisting of partitions of X (well ordered with respect to starrefining) for its uniformity. A space is called ω_μ-additive iff the intersection of fewer than ω_μ open sets is open again. Clearly every ω_μ-metrizable space is ω_μ-additive. A space is called ω_μ-compact iff every open cover has a subcover with fewer than ω_μ elements.

A first example of a group with cofinality ω_μ can be obtained by taking the product of the additive group of integers Z, i.e. $G = \prod_{i < \omega_\mu} Z_i$, where $Z_i = Z$, with co-ordinatewise addition and the lexicographic order. As a second example take P_μ, the set of all ordinals smaller than ω_μ with the so-called "natural sum" and "natural product" in the sense of HESSENBERG [4]. With W_μ denote the smallest field which contains P_μ. Then W_μ is an ordered field with cofinality ω_μ. An ω_μ-sequence converging to 0 is for example $\left\{\frac{1}{\alpha} : \alpha < \omega_\mu\right\}$. For $\mu = 0$ it is the field of rationals (cf [12], [13]). $|w|$ stands for max $(w, -w)$, $w \in W_\mu$. $\sigma(X)$ denotes the family of open sets in a topological space, $\sigma_\xi(X)$ the family of all G_{ω_ξ}-sets, i.e. all intersections of less or equal than ω_ξ many open sets.

2. Products

It is a well-known result that a product of topological spaces is metrizable iff each factor is so and only countable many factor spaces consist of more than one element. Further the product is completely metrizable iff each factor is so. For ω_μ-metrizable spaces the situation is quite different, since the Tychonoff-producttopology is not ω_μ-additive in general. P. NYIKOS [8] has proved that a box-product of fewer than ω_μ ω_μ-metrizable spaces is itself ω_μ-metrizable. With a modified notion of a producttopology due to I. I. PARAWITSCHENKO (see [6]) we can get further results.

DEFINITION: Let X_i be topological spaces for every $i<\omega_\nu$, and $X = \prod_{i<\omega_\nu} X_i$ the set-theoretical product. If ω_σ is an arbitrary initial ordinal, a base for the ω_σ-producttopology on X is the following:

$$\mathcal{B} = \left\{ \prod_{i<\omega_\nu} U_i : U_i \in \sigma(X_i),\ U_i \neq X_i \text{ only for fewer than } \omega_\sigma \text{ indices} \right\}.$$

For $\omega_\sigma = \omega_0$ we obtain the Tychonoff-producttopology, for $\omega_\sigma > \omega_\nu$ the box-topology. I. JUHÁSZ proved – assuming CH – that the ω_μ-producttopology of ω_μ ω_μ-metrizable spaces is ω_μ-metrizable (cf. [6]); M. M. ČOBAN observed that the assumption of CH can be deleted [2]. The results of NYIKOS and JUHÁSZ can be reformulated as follows: the ω_μ-producttopology of less or equal ω_μ many ω_μ-metrizable spaces is itself ω_μ-metrizable. The following theorem now asserts that except one case concerning discrete spaces the previous statement represents the only case in which such a product of ω_μ-metrizable spaces can be ω_μ-metrizable.

THEOREM 2.1.: *Let X_i be non-discrete topological spaces for every $i<A$ (A is an arbitrary cardinal) and Z_j be discrete (ergo ω_μ-metrizable) spaces with card $Z_j>1$ for all $j<B$. Let $Y = \prod_{i<A} X_i \times \prod_{j<B} Z_j$ be the set-theoretic product equipped with an ω_σ-producttopology. Then Y is ω_μ-metrizable iff each factor is so and one of the following conditions holds:*

1. $\omega_\sigma = \omega_\mu$ and $A \leq \omega_\mu$ and $B \leq \omega_\mu$,
2. $\omega_\sigma > \omega_\mu$ and $A < \omega_\mu$ and $B < \omega_\sigma$,
3. $\omega_\sigma < \omega_\mu$ and $A < \omega_\sigma$ and $B < \omega_\sigma$.

PROOF: For sufficiency let Y be ω_μ-metrizable. Then X_i is a subspace of Y and hence ω_μ-metrizable. Further we show:

Case 1. Let $\omega_\sigma = \omega_\mu$. If $A>\omega_\mu$, take $y = ((x^i),(y^j)) \in Y$ and by ω_μ-metrizability of Y let $\{U_\alpha : \alpha<\omega_\mu\}$ be a neighbourhood base at y. Without loss of generality $U_\alpha = \prod_{i<A} U_\alpha^i \times \prod_{j<B} V_\alpha^j$ and by definition the cardinality of $\Gamma_\alpha = \{i : U_\alpha^i \neq X_i\}$ is less than ω_μ for all $\alpha<\omega_\mu$. Therefore $\left|\bigcup_{\alpha<\omega_\mu} \Gamma_\alpha\right| \leq \omega_\mu$. Since $A>\omega_\mu$, there is an index $k \notin \bigcup_{\alpha<\omega_\mu} \Gamma_\alpha$ and $k<A$. Take now an open neighbourhood W_k of x^k for which $W_k \neq X_k$. $W = \prod_{j\neq k} X_j \times W_k \times \prod_j Z_j$ is an open neighbourhood of y, but $U_\alpha \nsubseteq W$ for all $\alpha<\omega_\mu$. Consequently $A\leq\omega_\mu$. A similar argument shows $B\leq\omega_\mu$.

Case 3. Now let $\omega_\sigma<\omega_\mu$. Supposing $A\geq\omega_\sigma$ take $U^i \subsetneqq X_i$ for $i<A$. Define the open set $U_\alpha = \prod_{i<\alpha} U^i \times \prod_{\alpha\leq i<A} X_i \times \prod_{j<B} Z_j$ for all $\alpha<\omega_\sigma$. On the one hand the set $U = \bigcap_{\alpha<\omega_\sigma} U_\alpha$ is now open by ω_μ-additivity. On the other hand

$$U = \bigcap_{\alpha<\omega_\sigma} U_\alpha = \prod_{i<\omega_\sigma} U^i \times \prod_{\substack{\omega_\sigma \leq i \\ i<A}} X_i \times \prod_{j<B} Z_j$$

and therefore is not open in the ω_σ-producttopology. Consequently $A < \omega_\sigma$. A similar argument applies to show $B < \omega_\sigma$.

Case 2. At the end we have the case $\omega_\mu < \omega_\sigma$. First supposing $A \geq \omega_\mu$ we consider the subspace $Y' = \prod_{i<\omega_\mu} X_i$. This subspace of Y carries the box-topology since $\omega_\mu < \omega_\sigma$, and is clearly ω_μ-metrizable. Now let $y = (x^i) \in Y'$ be a point, such that $x^i \in X_i$ is not isolated for all $i < \omega_\mu$ (recall every X_i is not discrete). Now again choose a linearly ordered neighbourhoodbase of the form $\left\{\prod_{i<\omega_\mu} U^i_\alpha : \alpha < \omega_\mu\right\}$. Clearly $\{U^i_\alpha : \alpha < \omega_\mu\}$ is a neighbourhoodbase at x^i and since x^i is not isolated there is some open neighbourhood $V_i \supsetneq \bigcap_{\alpha \leq i} U^i_\alpha$ of x^i in X_i for every $i < \omega_\mu$. $V = \prod_{i<\omega_\mu} V_i$ is a neighbourhood of y in Y' and consequently there is an $\prod_{i<\omega_\mu} U^i_\beta \subset V$, which implies $U^i_\beta \subseteq V_i$ for all i; choose an $i > \beta$, then we have $U^i_\beta \subseteq V_i \subsetneq \bigcap_{\alpha \leq i} U^i_\alpha \subseteq U^i_\beta$ for this i, which is a contradiction. So $A < \omega_\mu$ must hold. Suppose now $B \geq \omega_\sigma$ still holds. Then we consider the subspace $Z = \prod_{j<\omega_\sigma} Z_j$. The subspace topology coincides with the ω_σ-producttopology and again Z is ω_μ-metrizable. If cof $(\omega_\sigma) < \omega_\mu$, take for every $\beta <$ cof (ω_σ) a $\gamma_\beta < \omega_\sigma$ such that $\sup_\beta \gamma_\beta = \omega_\sigma$. Let $\{U_\alpha : \alpha < \omega_\mu\}$ be a linearly ordered neighbourhood base for $z \in Z$, of the form $U_\alpha = \prod_{i<\omega_\sigma} U^i_\alpha$. Now for all γ_β there exist α_β such that $U^i_{\alpha_\beta} \neq Z_i$ for at least γ_β many indices i. Now put $U = \bigcap_{\beta < \mathrm{cof}\, \omega_\sigma} U_{\alpha_\beta}$. U is by ω_μ-additivity open in Z. On the other hand U consists of at least ω_σ ($= \sup \gamma_\beta$) factors $\neq Z_i$ and therefore cannot be open in the ω_σ-producttopology. So we must have cof $(\omega_\sigma) \geq \omega_\mu$. Denote now with δ_α the set of indices i such that $U^i_\alpha \neq Z_i$. Then card $\delta_\alpha < \omega_\sigma$. Since $U_\alpha \subseteq U_\beta$ if $\alpha > \beta$, we have $\delta_\alpha \supseteq \delta_\beta$. Choose $i_\alpha \notin \delta_\alpha$ for every $\alpha < \omega_\mu$ with the additional property $i_\alpha > i_{\alpha'}$ if $\alpha > \alpha'$. Clearly $U^{i_\alpha}_\alpha = Z_{i_\alpha}$. Put now

$$V = \prod_{\substack{j \notin \{i_\alpha : \alpha < \omega_\mu\},\\ j < \omega_\sigma}} Z_j \times \prod_{\alpha < \omega_\mu} \{z^{i_\alpha}\}$$

where $z = (z^i)$. Then V is open and contains $z = (z^i)$. But no U_α is a subset of V since $U^{i_\alpha}_\alpha = Z_{i_\alpha}$ and card $Z_{i_\alpha} > 1$. Therefore $B < \omega_\sigma$.

To show necessity one only has to use the just mentioned theorems of NYIKOS and JUHÁSZ.

Now as a corollary we can describe the space A^{ω_μ} as a product of discrete spaces.

COROLLARY: For regular ω_μ (A^{ω_μ}, ν) is homeomorphic to the product $\prod_{i<\omega_\mu} A_i$ with the ω_μ-producttopology, where $A_i = A$ for every i with the discrete topology. For singular ω_μ the topology ν is weaker than the ω_μ-producttopology.

Therefore for $\mu>0$ every ω_μ-metrizable space is embeddable in an ω_μ-product of discrete spaces.

For products of spaces metrizable over groups with different cofinality, the situation looks quite different. If X is ω_μ-metrizable and Y ω_ν-metrizable ($\omega_\mu \neq \omega_\nu$), then $X\times Y$ is not ω_σ-metrizable for any ω_σ, except X or Y is discrete. J. E. VAUGHN has given an example of a product of a metric space with an ω_μ-metric one, which is not even normal, see [16]. Searching for conditions ensuring paracompactness we obtain the following:

THEOREM 2.2.: *Let X and Y be paracompact spaces and Y ω_ν-compact. If X has characteristic $\geq \omega_\nu$, then $X\times Y$ is paracompact.*

PROOF: Take an open cover $\mathcal{O}$ of $X\times Y$. Without loss of generality we may assume that $\mathcal{O}$ consists of sets of the form $U\times V$ with $U\subseteq X$, $V\subseteq Y$. Since Y is ω_ν-compact, there is a subcover $\{U^x_\alpha\times V^x_\alpha : \alpha\leq\beta_x\}$ of $\{x\}\times Y$, $\beta_x<\omega_\nu$, for every $x\in X$. Because of char $(X)\geq\omega_\nu>\beta_x$ there is an open neighbourhood $U(x)\subseteq X$ satisfying $U(x)\subseteq U^x_\alpha$ for all $\alpha\leq\beta_x$ (for all $x\in X$). The open cover $\mathcal{U}=\{U(x): x\in X\}$ of X is now refined by an open, locally finite cover $\mathcal{U}'$. For every $U'\in\mathcal{U}'$ choose $U(x)\in\mathcal{U}$ such that $U'\subseteq U(x)$ and denote this x by $x=f(U')$. Also for all $x\in X$ there is an open, locally finite cover $\mathcal{W}^x$ of Y refining $\{V^x_\alpha : \alpha\leq\beta_x\}$. Define now the family $\mathcal{A}=\bigcup_{U'\in\mathcal{U}'}\{U'\times W$: $W\in\mathcal{W}^x$ with $x=f(U')\}$. Clearly $\mathcal{A}$ is an open cover of $X\times Y$ and refines $\mathcal{O}$. We claim $\mathcal{A}$ to be locally finite. Take an $(x,y)\in X\times Y$, then we can find an open neighbourhood 0 of x, such that only finite many elements $U'_i\in\mathcal{U}'$ meet 0. The point (x,y) is therefore contained at most in the sets of the form $U'_i\times$ $\times W$ with $W\in\mathcal{W}^{f(U'_i)}$. But for every index i there is a neighbourhood V_i of y in Y which meets only finite many members $W\in\mathcal{W}^{f(U'_i)}$. Let $V:=\bigcap V_i$. Then V is an open set containing y and meeting only finite many elements of $\mathcal{W}^{f(U'_i)}$ for every i. Consequently $0\times V$ is an open neighbourhood of (x,y), which is intersected only by finite many elements $U'_i\times W$ with $W\in\mathcal{W}^{f(U'_i)}$. Therefore $\mathcal{A}$ is locally finite.

COROLLARY: If X is ω_μ-metrizable and Y is ω_o-metrizable and ω_ν-compact, then $X\times Y$ is paracompact if $\omega_\nu\leq\omega_\mu$.

REMARK: If we demand char $(X)=\omega_\nu$, we can weaken ω_ν-compact to $[\omega_\nu,\omega_\nu]$-compact. See VAUGHN [16].

3. ω_μ-complete spaces

We now turn to products and subspaces of ω_μ-complete spaces. Lavrentieff's theorem will be generalized and a characterization of ω_μ-completely ω_μ-metrizable spaces will be given. An ω_μ-metric space (X, d) is called ω_μ-complete if every Cauchy- ω_μ-sequence converges (a Cauchy- ω_μ-sequence is an ω_μ-sequence $(x_\alpha)_{\alpha<\omega_\mu}$ such that for every $a\in G$, $a>0$ there is $\beta<\omega_\mu$ and $\alpha,\alpha'>\beta$ implies $d(x_\alpha, x_{\alpha'})<a$). STEVENSON and THRON proved that an ω_μ-metric space (X, d) is ω_μ-complete iff $(X, \mathcal{U}_d)$ is complete in the uniform sense.

H. C. REICHEL has shown in [10] that A^{ω_μ} is ω_μ-complete by showing that $(A^{\omega_\mu}, \mathcal{U}_r)$ is complete, where $\mathcal{U}_r$ is the natural uniformity associated with the natural topology r. He also has shown that an ω_μ-metric space is ω_μ-complete iff it is isomorphic with a closed subspace of A^{ω_μ}.

THEOREM 3.1.: *Let X_i be ω_μ-metrizable spaces and let $\prod_{i<A} X_i$ carry such an ω_σ-producttopology that it is ω_μ-metrizable* (cf. Th. 2.1.). *Then every X_i is ω_μ-completely ω_μ-metrizable iff $\prod_{i<A} X_i$ is so.*

PROOF: Since every X_i is a closed subset of the product, the reverse implication is easy. To prove the other one we observe that by Theorem 2.1. it suffices to investigate only the case where $A \leq \omega_\mu$ and the product carries the ω_μ-producttopology. Then every X_i has a compatible, complete uniformity $\mathcal{U}_i$ with a linearly ordered base. Define now in an obvious way the ω_μ-productuniformity $\mathcal{U}$. $\mathcal{U}$ is compatible with the ω_μ-producttopology and has a linearly ordered base. To show completeness, take a Cauchy-net $(x_\alpha)_{\alpha<\omega_\mu}$ relative to $\mathcal{U}$ (it suffices to consider only nets with index-set ω_μ, since $\mathcal{U}$ can be generated by an ω_μ-metric). Then the projection of (x_α) on X_i, i.e. (x_α^i) is a Cauchy-net relative to $\mathcal{U}_i$ and therefore converges to x^i. Now (x_α) converges to $x = (x^i)$.

REMARK: Here we have used the fact that co-ordinatewise convergence of an ω_μ-sequence implies the convergence in the ω_μ-producttopology since all spaces are ω_μ-metrizable.

In the following we investigate the properties of subspaces of ω_μ-complete spaces. Similar to the metric case (i.e. $\mu = 0$) $G_{\delta,\mu}$-subsets are exactly the ω_μ-completely ω_μ-metrizable ones. To prove this we need some lemmata and definitions.

DEFINITION 3.1.: Let (Y, d) be an ω_μ-metric space with $d: Y^2 \to G$, where $d(Y^2) = \{s_\alpha : \alpha < \omega_\mu\}$ and s_α converging monotonically to 0 in G (if $\mu = 0$, we drop the condition for $d(Y^2)$). For a function f from a set $A \subseteq X$ into Y and for $U \subseteq X$ we define $\operatorname{osc}(f, U) = \sup\{d(f(x), f(y)) : x, y \in U \cap A\}$ if $U \cap A \neq \emptyset$, and $\operatorname{osc}(f, U) = s_1$ if $U \cap A = \emptyset$. If X is a topological space and $x \in \overline{A}$, then $\operatorname{osc}(f, x) = \inf \operatorname{osc}(f, U)$ where the infimum is taken over all neighbourhoods U of x.

REMARK: The sup (inf) in the definition is to be understood as supremum (infimum) in the set $\{s_\alpha : \alpha < \omega_\mu\}$ and not in G. Then it always exists, since the s_α are well ordered in the reverse order of the group G. For the case $\mu = 0$ we avoid such problems by taking $G = R$.

LEMMA 3.2.: Let X be ω_μ-metrizable, Y ω_μ-completely ω_μ-metrizable. Let A be an arbitrary subset of X and $f: A \to Y$ a continuous function. Then there exists a continuous extension f^* of f over a $G_{\delta,\mu}$-set $A^* \subseteq X$, where $A \subseteq A^* \subseteq \overline{A}$.

PROOF: By Proposition 1.1. find an ω_μ-metric d, which has all properties mentioned in Definition 3.1. Now mimic the proof of the case $\mu = 0$, e.g. in [17], p. 177.

LEMMA 3.3.: Let X and Y be ω_μ-completely ω_μ-metrizable spaces. Let $h: A \subseteq X \to B \subseteq Y$ be a homeomorphism. Then there are two $G_{\delta,\mu}$-sets $A^* \subseteq X$ and $B^* \subseteq Y$ with $A \subseteq A^* \subseteq \bar{A}$ and $B \subseteq B^* \subseteq \bar{B}$ such that h can be extended to a homeomorphism h^* from A^* to B^*.

(For $\mu = 0$ we get the theorem of Lavrentieff.)

PROOF: Similar to the proof of Theorem 24.9. in [17].

THEOREM 3.4.: *Let G be a subset of an ω_μ-metric space (X, d). If X is ω_μ-complete and G is a $G_{\delta,\mu}$-set in X, then G is ω_μ-completely ω_μ-metrizable. Conversely, if G is ω_μ-completely ω_μ-metrizable it is a $G_{\delta,\mu}$-set in X.*

PROOF: For $\mu = 0$ see [17]. Therefore $\mu > 0$. First let G be open in X. Because of Proposition 1.1., we may assume that the ω_μ-metric d on X has as range W_μ and additionally $d(X^2) = \left\{\frac{1}{\xi} : \xi < \omega_\mu\right\} \subseteq W_\mu$. For $A \subseteq X$ we define: $d(x, A) = \max\left\{\frac{1}{\alpha} : \frac{1}{\alpha} < d(x, y) \text{ for all } y \in A\right\}$ if this set is not void and $d(x, A) = 0$ otherwise. The maximum exists, since $\left\{\frac{1}{\alpha} : \alpha < \omega_\mu\right\}$ is well ordered in the reverse order of the field W_μ. Now define a continuous function f from G to W_μ by $f(x) = \dfrac{1}{d(x, X-G)}$. Then $d^*(x, y) = d(x, y) + |f(x) - f(y)|$ is an ω_μ-metric on G compatible with d, relative to which G is ω_μ-complete. To show this, take a Cauchy-ω_μ-sequence (x_γ) relative to d,* i.e. for all $\frac{1}{\alpha} \in W_\mu$ there exists $\alpha' < \omega_\mu$ such that for $\beta, \gamma > \alpha'$ $|f(x_\beta) - f(x_\gamma)| < \frac{1}{\alpha}$. With other words

$$\left|\frac{1}{d(x_\beta, X-G)} - \frac{1}{d(x_\gamma, X-G)}\right| < \frac{1}{\alpha}\,.$$

Therefore $d(x_\beta, X-G) = \frac{1}{\alpha_\beta}$ must be bounded away from zero, otherwise $|\alpha_\beta - \alpha_\gamma| < \frac{1}{\alpha}$ for fixed $\gamma > \alpha'$ and *all* $\beta > \alpha'$, which is impossible. Hence $\{x_\alpha : \alpha < \omega_\mu\} \subseteq M_\varepsilon := \{x \in X : d(x, X-G) \geq \varepsilon\}$. M_ε is closed and since $d \leq d^*$, (x_α) is also Cauchy relative to d. Therefore $(x_\alpha) \to x \in M_\varepsilon$, and G is ω_μ-complete. If G is now a $G_{\delta,\mu}$-set, i.e. $G = \bigcap_{\alpha < \omega_\mu} H_\alpha$, H_α open, then G is homeomorphic to the diagonal in the product $\prod_{\alpha < \omega_\mu} H_\alpha$ when the product carries the ω_μ-product-topology. Now the diagonal is closed in the product and each H_α is ω_μ-completely ω_μ-metrizable as just shown.

To prove the converse proceed as in [17] p. 179 f, using Lemma 3.3. We have now proved the following characterization of ω_μ-completely ω_μ-metrizable spaces:

COROLLARY 3.5.: Let X be an ω_μ-metric space. The following are equivalent:

(a) X is ω_μ-completely ω_μ-metrizable
(b) X is $G_{\delta,\mu}$ in an ω_μ-completion $\hat{X}$
(c) X is $G_{\delta,\mu}$ in every ω_μ-metrizable space, in which X can be embedded.

It is well-known that in the case $\mu = 0$ the following is also equivalent to (a), (b), (c):

(d′) X is G_δ in βX
(e′) X is G_δ in every Tychonoff space in which X is embeddable as a dense subspace.

In the next theorem we will show that analogous statements as (d′) and (e′) hold also for $\mu > 0$.

THEOREM 3.6.: *Let X be an ω_μ-metrizable space. Then the following statements are equivalent to every one of* (a), (b), (c):

(d) X is $G_{\delta,\mu}$ in βX
(e) *X is $G_{\delta,\mu}$ in every Tychonoff space in which X is densely embeddable*
(f) *X is $G_{\delta,\mu}$ in one K, where K is an arbitrary compactification of X.*

For $\mu > 0$ additionally:

(g) *X is $G_{\delta,\mu}$ in a suitable A^{ω_μ}.*

PROOF: (a)→(d): If $\mu = 0$ we have finished by Corollary 3.5. Therefore assume $\mu > 0$. Thus X has a complete uniformity $\mathcal{U}$ with a linearly ordered base, say $\{V_\alpha : \alpha < \omega_\mu\}$, of entourages. Let D be a set of pseudometrics, which generate the same uniformity. Now choose a subset of D in the following way: take $\varrho_1 \in D$ arbitrarily. If ϱ_α is choosen for $\alpha < \gamma < \omega_\mu$ take $\varrho_\gamma \in D$ such that

$$U_1^{\varrho_\gamma} \subseteq \bigcap_{\substack{n=1 \\ \alpha<\gamma}}^{\infty} U_{1/n}^{\varrho_\alpha} \cap V_\gamma ,$$

where $U_\varepsilon^\varrho = \{(x, y) : \varrho(x, y) < \varepsilon\}$. Then clearly $U_{1/n}^{\varrho_\alpha} \subseteq U_{1/m}^{\varrho_\beta}$ for $\alpha > \beta$ and also for $\alpha = \beta$ if $n \geq m$. The set $\{\varrho_\alpha : \alpha < \omega_\mu\}$ also generates the uniformity. From the pseudometric space (X, ϱ_α) we can obtain the metric identification $(X_\alpha^*, \varrho_\alpha^*)$. The projection from X to X_α^* is denoted by h_α, i.e. $h_\alpha(x) = h_\alpha(y)$ iff $\varrho_\alpha(x, y) = 0$. Now consider the following diagram:

$$\begin{tikzcd}
(X,\tau_{\mathcal{U}}) \arrow[r, "id_X"] \arrow[d, hook, "i"'] & (X,\varrho_\alpha) \arrow[r, "h_\alpha"] & (X^*_\alpha,\varrho^*_\alpha) \arrow[r, hook, "e_\alpha"] & (\hat{X}^*_\alpha,\hat{\varrho}^*_\alpha) \arrow[d, hook, "i_\alpha"] \\
\beta X \arrow[rrr, dashed, "F_\alpha"] & & & \beta\hat{X}^*_\alpha
\end{tikzcd}$$

Fig. 1.

$(\hat{X}^*_\alpha, \hat{\varrho}^*_\alpha)$ is the metric completion of $(X^*_\alpha, \varrho^*_\alpha)$ and e_α the corresponding embedding. i resp. i_α are the embeddings of X resp. $\hat{X}^*_\alpha$ in their Stone-Čech-compactifications. F_α is the extension of $i_\alpha \circ e_\alpha \circ h_\alpha \circ id_X$ from X to βX.

Since $\hat{X}^*_\alpha$ is complete, $i_\alpha(\hat{X}^*_\alpha)$ is a G_δ-set in $\beta\hat{X}^*_\alpha$. Consequently $A_\alpha :=$ $:= F_\alpha^{-1}(i_\alpha(\hat{X}^*_\alpha))$ is again a G_δ-set in βX. As it is easily seen $i(X) \subseteq \bigcap_{\alpha<\omega_\mu} A_\alpha$ and this intersection is a $G_{\delta,\mu}$-set in βX. We will show that the reverse inclusion also holds. Suppose it would not, then there would be at least one $p \in \bigcap_{\alpha<\omega_\mu} A_\alpha$ with $p \notin i(X)$. But this means that for all $\alpha<\omega_\mu$ $F_\alpha(p) \in i_\alpha(\hat{X}^*_\alpha)$ and consequently there is exactly one $y_\alpha \in \hat{X}^*_\alpha$ with $F_\alpha(p) = i_\alpha(y_\alpha)$, because i_α is injective. For the further proof we need the following two diagrams for $\beta > \alpha$:

$$\begin{tikzcd}
 & (X,\tau_{\mathcal{U}}) \arrow[dl, "id_X"'] \arrow[dr, "id_X"] & \\
(X,\varrho_\alpha) \arrow[d, "h_\alpha"'] & & (X,\varrho_\beta) \arrow[ll, "id_X"'] \arrow[d, "h_\beta"] \\
(X^*_\alpha,\varrho^*_\alpha) \arrow[d, "e_\alpha"'] & & (X^*_\beta,\varrho^*_\beta) \arrow[ll, dashed, "\bar{f}_{\beta\alpha}"'] \arrow[d, "e_\beta"] \\
(\hat{X}^*_\alpha,\hat{\varrho}^*_\alpha) & & (\hat{X}^*_\beta,\hat{\varrho}^*_\beta) \arrow[ll, dashed, "g_{\beta\alpha}"']
\end{tikzcd}$$

Fig. 2.

Herein $\bar{f}_{\beta\alpha}$ is defined as follows: $\bar{f}_{\beta\alpha}(x^*) := h_\alpha(x)$, where $x^* \in X^*_\beta$ and $x \in h_\beta^{-1}(x^*)$. As it is easily checked $\bar{f}_{\beta\alpha}$ is well defined and uniformly continuous. Therefore the extension of $\bar{f}_{\beta\alpha}$ on $\hat{X}^*_\beta$ exists and is denoted by $g_{\beta\alpha}$. The third diagram looks like:

$$
\begin{tikzcd}
\hat{X}^*_\alpha & \hat{X}^*_\beta \arrow[l, "g_{\beta\alpha}"'] & X \arrow[l, "G_\beta"'] \arrow[ll, bend right, "G_\alpha"'] \\
\\
\beta\hat{X}^*_\alpha \arrow[from=1-1, hook, "i_\alpha"'] & \beta\hat{X}^*_\beta \arrow[l, dashed, "f_{\beta\alpha}"'] \arrow[from=1-2, hook, "i_\beta"'] & \beta X \arrow[l, "F_\beta"'] \arrow[ll, bend left, "F_\alpha"] \arrow[from=1-3, hook, "i"]
\end{tikzcd}
$$

Fig. 3.

Here $G_\alpha := e_\alpha \circ h_\alpha \circ id_X$ and $f_{\beta\alpha}$ is the extension of $g_{\beta\alpha}$ to the Stone-Čech-compactification. The subdiagram consisting of G_α, G_β, $g_{\beta\alpha}$ commutes because the second diagram does. The subdiagram F_α, F_β, $f_{\beta\alpha}$ commutes since these functions are extensions of G_α, G_β, $g_{\beta\alpha}$. Now we return to show that $\bigcap_{\alpha<\omega_\mu} A_\alpha \subseteq i(X)$ holds. We have just mentioned that otherwise we would have a point $p \notin i(X)$ and $F_\alpha(p) = i_\alpha(y_\alpha)$ for exactly one $y_\alpha \in \hat{X}^*_\alpha$ and for all $\alpha < \omega_\mu$. Then we have $i_\alpha(y_\alpha) = F_\alpha(p) = f_{\beta\alpha}(F_\beta(p)) = f_{\beta\alpha}\left(i_\beta(y_\beta)\right) = i_\alpha\left(g_{\beta\alpha}(y_\beta)\right)$ for all $\beta > \alpha$ and therefore $y_\alpha = g_{\beta\alpha}(y_\beta)$. For every $\beta < \omega_\mu$ choose $x_\beta \in X$ such that $\hat{\varrho}^*_\beta(G_\beta(x_\beta), y_\beta) < \frac{1}{3}$, which is possible since $G_\beta(X)$ is dense in $\hat{X}^*_\beta$. We show that $(x_\beta)_{\beta<\omega_\mu}$ is a Cauchy-ω_μ-net in $(X, \mathcal{U})$. Take $\alpha < \omega_\mu$, then it suffices to show that $\varrho_\alpha(x_\beta, x_{\beta'}) < 1$ for finally many β, β'. To show this take β, $\beta' > \alpha$. Then we have $\hat{\varrho}^*_\beta(G_\beta(x_\beta), y_\beta) < \frac{1}{3}$ and $\hat{\varrho}^*_{\beta'}\left(G_{\beta'}(x_{\beta'}), y_{\beta'}\right) < \frac{1}{3}$. Since $\varrho_\beta(x, y) < 1$ implies $\varrho_\alpha(x, y) = 0$ for $\beta > \alpha$ by construction of the ϱ's, we get for β, $\beta' > \alpha$

$$(*) \qquad \hat{\varrho}^*_\alpha\left(g_{\beta\alpha}(G_\beta(x_\beta)), g_{\beta\alpha}(y_\beta)\right) = 0$$

and the same for β' instead of β. By the fact that $g_{\beta\alpha}(y_\beta) = y_\alpha = g_{\beta'\alpha}(y_{\beta'})$ we get $\hat{\varrho}^*_\alpha(G_\alpha(x_\beta), G_\alpha(x_{\beta'})) = 0$. But this means $\varrho_\alpha(x_\beta, x_{\beta'}) = 0$ and therefore (x_β) is a Cauchy-net in $(X, \mathcal{U})$. Now it must converge to some point, say $x \in X$. Then clearly $\varrho_\alpha(x_\beta, x) = 0$ for $\beta > \alpha$ and consequently $\hat{\varrho}^*_\alpha(G_\alpha(x_\beta), G_\alpha(x)) = 0$; now using $(*)$ we obtain $\hat{\varrho}^*_\alpha(G_\alpha(x), y_\alpha) = 0$, which means $G_\alpha(x) = y_\alpha$. This implies $i_\alpha(G_\alpha(x)) = i_\alpha(y_\alpha) = F_\alpha(p)$. Since $i_\alpha \circ G_\alpha = F_\alpha \circ i$ we get $F_\alpha(i(x)) = F_\alpha(p)$ for all $\alpha < \omega_\mu$ and for this fixed $x \in X$. But this is impossible: take U and V as disjoint neighbourhoods of $i(x)$ and p in βX. Then there is an $\alpha < \omega_\mu$ such that $B^{\varrho_\alpha}(x, \varepsilon_\alpha) := \{y \in X : \varrho_\alpha(x, y) < \varepsilon_\alpha\} \subseteq i^{-1}(U)$. Therefore $i_\alpha\left(G_\alpha\left(B^{\varrho_\alpha}(x, \varepsilon_\alpha)\right)\right)$ is a subspace-neighbourhood of $i_\alpha(G_\alpha(x)) =$

$:= F_\alpha(i(x))$ in $i_\alpha(\hat{X}_\alpha^*) \subseteq \beta\hat{X}_\alpha^*$. For all open $V' \subseteq V$ and $p \in V'$ there is an $i(z) \in V'$ since $i(X)$ is dense in βX. This $i(z)$ has the property that $F_\alpha(i(z))$ is no element of $i_\alpha\Big(G_\alpha(B^{\varrho_\alpha}(x, \varepsilon_\alpha))\Big)$, because $F_\alpha(i(z)) = i_\alpha(G_\alpha(z))$ and $\varrho_\alpha(x, z) \geq \varepsilon_\alpha$, which implies $i_\alpha(G_\alpha(z)) \notin i_\alpha\Big(G_\alpha(B^{\varrho_\alpha}(x, \varepsilon_\alpha))\Big)$. But this means that F_α would not be continuous. This is a contradiction and hence $\bigcap\limits_{\alpha < \omega_\mu} A_\alpha = i(X)$.

(f)→(d): Let X be a G_{ω_μ}-set in one compactification K and f the corresponding embedding. Denote with f^* the extension of f to βX. Then $X \subseteq f^{*-1}(f(X))$ and $f^{*-1}(f(X))$ is clearly a G_{ω_μ}-set in βX, since $f(X)$ is so in K. The inclusion $f^{*-1}(f(X)) \subseteq X$ is proved by showing that no $p \in \beta X - X$ has $f^*(p) \in f(X)$. But this follows from $f^*(\beta X - X) \subseteq K - f^*(X)$ (cf. [17], p. 138.)

The implications (d)→(e), (e)→(c), (c)→(f) and for $\mu > 0$ (c)→(g), (g)→(a) are easy modifications of the proofs for $\mu = 0$ or are obvious.

4. ω_μ-compact spaces

It is a well-known fact that every compact space has exactly one compatible uniformity, but the converse is false. Therefore it is just natural to ask what spaces look like which have only one uniformity, which has a linearly ordered base. An answer is given in [10], where it is shown that a topological space is a compact metric one iff it admits exactly one uniformity, and this uniformity has a linearly ordered base. The next step is to investigate the structure of spaces possibly having more than one uniformity but having only one which has a linearly ordered base. Recall that if a topological space has compatible uniformities with linearly ordered bases, then all these bases (even those from different uniformities) have the same cofinality except the case that X is discrete (cf. p. 2).

THEOREM 4.1.: *An ω_μ-compact T_2-space has at most one compatible uniformity with a linearly ordered base of cofinality ω_μ. Conversely, a T_2-space which admits only one uniformity $\mathcal{U}$ with a linearly ordered base of cofinality ω_μ (but possibly other compatible uniformities), is ω_μ-compact.*

PROOF: For the first part see [10]. Now we prove the converse. First we consider the case $\mu > 0$. With $\mathcal{B}$ we denote a linearly ordered base for $\mathcal{U}$ with $\mathcal{B} = \{\mathcal{B}_\alpha : \alpha < \omega_\mu\}$ and $\mathcal{B}_\alpha < \mathcal{B}_\beta$ for $\alpha > \beta$. The symbol $<$ stands for "refining". Furthermore $\mathcal{B}_\alpha$ can be taken as a partition of X for every $\alpha < \omega_\mu$. For later use we remark that the cofinality of $\mathcal{B}$ is clearly ω_μ and that we can assign in a unique way an ω_μ-sequence of sets $B_\alpha(x) \in \mathcal{B}_\alpha$ to every $x \in X$, where $\{B_\alpha(x) : \alpha < \omega_\mu\}$ is a neighbourhood base at x.

Case 1: X is not discrete.

X' denotes the set of accumulation points of X and $P_\alpha = \{x \in X: x$ is isolated and $\{x\} \notin \mathcal{B}_\alpha\}$ for $\alpha < \omega_\mu$. Clearly $X = X' \cup \bigcup\limits_{\alpha < \omega_\mu} P_\alpha \cup \{x: \{x\} \in \mathcal{B}_1\}$. There are now two further cases to distinguish between:

(A) $$|X - (P_\alpha \cup X')| < \omega_\mu \quad \text{for all} \quad \alpha < \omega_\mu .$$

Suppose now X' were not ω_μ-compact. Then there is an ω_μ-sequence $(z_\alpha)_{\alpha<\omega_\mu}\subseteq X'$ with no accumulation point. Therefore we can find pairwise disjoint neighbourhoods $U(z_\alpha)$ in X. Choose $\gamma_\alpha<\omega_\mu$ such that $B_{\gamma_\alpha}(z_\alpha)\subseteq U(z_\alpha)$ and $B_{\gamma_\alpha}(z_\alpha)\subseteq B_\alpha(z_\alpha)$. $\bigcup\limits_{\alpha<\omega_\mu} B_{\gamma_\alpha}(z_\alpha)$ is closed: take $x_\beta\in\bigcup\limits_{\alpha<\omega_\mu} B_{\gamma_\alpha}(z_\alpha)$ and $x_\beta\to x$. Then either $\{x_\beta:\beta<\omega_\mu\}\subseteq\bigcup\limits_{\alpha<\delta} B_{\gamma_\alpha}(z_\alpha)$ for some $\delta<\omega_\mu$, or $x_\beta\in B_{\gamma_{\alpha_\beta}}(z_{\alpha_\beta})$ with γ_{α_β} cofinal in ω_μ. In the first case $\bigcup\limits_{\alpha<\delta} B_{\gamma_\alpha}(z_\alpha)$ is closed and therefore x lies in it. In the second case we have $d(x_\beta, z_{\alpha_\beta})\to 0$ in a compatible ω_μ-metric d and this would yield $z_{\alpha_\beta}\to x$, which contradicts the choice of (z_α). Define now

$$\mathcal{W}=\{B_{\gamma_\alpha}(z_\alpha):\alpha<\omega_\mu\}\cup\Big\{X-\bigcup_{\alpha<\omega_\mu} B_{\gamma_\alpha}(z_\alpha)\Big\},$$

which is an open partition of X. Define $\mathcal{B}'_\alpha=\mathcal{B}_\alpha\wedge\mathcal{W}$:

(a) $\{\mathcal{B}'_\alpha:\alpha<\omega_\mu\}$ is a linearly ordered base for a uniformity $\mathcal{U}'$ since all $\mathcal{B}'_\alpha$ are partitions of X and $\mathcal{B}'_\alpha<\mathcal{B}'_\beta$ if $\alpha>\beta$. Clearly $\mathcal{U}'$ induces the same topology as $\mathcal{U}$ does.

(b) The cofinality of this base is ω_μ, since X is not discrete by assumption.

(c) $\mathcal{U}\neq\mathcal{U}'$. Otherwise we would have a $\tau<\omega_\mu$ such that $\mathcal{B}_\tau<\mathcal{B}'_1$. But then $B_\tau(z_\tau)\subseteq B_{\gamma_\tau}(z_\tau)$ which contradicts the construction of γ_τ.

We have now proved that X' is ω_μ-compact. Assuming now that X is not ω_μ-compact there exists an ω_μ-sequence $(x_\alpha)_{\alpha<\omega_\mu}$ without accumulation point. Since X' is ω_μ-compact, there cannot be a cofinal subnet of (x_α) which lies in X'. Hence we may assume $(x_\alpha)\subseteq X-X'$. But (x_α) cannot be a subset of a fixed $X-(P_\alpha\cup X')$, since $|X-(P_\alpha\cup X')|<\omega_\mu$ would imply that (x_α) were at last constant and had therefore an accumulation point. Consequently we have for all $\alpha<\omega_\mu$ a $\beta_\alpha<\omega_\mu$ with $x_{\beta_\alpha}\in P_\alpha$. Now we define

$$\mathcal{W}'=\{\{x_{\beta_\alpha}\}:\alpha<\omega_\mu\}\cup\Big\{X-\bigcup_{\alpha<\omega_\mu}\{x_{\beta_\alpha}\}\Big\}.$$

Then this is an open partition of X. Again we put $\mathcal{B}''_\tau=\mathcal{B}_\tau\wedge\mathcal{W}'$ and get:

(a) $\{\mathcal{B}''_\tau:\tau<\omega_\mu\}$ is a linearly ordered base for a compatible uniformity $\mathcal{U}''$.

(b) The cofinality of this base is ω_μ.

(c) $\mathcal{U}\neq\mathcal{U}''$. Otherwise we would have a $\tau<\omega_\mu$ such that $\mathcal{B}_\tau<\mathcal{B}''_1$, i.e. $B_\tau(x_{\beta_\tau})\subseteq\{x_{\beta_\tau}\}$ which implies $\{x_{\beta_\tau}\}\in\mathcal{B}_\tau$ and therefore $x_{\beta_\tau}\notin P_\tau$, which is a contradiction.

Therefore X is ω_μ-compact.

(B) $$|X-(P_\alpha\cup X')|\geq\omega_\mu \quad\text{for one}\quad \alpha<\omega_\mu.$$

Then $X-(P_\alpha\cup X')$ is a clopen subset of X. Take two disjoint subsets E resp. F of $X-(P_\alpha\cup X')$ with card E = card $F=\omega_\mu$. We write $E=\{x_\tau:\tau<\omega_\mu\}$

and $F = \{y_\tau : \tau < \omega_\mu\}$. We define $\mathcal{B}'_\tau = (\mathcal{B}_\tau \wedge \{X - E\}) \cup \{E\}$ and $\mathcal{B}''_\tau = (\mathcal{B}_\tau \wedge \{X - F\}) \cup \{F\}$ and $\mathcal{B}^E_\tau = \mathcal{B}'_\tau \wedge (\{\{x_\beta\} : \beta < \tau\} \cup \{X - \{x_\beta : \beta < \tau\}\})$. $\mathcal{B}^F_\tau$ is defined in the same way as $\mathcal{B}^E_\tau$ only with $\mathcal{B}''_\tau$ instead of $\mathcal{B}'_\tau$ and y_β instead of x_β. Clearly $\{\mathcal{B}^E_\tau : \tau < \omega_\mu\}$ resp. $\{\mathcal{B}^F_\tau : \tau < \omega_\mu\}$ generate two different uniformities, each with a linearly ordered base of cofinality ω_μ. Both uniformities induce the given topology on X, but this is a contradiction, therefore case 1 (B) is impossible.

Case 2: X is discrete.

If card $X < \omega_\mu$ then X is ω_μ-compact. The case card $X \geq \omega_\mu$ is treated in the same way as 1 (B).

It only remains to consider the case $\mu = 0$. It suffices to show that X is complete in every metric which is compatible with the given topology.[4] Suppose there is a metric d on X, which is not complete. Then we have a d-Cauchy-sequence (x_n) in X without limit. Define $\varrho(x_n, x_m) = |n-m|$. This is a metric on $\{x_n : n \in N\}$ and can be extended to a metric on X, which is equivalent to d (cf. [17], p. 165). But clearly $\mathcal{U}_d \neq \mathcal{U}_\varrho$ and both are uniformities with a linearly ordered base of cofinality ω_0. This is a contradiction, and hence X is compact. As a by-product we got a characterization of compact metric spaces:

Corollary 4.1.1.: A topological space is a compact metric one iff it has only one compatible (separated) uniformity with a linearly ordered base of cofinality ω_0 (i.e. only one uniformity with a countable base).

Remark: It is in a way surprising, that in the case $\mu = 0$ one can from the fact, that there is only one uniformity with a linearly ordered base, conclude, that there is only one uniformity in toto. For $\mu > 0$ this is not true. Otherwise it would follow that X were metrizable by the theorem in [10] just mentioned at the beginning of this chapter.

In the preceding proof we have used the fact that a metrizable space is compact iff it is complete in every metric compatible with the given topology. We now give an analogous result for ω_μ-metrizable spaces.

Theorem 4.2.: *An ω_μ-metrizable space is ω_μ-compact iff it is ω_μ-complete in every ω_μ-metric compatible with the topology.*

Proof: For $\mu = 0$ see [7]. Therefore $\mu > 0$. One direction is obvious-For necessity assume X to be not ω_μ-compact, but nevertheless to be ω_μ-complete in every ω_μ-metric, i.e. every compatible uniformity with a linearly ordered base of cofinality ω_μ is complete.

(A) X is not discrete.

Since X is assumed to be not ω_μ-compact there is an ω_μ-sequence $(z_\alpha)_{\alpha < \omega_\mu}$ in X without an accumulation point. Let $\{\mathcal{B}_\alpha : \alpha < \omega_\mu\}$ be a linearly ordered base for a compatible uniformity $\mathcal{U}$, where every $\mathcal{B}_\alpha$ is an open partition of X.

[4] A metrizable space is compact iff it is complete in every metric compatible with the topology. For a proof see e.g. [7], [17] p. 183.

Since (z_α) has no accumulation point and X is ω_μ-metrizable, there are sets $B_{\beta_\alpha} \in \mathcal{B}_{\beta_\alpha}$ with $z_\alpha \in B_{\beta_\alpha}$ and $z_\gamma \notin B_{\beta_\alpha}$ for $\gamma \neq \alpha$. We now put $\mathcal{A}_\alpha = \{B \in \mathcal{B}_\alpha : z_\gamma \notin B \text{ for } \gamma > \alpha\}$ and $\mathcal{B}'_\alpha = \mathcal{A}_\alpha \cup \left\{ \bigcup_{B \in \mathcal{B}_\alpha \setminus \mathcal{A}_\alpha} B \right\}$. Then we have:

1. $\mathcal{B}'_\alpha$ is an open partition of X with $\mathcal{B}'_\beta < \mathcal{B}'_\alpha$ for $\beta > \alpha$. Therefore they constitute a linearly ordered base for a uniformity $\mathcal{U}'$.
2. $\tau_{\mathcal{U}} = \tau_{\mathcal{U}'}$. This can be seen as follows: clearly $\tau_{\mathcal{U}'} \subseteq \tau_{\mathcal{U}}$ since every $\mathcal{B}'_\alpha$ consists of $\tau_{\mathcal{U}}$-open sets. On the other hand every $x \in X$ lies at most in one B_{β_α}, and for $\gamma > \beta_\alpha$ the sets in $\mathcal{B}'_\gamma$ which contain x are the same as the sets in $\mathcal{B}_\gamma$. Hence $\tau_{\mathcal{U}'} \supseteq \tau_{\mathcal{U}}$.
3. The cofinality of the base $\{\mathcal{B}'_\alpha : \alpha < \omega_\mu\}$ is clearly ω_μ. Furthermore (z_α) is a Cauchy-net in $\mathcal{U}'$ and, since $\mathcal{U}'$ must be complete, (z_α) must converge, which yields a contradiction.

(B) X is discrete.

The assumption that X is not ω_μ-compact implies card $X \geq \omega_\mu$. Choose an ω_μ-sequence (z_α) with $z_\alpha \neq z_\beta$ for $\alpha \neq \beta$. Then define a linearly ordered base of cofinality ω_μ for a uniformity $\mathcal{U}'$ which induces the discrete topology on X as follows: $\mathcal{B}'_\alpha = \{\{x\} : x \neq z_\gamma \text{ for all } \gamma < \omega_\mu\} \cup \{\{z_\beta\} : \beta < \alpha\} \cup \{\{z_\beta : \beta \geq \alpha\}\}$. Clearly (z_α) is a Cauchy-net in $\mathcal{U}'$ and must therefore converge, which contradicts the construction of the net (z_α).

Now we give a characterization of ω_μ-compact spaces in terms of embeddings in Hausdorff spaces.

THEOREM 4.3.: *Let X be a regular T_2-space. X is ω_μ-compact iff the following is true: if X is embedded in an T_2-space Y, then for $p \in Y - X$ we have a $\xi < \mu$, such that p is contained in a $G_{\xi,\xi}$-set in Y, which is disjoint with X.*

REMARK: For $\mu = 0$ this is to be interpreted as "... p is contained in an open set in Y ...". The proof is an easy extension of the proof for the case $\mu = 0$.

A generalization of the fact that every compact and also every regular Lindelöf space is paracompact is now given.

THEOREM 4.4.: *If X is an ω_μ-compact T_2-space and ω_ξ-additive for all $\xi < \mu$, then X is paracompact.*

PROOF: By a well-known result of MICHAEL it suffices to show that every open cover $\mathcal{U}$ of X with $|\mathcal{U}| < \omega_\mu$ has a closure preserving refinement. Let $\mathcal{V}$ be a cover of X such that $|\mathcal{V}| < \omega_\mu$ and $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$. Then there is a $\xi < \mu$ such that $\mathcal{U} = \{U_\alpha : \alpha < \omega_\xi\}$ and $\mathcal{V} = \{V_\beta : \beta < \omega_\xi\}$ (with possible repetitions). Then put for $\alpha < \omega_\xi$ $W_\alpha = U_\alpha \setminus \bigcup \{\overline{V}_\beta : \beta < \alpha \ \& \ \exists \gamma < \alpha \ (\overline{V}_\beta \subseteq U_\gamma)\}$. By ω_ξ-additivity W_α is open, clearly $\mathcal{W} = \{W_\alpha : \alpha < \omega_\xi\}$ covers X, refines $\mathcal{U}$, and is locally $< \omega_\xi$ because if $x \in \overline{V}_\beta \subseteq U_\gamma$ and $\beta, \gamma < \alpha$ then $V_\beta \cap W_\alpha = \emptyset$. But then $\mathcal{W}$ is closure preserving as X is ω_ξ-additive.

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KREISSPIEGELUNG IN METRISCHEN AFFINEN EBENEN UND IHRE KONSTRUKTIVE DARSTELLUNG UNTER BESONDERER BERÜCKSICHTIGUNG VON ENDLICHKEIT

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In der vorliegenden Note wird von metrischen affinen Ebenen ausgegangen, die euklidische und minkowskische (pseudo-euklidische) Ebenen umfassen und auf synthetisch konstruktive Weise mit Orthogonalitätsrelationen begründet sind. Unter Kreisspiegelungen werden Abbildungen verstanden, die gewissen einfachen und aus der Elementargeometrie bekannten Eigenschaften genügen. Für sie werden eine Reihe von Eigenschaften und konstruktive Darstellungen gegeben, wobei die fehlende Beweglichkeit, die Existenz isotroper (d. h. selbstorthogonaler) Geraden und die Einschränkung auf Endlichkeit hier die Besonderheiten sind. Isotrope Elemente ermöglichen gerade eine sehr einfache konstruktive Darstellung der Kreisspiegelung.

Die Note ist als ein Beitrag zu metrischen affinen Ebenen angelegt. Auf Bezüge zu Möbiusschen Kreisebenen wird hingewiesen. Ferner sei bemerkt, daß modelltheoretische Aspekte beim Aufbau der Geometrie auf der Kugel von SCHREIBER [9] zu dieser Arbeit anregten.

1. Metrische affine Ebenen, Kreise

Wir legen im folgenden *pappossche Ebenen* $(\mathcal{P}, \mathcal{G})$ mit *Fano-Aussage* zugrunde, d. h. affine Ebenen über einem kommutativen Körper K der Charakteristik $\neq 2$. Bezeichnungen sind: $A, B, C, \ldots$ $(\in \mathcal{P})$ für *Punkte*; $a, b, c, \ldots$ $(\in \mathcal{G}$ mit $a, b, c, \ldots \subset \mathcal{P})$ für *Geraden*; $a \| b$ für die *Parallelität*; AB für die *Verbindungsgerade*. In der analytischen Darstellung identifizieren wir die Gerade $\{(x, y)\colon ux+vy+w = 0;\ u, v, w \in K, (u, v) \neq (0, 0)\}$ zur Vereinfachung mit (u, v, w).

Unter einer *metrischen affinen Ebene* wird hier eine pappossche Ebene (Char$\neq 2$) mit einer *Orthogonalitätsrelation* $\perp$ verstanden, die als binäre Relation in $\mathcal{G}$ durch folgende Eigenschaften bestimmt ist:

Invarianz gegenüber Parallelität; Eindeutigkeit der Lote; Höhenschließungssatz (Ist ABC ein Dreieck und ist a eine Gerade durch A mit $BC \perp a$

und b eine Gerade durch B mit $CA \perp b$, und schneiden sich a und b in einem Punkt $H \neq C$, so ist $AB \perp CH$); *Reichhaltigkeitsforderung* (Es gibt Geraden a_1, b_1, a_2, b_2 mit $a_1, b_1 \nparallel a_2, b_2$ und $a_i \perp b_i$, $i = 1, 2$).

Diese axiomatische Charakterisierung metrischer affiner Ebenen kann wesentlich abgeschwächt werden; so ist bereits in Translationsebenen mit einer Orthogonalitätsrelation der affine Satz von Pappos ableitbar. (Näheres dazu und weitere Zusammenhänge findet man in [7], [8a] und anderen Arbeiten des Verfassers.)

Die Orthogonalitätsrelationen lassen sich (nach [7], S. 28) analytisch durch eine quadratische Form beschreiben. Es gilt

(1) $(u, v, w) \perp (u', v', w') \Leftrightarrow uu' + \lambda vv' = 0$, wobei $\lambda (\neq 0) \in K$ eine Konstante ist.

Isotrope, d. h. selbstorthogonale Geraden sind nicht ausgeschlossen. Vielmehr umfassen die hier vorgestellten metrischen affinen Ebenen gerade die *euklidischen* und *minkowskischen (pseudo-euklidischen)* Ebenen. Letztere sind dadurch ausgezeichnet daß sie (genau zwei) isotrope Richtungen besitzen.

Für konstruktive Darstellungen ist von grundlegender Bedeutung, daß sich nach dem Höhenschließungssatz eine Orthogonalitätsrelation konstruktiv beschreiben läßt. Es gilt

(2) *In einer papposschen Ebene (Char $\neq 2$) gibt es zu je zwei vorgegebenen Geradenpaaren (a_1, b_1), (a_2, b_2) mit $a_1, b_1 \nparallel a_2, b_2$ genau eine Orthogonalitätsrelation $\perp$ mit $a_i \perp b_i$ $(i = 1, 2)$.*

So erhält man für $a_1 = b_1$ und $a_2 = b_2$ gerade eine minkowskische Orthogonalitätsrelation, und für ihre konstruktive Darstellung aus dieser Vorgabe kommt man bereits mit einer einfachen Parallelogrammkonstruktion aus, wie Abb. 1 zeigt.

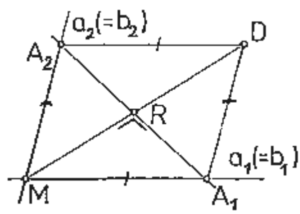

Abb. 1.

Ein *Kreis um M durch A* $(\neq M)$ — kurz $\mathcal{K}(M, A)$ — sei die Menge aller Punkte P für die $P = A$ oder das Mittellot von (P, A) durch M geht.

Es gilt:

Durch je drei nicht kollineare Punkte geht genau ein Kreis;

$$\mathcal{K}(M_1, A_1) = \mathcal{K}(M_2, A_2) \Leftrightarrow M_1 = M_2 \wedge A_2 \in \mathcal{K}(M_1, A_1) \wedge A_2 \neq M_1 .$$

Insbesondere heißt $\mathcal{K}(M, A)$ *isotrop*, falls M selbst zum Kreis gehört. Er besteht gerade aus den beiden isotropen Geraden durch M. (Er existiert nur in einer minkowskischen Ebene; so ist in Abb. 1. $\mathcal{K}(M, A_1) = a_1 \cup a_2$.) Mit der konstruktiven Darstellung der Orthogonalität ist auch die der Kreise gegeben:

(3) *Es ist $P \in \mathcal{K}(M, A) \setminus MA$ genau dann, wenn P Schnittpunkt einer von MA verschiedenen anisotropen Geraden durch A mit ihrem Lot durch B ist, wobei B durch $\overrightarrow{AM} = \overrightarrow{MB}$ bestimmt ist* (Abb. 2. (A, B) wird ein *Durchmesser* von $\mathcal{K}(M, A)$ genannt).

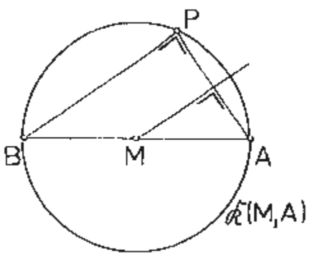

Abb. 2.

Dies gilt also auch für isotrope Kreise. Aus (3) ergibt sich ein Schnittverhalten für eine anisotrope Gerade mit einem Kreis wie in der klassischen Elementargeometrie; insbesondere folgt

(4) *Durch jeden Punkt P eines anisotropen Kreises $\mathcal{K}$ mit dem Mittelpunkt M geht genau eine Tangente t*, d. h. eine anisotrope Gerade, die mit $\mathcal{K}$ genau einen Punkt gemeinsam hat. *Es ist $t \perp MP$.*

Für isotrope Geraden gilt

(5) *Jeder anisotrope Kreis hat mit jeder isotropen Geraden, die nicht durch seinen Mittelpunkt geht, genau einen Punkt gemeinsam.*

Beweis. Die Eindeutigkeit ist nach (3) klar. Den Existenzbeweis führen wir in Hinblick auf die Zielstellung dieser Note konstruktiv. — Sei $\mathcal{K}(M, A)$ anisotrop, I_1 und I_2 die beiden isotropen Richtungen und $g \in I_1$ mit $M \notin g$; ferner kann $A \notin g$ vorausgesetzt werden. Dann schneidet g die isotrope Gerade aus I_2, die durch A geht, in einem Punkt C $(\neq M, A)$, und MC schneidet die durch A gehende isotrope Gerade aus I_1 in einem Punkt $D \notin AC$ (Abb. 3.) Für die vierte Ecke P im Parallelogramm $CADP$ gilt $CP = g \in I_1$, $CA \in I_2$; folglich ist $MC = DC$ Mittellot von (P, A) und damit $P \in \mathcal{K}(M, A)$. ∎

Im endlichen Fall sei n die Ordnung der Ebene, d. h. die Anzahl der Punkte auf einer Geraden; dann ist $n = p^r$ $(r \geqq 1$, p prim) und $|\mathcal{P}| = n^2$, $|\mathcal{G}| = n^2+n$. Nach (2) gibt es $\frac{n}{2}(n-1)$ euklidische und $\frac{n}{2}(n+1)$ minkowskische Orthogonalitätsrelationen in einer endlichen papposschen Ebene

(Char $\neq 2$, also n ungerade). Ein anisotroper Kreis besteht in einer euklidischen Ebene aus $n+1$ und in einer minkowskischen Ebene aus $n-1$ Punkten, von denen keine drei kollinear sind. In euklidischen Ebenen ist also jeder Kreis ein *Oval*. Umgekehrt gibt es zu jedem Oval $\mathcal{K}$ eine (und nur eine) Orthogonalitätsrelation derart, daß $\mathcal{K}$ Kreis ist (siehe [8a], S. 77).

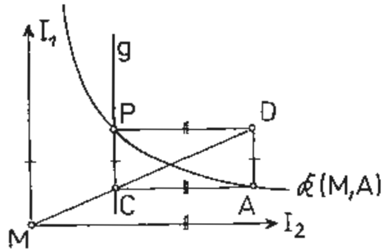

Abb. 3.

2. Spiegelung an einem Kreis

Im folgenden betrachten wir einen Kreis $\mathcal{K}$ mit dem Mittelpunkt M und erklären

$$I_M := \begin{cases} \{M\}, \text{ falls } (\mathcal{P}, \mathcal{G}) \text{ euklidisch} \\ \text{die Vereinigungsmenge der isotropen Geraden durch } M, \text{ falls } (\mathcal{P}, \mathcal{G}) \\ \text{minkowskisch.} \end{cases}$$

Konstruktiven Vorstellungen von einer Spiegelung σ an $\mathcal{K}$ entsprechend gehen wir von folgenden Eigenschaften aus

(S1) $P \in \mathcal{K} \to P^\sigma = P$,
(S2) $P \neq M \to P^\sigma \in MP \setminus \{M\}$,
(S3) M, P, Q *nicht kollinear* $\wedge MP \perp PQ \Rightarrow MQ^\sigma \perp Q^\sigma P^\sigma$.

Es gilt der Satz

(6) (a) *Ist $\mathcal{K}$ anisotrop, so gibt es genau eine Abbildung σ vom $\mathcal{P} \setminus I_M$ auf sich mit den Eigenschaften* (S1)–(S3). *Sie ist eine Bijektion.*
(b) *Ist $\mathcal{K}$ isotrop, so gibt es keine Bijektion von $\mathcal{P}$ auf sich mit den Eigenschaften* (S1)–(S3).

Die in (6a) vorliegenden Abbildungen heißen nun *Kreisspiegelungen*.

Wir skizzieren einen analytischen Beweis für (6a); dabei kann von $M = (0, 0)$ und $A = (1, 0)$ ausgegangen werden. Auf der Grundlage von (1) erhält man zunächst

$$\mathcal{K}(M, A) = \{(x, y) \colon \lambda x^2 + y^2 = \lambda\}.$$

Die Voraussetzung $P \in \mathcal{P} \setminus I_M$ ist äquivalent mit $\lambda x^2 + y^2 \neq 0$, wobei $P = (x, y)$. Nach einigen Rechnungen ergeben sich für das Bild $P^\sigma = (x', y')$ des Punktes P bei einer Abbildung σ von $\mathcal{P} \setminus I_M$ in sich, die die Bedingungen (S1)–(S3) erfüllt, die Beziehungen

(7) $$x' = \frac{x\lambda}{\lambda x^2 + y^2}, \quad y' = \frac{y\lambda}{\lambda x^2 + y^2}.$$

Damit ist die Existenz- und Eindeutigkeitsaussage sowie die behauptete Eineindeutigkeit in (6a) klar.

Diese Bijektion kann auf Punkte $Q(\neq M)$ aus I_M – falls sie existieren – nicht erweitert werden, wie folgende Betrachtung zeigt. Der Kreis mit dem Durchmesser (M, Q) ist isotrop und schneidet nach (5) den anisotropen Kreis $\mathcal{K}$ in einem Punkt $P \notin MQ$, für den $MP \perp PQ$ wegen (3) gilt. Nun wäre aber $MQ^\sigma \perp Q^\sigma P$ und damit $P \in MQ^\sigma = MQ$. Zum Beweis von (6b) sei $\mathcal{K}$ isotrop und $R \in \mathcal{P} \setminus \mathcal{K}$, und es gäbe eine Bijektion σ von $\mathcal{P}$ auf sich, für die (S1)–(S3) gilt. Die Orthogonale zu MR durch R schneidet $\mathcal{K}$ in Punkten A_1, $A_2 \neq M$; siehe Abb. 1. Dann ist nach (S1)–(S3) sofort $R^\sigma = M$ im Widerspruch zur Voraussetzung für σ. ∎

Die noch folgenden konstruktiven Darstellungen ergeben auch die Möglichkeit eines synthetisch-konstruktiven Beweises für (6a).

3. Einige Eigenschaften

Für eine Kreisspiegelung σ ist anhand von (7) einsichtig

(S1)′ $P^\sigma = P \rightarrow P \in \mathcal{K}$ (Ergänzung zu (S1))

(S4) *σ ist involutorisch* (d. h. $\sigma \neq 1$ und $\sigma\sigma = 1$).

Auch dafür wie für das Folgende bieten sich zum Teil später synthetische Beweise mit Hilfe der konstruktiven Darstellungen an.

(S5) *Ist g eine Gerade, so ist*

(a) $(g \setminus \{M\})^\sigma = g \setminus \{M\}$, *falls $g \ni M$ und g anisotrop*

(b) $(g \setminus I_M)^\sigma \cup \{M\}$ *ein (anisotroper) Kreis, falls $g \not\ni M$ und g anisotrop*

(c) $(g \setminus I_M)^\sigma = h \setminus I_M$, *falls $g \not\ni M$, g isotrop und h diejenige isotrope Gerade ist, die sich mit g auf $\mathcal{K}$ schneidet.*

Beweis. Die Eigenschaft (a) ist nach (S2) sofort klar. (b) folgt aus (S3) in bekannter Weise, und (c) bestätigt man mit (7) durch Rechnung. ∎

(S6) *Ist $P \notin I_M$ und $P^\sigma \neq P$, dann gilt*

$$(\mathcal{K}_1 \setminus I_M)^\sigma = \mathcal{K}_1 \setminus I_M \text{ für jeden Kreis } \mathcal{K}_1 \text{ mit } P, P^\sigma \in \mathcal{K}_1.$$

Zum Beweis nur einige Anmerkungen. Wegen $P \notin I_M$ ist $M \notin \mathcal{K}_1$. Ist der Kreis $\mathcal{K}_1$ isotrop (was nicht ausgeschlossen ist), dann ergibt sich (S6) sofort aus (S5, c). Für anisotrope Kreise $\mathcal{K}_1$ bestätigt man die Behauptung nach einigen Rechnungen.

Falls MM_1 den Kreis $\mathcal{K}_1$ schneidet – wobei M_1 der Mittelpunkt von $\mathcal{K}_1$ ist – genügt die Aussage

(S7) *Wenn $P, Q, R \notin I_M$, $Q \in MP$, $R \notin MP$ und $PR \perp QR$, so $P^\sigma R^\sigma \perp Q^\sigma R^\sigma$,*

die sich einfacher bestätigen läßt.

Aus dieser Eigenschaft folgt überdies in bekannter Weise der im Vergleich zu (S6) allgemeinere Satz, daß das *Bild eines Kreises* $\mathcal{K}_1$ (mit $M \notin \mathcal{K}_1$, der von MM_1 geschnitten wird), *wieder ein Kreis* ist, wenn von Punkten aus I_M abgesehen wird *(Kreisinvarianz)*.

Da in der vorliegenden metrischen Geometrie eine freie Beweglichkeit nicht bestehen muß, schneidet nicht unbedingt jede anisotrope Gerade g durch den Mittelpunkt M eines anisotropen Kreies den Kreis selbst. Die Relation $g \sim h$: $\Leftrightarrow$ es gibt Punkte M, $A \in h$ derart, daß die Parallele zu g durch M den Kreis $\mathcal{K}(M, A)$ schneidet bedeutet für anisotrope Geraden, daß g in h *beweglich* ist. Sie ist eine Äquivalenzrelation und stimmt mit der *Kommensurabilität* aus [7], [8a] überein. In endlichen Ebenen gibt es genau zwei Äquivalenzklassen.

Darauf aufbauend wird ein anisotroper Kreis $\mathcal{K}_1$ *ähnlich zu* einem anisotropen Kreis $\mathcal{K}_2$ genannt — $\mathcal{K}_1 \sim \mathcal{K}_2$ —, wenn es zu $A \in \mathcal{K}_1$ einen Punkt $B \in \mathcal{K}_2$ derart gibt, daß $M_1A \sim M_2B$ ist, wobei M_1, M_2 die Mittelpunkte dieser Kreise sind. (In der Tat ist dies eine Ähnlichkeit im klassischen Sinn, nämlich als Produkt eine Bewegung und einer Dehnung.) In endlichen Ebenen hat diese Ähnlichkeit $\sim$ zwei Äquivalenzklassen. Das bestätigt folgende einfache Überlegung. Ist $P \notin I_M$, so gibt es offenbar $\frac{n-1}{2}$ anisotrope Kreise um M, die MP schneiden; diese erfassen bei euklidischer Metrik

$$\frac{n-1}{2}(n+1) = \frac{n^2-1}{2} = \frac{1}{2}\,|\mathcal{P}\backslash I_M|$$

Punkte und bei minkowskischer Metrik

$$\frac{n-1}{2}(n-1) = \frac{n^2-2n+1}{2} = \frac{1}{2}\,|\mathcal{P}\backslash I_M|$$

Punkte. Folglich gibt es noch $\frac{n-1}{2}$ anisotrope Kreise um M, die MP nicht schneiden, aber aus Mächtigkeitsgründen selbst wieder in einer Klasse bezüglich $\sim$ liegen müssen.

Für endliche Ebenen ist nun einsichtig

(S8) *Sind $\mathcal{K}_1$ und $\mathcal{K}_2$ anisotrope Kreise und $(\mathcal{K}_1\backslash I_M)^\sigma = (\mathcal{K}_2\backslash I_M)$, so ist $\mathcal{K}_1 \sim \mathcal{K}_2$.*

Durch (S6) werden Kreise ausgezeichnet, die bei der Spiegelung an $\mathcal{K}$ in sich übergehen. Speziell kann man nach derartigen Kreisen fragen, die kozentrisch zu $\mathcal{K}$ sind. Für endliche Ebenen ist bemerkenswert

(S9) *Es gibt genau einen zu $\mathcal{K}$ kozentrischen Kreis $\mathcal{K}_1$ $(\neq \mathcal{K})$ mit $\mathcal{K}_1^\sigma = \mathcal{K}_1$. Überdies gibt es ohne Einschränkung auf Endlichkeit einen derartigen Kreis $\mathcal{K}_1$ mit $\mathcal{K}_1 \sim \mathcal{K}$ genau dann, wenn -1 quadratisch im Koordinatenkörper ist.*

Beweis. Für eine Kreis $\mathcal{K}_1$ mit dieser Eigenschaft muß der Mittelpunkt M von $\mathcal{K}$ auch Mittelpunkt von (P, P^σ) für alle Punkte $P \in \mathcal{K}_1$ sein. Aus (7) ergibt sich dann für $\mathcal{K}_1$ die Gleichung

$$\lambda x^2 + y^2 = -\lambda$$

d. h. die Einzigkeit besteht für jede metrische affine Ebene.

Die Existenz folgt für endliche Ebenen aus dem simplen Sachverhalt, daß bei der Spiegelung an $\mathcal{K}$ die Kreise um M paarweise einander zugeordnet werden; wegen $\mathcal{K}^\sigma = \mathcal{K}$ und $2 \nmid (n-1)-1$ muß es deshalb noch einen Kreis $\mathcal{K}_1$ mit $\mathcal{K}_1^\sigma = \mathcal{K}_1$ geben.

Die Existenz besteht auch in gewissen Ebenen, die nicht endlich sind. Die Existenz eines kozentrischen Kreises $\mathcal{K}_1 \neq \mathcal{K}$ mit $\mathcal{K}_1^\sigma = \mathcal{K}_1$ und $\mathcal{K}_1 \sim \mathcal{K}$ ist gleichwertig damit, daß es eine Dehnung

$$x \mapsto xu, \quad y \mapsto yu \quad (u \in K)$$

gibt, die $\mathcal{K}$ (mit der Gleichung $\lambda x^2 + y^2 = \lambda$) in $\mathcal{K}_1$ mit der Gleichung $\lambda x^2 + y^2 = -\lambda$ überführt, also äquivalent damit, daß es im Koordinatenkörper K ein u mit $u^2 = -1$ gibt. ∎

4. Konstruktive Darstellungen

4.1. Eine *klassischen elementare Konstruktion* des Bildes eines von M verschiedenen Punktes $P \in \mathcal{K}$ bei der Spiegelung σ an $\mathcal{K}(M, A)$ legt die Tangenten von P an den Kreis $\mathcal{K}$ und bestimmt P^σ als Mittelpunkt der Tangentenberührungspunkte oder bringt — falls dies nicht geht — die Orthogonale zu MP durch P zum Schnitt mit dem Kreis und bestimmt P^σ als Schnitt der Tangenten durch diese Kreispunkte (Abb. 4.). Diese Konstruktion ist hier auf Grund der Eigenschaften (S1)—(S3) gerechtfertigt. Es bleibt aber offen, ob sie stets ausführbar ist.

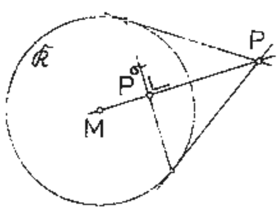

Abb. 4.

Wir betrachten dazu folgende Punkte:

P heißt äußerer *Punkt* von $\mathcal{K}$ genau dann, wenn P Schnittpunkt zweier Tangenten von $\mathcal{K}$ ist; $\mathcal{P}_{ex}$ bezeichne die Menge aller dieser Punkte. Die Punkte von $\mathcal{P} \setminus (\mathcal{P}_{ex} \cup \mathcal{K})$ heißen *innere Punkte* ($\mathcal{P}_{in}$). Von Włodarski [10] werden verschiedene Auffassungen des „Inneren von $\mathcal{K}$" und ihre Beziehungen zueinander studiert; $\mathcal{P}_{in} \setminus I_M$. entspricht $W_3(\mathcal{K})$ in [10]. Ferner heißt P *Sehnenmittelpunkt* von $\mathcal{K}$ genau dann, wenn er Mittelpunkt zweier verschiedener Punkte von $\mathcal{K}$ ist; die Menge dieser Punkte wird mit $\mathcal{P}_s$ bezeichnet.

Die obige Konstruktion und damit die Kreisspiegelung σ stiftet nun offenbar eine Bijektion von $\mathcal{P}_{ex}$ auf $\mathcal{P}_s\setminus\{M\}$; also ist $|\mathcal{P}_{ex}| = |\mathcal{P}_s\setminus\{M\}|$.

Für endliche Ebenen der Ordnung n ergeben sich folgende Mächtigkeiten

	$\lvert\mathcal{P}_{ex}\rvert$	$\lvert\mathcal{P}_s\rvert$	$\lvert\mathcal{P}_{in}\rvert$
euklidische Ebene	$\frac{(n-1)(n+1)}{2}$	$\frac{n^2+1}{2}$	$\frac{n^2-2n-1}{2}$
minkowskische Ebene	$\frac{(n-1)(n-3)}{2}$	$\frac{n^2-4n+5}{2}$	$\frac{n^2+2n-1}{2}$

Es ist nicht jeder Punkt von $\mathcal{P}\setminus(\mathcal{K}\cup I_M)$ ein Punkt aus $\mathcal{P}_{ex}$ oder aus $\mathcal{P}_s\setminus\{M\}$. Wir betrachten dazu als Beispiel in einer euklidischen Ebene eine Gerade g durch den Mittelpunkt $M = (0, 0)$ von $\mathcal{K}$, die $\mathcal{K}$ schneidet, ohne auf die für sich interessante Frage nach der Verteilung von $\mathcal{P}_{ex}$, $\mathcal{P}_s$ und $\mathcal{P}_{in}$ weiter einzugehen.

Für eine analytische Betrachtung kann ohne Beschränkung der Allgemeinheit angenommen werden, daß $(1, 0)$ auf g liegt. Anhand von Abb. 4. bestätigt man durch Rechnung, daß

$$g_{ex} := \left\{\left(\frac{\lambda+a^2}{\lambda-a^2}, 0\right) : a\in K,\ a\neq 0,\ a^2\neq\lambda\right\}$$

die Menge aller äußeren Punke und

$$g_s := \left\{\left(\frac{\lambda-a^2}{\lambda+a^2}, 0\right) : a\in K,\ a\neq 0\right\}$$

die Menge aller Sehnenmittelpunkte (von zu g orthogonalen Sehnen) auf g ist; dabei gilt $\lambda+a^2\neq 0$, da die Ebene euklidisch ist. Nun ist $g_{ex}\cap g_s\neq\emptyset$ gleichwertig damit, daß es von 0 verschiedene Elemente $a, b\in K$ mit $(\lambda-a^2)\cdot\cdot(\lambda-b^2) = (\lambda+b^2)\cdot(\lambda+a^2)$, d. h. mit $a^2 = -b^2$ gibt.

Demnach folgt aus $g_{ex}\cap g_s\neq\emptyset$, daß -1 quadratisch in K ist. Dann gibt es aber in K zu jedem $a\neq 0$ ein $b\neq 0$ mit $a^2 = -b^2$. Außerdem ist dann $a^2\neq\lambda$, da $a^2\neq-\lambda$ gilt. Demnach folgt aus $g_{ex}\cap g_s\neq\emptyset$ bereits $g_{ex} = g_s$. Die Menge g_s besteht also entweder nur aus äußeren oder inneren Punkten.

Die obige Konstruktion liefert deshalb nicht unmittelbar alle Bilder der Kreisspiegelung. Man kann sie jedoch leicht vervollständigen. Ist $P\notin\mathcal{P}_{ex}\cup P_s\cup I_M$, so schneidet das Lot h zu MP durch P (wenigstens) eine Tangente von $\mathcal{K}$; h enthält also wenigstens einen äußeren Punkt Q. Das Bild von Q bestimmt man nun nach der obigen Konstruktion, und P^σ ist schließlich nach (S3) der Schnitt des Lotes zu MQ^σ durch Q^σ mit der Geraden MP.

4.2. Eine weitere konstruktive Darstellung erhält man über *Steinersche Kegelschnitte*. Nach (3) und dem Höhenschließungssatz ist jeder anisotrope

Kreis $\mathcal{K}$ ein Schnitt zweier echter projektiver Geradenbüschel, also ein Steinerscher Kegelschnitt (in der projektiven Einbettung der Ebene). Er ist elliptisch oder hyperbolisch je nachdem, ob die Ebene euklidisch oder minkowskisch ist. Nach einem Satz der projektiven Geometrie (siehe u. a. LENZ [4], S. 62) gilt: *Zu jedem Punkt $P \notin \mathcal{K} \cup I_M$ gibt es genau eine Gerade p, so daß die involutorische (P, p)-Homologie $\mathcal{K}$ in sich abbildet. Überdies ist p nicht Tangente von $\mathcal{K}$, $M \notin p$ und – wie man* durch Orthogonalspiegelung an PM erkennt – $p \perp PM$. Die Gerade p läßt sich konstruktiv durch Sekanten von $\mathcal{K}$ durch den Punkt P bestimmen (vgl. Abb. 5.). Man bestätigt leicht, daß die Abbildung

$$\tau : \mathcal{P}\backslash I_M \to \mathcal{P}\backslash I_M \text{ vermöge}$$

$$P \mapsto \begin{cases} \text{Schnittpunkt von } p,\ PM;\ \text{falls } P \notin \mathcal{K} \\ P;\ \text{falls } P \in \mathcal{K} \end{cases}$$

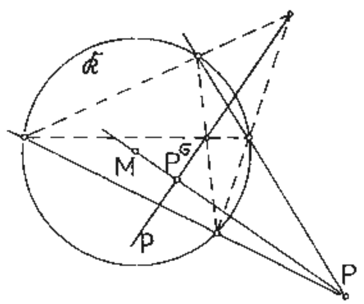

Abb. 5.

die Eigenschaften (S1)–(S3) erfüllt, und nach (6a) ist deshalb τ die Spiegelung an $\mathcal{K}$.

Die vorgestellte Konstruktion arbeitet uneingeschränkt für jeden Punkt $P \notin I_M$.

4.3. Durch die *Inversionen*, die MOLNÁR in [5] mit spiegelungsgeometrischen Methoden auf der Idealebene metrischer Ebenen (im Sinne von BACHMANN [1]) behandelt, bietet sich eine weitere konstruktive Darstellung der Kreisspiegelung an. In [5] sind jedoch die minkowskischen Ebenen ausgeschlossen. Anhand der Kreisdefinition kann jeder Kreis $\mathcal{K}(M, A)$ als Menge aller Bilder von A (als *Orbit* von A) bezüglich der Orthogonalspiegelungen an den anisotropen Geraden durch M aufgefaßt werden. Wir erhalten damit Anschluß an die spiegelunggeometrische Darstellung bei MOLNÁR. Entsprechend dieser wird unter einer *Inversion* α mit dem Zentrum M eine Abbildung von $\mathcal{P}\backslash I_M$ auf sich verstanden, bei der alle Punkte $P, Q, R \notin I_M$ mit $P \neq Q$, $Q \in MP$ gilt, daß $P^\alpha = Q$, $Q^\alpha = P$ und das Bild R^α mit dem Bild von R bei der Spiegelung an dem Lot von dem Mittelpunkt des Kreises durch P, Q, R auf die Gerade MR übereinstimmt (Abb. 6.). Entsprechend den Methoden in [5] kann auch für minkowskische Ebenen gezeigt werden, daß es zu je drei verschiedenen kollinearen Punkten M, P, Q genau eine Inversion α mit dem Zentrum M gibt, bei der $P^\alpha = Q$ und $Q^\alpha = P$

ist. Die in Abb. 6. vorgestellte Konstruktion ist nun für eine konstruktive Darstellung der Kreisspiegelung sofort nutzbar.

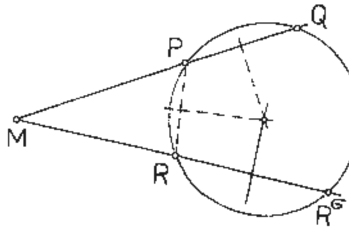

Abb. 6.

Denn, nach (S6) ist jede Kreisspiegelung eine Inversion. Umgekehrt erfüllt jede Inversion mit dem Zentrum M die Eigenschaften (S2) und (S3). Besitzt sie überdies noch einen Fixpunkt F, so ist auch das Bild von F bei der Orthogonalspiegelung an einer anisotropen Geraden durch M ein Fixpunkt von α, d. h. $\mathcal{K}(M, F)$ bleibt punktweise fest. Demnach ist *eine Inversion genau dann eine Kreisspiegelung, wenn sie einen Fixpunkt besitzt.*

Überdies gilt

(8) *In endlichen Ebenen ist jede Inversion eine Kreisspiegelung.*

Denn um einen Punkt M gibt es — wie wir bereits feststellten — $n-1$ Kreise; diese bilden einen Punkt $P \notin I_M$ auf $n-1$ verschiedene Punkte im $MP \setminus \{M\}$ ab. Wegen $|PM \setminus \{M\}| = n-1$ gibt es aber auch nur $n-1$ verschiedene Inversionen mit dem Zentrum M.

4.4. Es ist bekannt, daß in der minkowskischen Geometrie die isotropen Elemente Ausnahmen bei Aussagen bilden und damit eine glatte Formulierung von Resultaten stören. Bekannt sind aber auch eine Reihe von Beispielen, in denen die isotropen Elemente gerade zu einer vereinfachten Beschreibung von Sachverhalten beitragen können; so auch in spiegelungsgeometrischen Darstellungen. (Dort arbeitet man u. a. mit „unverbindbaren Punkten".)

Auch hier besteht eine derartige Möglichkeit. Der Satz (S5, c) bietet eine *vereinfachte konstruktive Darstellung der Kreisspiegelung in minkowskischen Ebenen.*

Man legt durch $P \notin I_M$ die beiden isotropen Geraden g_1 und g_2; diese schneiden nach (5) den Kreis $\mathcal{K}$ in Punkte P_1 und P_2 (Abb, 7.). Auf Grund der Eigenschaft (S5, c) ist das Bild g_1^{σ} die Parallele zu g_2 durch P_1 und das Bild g_2^{σ} die Parallele zu g_1 durch P_2. Demnach ist P^{σ} derjenige Punkt, für den $P_1 P P_2 P^{\sigma}$ Parallelogramm ist; überdies liegt er auf PM. Bei vorliegendem Kreis genügt somit eine einfache Parallelogrammkonstruktion.

Zum Abschluß einige

Anmerkungen

1. Ergänzt man eine euklidische Ebene durch einen Idealpunkt als gemeinsamen Punkt aller (anisotropen) Geraden und erklärt alle Kreise und (anisotropen) Geraden als *Zirkel*, so erfüllt diese Inzidenzstruktur der Punkte

und Zirkel die Axiome einer *inversen* oder *Möbius-Ebene*. (Vgl. DEMBOWSKI [2] KÁRTESZI [3]; die Bedingungen (1), (2) in [2], S. 252/253 schließen eine möglichst entsprechende Einbeziehung minkowskischer Ebenen aus.)

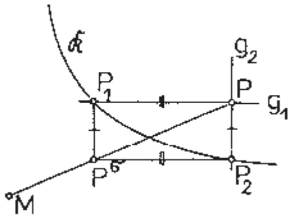

Abb. 7.

2. Es liegt nahe, Eigenschaften der Kreisspiegelungen bereitzustellen, die in Anlehung an die Arbeit [6] von MOLNÁR einen axiomatischen Aufbau der Kreisebene über einem beliebigen Körper oder auch speziell über einem endlichen Körper (Char $\neq 2$) auf der Grundlage von Spiegelungen ermöglichen. Bei Endlichkeit kann dabei die Eigenschaft (S9) möglicherweise recht nützlich sein. In [6] wird eine solche Darstellung für Kreisebenen über pythagoreischen Körpern gegeben.

3. Büschel, Orthogonalitätsrelationene und Kreise stehen in einem engen Zusammenhang (siehe u. a. [8a]; KLOTZEK/QUAISSER in *Math. Nachr.*, **58** (1973); *Klotzek* in *Wiss. Z. PH Potsdam*, **21** (1977)). Eine einheitliche Behandlung euklidischer und minkowskischer Ebenen hier in dieser Note läßt eine mögliche Einbeziehung minkowskischer Ebenen im Rahmen der Arbeit [5] von MOLNÁR erwarten.

Für freundliche Hinweise möchte ich Herrn EMIL MOLNÁR danken.

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ON EXPRESSING $\sqrt{p}$ AS A RATIONAL LINEAR COMBINATION OF COSINES OF ANGLES WHICH ARE RATIONAL MULTIPLES OF π

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§ 1. Introduction. Recently CONWAY and JONES [1] have given an effective method of solving equations of the type

$$c_1 \cos \pi d_1 + \ldots + c_r \cos \pi d_r = E \tag{1.1}$$

where all the variables involved are rational. They explicitly give all the solutions of (1.1) for $r = 4$ (theorem 7 of [1]) and it is fairly easy to extend the result to $r = 5, 6, \ldots$; the calculations, however, get more tedious as n gets large.

In books on elementary trigonometry one sees a large number of formulae in which the right hand side of (1.1) is a quadratic irrational. Indeed for any integer $n>0$, $\sqrt{n}$ can always be written as a rational linear combination of cosines of angles that are rational multiples of π (since $\sqrt{n} \in$ the maximal real subfield of $Q(e^{2\pi i/4n})$). Let $\lambda = \lambda(\sqrt{n})$ denote the least number of cosines needed to express $\sqrt{n}$ in the above way:

$$\sqrt{n} = c_1 \cos \pi d_1 + \ldots + c_\lambda \cos \pi d_\lambda \tag{1.2}$$

(c_j, d_j rational). The object of this paper is to determine λ in the case when n is a prime number p and to give all solutions of (1.2) for $n = p$. In **§ 2** we shall consider rational linear combinations of

(i): $\sin \pi/p, \sin 2\pi/p, \ldots, \sin (p-1)\pi/p$ and
(ii): $\cos \pi/p, \cos 2\pi/p, \ldots, \cos (p-1)\pi/p$,

which lie in a quadratic extension of Q. In **§ 3** we solve the equation

$$c_1 \cos \pi d_1 + \ldots + c_r \cos \pi d_r = c + d\sqrt{p} \tag{1.3}$$

(c_j, d_j, c, d rational, $d \neq 0$ and r minimal) and show that the results of **§ 2** already give all the solutions of (1.3). In **§ 4** we determine $\lambda(\sqrt{p})$ and solve

(1.2) completely for $n = p$. The relation between $\lambda(\sqrt{p})$ and the minimal r of (1.3) will appear in the text. Finally in **§ 5** we shall completely solve the equation

$$c_1 \cos \pi d_1 + \ldots + c_l \cos \pi d_l = c + d\sqrt{p} \quad (c_j, d_j, c, d \in Q, \ d \neq 0)$$

for $l = 1, 2, 3$ and make some remarks.

In this paper we have exclusively looked at the case $n = p$ (a prime). The more difficult problem of expressing $\sqrt{n}$ (n composite) as a rational linear combination of cosines in a minimal way (i.e. involving the least number of cosines) will be looked at in a later paper.

It is easy to see **(§ 4)** the impossibility of expressing $\sqrt[m]{n}$ ($m > 2$, n positive integer) as a rational linear combination of cosines, thereby making the results of **§ 3** and **§ 4** more satisfying.

§ 2. Let p be an odd prime. In this paragraph we shall be considering rational linear combinations of

(i) $\sin \pi/p, \sin 2\pi/p, \ldots, \sin (p-1)\pi/p$ and
(ii) $\cos \pi/p, \cos 2\pi/p, \ldots, \cos (p-1)\pi/p$,

which are either rational or lie in some quadratic extension of the rationals. Since $\sin j\pi/p = \sin (p-j)\pi/p$ and $\cos j\pi/p = -\cos (p-j)\pi/p$, we may actually only consider rational linear combinations of

(i)′ $\sin \pi/p, \sin 2\pi/p, \ldots, \sin \frac{1}{2}(p-1)\pi/p$ and

(ii)′ $\cos \pi/p, \cos 2\pi/p, \ldots, \cos \frac{1}{2}(p-1)\pi/p$.

Thus we want to completely solve the two equations

$$\sum_{j=1}^{(p-1)/2} a_j \sin j\pi/p = c + d\sqrt{n} \tag{2.1}$$

$$\sum_{j=1}^{(p-1)/2} a_j \cos j\pi/p = c + d\sqrt{n} \tag{2.2}$$

where $a_j, c, d \in Q$ and n is a square-free positive integer. We have

THEOREM 1. (i) *If $d \neq 0$, neither* (2.1) *nor* (2.2) *can have a solution unless $n = p$. For $n = p$ then, we have*

(ii) *If $p \equiv 1$ (mod 4) then* (2.1) *still has no solution for any p except the trivial one with $c = d = a_j = 0$ (for all j) while* (2.2) *is solvable uniquely for every $c, d \in Q$, the solution being*

$$a_j = (-1)^j \{-2c + 2d\,(2j/p)\},$$

(iii) *If $p \equiv 3$ (mod 4) then* (2.1) *has a solution if and only if $c = 0$ and then the solution is unique for every $d \in Q$ and is given by $a_j = (-1)^j 2d\,(2j/p)$.*

while (2.2) *is solvable if and only if* $d = 0$ *and then uniquely for every* $c \in Q$ *the solution being* $a_j = (-1)^{j-1} 2c$.

We first prove the following

LEMMA 1. *Let* R *denote quadratic residues mod* p *and* N *the non-residues, then* $a_1 \zeta + a_2 \zeta^2 + \ldots + a_{p-1} \zeta^{p-1} = c + d\sqrt{(-1)^{(p-1)/2} p}$ $(a_j \in \mathbf{Q})$ *if and only if* $a_N = -c-d$, $a_R = d-c$, *where* $\zeta = e^{2\pi i/p}$.

PROOF. First if $a_N = -c-d$, $a_R = d-c$ then $a_1 \zeta + \ldots + a_{p-1} \zeta^{q-1} = (d-c) \sum \zeta^R - (c+d) \sum \zeta^N = (d-c) \sum \zeta^R - (c+d)(-1 - \sum \zeta^R)$ (since $1 + \sum \zeta^R + \sum \zeta^N = 0$) $= c + d(1 + 2\sum \zeta^R) = c + d\sqrt{(-1)^{(p-1)/2} p}$ by the Gauss sum as required. Conversely the linear independence of $\zeta, \zeta^2, \ldots, \zeta^{p-1}$ proves the result.

PROOF OF THEOREM 1. (i) Let $\zeta = e^{2\pi i/p}$. Then

$$\sin j\pi/p = \begin{cases} (\zeta^{j/2} - \zeta^{-j/2})/2i & \text{if } j \text{ is even} \\ -(\zeta^{(j+p)/2} - \zeta^{-(j+p)/2})/2i & \text{if } j \text{ is odd} \end{cases} \tag{2.3}$$

and

$$\cos j\pi/p = \begin{cases} (\zeta^{j/2} + \zeta^{-j/2})/2 & \text{if } j \text{ is even} \\ -(\zeta^{(j+p)/2} + \zeta^{-(j+p)/2})/2 & \text{if } j \text{ is odd.} \end{cases} \tag{2.3$'$}$$

It follows that the left hand side of (2.1) $\in Q(\zeta, i) \cap \mathbf{R}$. But the only quadratic fields contained in $Q(\zeta, i)$ are $Q(\sqrt{\pm p})$ and $Q(i)$ (indeed that contained in $Q(\zeta)$ is $Q(\sqrt{(-1)^{(p-1)/2} p})$), so in the right hand side of (2.1), $n = p$ as required. Similarly for (2.2). This proves (i).

(ii) Here i times the left hand side of (2.1) $\in Q(\zeta)$ (by (2.3)) and the right hand side $\in Q(\zeta)$ too (using the Gauss sum for $\sqrt{p}$). Since $i \notin Q(\zeta)$ so the left and right sides of (2.1) must both be 0. Hence $c = d = 0$ and $a_j = 0$ for all j by the linear independence of $\zeta, \zeta^2, \ldots, \zeta^{p-1}$.

As for (2.2) write the cosines in terms of powers of ζ using (2.3)$'$ and applying lemma 1, the result follows.

(iii) Here i times the left hand side of (2.1) $\in Q(\zeta)$ by (2.3). Therefore $i(c + d\sqrt{p}) \in Q(\zeta)$, i.e. $ic + d\sqrt{-p} \in Q(\zeta)$ i.e. $ic \in Q(\zeta)$. Since $i \notin Q(\zeta)$ it follows that $c = 0$. To get the solution explicitly use (2.3) and lemma 1.

Finally for (2.2) the left hand side $\in Q(\zeta)$, therefore $c + d\sqrt{p} \in Q(\zeta)$ i.e. $d\sqrt{p} \in Q(\zeta)$. Since $\sqrt{p} \notin Q(\zeta)$ so $d = 0$. To get the solution explicitly use (2.3)$'$ and lemma 1. This completes the proof of theorem 1.

§ 3. Throughout this paragraph we shall be using the notation and results of [1]. Our object is to completely solve the equation

$$A_1 \cos \pi a_1/b_1 + \ldots + A_l \cos \pi a_l/b_l = c + d\sqrt{p} \tag{3.1}$$

a_j, b_j positive integers, $(a_j, b_j) = 1$, A_j, c, d rational, $d \neq 0$ and l minimal in the sense that no subsum is of the type $x + y\sqrt{p}$ (x, y rational, $y \neq 0$). We have the following

LEMMA 2. *No b_j can have an odd square factor.*

PROOF. By using the expression for $\sqrt{(-1)^{(p-1)/2}p}$ as the Gauss sum we can write the right hand side of (3.1) as a linear combination of roots of unity which are $\sim$ (equivalent to) 1 or i (in the notation of [1], we say 2 roots of unity η and ζ are equivalnet if $o(\eta/\zeta)$ is square-free). Now it is easy to see that $\zeta \sim 1$ or i if and only if $\zeta^{-1} \sim 1$ or i.

Let $\eta_j = e^{\pi i/b_j}$ so that $\cos \pi a_j/b_j = (\eta_j^{a_j} + \eta_j^{-a_j})/2$. Suppose for some j (say $j=1$) $q^2|b_1$ (q an odd prime). By theorem 1 of [1] the sum of the terms in the left hand side involving roots of unity equivalent to 1 or i must be equal to the right hand side. The above considerations then give us a subsum of $\sum_{k=2}^{l} A_k \cos \pi a_k/b_k$ which equals $c + d\sqrt{p}$ and this contradicts the minimality of l. This proves lemma 2.

Now write $\eta_j^{a_j} = \zeta^{e_j}\zeta_j$ ($\zeta = e^{2\pi i/p}$) where $(p, o(\zeta_j)) = 1$. This is possible (for either $b_j = pb'_j$ $(p \nmid b'_j)$ or $(p, b_j) = 1$. In the latter case $e_j = 0$ will do. In the former case write $1 = xp + yb'_j$ with y even. Then

$$\eta_j^{a_j} = (e^{\pi i a_j/pb'_j})^{xp+yb'_j} = e^{\pi i a_j x/b'_j} e^{\pi i a_j y/p} = \zeta_j \zeta^{a_j \cdot y/2} = \zeta^{e_j}\zeta_j$$

as required).

Now insert this in the relation (3.3) or (3.3)' as the case may be and we get: for $p \equiv 3 \pmod 4$

$$\text{(3.4)} \qquad \sum_{j=1}^{l} (A_j/2)\zeta_j\zeta^{e_j} + \sum_{j=1}^{l} (A_j/2)\bar{\zeta}_j\zeta^{-e_j} + \sum_{j=1}^{p-1} (j/p)\, d\, i\, \zeta^j - c = 0$$

and for $p \equiv 1 \pmod 4$

$$\text{(3.4)}' \qquad \sum_{j=1}^{l} (A_j/2)\zeta_j\zeta^{e_j} + \sum_{j=1}^{l} (A_j/2)\bar{\zeta}_j\zeta^{-e_j} - \sum_{j=1}^{p-1} (j/p)\, d\, \zeta^j - c = 0\,.$$

We then have the following

THEOREM 2. (i) *Let $p \equiv 3$ (mod 4). Suppose* (3.1) *has a solution with $d \neq 0$ and suppose no subsum is of the type $c_1 + d_1\sqrt{p}$. Then*

$$p \leq 2l + 1\,.$$

(ii) *Let $p \equiv 1$ (mod 4). Hypothesis as in* (i), *then*

$$p \leq 4l + 1\,.$$

PROOF. (i) Suppose $p > 2l+1$. If a solution of (3.1) exists we can get a relation like (3.4). Write it as

$$\text{(3.5)} \qquad S_0 + S_1\zeta + \ldots + S_{p-1}\zeta^{p-1}\,.$$

Where each S_j is a rational linear combination of roots of unity having order prime to p. Since $p > 2l+1$ there exists an r (some residue mod p) such that $r \neq 0, \pm e_1, \pm e_2, \ldots, \pm e_l$ and then $-r$ (i.e. $p-r$) $\neq 0, \pm e_1, \pm e_2, \ldots, \pm e_l$

either. For such an r we have $S_r = (r/p)\, id$ (compare (3.4) with (3.5)) while $S_{p-r} = \left(\frac{p-r}{p}\right) id = -(r/p)\, id$ since $p \equiv 3 \pmod 4$. But by lemma 1 of [1], page 233, all the S_j are equal. Hence $d = 0$, a contradiction. Thus for a solution to exist, $p \leq 2l+1$.

(ii) Suppose $p > 4l+1$. Existence of a solution implies the relation (3.4)′ which we write as (3.5). Since $p > 4l+1$ there exists an r (a quadratic residue mod p) and an n (a quadratic non-residue mod p) both avoiding the set $\{0, \pm e_1, \ldots, \pm e_l\}$. For such an r and n, we get, on comparing (3.4)′ and (3.5)

$$S_r = -(r/p)\, d = -d\,,$$

$$S_n = -(n/p)\, d = d\,.$$

Again by Lemma 1 of [1] all the S_j are equal and so $d = 0$, a contradiction. This completes the proof of the theorem.

Our main result is the following

THEOREM 3. (i) *Let $p \equiv 3$ (mod 4) and let $p = 2l+1$. Then the equation* (3.1) *has a unique solution (up to a constant factor) viz that given by theorem* 1 *with the sines converted into cosines.*

(ii) *Let $p \equiv 1$ (mod 4) and let $p = 4l+1$. Then the equation* (3.1) *has exactly* 2 *solutions (up to a constant factor) viz the ones given by theorem* 1 *with $c = d$ and $c = -d$ in the cosine solution.*

PROOF. (i) The sine solution of theorem 1 converted into cosines is certainly one solution of (3.1). We must show that any other solution is the same as this solution. Existence of a solution gives the equation (3.4) written as (3.5). Then as sets

$$\{0, \pm e_1, \pm e_2, \ldots, \pm e_l\} = \{0, 1, \ldots, p-1\}\,,$$

for if the left hand side is strictly contained in the right hand side then the argument of the last result would imply $di = -di$ i.e. $d = 0$, a contradiction. Thus by renaming the A_j we may assume without loss of generality that $e_j = j$, $-e_j = -j$, so that $\cos \pi a_j/b_j = (\zeta^j \zeta_j + \bar{\zeta}^j \bar{\zeta}_j)/2$. Further by Lemma 1 of [1] we have $S_0 = S_1 = \ldots = S_{p-1}$. Thus comparing (3.4) with (3.5) we get

$$-c = S_0 = S_j = (j/p)\, id + \frac{1}{2} A_j \zeta_j \quad (1 \leq j \leq (p-1)/2)\,.$$

This gives $\zeta_j \in Q(i)$, but $\zeta_j \notin Q$ since $d \neq 0$, hence, being a root of unity, $\zeta_j = \pm i$. It follows that $c = 0$ and $\zeta_j = -2id(j/p)/A_j$. Consequently $A_j \cos \pi a_j/b_j = 2d(jp) \sin 2\pi j/p$. Hence (3.1) becomes

$$\sum_{j=1}^{(p-1)/2} 2d\,(j/p) \sin 2\pi j/p = d\sqrt{p}, \quad \text{i.e.} \quad \sum_{j=1}^{(p-1)/2} (j/p) \sin 2\pi j/p = \frac{1}{2}\sqrt{p}\,,$$

which is just the sine equation of theorem 1 for the case $p \equiv 3$ (mod 4).

(ii) In this case for $c = d$ and $c = -d$, in the solution of the cosine equation of theorem 1, there are exactly $(p-1)/4$ non-zero terms on the left hand side of (2.2) because of the following simple (proof omitted)

LEMMA 3. *If $p \equiv 1 \pmod 4$ then exactly one half of the numbers $2j$ ($j = 1, 2, \ldots, (p-1)/2$) are quadratic residues and the remaining half are quadratic non-residues mod p.*

This gives us the two solutions of (3.1) stated in theorem 3. We must show that any other solution coincides with one of these two. The existence of a solution gives the equation (3.4)$'$ written as (3.5). Then the set $A = \{0, \pm e_1, \pm e_2, \ldots, \pm e_l\}$ equals

either $\{0$, all the quadratic residues R mod $p\}$,

or $\{0$, all the quadratic non-residues N mod $p\}$.

For otherwise there exists an R_0 and an N_0 both avoiding the set A. But then $S_{R_0} = S_{N_0}$, i.e. $-(R_0/p)\,d = -(N_0/p)\,d$, i.e. $-d = d$ giving $d = 0$, a contradiction.

Suppose first that $A = \{0$, all $R\}$, say $e_j = R_j$, $-e_j = -R_j$ ($j = 1, 2, \ldots, (p-1)/4$). Then (3.4)$'$ becomes

$$\sum_{j=1}^{(p-1)/4} \frac{1}{2} A_j \zeta_j \zeta^{R_j} + \sum_{j=1}^{(p-1)/4} \frac{1}{2} A_j \bar{\zeta}_j \zeta^{-R_j} - \sum_{j=1}^{p-1} (j/p)\, d\, \zeta^j - c = 0\,.$$

Comparison with (3.5) gives the following equations:

$$S_0 = -c,\ S_{R_j} = \frac{1}{2} A_j \zeta_j - d,\ S_{p-R_j} = \frac{1}{2} A_j \bar{\zeta}_j - d,\ S_N = -(N/p)\,d = d$$

with obvious notation. Then by lemma 1 of [1] all these S_j are equal to each other. This gives $\zeta_j = \bar{\zeta}_j = 4d/A_j$ and $d = -c$. Then (3.1) becomes

$$\sum_{j=1}^{(p-1)/4} 4d \cos 2\pi R_j/p = d(-1+\sqrt{p}) \quad \text{i.e.}$$

$$\text{(3.6)} \qquad \sum_{j=1}^{(p-1)/4} \cos 2\pi R_j/p = (-1+\sqrt{p})/4\,.$$

This we claim is the same as the solution of (2.2) given in theorem 1 with $c = -d$. First of all we shall make the angles tally. Then the uniqueness of the solution in theorem 1 will imply the result.

The solution of (2.2) for $c = -d$ according to theorem I is

$$\text{(3.7)} \qquad \sum_{j=1}^{(p-1)/2} (-1)^j 2\{1+(2j/p)\} \cos \pi j/p = -1+\sqrt{p}\,.$$

In one half of these terms j is even $= 2j_1$ say. The corresponding term equals $2\{1+(j_1/p)\} \cos(\pi\, 2j_1/p)$, $(1 \le j_1 \le (p-1)/4)$. In the remaining half j is odd and so $p-j$ is even $= 2j_2$ say. The corresponding term equals

$$-2\{1+((2p-4j_2)/p)\}\cos(\pi(p-2j_2)/p)),\ \ ((p+3)/4\leq j_2\leq(p-1)/2),$$
$$=2\{1+(j_2/p)\}\cos\pi\, 2j_2/p$$

where

$$(p-1)/4+1\leq j_2\leq(p-1)/2.$$

Thus the left hand side of (3.7) is

$$\sum_{j=1}^{(p-1)/2} 2\{1+(j/p)\}\cos 2\pi\, j/p = \quad 1+\sqrt{p} \quad \text{i.e.} \quad \sum\cos\ 2\pi\, j/p=(-1+\sqrt{p})/4,$$

where the sum is taken over $j=$ the quadratic residues up to $(p-1)/2$. By lemma 3 this last sum is just (3.6).

The second possibility is $A=\{0$, all N (the non-residues)$\}$. Then by a reasoning exactly as above we get

$$\sum_{j=1}^{(p-1)/4}\cos 2\pi\, N_j/p = -(1+\sqrt{p})/4,$$

which as before is precisely the solution of (2.2) given in theorem 1 with $c=d$. This completes the proof of theorem 3.

Just for completeness we say a word about $p=2$. In this case (3.1) becomes $A_1\cos\pi\, a_1/b_1+\ldots+A_t\cos\pi\, a_t/b_t=c+d\sqrt{2}$. But $-2d\cos\pi/4= =-d\sqrt{2}$. Adding we get $A_1\cos\pi\, a_1/b_1+\ldots+A_t\cos\pi\, a_t/b_t-2d\cos\pi/4=c$ and this is taken care of by the results of [1].

§ 4. We first prove the following

THEOREM 4.

$$\lambda(\sqrt{p})=\begin{cases}(p-1)/2 & \textit{if}\ \ p\equiv 3\ (\textit{mod}\ 4),\\ (p+3)/4 & \textit{if}\ \ p\equiv 1\ (\textit{mod}\ 4).\end{cases}$$

PROOF. From the results of **§ 3** we see that for $p\equiv 3$ (mod 4), if $p>2l+1$ then (3.1) has no solution while if $p=2l+1$ then there is a unique solution in which necessarily $c=0$. Thus $\sqrt{p}$ may be expressed as a rational linear combination of $(p-1)/2$ (and no fewer) cosines, so $\lambda(\sqrt{p})=(p-1)/2$ in this case. Similarly for $p\equiv 1$ (mod 4) we can write $(-1+\sqrt{p})/4$ as a rational linear combination of $(p-1)/4$ cosines. Hence $\lambda(\sqrt{p})\leq 1+(p-1)/4$. But by theorem 3, $\lambda(\sqrt{p})>(p-1)/4$, hence $\lambda(\sqrt{p})=(p+3)/4$ in this case.

We shall now determine all possible solutions of (1.2) for $n=p$.

THEOREM 5. (i) *For* $p\equiv 3$ (*mod* 4), (1.2) *has a unique solution given by*

$$\sqrt{p}=2\sum_{j=1}^{(p-1)/2}(-1)^j\left(\frac{2j}{p}\right)\cos\pi\, j/p,$$

(ii) For $p\equiv 1$ (mod 4), (1.2) *has exactly two solutions given by*

$$\sqrt{p}=\quad 2\cos\pi/3+4\sum_{1\leq j\leq\frac{p-1}{2},\ (2j/p)=1}(-1)^j\cos\pi\, j/p,$$

and

$$\sqrt{p} = -2\cos\pi/3 - 4 \sum_{1 \le j \le \frac{p-1}{2},\ (2j/p)=-1} (-1)^j \cos\pi j/p\,,$$

unless $p = 5$, *in which case* (1.2) *has one further solution viz*

$$\sqrt{5} = 2\cos\pi/5 + 2\cos 2\pi/5\,.$$

PROOF. (i) has already been proved in theorem 3, part (i). As for (ii) let $\sqrt{p} = \sum_{j=1}^{(p+3)/4} c_j \cos\pi\, a_j/b_j$. Here we may suppose that none of the $\cos\pi\, a_j/b_j$ is rational for otherwise the result has already been proved in theorem 3, part (ii). Let $\eta_j = e^{\pi i/b_j}$ and $\zeta = e^{2\pi i/p}$, then

$$\sum_{j=1}^{(p+3)/4} c_j\,(\eta_j^{a_j} + \eta_j^{-a_j})/2 = \sqrt{p} = \sum_{x(p)} \zeta^{x^2} \tag{4.1}$$

by the Gauss sum.

Notice that $\zeta^{x^2} \sim 1$ for all x and $\eta_j^{a_j} \sim 1$ if and only if $\eta_j^{-a_j} \sim 1$. Splitting the above relation into equivalence classes and applying theorem 1 of [1], we get

$$\sum_{\eta_j^{a_j} \sim 1,\ j=1}^{(p+3)/4} c_j\,(\eta_j^{a_j} + \eta_j^{-a_j})/2 = \sqrt{p}\,.$$

Since $\lambda(\sqrt{p}) = (p+3)/4$ so $\eta_j^{a_j} \sim 1$ for all j i.e. $o(\eta_j^{a_j})$ is square-free, so we can write $\eta_j^{a_j} = \zeta^{e_j} \cdot \zeta_j$ where $0 \le e_j \le p-1$ and ζ_j is a root of unity whose order is prime to p. Now write (4.1) in the form

$$\sqrt{p} = S_0 + S_1\,\zeta + \ldots + S_{p-1}\,\zeta^{p-1} \tag{4.2}$$

where each S_j is a linear combination of roots of unity whose order is prime to p. Further $S_{p-j} = \bar{S}_j$ and $\bar{S}_0 = S_0$ even as expressions (and so certainly in value as complex numbers). Also (denoting the number of terms in S_j by $|S_j|$)

$$\sum_{j=0}^{p-1} |S_j| = (p+3)/2 \quad ((\text{by } (4.1) \text{ and } (4.2))\,. \tag{4.3}$$

In (4.2) write $\sqrt{p} = 1 + 2\sum_{(R/p)=1} \zeta^R$ and apply lemma 1 of [1]; we get $S_0 - 1 = S_R - 2 = S_N$ (where $(N/p) = -1$). If the value of $S_N \ne 0, -1, -2$ then $|S_j| \ge 1$ for all j and by (2.3) $(p+3)/2 \ge p$ i.e. $p \le 3$ which is not possible. So let $S_N = 0, -1$, or -2 i.e. $S_0 = 1, 0, -1$. First let $S_0 = 1$ or -1. Then $S_0 = a$ proper subsum of (1.2) that is rational. In view of theorem 2, part (ii), $S_0 = a\cos\pi/3$. Hence by theorem 3, part (ii), there are only 2 such solutions of (1.2) as required. Finally let $S_0 = 0$. Then by the minimality of (1.2), S_0 is the empty expression. Hence value of $S_N = -1$ and value of

$S_R = 1$ and so $\dfrac{p+3}{2} = \sum_{j=0}^{p-1} |S_j| \geq p-1$, i.e. $p \leq 5$. For $p = 5$ we must have $|S_j| = 1$ for $j = 1, 2, \ldots, p-1$ and then $S_N = -1$, $S_R = 1$ even as expressions. Thus (1.2) is $\sqrt{5} = \zeta - \zeta^2 - \zeta^3 - \zeta^4$, $\zeta = e^{2\pi i/5}$ and so (1.2) is $\sqrt{5} = 2\cos\pi/5 + 2\cos 2\pi/5$. This completes the proof.

§ 5. Examples and some remarks

(a). Here we shall solve (3.1) for small values of l. First let $l = 1$. Our equation is then $A\cos\pi\, a/b = c + d\sqrt{p}$. By our results this has no solution if

$$p > \begin{cases} 2l+1\ (p \equiv 3 \pmod 4)\,, \\ 4l+1\ (p \equiv 1 \pmod 4)\,. \end{cases}$$

Thus solution is possible only if $p = 2, 3, 5$. Indeed in this easy case we may just as easily look at the equation $A\cos 2\pi\, a/b = c' + d'\sqrt{n}$ (n composite allowed) i.e. on dividing by A the equation $\cos 2\pi\, a/b = c + d\sqrt{n}$ [c, $d \in Q$, n, a, $b \in \mathbf{Z}$, $n > 1$, $(a, b) = 1$]. Since $\cos 2\pi/b = (e^{2\pi i/b} + e^{-2\pi i/b})/2$ so $\cos 2\pi/b$ and so also $\cos 2\pi\, a/b$ belongs to the maximal real subfield of $Q(e^{2\pi i}/b)$. Hence $\cos 2\pi\, a/b$ is of degree $\varphi(b)/2$ over Q. But $\cos 2\pi\, a/b = c + d\sqrt{n}$ which is of degree 2 over Q. It follows that $\varphi(b)/2 = 2$ giving $b = 5, 8, 10, 12$. The respective solutions with angles lying between 0 and $\pi/2$ are $\cos\pi/5 = (1+\sqrt{5})/4$, $\cos 2\pi/5 = (-1+\sqrt{5})/4$, $\cos\pi/4 = \sqrt{2}/2$ and $\cos\pi/6 = \sqrt{3}/2$ and no others.

Next let $l = 2$. Our equation now is $A\cos\pi\, a_1/b_1 + B\cos\pi\, a_2/b_2 = c + d\sqrt{p}$. Solution is possible only if $p = 2, 3, 5$. On dividing by d we get $A\cos\pi\, a_1/b_1 + B\cos\pi\, a_2/b_2 = c + \sqrt{p}$. First let $p = 2$. Subtract from this the equation $2\cos\pi/4 = \sqrt{2}$ and we get $A\cos\pi\, a_1/b_1 + B\cos\pi\, a_2/b_2 - 2\cos\pi/4 = c$. This equation has been solved in [1] (theorem 7). There is only one solution viz $\pi/3 - \Theta = \pi/4$ i.e. $\Theta = \pi/12$ and then $\pi/3 + \Theta = 5\pi/12$. Hence $\cos\pi/12 - \cos 5\pi/12 - \cos\pi/4 = 0$ i.e. $\cos\pi/12 - \cos 5\pi/12 = \sqrt{2}/2$. Similarly for $p = 3$ use $2\cos\pi/6 = \sqrt{3}$ and for $p = 5$ use $\cos\pi/5 = (1+\sqrt{5})/4$, $\cos 2\pi/5 = (-1+\sqrt{5})/4$. For $p = 3$ we get no solution; for $p = 5$ we get the following two solutions: $\cos\pi/15 - \cos 4\pi/15 = (-1+\sqrt{5})/4$ and $\cos 2\pi/15 - \cos 7\pi/15 = (1+\sqrt{5})/4$. We proceed likewise for $l = 3$. Our results are collected in

THEOREM 4. *All the solutions of* (3.1) *for* $l = 1, 2, 3$ *are the following*

$l = 1$: $4\cos\pi/5 = 1+\sqrt{5}$, $4\cos 2\pi/5 = -1+\sqrt{5}$, $2\cos\pi/4 = \sqrt{2}$, $2\cos\pi/6 = \sqrt{3}$,

$l = 2$: $2\cos\pi/12 - 2\cos 5\pi/12 = \sqrt{2}$,
$4\cos\pi/15 - 4\cos 4\pi/15 = -1+\sqrt{5}$, $4\cos 2\pi/15 - 4\cos 7\pi/15 = 1+\sqrt{5}$,

$$l=3:\ 2\cos\pi/14+2\cos 3\pi/14-2\cos 5\pi/14=\sqrt{7},$$
$$4\cos\pi/13+4\cos 3\pi/13-4\cos 4\pi/13=1+\sqrt{13},$$
$$4\cos 2\pi/13-4\cos 5\pi/13+4\cos 6\pi/13=-1+\sqrt{13}.$$

(b). We remark here that no other pure surd $\sqrt[n]{a}$ ($n>2$, $a>1$) can be written as a rational linear combination of cosines (we assume of course that a is not a p^{th} power if $p|n$). Write $\alpha=\sqrt[n]{a}$. We split 2 cases.

Case 1. n is not a power of 2. Then there exists an odd prime p with $p|n$. Let $\beta=\alpha^{n/p}=\sqrt[p]{a}$. Then the Galois group of the splitting field of X^p-a (i.e. of $Q(\beta, e^{2\pi i/p})$ over Q) is isomorphic to the group of transformations $z\to cz+d$ (c, d integers mod p, $c\not\equiv 0$) and this group is non-Abelian. Thus β does not lie in any Abelian extension of Q and so not in any cyclotomic field; thus nor does α. It follows that α can not be written in terms of cosines.

Case 2. n is a power of 2. Since $n>2$ so $4|n$. Here let $\beta=\alpha^{n/4}=\sqrt[4]{a}$. Now X^4-a is irreducible over $Q(i)$. Let σ be the automorphism of $Q(\beta, i)$ given by $\sigma(\beta)=\beta i$, $\sigma(i)=i$ and τ by $\tau(\beta)=\beta$, $\tau(i)=-i$. Then $\sigma\tau\neq\tau\sigma$ and so $\beta\notin$ any cyclotomic field, so nor does α.

References

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ON EXPRESSING A QUADRATIC IRRATIONAL AS A RATIONAL LINEAR COMBINATION OF ROOTS OF UNITY

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§ 1. Introduction and some lemmas

For any integer n (since the field $Q(\sqrt{n})\subset$ some cyclotomic field) $\sqrt{n}$ can be expressed as a rational linear combination of roots of unity:

$$\text{(I)}\qquad \sqrt{n}=c_1\zeta_1+\ldots+c_k\zeta_k .$$

We define $\mu(\sqrt{n})$ to be the least value of k for which (I) holds. Clearly $\mu(\sqrt{-n})=\mu(\sqrt{n})$ and so we may always take $n>0$. In [2] we defined $\lambda(\sqrt{n})$ to be the least k for which the equation

$$\text{(II)}\qquad \sqrt{n}=c_1\cos a_1\pi+\ldots+c_k\cos a_k\pi$$

has a solution with c_j, $a_j\in Q$. The object of this paper is to get bounds for $\mu(\sqrt{n})$ and to determine the exact value of $\mu(\sqrt{n})$ for $n=p$ and pq (p, q distinct primes). We also relate the quantities λ and μ and determine the exact value of $\lambda(\sqrt{n})$ for $n=2p$, $3p$ (p prime >3) and pq (p, q distinct primes both $\equiv 1 \pmod 4$). In a later paper we give all the solutions of (II) with k minimal (i.e. $k=\lambda(\sqrt{n})$) for $n=2p$, $3p$ (p prime >3) and pq (p, q distinct primes both $\equiv 1 \pmod 4$).

For $n>0$ let

$$\text{(1.1)}\qquad \sqrt{n}=\sum_{j=1}^{\mu(\sqrt{n})} c_j\zeta_j$$

be an expression for $\sqrt{n}$ as a rational linear combination of roots of unity involving the least number $\mu(\sqrt{n})$ of terms. Write μ for $\mu(\sqrt{n})$. If S is a rational linear combination of roots of unity then we denote by $|S|$ the number of terms in S. We shall use the following expression for $\sqrt{n}$ as the Gauss sum:

$$(1.2)\quad \sqrt{n} = \begin{cases} \sum\limits_{h(\bmod n)} \zeta^{h^2} & \text{if} \quad n\equiv 1 \pmod 4, \quad \zeta = e^{2\pi i/n} \\ -i\sum\limits_{h(\bmod n)} \zeta^{h^2} & \text{if} \quad n\equiv 3 \pmod 4, \quad \zeta = e^{2\pi i/n} \\ \sum\limits_{h(\bmod m)} (\xi^{h^2}\eta - \xi^{h^2}\eta^3) & \text{if} \quad n = 2m \text{ with } m\equiv 1 \pmod 4 \\ \sum\limits_{h(\bmod m)} (-\xi^{h^2}\eta - \xi^{h^2}\eta^3) & \text{if} \quad n = 2m \text{ with } m\equiv 3 \pmod 4 \end{cases}$$

where $\xi = e^{2\pi i/m}$, $\eta = e^{2\pi i/8}$.

Following Conway and Jones [1] we say that two roots of unity α, β are equivalent if $o(\alpha/\beta)$ is square-free. We have the following

Lemma 1. *In* (1.1), *for no j, can the order $b_j = o(\zeta_j)$ be divisible by an odd square factor. Moreover*

$$\begin{aligned} 2^3 \| b_j \quad &\textit{if} \quad n\equiv 2 \pmod 4, \\ 2^2 \| b_j \quad &\textit{if} \quad n\equiv 3 \pmod 4, \\ 4 \nmid b_j \quad &\textit{if} \quad n\equiv 1 \pmod 4, \end{aligned}$$

indeed if $n\equiv 1 \pmod 4$ *we can make b_j odd by changing ζ_j to $-\zeta_j$ (if necessary).*

Proof. Equate the expressions in (1.1) and (1.2) for $\sqrt{n}$ and split into equivalence classes. Theorem 1 of [1] then gives

$$\sum_{\substack{j=1 \\ \zeta_j\sim 1}}^{n(\sqrt{n})} c_j\zeta_j - \sum_{h(\bmod n)} \zeta^{h^2} = 0 \quad \text{if} \quad n\equiv 1 \pmod 4$$

i.e.

$$\sum_{\substack{j=1 \\ \zeta_j\sim 1}}^{n(\sqrt{n})} c_j\zeta_j = \sqrt{n}\,.$$

But in (1.1) the sum was minimal hence each $\zeta_j \sim 1$, i.e. $o(\zeta_j)$ is square-free as required.

Similarly if $n\equiv 3 \pmod 4$ one gets the relation

$$\sum_{\substack{j=1 \\ \zeta_j\sim i}}^{n(\sqrt{n})} c_j\zeta_j = \sqrt{n}$$

and result again follows. For n even we get the relation

$$\sum_{\substack{j=1 \\ \zeta_j\sim \eta \text{ or } \eta^3}}^{n(\sqrt{n})} c_j\zeta_j = \sqrt{n}$$

and result follows as above.

LEMMA 2. *In* (1.1) *for an odd prime q if $q \nmid n$ then $q \nmid b_j$ (the order $o(\zeta_j)$ of ζ_j) for any j.*

PROOF. Write $\zeta_j = \eta_j \cdot \varrho^{e_j}$ where $\varrho = e^{2\pi i/q}$ and η_j is a root of unity whose order is not a multiple of q (i.e. $e_j = 0, 1, \ldots, q-1$). This is certainly possible for we may take $e_j = 0$, $\eta_j = \zeta_j$ if $q \nmid b_j$, otherwise write $b_j = qd_j$ $(q \nmid d_j)$ and then $1 = xq + yd_j$ giving $\zeta_j = e^{2\pi i a_j/b_j}$ (say) $= e^{2\pi i a_j (xq+yd_j)/b_j} = e^{2\pi i a_j x/d_j} \cdot e^{2\pi i a_j y/q} = \eta_j \varrho^{e_j}$ as required. Now by renaming the ζ_j we may suppose without loss of generality that $q \nmid b_j$ for $j = 1, 2, \ldots, r$ and we have to prove that $r = \mu\sqrt{n}$ ($= \mu$ say). Substituting for the ζ_j in (1.1) gives

$$c_1 \eta_1 + \ldots + c_r \eta_r + c_{r+1} \eta_{r+1} \varrho^{e_{r+1}} + \ldots + c_\mu \eta_\mu \varrho^{e_\mu} = \sqrt{n} = S,$$

where S is given by (1.2). On collecting various coefficients in powers of ϱ the above becomes $S_0 + S_1 \varrho + \ldots + S_{q-1} \varrho^{q-1} = S$.

Then by lemma 1 of [1] we have the following relations:

$$S_0 - S = S_1 = S_2 = \ldots = S_{q-1}.$$

It follows that $S_0 - S_1 = S$, i.e. $S_0 - S_1 = \sqrt{n}$. But unless $S_2 = S_3 = \ldots = S_{q-1} = 0$ we see that $S_0 - S_1$ has lesser number of terms in it than $S_0 + S_1 \varrho + \ldots + S_{q-1} \varrho^{q-1}$ has and this is a contradiction to the minimality of μ. Hence $S_2 = S_3 = \ldots = S_{q-1} = 0$. But $S_1 = S_2$. So $S_j = 0$ for $j = 1, 2, \ldots, q-1$. Thus $S_0 = S$ and we get $r = \mu$ as required.

REMARK. Lemmas 1 and 2 combined essentially tell us that one need not look beyond the field $Q(e^{2\pi i/4n})$ to obtain a minimal representation for $\sqrt{n}$ (n square-free).

The two lemmas above immediately imply our first result viz.

THEOREM 1.

$$\mu(\sqrt{p}) = \begin{cases} (p+1)/2 & \textit{if} \quad p \textit{ is an odd prime}, \\ 2 & \textit{if} \quad p = 2. \end{cases}$$

PROOF. Since $\sqrt{2}$ is itself not a rational multiple of a root of unity but can be written as a rational linear combination of two roots of unity: $\sqrt{2} = 2\cos\pi/4 = 2e^{\pi i/4} + 2e^{-\pi i/4}$, it follows that $\mu(\sqrt{2}) = 2$.

Let $p^* = (-1)^{(p-1)/2} p$ and write $\sqrt{p^*} = \sum_{j=1}^{\mu} c_j \zeta_j$ in the minimal way. By lemmas 1 and 2 $o(\zeta_j) = p$ or 1 for all j (where we change ζ_j to $-\zeta_j$ if necessary and absorb the minus sign in a_j). It follows that each ζ_j is a power of $\zeta = e^{2\pi i/p}$. But the only relations giving $\sqrt{p^*}$ in this way are

$$\sqrt{p^*} = 1 + 2 \sum_{(R/p)=1} \zeta^R + A(1 + \zeta + \ldots + \zeta^{p-1})$$

ond in this the least number of terms occurs on the right hand side if and anly if $A = 0$ or -2, the number then being $(p+1)/2$ as required.

§ 2. Bounds for $\mu(\sqrt{n})$ and $\mu(\sqrt{2n})$, n odd

The object of this section is the following

THEOREM 2. *Let $n = p_1 p_2 \ldots p_s$ be odd where $p_1 < p_2 < \ldots < p_s$, then*

(i) $\frac{p_1+1}{2} \cdot \frac{p_2}{2} \ldots \frac{p_s}{2} \le \mu(\sqrt{n}) \le \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \ldots \frac{p_s+1}{2}$,

(ii) $\mu(\sqrt{2n}) = 2\mu(\sqrt{n})$.

We first prove the following

LEMMA 3. *Let $n = pm$ $(p \nmid m)$ where p is an odd prime, m is odd then*

$$\frac{p+1}{2}\mu(\sqrt{m}) \ge \mu(\sqrt{pm}) \ge \frac{p}{2}\mu(\sqrt{m}).$$

PROOF. The left hand inequality is trivial: Write $\sqrt{p}$ and $\sqrt{m}$ in a minimal way involving $\mu(\sqrt{p}) = (p+1)/2$ and $\mu(\sqrt{m})$ terms respectively. Multiplying we get an expression for $\sqrt{pm}$ involving $\frac{p+1}{2} \cdot \mu(\sqrt{m})$ terms. This gives $\mu(\sqrt{pm}) \le \frac{p+1}{2} \cdot \mu(\sqrt{m})$ as required.

Now write $\sqrt{pm} = \sum_{j=1}^{\mu(\sqrt{pm})} c_j \zeta_j$ in the minimal way. Here write

$$\sqrt{pm} = \begin{cases} \sqrt{p} \cdot \sqrt{m} & \text{if} \quad p \equiv 1 \pmod 4, \\ -i\sqrt{-p} \cdot \sqrt{m} & \text{if} \quad p \equiv 3 \pmod 4. \end{cases}$$

Then write $\sqrt{m}$ as a rational linear combination, S say, of roots of unity involving the least possible number of terms. Then

$$\sqrt{pm} = \begin{cases} S\left(1+2\sum_{(R/p)=1} \zeta^R\right) & \text{if} \quad p \equiv 1 \pmod 4, \\ -iS\left(1+2\sum_{(R/p)=1} \zeta^R\right) & \text{if} \quad p \equiv 3 \pmod 4, \ \text{(where } \zeta = e^{2\pi i/p}). \end{cases}$$

Now since S is minimal both S and $-iS$ involve $\mu(\sqrt{m})$ terms. Write S^* to mean S if $p \equiv 1 \pmod 4$ and $-iS$ if $p \equiv 3 \pmod 4$, Then

$$\sqrt{pm} = \sum_{j=1}^{\mu(\sqrt{pm})} c_j \zeta_j = S^* + \sum_{(R/p)=1} 2S^* \zeta^R.$$

Now write

$$\sum_{j=1}^{\mu(\sqrt{pm})} c_j \zeta_j \quad \text{as} \quad S_0 + S_1 \zeta + \ldots + S_{p-1} \zeta^{p-1}.$$

where all the S_j involve roots of unity whose order is prime to p (see the proof of lemma 2). Then $S_0+S_1\zeta+\ldots+S_{p-1}\zeta^{p-1}=S^*+2S^*\sum_{(R/p)=1}\zeta^R$, i.e. $(S_0-S^*)+\sum_R(S_R-2S^*)\zeta^R+\sum_N S_N\zeta^N=0$, where $(R/p)=1$ and $(N/p)=-1$. By lemma 1 of [1] we get $S_0-S^*=S_R-2S^*=S_N$, i.e. $S_R-S_N=2S^*$, $S_0-S_R=-S^*$, $S_0-S_N=S^*$. These imply the following relations

(i) $|S_R|+|S_N|\geqq|S_R-S_N|\geqq|2S^*|=\mu(\sqrt{m})$,

(ii) $|S_0|+|S_R|\geqq|S_0-S_R|\geqq|-S^*|=\mu(\sqrt{m})$,

(iii) $|S_0|+|S_N|\geqq|S_0-S_N|\geqq|S^*|=\mu(\sqrt{m})$.

Now

$$\mu(\sqrt{pm})=\sum_{j=0}^{p-1}|S_j|=$$

$$=\frac{1}{2}\Big\{(|S_0|+|S_1|)+\Big(\sum_R|S_R|+\sum_N|S_N|\Big)+\Big(|S_0|+\sum_{R\neq 1}|S_R|+\sum_N|S_N|\Big)\Big\}\geqq$$

$$\geqq\frac{1}{2}\left[\mu(\sqrt{m})+\mu(\sqrt{m})\cdot\frac{p-1}{2}+\mu(\sqrt{m})\cdot\frac{p-1}{2}\right]$$

by (i), (ii) and (iii) above, and this $=\frac{p}{2}\cdot\mu(\sqrt{m})$. This completes the proof of lemma 3.

Proof of Theorem 2. (i) is an immediate consequence of lemma 3. To prove (ii) we proceed as follows. Write $\sqrt{2n}$ as a rational linear combination of roots of unity in the minimal way involving $\mu(\sqrt{2n})=\mu$ terms:

$$\sum_{j=1}^{\mu}c_j\zeta_j=\sqrt{2n}=\sqrt{2}\sqrt{n}=S(\eta-\eta^3) \tag{2.1}$$

where $\eta=e^{2\pi i/8}$ and S is the expression for $\sqrt{n}$ given in (1.2). This expression for S involves roots of unity which are either all ~ 1 or all $\sim i$. Now split (2.1) into equivalence classes and get either

$$\sum_{\substack{j=1\\ \zeta_j\sim\eta}}^{\mu}c_j\zeta_j=S\eta,\qquad \sum_{\substack{j=1\\ \zeta_j\sim\eta^3}}^{\mu}c_j\zeta_j=-S\eta^3$$

or

$$\sum_{\substack{j=1\\ \zeta_j\sim\eta}}^{\mu}c_j\zeta_j=-S\eta^3,\qquad \sum_{\substack{j=1\\ \zeta_j\sim\eta^3}}^{\mu}c_j\zeta_j=S\eta.$$

Since $S=\sqrt{n}$ (in value) so in either case both $\sum_{\zeta_j\sim\eta}1$ and $\sum_{\zeta_j\sim\eta^3}1$ are $\geqq\mu(\sqrt{n})$. Hence $\mu\geqq 2\mu(\sqrt{n})$. But since $\sqrt{2n}=\sqrt{n}(\eta-\eta^3)$ and $\sqrt{n}$ can be written as a rational linear combination of $\mu(\sqrt{n})$ roots of unity so $\mu\leqq 2\mu(\sqrt{n})$, This completes the proof.

§ 3. The exact value of $\mu(\sqrt{n})$ in some special cases

THEOREM 3.

(i) $\mu(\sqrt{2p}) = p+1$ (*p odd prime*),

(ii) $\mu(\sqrt{3p}) = p$ (*p prime* >3),

(iii) $\mu(\sqrt{pq}) = \frac{p+1}{2} \cdot \frac{q+1}{2}$ (*p, q primes with* $3<p<q$).

PROOF. (i) By theoren 2, (ii) $\mu(\sqrt{2p}) = 2\mu(\sqrt{p}) = 2((p+1)/2)$ (by theorem 1) $= p+1$ as required.

(ii). By theoren 2 we get

$$p = \frac{3+1}{2} \cdot \frac{p}{2} \le \mu(\sqrt{3p}) \le \frac{3+1}{2} \cdot \frac{p+1}{2} = p+1 .$$

It follows that $\mu(\sqrt{3p})$ is either equal to p or equal to $p+1$. We now explicitly exhibit $\sqrt{3p}$ as a rational linear combination of roots of unity having p terms. Let $\omega = (-1+\sqrt{-3})/2$, $\zeta = e^{2\pi i/p}$, and let

$$X = -1 + \sum_{(R/p)=1} 2\omega\, \zeta^R + \sum_{(N/p)=-1} 2\omega^2 \zeta^N .$$

Now $\sqrt{-3} = 2\omega+1 = -1-2\omega^2 = \omega-\omega^2$. Hence

$$X = X+0 = X+\Big(1+\sum_R \zeta^R+\sum_N \zeta^N\Big) = \sum_R (1+2\omega)\,\zeta^R+\sum_N (1+2\omega^2)\,\zeta^N$$

(by using the definition of X)

$$X = \sqrt{-3}\,\Big(\sum_R \zeta^R-\sum_R \zeta^N\Big) = \sqrt{-3}\,\sqrt{(-1)^{p-1/2}\,p}$$

(by the Gauss sum)

$$X = \begin{cases} -\sqrt{3p} & \text{if} \quad p\equiv 3 \pmod 4, \\ i\sqrt{3p} & \text{if} \quad p\equiv 1 \pmod 4 . \end{cases}$$

Thus

$$\sqrt{3p} = \begin{cases} -X & \text{if} \quad p\equiv 3 \pmod 4, \\ -iX & \text{if} \quad p\equiv 1 \pmod 4 . \end{cases}$$

But X and $-iX$ are both rational linear combinations of roots of unity involving p terms. This proves (ii).

(iii). Let $\varrho = e^{2\pi i/q}$, $\eta = e^{2\pi i/p}$. Then as in the proof of lemma 3 we arrive at the equation

$$S_0+S_1\,\varrho+\ldots+S_{q-1}\,\varrho^{q-1} = S^*+\sum_{(R/q)=1} 2S^*\cdot\varrho^R \tag{3.1}$$

where $S^* = 1$ or $-i$ times $\left(1+2\sum_{(R'/p)=1}\eta^{R'}\right)$. Both sides of (3.1) $=\sqrt{p}$ in value and as before we get the equations

$$S_0 - S^* = S_R - 2S^* = S_N. \tag{3.2}$$

Then $S_R - S_N = 2S^* = 2\sqrt{p}$ in value. We see that $\sqrt{p}$ is represented as a rational linear combination of roots of unity by $S_R - S_N$ and hence

$$|S_R| + |S_N| \geq (p+1)/2. \tag{3.3}$$

Case 1: $|S_R| + |S_N| > (p+1)/2$ for all R, N.

This means that $|S_R| + |S_N| \geq (p+3)/2$. Then by (3.2)

$$\mu(\sqrt{pq}) = |S_0| + \sum_{j=1}^{(q-1)/2} \{|S_{R_j}| + |S_{N_j}|\} \geq |S_0| + \frac{q-1}{2}\cdot\frac{p+3}{2} >$$

$$\geq \frac{q-1}{2}\cdot\frac{p+3}{2} = \frac{pq+3q-p-3}{4} > \frac{pq+p+q+1}{4} \text{ (since } q \geq p+2) =$$

$$= \frac{p+1}{2}\cdot\frac{q+1}{2}.$$

The required result follows by the right hand inequality of theorem 2, part (i).

Case 2: There is an R_0, N_0 such that $|S_{R_0}| + |S_{N_0}| = (p+1)/2$.

For this pair (as for the others) $S_{R_0} - S_{N_0} = 2\sqrt{p}$, i.e. $\sqrt{p}$ is represented as a rational linear combination of roots of unity by $S_{R_0} - S_{N_0}$ and by the hypothesis of case 2, this is the minimal way. So there is no common root of unity in S_{R_0} and S_{N_0} (even with different coefficients) otherwise these would be joined up into one term in $S_{R_0} - S_{N_0}$ to give $\sqrt{p}$ as a rational linear combination of roots of unity with a smaller length than $(p+1)/2$.

Now (3.2) implies $2S_0 = S_R + S_N$. In particular we have $2S_0 = S_{R_0} + S_{N_0}$ (in value) and then $2S_0 = S_{R_0} + S_{N_0} + A(1+\eta+\ldots+\eta^{p-1})$ for a suitable rational number A, not merely in value but even as expressions. Now we have seen above that all roots of unity in S_{R_0} and S_{N_0} are different, so even if all the terms of S_{R_0} and S_{N_0} cancel out by $A(1+\eta+\ldots+\eta^{p-1})$ there remain $p-(p+1)/2 = (p-1)/2$ terms in $2S_0$, since $|S_{R_0}| + |S_{N_0}| = (p+1)/2$. Thus in any event $|S_0| \geq (p-1)/2$ with equality if and only if all terms in $S_{R_0} + S_{N_0}$ cancel out by $A(1+\eta+\ldots+\eta^{p-1})$ i.e. if and only if each coefficient in $S_{R_0} + S_{N_0}$ is equal to $-A$ (and all roots of unity in S_{R_0} and S_{N_0} are powers of η). In all other events $|S_0| > (p-1)/2$ and so $> (p+1)/2$ and then by (3.1) we have

$$\mu(\sqrt{pq}) = |S_0| + \sum_{j=1}^{(q-1)/2} \{|S_{R_j}| + |S_{N_j}|\} \geq \frac{p+1}{2} + \frac{q-1}{2}\cdot\frac{p+1}{2}$$

(by (3.4))

$$= \frac{p+1}{2}\cdot\frac{q+1}{2}.$$

We look at the worst event when $|S_0| = (p-1)/2$. We claim that $|S_{R_0}| \neq 0$, $|S_{N_0}| \neq 0$. For if $S_{R_0} = 0$ then since $S_R - S_0 = S^* = \sqrt{p}$ or $i\sqrt{p}$ we get $-S_0$ representing $\sqrt{p}$ or $i\sqrt{p}$, so $|S_0| > \mu(\sqrt{p}) = (p+1)/2$, a contradiction. Similarly $|S_{N_0}| \neq 0$. Also $S_{R_0} - S_{N_0} = 2\sqrt{p}$ and $|S_{R_0}| + |S_{N_0}| = (p+1)/2$. Thus $S_{R_0} - S_{N_0}$ is a minimal way of representing $\sqrt{p}$ and the coefficients in S_{R_0}, S_{N_0} are all equal to $-A$. But this is impossible by (i) of the following

Lemma 4. (i) *Let $p > 3$ be an odd prime. Then there are only two ways of representing $\sqrt{p}$ as a rational linear combination of roots of unity in the minimal way viz*

$$\sqrt{p} \quad \text{or} \quad i\sqrt{p} = 1 + 2 \sum_{(R/p)=1} \eta^R = -1 - 2 \sum_{(N/p)=-1} \eta^N, \quad \text{where} \quad \eta = e^{2\pi i/p},$$

(ii) *There are only three ways of representing $\sqrt{3}$ (or $\sqrt{-3}$ for convenience) in the minimal way as a rational linear combination of roots of unity viz*

$$\sqrt{-3} = \omega - \omega^2 = 1 + 2\omega = -1 - 2\omega^2, \quad \text{where} \quad \omega = e^{2\pi i/3},$$

(iii) *there is a unique way of representing $\sqrt{2}$ as a rational linear combination of roots of unity in the minimal way viz*

$$\sqrt{2} = \gamma + \gamma^{-1}, \quad \text{where} \quad \gamma = e^{2\pi i/8}.$$

Proof. Let p be an odd prime (including 3) and let $p^* = (-1)^{(p-1)/2} p$. Let $\sqrt{p^*} = \sum_{j=1}^{(p+1)/2} c_j \zeta_j$ be a minimal representation of $\sqrt{p^*}$. By lemma 1 $o(\zeta_j) = p$ or 1 for each j i.e. the ζ_j above are all p^{th} roots of unity i.e. powers of η and so any such relation is of the type

$$1 + 2 \sum_{(R/p)=1} \eta^R + A(1 + \eta + \ldots + \eta^{p-1}).$$

This has $(p+1)/2$ terms in it if $A = 0$ or -2 (the two ways mentioned in (i)). For all other values of A the number of terms is more than $(p+1)/2$ except when $p = 3$ and $A = -1$, when the number of terms equals $p-1$ which equals $(p+1)/2$ (since $p = 3$). Thus only for $p = 3$ we get the third way mentioned in (ii). This proves (i) and (ii). To prove (iii) we note that $\sqrt{2} = \gamma + \gamma^{-1}$ is certainly one way. Let also $\sqrt{2} = c_1 \zeta_1 + c_2 \zeta_2 = \gamma + \gamma^{-1}$ be another way. Multiply by γ and get $c_1(\gamma\zeta_1) + c_2(\gamma\zeta_2) = 1 + \gamma^2$ i.e. say $c_1 \zeta_1' + c_2 \zeta_2' = 1 + i$. Here $1 \not\sim i$ since $o(1/i) = 4$ is not square-free. It follows that ζ_1' and ζ_2' are in the class of 1 and i respectively since both ζ_1' and ζ_2' can not belong to the class of 1 (nor to the calss of i similarly) because then only i would remain in the class of i and would give $i = 0$. Say then $\zeta_1' \sim 1$, $\zeta_2' \sim i$, i.e. $c_1 \zeta_1' = 1$, $c_2 \zeta_2' = i$, or $c_1 \gamma \zeta_1 = 1$, $c_2 \gamma \zeta_2 = i$, or $c_1 \zeta_1 = \gamma^{-1}$, $c_2 \zeta_2 = \gamma^{-1} i = \gamma$ as required. This completes the proof of lemma 4.

To complete the proof of our theorem 3 we note that in both the representations of $\sqrt{p}$ given in (i) of lemma 4, the coefficients are not all the same. This shows that the worst case of the theorem under discussion never actually occurs. This proves theorem 3.

§ 4. λ and μ

We prove the following

THEOREM 4.

(i) $\lambda(\sqrt{n}) \leq \mu(\sqrt{n}) \leq 2\lambda(\sqrt{n})$,

(ii) $\lambda(\sqrt{2p}) = (p+1)/2$ (p *prime* >3),

(iii) $\lambda(\sqrt{3p}) = (p+1)/2$ (p *prime* >3),

(iv) $\lambda(\sqrt{pq}) = \left(1+\frac{p+1}{2}\cdot\frac{q+1}{2}\right)/2$ (p, q *primes satisfying* $3<p<q$, $p \equiv q \equiv 1 \pmod 4$).

PROOF. (i) Write $\sqrt{n} = \sum_{j=1}^{\lambda(\sqrt{n})} c_j \cos A_j$ in the minimal way

$$\sqrt{n} = \sum_{j=1}^{\lambda(\sqrt{n})} \frac{c_j}{2}(\zeta_j + \bar{\zeta}_j)$$

where $\zeta_j = e^{iA_j}$, which is a rational linear combination of roots of unity involving 2λ terms. It follows that $\mu \leq 2\lambda$.

Now write $\sqrt{n} = \sum_{j=1}^{\mu(\sqrt{n})} c_j \zeta_j$ in the minimal way. Take complex conjugates: $\sqrt{n} = \sum_{j=1}^{\mu(\sqrt{n})} c_j \bar{\zeta}_j$. Adding these two we get $\sqrt{n} = \sum_{j=1}^{\mu(\sqrt{n})} c_j \cos A_j$ which is a rational linear combination of cosines involving μ terms. It follows that $\lambda \leq \mu$. This proves (i).

(ii) We have $\lambda(\sqrt{2p}) \geq \frac{1}{2}\mu(\sqrt{2p}) = \frac{1}{2} 2\mu(\sqrt{p}) = (p+1)/2$. It remains to exhibit a rational linear combination of $(p+1)/2$ cosines giving $\sqrt{2p}$.

Case 1: $p \equiv 1 \pmod 4$. By the results proved in [2], $(-1+\sqrt{p})/4$ can be written as a rational linear combination of $(p-1)/4$ cosines. Thus $\sqrt{p} = 1+4(\cos\Theta_1 + \ldots + \cos\Theta_{(p-1)/4})$ and $\sqrt{2} = 2\cos\pi/4$. Multiplying we get $\sqrt{2p} = 2\cos\pi/4 + 4[\cos(\Theta_1+\pi/4)+\cos(\Theta_1-\pi/4)] + \ldots + 4[\cos(\Theta_{(p-1)/4} + \pi/4) + \cos(\Theta_{(p-1)/4} - \pi/4)]$. It follows that $\lambda(\sqrt{2p}) \leq 1+(p-1)/2 = (p+1)/2$. This does case 1.

Case 2: $p \equiv 3 \pmod 4$. Let $\zeta = e^{2\pi i/p}$, $\eta = e^{2\pi i/8}$ so that $\eta - \eta^7 = i\sqrt{2}$. Let $X = \eta + \eta^7 + 2\sum \zeta^R \eta + 2\sum \overline{\zeta^R \eta} = \eta + \eta^7 + \eta(\sqrt{-p}-1) + \eta^7(-\sqrt{-p}-1)$ (since $2\sum \zeta^R = \sqrt{-p}-1$ by the Gauss sum) $= \sqrt{-p}(\eta - \eta^7) = \sqrt{-p}(i\sqrt{2})$. Hence $X^2 = 2p$ and so $X = \pm\sqrt{2p}$. But the definition of X already exhibits X as a rational linear combination of $1+(p-1)/2$ cosines. It follows that $\lambda(\sqrt{2p}) \leq (p+1)/2$. This does case (ii).

(iii) By theorem 3, $\mu(\sqrt{3p}) = p$ and so $\lambda(\sqrt{3p}) \geq p/2$ (by (i)) and therefore $\geq (p+1)/2$. It remains to exhibit one representation of $\sqrt{3p}$ as a rational linear combination of $(p+1)/2$ cosines. If $p \equiv 1 \pmod 4$ we proceed exactly as in case I of (ii) with $\sqrt{3} = 2\cos \pi/6$ (instead of $\sqrt{2} = 2\cos \pi/4$) and get the result. For the case $p \equiv 3 \pmod 4$ let $\zeta = e^{2\pi i/p}$, $\omega = e^{2\pi i/3}$ and let $X = -1 + 2\sum_R \omega\,\zeta^R + 2\sum_R \omega^2\,\zeta^R$. Then we have shown in theorem 3 that $X = -\sqrt{3p}$. On the other hand $X = -1 + 2\sum_R \{\omega\,\zeta^R + \overline{\omega}\,\zeta^R\}$ since the set of all N is equal to the set of all $-R$ since $p \equiv 3 \pmod 4$, and this

$$= -1 + 2 \sum_{(R/p)=1} 2\cos \Theta_R$$

say where $\Theta_R = 2\pi(p+3R)/3p$. Thus

$$\sqrt{3p} = 1 - 4 \sum_{(R/p)=1} \cos \Theta_R$$

giving $\sqrt{3p}$ as a rational linear combination of $(p+1)/2$ cosines as required.

(iv) By the results of [2], since $p \equiv q \equiv 1 \pmod 4$, we have

$$\sqrt{p} = 1 + 4(\cos A_1 + \ldots + \cos A_{(p-1)/4}),$$

$$\sqrt{q} = 1 + 4(\cos B_1 + \ldots + \cos B_{(q-1)/4}).$$

Multiplying we get

$$\sqrt{pq} = 1 + 4(\cos A_1 + \ldots + \cos A_{(p-1)/4}) + 4(\cos B_1 + \ldots + \cos B_{(q-1)/4}) + \\ + 8 \sum_{k=1}^{(q-1)/4} \sum_{j=1}^{(p-1)/4} \cos(A_j - B_k) + \cos(A_j + B_k).$$

It follows that

$$\lambda(\sqrt{pq}) \leq 1 + (p-1)/4 + (q-1)/4 + [2(p-1)/4]\,[(q-1)/4] = \\ = \frac{1}{2}\left(1 + \frac{p+1}{2} \cdot \frac{q+1}{2}\right).$$

On the other hand

$$\lambda(\sqrt{pq}) \geq \frac{1}{2}\,\mu(\sqrt{pq}) = \frac{1}{2}\left(\frac{p+1}{2} \cdot \frac{q+1}{2}\right) = \frac{1}{2}\,m$$

say, where m is odd. Hence, being an integer

$$\lambda(\sqrt{pq}) \geq (m+1)/2 = \frac{1}{2}\left(1 + \frac{p+1}{2} \cdot \frac{q+1}{2}\right).$$

This proves (iv).

APPENDIX. We just wish to mention here that the proof of theorem 1 of [1] appears to be erroneous as the following example shows: Let $n=24$, then $a=6$, $b=4$, $\omega=e^{2\pi i/24}$ and $\Omega=\omega^6=e^{2\pi i/4}$ in the notation of [1]. Also S_i is a rational linear combination of ω^i, ω^{4+i}, ω^{8+i}, ω^{12+i}, ω^{16+i}, ω^{20+i} $(i=0,1,2,3)$. The automorphism $\sigma\colon \omega\to\omega^{a+1}=\omega\Omega$ of $Q(\omega)/Q$ transforms $S=S_0+S_1+S_2+S_3$ to $\sigma(S)=S_0+\Omega S_1+\Omega^2 S_2+\Omega^3 S_3$. A second application of σ to S gives $\sigma^2(S)=S_0+S_1+S_2+S_3=S$ and we get nowhere if we follow up the proof of theorem 1 of [1]. A correct proof goes as follows:

Each n^{th} root of unity can be written as ω^{i+bk}, $0\le i\le b-1$, $0\le k\le a-1$ and since $(\omega^{i+bk}/\omega^{i+bk'})^a=1$, $o(\omega^i/\omega^j)>a$ for $0\le i,\ j\le b-1$ $(i\ne j)$ so $\omega^{i+bk}\sim$ $\sim\omega^{i+bk'}$ and $\omega^i\not\sim\omega^j$. Hence each n^{th} root of unity is $\sim$ precisely one of ω^i $(0\le i\le b-1)$.

Now let S be a rational linear combination of the n^{th} roots of unity. Write $S=S_0+S_1+\ldots+S_{b-1}$ (splitting into equivalence classes), where each S_j is a rational linear combination of the ω^{j+bk} $(0\le k<a-1)$. Here $\nu(S_j)=a_j\omega^j$ say where $a_j\in Q(\omega^b)$. Notice that $[Q(\omega):Q(\omega^b)]=\varphi(n)/\varphi(a)=b$. It follows that $1,\omega,\ldots,\omega^{b-1}$ are linearly independent over $Q(\omega^b)$. Hence $\nu(S)=0$ implies $\sum_{j=0}^{b-1} a_j\omega^j=0$ which implies each $a_j=0$ which implies that $\nu(S_j)=0$ for all j. This completes the proof.

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RADICAL PROPERTIES DEFINED BY THE ABSENCE OF FREE SUBOBJECTS

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Previously [5], [6] we have considered radical properties of rings defined "locally" by polynomial identities (see also [13] and [20]). Observing that the requirement that subsets satisfy identities implies the absence of free subrings, we are led to ask when the class of rings having no free subrings of some or all ranks is a radical class. This question, suitably modified, is meaningful in more general situations, e.g. for groups. For this reason the present investigation is carried out for universal classes which are varieties of multioperator groups in the sense of HIGGINS [9]. In this way not only are rings (of various kinds) and groups (of various kinds) treated, but also, *inter alia*, algebras and modules.

The discussion in full generality **(§ 1)** is restricted to objects having no free one-generator subobjects (or, equivalently, no free subobjects at all) and some conditions are pointed out which imply that such objects form a radical class. It is convenient to consider at the same time, and in the same way, the class of objects without non-zero projective subobjects. We also give characterizations of these classes in a number of particular instances.

In § 2 we consider analogous questions for the classes

$$\mathcal{R}_\alpha = \{A \mid A \text{ has no free subobjects of rank } \alpha\}$$

for various cardinal numbers α, in the universal classes of all algebras over a field, Lie algebras over a field, groups, associative rings and alternative rings.

We note that CRAMER [3] has examined boolean algebras from a vaguely similar viewpoint.

§ 1. A *multioperator group* is a group G, additively written, with a set Ω of other finitary operations subject to the constraint that $\omega(0, 0, \ldots, 0) = 0$ for every $\omega \in \Omega$. Thus, for instance, when $\Omega = \{\cdot\}$, where $\cdot$ is binary and is both left and right distributive over $+$, we get rings and when $\Omega = \emptyset$ we get groups. (Implicitly, then, we are writing groups additively, but when we wish to single out the universal class of groups for special mention we shall use multiplicative notation.) Normal subobjects will be indicated by the symbol $\lhd$.

Our universal class will be a variety $\mathcal{W}$ of Ω-groups generally satisfying the following condition.

(V1) If B is a non-zero subobject of a projective object of $\mathcal{W}$, then B contains a non-zero projective subobject.

LEMMA 1.1 (V1) *is implied by the condition*

(V2) If B is a non-zero subobject of a free object of $\mathcal{W}$, then B contains a free subobject.

PROOF. If $0 \neq B \subseteq P$ and P is projective, then P is a retract of a free object F, whence we can assume that $P \subseteq F$. Then P contains a free (and hence non-zero projective) subobject. ▌

We mention a few examples. Associative rings satisfy (V2), since free rings have no zero-divisors and hence non-zero one-generator subrings of free rings are isomorphic to $x\,Z[x]$, i.e. are free. In the same way associative algebras over any integral domain satisfy (V2). A *Schreier variety* is a variety in which non-zero subobjects of free objects are free. Clearly Schreier varieties satisfy (V2). Examples of Schreier varieties are the varieties of groups [15], algebras (not necessarily associative) over a field [12], [22], Lie algebras over a field [17], [23], commutative and anticommutative algebras over a field [18]. The class of (left, unital) modules over a ring R is a Schreier variety if and only if R is a (left) fir ([2] p. 47). The class of modules over a semihereditary ring R satisfies (V1) but not necessarily (V2) (e.g. when R is semi-simple artinian).

In our universal variety $\mathcal{W}$, we consider the following classes:

$$\mathcal{R}_P = \{A \mid A \text{ has no non-zero projective subobjects}\};$$

$$\mathcal{R}_F = \{A \mid A \text{ has no free subobjects}\}.$$

PROPOSITION 1.2. *If $\mathcal{W}$ satisfies* (V1), *then $\mathcal{R}_P$ is a radical class.*

PROOF. If $A \in \mathcal{R}_P$, $I \lhd A$ and A/I has a projective subobject P, then the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ A \arrow[r] & A/I \end{tikzcd}$$

Fig. 1.

can be completed commutatively, whence A has a subobject isomorphic to P, $P=0$ and $A/I \in \mathcal{R}_P$. If $J \lhd R$ and if J and R/J are in $\mathcal{R}_P$, suppose R has a projective subobject L. If $L \cap J \neq 0$, then by (V1) $L \cap J$, and hence J, has a non-zero projective subobject. This is impossible, so $L \cap J = 0$. But then $L \cong L/L \cap J \cong (L+J)/J \subseteq R/J$. Since R/J is in $\mathcal{R}_P$, we have $L=0$, so R is in $\mathcal{R}_P$. Finally, if B is the union of a chain $\{B_\lambda \mid \lambda \in \Lambda\}$ of normal subobjects from $\mathcal{R}_P$, then for any projective subobject M of B we have $M = \bigcup_\Lambda M \cap B_\lambda$. Since $M \cap B_\lambda \subseteq B_\lambda \in \mathcal{R}_P$, $M \cap B_\lambda$ has no non-zero projective subbojects. But then, by (V1), $M \cap B_\lambda = 0$. Hence $M=0$ and B is in $\mathcal{R}_P$. ▮

It is clear that $\mathcal{R}_F$ is the class of objects with no one-generator free subobjects. We want a condition which makes $\mathcal{R}_F$ a radical class.

Proposition 1.3. *If $\mathfrak{W}$ satisfies*

(V3) Every non-zero subobject of a one-generator free object has a free subobject,

then $\mathcal{R}_F$ is a radical class.

The proof is like that of Proposition 1.2. Clearly (V2) implies (V3). However, the variety of alternative rings satisfies (V3), but not (V2); see [10], [16].

Proposition 1.4. *If $\mathfrak{W}$ satisfies* (V2), *then $\mathcal{R}_F = \mathcal{R}_P$.*

Proof. In any case, $\mathcal{R}_P \subseteq \mathcal{R}_F$. If (V2) is satisfied, $A \in \mathcal{R}_F$ and P is a projective subobject of A, then P, if not zero, being a subobject of a free object, has a free subobject and hence A does – contradiction. Hence A is in $\mathcal{R}_P$ and $\mathcal{R}_F \subseteq \mathcal{R}_P$. ▮

The following examples may help to put the foregoing results into perspective.

Example 1.5. When $\mathfrak{W}$ is the class of Z_4-modules, $\mathcal{R}_P$ contains $2Z_4$ and $Z_4/2Z_4$, but not Z_4, so $\mathcal{R}_P$ is not a radical class.

Example 1.6. An *autodistributive algebra* is a (not necessarily associative) algebra over the two element field satisfying the identities

$$x(yz) = (xy)(xz); \quad (xy)z = (xz)(yz).$$

Every autodistributive algebra has the form $A \oplus B$ where A satisfies the identities $x(yz) = 0 = (xy)z$ and B satisfies $x^2 = x$ (see [7] or [11]). The one-generator free algebra in this case is isomorphic to

$$(xZ_2[x]/x^3Z_2[x]) \oplus Z_2 .$$

Both factors are in $\mathcal{R}_F$, so $\mathcal{R}_F$ is not closed under extensions and therefore is not a radical class. In the decomposition $A \oplus B$ mentioned above, B actually consists of all the idempotents and so, if non-zero, contains a copy of Z_2. Thus all algebras in $\mathcal{R}_P$ satisfy the identities $x(yz) = 0 = (xy)z$. If A has an element a for which $a^3 = 0 \neq a^2$, then A contains a copy of $xZ_2[x]/x^3Z_2[x]$, and conversely. Thus we have

$$\mathcal{R}_P = \{A \mid a, b, c \in A \Rightarrow (ab)c = 0 = a(bc); \quad a^2 = 0\}$$
$$= \{A \mid a \in A \Rightarrow a^2 = 0\}.$$

This isn't closed under extensions either, as one sees by considering the Z_2-algebra spanned by $\{u, v\}$ with $u^2 = v$ and all other products zero.

EXAMPLE 1.7. If R is a semi-simple artinian ring, then in the class of R-modules, $\mathcal{R}_F$ is not a radical class while $\mathcal{R}_P = \{0\}$ is.

We turn now to an examination of what $\mathcal{R}_P$ and $\mathcal{R}_F$ look like for more orthodox choices of $\mathcal{W}$.

EXAMPLE 1.8. When $\mathcal{W}$ is a variety of rings or algebras, $\mathcal{R}_F$ is the class of algebraic algebras, i.e. denoting a free $\mathcal{W}$-algebra on one generator by F_1, we have

$$\mathcal{R}_F = \{A \in \mathcal{W} \mid a \in A \Rightarrow \exists \alpha \in F_1, \ \alpha \neq 0, \ \text{with} \ \alpha(a) = 0\}.$$

To see this, we just observe that the subalgebra generated by a is isomorphic to $F_1/I(a)$, where $I(a) = \{\beta \in F_1 \mid \beta(a) = 0\}$. If $a \in A \in \mathcal{R}_F$, then $F_1/I(a)$ is not free, so $I(a) \neq 0$. Conversely, if $A \notin \mathcal{R}_F$, then some $b \in A$ generates a free subalgebra $\langle b \rangle$. But there is an isomorphism $\langle b \rangle \to F_1$ given by $b \mapsto x$ (a generator of F_1). If $\beta \in I(b)$, then $0 = \beta(b) \mapsto \beta(x) = \beta$. Hence $I(b) = 0$ and A is not algebraic.

EXAMPLE 1.9. When $\mathcal{W}$ is the class of all groups, we have

$$\mathcal{R}_F = \mathcal{R}_P = \{G \mid g \in G \Rightarrow 0(g) \text{ is finite}\},$$

since belonging to $\mathcal{R}_F$ ($= \mathcal{R}_P$ since we're dealing with a Schreier variety) is equivalent to having no rank-one free subgroup, i.e. no infinite cyclic subgroup.

EXAMPLE 1.10. Let R be a ring (associative) with identity such that the class of (left, unital) R-modules satisfies (V1). Then $\mathcal{R}_P$ is a radical class. Let L be a left ideal of R such that $R/L \in \mathcal{R}_P$. If M is a left ideal of R and $M \cap L = 0$, then R/L contains a copy, $(M+L)/L$, of M. Since M is contained in the projective module R, either $M = 0$ or M has a non-zero projective submodule. The latter alternative is incompatible with the membership of R/L in $\mathcal{R}_P$, so $M = 0$. Thus L is essential. If, conversely, L is an essential left ideal, then any projective submodule of R/L lifts back to a projective left ideal not intersecting L, so is zero. Thus R/L is in $\mathcal{R}_P$. This proves that $\mathcal{R}_P$ is the Goldie torsion class and R is non-singular (see, e.g., [8]). In particular for abelian groups, we get ordinary torsion (more obviously).

In general, the $\mathcal{R}_P$-semi-simple objects are those in which every non-zero normal subobject has a non-zero projective subobject. Semi-simple classes of modules in which every non-zero module has a non-zero projective submodule have been studied by TEPLY [21].

We note that even in such a well-behaved universal variety as the class of abelian groups, the class of objects without non-zero injective homomorphic images is not a semi-simple class, so there seems to be little prospect for dualizing the results of this section.

§ 2. For each cardinal number α, let

$$\mathcal{R}_\alpha = \{A \in \mathcal{W} \mid A \text{ has no free subobject of rank } \alpha\}.$$

Clearly $\mathcal{R}_\alpha \subseteq \mathcal{R}_\beta$ whenever $\alpha \leq \beta$. In this section we shall consider two questions:

(1) When is $\mathcal{R}_\alpha$ a radical class?
(2) When is $\mathcal{R}_\alpha = \mathcal{R}_\beta$?

Note that $\mathcal{R}_1 = \mathcal{R}_F$ as defined in § 1.

Lemma 2.1. *Let $\mathcal{W}$ be the variety of* (i) *all algebras over a field,* (ii) *Lie algebras over a field or* (iii) *groups.*

If F is a free object of infinite rank and $0 \neq I \lhd F$, then $I \cong F$.

Proof. (i) Let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a free set of generators for F, $\{y_\gamma \mid \gamma \in \Gamma\}$ a free set for I. Suppose $|\Gamma| < |\Lambda|$. Let

$$\Lambda_0 = \{\lambda \in \Lambda \mid x_\lambda \text{ is involved in the representation of some } y_\gamma\}.$$

Then clearly $|\Lambda_0| = \aleph_0 |\Gamma| < |\Lambda|$, so we can choose $\bar\lambda \in \Lambda \setminus \Lambda_0$. But $x_{\bar\lambda} y_\gamma \in I$ for each γ, so each $x_{\bar\lambda} y_\gamma$ can be expressed in terms of the x_λ for $\lambda \in \Lambda_0$, while $x_{\bar\lambda} y_\gamma$ involves $x_{\bar\lambda}$ as well as various x_λ with $\lambda \in \Lambda_0$. This is impossible, so $|\Gamma| = |\Lambda|$ and $I \cong F$.

(ii) Now let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a free generating set for a free Lie algebra F, $\{y_\gamma \mid \gamma \in \Gamma\}$ a free generating set for a non-zero ideal I. Let Λ_0, $\bar\lambda$ be defined as in (i). Let F_a be the free associative algebra on $\{x_\lambda \mid \lambda \in \Lambda\}$, F_a^- the Lie algebra on F_a defined by commutation. Then we have a homomorphism $f: F \to F_a^-$ given by $f(x_\lambda) = x_\lambda \ \forall \lambda$. In F, we have $x_{\bar\lambda} y_\gamma$ expressed in terms of the x_λ, $\lambda \in \Lambda_0$ so in F_a we have $x_{\bar\lambda} y_\gamma - y_\gamma x_{\bar\lambda}$ expressed in terms of these x_λ. This is impossible.

(iii) Proceed as above, but consider $x_{\bar\lambda} y_\gamma x_{\bar\lambda}^{-1}$. ∎

(Case (iii) of Lemma 2.1 was noted by Shmel'kin [19]. We are grateful to Frank Harris for explaining (iii) and thereby suggesting (i) and (ii).)

Recall that an object A is *unequivocal* [4] if $\mathcal{R}(A) = A$ or 0 for every radical class $\mathcal{R}$.

Theorem 2.2. *Free algebras of infinite rank, free Lie algebras and free groups are unequivocal.*

Proof. Lemma 2.1 takes care of this infinite rank cases: if F is free and $\mathcal{R}$ is a radical class, then $\mathcal{R}(F) = 0$ or $F \cong \mathcal{R}(F) \in \mathcal{R}$. The finite rank case for groups was proved by Shmel'kin [19]. Consider, therefore, a free Lie algebra F_n of finite rank n and a radical class $\mathcal{R}$. If $0 \subsetneqq \mathcal{R}(F_n) \neq F_n$, then by Theorem 3 of Baumslag [1], $\mathcal{R}(F_n)$ is not finitely generated, so its rank is $\aleph_0$. But then F_n is a homomorphic image of $\mathcal{R}(F_n)$, so $F_n \in \mathcal{R}$ — contradiction. Thus F_n is unequivocal. ∎

We now show that for the three varieties we have been discussing, the classes $\mathcal{R}_\alpha$ are always radical classes.

THEOREM 2.3. (i) *In the variety of all algebras over a field, each* $\mathcal{R}_\alpha$ *is a radical class and we have* $\mathcal{R}_F = \mathcal{R}_1 = \mathcal{R}_2 = \dots = \mathcal{R}_{\aleph_0}$; $\mathcal{R}_\alpha \subsetneq \mathcal{R}_\beta$ *if* $\alpha < \beta$ *and* α *is infinite.*

(ii) *In the variety of all Lie algebras over a field, each* $\mathcal{R}_\alpha$ *is a radical class and* $\mathcal{R}_F = \mathcal{R}_1 \subsetneq \mathcal{R}_2 = \mathcal{R}_3 = \dots = \mathcal{R}_{\aleph_0}$; $\mathcal{R}_\alpha \subsetneq \mathcal{R}_\beta$ *if* $\alpha < \beta$ *and* α *is infinite.*

(iii) *In the variety of all groups, each* $\mathcal{R}_\alpha$ *is a radical class and* $\mathcal{R}_F = \mathcal{R}_1 \subsetneq \mathcal{R}_2 = \mathcal{R}_3 = \dots = \mathcal{R}_{\aleph_0}$; $\mathcal{R}_\alpha \subsetneq \mathcal{R}_\beta$ *if* $\alpha < \beta$ *and* α *is infinite.*

PROOF. (i): By Proposition 1.3, $\mathcal{R}_F = \mathcal{R}_1$ is a radical class. KUROSH ([12], p. 244) has shown that a free algebra on one generator has a subalgebra that is freely generated by a set of cardinality $\aleph_0$. Thus $\mathcal{R}_{\aleph_0} \subseteq \mathcal{R}_1$. If now α is infinite, then clearly $\mathcal{R}_\alpha$ is homomorphically closed (cf. the proof of Proposition 1.2). If $I \triangleleft A$ and both I and A/I are in $\mathcal{R}_\alpha$, suppose F_α is a free subalgebra of A with rank α. Then $F_\alpha \cap I$ is a free ideal of I and since I is in $\mathcal{R}_\alpha$, $F_\alpha \cap I$ can't have rank α. By Lemma 2.1, $F_\alpha \cap I = 0$. But then

$$F_\alpha \cong F_\alpha/(F_\alpha \cap I) \cong (F_\alpha + I)/I \subseteq A/I\,,$$

while A/I is in $\mathcal{R}_\alpha$ — contradiction. Thus A is in $\mathcal{R}_\alpha$. Finally, if R is a union of a chain $\{J_\lambda | \lambda \in \Lambda\}$ of $\mathcal{R}_\alpha$ — ideals, then for any free subalgebra F_α of rank α, we have $J_\lambda \cap F_\alpha \triangleleft J_\lambda \in \mathcal{R}_\alpha$ for each λ, so by Lemma 2.1, $J_\lambda \cap F_\alpha = 0$ for each λ. But this means that

$$F_\alpha = \Big(\bigcup_\Lambda J_\lambda\Big) \cap F_\alpha = \bigcup_\Lambda (J_\lambda \cap F_\alpha) = 0\,.$$

Again we have a contradiction, so R is in $\mathcal{R}_\alpha$ and $\mathcal{R}_\alpha$ is a radical class.

For free algebras of infinite rank, the rank and (vector space) dimension are equal. Thus if F_α is a free algebra of infinite rank α, then F_α has no subspace of dimension $\beta > \alpha$ and hence no free subalgebra of rank $\beta > \alpha$. Thus $F_\alpha \in \mathcal{R}_\beta \setminus \mathcal{R}_\alpha$.

(ii): By Proposition 1.3, $\mathcal{R}_F = \mathcal{R}_1$ is a radical class. In a one-generator free Lie algebra F_1 all products are zero, so $F_1 \in \mathcal{R}_2 \setminus \mathcal{R}_1$.

Let F_2 be a free Lie algebra of rank 2. Then $F_2^2 \neq 0$ and $F_2 \neq F_2^2$, so by [1], Theorem 3, F_2^2 has infinite rank. It follows that $\mathcal{R}_{\aleph_0} \subseteq \mathcal{R}_2$. The rest of the proof is like that of (i).

(iii): By Proposition 1.3, $\mathcal{R}_F = \mathcal{R}_1$ is a radical class. Free groups of rank 1 are abelian, and hence in $\mathcal{R}_2 \setminus \mathcal{R}_1$. Free groups of rank 2 have infinite-rank free subgroups (see, e.g. [14], p. 155), so $\mathcal{R}_{\aleph_0} \subseteq \mathcal{R}_2$. Observing that for free groups of infinite rank, order and rank are equal, we can now complete the proof by arguments analogous to those used for (i). ∎

We conclude with a discussion of some of the classes $\mathcal{R}_\alpha$ for associative and alternative rings.

THEOREM 2.4. *For associative rings,* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are radical classes and*

$$\mathcal{R}_F = \mathcal{R}_1 \subsetneq \mathcal{R}_2 = \mathcal{R}_3 = \dots = \mathcal{R}_{\aleph_0}\,.$$

PROOF. By Proposition 1.3, $\mathcal{R}_F = \mathcal{R}_1$ is a radical class. Since one-generator free rings are commutative and therefore do not have free subrings with rank greater than one, we have $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Let F_2 be free on $\{x, y\}$. Then as noted by Cohn ([2], p. 33) for associative algebras over a field, $\{x, xy, xy^2, xy^3, \ldots\}$ is a free generating set. Thus F_2 has free subrings of infinite rank and so $\mathcal{R}_2 = \mathcal{R}_{\aleph_0}$.

By a familiar argument, $\mathcal{R}_2 = \mathcal{R}_{\aleph_0}$ is homomorphically closed. If $I \triangleleft A$ and $\mathcal{R}_2$ contains I and A/I, suppose A has a free subring F of rank $\aleph_0$. If $F \cap I \neq 0$, choose a non-zero element u in $F \cap I$. Then $uF \neq Fu$, so we can choose $v \in F$ such that $uv \neq vu$. Then $u^2 v \neq uvu$ so by a result of Cohn ([2], p. 244), the non-commuting elements u, uv generate a free subring of I. This contradicts our assumption that I is in $\mathcal{R}_2$. Hence $F \cap I = 0$. But then F is isomorphic to a subring of $A/I \in \mathcal{R}_2$ — again a contradiction. Thus A is in $\mathcal{R}_2$. If now R is the union of a chain $\{J_\lambda | \lambda \in \Lambda\}$ of $\mathcal{R}_2$ — ideals, any rank — $\aleph_0$ free subring F of R is the union of the ideals $F \cap J_\lambda$, and each $F \cap J_\lambda$ has no free subrings of rank 2. Let $F \cap J_{\lambda_0} \neq 0$. If $u \in F \cap J_{\lambda_0}$ and $u \neq 0$, then as above, there is an element v in F such that $uv \neq vu$, and then $F \cap J_{\lambda_0}$ contains the free subring generated by $\{u, uv\}$. This means that $F \cap J_\lambda = 0$ for each λ, which is impossible, so R has no free subrings of rank $\aleph_0$, i.e. R is in $\mathcal{R}_{\aleph_0} = \mathcal{R}_2$. ▌

Proposition 2.5. *In any variety of rings, $\mathcal{R}_2$ is the class of rings A such that for every $a, b \in A$ there is a non-zero polynomial $\alpha = \alpha(x, y)$ in the free ring F_2 on $\{x, y\}$ for which $\alpha(a, b) = 0$.*

Proof. (Cf. Example 1.8.) For $a, b \in A$, let

$$I(a, b) = \{\beta \in F_2 | \beta(a, b) = 0\}.$$

There is a surjective homomorphism $F_2 \to \langle a, b \rangle$ (the subring generated by a and b) given by $\gamma(x, y) \mapsto \gamma(a, b)$. If $A \in \mathcal{R}_2$ then $\langle a, b \rangle$ is not isomorphic to F_2, so $I(a, b) \neq 0$. Conversely, if $I(a, b) \neq 0$ for every a, b, then no $\{a, b\}$ can be a free generating set, so A is in $\mathcal{R}_2$. ▌

Since free alternative rings on two generators are associative, by invoking the transfer theorem (Theorem 3.1) of 6, we get

Corollary 2.6. *For alternative rings, $\mathcal{R}_2$ is a radical class.* ▌

Our final result gives some further information about alternative rings.

Theorem 2.7. *For alternative rings, $\mathcal{R}_1 \subsetneqq \mathcal{R}_2$; $\mathcal{R}_1$ and $\mathcal{R}_2$ are radical classes, while $\mathcal{R}_4, \mathcal{R}_5, \ldots, \mathcal{R}_{\aleph_0}$ are not. In particular, $\mathcal{R}_{\aleph_0} \neq \mathcal{R}_2$.*

Proof. Proposition 1.3 and Corollary 2.6 take care of $\mathcal{R}_1$ and $\mathcal{R}_2$ ($\mathcal{R}_2 \neq \mathcal{R}_1$ as in the associative case). Humm and Kleinfeld [10] have shown that free alternative rings of rank ≥ 4 contain nilpotent ideals. Subsequently Shestakov [16] has shown that for such free rings F the Jacobson radical $J(F)$ is nil and $F/J(F)$ has no nilpotent elements. Thus $J(F)$ has no free subrings, while any free subring of $F/J(F)$ must have rank < 3, and therefore $F \in \mathcal{R}_4 \subseteq \mathcal{R}_{\aleph_0}$. Taking $4, 5, \ldots, \aleph_0$ as the rank of F, we see that $\mathcal{R}_4, \mathcal{R}_5, \ldots, \mathcal{R}_{\aleph_0}$ are not closed under extensions. ▌

There is a widely-held view (supported by a great deal of evidence) that radical theory is "the same" for alternative as for associative rings. Theorem 2.7 provides a marginal counterexample.

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INVARIANCE OF MULTIPLICITY (VIELFALT) IN WALSH FOURIER ANALYSIS

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1. Introduction. Originally the concept of multiplicity (Vielfalt) was used to describe the fact that many of the Walsh Fourier coefficients of polynomials are equal to zero. More precisely, P. WEISS [5] showed that polynomials of degree n have multiplicity n in every Walsh Fourier system defined by a bounded sequence. The concept of multiplicity itself is derived from the construction of Walsh functions by products of Rademacher functions.

In H. GRUBER [3], [4] the background of these facts was analized further and it was proved that for functions having multiplicity n certain n'th differences are constant. With this result it was not only possible to formulate the main result of P. WEISS [5] in a more general manner, but also to discuss the question whether there are other functions in $L^1[0, 1]$, besides the polynomials, having the same multiplicity in all Walsh Fourier systems.

By solving these difference equations simultaneously for all Walsh Fourier systems we show in this paper that the polynomials are, in fact, the only functions whose multiplicities are independent of the Walsh Fourier system. In fact, we do not need all Walsh Fourier systems for our proof.

A very detailed treatment of the discussions of this paper can be found in H. GRUBER [3].

2. Preliminary definitions, the theorem on differences. Let

$$F := (\alpha(0),\ \alpha(1),\ \ldots),$$
(1)
$$G := (\beta(0),\ \beta(1),\ \ldots),\ \ldots$$

be sequences of positive integers greater than one. Then, to F, we have the products

(2) $$A_F(0) := 1, \quad A_F(k) := \alpha(0)\,\alpha(1)\ldots\alpha(k-1), \quad k = 1, 2, \ldots,$$

and a system Ψ_F of (generalized) Walsh functions $\psi_{F,m}$ and of (generalized) Rademacher functions $\varphi_{F,n}$, $m, n = 0, 1, \ldots,$ as defined in H. GRUBER [4].

Additionally we construct the subsequences

$$F_t := (\alpha(t),\ \alpha(t+1),\ \ldots), \quad t = 0, 1, \ldots, \tag{3}$$

so that

$$A_{F_t}(s) = \frac{A_F(s+t)}{A_F(t)}, \quad s, t = 0, 1, \ldots . \tag{4}$$

To every subsequence of a sequence we are able to define a corresponding Walsh Fourier system.

For the multiplicity $V_F(f)$ of a function $f \in L^1[0, 1]$ we also use the same definition as in H. GRUBER [4], but we emphasize that, in general, the multiplicity of f depends on the Walsh Fourier system.

To formulate the theorem on differences of H. GRUBER [4], we use a slight (and trivial) modification of the definition of the n-tuples Q_n^ν, namely

$$\begin{gathered} Q_n^\nu := \{Q \mid Q = (q_1, q_2, \ldots, q_\nu) \in \mathbf{N}^\nu, \quad 0 < q_1 < q_2 < \ldots < q_\nu = n\}, \\ \nu = 1, 2, \ldots, n \geq \nu \end{gathered} \tag{5}$$

and

$$Q_n^0 := 0, \quad n = 0, 1, \ldots .$$

Furthermore, to every n-tuple $Q \in Q_n^\nu$, $\nu = 0, 1, \ldots, n \geq \nu$, and F we define a set of n-tuples

$$\begin{gathered} P_F(Q) := \{P \mid P = (p_1, p_2, \ldots, p_\nu) \in \mathbf{N}^\nu, \quad 1 \leq p_s < \alpha(q_s - 1)\}, \\ \nu = 1, 2, \ldots, \\ P_F(Q) := \emptyset \quad \text{for} \quad \nu = 0. \end{gathered} \tag{6}$$

Now, with these preliminaries, we can state (H. GRUBER [4])

THEOREM 1. *For every function $f \in L^1[0, 1]$, for every sequence F and for every ν, $\nu = 0, 1, \ldots, f$ has maximal multiplicity ν with respect to the system Ψ_F if and only if*

$$\forall\, n, \quad n \geq \nu, \quad \forall\, Q \in Q_n^\nu, \quad \forall\, P \in P_F(Q)$$

$$\Delta^\nu\left(\frac{p_1}{A_F(q_1)}, \frac{p_2}{A_F(q_2)}, \ldots, \frac{p_\nu}{A_F(q_\nu)}\right) f(x) = k(n, Q, P, f)$$

$$\textit{for almost all } x \in \left[0, \frac{1}{A_F(n)}\right], \quad k(n, Q, P, f) \in \mathbf{C}, ^1$$

i.e. the differences do not depend on x.

[1] We use $\Delta^r(a_1, \ldots, a_r)\, g(x)$ instead of the more usual notation $\underset{a_r}{\Delta} \ldots \underset{a_1}{\Delta}\, g(x)$ for an r-fold difference over g.

3. Invariance of multiplicity. 3.1 Functions having the same multiplicity for all Walsh Fourier systems: We shall now formulate our theorem and, after establishing some auxiliary results in 3.2, give the proof in 3.3.

THEOREM 2. *The only functions in $L^1[0,1]$ which have the same multiplicity n for all Walsh Fourier systems are the polynomials of degree n, $n\in\mathbf{N}$.*[2]

3.2 Some auxiliary results: Firstly, we give a short lemma for complex valued functions which are almost everywhere periodic up to constants k_1 and k_2 with respect to the periods ϱ_1 and ϱ_2 in the intervals $[a_1, b_1)$, $[a_2, b_2)$ respectively. We conclude that such a function is also periodic in an additional interval. More precisely:

LEMMA 1. For

$$a_1 < b_1,\ \ a_2 < b_2,\ \ |\varrho_1| > \varrho_2 > 0$$

$$\{\Delta^1(\varrho_1)f(x) = k_1, \quad x\in[a_1, b_1) \quad \text{and} \quad \Delta^1(\varrho_2)f(x) = k_2, \quad x\in[a_2, b_2)\}\Rightarrow$$
$$\Rightarrow \Delta^1(\varrho_2)f(x) = k_2, \quad x\in[\max\{a_1, a_2\}+\varrho_1, \min\{b_2, b_1-\varrho_2\}+\varrho_1).$$

We use this lemma to show that the difference equations of Theorem 1 for multiplicity 1 hold also on some additional intervals. For this purpose we establish Lemma 2 below, which we state in a slightly more general manner for later applications.

LEMMA 2. Let F be a sequence as in (1), $f\in L^1[0,1]$ and δ a positive real number. Then we have

$$\left\{\forall n\geqq 1,\ \Delta^1\left(\frac{p}{A_F(n)}\right)f(x) = k(n,p,f), \quad 1\leqq p<\alpha(n-1),\right.$$
$$\left.x\in\left[0,\ \frac{1}{A_F(n)}-\delta\right]\right\}\Rightarrow$$
$$\Rightarrow\left\{\forall n\geqq 1,\ \Delta^1\left(\frac{p}{A_F(n)}\right)f(x) = k(n,p,f), \quad 1\leqq p<\alpha(n-1),\right.$$
$$\left.x\in\left[\frac{t}{A_F(n-1)},\ \frac{t}{A_F(n-1)}+\frac{1}{A_F(n)}-\delta\right],\quad t=0,1,\ldots,A_F(n-1)-1\right\}.$$

PROOF: For every t there exists a representation

$$t = p_1\alpha(1)\ldots\alpha(n-1)+p_2\alpha(2)\ldots\alpha(n-1)+\ldots+p_n, \tag{7}$$
$$0\leqq p_i<\alpha(i-1), \quad i=1,2,\ldots,n-1.$$

The case $t=0$ is trivial. Let

$$t = p_i\alpha(i)\ldots\alpha(n-1)+p_{i+1}\alpha(i+1)\ldots\alpha(n-1)+\ldots+p_n,$$
$$i\in\{1,2,\ldots,n\}, \quad p_i\neq 0,$$

[2] Since we shall deal with L^1 functions, this and following statements are to be understood in the L^1-sense i.e. almost everywhere.

and

$$t' = t - p_i\,\alpha(i) \ldots \alpha(n-1)\,.$$

We suppose that the result has been proved for t' and show that it then holds for t. But with Lemma 1 this follows immediately from the induction hypothesis and the fact that there exists an i with

$$\Delta^1\left(\frac{p_i\,\alpha(i)\ldots\alpha(n-1)}{A_F(n-1)}\right)f(x) = \Delta^1\left(\frac{p_i}{A_F(i)}\right)f(x) = k(i,p,f)\,, \tag{8}$$

$$0 < p_i < \alpha(i-1),\ x\in\left[0, \frac{1}{A_F(i)} - \delta\right].$$

Applying Lemma 2 to the special case of multiplicity 1 in Theorem 1, we get

COMMENT 1. A function $f\in L^1[0,1]$ has multiplicity 1 with respect to the Walsh Fourier system Ψ_F if and only if

$$\forall\, n,\ n = 1, 2, \ldots, \quad \forall\, p,\ 1 \leqq p < \alpha(n-1)$$

$$\Delta^1\left(\frac{p}{A_F(n)}\right)f(x) = k(n,p,f)\,,$$

$$x\in\left[\frac{t}{A_F(n-1)}, \frac{t}{A_F(n-1)} + \frac{1}{A_F(n)}\right], \quad t = 0, 1, \ldots, A_F(n-1)-1\,,$$

$$k(n,p,f)\in\mathbf{C}\,.$$

In Comment 1 we have simplified the representation of the difference equations of Theorem 1 in an appropriate manner.

In the proof of Theorem 2 we use very special systems of Walsh functions and combine their specific properties to resolve our question. Thus we need an additional lemma to expand the domains of validity of difference conditions:

LEMMA 3. For $f\in L^1\,[0,1]$, for all n, $n = 1, 2, \ldots$ and for $0 \leqq \delta < \min\left\{\frac{1}{48}, \frac{1}{3\cdot 2^n}\right\}$

$$\left\{\Delta^1\left(\frac{1}{2^m}\right)f(x) = k_{1,m}, \quad x\in\left[\frac{t}{2^{m-1}}, \frac{t}{2^{m-1}} + \frac{1}{2^m} - \delta\right],\right.$$

$$\left. k_{1,m}\in\mathbf{C}, \quad t = 0, 1, \ldots, 2^{m-1}-1, \quad m = 1, 2, \ldots, \max\{4, n\}\right.$$

and

$$\left.\Delta^1\left(\frac{l}{3}\right)f(x) = k_{2,l}, \quad x\in\left[0, \frac{1}{3} - \delta\right], \quad k_{2,l}\in\mathbf{C}, \quad l = 1,2\right\} \Rightarrow$$

$$\left\{\Delta^1\left(\frac{1}{2^n}\right)f(x) = k_{1,0}\cdot\frac{1}{2^n},\ x\in\left[0, 1 - \frac{1}{2^n} - \delta\right], \quad k_{1,0} = 2k_{1,1}\right\}.$$

PROOF: First let $n = 4$. Then we have the conditions

$$(9) \quad \Delta^1\left(\frac{1}{2^4}\right)f(x) = k_{1,4}, \quad x \in \left[\frac{t}{8}, \frac{t}{8} + \frac{1}{16} - \delta\right), \quad t = 0, 1, \ldots, 7,$$

and with a slight modification

$$(10) \qquad \Delta^1\left(-\frac{1}{3}\right)f(x) = -k_{2,1}, \quad x \in \left[\frac{1}{3}, \frac{2}{3} - \delta\right)$$

and

$$(11) \qquad \Delta^1\left(-\frac{2}{3}\right)f(x) = -k_{2,2}, \quad x \in \left[\frac{2}{3}, 1 - \delta\right).$$

With Lemma 1 we get from (9) and (10) that the difference conditions (9) are also valid in the intervals

$$\left[\frac{6}{16} - \frac{1}{3}, \frac{7}{16} - \frac{1}{3} - \delta\right) \quad \text{and} \quad \left[\frac{8}{16} - \frac{1}{3}, \frac{9}{16} - \frac{1}{3} - \delta\right),$$

and, similarly, by (11) instead of (10), that they are valid in

$$\left[\frac{12}{16} - \frac{2}{3}, \frac{13}{16} - \frac{2}{3} - \delta\right) \quad \text{and} \quad \left[\frac{14}{16} - \frac{2}{3}, \frac{15}{16} - \frac{2}{3} - \delta\right).$$

Together with the original domain of validity of (9) these intervals cover $\left[0, \frac{5}{16}\right)$. We may now use (10) and (11) again and the hypotheses of the lemma for $m = 1,2$ to prove the validity of (9) in the entire domain. If we finally put

$$(12) \qquad 4\,k_{1,4} =: k_{1,0},$$

we see that the conclusion of Lemma 3 holds for $n = 4$. The case $n < 4$ follows then by simple addition.

Now our induction hypothesis is

$$(13) \qquad f\left(x + \frac{1}{2^n}\right) - f(x) = \frac{1}{2^n} k_{1,0}, \quad x \in \left[0, 1 - \frac{1}{2^n} - \delta\right),$$

$n \geqq 4$, and we have

$$(14) \qquad f\left(x + \frac{1}{2^{n+1}}\right) - f(x) = k_{1,n+1}, \quad x \in \left[\frac{t}{2^n}, \frac{t}{2^n} + \frac{1}{2^{n+1}} - \delta\right),$$

$t = 0, 1, \ldots, 2^n - 1$. For any t we subtract now (14) from (13) and get

$$(15) \qquad f\left(x + \frac{1}{2^n}\right) - f\left(x + \frac{1}{2^{n+1}}\right) = \frac{1}{2^n} k_{1,0} - k_{1,n+1},$$

$$x \in \left[\frac{t}{2^n}, \frac{t}{2^n} + \frac{1}{2^{n+1}} - \delta\right],$$

which may also be written as

$$f\left(x + \frac{1}{2^{n+1}}\right) - f(x) = \frac{1}{2^n} k_{1,0} - k_{1,n+1}, \tag{16}$$

$$x \in \left[\frac{t}{2^n} + \frac{1}{2^{n+1}}, \frac{t+1}{2^n} - \delta\right].$$

By shifting the domains of validity as in Lemma 1 with (10) and (11), one sees that there are non-trivial intervals, in which the conditions of (14) and (16) both hold. We conclude that

$$k_{1,n+1} = \frac{1}{2^n} k_{1,0} - k_{1,n+1}. \tag{17}$$

Now, by careful use of Lemma 1, we can complete the proof.

3.3 Proof of Theorem 2. The fact that a polynomial of degree n has multiplicity n has already been mentioned; see H. Gruber [4].

Furthermore, it is trivial that the constants are the only functions having multiplicity zero in any Walsh Fourier system.

To treat the case of multiplicity one we choose the sequences $F = (2, 2, 2, \ldots)$ and $G = (3, 2, 2, \ldots)$, and postulate that $f \in L^1\,[0, 1]$ has multiplicity one with respect to the Walsh Fourier systems defined by these two sequences. By Theorem 1 respectively Comment 1 it follows that for f the hypotheses of Lemma 3 with $\delta = 0$ hold, and therefore:

$$f\left(x + \frac{1}{2^n}\right) - f(x) = k \cdot \frac{1}{2^n}, \quad x \in \left[0, 1 - \frac{1}{2^n}\right), \quad k \in \mathbf{C}, \quad n = 1, 2, \ldots. \tag{18}$$

By addition of a finite number of equations (18) we get

$$f(x + \Delta x) - f(x) = k \cdot \Delta x, \tag{19}$$

for $x \in [0,1)$, Δx a positive dyadic rational with $x + \Delta x \in [0,1)$.

Hence for

$$f^*(x) := f(x) - k \cdot x \tag{20}$$

we have the equation

$$f^*(x + \Delta x) - f^*(x) = 0. \tag{21}$$

By Aczél [1], respectively the literature cited there, it follows that

$$f^* \equiv c, \quad c \in \mathbf{C}. \tag{22}$$

For a thorough discussion, see also H. Gruber [3]. This establishes our result.

Now let $f \in L^1[0,1]$ be of multiplicity $\nu+1$ in every Walsh Fourier system. By Theorem 1 we have:

For any sequence F

$$\forall n, \quad n \geqq \nu+1, \quad \forall Q \in Q_n^{\nu+1}, \quad \forall P \in P_F(Q)$$

$$(23)\qquad \Delta^\nu\left(\frac{p_2}{A_F(q_2)}, \frac{p_3}{A_F(q_3)}, \ldots, \frac{p_{\nu+1}}{A_F(q_{\nu+1})}\right)\left(\Delta^1\left(\frac{p_1}{A_F(q_1)}\right)f(x)\right) =$$
$$= k(n,Q,P,f), \quad x \in \left[0, \frac{1}{A_F(n)}\right), \quad k(n,Q,P,f) \in \mathbf{C}.$$

Here we have used the commutativity of difference operations to modify the representation of the theorem. We now substitute in (23) the subsequences F_r, $r = 1, 2, \ldots$, of F as described in (3) to get:

For any sequence F, for all $r = 1, 2, \ldots$ and for all $1 \leqq s < \alpha(r-1)$

$$\forall m \geqq \nu, \quad \forall Q \in Q_m^\nu, \quad \forall P \in P_{F_r}(Q)$$

$$(24)\qquad \Delta^\nu\left(\frac{p_2}{A_F(r)\,A_{F_r}(q_2)}, \ldots, \frac{p_\nu}{A_F(r)\,A_{F_r}(q_\nu)}\right)\left(\Delta^1\left(\frac{s}{A_F(r)}\right)f(x)\right) =$$
$$= k\left(m, Q, P, \Delta^1\left(\frac{s}{A_F(r)}\right)f\right), \quad x \in \left[0, \frac{1}{A_F(r)\,A_{F_r}(m)}\right).$$

These are now exactly the conditions that the functions $\Delta^1\left(\frac{s}{A_F(r)}\right)f$, in the interval $\left[0, \frac{1}{A_F(r)}\right)$, have multiplicity ν with respect to the Walsh Fourier systems defined by the sequences F_r.

As F ranges over all sequences, so does F_r. Therefore, our induction hypothesis is that these first differences of f are polynomials of degree n in x. By Aczél [2] it follows that:

For all sequences F, for all $r = 1, 2, \ldots$ and for all $1 \leqq s < \alpha(r-1)$

$$(25)\qquad \Delta^\nu(h_1, h_2, \ldots, h_\nu)\left(\Delta^1\left(\frac{s}{A_F(r)}\right)f(x)\right) = k(s,r,f)\cdot\prod_{i=1}^{\nu} h_i\,,$$

in

$$0 \leqq x < \frac{1}{A_F(r)} - \sum_{i=1}^{\nu} h_i \quad (h_i \geqq 0).$$

Once more we choose special sequences F; firstly we take those with the first r members equal to two ($r = 1, 2, \ldots$), and one whose first term is three. Then, from (25) we get

$$(26)\qquad \Delta^\nu(h_1, \ldots, h_\nu)\left(\Delta^1\left(\frac{1}{2^r}\right)f(x)\right) = k(1,r,f)\cdot\prod_{i=1}^{\nu} h_i\,,$$

$$0 \leq x < \frac{1}{2^r} - \sum_{i=1}^{\nu} h_i, \quad r = 1, 2, \ldots$$

and

(27) $$\Delta^{\nu}(h_1, \ldots, h_\nu)\left[\Delta^1\binom{p}{3} f(x)\right] = k'(p, 1, f) \prod_{i=1}^{\nu} h_i,$$

$$0 \leq x < \frac{1}{3} - \sum_{i=1}^{\nu} h_i, \quad p = 1, 2.$$

Let

(28) $$f_{\mathbf{h}} := \Delta^{\nu}(h_1, h_2, \ldots, h_\nu) f.$$

By interchanging the difference operations in (26) and (27), using (28) and applying Lemma 2 followed by Lemma 3, we get

(29) $$\Delta^1\left(\frac{1}{2^n}\right) f_{\mathbf{h}}(x) = \frac{1}{2^n} k(f) \cdot \prod_{i=1}^{\nu} h_i,$$

$$x \in \left[0, 1 - \frac{1}{2^n} - \sum_{i=1}^{\nu} h_i\right), \quad n = 1, 2, \ldots,$$

for small h_i.

$f_{\mathbf{h}}$ is an integrable function, and so we can conclude as above that $f_{\mathbf{h}}$ is a polynomial of degree one in x, i.e.

(30) $$f_{\mathbf{h}}(x) = k(f) \cdot x \cdot \prod_{i=1}^{\nu} h_i + k'(f_{\mathbf{h}}), \quad x \in \left[0, 1 - \sum_{i=1}^{\nu} h_i\right),$$

or

(31) $$\Delta^{\nu+1}(h_1, \ldots, h_{\nu+1}) f(x) = k(f) \prod_{i=1}^{\nu+1} h_i, \quad x \in \left[0, 1 - \sum_{i=1}^{\nu+1} h_i\right).$$

The most general integrable solutions of (31) are the polynomials of degree $\nu+1$, as may bee seen by successive separation of powers of x, for example. See also Aczél [2].

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AN APPLICATION OF THE METHOD OF MONOTONE OPERATORS TO NON-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS IN UNBOUNDED DOMAINS

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The purpose of this paper is to deal with the non-linear elliptic problem

$$L(x, D)(u) \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x), \quad |\gamma| \leq m:$$

$$D^\omega u|_\Gamma = 0, \quad |\omega| \leq m-1$$

in an unbounded domain. In the first part of this paper it will be shown that under certain conditions the problem has a unique solution. Imposing some other conditions in the second part, it will be proved that the problem has at least one solution. An analogous problem has been treated for bounded domains in [1] and we shall give a generalization for unbounded domains with a similar treatment.

Notice that in [2], F. E. BROWDER considered a similar problem. In our work some conditions on the differential operator are more general than in [2]. Non-linear elliptic problems in bounded domains were considered in some works before e.g. in [3]–[9].

The notations and symbols in this paper were generally used in [1]. In solving this problem, the method of monotone operators will be used. As applications, two examples satisfying the main theorem of the first part and an example for the second part, will be given.

§. 1. Let G be an unbounded domain in the real n-dimensional Euclidean space $\mathbf{R}^n$. Consider the non-linear elliptic problem (in this paper we use the notation $A_\alpha(x, D^\gamma u)$ for the operator

$$A_\alpha\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, D^\gamma u, \ldots\right),$$

$|\gamma| \leq m$)

(1) $$L(x, D)(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x), \quad |\gamma| \leq m;$$

(2) $$D^{\omega} u|_{\Gamma} = 0, \quad |\omega| \le m-1$$

where

$$D^{\gamma} = \frac{\partial^{\gamma_1+\gamma_2+\ldots+\gamma_n}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \ldots \partial x_n^{\gamma_n}},$$

Γ is the boundary of G and $L(x, D)$ satisfies the following conditions:

(I) $A_{\alpha}(x, \xi_{\gamma})$ are defined for all $x \in G$ and for all $\xi_{\gamma} \in \mathbf{R}^n$: $A_{\alpha}(x, \xi_{\gamma})$ are continuous in x and ξ_{γ} and satisfy the estimation:

(3) $$|A_{\alpha}(x, \xi_{\gamma})| < K \Big(\sum_{|\gamma| \le m} |\xi_{\gamma}|^{p-1} + g(x, \xi_{\gamma}) \Big), \quad \forall\, |\alpha| \le m;$$

where $p > 1$, $K > 0$ are constans and g is a measurable function such that

(4) $$\|u\|^{p}_{W^m_{0,p}(G)} \le c_1 \quad \text{implies that} \quad \int_G [g(x, D^{\gamma} u)]^{p'} dx \le c_2,$$

c_1, c_2 are constans and p, p' are the usual conjugates with $\frac{1}{p} + \frac{1}{p'} = 1$.

Here $W^m_p(G)$ is the completion of the space $C^m(G)$ of continuous functions v defined on $\bar{G}$ with continuous derivates up to order m such that the norm

(5) $$\|v\|_{W^m_p(G)} = \sum_{|\alpha| \le m} \left[\int_G |D^{\alpha} v|^p dx \right]^{\frac{1}{p}}$$

is finite. $W^m_{0,p}(G)$ is[1] the completion of the space $C^m_0(G)$ of continuous functions defined on G with compact support contained in G and continuous derivatives up to order m and the norm on it is defined by (5).

(II) Condition of strong ellipticity:

For all $u, v \in W^m_{0,p}(G)$ the following inequality is fulfilled:

(6) $$\operatorname{Re} \langle L(x, D)(u) - L(x, D)(v), u - v \rangle_G \equiv$$
$$\equiv \sum_{|\alpha| \le m} \operatorname{Re} \langle A_{\alpha}(x, D^{\gamma} u) - A_{\alpha}(x, D^{\gamma} v), D^{\alpha}(u - v) \rangle_G \ge$$
$$\ge a_1 \|u - v\|^{p}_{W^m_{0,p}(G)},$$

where

(7) $$\langle L(x, D)(u), v \rangle_G = \sum_{|\alpha| \le m} \int_G A_{\alpha}(x, D^{\gamma} u) D^{\alpha} \bar{v}\, dx.$$

[1] Further — owing to typographical reasons — this expression has in index the following form: $W^m_{0,p}(G)$.

Let us mention that in the following examples condition (4) is fulfilled:

a) If $g(x, \xi_\gamma)$ is a function of x only where $g \in L^{p'}(G)$.

b) If $g(x, \xi_\gamma) = q(x) + h(\xi_\gamma)$ where $q \in L^{p'}(G)$ and $\int_G |h(D^\gamma u)|^{p'} dx \leq c_2$ provided that $u \in W^m_{0,p}(G)$, $\|u\|^p_{W^m_{0,p}(G)} \leq c_1$ e.g. $|h(\xi_\gamma)| \leq c\,|\xi_\gamma|^\varrho$ and $\varrho p' = p$.

c) If $g(x, \xi_\gamma) = q(x) \cdot h(\xi_\gamma)$ where $|h(\xi_\gamma)| \leq c|\xi_\gamma|^\varrho$ and $q(x) \in L_{\frac{pp'}{p-\varrho p'}}(G)$.

Let X be a reflexive separable Banach space, X^* its conjugate space, $A: X \to X^*$ a (non-linear) operator.

DEFINITION 1. A is said to be a monotone operator if it satisfies

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \geq 0 \quad \forall\, u_1, u_2 \in D(A). \tag{8}$$

DEFINITION 2. A is called a strictly monotone operator if besides (8) it satisfies the following condition:

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \equiv 0 \quad \text{iff} \quad u_1 \equiv u_2. \tag{9}$$

DEFINITION 3. A is said to be a coercive operator if it satisfies

$$\lim_{\substack{\|u\| \to +\infty \\ u \in D(A)}} \frac{\operatorname{Re}\langle Au, u \rangle}{\|u\|} = +\infty. \tag{10}$$

THEOREM 1. *If conditions* (I) *and* (II) *are fulfilled, then* (i) $L(x, D)(u)$ *defined by* (7) *is a conjugate linear continuous functional on* $W^m_{0,p}(G)$.

(ii) $L(x, D): W^m_{0,p}(G) \to W_{p'}{}^{-m}(G)$ *is bounded and continuous in finite dimension.*

(iii) $L(x, D)$ *is strictly monotone and coercive.*

PROOF. It is easy to check the conjugate linearity of $L(x, D)(u)$ and hence we show its continuity only.

From (7) applying Hölder's inequality we obtain

$$|\langle L(x, D)(u), v \rangle| \leq \sum_{|\alpha| \leq m} \left[\int_G A_\alpha(x, D^\gamma u)|^{p'} dx \right]^{\frac{1}{p'}} \left[\int_G |D^\alpha v|^p dx \right]^{\frac{1}{p}}. \tag{11}$$

Furthermore, from (3) and (4), we get

$$\left[\int_G |A_\alpha(x, D^\gamma u)|^{p'} dx \right]^{\frac{1}{p'}} \leq [c' \|u\|^p_{W^m_{0,p}(G)} + c'']^{\frac{1}{p'}} \tag{12}$$

provided that

$$\|u\|^p_{W^m_{0,p}(G)} \leq c_1.$$

since $u \in W^m_{0,p}(G)$ we have

$$\left[\int_G |A_\alpha(x, D^\gamma u)|^{p'}\,dx\right]^{\frac{1}{p'}} < \infty, \quad \forall\, |\alpha| \le m\,.$$

In view of (11), $L(x, D)(u)$ is a continuous functional.

Next, we prove that the operator $L(x, D)$ is a bounded operator, i.e. $L(x, D)$ maps any bounded set onto a bounded set. Let X_0 be a bounded subset of $W^m_{0,p}(G)$, i.e. suppose that for any $u \in X_0$, $\|u\| \le c$, where c is a constant. By definition

$$\|L(x, D)(u)\|_{W^{-m}_{p'}(G)} = \sup_{\|v\|_{W^m_{0,p}(G)} \le 1} |\langle L(x, D)(u), v\rangle|\,,$$

where $W^{-m}_{p'}(G)$ is the dual space of the space $W^m_{0,p}(G)$; therefore

$$\|L(x, D)(u)\|_{W^{-m}_{p'}(G)} \le \sup_{\|v\|_{W^m_{0,p}(G)} \le 1} \sum_{|\alpha| \le m} \int_G |A_\alpha(x, D^\gamma u)\, D^\alpha \bar{v}|\,dx \le$$

$$\le \sup_{\|v\|_{W^m_{0,p}(G)} \le 1} \sum_{|\alpha| \le m} \left[\int_G |A_\alpha(x, D^\gamma u)|^{p'}\,dx\right]^{\frac{1}{p'}} \left[\int_G |D^\alpha v|^p\,dx\right]^{\frac{1}{p}}.$$

Since $\|v\|_{W^m_{0,p}(G)} \le 1$, by use of (12) and $u \in X_0$, we conclude that

$$\|L(x, D)(u)\|_{W^{-m}_{p'}(G)} \le c^*\,,$$

where c^* is a constant.

To show that $L(x, D)$ is continuous in finite dimension, consider for all $u_1, u_2, \dots, u_n \in W^m_{0,p}(G)$, $w \in W^m_{0,p}(G)$, a sequence $\{c^{(k)}\}$ converging to $c^{(0)}$, where $c^{(k)} = (c_1^{(k)}, c_2^{(k)}, \dots, c_n^{(k)}) \in \mathbf{R}^n$ for $k = 0, 1, 2, \dots$. We have to prove that

$$\langle L\left(x, D^\gamma(c_1^{(k)} u_1 + c_2^{(k)} u_2 + \dots + c_n^{(k)} u_n)\right), w\rangle \to$$
$$\to \langle L\left(x, D^\gamma(c_1^{(0)} u_1 + c_2^{(0)} u_2 + \dots + c_n^{(0)} u_n)\right), w\rangle \quad \text{as} \quad k \to \infty\,.$$

Since $\lim\limits_{k\to\infty} (c_1^{(k)} u_1 + \dots + c_n^{(k)} u_n) = c_1^{(0)} u_1 + c_2^{(0)} u_2 + \dots + c_n^{(0)} u_n$ and A_α is continuous, we have at any fixed point that

$$A_\alpha\left(x, D^\gamma(c_1^{(k)} u_1 + \dots + c_n^{(k)} u_n)\right) D^\alpha w \to A_\alpha\left(x, D^\gamma(c_1^{(0)} u_1 + \dots + c_n^{(0)} u_n)\right) D^\alpha w$$
$$\text{as} \quad k \to \infty\,. \tag{13}$$

On the other hand, from condition (I), we have

$$|A_\alpha\left(x, D^\gamma(c_1^{(k)} u_1 + c_2^{(k)} u_2 + \dots + c_n^{(k)} u_n)\right) D^\alpha w| \le$$
$$\le K \sum_{|\gamma| \le m} \left(|c_1^{(k)} D^\gamma u_1 + \dots + c_n^{(k)} D^\gamma u_n|^{p-1} + g\left(x, D^\gamma(c_1^{(k)} u_1 + \dots + c_n^{(k)} u_n)\right)\right) \cdot$$
$$\cdot |D^\alpha w|\,.$$

Since $\{c^{(k)}\}\to c^{(0)}$, there exists a constant $l>1$ such that

$$|c_j^{(k)}|\le l \quad \text{for all} \quad k \quad \text{and} \quad j.$$

Thus

$$|A_\alpha(x, D^\gamma(c_1^{(k)}u_1+\ldots+c_n^{(k)}u_n))\,D^\alpha w|\le \tag{14}$$

$$\le Kl^{p-1}\Big(\sum_{|\gamma|\le m}[\,|D^\gamma u_1|+\ldots+|D^\gamma u_n|\,]^{p-1}+g\Big)\,|D^\alpha w|\,,$$

where the right hand side does not depend on k and it is integrable, because

$$\int_G Kl^{p-1}\sum_{|\alpha|\le m}\left(\sum_{|\gamma|\le m}[\,|D^\gamma u_1|+\ldots+|D^\gamma u_n|\,]^{p-1}+g\right)|D^\alpha w|\le$$

$$\le c'\left[c_3(\|u_1\|^p+\|u_2\|^p+\ldots+\|u_n\|^p+c_2)\right]^{\frac{1}{p'}}\|w\|_{W^m_{0,p}(G)}<\infty\,.$$

In view of (13), (14) from Lebesgue's theorem it follows that $L(x, D)$ is a continuous operator in finite dimension.

In order to show the monotonicity of $L(x, D)$, we begin with

$$\langle L(x, D)(u)-L(x, D)(v),\, u-v\rangle=$$

$$=\sum_{|\alpha|\le m}\int_G\{A_\alpha(x, D^\gamma u)\,D^\alpha(\overline{u}-\overline{v})-A_\alpha(x, D^\gamma v)\,D^\alpha(\overline{u}-\overline{v})\}\,dx\,.$$

From condition (II), we obtain

$$\mathrm{Re}\,\langle L(x, D)(u)-L(x, D)\,v,\, u-v\rangle\ge a_1\,\|u-v\|^p_{W^m_{0,p}(G)}\,; \tag{15}$$

and we are led to the strict monotonicity of $L(x, D)$.

The coercivity of $L(x, D)$ will follow from the following

LEMMA. Suppose that conditions (I) and (II) are fulfilled, then for all $u\in W^m_{0,p}(G)$, the following estimate holds:

$$Re\,\langle L(x, D)(u),\, u\rangle_G\equiv\sum_{|\alpha|\le m}Re\,\langle A_\alpha(x, D^\gamma u),\, D^\alpha u\rangle_G\ge \tag{16}$$

$$\ge a_2\,\|u\|^p_{W^m_{0,p}(G)}-K\,,$$

where $a_2>0$, $K\ge 0$ are constants.

PROOF OF THE LEMMA.

a) If $A_\alpha(x, 0)\equiv 0$ for all $|\alpha|\le m$, then the lemma follows from condition (II) by choosing $v(x)\equiv 0$ $(a_2=a_1,\ K=0)$.

b) Suppose that there exists at least one $A_\alpha(x, 0)\not\equiv 0$. Then condition (II) implies that

$$Re\sum_{|\alpha|\le m}\langle A_\alpha(x, D^\gamma u)-A_\alpha(x, 0),\, D^\alpha u\rangle_G\ge a_1\,\|u\|^p_{W^m_{0,p}(G)}\,. \tag{17}$$

From condition (I) we have $|A_\alpha(x, 0)| \leq Kg(x, 0)$. Thus

$$|\langle A_\alpha(x, 0), D^\alpha u\rangle_G \leq K\langle g(x, 0), |D^\alpha u|\rangle_G =$$
$$= K\int_G g(x, 0)\,|D^\alpha u|\,dx\,.$$

Put $g(x, 0) = g_1(x)$ and apply the following inequality:

$$\int_G g_1\,|D^\alpha u|\,dx \leq \frac{\varepsilon^p}{p}\,\|u\|^p_{W^m_{0,p}(G)} + \frac{1}{p'\,\varepsilon^{p'}}\,\|g_1\|^{p'}_{L^{p'}(G)}\,, \tag{18}$$

for $\varepsilon > 0$, $|\alpha| \leq m$.

Choosing $\varepsilon > 0$ sufficiently small, (17) implies (16), as $g_1 \in L^{p'}(G)$ [see (4)].

The proof of inequality (18) runs as follows: In the integral

$$\int_G g_1\,|D^\alpha u|\,dx\,,$$

put

$$v = D^\alpha u \in L^p(G)$$

and

$$\int_G g_1\,|v|\,dx = \int_G |\varepsilon v|\,\frac{g_1}{\varepsilon}\,dx\,.$$

As we have

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}\,, \qquad \frac{1}{p} + \frac{1}{p'} = 1\,,$$

we get

$$\int_G g_1\,|v|\,dx \leq \int_G \frac{|\varepsilon v|^p}{p}\,dx + \frac{1}{p'}\int_G \left|\frac{g_1}{\varepsilon}\right|^{p'}dx\,.$$

Thus we get (18) finally.

From (16) we can see that

$$\lim_{\|u\|_{W^m_{0,p}(G)}\to+\infty} \frac{\operatorname{Re}\langle L(x, D)(u), u\rangle}{\|u\|_{W^m_{0,p}(G)}} \geq \lim_{\|u\|\to+\infty} \frac{a_3\,\|u\|^p - K}{\|u\|} = +\infty\,.$$

Thus $L(x, D)$ is coercive.

Now we shall give an algebraic formulation of condition (II).

THEOREM 2. *Let $p \geq 2$ and let $A_\alpha(x, \xi_\gamma)$ be positive and continuously differentiable with respect to ξ_γ. For all complex ξ_γ, η_γ, suppose that*

$$Re \sum_{|\alpha|\leq m} A_{\alpha\beta}(x, \xi_\gamma)\,\eta_\alpha\,\bar\eta_\beta \geq a_4 \sum_{|\alpha|=m} |\xi_\alpha|^{p-2}\,|\eta_\alpha|^2\,, \tag{19}$$

where $a_4 > 0$ and $A_{\alpha\beta}(x, \xi_\gamma) = \dfrac{\partial}{\partial\xi_\beta} A_\alpha(x, \xi_\gamma)$. Then

$$Re\,\langle L(x, D)(u) - L(x, D)(v), u - v\rangle_G \geq a_0\,\|u - v\|^p_{W^m_{0,p}(G)}\,, \tag{20}$$

where $a_0 > 0$, which is the same as condition (II).

The proof of this theorem is exactly the same as that for the bounded case (see [1]).

THEOREM 3. *The problem* (1), (2) *has one and only one solution if the operator* $L(x, D)$ *satisfies* (i), (ii) *and* (iii) *in theorem* 1.

The proof of this theorem follows immediatly from the method of monotone operators (see [1]).

REMARK. We can generalize the problem (1), (2) for the case when the boundary conditions are not homogeneous:

$$D^{\omega} u|_{\Gamma} = f_{\omega}(x') \quad |\omega| \leq m-1, \quad x' \in \Gamma \tag{21}$$

where f_{ω} are such that there exist functions $f \in W_p^m(G)$ such that $D^{\omega} f|_{\Gamma} = f_{\omega}$. By putting $u-f$ instead of u in the solution of the problem (1), (2), we can get a solution of the problem (1), (21) such that $u-f \in W_{0,p}^m(G)$. We seek for the solution $u \in W_p^m(f)$, where $W_p^m(f)$ is the set of all functions $u = f+Z$ such that $Z \in W_{0,p}^m(G)$ and $f \in W_p^m(G)$. Thus it can be proved that problem (1), (21) has one and only one solution.

We shall illustrate our considerations by same examples.

EXAMPLE 1. Consider the differential operator

$$L(x, D)(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} \left(f(x) \, |D^{\alpha} u|^{p-2} D^{\alpha} u \right),$$

where f is a measurable function such that $c_2 \geq f(x) \geq c_1 > 0$, ($c_1$, c_2 are constants).

We show that $L(x, D)$ satisfies conditions (I) and (II) with $p \geq 2$.

$$A_{\alpha}(x, \xi_{\gamma}) = f(x) \, |\xi_{\alpha}|^{p-2} \xi_{\alpha},$$

$$|A_{\alpha}(x, \xi_{\gamma})| = |f(x)| \cdot |\xi_{\alpha}|^{p-1} \leq K \Big(\sum_{|\gamma| \leq m} |\xi_{\gamma}|^{p-1} \Big), \quad p > 1,$$

i.e. it satisfies condition (I) with $g(x, \xi_{\gamma}) = 0$.

On the other hand,

$$A_{\alpha\beta}(x, \xi_{\gamma}) = f(x) \frac{\partial}{\partial \xi_{\beta}} \left(|\xi_{\alpha}|^{p-2} \xi_{\alpha} \right).$$

Thus

$$A_{\alpha\alpha}(x, \xi_{\gamma}) = (p-1) f(x) \, |\xi_{\alpha}|^{p-2}, \quad A_{\alpha\beta} = 0 \quad \text{for all} \quad \alpha \neq \beta.$$

Hence

$$Re \sum_{|\alpha| \leq m} A_{\alpha\beta}(x, \xi_{\gamma}) \eta_{\alpha} \bar{\eta}_{\beta} = (p-1) f(x) \, Re \sum_{|\alpha| \leq m} |\xi_{\alpha}|^{p-2} |\eta_{\alpha}|^2 \geq$$

$$\geq a \sum_{|\alpha| = m} |\xi_{\alpha}|^{p-2} \cdot |\eta_{\alpha}|^2,$$

where $a > 0$, $p \geq 2$, i.e. it satisfies (19). Thus $L(x, D)$ satisfies conditions (I) and (II).

For $f(x)=1$, we obtain that

$$L(x,D)(u)=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(|D^{\alpha}u|^{p-2}D^{\alpha}u).$$

which also satisfies conditions (I) and (II).

Example 2. Consider the differential operator

$$L(x,D)(u)=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(f(x)\,|D^{\alpha}u|^{p-2}D^{\alpha}u)+$$
$$+\sum_{\substack{|\alpha|\leq m\\|\gamma|\leq m}}(-1)^{|\alpha|}D^{\alpha}(b_{\alpha\gamma}(x)\,D^{\gamma}u),$$

where f is a measurable function such that $c_2\geq f(x)\geq c_1>0$, (c_1, c_2 are constants); $b_{\alpha\gamma}(x)\in L^{p'}(G)$ having compact support contained in G.

We are going to prove that this operator satisfies conditions (I) and (II).

$$A_{\alpha}(x,\xi_{\gamma})=f(x)\,|\xi_{\alpha}|^{p-2}\,\xi_{\alpha}+\sum_{|\gamma|\leq m}b_{\alpha\gamma}\,\xi_{\gamma}.$$

We have

$$|A_{\alpha}(x,\xi_{\gamma})|=|f(x)|\,|\xi_{\alpha}|^{p-1}+\sum_{|\gamma|\leq m}|b_{\alpha\gamma}(x)|\cdot|\xi_{\gamma}|\leq$$
$$\leq c_2\left[\sum_{|\gamma|\leq m}|\xi_{\gamma}|^{p-1}+\frac{1}{c_2}\sum_{|\gamma|\leq m}|b_{\alpha\gamma}(x)|\cdot|\xi_{\gamma}|\right],\quad p\geq 2.$$

Thus, our operator satisfies condition (I) with

$$g(x,\xi_{\gamma})=\frac{1}{c_2}\sum_{|\gamma|\leq m}|b_{\alpha\gamma}(x)|\cdot|\xi_{\gamma}|.$$

Now we need to show that this function $g(x,\xi_{\gamma})$ satisfies (4).

Consider

$$\int_G|b_{\alpha\gamma}(x)|^{p'}\,|D^{\gamma}u|^{p'}\,dx=\int_{|D^{\gamma}u|\leq 1}|b_{\alpha\gamma}(x)|^{p'}\,|D^{\gamma}u|^{p'}\,dx+$$
$$+\int_{|D^{\gamma}u|>1}|b_{\alpha\gamma}(x)|^{p'}\,|D^{\gamma}u|^{p'}\,dx.$$

Since $p\geq 2$, hence $p'\leq 2$; and we have

$$|D^{\gamma}u|^{p'}\leq\begin{cases}1, & \text{if}\quad |D^{\gamma}u|\leq 1\\ |D^{\gamma}u|^{p}, & \text{if}\quad |D^{\gamma}u|>1.\end{cases}$$

Hence

$$\int_G|b_{\alpha\gamma}(x)|^{p'}\,|D^{\gamma}u|^{p'}\,dx\leq\int_G|b_{\alpha\gamma}(x)|^{p'}\,dx+\int_G|b_{\alpha\gamma}(x)|^{p'}\,|D^{\gamma}u|^{p}\,dx.$$

As each $b_{\alpha\gamma}(x)\in L^{p'}(G)$ and has a compact support, we have

$$|b_{\alpha\gamma}(x)|^{p'}\leq c\quad\text{and}\quad\int_G|b_{\alpha\gamma}(x)|^{p'}\,dx\leq c'.$$

Thus

$$\int_G |b_{\alpha\gamma}(x)|^{p'} |D^\gamma u|^{p'} dx \le c' + c\,\|u\|^p_{W^m_{0,p}(G)}.$$

If

$$\|u\|^p_{W^m_{0,p}(G)} \le c_1' \quad \text{then} \quad \int_G |b_{\alpha\gamma}(x)|^{p'} |D^\gamma u|^{p'} dx \le c_2',$$

i.e. condition (4) is fulfilled.

Further, we have

$$A_{\alpha\beta}(x, \xi_\gamma) = \begin{cases} (p-1) f(x)\,|\xi_\beta|^{p-2} + b_{\beta\beta}, & \alpha = \beta \\ b_{\alpha\beta} & \alpha \ne \beta\,. \end{cases}$$

Thus

$$Re \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} A_{\alpha\beta}(x, \xi_\gamma)\,\eta_\alpha \bar\eta_\beta \ge c_1 (p-1)\, Re \sum_{|\beta|\le m} |\xi_\beta|^{p-2}\, |\eta_\beta|^2 + Re \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} b_{\alpha\beta}\, \eta_\alpha \bar\eta_\beta\,.$$

This satisfies (19) whenever

$$Re \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} b_{\alpha\beta}\,\eta_\alpha \bar\eta_\beta \ge 0\,, \tag{22}$$

i.e. $L(x, D)$ satisfies condition (I) and (II) provided that (22) is fulfilled.

Also for $f(x) = 1$ we obtain that

$$L(x, D)(u) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (|D^\alpha u|^{p-2} D^\alpha u) + \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} (-1)^{|\alpha|} D^\alpha \big(b_{\alpha\gamma}(x) D^\gamma u\big)$$

satisfies conditions (I) and (II) if $b_{\alpha\gamma}(x) \in L^{p'}(G)$ having compact support contained in G and (22) is fulfilled.

§. 2. Consider the non-linear elliptic problem defined by (1), (2) in unbounded domain G such that $L(x, D)$ satisfies the conditions:

(III) $A_\alpha(x, \xi_\gamma)$ satisfies condition (3).

(IV) $L(x, D)$ satisfies condition (16) for all $u \in W^m_{0,p}(G)$.

(V) Condition of semibounded variation of $L(x, D)(u)$:

For all $u, v \in W^m_{0,p}(G)$ such that $\|u\|_{W^m_{0,p}(G)} \le R$, $\|v\|_{W^m_{0,p}(G)} \le R$ ($R > 0$ is an arbitrary constant) it is true that

$$Re\,\langle L(x, D)(u) - L(x, D)(v), u - v\rangle_G \ge -c\big(R, \|(u-v)\,\psi\|_{W^{m-1}_{0,p}(G)}\big) \tag{23}$$

where $\psi \in C_0^\infty(G_1)$, $G_1 \subset G$ is compact set and $c(R, \varrho)$ is a continuous function such that for all fixed R, ϱ

$$\lim_{\xi \to +0} \frac{c(R, \xi\varrho)}{\xi} = 0\,.$$

THEOREM 4. *If conditions* III, IV, V *are fulfilled, then for all* $h \in W^{-m}_{p'}(G)$, *the problem* (1), (2) *has at least one solution* $u(x) \in W^m_{0,p}(G)$.

In order to prove theorem 4, we shall use the following theorem which is similar to theorem 1. 2. of [1].

THEOREM 5. *Let* $X = W^m_{0,p}(G)$, *suppose that the operator* $A: X \to X^*$ *is coercive, bounded, continuous in finite dimension, and the operator* A *has the semibounded variation property, i.e. for every* $u, v \in X$ *such that* $\|u\|_X \le R$ *and* $\|v\|_X \le R$, *it is true that*

$$Re \langle A(u) - A(v), u - v \rangle_G \ge -c \left(R \, \|(u-v)\,\psi\|_{W^{m-1}_{0,p}(G)}\right), \tag{24}$$

where $\psi \in C^\infty_0(G_1)$ $G_1 \subset G$ *is compact set and* $c(R, \varrho)$ *is a continuous non-negative function such that for all fixed* $R \ge 0$, $\varrho \ge 0$

$$\lim_{\xi \to +0} \frac{c(R, \xi\varrho)}{\xi} = 0 .$$

Then for all $h \in X^*$ *the equation* $A(u) = h$ *has at least one solution.*

PROOF. Let v_i $(i = 1, 2, \ldots)$ be a system of linearly independent elements of X such that linear combinations of them are dense in X. Denote by X_k the set of linear combinations of $v_1, v_2, \ldots, v_k$. The approximate solution u_k of equation $A(u) = h$ is an element $u_k \in X_k$ such that

$$\langle A(u_k), v_j \rangle = \langle h, v_j \rangle, \quad j = 1, 2, \ldots, k$$

i.e.

$$\langle A(u_k), v \rangle = \langle h, v \rangle, \quad \forall v \in X_k . \tag{25}$$

The existence of u_k and the inequality $\|u_k\| \le C_1$ can be proved in the same way as in theorem 1. 1. of [1].

In order to prove that $A(u) = h$ we use the condition of semibounded variation property (24), where $R \ge C_1 \ge \|u_k\|_X > 0$.

Since u_k is bounded in $W^m_{0,p}(G)$, then $\{u_k\}$ has a subsequence $\{u'_k\}$ such that u'_k converges to u weakly in $W^m_{0,p}(G)$. Since u'_k is bounded in $W^m_{0,p}(G)$, then $\psi u'_k$ is bounded in $W^m_{0,p}(G_1)$. We know that the imbedding of the space $W^m_{0,p}(G_1)$ into the space $W^{m-1}_{0,p}(G_1)$ is compact, then there exists a subsequence $\{\psi u''_k\}$ of the sequence $\{\psi u'_k\}$ which converges in $W^{m-1}_{0,p}(G_1)$ to a function u^*. Since $W^m_{0,p}(G) \subset W^{m-1}_{0,p}(G)$, then $W^{-m}_{p'}(G) \supset W^{1-m}_{p'}(G)$. The fact that u'_k converges weakly in $W^m_{0,p}(G)$ to u, means that for all functional $F \in W^{-m}_{p'}(G)$, $F(u'_k)$ converges to $F(u)$ and from this follows that for all functional $F \in W^{1-m}_{p'}(G)$, $F(u'_k)$ converges to $F(u)$, i.e. u'_k converges weakly to u in the space $W^{m-1}_{0,p}(G)$. From the above discussion it follows that $\psi u'_k$ converges weakly to ψu in $W^{m-1}_{0,p}(G_1)$. As $\psi u''_k$ converges weakly to u^* in $W^{m-1}_{0,p}(G_1)$, it follows that $\psi u = u^*$. Then $\psi u''_k$ converges to ψu in $W^{m-1}_{0,p}(G_1)$.

From (24) and (25) we have:

$$Re\left[\langle h, u''_k \rangle - \langle A(u''_k), v \rangle - \langle A(v), u''_k - v \rangle\right] \ge$$
$$\ge -c \left(R, \|(u''_k - v)\,\psi\|_{W^{m-1}_{0,p}(G)}\right).$$

As $A(u_k'')$ is bounded in X^*, so there exists a subsequence $A(u_k''')$ which converges weakly in X^*. From (25) it is easy to see that the limit must be equal to h. Then we get as $k \to \infty$

$$Re\langle h - A(v), u - v\rangle \geq -c\left(R, \|(u-v)\,\varphi\|_{W^{m-1}_{0,p}(G)}\right).$$

If we write here $v = u - \xi w$, where $w \in X$, $\|w\|_X \leq R$, and $\xi \to +0$, then by use of the continuity of A in the finite dimension and the properties of the function $c(R, \varrho)$ we get:

$$Re\langle h - A(u), w\rangle \geq 0. \tag{26}$$

Since $R \geq c_1$ and $w \in X$ are arbitrary, inequality (26) is possible only if $A(u) = h$.

Proof of theorem 4. It is based on theorem 5. From condition (IV) it follows that $L(x, D)$ is coercive, from condition (III) it follows that $L(x, D)$ is bounded and continuous in finite dimension and condition (V) implies that $L(x, D)$ has the semibounded variation property, i.e. conditions (III), (IV), (V) of theorem 4 satisfy the conditions of theorem 5. Then problem (1), (2) has at least one solution $u \in W^m_{0,p}(G)$.

An algebraic formulation of condition (V).

Consider the operator

$$L(x, D)(u) := \sum_{|\alpha|=m} (-1)^m D^\alpha \left(E_\alpha(x, D^\beta u) + B_\alpha(x, D^\gamma u)\right) + \\ + \sum_{|\delta| \leq m-1} (-1)^{|\delta|} D^\delta T_\delta(x, D^\tau u), \tag{27}$$

$|\beta| = m$, $|\gamma| \leq m$, $|\tau| \leq m$, where $E_\alpha(x, \xi_\beta)$, $B_\alpha(x, \xi_\gamma)$, $T_\delta(x, \xi_\tau)$ are continuous functions and both $B_\alpha(x, \xi_\gamma)$, $T_\delta(x, \xi_\tau)$ are equal to zero if x is outside of G_2, where $\bar{G}_2 \subset G$ is compact.

Theorem 6. *Suppose that*

a) $|E_\alpha(x, \xi_\beta)| \leq K\left(\sum_{|\beta|=m} |\xi_\beta|^{p-1} + g(x, \xi_\beta)\right)$, $p \geq 2$; *and* $g(x, \xi_\beta)$ *satisfies condition* (4).

b) The operator

$$L_0(x, D)(u) = (-1)^m \sum_{|\alpha|=m} D^\alpha E_\alpha(x, D^\beta u)$$

is strongly elliptic.

c) $B_\alpha(x, \xi_\gamma)$, $T_\delta(x, \xi_\tau)$ *are differentiable with recept to* ξ_γ, ξ_τ *and*

$$|B_\alpha(x, \xi_\gamma)| \leq K f(x)\left(\sum_{|\gamma| \leq m} |\xi_\gamma|^q + 1\right),$$

$$|T_\delta(x, \xi_\tau)| \leq K\, l(x)\left(\sum_{|\tau| \leq m} |\xi_\tau|^q + 1\right),$$

$$|B_{\alpha\gamma}(x, \xi_\gamma| \equiv \left|\frac{\partial B_\alpha(x, \xi_\gamma)}{\partial \xi_\gamma}\right| \leq K f(x)\left(\sum_{|\gamma| \leq m} |\xi_\gamma|^{q-1} + 1\right),$$

$$|T_{\delta\tau}(x, \xi_\tau)| \equiv \left|\frac{\partial T_\delta(x, \xi_\tau)}{\partial \xi_\tau}\right| \leq K\, I(x) \Big(\sum_{|\tau| \leq m-1} |\xi_\tau|^{q-1} + 1\Big),$$

$1 \leq q < p-1$, *where* $f(x)$ *and* $I(x)$ *are measurable functions such that*

$$(28) \qquad \begin{cases} \displaystyle\int_G (f(x))^{\frac{p}{p-q-1}}\, dx < \infty\,, \\ \displaystyle\int_G (I(x))^{\frac{p}{p-q-1}}\, dx < \infty \end{cases}$$

Then the operator $L(x, D)$ *defined by* (27) *satisfies the conditions of theorem* 4.

PROOF. From *a), c)* we get condition (III).

As a consequence of strong ellipticity of $L_0(x, D)(u)$ and by use of the lemma in the first part we have

$$(29) \qquad Re\,\langle L_0(x, D)(u), u\rangle_G \equiv Re \sum_{|\alpha|=m} \langle E_\alpha(x, D^\beta u), D^\alpha u\rangle_G \geq$$

$$\geq a_1\, \|u\|^p_{W^m_{0,p}(G)} - K\,.$$

where $a_1 > 0$, $K > 0$ are constants. From this we get

$$Re\,\langle L(x, D)(u), u\rangle_G \geq a_1\, \|u\|^p_{W^m_{0,p}(G)} - K +$$

$$+ Re \sum_{|\alpha|=m} \langle B_\alpha(x, D^\gamma u), D^\alpha u\rangle_G + Re \sum_{|\delta| \leq m-1} \langle T_\delta(x, D^\tau u), D^\delta u\rangle_G\,.$$

Condition *c)* indicates that all terms in the sums are of powers of $D^\gamma u$ less than or equal to $q+1 < p$. From this, by use of Young's inequality with power $\dfrac{p}{q+1}$ and with small ε, we get

$$Re\,\langle L(x, D)(u), u\rangle_G \geq a'\, \|u\|^p_{W^m_{0,p}(G)} - K'\,,$$

where $a' > 0$, $K' > 0$ are constants, which proves condition (IV).

From condition *b)* we have

$$(30) \qquad Re\,\langle L(x, D)(u) - L(x, D)(v), u - v\rangle_G \geq a_0\, \|u - v\|^p_{W^m_{0,p}(G)} +$$

$$+ Re \Big[\sum_{|\alpha|=m} \langle B_\alpha(x, D^\gamma u) - B_\alpha(x, D^\gamma v), D^\alpha(u - v)\rangle_G +$$

$$+ \sum_{|\delta| \leq m-1} \langle T_\delta(x, D^\tau u) - T_\delta(x, D^\tau v), D^\delta(u - v)\rangle_G\Big] \equiv$$

$$\equiv a_0\, \|u - v\|^p_{W^m_{0,p}(G)} + Re\,[I_1 + I_2]\,.$$

Furthermore

$$I_1 = \sum_{\substack{|\alpha|=m \\ |\gamma|\leq m}} \left\langle \int_0^1 B_{\alpha\gamma}\big(x, D^\gamma v + \tau D^\gamma (u-v)\big)\, d\tau\, D^\gamma (u-v), D^\alpha (u-v) \right\rangle_G \equiv$$

$$\equiv \sum_{\substack{|\alpha|=m \\ |\gamma|\leq m}} \langle b_{\alpha\gamma}(u, v)\, D^\gamma (u-v), D^\alpha (u-v)\rangle_G .$$

Condition *c)* implies that $b_{\alpha\gamma}(u, v)$ increase as $(D^\gamma u)^{p-2}$, $(D^\gamma v)^{p-2}$ and then $b_{\alpha\gamma}(u, v) \in L_{\frac{p}{p-2}}(G)$ if $u \in W^m_{0,p}(G)$, $v \in W^m_{0,p}(G)$. Furthermore if $\|u\|_{W^m_{0,p}(G)} \leq R$ and $\|v\|_{W^m_{0,p}(G)} \leq R$, then $b_{\alpha\gamma}(u, v)$ is a bounded set of functions in $L_{\frac{p}{p-2}}(G)$, because we have

$$b_{\alpha\gamma}(u, v) = \int_0^1 B_{\alpha\gamma}\big(x, D^\gamma v + \tau D^\gamma (u-v)\big)\, d\tau$$

and

$$|B_{\alpha\gamma}(x, \xi_\gamma)| \leq K f(x) \Big(\sum_{|\gamma|\leq m-1} |\xi_\gamma|^{q-1} + 1 \Big) .$$

Therefore

$$|B_{\alpha\gamma}\big(x, D^\gamma v + \tau D^\gamma (u-v)\big)| \leq K f(x) \Big(\sum_{|\gamma|\leq m-1} |D^\gamma v + \tau D^\gamma (u-v)|^{q-1} + 1 \Big) .$$

Hence

$$|b_{\alpha\gamma}(u, v)| \leq K_1 f(x) \int_0^1 \Big[\sum_{|\gamma|\leq m-1} \big(|D^\gamma v|^{q-1} + \tau^{q-1} |D^\gamma (u-v)|^{q-1} + 1 \big) \Big] d\tau \leq$$

$$\leq K_1' f(x) \sum_{|\gamma|\leq m-1} \big(|D^\gamma v|^{q-1} + |D^\gamma u|^{q-1} + 1 \big) .$$

Thus

$$|b_{\alpha\gamma}(u, v)|^{\frac{p}{p-2}} \leq K_2 f(x)^{\frac{p}{p-2}} \sum_{|\gamma|\leq m-1} \Big(|D^\gamma v|^{\frac{p(q-1)}{p-2}} + |D^\gamma u|^{\frac{p(q-1)}{p-2}} + 1 \Big) .$$

Then due to Hölder's inequality

$$\{ b_{\alpha\gamma}(u, v) : \|u\|_{W^m_{0,p}(G)} \leq R,\ \|v\|_{W^m_{0,p}(G)} \leq R \}$$

is bounded in $L_{\frac{p}{p-2}}(G)$. Therefore, by using Young's inequality we obtain:

$$|\langle b_{\alpha\gamma}(u, v)\, D^\gamma (u-v), D^\alpha (u-v)\rangle_G| \leq$$

$$\leq \int_G \left| \frac{1}{\varepsilon} b_{\alpha\gamma}(u, v)\, D^\alpha (u-v) \right| \cdot |\varepsilon D^\alpha (u-v)|\, dx \leq$$

$$\leq \frac{p-1}{p} \int_{G_2} \left| \frac{1}{\varepsilon} b_{\alpha\gamma}(u, v)\, D^\gamma (u-v) \right|^{\frac{p}{p-2}} dx + \frac{1}{p} \int_G |\varepsilon D^\alpha (u-v)|^p\, dx .$$

(Here we have used the fact that $B_{\alpha\gamma}(x, \xi_\gamma)$ is equal to zero if x is outside of G_2.)

But we have

$$\frac{1}{p} \int_G \sum_{|\alpha|=m} \varepsilon^p |D^\alpha (u-v)|^p \, dx \le \varepsilon^p \|u-v\|^p_{W^m_{0,p}(G)} .$$

and

$$\frac{p-1}{p} \left(\frac{1}{\varepsilon}\right)^{\frac{p}{p-1}} \int_{G_2} |b_{\alpha\gamma}(u, v) D^\gamma (u-v)|^{\frac{p}{p-1}} dx \le$$

$$\le \frac{p-1}{p} \left(\frac{1}{\varepsilon}\right)^{\frac{p}{p-1}} \int_{G_2} |b_{\alpha\gamma}(u, v)|^{\frac{p}{p-1}} |D^\gamma (u-v)|^{\frac{p}{p-1}} dx .$$

Using Hölder's inequality we get

$$\int_{G_2} |b_{\alpha\gamma}(u, v)|^{\frac{p}{p-1}} |D^\gamma (u-v)|^{\frac{p}{p-1}} dx \le$$

$$\le C \left\{ \int_{G_2} |b_{\alpha\gamma}(u, v)|^{\frac{p}{p-1}} dx \right\}^{\frac{p-1}{p-2}} \left\{ \int_{G_2} |D^\gamma (u-v)|^p \, dx \right\}^{\frac{1}{p-1}} .$$

Finally we obtain

$$|I_1| \le K_3 \varepsilon^p \|u-v\|^p_{W^m_{0,p}(G)} + K(\varepsilon, R) \|u-v\|^{\frac{p}{p-1}}_{W^{m-1}_{0,p}(G_2)} .$$

In the same way, I_2 can be estimated.

At the end for sufficiently small $\varepsilon > 0$, we get from (30) that

$$Re \langle L(x, D)(u) - L(x, D)(v), u - v \rangle_G \ge$$

$$\ge a_0' \|u-v\|^p_{W^m_{0,p}(G)} - K'(\varepsilon, R) \|u-v\|^{\frac{p}{p-1}}_{W^{m-1}_{0,p}(G_2)} ,$$

$a_0' > 0$, which implies that condition (V) is fulfilled with a function $\psi \in C_0^\infty (G_1)$, where $G_2 \subset G_1$ and $\psi = 1$ in G_2 and the theorem is proved.

We shall mention an example which satisfies the conditions of theorem 6.

EXAMPLE 3. Consider the operator

$$L(x, D)(u) = (-1)^m \sum_{|\alpha|=m} D^\alpha \left[|D^\alpha u|^{p-2} D^\alpha u + f_\alpha(x) \left(|D^{\alpha-1} u|^{q-1} + D^\alpha u \right) \right] +$$

$$+ \sum_{|\beta| \le m-1} (-1)^{|\beta|} D^\beta \left(\prod_{|\omega| \le m} a_{\beta\omega}(x) |D^\omega u|^{p_{\beta\omega}} \right) , \tag{31}$$

where

$$1 \le q < p-1, \quad \sum_{|\omega| \le m-1} p_{\beta\omega} \le q, \quad p_{\beta\omega} \ge 0, \quad a_{\beta\omega}, \quad f_\alpha \in C(G)$$

have compact support and G is an unbounded domain in $\mathbf{R}^n$.

We shall show that $L(x, D)$ defined by (31), and under these conditions, satisfies the conditions *a)*, *b)* and *c)* of theorem 6.

It is clear that

$$E_\alpha(x, D^\beta u) = |D^\alpha u|^{p-2} D^\alpha u, \quad p > 2;$$

$$B_\alpha(x, D^\gamma u) = f_\alpha(x)\,|D^{\alpha-1} u|^q + D^\alpha u, \quad q-1 < p-2;$$

$$T_\delta(x, D^\tau u) = \prod_{|\omega| \leq m} a_{\delta\omega}(x)\,|D^\omega u|^{p_{\delta\omega}}.$$

For condition *a)*

$$|E_\alpha(x, \xi_\beta)| = |\xi_\alpha|^{p-1} \leq K \sum_{|\beta|=m} |\xi_\beta|^{p-1}, \quad p > 2$$

i.e. $E_\alpha(x, \xi_\beta)$ satisfies condition *a)* with $g(x, \xi_\beta) = 0$.

For condition *b)*, it is clear that the operator

$$L_0(x, D)(u) = (-1)^m \sum_{|\alpha|=m} D^\alpha (|D^\alpha u|^{p-2} D^\alpha u)$$

is strongly elliptic operator.

For condition *c)*, it is easy to see that $B_\alpha(x, \xi_\gamma)$ and $T_\delta(x, \xi_\tau)$ are differentiable with respect to ξ_γ and ξ_τ and

$$|B_\alpha(x, \xi_\gamma)| = |f_\alpha(x)|\,(|\xi_{\alpha-1}|^q + |\xi_\alpha|) \leq K f(x) \Big(\sum_{|\gamma| \leq m} |\xi_\gamma|^q + 1\Big),$$

where K is a constant, and f satisfies (28).

Also,

$$|T_\delta(x, \xi_\tau)| = \prod_{|\omega| \leq m} |a_{\delta\omega}(x)| \cdot |\xi_\omega|^{p_{\delta\omega}} \leq K\, I\,(x) \Big(\sum_{|\tau| \leq m} |\xi_\tau|^q + 1\Big),$$

where K is a constant, and I satisfies (28).

Recall that

$$B_\alpha(x, \xi_\gamma) = f_\alpha(x)\,(|\xi_{\alpha-1}|^q + \xi_\alpha),$$

then

$$B_{\alpha\alpha} = f_\alpha(x), \quad B_{\alpha,\alpha-1} = \begin{cases} f_\alpha(x)\, q\, \xi_{\alpha-1}^{q-1}, & \xi_{\alpha-1} > 0 \\ -f_\alpha(x)\, q\, (-\xi_{\alpha-1})^{q-1}, & \xi_{\alpha-1} < 0. \end{cases}$$

Thus

$$|B_{\alpha\gamma}(x, \xi_\gamma)| \leq |q|\ |f_\alpha(x)|\ |\xi_{\alpha-1}|^{q-1} + |f_\alpha(x)| \leq$$

$$\leq K f(x) \Big(\sum_{|\gamma| \leq m-1} |\xi_\gamma|^{q-1} + 1\Big).$$

Also for

$$T_\delta(x, \xi_\tau) = \prod_{|\omega| \leq m} a_{\delta\omega}(x)\,|\xi_\omega|^{p_{\delta\omega}}.$$

we have

$$T_{\delta\tau}(x, \xi_\tau) = \pm p_{\delta\tau} \Bigg[\prod_{\substack{|\omega| \leq m \\ |\omega| \neq \tau}} a_{\alpha\omega}(x)\,|\xi_\omega|^{p_{\delta\omega}}\Bigg] a_{\delta\tau}(x)\,|\xi_\tau|^{p_{\delta\tau}-1},$$

therefore,

$$|T_{\delta\tau}(x, \xi_\tau)| = |p_{\delta\tau}| \cdot |a_{\delta\tau}(x)| \cdot |\xi_\tau|^{p_{\delta\tau}-1} \cdot \left[\prod_{\substack{|\omega| \leq m \\ |\omega| \neq \tau}} |a_{\delta\omega}(x)| \cdot |\xi_\omega|^{p_{\delta\omega}}\right] \leq$$

$$\leq K\, I(x) \left(\sum_{|\tau| \leq m} |\xi_\tau|^{q-1} + 1\right),$$

provided that $p_{\delta\tau} \geq 1$ or $p_{\delta\tau} = 0$. i.e. the operator given by (31) is an example for theorem 6. provided that $p_{\delta\tau} \geq 1$ or $p_{\delta\tau} = 0$.

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ÜBER MITTELWERTE MULTIPLIKATIVER ZAHLENTHEORETISCHER FUNKTIONEN

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1. Bezeichnungen. Eine Funktion $f: \mathbf{N} \to \mathbf{C}$ heißt multiplikativ, wenn $f(nm) = f(n)f(m)$ für alle teilerfremden natürlichen Zahlen n und m gilt. f hat einen Mittelwert $M(f)$, wenn

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)$$

existiert und gleich $M(f)$ ist. Weiter sei

$$\mathcal{G} = \left\{ f: \mathbf{N} \to \mathbf{C},\ f \text{ multiplikativ},\ \sum_p \frac{|f(p)|^2}{p^2} < \infty \text{ und } \sum_p \sum_{k \geq 2} \frac{|f(p^k)|}{p^k} < \infty \right\},$$

wobei p die Folge der Primzahlen durchläuft,

$$\varphi_f(p, s) = \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}$$

und $\mathcal{G}^* = \{f \in \mathcal{G},\ \varphi_f(p, s) \neq 0$ für $\operatorname{Re} s \geq 1\}$. Zwei Funktionen f und g heißen benachbart, wenn

$$\sum_p \frac{|f(p) - g(p)|}{p} < \infty$$

ist. Benachbarte Funktionen wurden in [6] untersucht. Dort wurde u. a. gezeigt:

Seien $f \in \mathcal{G}$ und $g \in \mathcal{G}^*$ benachbart. Existiert dann $M(g)$, so existiert auch $M(f)$.

Dieses Argument werden wir im folgenden mehrfach verwenden.

2. Ergebnisse. Notwendige und hinreichende Bedingungen für die Existenz von Mittelwerten für gewisse Klassen multiplikativer Funktionen wurden schon von verschiedenen Autoren untersucht:

H. Delange zeigte für multiplikative Funktionen f mit $|f| \leq 1$ ([2]): *f hat dann und nur dann einen Mittelwert $M(f) \neq 0$, wenn*

$$\sum_p \frac{1-f(p)}{p} \tag{1}$$

konvergiert und für ein $\nu \geq 1$ ist $f(2^\nu) \neq -1$.

P. D. T. A. Elliott zeigte ([4]):

Die multiplikative Funktion f hat einen Mittelwert ungleich Null und es gilt

$$\limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^2 < \infty \tag{2}$$

dann und nur dann, wenn die Reihen

$$\sum_p \frac{1-f(p)}{p}, \quad \sum_p \frac{|1-f(p)|^2}{p} \quad \text{und} \quad \sum_p \sum_{k \geq 2} \frac{|f(p^k)|^2}{p^k}$$

konvergieren und für jede Primzahl p

$$\sum_{k=1}^{\infty} \frac{f(p^k)}{p^k} \neq -1 \tag{4}$$

ist.

Dieser Satz wurde von H. Daboussi verallgemeinert zu ([1]):

Sei $\lambda > 1$ und f eine multiplikative Funktion. Dann existiert $M(f) \neq 0$ und es gilt

$$\limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^\lambda < \infty \tag{5}$$

dann und nur dann, wenn die Reihen

$$\sum_p \frac{1-f(p)}{p}, \quad \sum_{|f(p)| \leq 3/2} \frac{|1-f(p)|^2}{p}, \tag{6}$$

$$\sum_{|f(p)| > 3/2} \frac{|f(p)|^\lambda}{p}$$

und

$$\sum_p \sum_{k \geq 2} \frac{|f(p^k)|^\lambda}{p^k} \tag{7}$$

konvergieren und für jede Primzahl p ist

$$\sum_{k=1}^{\infty} \frac{f(p^k)}{p^k} \neq -1\,. \tag{8}$$

Die Bedingung (6) dieses Satzes ist äquivalent zur Konvergenz von

$$\sum_p \frac{1-f(p)}{p} \quad \text{und} \quad \sum_p \frac{1-|f(p)|^\lambda}{p} \tag{9}$$

(vgl. Abschnitt 3). In dieser Formulierung wird die unwesentliche Konstante 3/2 vermieden.

Mit einem bekannten Ergebnis von E. WIRSING ([8]) würde die Existenz von $M(|f|^\lambda)$ sofort aus (7) und der Konvergenz von

$$\sum_p \frac{1-|f(p)|^\lambda}{p}$$

folgen, wenn noch $|f(p)|^\lambda = O(1)$ gelten würde. Mit dem oben zitierten Benachbarkeitsargument läßt sich diese letzte Bedingung etwa zu

$$\sum_{|f(p)|>3/2} \frac{|f(p)|^\lambda}{p} < \infty$$

abschwächen, so daß sich dieser Teil der Behauptung also auch mit dem Satz von Wirsing beweisen läßt.

Zum Beweis der Existenz von $M(f)$ benutzt ELLIOTT ähnliche Methoden, wie sie von G. HALÁSZ in [5] verwendet werden. Wir wollen nun zeigen, wie man die Existenz von $M(f)$ direkt aus dem dort bewiesenen Ergebnis herleiten kann. Es genügen dazu etwas schwächere Bedingungen als (6) und (7) (vgl. Abschnitt 4). Wir zeigen:

SATZ 1. *Sei $f:\mathbf{N}\to\mathbf{C}$ multiplikativ mit*

$$\sum_p \sum_{k\geq 2} \frac{|f(p^k)|}{p^k} < \infty\,, \tag{10}$$

$$\sum_p \frac{1-f(p)}{p} \tag{11}$$

konvergiert und es existiert eine Konstante $c<1$ mit

$$\sum_{\substack{p \\ |f(p)-1|>c}} \frac{|f(p)-1|}{p} < \infty\,. \tag{12}$$

Dann existiert $M(f)$.

Die Voraussetzungen dieses Satzes sind natürlich nicht mehr notwendig für die Existenz von $M(f)$. Dies erkennt man z. B. aus der folgenden Verallgemeinerung.

SATZ 2. *Sei $\Psi:\mathbf{N}\to\mathbf{C}$ periodisch mit Periode k, $f:\mathbf{N}\to\mathbf{C}$ multiplikativ und es gelte*

$$\frac{1}{\varphi(k)} \sum_{\substack{a \bmod k \\ (a,k)=1}} \Psi(a) = 1\,, \tag{13}$$

(14)
$$\sum_p \sum_{m\geq 2} \frac{|f(p^m)|}{p^m} < \infty ,$$

(15)
$$\sum_p \frac{f(p)-\Psi(p)}{p}$$

konvergiert und es existiert eine Konstante $c<1$ mit

(16)
$$\sum_{\substack{p \\ |f(p)-\Psi(p)|>c}} \frac{|f(p)-\Psi(p)|}{p} < \infty .$$

Dann existiert $M(f)$.

Auch hier sind die Voraussetzungen sicher nicht notwendig. Interessant wäre wohl auch die Frage, wieweit sich die etwas unschöne Voraussetzung $c<1$ abschwächen läßt.

3. Äquivalenz von (6) **und** (9). Hier benutzen wir ein Lemma aus der oben zitierten Arbeit [1] von Daboussi:

Lemma 1. *Sei $\alpha>1$ und $z\in\mathbf{C}$. Dann ist $|z|^\alpha-1+\alpha(1-\operatorname{Re} z)$ stets positiv und es gibt nur von α abhängige Konstanten $c_1,\ldots,c_4>0$ mit*

(17)
$$c_1|z-1|^2 \leq |z|^\alpha - 1 + \alpha(1-\operatorname{Re} z) \leq c_2|z-1|^2 \quad \text{für} \quad |z|\leq\frac{3}{2}$$

und

(18)
$$c_3|z|^\alpha < |z|^\alpha - 1 + \alpha(1-\operatorname{Re} z) \leq c_4|z|^\alpha \quad \text{für} \quad |z|>\frac{3}{2}.$$

Es gelte zunächst (6). Für $|f(p)|\leq\frac{3}{2}$ haben wir dann

$$0\leq \frac{|f(p)|^\lambda-1}{p}+\lambda\frac{1-\operatorname{Re} f(p)}{p} \ll \frac{|1-f(p)|^2}{p}.$$

Hieraus folgt

$$\sum_{|f(p)|\leq 3/2} \frac{|f(p)|^\lambda-1}{p}+\lambda\sum_{|f(p)|\leq 3/2}\frac{1-\operatorname{Re} f(p)}{p} \ll \sum_{|f(p)|\leq 3/2}\frac{|1-f(p)|^2}{p} < \infty .$$

Da $\sum_p \frac{1-\operatorname{Re} f(p)}{p}$ konvergiert und

$$\sum_{|f(p)|>3/2}\frac{1}{p}\leq\sum_{|f(p)|>3/2}\frac{|f(p)|}{p}\leq\sum_{|f(p)|>3/2}\frac{|f(p)|^\lambda}{p}<\infty$$

ist, konvergiert auch

$$\sum_{|f(p)|\leqq 3/2} \frac{1-\operatorname{Re} f(p)}{p}$$

und wir erhalten die Konvergenz von

$$\sum_{|f(p)|\leqq 3/2} \frac{|f(p)|^{\lambda}-1}{p}.$$

Da aber auch

$$\sum_{|f(p)|>3/2} \frac{1}{p} \quad \text{und} \quad \sum_{|f(p)|>3/2} \frac{|f(p)|^{\lambda}}{p}$$

konvergieren, muß auch

$$\sum_{p} \frac{|f(p)|^{\lambda}-1}{p}$$

konvergent sein.

Es gelte nun umgekehrt (9). Dann ist zunächst

$$\sum_{|f(p)|\leqq 3/2} \frac{|f(p)-1|^2}{p} \ll \sum_{|f(p)|\leqq 3/2} \left(\frac{|f(p)|^{\lambda}-1}{p} + \lambda \frac{1-\operatorname{Re} f(p)}{p}\right) \leq$$

$$\leq \sum_{p} \frac{|f(p)|^{\lambda}-1}{p} + \lambda \sum_{p} \frac{1-\operatorname{Re} f(p)}{p} < \infty .$$

Weiter haben wir auch

$$\sum_{|f(p)|>3/2} \frac{|f(p)|^{\lambda}}{p} \ll \sum_{|f(p)|>3/2} \left(\frac{|f(p)|^{\lambda}-1}{p} + \lambda \frac{1-\operatorname{Re} f(p)}{p}\right) \leq$$

$$\leq \sum_{p} \frac{|f(p)|^{\lambda}-1}{p} + \lambda \sum_{p} \frac{1-\operatorname{Re} f(p)}{p} < \infty .$$

4a. Aus $|f(p)|\leq 1$ und der Konvergenz von (1) **folgt** (12) (für jede Konstante c im Intervall $0<c<1$).

Wegen $|f(p)|\leq 1$ ist $1-\operatorname{Re} f(p)\geq 0$, also

$$\sum_{p} \frac{1-\operatorname{Re} f(p)}{p}$$

absolut konvergent. Hieraus folgt die Konvergenz von

$$\sum_{\operatorname{Re} f(p)<1-c/2} \frac{1}{p} \quad \text{und} \quad \sum_{|\operatorname{Im} f(p)|>c/2} \frac{1}{p}.$$

Nun ist (für $|f(p)|\leq 1$)

$$\sum_{\substack{p\\ |1-f(p)|>c}} \frac{|1-f(p)|}{p} \ll \sum_{\substack{p\\ |\operatorname{Im} f(p)|>c/2}} \frac{1}{p} + \sum_{\substack{p\\ |1-\operatorname{Re} f(p)|>c/2}} \frac{1}{p} < \infty .$$

4b. Aus (6) **folgt** (12) (für jedes c im Intervall $0<c<1$).
Für $|f(p)| \le 3/2$ und $|1-f(p)|>c$ ist

$$|1-f(p)| \le \frac{1}{c}\, |1-f(p)|^2 ,$$

also

$$\sum_{\substack{p \\ |1-f(p)|>c \\ |f(p)| \le 3/2}} \frac{|1-f(p)|}{p} \ll \sum_{|f(p)| \le 3/2} \frac{|1-f(p)|^2}{p} < \infty .$$

Für $|f(p)|>3/2$ ist

$$|1-f(p)| \ll |f(p)| \ll |f(p)|^{\lambda} ,$$

also

$$\sum_{\substack{p \\ |1-f(p)|>c \\ |f(p)|>3/2}} \frac{|1-f(p)|}{p} \ll \sum_{\substack{p \\ |f(p)|>3/2}} \frac{|f(p)|^{\lambda}}{p} < \infty .$$

4c. Aus (6) **folgt**

$$\sum_{\substack{p \\ |1-|f(p)|^{\lambda}|>c}} \frac{|1-|f(p)|^{\lambda}|}{p} < \infty ,$$

d. h. Bedingung (12) für die Funktion $|f|^{\lambda}$.

Ist $|f(p)|>3/2$, so ist $|1-|f(p)|^{\lambda}| \ll |f(p)|^{\lambda}$, ist $|f(p)| \le 3/2$ und $|1-|f(p)|^{\lambda}|>c$, so ist $|1-|f(p)|^{\lambda}| \ll 1 \ll |1-f(p)|^2$. Die Behauptung folgt deshalb genau wie in **4b.**

5. Beweis von Satz 1. Der Beweis von Satz 1 beruht hauptsächlich auf dem folgenden Satz von G. Halász:

Lemma 2. *Sei f multiplikativ und gelte*

(19) $$f(p)=O(1),$$

(20) $$\sum_p \sum_{m \ge 2} \frac{|f(p^m)|}{p^m} < \infty ,$$

(21) $$q_f(p, s) \neq 0$$

für $\mathrm{Re}\, s=1$ *für alle Primzahlen* p *und*

(22) $$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{C}{s-1} + O\left(\frac{|s|}{\sigma-1}\right)$$

gleichmäßig für $\mathrm{Re}\, s=\sigma>1$.

Dann existiert $M(f)$.

Wir wollen nun aus den Voraussetzungen von Satz 1 die Voraussetzungen von Lemma 2 herleiten. (Man vergleiche hierzu auch [7].) Zunächst zeigen wir

Lemma 3. *Sei h multiplikativ, $h(p^m)=0$ für $m\geq 2$, $\sum_p \frac{h(p)}{p}$ konvergent und es existiere eine Konstante $c<1$ mit $|h(p)|\leq c$. Dann gilt für $\sigma\to 1+$ gleichmäßig in $t=\operatorname{Im} s$ mit einer Konstanten D die Beziehung*

$$H(s):=\sum_{n=1}^{\infty}\frac{h(n)}{n^s}=D+O\left(\frac{|s-1|}{\sigma-1}\right).$$

Für $|\operatorname{Im} s|\geq 1/2$ gilt gleichmäßig in $t=\operatorname{Im} s$

$$H(s)=O\left(\frac{1}{(\sigma-1)^c}\right).$$

Beweis. Für $\operatorname{Re} s>1$ ist

$$H(s)=H_1(s)\exp\left(\sum_p\frac{h(p)}{p^s}\right)$$

mit

$$H_1(s)=\prod_p\left(1+\frac{h(p)}{p^s}\right)\exp\left(-\frac{h(p)}{p^s}\right).$$

Wegen $|h(p)|\leq c<1$ ist $H_1(s)$ holomorph in $\operatorname{Re} s>1/2$ und in $\operatorname{Re} s\geq 1$ gleichmäßig beschränkt.

Hieraus folgt

$$|H(s)|\leq O(1)\exp\left(\sum_p\frac{c}{p^\sigma}\right)=O\left(\frac{1}{(\sigma-1)^c}\right)$$

gleichmäßig in t für $\sigma\to 1+$. Damit ist die zweite Behauptung schon gezeigt.

Sei nun $\alpha+c<1$, $\alpha>0$. Dann gilt für $t\geq(\sigma-1)^\alpha$

$$\frac{1}{(\sigma-1)^c}=o\left(\frac{|t|}{\sigma-1}\right)$$

also

$$H(s)=D+o\left(\frac{|s-1|}{\sigma-1}\right)$$

gleichmäßig in $|t|\geq(\sigma-1)^\alpha$. $\left(\text{In diesem Bereich ist jedes } D \text{ ein } o\left(\frac{|s-1|}{\sigma-1}\right).\right)$

Im Bereich $|t|\leq(\sigma-1)^\alpha$ ist für $\sigma\to 1+$

$$H_1(s)=H_1(1)+o(1)$$

wegen der Stetigkeit von $H_1(s)$ im Punkt $s=1$.

Es genügt somit in $|t|\leq(\sigma-1)^\alpha$ die Beziehung

$$\exp\left(\sum_p\frac{h(p)}{p^s}\right)=D'+o\left(\frac{|s-1|}{\sigma-1}\right)\tag{23}$$

für $\sigma\to 1+$ zu zeigen.

Man wähle eine große positive Konstante M. Wegen der Konvergenz der Reihe $\sum\limits_p \frac{h(p)}{p^s}$ ist nach dem Stetigkeitssatz für Dirichletreihen

$$\sum_p \frac{h(p)}{p^s} = D' + o(1)$$

im Winkelbereich $|t| < M(\sigma - 1)$, d. h. in diesem Bereich gilt (23).

Es bleibt der Bereich

$$M(\sigma - 1) \leq |t| \leq (\sigma - 1)^{\frac{1}{2}}.$$

Dort vergleichen wir

$$u_1(s) = \exp\left(\sum_p \frac{h(p)}{p^s}\right)$$

mit

$$u_2(\sigma) = \exp\left(\sum_p \frac{h(p)}{p^\sigma}\right)$$

und erhalten

$$\frac{u_1(s)}{u_2(\sigma)} = \exp\left(\sum_p \frac{1}{p^\sigma}\left(-1 + \operatorname{Re}\frac{h(p)}{p^{it}}\right)\right).$$

Nun ist

$$\left|\operatorname{Re}\frac{h(p)}{p^{it}}\right| \leq |h(p)| < 1 .$$

Deshalb ist bei festem t die rechtsstehende Funktion monoton fallend für $\sigma \to 1+$. Der Quotient wird also maximal, wenn σ bei festem t auf dem Rand des Winkelraumes $|t| \leq M(\sigma - 1)$ gewählt wird, d. h. für $\sigma = \sigma_1 = 1 + \frac{|t|}{M}$.

Es gilt also

$$|u_1(s)| \leq u_2(\sigma) \frac{u_1(\sigma_1 + it)}{u_2(\sigma_1)} .$$

Mit von M unabhängigen O-Konstanten ist

$$\frac{u_2(\sigma)}{u_2(\sigma_1)} = O\left(\frac{\zeta(\sigma)}{\zeta(\sigma_1)}\right) = O\left(\frac{1}{\sigma - 1} \cdot \frac{|t|}{M}\right) = O\left(\frac{|s-1|}{\sigma - 1} \cdot \frac{1}{M}\right).$$

Weiter ist

$$|u_1(\sigma_1 + it)| = |D'| + o_M(1) < |D'| + 1$$

wenn σ genügend nahe bei 1 liegt. Damit ist also im betrachteten Bereich

$$|u_1(s) - D'| < |u_1(s)| + |D'| \leq O\left(\frac{|s-1|}{\sigma - 1} \cdot \frac{1}{M}\right) + |D'| \frac{|s-1|}{M(\sigma - 1)} .$$

Hiermit ist das Lemma 3 bewiesen.

Mit Hilfe von Lemma 3 sollen nun für eine multiplikative Funktion f die Voraussetzungen von Lemma 2, insbesondere die Asymptotik (22), nachgewiesen werden, falls die Funktion f sich „genügend wenig" von einer Funktion g unterscheidet, für deren Dirichletreihe eine „gute" Asymptotik bekannt ist. Wir zeigen

LEMMA 4. *Seien f und g multiplikative Funktionen mit*

$$\sum_p \sum_{m \geq 2} \frac{|f(p^m)|}{p^m} < \infty . \tag{24}$$

$$f(p) = O(1), \tag{25}$$

$$\varphi_f(p, s) \neq 0 \quad \textit{für} \quad \operatorname{Re} s = 1 , \tag{26}$$

$$\sum_p \frac{f(p) - g(p)}{p^s} \tag{27}$$

konvergiert es existiert eine Konstante $c < 1$ mit

$$|f(p) - g(p)| \leq c , \tag{28}$$

$$g \neq 0 \quad \textit{und} \quad (f * g^*)(p^m) = 0 \tag{29}$$

für alle $m \geq 2$ und alle Primzahlen p und

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \frac{D''}{s-1} + O(|s|) . \tag{30}$$

Dann existiert $M(f)$.

(Hier bezeichnet g^* die bezüglich der Faltung $*$ zu g inverse Funktion).

BEWEIS. Sei $h = f * g^* \cdot h$ erfüllt die Voraussetzungen von Lemma 3. Damit gilt

$$H(s) = O \frac{1}{(\sigma - 1)^c} \quad \text{für} \quad |t| \geq \frac{1}{2} \tag{31}$$

und

$$H(s) = D + o\left(\frac{|s-1|}{\sigma - 1}\right) \quad \text{für} \quad |t| \leq \frac{1}{2} . \tag{32}$$

Aus $F(s) = H(s) G(s)$ folgt damit

$$F(s) = O\left(\frac{|s|}{(\sigma - 1)^c}\right) = o\left(\frac{|s|}{\sigma - 1}\right) \quad \text{für} \quad |t| \geq \frac{1}{2} \tag{33}$$

und

$$F(s) = \frac{DD''}{s-1} + o\left(\frac{1}{\sigma - 1}\right) \quad \text{für} \quad |t| \leq \frac{1}{2} . \tag{34}$$

f erfüllt somit die Voraussetzungen von Lemma 2 und wir erhalten die Behauptung.

Satz 1 ergibt sich nun sofort mit dem oben zitierten Benachbarkeitsargument aus [6]. Wir setzen in Lemma 4 $g \equiv 1$ und wählen f_0 multiplikativ mit

$$f_0(p) = \begin{cases} f(p) & \text{für} \quad |f(p)-1| \leq c, \ p > 2 \\ 1 & \text{sonst} \end{cases}$$

$$f_0(p^m) = f_0(p) \quad \text{für} \quad m \geq 2 .$$

Nach Lemma 4 existiert dann $M(f_0)$. Weiter sind $f_0 \in \mathcal{G}^*$ und $f \in \mathcal{G}$ benachbart. Deshalb existiert auch $M(f_0)$.

6. Beweis von Satz 2. Zum Beweis von Satz 2 müssen wir nur in Lemma 4 eine andere Funktion g wählen. Seien χ_j, $j = 1, -, \varphi(k)$ die Charaktere mod k, χ_1 der Hauptcharakter. Wir setzen

$$\alpha_j = \frac{1}{\varphi(k)} \sum_{\substack{a \bmod k \\ (a,k)=1}} \Psi(a) \overline{\chi_j(a)} .$$

Dann gilt für $(n, k) = 1$

$$\Psi(n) = \sum_{j=1}^{\varphi(k)} \alpha_j \chi_j(n) \quad \text{und} \quad \alpha_1 = 1 .$$

Sei

$$A = L \sum_{1 \leq j \leq \varphi(k)} |\alpha_j| ,$$

wobei die Konstante L später genügend groß gewählt wird und

$$k' = k \cdot \prod_{\substack{p \leq A \\ p \nmid k}} p .$$

Wir definieren die Charaktere χ'_j, $j = 1, -, \varphi(k)$, mod k' durch

$$\chi'_j(a) = \begin{cases} \chi_j(a) & \text{für} \quad (a, k') = 1 \\ 0 & \text{sonst} \end{cases} .$$

Dann gilt wieder

$$\Psi(n) = \sum_{j=1}^{\varphi(k)} \alpha_j \chi'_j(n) \quad \text{für} \quad (n, k') = 1 .$$

Wir definieren nun weiter die vollständig multiplikativen Funktionen g_j für $j = 1, -, \varphi(k)$ durch $g_j(p) = \alpha_j \chi_j(p)$. Sei weiter (für $\operatorname{Re} s > 1$)

$$G_j(s) = \sum_{n=1}^{\infty} \frac{g_j(n)}{n^s} .$$

Delange zeigte in [3] für $j \geq 2$:

$G_j(s)$ ist holomorph in $\operatorname{Re} s \geq 1-\varepsilon$ für ein geeignetes $\varepsilon > 0$ und

$$G_j(s) = O\left((\log|s|)^{d_j}\right) \quad \text{für} \quad \operatorname{Re} s \geq 1, \ |\operatorname{Im} s| \geq \frac{1}{2}.$$

Weiter ist $G_1(s) = L(s, \chi_1)$.

Sei nun $g_0 = g_1 * \ldots * g_{\varphi(k)}$. Wegen

$$\left|\frac{\alpha_j \chi_j(p)}{p}\right| < \frac{1}{L}$$

sind alle $g_j \in \mathcal{G}^*$, also ist auch $g_0 \in \mathcal{G}^*$. Sei g die durch $g(p) = g_0(p)$ definierte vollständig multiplikative Funktion. Wegen

$$\left|\frac{g(p)}{p}\right| = \left|\sum_{j=1}^{\varphi(k)} \frac{g_j(p)}{p}\right| \leq \frac{\varphi(k)}{L}$$

ist auch $g \in \mathcal{G}^*$, falls L genügend groß gewählt wird. Außerdem sind natürlich g und g_0 benachbart. Nach [6] unterscheiden sich deshalb ihre Dirichletreihen nur zum einen in $\operatorname{Re} s \geq 1$ beschränkten und stetigen Faktor. Somit erfüllt auch g die Voraussetzungen von Lemma 4.

Schließlich sei noch die multiplikative Funktion f_0 definiert durch

$$f_0(p) = \begin{cases} f(p) & \text{für} \quad |f(p) - \Psi(p)| \leq c, \ p > A, \ p \neq k \\ g(p) & \text{sonst} \end{cases}$$

und $f_0(p^m) = f_0(p)\,(g(p))^{m-1}$ für $m \geq 2$.

f_0 erfüllt die Voraussetzungen (für f) von Lemma 4:

Es ist

$$\varphi_{f_0}(p, s) = \varphi_g(p, s)$$

oder

$$\varphi_{f_0}(p, s) = 1 + \frac{f_0(p)}{p^s}\, \varphi_g(p, s).$$

Dabei ist

$$\left(1 - \frac{1}{L-1}\right)^{\varphi(k)} < |\varphi_g(p, s)| < \left(1 + \frac{1}{L-1}\right)^{\varphi(k)}.$$

Hieraus folgt $f_0 \in \mathcal{G}^*$, falls L genügend groß gewählt wurde. Weiter ist auch

$$(f_0 * g)(p^m) = f_0(p^m) - f_0(p^{m-1})\, g(p) = 0$$

für $m \geq 2$. Nach Lemma 4 existiert also $M(f_0)$ und mit dem üblichen Benachbarkeitsargument folgt hieraus die Existenz von $M(f)$.

7. Satz 3. Die in Lemma 4 geforderte Asymptotik für die multiplikative Funktion läßt sich natürlich auch aus einer „genügend guten" Asymptotik für $\sum_{n \leq x} g(n)$ herleiten. Man zeigt so

Satz 3. *Seien f und g multiplikative Funktionen mit*

$$\sum_{n \leq x} g(n) = Cx + O\left(\frac{x}{(\log x)^{\gamma}}\right), \quad \gamma > 1 \tag{35}$$

$$g(p) = O(1), \quad \sum_{p} \sum_{m \geq 2} \frac{|g(p^m)|}{p^m} < \infty, \tag{36}$$

$$\varphi_g(p, s) \neq 0 \quad in \quad \operatorname{Re} s \geq 1,$$

$$\sum_{p} \sum_{m \geq 2} \frac{|f(p^m)|}{p^m} < \infty, \tag{37}$$

$$\sum_{p} \frac{f(p) - g(p)}{p} \tag{38}$$

konvergiert, es existiert ein $c < 1$ mit

$$\sum_{\substack{p \\ |f(p) - g(p)| > c}} \frac{|f(p) - g(p)|}{p} < \infty. \tag{39}$$

Dann existiert $M(f)$.

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PERIODIC SOLUTIONS OF CERTAIN FIFTH ORDER DIFFERENTIAL EQUATIONS

By

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1. Introduction. Consider the fifth order differential equation:

$$x^{(5)}+a_1 x^{(4)}+a_2 \dddot{x}+\alpha_3 \ddot{x}+a_4 \dot{x}+a_5 x = 0 \tag{1.1}$$

in which $a_1, a_2, \ldots, a_5$ are all constants. Our arguments in [1] and [2] show clearly that if

$$a_4 > \frac{1}{4} a_2^2,\ a_5 \neq 0 \tag{1.2}$$

or if

$$a_1 \neq 0,\ a_5 \operatorname{sgn} a_1 > \frac{1}{4} a_3^2 \, |a_1|^{-1} \tag{1.3}$$

then the auxiliary equation corresponding to (1.1) has no purely imaginary roots whatever. Thus if (1.2) or (1.3) holds then (1.1) has no non-trivial periodic solutions. By the general theory this implies, in turn, that the perturbed equation

$$x^{(5)}+a_1 x^{(4)}+a_2 \dddot{x}+a_3 \ddot{x}+a_4 \dot{x}+a_5 x = p(t) \tag{1.4}$$

in which $p(\neq 0)$ is any continuous ω-periodic function of t does have an ω-periodic solution subject to (1.2) or (1.3). The object of the present paper is to extend this result for (1.4) to equations in which $a_1, a_2, \ldots, a_5$ are not all constants.

2. Statement of the results. We shall be concerned with the two equations:

$$\begin{aligned} x^{(5)}+a_1 x^{(4)}+f_2(\ddot{x})\dddot{x}+f_3(\dot{x})\ddot{x}+f_4(x)\dot{x}+f_5(x) &= \\ &= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \end{aligned} \tag{2.1}$$

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + g_3(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\, \ddot{x} + g_4(x)\, \dot{x} + g_5(x) =$$

$$= p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \tag{2.2}$$

in which a_1, a_2 are constants as before. The functions $f_2, f_3, f_4, f_5, g_3, g_4, g_5$ and p are continuous functions depending only on the arguments shown with g_3 and p ω-periodic in t, that is $g_3(t, x_1, \ldots, x_5) = g_3(t+\omega, x_1, \ldots, x_5)$ and $p(t, x_1, \ldots, x_5) = p(t+\omega, x_1, \ldots, x_5)$ for some $\omega > 0$ and arbitrary t, $x_1, \ldots, x_5$. We shall establish here the following theorems:

THEOREM 1. *Suppose that*

(i) *there exists a constant* $a_2 \geq 0$ *such that*

$$|f_2(x_3)| \leq a_2 \quad \textit{for all} \quad x_3, \tag{2.3}$$

$$a_4 \equiv \inf_{x_1} f_4(x_1) > \frac{1}{4} a_2^2\,; \tag{2.4}$$

(ii) *there exist constants* $A_0 \geq 0$, $A_1 \geq 0$ *such that*

$$|p(t, x_1, \ldots, x_5)| \leq A_0 + A_1 (|x_2| + |x_3|) \tag{2.5}$$

for all $t, x_1, \ldots, x_5$;

(iii) f_5 *satisfies either*

$$f_5(x_1) \operatorname{sgn} x_1 \to +\infty \quad \textit{as} \quad |x_1| \to \infty, \tag{2.6}$$

or

$$f_5(x_1) \operatorname{sgn} x_1 \to -\infty \quad \textit{as} \quad |x_1| \to \infty. \tag{2.7}$$

Then there exists a constant $\varepsilon_0 > 0$ *such that* (2.1) *admits of at least one* ω*-periodic solution, for all arbitrary* a_1 *and* f_3, *if* $A_1 < \varepsilon_0$.

The conditions here can be seen to be a generalization of (1.2).

THEOREM 2. *Given the equation* (2.2) *suppose that* $a_1 \neq 0$ *and that*

(i) *there exists a constant* $a_3 \geq 0$ *such that*

$$|g_3(t, x_1, \ldots, x_5)| \leq a_3 \quad \textit{for all} \quad t, x_1, \ldots, x_5, \tag{2.8}$$

$$\alpha_5 \equiv \inf_{|x_1| \geq 1} x_1^{-1} g_5(x_1) \operatorname{sgn} a_1 > \frac{1}{4} a_3^2 |a_1|^{-1}\,; \tag{2.9}$$

(ii) *there exist constants* $A_0 \geq 0$, $A_1 \geq 0$ *such that*

$$|p(t, x_1, \ldots, x_5)| \leq A_0 + A_1 (|x_1| + |x_2| + |x_3|) \tag{2.10}$$

for all $t, x_1, \ldots, x_5$.

Then there exists a constant $\varepsilon_0 \geq 0$ *such that the equation* (2.2) *admits of at least one* ω*-periodic solution, for all arbitrary* a_2 *and* g_4, *if* $A_1 \leq \varepsilon_0$.

Note that the conditions here generalize (1.3).

3. Some useful preliminaries on the proofs. Our proofs will be by the Leray-Schauder technique, with the equation (2.1) embedded in the parameter-dependent equation:

$$x^{(5)}+a_1 x^{(4)}+\{(1-\lambda) a_2+\lambda f_2(\ddot{x})\} \dddot{x}+\lambda f_3(\dot{x}) \ddot{x}+ \\ +\{(1-\lambda) a_4+\lambda f_4(x)\} \dot{x}+(1-\lambda) cx+\lambda f_5(x) = \lambda p(t, x, \dot{x}, \ldots, x^{(4)}) \tag{3.1}$$

and (2.2) embedded in the equation

$$x^{(5)}+a_1 x^{(4)}+a_2 \dddot{x}+\{(1-\lambda) a_3+\lambda g_3(t, x, \dot{x}, \ldots, x^{(4)})\} \ddot{x}+ \\ +\lambda g_4(x) \dot{x}+(1-\lambda) A_5 x+\lambda g_5(x) = \lambda p(t, x, \dot{x}, \ldots, x^{(4)}) . \tag{3.2}$$

In either case the parameter λ is restricted to the range $0 \leq \lambda \leq 1$. The c in (3.1) is an arbitrary constant which shall be fixed positive or negative according as the f_5 in (2.1) is subject to (2.6) or (2.7). The constant A_5 in (3.2) is defined by

$$A_5 = \alpha_5 \operatorname{sgn} a_1 .$$

Note that, for $\lambda = 1$ (3.1) reduces to (2.1) and (3.2) to (2.2). Also, for $\lambda = 0$ (3.1) reduces to

$$x^{(5)}+a_1 x^{(4)}+a_2 \dddot{x}+a_4 \dot{x}+cx = 0 \tag{3.3}$$

while (3.2) reduces to

$$x^{(5)}+a_1 x^{(4)}+a_2 \dddot{x}+a_3 \ddot{x}+A_5 x = 0 . \tag{3.4}$$

Since $c \neq 0$ and $a_4 > \frac{1}{4} a_2^2$ (see (2.4)) the auxiliary equation corresponding to (3.3), that is

$$r^5+a_1 r^4+a_2 r^3+a_4 r+c = 0 ,$$

has no purely imaginary roots and thus (3.3) definitely has no non-trivial ω-periodic solutions. Analogously, because of the condition: $\alpha_5 > \frac{1}{4} a_3^2 |a_1|^{-1}$ (See (2.9)), (3.4) has no non-trivial ω-periodic solutions. Thus Theorem 1 (or Theorem 2) will follow from the usual fixed point considerations (see for example [3; Theorem 1.39]) if it can be shown that there is a constant D whose magnitude is independent of $\lambda\,(0 \leq \lambda \leq 1)$ such that, if $x(t)$ is any ω-periodic solution of (3.1) (or (3.2)), then

$$|x(t)| \leq D, \quad |\dot{x}(t)| \leq D, \quad |\ddot{x}(t)| \leq D, \quad |\dddot{x}(t)| \leq D, \quad |x^{(4)}(t)| \leq D \tag{3.5}$$

for all $t \in [0, \omega]$. Note that the t-range here may be replaced by $[T, T+\omega]$ (arbitrary T) since we are dealing with an ω-periodic $x(t)$.

4. Some remarks on notation. Throughout what follows, D's with or without subscripts denote finite positive constants whose magnitudes depend on $c, a_1, a_2, a_4, A_0, f_3, f_4$ and f_5 (in the context of Theorem 1) or on a_1, a_2, a_3,

A_0, g_4 and g_5 (in the context of Theorem 2). The D's are all independent of λ. Finally a D without a subscript is not necessarily the same each time it occurs, but the numbered D's: $D_0, D_1, \ldots$ retain a fixed identity throughout.

5. Proof of Theorem 1. Assume now that hypotheses (i) and (ii) of the theorem hold, with f_5 subject to (2.6). The relevant parameter-dependent equation is then (3.1) with the constant c fixed *positive* but otherwise arbitrary. We shall set

$$(5.1)\qquad f_{2,\lambda}(\ddot{x}) \equiv (1-\lambda)\, a_2 + f_2(\ddot{x}), \quad f_{4,\lambda}(x) \equiv (1-\lambda)\, a_4 + \lambda f_4(x),$$
$$f_{5,\lambda}(x) = (1-\lambda)\, cx + \lambda f_5(x)$$

so as to be able to write (3.1) in the more compact form:

$$x^{(5)} + a_1 x^{(4)} + f_{2,\lambda}(\ddot{x})\, \dddot{x} + \lambda f_3(\dot{x})\, \ddot{x} + f_{4,\lambda}(x)\, \dot{x} + f_{5,\lambda}(x) = \lambda\, p(t, x, \ldots, x^{(4)}).$$

In what follows in the rest of this paragraph $x = x(t)$ is an arbitrary ω-periodic solution of (5.1). We will now show that $x(t)$ satisfies (3.5) if A_1 is sufficiently small.

Our main tool in the verification of (3.5) for $x(t)$ is the function

$$V(t) \equiv -\dot{x}\,(x^{(4)} + a_1 \dddot{x}) + \ddot{x}\,\dddot{x} + \frac{1}{2}\, a_1 \ddot{x}^2 - \lambda \int_0^{\dot{x}} y f_3(y)\, dy - \int_0^{x} f_{5,\lambda}(s)\, ds.$$

An elementary differentiation will show that

$$(5.2)\qquad \dot{V} = U_0 - \lambda \dot{x}\, p(t, x, \ldots, x^{(4)})$$

where

$$U_0 = \dddot{x}^2 + f_{4,\lambda}(x)\dot{x}^2 + f_{2,\lambda}(\ddot{x})\, \dot{x}\, \dddot{x}.$$

By (2.3) and (2.4),

$$|f_{2,\lambda}(\ddot{x})| \leq a_2, \quad f_{4,\lambda}(x) \geq a_4$$

so that

$$(5.3)\qquad U_0 \geq \dddot{x}^2 + a_4 \dot{x}^2 - a_2\, |\dot{x}|\, |\dddot{x}|,$$

from which it is now not difficult to verify that

$$(5.4)\qquad U_0 \geq D_0\,(\dot{x}^2 + \dddot{x}^2),$$

for sufficiently small D_0. For, suppose for example that $D_0 < 1$, then, by (5.3),

$$U_0 - D_0\,(\dot{x}^2 + \dddot{x}^2) \geq (1-D_0)\, \dddot{x}^2 + (a_4 - D_0)\, \dot{x}^2 - a_2\, |\dot{x}|\, |\dddot{x}| \equiv$$
$$\equiv (1-D_0)\left\{|\dddot{x}| - \frac{a_2\, |\dot{x}|}{2\,(1-D_0)}\right\}^2 + \frac{1}{4}(1-D_0)^{-1}\, U_1 \dot{x}^2 \geq \frac{1}{4}\,(1-D_0)^{-1}\, U_1 \dot{x}^2$$

where

$$U_1 = (4a_4 - a_2^2) - 4D_0\,(1+a_4) + 4D_0^2.$$

However, by (2.4), $4a_1 - a_2^2 > 0$ and so U_1 is strictly positive if, say

$$0 < D_0 < \frac{1}{8}\,(4a_4 - a_2^2)\,(1+a_4)^{-1}\,.$$

Thus $U_0 - D_0(\dot{x}^2 + \dddot{x}^2) \geq 0$ if D_0 is sufficiently small, which gives (5.4) and hence, by (5.2) and (2.5), leads to the estimate:

$$\text{(5.5)}\quad \begin{cases} \dot{V} \geq D_0(\dddot{x}^2 + \dot{x}^2) - \{A_0\,|\dot{x}| + A_1\,|\dot{x}|\,[|\dot{x}| + |\ddot{x}|]\} \geq D_0(\dddot{x}^2 + \dot{x}^2) - \\ - A_0\,|\dot{x}| - \frac{1}{2}\,A_1(3\dot{x}^2 + \ddot{x}^2) = D_0\,\dddot{x}^2 + \left(D_0 - \frac{1}{2}\,3A_1\right)\dot{x}^2 - A_0\,|\dot{x}| - \\ \qquad - \frac{1}{2}\,A_1\,\ddot{x}^2 \geq D_1(\dddot{x}^2 + \dot{x}^2) - \frac{1}{2}\,A_1\,\ddot{x}^2 - D_2 \end{cases}$$

for some D_1, D_2 if A_1 is fixed sufficiently small, say $A_1 \leq D_3$.

Because of the (assumed) ω-periodicity of x, we have, on integrating (5.5), that

$$\text{(5.6)}\qquad 0 \geq D_1 \int_0^\omega (\dddot{x}^2 + \dot{x}^2)\,dt - \frac{1}{2}\,A_1 \int_0^\omega \ddot{x}^2\,dt - D_2\,\omega\,.$$

Combined with the inequality

$$\text{(5.7)}\qquad \int_0^\omega \ddot{x}^2\,dt \leq \frac{1}{4}\,\omega^2\pi^{-2} \int_0^\omega \dddot{x}^2\,dt\,,$$

which can be verified by substituting the Fourier expansions of $\ddot{x}$ and $\dddot{x}$ in (5.7), (5.6) leads to the estimate

$$\left(D_1 - \frac{1}{8}\,\omega^2\pi^{-2}A_1\right)\int_0^\omega \dddot{x}^2\,dt + D_1 \int_0^\omega \dot{x}^2\,dt \leq D_2\,\omega\,.$$

Hence, if A_1 is further fixed such that

$$A_1\,\omega^2\pi^{-2} \leq 4D_1$$

as we assume henceforth, then

$$\frac{1}{2}\,D_1 \int_0^\omega (\dddot{x}^2 + \dot{x}^2)\,dt \leq D_2\,\omega\,.$$

In particular

$$\text{(5.8)}\qquad \int_0^\omega \dddot{x}^2\,dt \leq D_3\,.$$

Considering now the identity:

$$\ddot{x}(t) = \ddot{x}(T_1) + \int_{T_1}^{t} \dddot{x}(s)\,ds$$

with T_1 fixed (as is possible in view of the periodicity condition $\dot{x}(0) = \dot{x}(\omega)$) such that $\ddot{x}(T_1) = 0$, we have that

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \int_0^{\omega} |\dddot{x}(s)|\,ds \leq \omega^{\frac{1}{2}} \left(\int_0^{\omega} \dddot{x}^2(s)\,ds \right)^{\frac{1}{2}}$$

by Schwarz's inequality. Thus (5.8) implies that

(5.9) $$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq D_4 .$$

From this, on referring to the identity

$$\dot{x}(t) = \dot{x}(T_2) + \int_{T_2}^{t} \ddot{x}(s)\,ds$$

with T_2 chosen such that $\dot{x}(T_2) = 0$ (the choice being possible in view of the periodicity condition $x(0) = x(\omega)$), we have that

(5.10) $$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq D_1\,\omega .$$

To obtain an estimate for $|x(t)|$ first note that, because of the ω-periodicity of x, integration of both sides of (5.1) yields the result

(5.11) $$\int_0^{\omega} \{f_{5,\lambda}(x) - \lambda\, p(t, x, x, \ldots x^{(4)})\}\,dt = 0 .$$

But, by (2.5), (5.9) and (5.10),

(5.12) $$|\lambda\, p(t, x, \ldots, x^{(4)})| \leq D_1$$

for some D_1. Also, since $c > 0$ and f_5 is subject to (2.6), there clearly exists D_3 such that

(5.13) $$f_{5,\lambda}(x)\operatorname{sgn} x > D_4 \quad \text{for} \quad |x| \geq D_5 .$$

Because of (5.11), (5.12) and (5.13) it is plain that $|x(T_3)| < D_5$ for some T_3. Hence

(5.14) $$\max_{0 \leq t \leq \omega} |x(t)| \leq |x(T_3)| + \int_0^{\omega} |\dot{x}(s)|\,ds \leq D_5 + D_4\,\omega^2 .$$

by (5.10).

It remains now to obtain estimates for $|\dddot{x}(t)|$ and $|x^{(4)}(t)|$ in order to complete our verification of (3.5). For this, note that if (5.1) is written as:

$$x^{(5)} + a_1 x^{(4)} = Q_1 \tag{5.15}$$

the function Q_1, by virtue of (2.3), (2.5), (5.9), (5.10) and (5.14) would satisfy

$$|Q_1| \leq D_6(|\dddot{x}| + 1)\,. \tag{5.16}$$

Thus, if we multiply both sides of (5.15) by $x^{(5)}$ and integrate from $t = 0$ to $t = \omega$, we shall have, x being ω-periodic, that

$$\int_0^\omega \{x^{(5)}\}^2\,dt \leq D_6\left(\int_0^\omega |\dddot{x}|\,|x^{(5)}|\,dt + \int_0^\omega |x^{(5)}|\,dt\right),$$

or, on applying Schwarz's inequality, that

$$\int_0^\omega \{x^{(5)}\}^2\,dt \leq D_6\left[\int_0^\omega \{x^{(5)}\}^2\,dt\right]^{\frac{1}{2}}\left[\omega^{\frac{1}{2}} + \left\{\int_0^\omega \dddot{x}^2\,dt\right\}^{\frac{1}{2}}\right] \leq$$

$$\leq D_7\left[\int_0^\omega \{x^{(5)}\}^2\,dt\right]^{\frac{1}{2}},$$

by (5.8). Hence

$$\int_0^\omega \{x^{(5)}\}^2\,dt \leq D_8\,. \tag{5.17}$$

Since $x^{(4)}(T_1) = 0$ for some T_1, it follows from the identity

$$x^{(4)}(t) = x^{(4)}(T_1) + \int_T^t x^{(5)}(s)\,ds$$

and the result (5.17), in the usual manner, that

$$|x^{(4)}(t)| \leq \omega^{\frac{1}{2}} D_8^{\frac{1}{2}} \quad (0 \leq t \leq \omega)\,. \tag{5.18}$$

In turn (5.18), combined with the identity

$$\dddot{x}(t) = \dddot{x}(T_5) + \int_{T_5}^t x^{(4)}(s)\,ds$$

with T_5 chosen such that $\dddot{x}(T_5) = 0$, implies that

$$|\dddot{x}(t)| \leq \omega^{\frac{3}{2}} D_8^{\frac{1}{2}} \quad (0 \leq t \leq \omega)\,. \tag{5.19}$$

The results (5.9), (5.10), (5.14), (5.18) and (5.19) fully verify (3.5) for the arbitrarily chosen ω-periodic solution $x(t)$ of (5.1). The theorem now follows, as was pointed out in **§3**, for the case f_5 subject to (2.6).

For the case f_5 subject to (2.7) the same method as before applies except only that the constant c in (3.1) will now be fixed strictly negative. The estimates (5.9) and (5.10) are in any case independent of c or the restriction (1.5) and are thus valid here. The choice of a negative c indeed only comes in for the sole purpose of establishing from (5.11) that

$$|x(T)| \leq D \quad \text{for some} \quad T. \tag{5.20}$$

This is secured here by the fact that if $c<0$ then $f_{5,\lambda}(x) \operatorname{sgn} x \to -\infty$ as $|x| \to \infty$. With (5.20) assured, (5.14), (5.18) and (5.19) can now follow exactly as before, so that the theorem also holds when f_5 is subject to (2.7).

6. Proof of Theorem 2. The numbering of the D's will start afresh here in connection with the parameter-dependent equation (3.2), which we shall write in the more compact form:

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + g_{3,\lambda} \ddot{x} + \lambda g_4 \dot{x} + g_{5,\lambda} = \lambda p(t, x, \ldots, x^{(4)}) \qquad (0 \leq \lambda \leq 1) \tag{6.1}$$

where

$$g_{3,\lambda} \equiv (1-\lambda) a_3 + \lambda g_3(t, x, \ldots, x^{(4)}), \quad g_{5,\lambda} \equiv (1-\lambda) A_5 x + \lambda g_5(x).$$

Note that

$$|g_{3,\lambda}| \leq a_3 \tag{6.2}$$

by (2.8), and that

$$x^{-1} g_{5,\lambda} \operatorname{sgn} a_1 \geq \alpha_5 \quad (|x| \geq 1),$$

by (2.9), which in turn also implies that, for some D_0,

$$x g_{5,\lambda} \operatorname{sgn} a_1 \geq \alpha_5 x^2 - D_0 \quad \text{for all} \quad x. \tag{6.3}$$

Let $x = x(t)$ be any ω-periodic solution of (6.1). We shall now show that $x(t)$ satisfies (3.5). The main tool for this is the function

$$V = W \operatorname{sgn} a_1 \tag{6.4}$$

where

$$W = \dot{x}(\dddot{x} + a_1 \ddot{x}) - x(x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x}) + \frac{1}{2} a_2 \ddot{x}^2 - \lambda \int_0^x g_4(s)\, s\, ds.$$

Differentiating (6.4) with respect to t we have, in view of (6.1), that

$$V = U_2 - \lambda x p(t, x, \ldots, x^{(4)}) \operatorname{sgn} a_1 \tag{6.5}$$

where

$$\begin{aligned} U_2 &= |a_1| \ddot{x}^2 + x \ddot{x} g_{3,\lambda} \operatorname{sgn} a_1 + x g_{5,\lambda} \operatorname{sgn} a_1 \\ &\geq |a_1| \ddot{x}^2 + x \ddot{x} g_{3,\lambda} \operatorname{sgn} a_1 + \alpha_5 x^2 - D_0, \end{aligned} \tag{6.6}$$

by (6.3). It is readily verified from this that

$$U_2 \geq D_1(\ddot{x}^2+x^2)-D_0 \tag{6.7}$$

if D_1 is sufficiently small. Indeed assume for a start that $D_1 < |a_1|$ and set

$$U_3 \equiv U_2 - \{D_1(\ddot{x}^2+x^2)-D_0\}.$$

We have, by (6.6), that

$$\begin{aligned} U_3 &\geq |a_1|\,\ddot{x}^2 + x\ddot{x}\,g_{3,\lambda}\operatorname{sgn} a_1 + \alpha_5 x^2 - D_1(\ddot{x}^2+x^2), \\ &= (|a_1|-D_1)\left\{\ddot{x}+\frac{1}{2}(|a_1|-D_1)^{-1}xg_{3,\lambda}\operatorname{sgn} a_1\right\}^2 + \\ &\quad+\frac{1}{4}(|a_1|-D_1)^{-1}U_4x^2, \\ &\geq \frac{1}{4}(|a_1|-D_1)^{-1}U_4x^2, \end{aligned}$$

where

$$U_4 \equiv 4(|a_1|-D_1)(\alpha_5-D_1)-g_{3,\lambda}^2 \geq 4(|a_1|-D_1)(\alpha_5-D_1)-a_3^2,$$

by (6.2). Hence

$$U_4 \geq (4|a_1|\alpha_5-a_3^2)-4(|a_1|+\alpha_5)D_1+4D_1^2.$$

Since $4|a_1|\alpha_5-a_3^2>0$, by (2.9), it follows that $U_4>0$ if, for example,

$$D_1 \leq \frac{1}{4}(4|a_1|\alpha_5-a_3^2)(|a_1|+\alpha_5)^{-1}$$

as we assume henceforth; and hence $U_3 \geq 0$ which verifies (6.7). Since

$$|xp(t,x,\ldots,x^{(4)})| \leq A_0|x| + A_1(|x|+|\dot{x}|+|\ddot{x}|)|x|,$$

by (2.10), it is clear from (6.5) that

$$\left\{\begin{aligned} \dot{V} &\geq D_1(\ddot{x}^2+x^2)-A_0|x|-A_1(|x|+|\dot{x}|+|\ddot{x}|)|x|-D_0, \\ &\geq D_1(\ddot{x}^2+x^2)-A_0|x|-A_1\left(2x^2+\frac{1}{2}\dot{x}^2+\frac{1}{2}\ddot{x}^2\right)-D_0, \\ &\geq D_2(\ddot{x}^2+x^2)-\frac{1}{2}A_1\dot{x}^2-D_3, \end{aligned}\right. \tag{6.8}$$

for some sufficiently small D_2. Integrating (6.8) from $t=0$ to $t=\omega$ now gives

$$0 \geq D_2\int_0^\omega(\ddot{x}^2+x^2)\,dt-\frac{1}{2}A_1\int_0^\omega\dot{x}^2\,dt-D_3\,\omega, \tag{6.9}$$

so that, since

$$\int_0^\omega \dot{x}^2\, dt \leq \frac{1}{4}\, \omega^2 \pi^{-2} \int_0^\omega \ddot{x}^2\, dt\,,$$

we have that

$$\left(D_2 - \frac{1}{8}\, \omega^2 \pi^{-2}\right) \int_0^\omega \ddot{x}^2\, dt + D_2 \int_0^\omega x^2\, dt \leq D_3\, \omega\,.$$

Hence if, say, $A_1 \leq 4 D_2\, \omega^{-2} \pi^2$, which we assume, then

$$D_2 \int_0^\omega \ddot{x}^2\, dt + 2 D_2 \int_0^\omega x^2\, dt \leq 2 D_3\, \omega\,.$$

Thus

(6.10) $$\int_0^\omega x^2\, dt \leq D_4\,.$$

(6.11) $$\int_0^\omega \ddot{x}^2\, dt \leq D_5\,.$$

As before, a combination of (6.11) with the identity:

$$\dot{x}(t) = \dot{x}(T) + \int_T^t \ddot{x}(s)\, ds\,.$$

where T is fixed such that $\dot{x}(T) = 0$, leads to the estimate

(6.12) $$|\dot{x}(t)| \leq D_6 \equiv \omega^{\frac{1}{2}} D_5^{\frac{1}{2}} \quad (0 \leq t \leq \omega)\,.$$

Also (6.10) implies that $|x(T)| \leq D_4^{\frac{1}{2}}\, \omega^{-\frac{1}{2}}$ for some T, and therefore as usual, because of (6.12), that

(6.13) $$|x(t)| \leq D_7 \quad (0 \leq t \leq \omega)$$

where $D_7 = D_4^{\frac{1}{2}}\, \omega^{-\frac{1}{2}} + D_6\, \omega$.

To obtain bounds for $|\dddot{x}(t)|$ and $|\ddot{x}(t)|$ let us note that if (6.1) is set in the form

(6.14) $$x^{(5)} + a_1\, x^{(4)} + a_2\, \dddot{x} = Q_2$$

the function Q_2, by (2.10), (6.12) and (6.13), satisfies

$$|Q_2| \leq D_8 + (a_3 + A_1)\,|\ddot{x}|\,.$$

Thus if we multiply (6.14) by $x^{(4)}$ and integrate we shall obtain, x being ω-periodic, that

$$|a_1| \int_0^\omega \{x^{(4)}\}^2\,dt \leq \int_0^\omega |x^{(4)}|\,\{D_8 + (a_3 + A_1)\,|\ddot{x}|\}\,dt \leq$$

$$\leq D_8\,\omega^{\frac{1}{2}} \left(\int_0^\omega \{x^{(4)}\}^2\,dt\right)^{\frac{1}{2}} + (a_3 + A_1) \left(\int_0^\omega \{x^{(4)}\}^2\,dt\right)^{\frac{1}{2}} \left(\int_0^\omega \ddot{x}^2\,dt\right)^{\frac{1}{2}},$$

by Schwarz's inequality. Because of (6.11) the last inequality, with $A_1 \leq D$, definitely implies that

$$|a_1| \int_0^\omega \{x^{(4)}\}^2\,dt \leq D \left(\int_0^\omega \{x^{(4)}\}^2\,dt\right)^{\frac{1}{2}}$$

and therefore that

$$\int_0^\omega \{x^{(4)}\}^2\,dt \leq D\,.$$

From this the usual arguments can now be adduced for the estimates

$$(6.15) \qquad |\dddot{x}(t)| \leq D, \quad |\ddot{x}(t)| \leq D \quad (0 \leq t \leq \omega)\,,$$

and it remains only to estimate a bound for $|x^{(4)}(t)|$.

In order to arrive at this remaining estimate we shall recast (6.1) once again, but now in the form:

$$(6.16) \qquad x^{(5)} + a_1\,x^{(4)} = Q_3\,.$$

Because of (2.10), (6.12), (6.13) and (6.15) it is evident that

$$|Q_3| \leq D$$

so that, if we multiply (6.16) by $x^{(5)}$ and integrate we shall obtain that

$$\int_0^\omega \{x^{(5)}\}^2\,dt \leq D \int_0^\omega |x^{(5)}|\,dt \leq D \left(\int_0^\omega \{x^{(5)}\}^2\,dt\right)^{\frac{1}{2}}$$

by Schwarz's inequality. Hence

$$\int_0^{\omega} \{x^{(5)}\}^2 \, dt \leq D$$

from which we now have, in the same way as before, that

$$\max |x^{(4)}(t)| \leq D \quad 0 \leq t \leq \omega .$$

This concludes the verification of (3.5) for any ω-periodic solution of (3.2) if the A_1 in (2.10) is sufficiently small. Theorem 2 is thereby established.

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ALGEBRAIC INDEPENDENCE OF SOME NUMBERS II.

By

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To the memory of P. TURÁN

Let

$$\beta \in \mathbf{A}, \quad \operatorname{Im} \beta \neq 0, \quad \omega_1 = 2, \quad \omega_2 = 2\beta . \tag{1}$$

Let $\mathfrak{p}(z)$ be the Weierstrass $\mathfrak{p}$-function with fundamental periods ω_1 and ω_2. It satisfies the equation

$$y'^2 = 4y^3 - g_2 y - g_3 .$$

THEOREM. *Let ω be any of the three numbers 1, β and $1+\beta$, $\alpha \in \mathbf{A}$, $\log \alpha \neq 0$. Then among the five numbers*

$$\log \alpha, \quad \alpha^{\beta}, \quad \alpha^{\beta^2}, \quad \delta_0 = \mathfrak{p}(\omega), \quad \delta_2 = \mathfrak{p}''(\omega) \tag{2}$$

there are two algebraically independent.

NOTE 1. It is known, that $\mathfrak{p}'(\omega) = 0$, so each of the numbers δ_0 and δ_2 is algebraically dependent with g_2 and g_3. This implies the possibility of replacing δ_0 and δ_2 in the formulation of theorem with g_2 and g_3.

NOTE 2. It is possible to replace the number β^2 in the formulation of the theorem with $\beta_1 \in \mathbf{A}$ if we replace in the set (2) the number α^{β^2} with α^{β_1} and $\alpha^{\beta\beta_1}$. One can refuse that β be algebraic (or β and β_1) if one adds to the set (2) the number β (or β and β_1). It will be necessary to make the corresponding changes in the lemma 10.

NOTE 3. If $\deg \beta = 2$, then we can leave only three numbers: $\log \alpha$, α^{β} and one of the numbers δ (or one of the numbers g) in the formulation of the theorem.

NOTE 4. For $\deg \beta = 3$ the theorem (even without the requirement $\operatorname{Im} \beta \neq 0$) is the corollary of Gelfond's theorem [1], because in this case the numbers α^{β} and α^{β^2} are algebraically independent.

NOTE 5. The paper [3] presents formulations of theorems, which are similar to the formulation, given above. But the scheme of proof given there contains a serious mistake — the inequality (5) is wrong. But if we replace it with the true one, corresponding to the author's conditions, the further discussion of the work [3] become wrong. The complete proofs of the assertions of the paper [3] or the improvement of the mistake mentioned above is not yet published, as far as the author is informed.

The base of the proof is a well-known Gelfond's method.

Lemma 1 [2]. Let $\alpha_k \in \mathbf{A}$, $n_k = \deg \alpha_k$, $L_k = L(\alpha_k)$, $k = 1, \ldots, s$; $n = \deg \mathbf{Q}(\alpha_1, \ldots, \alpha_s)$,

$$\mathcal{P}(z_1, \ldots, z_s) \in \mathbf{Z}[z_1, z_s], \quad \deg_{z_k} \mathcal{P} = N_k, \quad L(\mathcal{P}) = L .$$

If

$$\mathcal{P}(\alpha_1, \ldots, \alpha_s) \neq 0 ,$$

then

$$|\mathcal{P}(\alpha_1, \ldots, \alpha_s)| \geq L^{1-n} \prod_{k=1}^{s} L_k^{-\frac{nN_k}{n_k}} .$$

Lemma 2 [5]. If

$$\mathcal{P}(z) = a_n \prod_{k=1}^{n} (z - \alpha_k) ,$$

then

$$|a_n| \prod_{k=1}^{n} \max(1, |\alpha_k|) \leq L(\mathcal{P}) .$$

Lemma 3 [1]. Let

$$\mathcal{P}(z) = \mathcal{P}_1(z) \ldots \mathcal{P}_s(z), \quad n = \deg \mathcal{P}(z) ,$$

then

$$2^n L(\mathcal{P}) \geq L(\mathcal{P}_1) \ldots L(\mathcal{P}_s) \geq L(\mathcal{P}) .$$

Lemma 4 [6]. Let $\omega, \omega_1 \in \mathbf{C}$,

$$\mathcal{P}(x, y) = \mathcal{P}_r(x) y^r + \ldots + \mathcal{P}_0(x) \in \mathbf{Z}[x, y], \quad \mathcal{P}_r(\omega) \neq 0, \quad \mathcal{P}(\omega, \omega_1) = 0 ,$$

then

$$\omega_1^{r+n} = \mathcal{P}_r(\omega)^{-n-1} \sum_{k=0}^{r-1} \mathcal{P}_{k,n}(\omega) \omega_1^k = \mathcal{P}_r(\omega)^{-n-1} Q_n(\omega, \omega_1) ,$$

$$\mathcal{P}_{k,n}(x) \in \mathbf{Z}[x], \quad L(Q_n) \leq L(\mathcal{P})^{n+1}, \quad \deg_x Q_n \leq (n+1) \deg_x \mathcal{P} .$$

Lemma 5. Let $M > m$, $a_{r,j} \in \mathbf{Z}$, $r = 1, \ldots, m$; $j = 1, \ldots, M$,

$$A = \max_{1 \leq r \leq m} \sum_{j=1}^{M} |a_{r,j}| ,$$

then there exists a set $x_1, \ldots, x_M \in \mathbf{Z}$, satisfying the conditions

$$\left.\begin{array}{l} a_{1,1} x_1 + \ldots + a_{1,M} x_M = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m,1} x_1 + \ldots + a_{m,M} x_M = 0 \end{array}\right|$$

$$0 < \max_{1 \le k \le M} |x_k| \le \left[(2A)^{\frac{m}{M-m}}\right] + 1 .$$

The proof is well-known.

LEMMA 6 [6]. Let $\wp(z)$ be a Weierstrass elliptic function, $n, \sigma \in \mathbf{N}$. There exists an absolute constant γ_0, such that

$$(\wp^n(z))^{(\sigma)} = \sum_{2a+3b+4c=2n+\sigma} A_{a,b,c,n,\sigma}\, \wp(z)^a\, \wp'(z)^b\, \wp''(z)^c ,$$

where a, b, c and $A_{a,b,c,n,\sigma}$ are non-negative integers and

$$\sum_{2a+3b+4c=2n+\sigma} A_{a,b,c,n,\sigma} \le \sigma!\, 2^n e^{\gamma_0 \sigma} . \tag{3}$$

It is known, that if $2\omega_1$ and $2\omega_2$ are fundamental periods of $\wp(z)$ and ω is one of the numbers ω_1, ω_2, $\omega_1+\omega_2$, then $\wp'(\omega) = 0$. Then

$$(\wp''(z))^{(\sigma)}_{z=\omega} = \sum_{2a+4c=2n+\sigma} A_{a,0,c,n,\sigma}\, \delta_0^a\, \delta_2^c, \quad \delta_0 = \wp(\omega), \quad \delta_2 = \wp''(\omega) . \tag{4}$$

LEMMA 7. Let ω, ω_1-be transcendental numbers, $P(x, y) \in \mathbf{Z}[x, y]$, $P \ne 0$, P-irreducible, $\deg_x P = \nu$, $\deg_y P = \nu_1$, $P(\omega, \omega_1) = 0$. Then there exist Δ_0, γ_0, γ, depending only on $P(x, y)$, ω, ω_1, such that if for $\zeta \in \mathbf{A}$ the inequality

$$|\omega - \zeta| < L^{-\Delta n}, \quad \Delta \ge \Delta_0, \quad \deg \zeta = n, \quad L(\zeta) = L , \tag{5}$$

holds then there exists $\zeta_1 \in \mathbf{A}$, which satisfies the inequalities

$$|\omega_1 - \zeta_1| < L^{-\gamma \Delta n}, \quad L_1 = L(\zeta_1) \le L^{\gamma_0} e^{\gamma_0 n} ,$$

$$\frac{n}{\nu} \le \deg \zeta_1 = n_1 < n \nu_1; \quad \deg \mathbf{Q}(\zeta, \zeta_1) \le n_1 \nu .$$

PROOF. In what follows $\gamma_1, \gamma_2, \ldots$ denote positive constants, not depending on Δ, n and L. We have

$$|P(\zeta, \omega_1)| = |P(\zeta, \omega_1) - P(\omega, \omega_1)| = \left| \int_\zeta^\omega P'_x(x, \omega_1)\, dx \right| \le \tag{6}$$

$$\le |\omega - \zeta| \max_{|x-\omega| \le 1} |P'_x(x, \omega_1)| = |\omega - \zeta|\, \gamma_1 .$$

The polynomial $P(\zeta, y) \ne 0$. Indeed, it depends on y, because $P(\omega, \omega_1) = 0$ and ω is transcendental, and all the coefficients of the powers of y in $P(x, y)$ cannot vanish at $x = \zeta$ because $P(x, y)$ is irreducible. Further, $P(\zeta, y)$ cannot be a monomial. Indeed, if $P(\zeta, y) = P_t(\zeta)\, y^t$, then we could obtain from (5) and (6) that

$$0 < |P_t(\zeta)|\, |\omega_1^t| < \gamma_1 L^{-\Delta n} \gamma_2$$

and it is impossible for sufficiently large Λ, as by lemma 1

$$|P_t(\zeta)| > L(P_t)^{1-n} L^{-\deg P_t} \geq \gamma_3^{-n} L^{-\nu}, \tag{7}$$

and so

$$P(\zeta, y) = y^{\tau} P_{\nu_0+\tau}(\zeta) \prod_{s=1}^{\nu_0} (y-\beta_s), \tag{8}$$

where $\nu_1 > \tau \geq 0$, $\nu_0 \geq 1$, $P_{\nu_0+\tau}(x) \in \mathbf{Z}[x]$. Let

$$\zeta_1 = \beta_1, \quad |\omega_1 - \beta_1| = \min_{1 \leq s \leq \nu_0} |\omega_1 - \beta_s|,$$

then from (5), (6), (7) and (8)

$$|\omega_1 - \zeta_1| \leq \{|P(\zeta, \omega_1)|\, |\omega_1^{-\tau}|\, |P_{\nu_0+\tau}(\zeta)^{-1}|\}^{\frac{1}{\nu_0}} \leq \tag{9}$$

$$\leq (\gamma_1 L^{-\Lambda n} \gamma_5 \gamma_3^n L^{\nu})^{\frac{1}{\nu_0}} = \gamma_4 L^{-\frac{n\Lambda-\nu}{\nu_1}}.$$

As $P(\zeta, \zeta_1) = 0$ then $\deg_{\mathbf{Q}(\zeta)} \zeta_1 < \nu_1$, i.e.

$$n \leq \deg \mathbf{Q}(\zeta, \zeta_1) \leq n\nu_1, \quad n_1 = \deg \zeta_1 \leq n\nu_1. \tag{10}$$

If a_n is the leading coefficient of the defining polynomial of the number ζ, $\zeta^{(1)} = \zeta, \ldots, \zeta^{(n)}$ are the conjugates to ζ then

$$R(y) = a_n^{\nu} \prod_{m=1}^{n} P(\zeta^{(m)}, y) \in \mathbf{Z}[y],$$

and by lemma 2

$$L(R) < L(P)^n a_n^{\nu} \prod_{m=1}^{n} \max(1, |\zeta^{(m)}|)^{\nu} \leq L(P)^n L(\zeta)^{\nu} = L^{\nu} e^{\gamma_6 n}.$$

From $R(\zeta_1) = 0$ it follows, that the defining polynomial $Q_0(y)$ of the number ζ_1 divides $R(y)$ and in this case, according to the well-known Gauss lemma

$$R(y) = Q_0(y) Q_1(y), \quad Q_1(y) \in \mathbf{Z}[y],$$

therefore $L(Q_1) \geq 1$ and by lemma 3

$$L_1 = L(\zeta_1) = L(Q_0) < L(R)\, 2^{n\nu_1} < L^{\nu} e^{\gamma_6 n} 2^{\nu_1 n} \leq L^{\gamma_0} e^{\gamma_0 n}.$$

We have only to give the lower estimate for n_1. From $P(\zeta, \zeta_1) = 0$ it follows

$$n = \deg \zeta \leq \nu n_1, \quad n_1 \geq n/\nu,$$

and the lemma is proved.

LEMMA 8. Let $n \geq 2$, $\zeta \in \mathbf{C}$,

$$P(z) = a_n \prod_{k=1}^{n} (z - \alpha_k), \quad a_n > 0, \quad P(z) \in \mathbf{Z}[z], \quad \alpha_i \neq \alpha_j, \quad i \neq j, \tag{11}$$

then

$$(12) \qquad |P(\zeta)| \geq \delta (n-1)^{\frac{1-n}{2}} L(P)^{2-n} 2^{1-n}, \quad \delta = \min_{1 \leq k \leq n} |\zeta - \alpha_k| .$$

PROOF. Let $\delta = |\alpha_1 - \zeta|$, then

$$|P(\zeta)| = a_n \delta \prod_{i=2}^{n} |\zeta - \alpha_i| \geq a_n \delta \prod_{i=2}^{n} \frac{|\zeta-\alpha_i| + |\zeta - \alpha_1|}{2} \geq$$

$$\geq a_n \delta\, 2^{1-n} \prod_{i=2}^{n} |\alpha_1 - \alpha_i| .$$

Let

$$a_n^{2n-2} \prod_{i=2}^{n} \prod_{j<i} |\alpha_i - \alpha_j|^2 = \mathcal{D} ,$$

then $\mathcal{D} \in \mathbf{Z}$, $\mathcal{D} \neq 0$ and

$$a_n \prod_{i=2}^{n} |\alpha_1 - \alpha_i| = \sqrt{\mathcal{D}}\, a_n^{2-n} \prod_{i=3}^{n} \prod_{2 \leq j < i} |\alpha_i - \alpha_j|^{-1} .$$

As

$$A = \prod_{i=3}^{n} \prod_{2 \leq j < i} (\alpha_i - \alpha_j) = \begin{vmatrix} 1 & \dots & \alpha_2^{n-2} \\ \cdot & \dots & \cdot \\ 1 & \dots & \alpha_n^{n-2} \end{vmatrix} ,$$

the Hadamard's inequality implies, that

$$|A| \leq \prod_{i=2}^{n} \sqrt{(n-1) \max(1, |\alpha_i|^{2n-4})} = (n-1)^{\frac{n-1}{2}} \prod_{i=2}^{n} \max(1, |\alpha_i|)^{n-2} .$$

Now by lemma 2

$$|P(\zeta)| \geq \delta\, 2^{1-n} \sqrt{\mathcal{D}}\, a_n^{2-n} (n-1)^{\frac{1-n}{2}} \prod_{i=2}^{n} \max(1, |\alpha_i|)^{2-n} \geq$$

$$\geq \delta\, 2^{1-n} \sqrt{\mathcal{D}}\, (n-1)^{\frac{1-n}{2}} L(P)^{2-n} .$$

LEMMA 9 [1]. Let $\zeta \in \mathbf{C}$. Let $P_1(z), P_2(z) \in \mathbf{Z}[z]$, n_1, n_2, L_1, L_2-their degrees and lengths. If

$$|P_1(\zeta)| < 4^{-n_1 n_2} n_1^{-\frac{n_1}{2}} L_1^{2-n_1-n_2} L_2^{-n_1}, \quad |P_2(\zeta)| < 4^{-n_1 n_2} n_2^{-\frac{n_2}{2}} L_1^{-n_2} L_2^{2-n_1-n_2} ,$$

then $P_1(z)$ and $P_2(z)$ have a common zero.

LEMMA 10. Let $\alpha, \beta \in \mathbf{A}$, $\log \alpha \neq 0$. If each of the numbers α^β and α^{β^2} is algebraically dependent with the number $\log \alpha$, then there exists such a constant $A = A(\alpha, \beta) > 0$, that for any $\zeta \in \mathbf{A}$, $n = \deg \zeta$, $L = \max(L(\zeta), 3)$, the inequality

$$|\log \alpha - \zeta| \geq N^{-AN^2}, \quad N = n + \log L / \log \log L$$

holds.

Proof. Suppose, that for some $\zeta \in \mathbf{A}$ the inequality

$$|\log \alpha - \zeta| \leq N^{-A_0 N^2} \tag{13}$$

holds. We shall show, that then A_0 is less than some constant. By data there exist irreducible polynomials

$$\mathcal{P}(x, y) = \sum_{k=0}^{r} P_k(x) y^k, \quad P_r(x) \not\equiv 0; \quad Q(x, y) = \sum_{l=0}^{r_0} q_l(x) y^l, \quad q_{r_0}(x) \not\equiv 0, \tag{14}$$

such that

$$\mathcal{P}(\log \alpha, \alpha^{\beta}) = Q(\log \alpha, \alpha^{\beta^2}) = 0. \tag{15}$$

As

$$\log L \leq 2N \log N, \quad n \log L \leq 2N^2 \log N,$$

then for sufficiently large A_0 we can apply lemma 7 to each of the pairs of numbers α^{β}, $\log \alpha$ and α^{β^2}, $\log \alpha$, and thus there exist $\zeta_1, \zeta_2 \in \mathbf{A}$, satisfying the inequalities

$$|\alpha^{\beta} - \zeta_1|, \quad |\alpha^{\beta^2} - \zeta_2| \leq N^{-\gamma_1 A_0 N^2}, \tag{16}$$

$$n \gamma_3^{-1} \leq \deg \zeta_i \leq n \gamma_3, \quad i = 1, 2; \tag{17}$$

$$L(\zeta_1), L(\zeta_2) \leq L^{\gamma_2} e^{\gamma_2 n}, \quad \deg Q(\zeta_1, \zeta_2, \zeta) \leq \gamma_4 n$$

where $\gamma_1, \gamma_2, \ldots$ are positive constants, not depending on n, L, N, λ.

Let λ be sufficiently large, $N \in \mathbf{N}$,

$$q = X = [\lambda \sqrt{N}], \quad q_0 = [\lambda^2 N], \quad S = [\lambda N], \quad A_0 = \lambda^6, \tag{18}$$

$$\Omega(A, B) = ((s, x, y) : s, x, y \in \mathbf{Z},\ 0 \leq s \leq A,\ 0 \leq x,\ y \leq B),$$

$$g(z) = \sum_{k=0}^{q_0} \sum_{l, m=0}^{q} D_{k,l,m} z^k \alpha^{(l+m\beta)z}, \quad D_{k,l,m} = \sum_{t=0}^{q_0} D_{k,l,m,t} \log^t \alpha \tag{19}$$

where the numbers $D_{k,l,m,t}$ will be selected later. It is evident that

$$g^{(s)}(x + \beta y) = \sum_{k,l,m} D_{k,l,m} \sum_{\sigma=0}^{s} \frac{s!}{(s-\sigma)!} C_k^{\sigma} (x + \beta y)^{k-\sigma} \cdot \tag{20}$$

$$\cdot \log^{s-\sigma} \alpha\, (l + m\beta)^{s-\sigma} \alpha^{lx} (\alpha^{\beta})^{mx+ly} (\alpha^{\beta^2})^{my}.$$

For non-negative integers x and y the right-hand side of this equality is a polynomial in $\log \alpha$, α, β, α^{β} and α^{β^2}. With the help of lemma 4 equalities

(14), (15) and the defining polynomials of α and β we express the powers α^{β}, α^{β^2}, α and β, not less than ν, ν_0, $\nu_1 = \deg\alpha$ and $\nu_2 = \deg\beta$ correspondingly with the smaller powers. We have

$$g^{(s)}(x+\beta y) = \gamma_s^{-q_1} P_\nu(\log\alpha)^{-\gamma_6 qX} P_{\nu_0}(\log\alpha)^{-\gamma_7 qX} \cdot$$

$$\cdot \sum_{u=0}^{\nu_1-1} \sum_{v=0}^{\nu_2-1} \sum_{p=0}^{q_2} \sum_{r=0}^{\nu-1} \sum_{w=0}^{\nu_0-1} A_{s,x,y}^{u,v,p,r,w} \log^p\alpha \cdot \alpha^{u+r\beta+w\beta^2} \beta^v ,$$

where

$$q_1 = q_0 + S + qX \leq 3\lambda^2 N, \quad q_2 \leq S + q_1 + \gamma_6 qX \leq \gamma_7 \lambda^2 N, \tag{21}$$

$$(s, x, y) \in \Omega(S, X),$$

$$A_{s,x,y}^{u,v,p,r,w} = \sum_{k,l,m,t} B_{s,x,y}^{u,v,p,r,w}(k, l, m, t) D_{k,l,m,t}, \tag{22}$$

$$B_{s,x,y}^{u,\ldots,w}(k, l, m, t) \in \mathbf{Z}, \quad \sum_{k,l,m,t} \left|B_{s,x,y}^{u,\ldots,w}(k, \ldots, t)\right| \leq (\lambda N)^{\gamma_8 \lambda^2 N}.$$

By lemma 5 there exist $M = (q+1)^2 (q_0+1)^2$ numbers $D_{k,l,m,t}$ satisfying the $m = (S+1)(X+1)^2 \nu \nu_0 \nu_1 \nu_2 (q_2+1)$ conditions

$$A_{s,x,y}^{u,v,p,r,w} = 0, \quad (s, x, y) \in \Omega(S, X), \quad u = 0, \ldots, \nu_1 - 1, \quad v = 0, \ldots, \nu_2 - 1,$$

$$p = 0, \ldots, q_2+1, \quad r = 0, \ldots, \nu-1, \quad w = 0, \ldots, \nu_0 - 1, \tag{23}$$

while

$$D_{k,l,m,t} \in \mathbf{Z}, \quad 0 < \max|D_{k,l,m,t}| < 2(\lambda N)^{\gamma_8 \lambda^2 Nm/(M-m)} + 1.$$

Indeed, from (18) it follows that for $\lambda \geq \lambda_1$ also $M > m$ and

$$\frac{m}{M-m} \leq \frac{\gamma_9 \lambda^5 N^3}{\lambda^6 N^3 - \gamma_9 \lambda^5 N^3} < \frac{\gamma_{10}}{\lambda}, \quad \lambda \geq \lambda_2 \geq \lambda_1,$$

thus

$$0 < D = \max|D_{k,l,m,t}| \leq (\lambda N)^{\gamma_{11} \lambda N}. \tag{24}$$

Let

$$\zeta_{k,l,m}(z) = \sum_{t=0}^{q_0} D_{k,l,m,t} z^t, \quad k = 0, \ldots, q_0, \quad l, m = 0, \ldots, q,$$

and let $\psi(z)$ be the greatest common divisor of these polynomials. Put

$$\zeta_{k,l,m}(z) = \omega_{k,l,m}(z)\psi(z), \tag{25}$$

$$C_{k,l,m} = \omega_{k,l,m}(\log\alpha) = \sum_{t=0}^{q_3} C_{k,l,m,t} \log^t\alpha.$$

Then from (17), (19), (24) and lemma 3

$$C_{k,l,m,t} \in \mathbf{Z}, \quad 0 < \max|C_{k,l,m,t}| \leq (\lambda N)^{\gamma_{12} \lambda^2 N}, \tag{26}$$

$$q_3 \leq \lambda^2 N, \quad \lambda \geq \lambda_3 \geq \lambda_2.$$

Let

$$f(z) = \psi (\log \alpha)^{-1} g(z) = \sum_{k=0}^{q_0} \sum_{l,m=0}^{q} C_{k,l,m} z^k \alpha^{(l+m\beta)z} . \tag{27}$$

From (23) we obtain

$$f^{(s)}(x+\beta y) = 0, \quad (s, x, y) \in \Omega (S, X) .$$

Now from (18), (26), (27)

$$\max_{|z| \leq (1+|\beta|)\sqrt{\lambda^3 N+1}} |f(z)| = \left| \frac{1}{2\pi i} \oint_{|\tau|=\lambda^2 N} \prod_{x,y=0}^{X} \left(\frac{z-x-\beta y}{\tau-x-\beta y} \right)^{S+1} \frac{f(\tau)\, d\tau}{\tau - z} \right| \leq$$

$$\leq \lambda^2 N \left(\frac{(1+|\beta|)\sqrt{\lambda^2 N}+1+(1+|\beta|)\lambda\sqrt{N}}{\lambda^2 N-(1+|\beta|)\lambda\sqrt{N}} \right)^{(X+1)^2 (S+1)} \cdot$$

$$\cdot (q+1)^2 (q_0+1) D e^{\gamma_{13} q_1} (\lambda^2 N)^{q_0} e^{\gamma_{14} (1+|\beta|) q \lambda^2 N} \leq$$

$$\leq \left(\gamma_{15} \sqrt{\lambda N}\right)^{-\lambda^3 N^2} e^{\gamma_{16} \lambda^3 N^{3/2}} < (\lambda N)^{-\gamma_{17} \lambda^3 N^2}, \quad \lambda \geq \lambda_1 \geq \lambda_3 . \tag{28}$$

By Cauchy's formula this implies

$$\max_{\substack{|z| \leq \sqrt{\lambda^3 N}(1+|\beta|) \\ s \leq \lambda^2 N}} |f^{(s)}(z)| \leq (\lambda^2 N)^{\lambda^2 N} (\lambda N)^{-\gamma_{17} \lambda^3 N^2} < (\lambda N)^{-\gamma_{18} \lambda^3 N^2} ,$$

thus

$$|f^{(s)}(x+\beta y)| \leq (\lambda N)^{-\gamma_{18} \lambda^3 N^2}, \quad (s, x, y) \in \Omega (S_1, X_1) = \Omega_1 , \tag{29}$$

$$S_1 = \lambda^2 N, \quad X_1 = \left[\sqrt{\lambda^3 N}\right].$$

Let $\Phi_{s,x,y}$ be the number, which arives from $f^{(s)}(x+\beta y)$ on replacing $\log \alpha$, α^β and α^{β^2} with ζ, ζ_1 and ζ_2. Evidently, $\Phi_{s,x,y}$ is a polynomial with rational integral coefficients in the algebraic numbers α, β, ζ, ζ_1 and ζ_2. From (18), (26), (29) we obtain, that for $(s, x, y) \in \Omega_1$

$$L\, \Phi_{s,x,y} \leq (\lambda N)^{\gamma_{19} \lambda^2 N}, \quad \deg_\alpha \Phi_{s,x,y}, \quad \deg_\beta \Phi_{s,x,y} \leq \gamma_2 \lambda^{5/2} N , \tag{30}$$

$$\deg_\zeta \Phi_{s,x,y} \leq 2\lambda^2 N, \quad \deg_{\zeta_1} \Phi_{s,x,y}, \quad \deg_{\zeta_2} \Phi_{s,x,y} \leq \lambda^{5/2} N .$$

Let $L(\alpha)$, $L(\beta) \leq \gamma_{21}$. If $\Phi_{s,x,y} \neq 0$ then lemma 1, (1), (17), (30) give us

$$|\Phi_{s,x,y}| \geq \left((\lambda N)^{\gamma_{19} \lambda^2 N}\right)^{1-\gamma_1 n \deg \alpha \cdot \deg \beta} e^{-\gamma_{21} \gamma_1 n \gamma_{20} \lambda^{5/2} N} L(\zeta)^{-\gamma_{22} \lambda^2 N} \cdot$$

$$\cdot \left(L(\zeta_1) L(\zeta_2)\right)^{-\gamma_{23} \lambda^{5/2} N} \geq (\lambda N)^{-\gamma_{24} \lambda^{5/2} N^2}, \quad \lambda \geq \lambda_6 \geq \lambda_5, \quad (s, x, y) \in \Omega_1 . \tag{31}$$

On estimating with the help of (13) and (16) the numbers $\zeta^a \zeta_1^b \zeta_2^c - \log^a \alpha \cdot \alpha^{b\beta+c\beta^2}$ we obtain from (30) the inequalities

$$|\Phi_{s,x,y} - f^{(s)}(x+\beta y)| \leq N^{-\gamma_{25} \lambda^6 N^2}, \quad \lambda \geq \lambda_7 \geq \lambda_6, \quad (s, x, y) \in \Omega_1 . \tag{32}$$

Now from this and from (28)

$$(33)\qquad |\Phi_{s,x,y}| \leq N^{-\gamma_{25}\lambda^5 N^2} + (\lambda N)^{-\gamma_{18}\lambda^3 N^2} \leq (\lambda N)^{-\gamma_{26}\lambda^3 N^2}.$$

For $\lambda \geq \lambda_8 \geq \lambda_7$ the inequalities (31) and (33) are incompatible, therefore

$$\Phi_{s,x,y} = 0, \quad (s, x, y) \in \Omega_1, \quad \lambda \geq \lambda_8,$$

and this, together with (33) gives

$$|f^{(s)}(x+\beta y)| \leq N^{-\gamma_{25}\lambda^6 N^2}, \quad (s, x, y) \in \Omega_1, \quad \lambda \geq \lambda_8.$$

Using Hermite's interpolation formula we obtain

$$\max_{|z| \leq 2\sqrt{\lambda^3 N}} |f(z)| \leq \left| \frac{1}{2\pi i} \oint_{|\tau| = \lambda^2 N} \prod_{x,y=0}^{X_1} \left(\frac{z-x-\beta y}{\tau - x - \beta y}\right)^{S_1+1} \frac{f(\tau)\, d\tau}{\tau - z} \right| +$$

$$+ \left| \sum_{x,y=0}^{X_1} \sum_{s=0}^{S_1} \frac{f^{(s)}(x+\beta y)}{s!\, 2\pi i} \oint_{|x+\beta y - \varrho| = 0{,}5} \prod_{a,b=0}^{X_1} \left(\frac{z-a-b\beta}{\varrho - a - b\beta}\right)^{S_1+1} \frac{(\varrho - x - \beta y)^s}{\varrho - z} \cdot d\varrho \right| \leq$$

$$\leq \lambda^2 N \left(\frac{\sqrt{\lambda^3 N}\,(3+|\beta|)}{\lambda^2 N - \sqrt{\lambda^3 N} - 1} \right)^{\lambda^5 N^2} e^{\gamma_{16}\lambda^3 N^{3/2}} + N^{-\gamma_{25}\lambda^6 N^2} \left(4(1+|\beta|)\sqrt{\lambda^3 N}\right)^{\lambda^5 N^2} \leq$$

$$(34)\qquad \leq (\lambda N)^{-\gamma_{26}\lambda^5 N^2}, \quad \lambda \geq \lambda_9 \geq \lambda_8.$$

Let

$$(35)\qquad Q = \lambda^{5/2} N^{3/2}$$

then

$$\frac{(q+1)^2 (q+1) - 1}{\lambda^{5/2} N^{3/2}} < \frac{2\lambda^4 N^2}{\lambda^{5/2} N^{3/2}} = 2\sqrt{\lambda^3 N},$$

therefore, according to (34)

$$(36)\qquad |f(x/Q)| \leq (\lambda N)^{-\gamma_{26}\lambda^5 N^2}, \quad x = 0, 1, \ldots, (1+q)^2(1+q_0) - 1.$$

Now we use lemma 7 from [2]. Replacing $q_0 - 1$ with q_0 for

$$p = (q+1)^2, \quad \omega_l = \exp\left((l + m\beta)/Q\right); \quad l, m = 0, 1, \ldots, q,$$

$$A_{k,l,t} = C_{k,l,t} Q^{-k}, \quad Q_1 = (q+1)^2 (q_0+1) - 1,$$

we obtain

$$|C_{\kappa,\lambda,\mu}| \leq Q^k (q_0+1)(q+1)^2 \max_{0 \leq x \leq Q_1} \left| f\left(\frac{x}{Q}\right) \right| (q_0+1)!\, e^{(1+|\beta|)\, q Q_1/Q}.$$

$$(37)\qquad \cdot \max_{0 \leq t \leq q_0} \left| \frac{1}{t!} \left\{ \prod_{\substack{\varrho=1 \\ \varrho \neq \sigma}}^{p} (z - \omega_\varrho)^{-q_0} \right\}^{(t)}_{z=\omega_\sigma} \right|.$$

Let $|m_1-m_2|+|l_1-l_2|>0$. Then

$$\delta_{\sigma,\varrho}=|\omega_\sigma-\omega_\varrho|=\left|e^{\frac{l_2+m_2\beta}{Q}}\right|\left|e^{\frac{l_3+m_3\beta}{Q}}-1\right|;\quad l_2,m_2,|l_3|,|m_3|\le\lambda\sqrt{N}\,.$$

As $l, m=0, 1, \ldots, q$ then from (18), (35) and lemma 1 we have

$$\delta_{\sigma,\varrho}\ge\gamma_{27}\frac{|l_3+m_3\beta|}{Q}\ge\gamma_{27}Q^{-1}q^{-\gamma_{28}}\,.$$

Now

$$\left|\frac{1}{t!}\left\{\prod_{\substack{\varrho=1\\ \varrho\ne\sigma}}^{p}(z-\omega_\varrho)^{-q_0}\right\}^{(t)}_{z=\omega_\sigma}\right|\le$$

$$\le\left|\sum_{\substack{t_1+\ldots+t_p=t\\ t_\sigma=0}}\prod_{\substack{\varrho=1\\ \varrho=\varrho_\sigma}}^{p}\frac{q_0(q_0+1)\ldots(q_0+t_\varrho-1)}{t_\varrho!}(\omega_\sigma-\omega_\varrho)^{-q_0-t_\varrho}\right|\le$$

$$\le t^p\,2^{(p-1)q_0+t}\left(\frac{1}{\min\limits_{\varrho\ne\sigma}|\omega_\varrho-\omega_\sigma|}\right)^{q_0(p-1)+t}\le$$

$$\le q_0^p\,2^{pq_0}\left(Q_q^{\gamma_{28}}\gamma_{27}^{-1}\right)^{pq_0}\le(\lambda N)^{\gamma_{29}\lambda^4N^2}$$

and according to (25), (36) and (37)

$$|\omega_{k,l,m}(\log\alpha)|=|C_{k,l,m}|\le(\lambda N)^{-\gamma_{28}\lambda^5N^2}\,,\tag{38}$$

$$0\le k<q_0,\quad 0\le 1,\quad m\le q\,.$$

Let $\mathcal{D}(z)$ be the defining polynomial of ζ. Then, because of (13) and (18)

$$|\mathcal{D}(\log\alpha)|=|\mathcal{D}(\log\alpha)-\mathcal{D}(\zeta)|\le\tag{39}$$

$$\le|\log\alpha-\zeta|\,L(\mathcal{D}')\,(|\log\alpha|+1)^{n-1}\le N^{-\frac{1}{2}\lambda^6N^2}\,.$$

The inequalities (26), (38) and (39) imply, that the conditions of lemma 9 hold thus $\mathcal{D}(z)$ and $\omega_{k,l,m}(z)$ have a common zero and as $\mathcal{D}(z)$ is irreducible it divides all the polynomials $\omega_{k,l,m}(z)$. But these polynomials are relatively prime. This contradiction shows, that $\Lambda_0<\lambda_9^6$. We have only to put $\Lambda=\lambda_9^6$.

LEMMA 11. Let $m\ge 1$, $\omega_1,\ldots,\omega_m$ be transcendental numbers and each of the numbers $\omega_2,\ldots,\omega_m$ be algebraically dependent with ω_1. If $\Theta\in\mathbf{A}$,

$$P(z_0,z_1,\ldots,z_m)\in\mathbf{Z}[z_0,z_1,\ldots,z_m],\quad|P(\Theta,\omega_1,\ldots,\omega_m)|\le e^{-\Omega},\quad\Omega>0\,,$$

then there exists $Q(z_1)\in\mathbf{Z}[z_1]$ such, that

(1) $|Q(\omega_1)|\le e^{-\Omega}L(P)^{\gamma_0}e^{\gamma_1\deg P}$,
(2) $L(Q)\le L(P)^{\gamma_2}e^{\gamma_3\deg P}$,
(3) $\deg Q\le\gamma_4\deg P$,
(4) if $Q(\omega_1)=0$, then $P(\Theta,\omega_1,\ldots,\omega_m)=0$,

where $\gamma_0, \ldots, \gamma_4$ depend only on $\Theta, \omega_1, \ldots, \omega_m$, but do not depend on the polynomial P.

PROOF. By data, there exists a polynomial $\mathcal{P}(x, y) \in \mathbf{Z}[x, y]$, $\mathcal{P} \neq 0$, such that $\mathcal{P}(\omega_1, \omega_m) = 0$. Let $\mathbf{X}$ be the set of polynomials $R(z) \not\equiv 0$, $R(z) \in \mathbf{Z}[z, \Theta, \omega_1, \ldots, \omega_{m-1}]$ for which $R(\omega_m) = 0$. It is not empty, as $\mathcal{P}(\omega_1, z)$ belongs to it, therefore it contains a polynomial of the lowest degree. Let it be $R_0(z)$, $\deg R_0 = r$, $\omega_m^{(1)} = \omega_m, \ldots, \omega_m^{(r)}$ be it's zeros, and $r = r(\Theta, \omega_1, \ldots, \ldots, \omega_{m-1}) \neq 0$ be its leading coefficient. Then

$$\varrho_m = r^{rn_m} \prod_{\tau=1}^{r} P(\omega_m^{(\tau)}, \Omega, \omega_1, \ldots, \omega_{m-1}), \quad n_m = \deg_{z_m} P,$$

is a symmetrical polynomial in $\omega_m^{(1)}, \ldots, \omega_m^{(r)}$, therefore ϱ_m is a polynomial with rational integer coefficients in the arguments $\Theta, \omega_1, \ldots, \omega_{m-1}$. Replacing these arguments with $z_0, z_1, \ldots, z_{m-1}$ we obtain $Q_{m-1}(z_0, \ldots, z_{m-1})$ and if $Q_{m-1}(\Theta, \omega_1, \ldots, \omega_{m-1}) = 0$ then for some index τ_0

$$P(\Theta, \omega_1, \ldots, \omega_{m-1}, \omega_m^{(\tau_0)}) = 0$$

and because of the minimality of the degree of $R_0(z)$ this equality takes place also for $\tau = 1, \ldots, r$, i.e.

$$P(\Theta, \omega_1, \ldots, \omega_{m-1}, \omega_m) = 0.$$

If we produce all the eliminations, we obtain the polynomial $Q(z_1) \in \mathbf{Z}[z_1]$ (in the last step we eliminate Θ). Evidently (1), (2), (3) hold.

LEMMA 12. Let $P(z) \in \mathbf{Z}[z]$, $P \not\equiv 0$, $n_0 = \deg P$, $L_0 = L(P)$, $\zeta \in \mathbf{C}$. If

$$|P(\zeta)| < 16^{-n_0^2} n_0^{-n_0} L_0^{2-2n_0}$$

then there exists $Q(z) \in \mathbf{Z}[z]$ which divides $P(z)$ and is a power of an irreducible polynomial and satisfies the inequality

$$|Q(\zeta)| < |P(\zeta)| \, 16^{n_0^2} n_0^{n_0} L_0^{2n_0-2}.$$

PROOF. Let $P_1(z), \ldots, P_s(z) \in \mathbf{Z}[z]$, $(P_i, P_j) = 1$, $i \neq j$, $P_i(z)$ be a power of an irreducible polynomial, $m_i = \deg P_i$, $L_i = L(P_i)$, $i = 1, \ldots, s$,

$$P(z) = aP_1(z) \ldots P_s(z), \quad a \in \mathbf{Z}.$$

If the inequality

$$(41) \qquad |P_i(\zeta)| \geq 4^{-m_i(n_0-m_i)} m_i^{-m_i} L_i^{2-n_0} L(P/P_i)^{-m_i}, \quad i = 1, \ldots, s,$$

would take place, then by lemma 3 we should obtain that

$$|P(\zeta)| \geq \prod_{i=1}^{s} 4^{-m_i(n_0-n_i)} m_i^{-m_i} L_i^{2-n_0} L(P/P_i)^{-m_i} \geq$$

$$\geq 4^{-n_0^2} n_0^{-n_0} (2^{n_0} L_0)^{2-n_0} (2^{n_0} L_0)^{-n_0} > 16^{-n_0^2} n_0^{-n_0} L_0^{2-2n_0}.$$

Thus for at least one $i = i_0$ the inequality (41) is not valid. Let $Q(z) = P_{i_0}(z)$, $Q_1(z) = P(z)/Q(z)$, $n_1 = \deg Q(z)$, $n_2 = \deg Q_1(z)$. Then

$$|Q_1(\zeta)| \geq 4^{-n_1 n_2} n_2^{-n_2} (2^{n_0} L_0)^{2-n_1-n_2} (2^{n_0} L_0)^{-n_2} > 16^{-n_0^2} n_0^{-n_0} L_0^{2-2n_0}$$

because otherwise by lemma 9 Q and Q_1 would have a common zero. Now

$$|Q(\zeta)| = |P(\zeta)|\,|Q(\zeta)|^{-1} \leq 16^{n_0^2} n_0^{n_0} L_0^{2n_0-2} |P(\zeta)|\,.$$

Proof of the theorem. Suppose that the theorem is not valid, i.e. that there exist polynomials

$$(42)\qquad P_1(x, y),\ P_2(x, y),\ P_3(x, y),\ P_4(x, y) \in \mathbf{Z}[x, y],\quad P_1 P_2 P_3 P_4 \neq 0\,,$$

for which

$$(43)\quad P_1(\log\alpha, \alpha^{\beta}) = P_2(\log\alpha, \alpha^{\beta^2}) = P_3(\log\alpha, \delta_0) = P_4(\log\alpha, \delta_2) = 0\,.$$

Let

$$(44)\qquad N \in \mathbf{N},\quad S_{-1} = q_0 = [N^2 \log N],\quad q_1 = N,\quad q = [\log^2 N]\,,$$
$$X_{-1} = [\gamma N \log N]\,,$$

where γ is some positive number, which, as well as the numbers $\gamma_1, \gamma_2, \ldots$, arising later does not depend on N. Let also

$$f(z) = \sum_{k=0}^{q_0} \sum_{l,m=0}^{q_1} \sum_{n=0}^{q} C_{k,l,m,n}\, z^k \alpha^{(l+m\beta)z} \wp^n(z+\omega)\,,$$

$$C_{k,l,m,n} = \sum_{t=0}^{q_0} C_{k,l,m,n,t} \log^t \alpha\,,$$

where the numbers $C_{k,l,m,n,t}$ will be selected later.

For $x, y \in \mathbf{Z}$ with the help of (4) we obtain the equality

$$f^{(s)}(2x+2\beta y) = \sum_{k,l,m,n} C_{k,l,m,n} \sum_{\sigma_1+\sigma_2+\sigma_3=s} \frac{s!}{\sigma_2!\,\sigma_3!} C_k^{\sigma_1} (2x+2\beta y)^{k-\sigma_1}\cdot$$
$$\cdot (l+m\beta)^{\sigma_2} \log^{\sigma_2}\alpha \cdot \alpha^{2(l+m\beta)(x+\beta y)} \left(\wp^n(z)\right)^{(\sigma_3)}_{z=\omega} =$$
$$= \sum_{k,l,m,n} C_{k,l,m,n} \sum_{\sigma_1+\sigma_2+\sigma_3=s} \frac{s!}{\sigma_2!\,\sigma_3!} C_k^{\sigma_1} (2x+2\beta y)^{k-\sigma_1}\cdot$$
$$\cdot (l+m\beta)^{\sigma_2} \log^{\sigma_2}\alpha \cdot \alpha^{2lx+\beta(2mx+2ly)+\beta^2 2my} \sum_{2a+4c=2n+\sigma_3} A_{a,0,c,n,\sigma_3}\, \delta_0^a \delta_2^c\,.$$

(46)

Just like in the proof of lemma 10 with the help of the equalities (43), defining polynomials for α and β and lemma 4 we express the greater powers of α^{β}, α^{β^2}, δ_0, δ_2, α and β by the less ones. On denoting by $p_r(x)$, $r = 1, 2, 3, 4$, the leading coefficients of the polynomials $P_r(x, y)$, $r = 1, 2, 3, 4$, we obtain the equality

$$f^{(s)}(2x+2\beta y) = \gamma_3^{M_0} P_1(\log\alpha)^{-M_1} P_2(\log\alpha)^{-M_2} P_3(\log\alpha)^{-M_3} P_4(\log\alpha)^{-M_4}\cdot$$

$$\cdot\sum_{\tau=0}^{q_2}\sum_{\bar\tau} E_{s,x,y}^{\bar\tau,\tau}\log^{\tau}\alpha\cdot\alpha^{\tau_0+\beta\tau_1+\beta^2\tau_2}\delta_0^{\tau_3}\delta_2^{\tau_4}\beta^{\tau_5}.$$

Here the components $\tau_0, \tau_1, \ldots, \tau_5$ of the vector $\bar\tau$ are non-negative integers less than or equal to γ_4 and $q_2, M_0, \ldots, M_4$ are non-negative integers and

(47) $$q_2 \le \gamma_6 N^2 \log N, \quad M_k \le \gamma_5 N^2 \log N, \quad k = 0, \ldots, 4.$$

The $E_{s,x,y}^{\bar\tau,\tau}$ (their number is less than or equal to $m = (S_{-1}+1)\cdot(X_{-1}+1)q_2\gamma_4^6$) are the linear forms over $\mathbf{Z}$ in

$$M = (q_0+1)^2(q_1+1)^2(q+1)$$

parameters $C_{k,l,m,t}$. For sufficiently small $\gamma<1$ the inequality $M \ge 2m$ holds. We use lemma 5. In our case, according to (3), (44), (45), (46) and (47)

$$A \le N^{\gamma_7 N^2 \log N}.$$

therefore there exist numbers $C_{k,l,m,t}\in\mathbf{Z}$, satisfying the conditions

$$E_{s,x,y}^{\bar\tau,\tau} = 0, \quad 0\le\tau\le q_2; \quad 0\le\tau_0, \ldots, \tau_5\le\gamma_4; \quad (s,x,y)\in\Omega(S_{-1}, X_{-1}),$$

(48) $$0 < \max|C_{k,l,m,t}| \le 2N^{\gamma_7 N^2 \log N}+1.$$

Thus

(49) $$f^{(s)}(2x+2\beta y) = 0, \quad (s,x,y)\in\Omega\quad(S_{-1}, X_{-1}) = \Omega_{-1}.$$

Let

(50) $$X_0 = [N\log^{3/2}N/\log\log N], \quad S_0 = [N^2\log^{3/2}N/\log\log N],$$

(51) $$X_p = 2^p X_0, \quad S_p = 2^p S_0, \quad p = 0, 1, \ldots; \Omega_p = \Omega(S_p, X_p).$$

MAIN LEMMA. Let $p\ge 0$. If $N\ge N_0$ and

(52) $$f^{(s)}(2x+2\beta y) = 0, \quad (s,x,y)\in\Omega_{p-1},$$

then

$$f^{(s)}(2x+2\beta y) = 0, \quad (s,x,y)\in\Omega_p.$$

PROOF. Let $T_p = 2^{p+1}N^2$, $\mathcal{L}_p$ be a parallelogramm with apexes at the points $\pm 2T_p$ $\pm 2T_p\beta$

(53) $$F_{T_p}(z) = \prod_{r=-T_p}^{T_p}\sin^{2q}\frac{\pi}{2}(z-2r\beta+\omega), \quad \Phi_{T_p}(z) = F_{T_p}(z)f(z).$$

The function $F_{T_p}(z)$ has the order of zero equal to $2q$ at the points

$$z = 2r\beta-\omega-2h, \quad h = 0, \pm 1, \pm 2, \ldots; \quad r = 0, \pm 1, \ldots, \pm T_p,$$

thus the function $\Phi_{T_p}(z)$ is analytic on $\mathcal{L}_p$ and in the open domain $\mathcal{L}_p'$, bounded with $\mathcal{L}_p$. According to (52) and (53)

$$\Phi_{T_p}^{(s)}(2x+2\beta y) = 0, \quad (s, x, y) \in \Omega_{p-1},$$

thus for $z \in \mathcal{L}_p'$

$$\Phi_{T_p}(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}_p} \prod_{x, y=0}^{X_{p-1}} \left(\frac{z-2x-2\beta y}{\zeta-2x-2\beta y} \right)^{S_{p-1}+1} \cdot \frac{\Phi_{T_p}(\zeta)\, d\zeta}{\zeta - z}.$$

Because $\wp(z+\omega)$ is bounded on $\mathcal{L}_p$, this, together with (44), (45), (48), (50), (51), (53) gives us

$$\max_{|z| = (1+|\beta|) X_p+1} |\Phi_{T_p}(z)| \leq \gamma_8 T_p \left(\frac{(1+|\beta|)(X_p+1+X_{p-1})}{\gamma_9 T_p - (1+|\beta|) X_{p-1}} \right)^{(X_{p-1}+1)^2 (S_{p-1}+1)} \cdot$$

$$\cdot e^{\gamma_{10} q T_p^2} (\gamma_{12} T_p)^{\gamma_{11} N^2 \log N} e^{\gamma_{13} q_1 T_p} \leq e^{-\gamma_{14} 8^p X_{p-1}^2 S_{p-1} \log N + \gamma_{15} T_p^2 (\log N + p)^2},$$

(54)

Let $\varrho = \min(1, |\beta|/2, |\omega|/2)$. Then in the circle $\varrho \geq |z-2x-2\beta y|$, $x, y \in \mathbf{Z}$, each of the factors of $F_{T_p}(z)$ is bounded from below, therefore from (53) and (54) we deduce that

$$\max_{|z-2x-2\beta y| = \varrho} |f(z)| \leq e^{-\gamma_{14} 8^p X_{p-1}^2 S_{p-1} \log N + \gamma_{15} T_p^2 (\log N + p)^2 + \gamma_{16} T_p q},$$

$$|x|, |y| \leq X_p.$$

$$\varepsilon_p = \max_{(s, x, y) \in \Omega_p} |f^{(s)}(2x+2\beta y)| \leq s! \, \varrho^{-S_p} e^{-\gamma_{14} X_{p-1}^2 S_{p-1} \log N + \gamma_{17} T_p^2 (\log N + p)^2},$$

(55)

$$\varepsilon_0 \leq \exp\left(\gamma_{18} N^2 \log^2 N - \gamma_{14} (\gamma N \log N - 1)^2 (N^2 \log N - 1) \log N + \right.$$
$$\left. + \gamma_{17} 4N^4 \log^2 N \right) < \exp(-\gamma_{19} N^4 \log^4 N),$$

(56) $$\varepsilon_p < \exp(-\gamma_{20} 8^p N^4 \log^5 N), \quad p = 1, 2, \ldots.$$

Let $\mathbf{Q}(\alpha, \beta) = \mathbf{Q}(\Theta)$ while α and β may be be expressed by Θ with rational integer coefficients. Then $f^{(s)}(2x+2\beta y)$ is a polynomial with rational integer coefficients in $\log \alpha$, α^β, $\alpha^{\beta/2}$ δ_0, δ_2, Θ. From (4), (44), (46), (48), (50) and (51) it follows, that the length of this polynomial is less than or equal to

(57) $$S_p^{\gamma_{21} S_p} X_p^{\gamma_{22} q_0} \leq N^{\gamma_{23} 2^p (p + \log N)^{3/2} / \log \log N}$$

and its degrees in all the arguments do not exceed

$$2^p N^2 \log^{3/2} N / \log \log N.$$

Using lemma 11 we obtain a polynomial $Q(z) \in \mathbf{Z}[z]$ which, according to (55), (56) satisfies the conditions

$$|Q(\log \alpha)| \leq \exp(-\gamma_{25} 8^p N^4 \log^4 N),$$
$$\deg Q \leq \gamma_{26} 2^p N^2 \log^{3/2} N / \log \log N,$$
$$L(q) \leq \exp\left(\gamma_{27} 2^p N^2 (p + \log N)^{3/2} \log N / \log \log N\right).$$

We shall show, that $Q(\log\alpha)=0$. If it is not so, then we apply lemma 12 to $Q(z)$. The condition (40) holds, because

$$16^{-n_0^2}\, n_0^{-n_0}\, L_0^{2-2n_0} \geq 16^{-\gamma_{26}^2 4^p N^4 \log^3 N} (\gamma_{26}\, 2^p N^2 \log^{3/2} N)^{-\gamma_{26} 2^p N^2 \log^{3/2} N} \cdot$$

$$\cdot\, N^{-2\gamma_{27}\gamma_{26} 4^p N^4 (p+\log N)^3/\log\log N} > e^{-\gamma_{26} 8^p N^4 \log^4 N}, \quad N \geq N_1 .$$

Thus there exists a polynomial $Q_0(z)\in\mathbf{Z}[z]$ which is a power of an irreductible polynomial and

$$|Q_0(\log\alpha)| < \exp(-\gamma_{28}\, 8^p N^4 \log^4 N),$$

$$\deg Q_0 \leq \gamma_{26}\, 2^p N^2 \log^{3/2} N/\log\log N,$$

$$L(Q_0) \leq 2^{\deg Q} L(Q) \leq N^{\gamma_{29} 2^p N^2 (p+\log N)^{3/2}/\log\log N} .$$

Let $Q_0(z)=q(z)^r$, $q(z)\in\mathbf{Z}[z]$, $r\in\mathbf{N}$, $q(z)$ be irreducible. Then

$$|q(\log\alpha)| < \exp(-\gamma_{28}\, 8^p r^{-1} N^4 \log^4 N),$$

$$L(q) \leq (2^{\deg Q} L(Q))^{1/r} \leq N^{\gamma_{30} 2^p r^{-1} N^2 (p+\log N)^{3/2}/\log\log N}, \tag{58}$$

$$\deg q \leq \gamma_{26}\, 2^p r^{-1} N^2 \log^{3/2} N/\log\log N . \tag{59}$$

Let ζ be the zero nearest to $\log\alpha$ of $q(z)$. By lemma 10

$$|\log\alpha-\zeta| \geq (\deg q+\log L(q)/\log\log L(q))^{-A(\deg q+\log L(q)/\log\log(q))^2} . \tag{60}$$

At the same time, by lemma 8 the inequality

$$|\log\alpha-\zeta| < |q(\log\alpha)|\,(2\deg q)^{\deg q} L(q)^{\deg q} < e^{-\gamma_{31} 8^p r^{-1} N^4 \log^4 N}$$

holds, which is incompatible with (60) because of (58) and (59). Thus $Q(\log\alpha)=0$ and by lemma 11

$$f^{(s)}(2x+2\beta y)=0, \quad (s,x,y)\in\Omega_p .$$

Because of (49) for $p=0$ the conditions of the main lemma hold, therefore at all the points $2x+2\beta y$; $x,y=0,1,\ldots,$ all the derivatives of the function $f(z)$ vanish, and thus $f(z)\equiv 0$. But $\wp(z)$ and the functions $e^{(k+l\beta)z}$, $k,l=0,1,\ldots,q$, are algebraically independent over $\mathbf{R}(z)$ therefore all the numbers $C_{k,l,m}=0$, and because the number $\log\alpha$ is transcendental all the numbers $C_{k,l,m,t}=0$. This contradicts to the selection of them. The theorem is proved.

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SOME PROPERTIES OF THE BMO_Φ-SPACES

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1. In what follows we suppose the knowledge of the theory of the Young-functions as well as the theory of the Orlicz-spaces generated by Young-functions. ([1])

In [2] we have introduced the notion of the conditional $L_\Phi^{\mathcal{F}}$-norm for a Young-function Φ as follows:

Let X be a random variable on the probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F}\subset\mathcal{A}$ be an arbitrary σ-field. Consider the set of the random variables γ defined by

$$F_X^{\Phi,\mathcal{F}} = \left\{\gamma : \gamma > 0 \text{ a.e., } \mathcal{F}\text{-meaurable, } E\left(\Phi\left(\frac{|X|}{\gamma}\right)\Bigg|\mathcal{F}\right) \leq 1 \text{ a.e.}\right\}.$$

We say that $X\in L_\Phi^{\mathcal{F}}$ if $F_X^{\Phi,\mathcal{F}}$ is not empty and in this case we put

$$\|X\|_\Phi^{\mathcal{F}} = \text{ess. inf. } F_X^{\Phi,\mathcal{F}}$$

We define $\|X\|_\Phi^{\mathcal{F}} = +\infty$ if $F_X^{\Phi,\mathcal{F}}$ is empty.

The existence and the uniqueness of $\|X\|_\Phi^{\mathcal{F}}$ is not in question. The random variable $\|X\|_\Phi^{\mathcal{F}}$ has the properties of a norm a.e. in the sense that if $X\in L_\Phi^{\mathcal{F}}$ then

(a) for any real c we have $cX\in L_\Phi^{\mathcal{F}}$ and

$$\|cX\|_\Phi^{\mathcal{F}} = |c|\,\|X\|_\Phi^{\mathcal{F}} \text{ a.e.}$$

(b) $\|X\|_\Phi^{\mathcal{F}} = 0$ a.e. if and only if $X = 0$ a.e.

(c) if $Y\in L_\Phi^{\mathcal{F}}$ then $X+Y\in L_\Phi^{\mathcal{F}}$, moreover,

$$\|X+X\|_\Phi^{\mathcal{F}} \leq \|X\|_\Phi^{\mathcal{F}} + \|Y\|_\Phi^{\mathcal{F}} \text{ a.e.}$$

he property (a) is satisfied in the following more general situation, too:

(a′) let $Y\neq 0$ a.e. be an $\mathcal{F}$-measurable random variable and suppose that $X\in L_\Phi^{\mathcal{F}}$ then $XY\in L_\Phi^{\mathcal{F}}$, and $\|XY\|_\Phi^{\mathcal{F}}=|Y|\,\|X\|_\Phi^{\mathcal{F}}$ a.e.

These notions will be useful whenever treating some properties of the so-called BMO_Φ-spaces.

2. The following lemma, which is a direct generalization of an inequality of Garsia, will help us to establish an interesting inequality in the theory of the so-called BMO_Φ-spaces.

LEMMA 1. Let $(X_n, \mathcal{F}_n)$, $n\geq 1$, be a non-negative submartingale and let Φ be a Young-function whose conjugate junction Ψ has a finite power q. Then for arbitrary σ-field $\mathcal{F}\subset\mathcal{F}_1$ we have

$$\|X_n^*\|_\Phi^{\mathcal{F}}\leq q\,\|X_n\|_\Phi^{\mathcal{F}}\ \text{a.e.}$$

PROOF. Let $\gamma>0$ a.e. be an $\mathcal{F}$-measurable random variable. Then trivially $\left(\frac{X_n}{\gamma}, \mathcal{F}_n\right)$, $n\geq 1$, is also a non-negative submartingale. We have for arbitrary $\lambda>0$

$$\lambda P\left(\frac{X_n^*}{\gamma}\geq\lambda\,\Big|\,\mathcal{F}\right)\leq E\left(\int\limits_{\left\{\frac{X_n^*}{\gamma}>\lambda\right\}}\frac{X_n}{\gamma}\,dP\,\Big|\,\mathcal{F}\right),\quad \text{where}\quad X_n^*=\max_{1\leq k\leq n} X_k .$$

This inequality can be deduced in the same way as the classical inequality of Doob and it holds also with

$$X_k'=\min\left(\frac{X_k}{\gamma}, a\right),\quad k=1,2,\ldots$$

where $a>0$ is arbitrary but fixed constant. Thus

$$\lambda P(X_n'^*\geq\lambda|\mathcal{F})\leq E\left(\int\limits_{\{X'_n\geq\lambda\}}\frac{X_n}{\gamma}\,dP\,\Big|\,\mathcal{F}\right),$$

where $X_n'^*=\max\limits_{1\leq k\leq n} X_k'$. The conditional distribution is taken to be the regular version of this notion. Integrate this inequality with respect to the measure generated by $\varphi(\lambda)$, the right-hand side derivative of Φ. Using at the same time the Fubini theorem we get

$$E\left(\int_0^{X'^*_n}\lambda\,d\varphi(\lambda)\,\Big|\,\mathcal{F}\right)\leq E\left(\frac{X_n}{\gamma}\int_0^{X'^*_n}d\varphi(\lambda)\,\Big|\,\mathcal{F}\right)\quad\text{a.e.}$$

which can also be written in the form

$$E\Big(\Psi\left(\varphi(X_n'^*)\right)\big|\mathcal{F}\Big)\leq E\left(X_n\,\varphi(X_n'^*)|\mathcal{F}\right)\quad\text{a.e.}$$

Use the Young-inequality for the right-hand side to get

$$E\left(\Psi\left(\varphi\left(X_n'^*\right)\right)|\mathcal{F}\right) \le cE\left(\Phi\left(\frac{X_n}{c\gamma}\right)|\mathcal{F}\right) + cE\left(\Psi\left(\varphi\left(X_n'^*\right)\right)|\mathcal{F}\right) \quad \text{a.e.}$$

where $0 < c < 1$ is an arbitrary constant. This gives a.e.

$$(1-c)\, E\left(\Psi\left(\varphi\left(X_n'^*\right)\right)|\mathcal{F}\right) \le cE\left(\Phi\left(\frac{X_n}{c\gamma}\right)|\mathcal{F}\right).$$

Now let $a \uparrow +\infty$. Then we get

$$(1-c)\, E\left(\Psi\left(\varphi\left(\frac{X_n^*}{\gamma}\right)\right)|\mathcal{F}\right) \le cE\left(\Phi\left(\frac{X_n}{c\gamma}\right)|\mathcal{F}\right) \quad \text{a.e.}$$

Apply this with $c = 1/q$ and with cX_n instead of the submartingale X_n. Then

$$\frac{q-1}{q}\, E\left(\Psi\left(\varphi\left(\frac{X_n^*}{q\gamma}\right)\right)|\mathcal{F}\right) \le \frac{1}{q}\, E\left(\Phi\left(\frac{X_n}{\gamma}\right)|\mathcal{F}\right) \quad \text{a.e.}$$

Putting $\gamma = \|X_n\|_\Phi^{\mathcal{F}} + \varepsilon$, where $\varepsilon > 0$ is arbitrary constant, we get a.e.

$$(q-1)\, E\left(\Psi\left(\varphi\left(\frac{X_n^*}{q\gamma}\right)\right)|\mathcal{F}\right) \le E\left(\Phi\left(\frac{X_n}{\gamma}\right)|\mathcal{F}\right) \le 1\,.$$

Remark that

$$(q-1)\,\Psi\left(\varphi(X)\right) \ge \Phi(x)\,.$$

From this

$$E\left(\Phi\left(\frac{X_n^*}{q\left(\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon\right)}\right)|\mathcal{F}\right) \le 1\,.$$

This means that for arbitrary $\varepsilon > 0$ we have

$$\|X_n^*\|_\Phi^{\mathcal{F}} \le q\left(\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon\right)$$

and finally letting $\varepsilon \to 0$ we get a.e.

$$\|X_n^*\|_\Phi^{\mathcal{F}} \le q\, \|X_n\|_\Phi^{\mathcal{F}}\,.$$

This proves the assertion.

In the paper [2] the author has introduced together with J. MOGYORÓDI the definition of the BMO_Φ-spaces. These are the following:

Let $X \in L^1$ be a random variable and let $(\mathcal{F}_n\uparrow)$, $n \ge 0$, be an increasing sequence of σ-fields of events. We suppose that $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right) = \mathcal{A}$, the basic σ-field of the considered probability space $(\Omega, \mathcal{A}, P)$. Consider the martingale $X_n = E(X|\mathcal{F}_n)$, $n \ge 0$, where we suppose that $X_0 = 0$ a.e. Let Φ be an arbitrary Young-function. We say that $X \in BMO_\Phi$ if

$$\left\| \sup_{n \ge 1} \|X - X_{n-1}\|_\Phi^{\mathcal{F}_n} \right\|_\infty < +\infty$$

and in this case we define the BMO_Φ-norm of X to be

$$\|X\|_{BMO_\Phi} = \left\|\sup_{n\geq 1} \|X - X_{n-1}\|_\Phi^{\mathcal{F}_n}\right\|_\infty .$$

It is easily seen that $\|X\|_{BMO_\Phi}$ is really a norm.

Before applying the result of the preceding simple lemma to the BMO_Φ-spaces we have to add some minor remarks.

Consider the preceding regular martingale which corresponds to $X\in \in BMO_\Phi$. For arbitrary indices $n\geq k\geq 1$ let us also consider the random variable

$$\|X_n - X_{k-1}\|_\Phi^{\mathcal{F}_k} .$$

We show that this random variable increases a.e. in n, and its a.e. limit is

$$\|X - X_{k-1}\|_\Phi^{\mathcal{F}_k} .$$

In fact, with arbitrary $\mathcal{F}_k$-measurable and a.e. positive γ, the sequence

$$\Phi\left(\frac{|X_l - X_{k-1}|}{\gamma}\right), \quad l = k, k+1, \dots$$

if it is integrable, forms a non-negative submartingale. Consequently, for every $\gamma\in F^{\Phi,\mathcal{F}_k}_{X_{n+1}-X_{k-1}}$ and for arbitrary $n\geq k\geq 1$ we have

$$E\left(\Phi\left(\frac{|X_n - X_{k-1}|}{\gamma}\right)|\mathcal{F}_k\right) \leq E\left(\Phi\left(\frac{|X_{n+1} - X_{k-1}|}{\gamma}\right)|\mathcal{F}_k\right) \leq 1 \quad \text{a.e.}$$

It follows that

$$\|X_n - X_{k-1}\|_\Phi^{\mathcal{F}_k} \leq \|X_{n-1} - X_{k-1}\|_\Phi^{\mathcal{F}_k} \quad \text{a.e.}$$

Thus

$$\lim_{n\to+\infty} \|X_n - X_{k-1}\|_\Phi^{\mathcal{F}_k}$$

exists a.e.

Let $\gamma\in F^{\Phi,\mathcal{F}_k}_{X-X_{k-1}}$. Then using the Jensen inequality we have a.e. for $n\geq k\geq 1$

$$E\left(\Phi\left(\frac{|X_n - X_{k-1}|}{\gamma}\right)|\mathcal{F}_k\right) = E\left(\Phi\left(\frac{|E(X - X_{k-1}|\mathcal{F}_n)|}{\gamma}\right)|\mathcal{F}_k\right) \leq$$

$$\leq E\left(\Phi\left(\frac{E(|X - X_{k-1}|\,|\mathcal{F}_n)}{\gamma}\right)|\mathcal{F}_k\right) \leq E\left(\Phi\left(\frac{|X - X_{k-1}|}{\gamma}\right)|\mathcal{F}_k\right) \leq 1 \quad \text{a.e.}$$

We deduce that for arbitrary indices $n\geq k\geq 1$

$$\gamma \geq \|X_n - X_{k-1}\|_\Phi^{\mathcal{F}_k} \quad \text{a.e.}$$

So,

$$\gamma \geq \lim_{n\to+\infty} \|X_n - \dot{X}_{k-1}\|_\Phi^{\mathcal{F}_k}$$

and, consequently,

$$\|X-X_{k-1}\|_\Phi^{\mathcal{F}_k} \geq \lim_{n\to+\infty} \|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k}.$$

We prove now the converse inequality. Let for this purpose $\varepsilon>0$ be arbitrary. Then with arbitrary $A\in\mathcal{F}_k$ from the Beppo Levi theorem we obtain

$$\int_A E\left(\Phi\left(\frac{|X-X_{k-1}|}{\lim\limits_{n\to+\infty}\|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k}+\varepsilon}\right)|\mathcal{F}_k\right)dP=$$

$$=\lim_{l\to+\infty}\int_A E\left(\Phi\left(\frac{|X_l-X_{k-1}|}{\lim\limits_{n\to+\infty}\|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k}+\varepsilon}\right)|\mathcal{F}_k\right)dP\leq$$

$$\leq\limsup_{l\to+\infty}\int_A E\left(\Phi\left(\frac{|X_l-X_{k-1}|}{\|X_l-X_{k-1}\|_\Phi^{\mathcal{F}_k}+\varepsilon}\right)|\mathcal{F}_k\right)dP\leq P(A).$$

This means that with arbitrary $\varepsilon>0$

$$\|X-X_{k-1}\|_\Phi^{\mathcal{F}_k}\leq\lim_{n\to+\infty}\|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k}+\varepsilon \quad \text{a.e.}$$

and so

$$\|X-X_{k-1}\|_\Phi^{\mathcal{F}_k}\leq\lim_{n\to+\infty}\|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k} \quad \text{a.e.}$$

Comparing this with the opposite inequality obtained above we finally get

$$\|X-X_{k-1}\|_\Phi^{\mathcal{F}_k}=\lim_{n\to+\infty}\|X_n-X_{k-1}\|_\Phi^{\mathcal{F}_k} \quad \text{a.e.}$$

We are now in the position to formulate:

THEOREM 1. *Let Φ be a Young-function and suppose that its conjugate Ψ has finite power q. If $X\in BMO_\Phi$ then*

$$\|X\|_{BMO_\Phi}\leq\left\|\sup_{l\geq1}\left\|\sup_{k\geq l}|X_k-X_{l-1}|\right\|_\Phi^{\mathcal{F}_l}\right\|_\infty\leq q\,\|X\|_{BMO_\Phi}.$$

PROOF. Since

$$\sup_{l\geq1}\|X-X_l\|_\Phi^{\mathcal{F}_l}\leq\sup_{l\geq1}\left\|\sup_{k\geq l}|X_k-X_{l-1}|\right\|_\Phi^{\mathcal{F}_l},$$

the left-hand side is trivially valid. For the right-hand side by Lemma 1 we have for the non-negative submartingale $|X_k-X_{l-1}|$, $k\geq l$, that

$$\left\|\max_{l\leq k\leq n}|X_k-X_{l-1}|\right\|_\Phi^{\mathcal{F}_l}\leq q\,\|X_n-X_{l-1}\|_\Phi^{\mathcal{F}_l} \quad \text{a.e.}$$

Consequently, letting $n \to +\infty$ and taking into account the preceding remarks we get

$$\left\|\sup_{k \geq l} |X_k - X_{l-1}|\right\|_{\Phi}^{\mathcal{F}_l} \leq q \lim_{n \to +\infty} \|X_n - X_{l-1}\|_{\Phi}^{\mathcal{F}_l} = q\,\|X - X_{l-1}\|_{\Phi}^{\mathcal{F}_l} \quad \text{a.e.}$$

From this also follows that

$$\left\|\sup_{l \geq 1} \left\|\sup_{k \geq l} |X_k - X_{l-1}|\right\|_{\Phi}^{\mathcal{F}_l}\right\|_{\infty} \leq q\,\|X\|_{BMO_\Phi}.$$

This proves our statement.

REMARKS 1. It is easily seen that for arbitrary $n \geq l \geq 1$ we have

$$X_n^* - X_{l-1}^* \leq \sup_{n \geq k \geq l} |X_k - X_{l-1}|.$$

From this by Theorem 1 we deduce that

$$\|X_n^* - X_{l-1}^*\|_{\Phi}^{\mathcal{F}_l} \leq q\,\|X\|_{BMO_\Phi} \quad \text{a.e.}$$

Following GARSIA we can introduce the notion of the increasing $BMO_\Phi(B)$ sequence (cf. GARSIA [3], p. 66). According to this notion the preceding inequality can be formulated in the following manner:

If $X \in BMO_\Phi$ then the sequence of the maxima $\{X_n^*\}$ is an increasing $BMO_\Phi(q\|X\|_{BMO_\Phi})$-sequence, provided that q, the power of the conjugate function Ψ, is finite.

2. If both Φ and Ψ have finite power, then with some positive constants c_Φ and C_Φ depending only on Φ we have by Theorem 1. and by Theorem 9. of [2] that

$$c_\Phi\,\|X\|_{BMO_1} \leq \left\|\sup_{l \geq 1} \left\|\sup_{k \geq l} |X_k - X_{l-1}|\right\|_{\Phi}^{\mathcal{F}_l}\right\|_{\infty} \leq qC_\Phi\,\|X\|_{BMO_1},$$

since Theorem 9. of [2] proves for any $X \in BMO_\Phi$, where Φ has finite power, that

$$c_\Phi\,\|X\|_{BMO_1} \leq \|X\|_{BMO_\Phi} \leq C_\Phi\,\|X\|_{BMO_1}.$$

3. Under the conditions of the preceding theorem we also have

$$\|X_n^*\|_\Phi \leq q\,\|X\|_{BMO_\Phi}, \quad n \in N.$$

In fact, putting $l = 1$ in the first remark, we obtain

$$\|X_n^*\|_{\Phi}^{\mathcal{F}_1} \leq q\,\|X\|_{BMO_\Phi}.$$

By Proposition 2 of [4], we have for every $\varepsilon > 0$ arbitrary that

$$E\left(\Phi\left(\frac{X_n^*}{\|X_n^*\|_{\Phi}^{\mathcal{F}_1} + \varepsilon}\right) \Big| \mathcal{F}_1\right) \leq 1 \quad \text{a.e.}$$

and, consequently,

$$E\left(\Phi\left(\frac{X_n^*}{q\,\|X\|_{BMO_\Phi} + \varepsilon}\right) \Big| \mathcal{F}_1\right) \leq 1 \quad \text{a.e.}$$

Taking the expectation we get

$$E\left(\Phi\left(\frac{X_n^*}{q\,\|X\|_{BMO_\Phi}+\varepsilon}\right)\right)\le 1\,.$$

From this we deduce for every arbitrary $\varepsilon>0$ that

$$\|X_n^*\|_\Phi\le q\,\|X\|_{BMO_\Phi}+\varepsilon\,.$$

This proves the assertion.

3. Now we turn to the special case of independent martingale differences. Our aim is to generalize a result of MARCINKIEWICZ and ZYGMUND [5]. They had shown that in this case there is an absolute constant, which is convenient in all the maximal inequalities. They proved that the L_p-norm of the n-th maximum and that of the n-th partial sum itself are equivalent.

For this purpose, first we prove the inequality of BICKEL in a conditional form, which is convenient for our aim. (cf. [6]).

LEMMA 2. Let $Y_1, Y_2, \dots$ be independent and symmetrically distributed random variables with zero mean, and put $X_n=\sum_{i=1}^n Y_i$, $n\in N$. If $g(x)$ is a non-negative convex function defined on R then for any σ-field $\mathcal{F}\subset\mathcal{F}_1$ we have

$$E\left(\max_{1\le k\le n} g(X_k)|\mathcal{F}\right)\le 2E\left(g(X_n)|\mathcal{F}\right)\quad\text{a.e.},$$

where $\mathcal{F}_1$ is the σ-field generated by Y_1.

PROOF. It is enough to prove the inequality

$$P\left(\max_{1\le k\le n} g(X_k)\ge t|\mathcal{F}\right)\le 2P\left(g(X_n)\ge t|\mathcal{F}\right)\quad\text{a.e.}$$

since taking the regular version of both sides and integrating we get the required inequality.

Define the stopping time

$$\nu=\begin{cases}\inf\,(k: 1\le k\le n,\ g(X_k)\ge t), & \text{if}\quad \max\limits_{1\le k\le n} g(X_k)\ge t\,,\\ n+1, & \text{if}\quad \max\limits_{1\le k\le n} g(X_k)<t\,.\end{cases}$$

Then

$$P\left(\max_{1\le k\le n} g(X_k)\ge t|\mathcal{F}\right)=P(\nu\le n|\mathcal{F})=\sum_{k=1}^n P(\nu=k|\mathcal{F})=$$

$$=\sum_{k=1}^n P\big(\nu=k,\ g(X_n)-g(X_k)\ge 0|\mathcal{F}\big)+\sum_{k=1}^n P\big(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\big)\le$$

$$\le\sum_{k=1}^n P\big(\nu=k,\ g(X_n)\ge t|\mathcal{F}\big)+\sum_{k=1}^n P\big(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\big)\le$$

$$\le P\big(g(X_n)\ge t|\mathcal{F}\big)+\sum_{k=1}^n P\big(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\big)\quad\text{a.e.}$$

To obtain the asserted inequality it suffices to show that

$$\sum_{k=1}^{n} P\left(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\right)\leq\frac{1}{2}P(\nu\leq n|\mathcal{F})\quad\text{a.e.}$$

By using the support line theorem for convex functions with some $\lambda(x)$ we have for all $x, y\in R$

$$g(y)-g(x)\geq\lambda(x)(y-x).$$

Thus

$$\sum_{k=1}^{n} P\left(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\right)\leq\sum_{k=1}^{n} P\left(\nu=k,\ \lambda(X_k)(X_n-X_k)<0|\mathcal{F}\right).$$

This is equal to

$$\sum_{k=1}^{n}\{P\left(\nu=k,\ \lambda(X_k)<0,\ (X_n-X_k)>0|\mathcal{F}\right)+$$
$$+P\left(\nu=k,\ \lambda(X_k)>0,\ (X_n-X_k)<0|\mathcal{F}\right)\}=$$
$$=\sum_{k=1}^{n}\{P\left(\nu=k,\ \lambda(X_k)<0|\mathcal{F}\right)P(X_n-X_k>0)+$$
$$+P\left(\nu=k,\ \lambda(X_k)>0|\mathcal{F}\right)P(X_n-X_k<0)\}.$$

Here we have used the fact that the partial sums (X_n-X_k), $k=1,\dots,n$ are independent of $\mathcal{F}\subset\mathcal{F}_1$. On the other hand

$$P(X_n-X_k>0)<\frac{1}{2}\quad\text{and}\quad P(X_n-X_k<0)\leq\frac{1}{2}$$

since the X_i'-s are also symmetrically distributed. It follows that

$$\sum_{k=1}^{n} P\left(\nu=k,\ g(X_n)-g(X_k)<0|\mathcal{F}\right)\leq$$
$$\leq\frac{1}{2}\sum_{k=1}^{n} P\left(\nu=k,\lambda(X_k)\neq 0|\mathcal{F}\right)\leq\frac{1}{2}P(\nu\leq n|\mathcal{F})\quad\text{a.e.}$$

which proves the lemma.

Assertion 1. Let $Y_1, Y_2, \dots$ be independent random variables with zero mean, Φ be a Young-function with finite power p. Denote

$$X_n=\sum_{i=1}^{n} Y_i\quad\text{and}\quad X_n^*=\max_{1\leq k\leq n}|X_k|,\quad n\in N.$$

Then we have a.e.

$$E\left(\Phi(X_n^*)|\mathcal{F}\right)\leq 2^{p+1}\{E\left(\Phi(|X_n|)|\mathcal{F}\right)+E\left(\Phi(|X_n|)\right)\}\quad\text{a.e.},$$

where $\mathcal{F}\subset\mathcal{F}_1$ is an arbitrary σ-field and $\mathcal{F}_1$ is the σ-field generated by Y_1.

Proof. It is easily seen that

$$E\left(\Phi(X_n^*)|\mathcal{F}\right) = E\left(\max_{1\le k\le n}\Phi(|X_k|)|\mathcal{F}\right) \quad \text{a.e.}$$

Let Φ' be defined by the formula

$$\Phi'(X) = \begin{cases}\Phi(X), & \text{if} \quad X\ge 0\\ \Phi(-X), & \text{if} \quad X<0.\end{cases}$$

Then Φ' is a convex function on R, it is even and

$$\Phi'(X) = \Phi(|X|).$$

We have to prove that

$$E\left(\max_{1\le k\le n}\Phi'(X_k)|\mathcal{F}\right)\le 2^{p+1}\{E\left(\Phi'(X_n)|\mathcal{F}\right)+E\left(\Phi(|X_n|)\right)\} \quad \text{a.e.}$$

It is known that there exist random variables $Y_1', Y_2', \ldots, Y_n'$ such that Y_i and Y_i' have the same distribution function, $i = 1, \ldots, n$, $Y_1, \ldots, Y_n$, $Y_1', \ldots, Y_n'$ are independent and $Y_i - Y_i'$ are symmetrically distributed. By using Lemma 2 we have

$$E\left(\max_{1\le k\le n}\Phi'(S_k)|\mathcal{F}\right)\le 2E\left(\Phi'(S_n)|\mathcal{F}\right) \quad \text{a.e.} \tag{1}$$

where

$$S_n = \sum_{i=1}^{n} Z_i, \quad n\in N, \quad \text{and} \quad Z_i = Y_i - Y_i', \quad i = 1, 2, \ldots, n.$$

By the convexity of Φ' we have

$$\Phi'(S_n) = \Phi(|S_n|)\le\Phi\left(\left|\sum_{i=1}^{n} Y_i\right| + \left|\sum_{i=1}^{n} Y_i'\right|\right)\le$$

$$\le 2^p\left\{\Phi\left(\left|\sum_{i=1}^{n} Y_i\right|\right)+\Phi\left(\left|\sum_{i=1}^{n} Y_i'\right|\right)\right\}.$$

Consequently,

$$E\left(\Phi'(S_n)|\mathcal{F}\right)\le 2^p\left\{E\left(\Phi\left(\left|\sum_{i=1}^{n} Y_i\right|\right)|\mathcal{F}\right)+E\left(\Phi\left(\left|\sum_{i=1}^{n} Y_i'\right|\right)|\mathcal{F}\right)\right\} =$$

$$= 2^p\{E\left(\Phi(|X_n|)|\mathcal{F}\right)+E\left(\Phi(|X_n|)\right)\} \quad \text{a.e.} \tag{2}$$

since the partial sums $\sum_{i=1}^{n} Y_i'$ are independent of $\mathcal{F}$, further $\sum_{i=1}^{n} Y_i = X_n$ and $\sum_{i=1}^{n} Y_i'$ have the same distribution function. Also, we have a.e.

$$E\left(\max_{1\le k\le n}\Phi'(S_k)|\mathcal{F}\right) = E\left(E\left(\max_{1\le k\le n}\Phi'(S_k)|\mathcal{F}_n\right)|\mathcal{F}\right)\ge E\left(\max_{1\le k\le n}\Phi'(X_k)|\mathcal{F}\right)$$

where

$$\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$$

Here we have used the inequality

$$E\left(\max_{1\le k\le n} E(\Phi'(S_k)|\mathcal{F}_n)\right) \ge \max_{1\le k\le n} E(\Phi'(S_k)|\mathcal{F}_n)$$

and we have applied the Jensen inequality:

$$E(\Phi'(S_k)|\mathcal{F}_n) \ge \Phi'(E(S_k|\mathcal{F}_n)) = \Phi'\left(\sum_{i=1}^{k} Y_i\right) = \Phi'(X_k).$$

Then it follows that

$$(3) \qquad E\left(\max_{1\le k\le n} \Phi'(S_k)|\mathcal{F}\right) \ge E(\Phi(X_n^*)|\mathcal{F}) \quad \text{a.e.}$$

So, from (1), (2) and (3) we get

$$E(\Phi(X_n^*)|\mathcal{F}) \le 2^{p+2}\{E(\Phi(|X_n|)|\mathcal{F}) + E(\Phi(|X_n|))\} \quad \text{a.e.}$$

which proves the assertion.

Note that the preceding assertion is valid also for $p = 1$.

Assertion 2. Under the conditions of Assertion 1 we have

$$\|X_n^*\|_\Phi^{\mathcal{F}} \le 2^{p+1} \|X_n\|_\Phi^{\mathcal{F}} \quad \text{a.e.}$$

Proof. On the basis of Assertion 1 we have for every $\mathcal{F}$-measurable and a.e. positive random variable γ that

$$E\left(\Phi\left(\frac{X_n^*}{\gamma}\right)|\mathcal{F}\right) \le 2^{p+1}\left\{E\left(\Phi\left(\frac{|X_n|}{\gamma}\right)|\mathcal{F}\right) + E\left(\Phi\left(\frac{|X_n|}{\gamma}\right)\right)\right\} \quad \text{a.e.,}$$

since $(X_n/\gamma, \mathcal{F}_n)$ is also a martingale. Thus we have for $\varepsilon > 0$ that

$$E\left(\Phi\left(\frac{X_n^*}{\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon}\right)|\mathcal{F}\right) \le$$

$$\le 2^{p+1}\left\{E\left(\Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon}\right)|\mathcal{F}\right) + E\left(\Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon}\right)\right)\right\}$$

which gives that

$$E\left(\Phi\left(\frac{X_n^*}{(\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon)\, 2^{p+2}}\right)|\mathcal{F}\right) \le 1,$$

and, consequently,

$$\|X_n^*\|_\Phi^{\mathcal{F}} \le 2^{p+2}(\|X_n\|_\Phi^{\mathcal{F}} + \varepsilon) \quad \text{a.e.}$$

which proves the assertion.

Note that the preceding assertion is also valid for $p = 1$.

Theorem 2. *Let $X \in BMO_1$. If Φ has a finite power p and X is of the form*

$$X = \sum_{i=1}^{\infty} Y_i,$$

where $Y_1, Y_2, \ldots$ are independent random variables with expectation 0, *and* $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, $n = 1, 2, \ldots$; *then there are constants c_Φ and C_Φ such that*

$$c_\Phi \|X\|_{BMO_1} \leq \left\| \sup_{l \geq 1} \left\| \sup_{k \geq l} |X_k - X_{l-1}| \right\|_\Phi^{\mathcal{F}_l} \right\|_\infty \leq C_\Phi \|X\|_{BMO_1}.$$

PROOF. By using the last assertion and in the same way as in the proof of Theorem 1 it follows that:

$$\|X\|_{BMO_\Phi} \leq \left\| \sup_{l \geq 1} \left\| \sup_{k \geq l} |X_k - X_{l-1}| \right\|_\Phi^{\mathcal{F}_l} \right\|_\infty \leq 2^{p+2} \|X\|_{BMO_\Phi}.$$

Also, by using the result of Theorem 9. of [2], according to which

$$c_\Phi \|X\|_{BMO_1} \leq \|X\|_{BMO_\Phi} \leq C_\Phi \|X\|_{BMO_1},$$

the theorem is completely proved.

Note that the preceding theorem was proved by supposing only that p is finite.

Another consequence of Assertion 2 is the conditional Marcinkiewicz–Zygmund inequality.

THEOREM 3. *Let $Y_1, Y_2, \ldots,$ be independent random variables with zero mean. Denote $X_n = \sum_{i=1}^{n} Y_i$ and $X_n^* = \max_{1 \leq k \leq n} \|X_n\|$, $n \in N$. Then for any $p \geq 1$ we have with an absolute constant A that*

$$\|X_n^*\|_p^{\mathcal{F}} \leq A \|X_n\|_p^{\mathcal{F}} \quad \text{a.e.}$$

where $\mathcal{F} \subset \mathcal{F}_1$ is an arbitrary σ-field and $\mathcal{F}_1$ is the σ-field generated by Y_1.

The proof of this assertion can be made in the same way as in the classical case. (cf. [5], or [7]).

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ON THE $L_\Phi^{\mathcal{F}}$-SPACES WITH GENERAL YOUNG-FUNCTION

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1. In what follows we suppose the knowledge of the theory of the Young-functions as well as the theory of the Orlicz-spaces generated by Young-functions. ([1])

In [2] we have introduced the notion of the conditional $L_\Phi^{\mathcal{F}}$-norm for a Young-function Φ as follows:

Let X be a random variable on the probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F} \subset \mathcal{A}$ be an arbitrary σ-field. Consider the set of the random vraiables γ defined by

$$\mathcal{F}_X^{\Phi,\mathcal{F}} = \left\{\gamma : \gamma > 0 \quad \text{a.e.}, \quad \mathcal{F}\text{-measurable}, \quad E\left(\Phi\left(\frac{|X|}{\gamma}\right) \Big| \mathcal{F}\right) \leq 1 \quad \text{a.e.}\right\}.$$

We say that $X \in L_\Phi^{\mathcal{F}}$ if $\mathcal{F}_X^{\Phi,\mathcal{F}}$ is not empty and in this case we put

$$\|X\|_\Phi^{\mathcal{F}} = \operatorname{ess\,inf} \mathcal{F}_X^{\Phi,\mathcal{F}}.$$

We define $\|X\|_\Phi^{\mathcal{F}} = +\infty$ if $\mathcal{F}_X^{\Phi,\mathcal{F}}$ is empty.

The existence and the uniqueness of $\|X\|_\Phi^{\mathcal{F}}$ is not in question. The random variable $\|X\|_\Phi^{\mathcal{F}}$ has the properties of a norm a.e. in the sense that if $X \in L_\Phi^{\mathcal{F}}$ then

(a) for any real c we have $cX \in L_\Phi^{\mathcal{F}}$ and

$$\|cX\|_\Phi^{\mathcal{F}} = |c| \, \|X\|_\Phi^{\mathcal{F}} \quad \text{a.e.}$$

(b) $\|X\|_\Phi^{\mathcal{F}} = 0$ a.e. if and only if $X = 0$ a.e.

(c) if $Y \in L_\Phi^{\mathcal{F}}$ then $X + Y \in L_\Phi^{\mathcal{F}}$, moreover,

$$\|X+Y\|_\Phi^{\mathcal{F}} \leq \|X\|_\Phi^{\mathcal{F}} + \|Y\|_\Phi^{\mathcal{F}} \quad \text{a.e.}$$

The property (a) is satisfied in the following more general situation, too:

(a′) let $Y \neq 0$ a.e. be an $\mathcal{F}$-meaurable random variable and suppose that $X \in L_\Phi^{\mathcal{F}}$ then $XY \in L_\Phi^{\mathcal{F}}$, and $\|XY\|_\Phi^{\mathcal{F}} = |Y|\,\|X\|_\Phi^{\mathcal{F}}$ a.e.

2. In this paper we shall treat some properties of the $L_\Phi^{\mathcal{F}}$-spaces. Firstly, we prove the completeness of the $L_\Phi^{\mathcal{F}}$-spaces.

We say that $X \in L_1^{\mathcal{F}}$ if $\|X\|_1^{\mathcal{F}} = E(|X| \,|\, \mathcal{F})$ is finite a.e. It is easily seen that in case $X \in L_1^{\mathcal{F}}$ the random variable $\|X\|_1^{\mathcal{F}}$ is a norm a.e.

We also need the following

PROPOSITION 1. $L_\Phi^{\mathcal{F}} \subset L_1^{\mathcal{F}}$, moreover,

$$c_\Phi \|X\|_1^{\mathcal{F}} \leq \|X\|_\Phi^{\mathcal{F}}$$

where c_Φ is a constant depending only on Φ.

PROOF. Suppose that $X \in L_\Phi^{\mathcal{F}}$. Then there exists $\gamma \in \mathcal{F}_X^{\Phi,\mathcal{F}}$ such that

$$E\left(\Phi\left(\frac{|X|}{\gamma}\right) \Big| \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

Let $x_0 > 0$ be such a number for which $\varphi(x_0) > 0$ holds and put

$$c_\Phi = \frac{\varphi(x_0)}{1 + x_0\,\varphi(x_0)}.$$

Here φ denotes the right hand side derivative of Φ.

It is easily seen that

$$\Phi(x) = \int_0^x \varphi(t)\,dt \geq (x - x_0)^+ \varphi(x_0).$$

Consequently,

$$E\left(\Phi\left(\frac{|X|}{\gamma}\right) \Big| \mathcal{F}\right) \geq \varphi(x_0)\, E\left(\left(\frac{|X|}{\gamma} - x_0\right)^+ \Big| \mathcal{F}\right) \quad \text{a.e.}$$

Since

$$\frac{|X|}{\gamma} \leq \left(\frac{|X|}{\gamma} - x_0\right)^+ + x_0,$$

we get

$$E\left(\frac{|X|}{\gamma} \Big| \mathcal{F}\right) \leq E\left(\left(\frac{|X|}{\gamma} - x_0\right)^+ \Big| \mathcal{F}\right) + x_0,$$

and therefore,

$$E\left(\frac{|X|}{\gamma} \Big| \mathcal{F}\right) \leq \left(\frac{1}{\varphi(x_0)}\right) E\left(\Phi\left(\frac{|X|}{\gamma}\right) \Big| \mathcal{F}\right) + x_0 \leq \frac{1}{c_\Phi} \quad \text{a.e.}$$

We deduce that $E(|X| \,|\, \mathcal{F}) \leq \dfrac{\gamma}{c_\Phi}$ and so $X \in L_1^{\mathcal{F}}$. Consequently,

$$c_\Phi \|X\|_1^{\mathcal{F}} \leq \|X\|_\Phi^{\mathcal{F}},$$

and this proves the assertion.

We deduce the following

PROPOSITION 2. Let $X \in L_{\Phi}^{\mathcal{F}}$. Then for $\varepsilon > 0$ arbitrary $\|X\|_{\Phi}^{\mathcal{F}} + \varepsilon$ belongs to $\mathcal{F}_X^{\Phi,\mathcal{F}}$.

PROOF. Consider the decreasing sequence $\gamma_n \in \mathcal{F}_X^{\Phi,\mathcal{F}}$, $n = 1, 2, \ldots$ such that $\gamma_n \downarrow \|X\|_{\Phi}^{\mathcal{F}}$. Then for arbitrary $\varepsilon > 0$ we have that $\gamma_n + \varepsilon \in \mathcal{F}_X^{\Phi,\mathcal{F}}$. In fact, by the monotonicity of Φ

$$E\left(\Phi\left(\frac{|X|}{\gamma_n + \varepsilon}\right) | \mathcal{F}\right) \leq E\left(\left(\frac{|X|}{\gamma_n}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

and at the same time

$$\lim_{n \to +\infty} \gamma_n + \varepsilon = \|X\|_{\Phi}^{\mathcal{F}} + \varepsilon .$$

Then by the monotone convergence theorem for the conditional expectation we have

$$E\left(\Phi\left(\frac{|X|}{\|X\|_{\Phi}^{\mathcal{F}} + \varepsilon}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

which proves our proposition.

DEFINITION 1. Let $\{X_n\}$, $n \geq 1$, be a sequence of random variables belonging to $L_{\Phi}^{\mathcal{F}}$. We say that $X_n \to X$ in $L_{\Phi}^{\mathcal{F}}$ if

$$\|X_n - X\|_{\Phi}^{\mathcal{F}} \to 0 \quad \text{a.e.}$$

as $n \to +\infty$. Also, we say that $\{X_n\}$ is a Cauchy sequence in $L_{\Phi}^{\mathcal{F}}$ if

$$\|X_n - X_m\|_{\Phi}^{\mathcal{F}} \to 0 \quad \text{a.e.}$$

as $n, m \to +\infty$.

We are able to prove the completeness of $L_{\Phi}^{\mathcal{F}}$.

THEOREM 1. *The space $L_{\Phi}^{\mathcal{F}}$ is complete.*

PROOF. Let $\{X_n\}$ be a Cauchy sequence in $L_{\Phi}^{\mathcal{F}}$, $n \geq 1$. Then, as it can be seen, we can pick out a subsequence, $\{X_{n_p}\}$, $p \geq 1$, from the original sequence $\{X_n\}$ such that

$$\sum_{p=0}^{\infty} \|X_{n_{p+1}} - X_{n_p}\|_{\Phi}^{\mathcal{F}}$$

converges a.e. where $X_{n_0} = 0$ a.e. This trivially implies that the series

$$Y = \sum_{p=0}^{\infty} |X_{n_{p+1}} - X_{n_p}|$$

converges a.e. From this it will turn out that the random variable

$$X = \lim_{p \to +\infty} X_{n_p}$$

exists a.e. In fact, by the Fatou lemma and by Proposition 2 we have a.e. that

$$E\left(\Phi\left(\frac{Y}{\sum_{p=0}^{\infty}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}}+\varepsilon}\right)\Big|\mathcal{F}\right)\le$$

$$\le\liminf_{k\to+\infty}E\left(\Phi\left(\frac{\sum_{p=0}^{k-1}|X_{n_{p+1}}-X_{n_p}|}{\sum_{p=0}^{\infty}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}}+\varepsilon}\right)\Big|\mathcal{F}\right)\le$$

$$\le\liminf_{k\to+\infty}E\left(\Phi\left(\frac{\sum_{p=0}^{k-1}|X_{n_{p+1}}-X_{n_p}|}{\sum_{p=0}^{k-1}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}}+\varepsilon}\right)\Big|\mathcal{F}\right)\le 1$$

where $\varepsilon>0$ is arbitrary. If follows that $\|X\|_{\Phi}^{\mathcal{F}}$ is finite a.e. and

$$\|X\|_{\Phi}^{\mathcal{F}}\le\sum_{p=0}^{\infty}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}}+\varepsilon \quad \text{a.e.}$$

Consequently, letting $\varepsilon\to 0$ we get that

$$X\in L_{\Phi}^{\mathcal{F}} \quad \text{and} \quad \|X\|_{\Phi}^{\mathcal{F}}\le\sum_{p=0}^{\infty}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}} \quad \text{a.e.}$$

We see by similar argument that for arbitrary fixed $q\ge 1$ the inequality

$$\|X-X_{n_q}\|_{\Phi}^{\mathcal{F}}\le\sum_{p=q}^{\infty}\|X_{n_{p+1}}-X_{n_p}\|_{\Phi}^{\mathcal{F}} \quad \text{a.e.}$$

holds, which tends a.e. to 0 as $q\to+\infty$. Thus $X_{n_q}\to X$ in $L_{\Phi}^{\mathcal{F}}$ as $q\to+\infty$. Further, since

$$\|X-X_n\|_{\Phi}^{\mathcal{F}}\le\|X-X_{n_q}\|_{\Phi}^{\mathcal{F}}+\|X_n-X_{n_q}\|_{\Phi}^{\mathcal{F}},$$

it follows that

$$\|X-X_n\|_{\Phi}^{\mathcal{F}}\to 0 \quad \text{a.e.}$$

since by the Cauchy-property

$$\|X_n-X_{n_q}\|_{\Phi}^{\mathcal{F}}\to 0 \quad \text{a.e.}$$

and by what we have just proved

$$\|X-X_{n_q}\|_{\Phi}^{\mathcal{F}}\to 0 \quad \text{a.e.}$$

as $q\to+\infty$ and $n\to+\infty$. This proves the completeness of $L_{\Phi}^{\mathcal{F}}$.

Conversely, from the fact that for some $X \in L_\Phi^{\mathcal{F}}$ we have

$$\|X - X_n\|_\Phi^{\mathcal{F}} \to 0 \quad \text{a.e.} \quad \text{as} \quad n \to +\infty$$

it follows trivially that $\{X_n\}$ has the Cauchy-property in $L_\Phi^{\mathcal{F}}$.

3. In some cases we can prove in the $L_\Phi^{\mathcal{F}}$-spaces some martingale-theoretic results obtained in the Orlicz-spaces. The following assertion is analogous to that of [1], Theorem IX$-3-4$.

THEOREM 2. *Let $(X_n, \mathcal{F}_n)$, $n \geq 1$, be a martingale and suppose that $X_n \in L_\Phi^{\mathcal{F}}$, $n \geq 1$, with $\sigma = \sup_{n \geq 1} \|X_n\|_\Phi^{\mathcal{F}} \leq b$ a.e., where $\mathcal{F} \subset \mathcal{A}$ is arbitrary, and b is a positive constant. Then $(X_n, \mathcal{F}_n)$, $n \geq 1$, is a regular martingale.*

PROOF. To prove this assertion it suffices to verify the uniform integrability of the sequence $\{X_n\}$.

Let $a > 0$ be arbitrary. Then for any $n \geq 1$ with arbitrary $\varepsilon > 0$ we have the following estimates:

$$\int_{\{|X_n| \geq a\}} |X_n|\, dP = \int_{\left\{\frac{|X_n|}{\sigma+\varepsilon} \geq \frac{a}{\sigma+\varepsilon}\right\}} \frac{|X_n|}{\sigma+\varepsilon} (\sigma+\varepsilon)\, dP \leq a \int_{\{|X_n| \geq a\}} \frac{\Phi\left(\frac{|X_n|}{\sigma+\varepsilon}\right)}{\Phi\left(\frac{a}{\sigma+\varepsilon}\right)} dP \leq$$

$$\leq \frac{a}{\Phi\left(\frac{a}{b+\varepsilon}\right)} \int_{\{|X_n| \geq a\}} \Phi\left(\frac{|X_n|}{\sigma+\varepsilon}\right) dP \leq \frac{a}{\Phi\left(\frac{a}{b+\varepsilon}\right)} \int_{\{|X_n| \geq a\}} \Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}}+\varepsilon}\right) dP \leq$$

$$\leq \frac{a}{\Phi\left(\frac{a}{b+\varepsilon}\right)} E\left(E\left(\Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}}+\varepsilon}\right) \Big| \mathcal{F}\right)\right) \leq \frac{a}{\Phi\left(\frac{a}{b+\varepsilon}\right)} \to 0$$

as $a \to +\infty$, since by Proposition 2 we have

$$E\left(\Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}}+\varepsilon}\right) \Big| \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

Here we have used the fact that $\Phi(x)/x$ increases together with x and that

$$\lim_{x \to +\infty} \frac{\Phi(x)}{x} = +\infty .$$

This proves the assertion.

If we suppose that the martingale is regular and the Young-function Φ has finite power p then $\{X_n\}$ converges to its a.e. limit in $L_\Phi^{\mathcal{F}}$.

THEOREM 3. *Suppose $(X_n, \mathcal{F}_n)$ is a regular martingale such that $X_n \in L_\Phi^{\mathcal{F}}$ for all n, $\sigma = \sup_{n \geq 1} \|X_n\|_\Phi^{\mathcal{F}}$ is finite a.e. and Φ has the finite power p.*

Then $X_n \to X_\infty$ in $L_\Phi^{\mathcal{F}}$, where X_∞ denotes the a.e. limit of $\{X_n\}$, for all σ-fields $\mathcal{F}$ such that $\mathcal{F} \subset \mathcal{F}_1$.

Proof. Let $\mathcal{F} \subset \mathcal{A}$ be an arbitrary σ-field.

We first prove that $X_\infty \in L_\Phi^{\mathcal{F}}$. In fact, remark that σ is $\mathcal{F}$-measurable. Also, by the Fatou lemma and by Proposition 2, we have

$$E\left(\Phi\left(\frac{|X_\infty|}{\sigma+\varepsilon}\right)|\mathcal{F}\right) \le \liminf_{n\to+\infty} E\left(\Phi\left(\frac{|X_n|}{\sigma+\varepsilon}\right)|\mathcal{F}\right) \le$$

$$\le \liminf_{n\to+\infty} E\left(\Phi\left(\frac{|X_n|}{\|X_n\|_\Phi^{\mathcal{F}}+\varepsilon}\right)|\mathcal{F}\right) \le 1 \quad \text{a.e.}$$

where $\varepsilon > 0$ is arbitrary. From this it follows that

$$\|X_\infty\|_\Phi^{\mathcal{F}} \le \sigma+\varepsilon \quad \text{a.e.}$$

and, consequently,

$$\|X_\infty\|_\Phi^{\mathcal{F}} \le \sigma \quad \text{a.e.}$$

The proof is divided into two parts. Firstly, we suppose that there exists a constant $a>0$ such that $|X_\infty| \le a$ a.e. This implies trivially that $|X_n| \le a$ a.e. for all $n \ge 1$. For arbitrary $n>1$ consider the random variable $\|X_\infty - X_n\|_\Phi^{\mathcal{F}}$. We prove the existence of such a random variable $\gamma_n \in \mathcal{F}_{X_\infty - X_n}^{\Phi,\mathcal{F}}$ for which $\gamma_n \to 0$ a.e. as $n \to +\infty$. This trivially implies that $\|X_\infty - X_n\|_\Phi^{\mathcal{F}} \to 0$ a.e. as $n \to +\infty$. Let γ_n be defined by the formula

$$\gamma_n^p x_0^p = \varepsilon^p + (2a)^p P(|X_\infty - X_n|^p \ge \varepsilon^p|\mathcal{F}),$$

where p is the power of Φ, $x_0>0$ is such that $\Phi(x_0) = \dfrac{1}{2}$ and $\varepsilon>0$ is arbitrary. We see that

$$\lim_{n\to+\infty} \gamma_n = \frac{\varepsilon}{x_0} \quad \text{a.e.}$$

It is also easily seen that

$$E\left(\Phi\left(\frac{|X_\infty - X_n|}{\gamma_n}\right)|\mathcal{F}\right) \le 1 \quad \text{a.e.},$$

since

$$E\left(\Phi\left(\frac{|X_\infty - X_n|}{\gamma_n}\right)|\mathcal{F}\right) \le E\left(\Phi\left(\frac{|X_\infty - X_n|}{\gamma_n} I_{\left\{\frac{|X_\infty - X_n|}{\gamma_n} \ge x_0\right\}}\right)|\mathcal{F}\right) +$$

$$+ E\left(\Phi\left(\frac{|X_\infty - X_n|}{\gamma_n} I_{\left\{\frac{|X_\infty - X_n|}{\gamma_n} < x_0\right\}}\right)|\mathcal{F}\right) \le \frac{1}{2} + \frac{1}{2} E\left(\frac{|X_\infty - X_n|^p}{\gamma_n^p x_0^p}|\mathcal{F}\right)$$

a.e. and trivially

$$E(|X_\infty - X_n|^p | \mathcal{F}) \leq \gamma_n^p x_0^p \quad \text{a.e.}$$

This proves that

$$\gamma_n \in \mathcal{F}_{X_\infty - X_n}^{\Phi, \mathcal{F}}$$

and that

$$\|X_\infty - X_n\|_\Phi^{\mathcal{F}} \leq \gamma_n \quad \text{a.e.}$$

Since

$$\lim_{n\to+\infty} \gamma_n = \frac{\varepsilon}{x_0}$$

and $\varepsilon > 0$ is arbitrary we see that

$$\lim_{n\to+\infty} \|X_\infty - X_n\|_\Phi^{\mathcal{F}} = 0 \quad \text{a.e.}$$

Secondly, when X_∞ does not belong to L^∞ then let $a > 0$ be an arbitrary constant and define

$$X_\infty^* = \begin{cases} X_\infty, & \text{if} \quad |X_\infty| < a, \\ 0, & \text{if} \quad |X_\infty| \geq a. \end{cases}$$

Also, define

$$X_\infty^{**} = X_\infty - X_\infty^* .$$

Since

$$X_n = E(X_\infty | \mathcal{F}_n) = E(X_\infty^* | \mathcal{F}_n) + E(X_\infty^{**} | \mathcal{F}_n) = X_n^* + X_n^{**}$$

we have

$$\|X_\infty - X_n\|_\Phi^{\mathcal{F}} \leq \|X_\infty^* - X_n^*\|_\Phi^{\mathcal{F}} + \|X_\infty^{**} - X_n^{**}\|_\Phi^{\mathcal{F}} \quad \text{a.e,}$$

X_∞^* being bounded by a the same is true for $X_n^* = E(X_\infty^* | \mathcal{F}_n)$, $n = 1, 2, \ldots$. So, by what we proved in the first part of this assertion, we have that

$$\|X_\infty^* - X_n^*\|_\Phi^{\mathcal{F}} \to 0 \quad \text{a.e.} \quad \text{as} \quad n \to +\infty .$$

Note that, till now all was proved for general $\mathcal{F} \subset \mathcal{A}$.

But concerning the second term on the right-hand side we have for $\mathcal{F} \subset \mathcal{F}_1$ that

$$\|X_\infty^{**} - X_n^{**}\|_\Phi^{\mathcal{F}} \leq \|X_\infty^{**}\|_\Phi^{\mathcal{F}} + \|X_n^{**}\|_\Phi^{\mathcal{F}} \leq 2\|X_\infty^{**}\|_\Phi^{\mathcal{F}} \quad \text{a.e.}$$

since by Jensen's inequality we have for every $\gamma > 0$ a.e. which is $\mathcal{F}$-measurable that

$$E\left(\Phi\left(\frac{|E(X_\infty^{**} | \mathcal{F}_n)|}{\gamma}\right) \Big| \mathcal{F}\right) \leq E\left(\Phi\left(\frac{E(|X_\infty| \, I_{\{|X_\infty| \geq a\}} | \mathcal{F}_n)}{\gamma}\right) \Big| \mathcal{F}\right) \leq$$

$$\leq E\left(\Phi\left(\frac{|X_\infty| \, I_{\{|X_\infty| \geq a\}}}{\gamma}\right) \Big| \mathcal{F}\right) \leq E\left(\Phi\left(\frac{|X_\infty^{**}|}{\gamma}\right) \Big| \mathcal{F}\right) \quad \text{a.e.}$$

and, consequently, we have to show that the right-hand side tends to 0 a.e. as $a \to +\infty$. To this end we construct a random variable $\gamma(a)$ in the following way. Let for arbitrary $\varepsilon > 0$

$$\gamma^p(a) =$$

$$= \begin{cases} \dfrac{1}{a}, & \text{if} \quad E\left(\Phi\left(\dfrac{|X_\infty|}{\sigma+\varepsilon} I_{\{|X_\infty|\geq a\}}\right)|\mathcal{F}\right) = 0, \\ 2^p(\sigma+\varepsilon)^p E\left(\Phi\left(\dfrac{|X_\infty|}{\sigma+\varepsilon} I_{\{|X_\infty|\geq a\}}\right)|\mathcal{F}\right), & \text{if } E\left(\Phi\left(\dfrac{|X_\infty|}{\sigma+\varepsilon} I_{\{|X_\infty|\geq a\}}\right)|\mathcal{F}\right) > 0. \end{cases}$$

It is easily seen that $\gamma(a)$ is positive, $\mathcal{F}$-measurable and tends to 0 a.e. as $a \to +\infty$. With this γ we have a.e. that

$$E\left(\Phi\left(\frac{|X_\infty^{**}|}{\gamma}\right)|\mathcal{F}\right) = E\left(\Phi\left(\frac{|X_\infty|}{\gamma} I_{\left\{|X_\infty|\geq a,\ \frac{2(\sigma+\varepsilon)}{\gamma}\leq 1\right\}}\right)|\mathcal{F}\right) +$$

$$+ E\left(\Phi\left(\frac{|X_\infty|}{\gamma} I_{\left\{|X_\infty|\geq a,\ \frac{2(\sigma+\varepsilon)}{\gamma}>1\right\}}\right)|\mathcal{F}\right) \leq \frac{1}{2} E\left(\Phi\left(\frac{|X_\infty|}{\sigma+\varepsilon}\right)|\mathcal{F}\right) +$$

$$+ E\left(\Phi\left(\frac{|X_\infty|\, 2(\sigma+\varepsilon)}{\gamma \cdot 2(\sigma+\varepsilon)} I_{\left\{|X_\infty|\geq a,\ \frac{2(\sigma+\varepsilon)}{\gamma}>1\right\}}\right)|\mathcal{F}\right) \leq$$

$$\leq \frac{1}{2} + \frac{1}{2} \cdot \frac{2^p(\sigma+\varepsilon)^p}{\gamma^p} E\left(\Phi\left(\frac{|X_\infty|}{\sigma+\varepsilon} I_{\{|X_\infty|\geq a\}}\right)|\mathcal{F}\right) \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Then $\gamma \in \mathcal{F}^{\Phi,\mathcal{F}}_{X_\infty^{**}}$ and consequently,

$$\|X_n^{**} - X_\infty^{**}\|_\Phi^{\mathcal{F}} \leq 2\gamma, \quad \text{a.e.}$$

where γ is defined above. This proves the assertion.

REMARK. Note that when supposing X_∞ to be bounded by a positive constant a in the first part of the preceding proof we have not used the assumption that $\mathcal{F} \subset \mathcal{F}_1$.

Consequently, the following assertion is also true.

COROLLARY 1. Let $(X_n, \mathcal{F}_n)$ be a martingale such that $|X_n| \leq a$ where a is a positive constant. Then with any Young-function Φ having the finite power p we have that $(X_n, \mathcal{F}_n)$ is regular, $X_n \in L_\Phi^{\mathcal{F}}$ where $\mathcal{F} \subset \mathcal{A}$ is an arbitrary σ-field, and, X_n converges also in the $L_\Phi^{\mathcal{F}}$-sense to its a.e. limit.

References

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[2] J. MOGYORÓDI and N. L. BASSILY, On the BMO_Φ-spaces with general Young-function, Submitted to *Periodica Mathematica Hungarica*.

ON THE OSCILLATION OF MARTINGALES

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1. Let $1 \leqq p < +\infty$ be a power and let $(X_n, \mathcal{F}_n)$, $n \geqq 0$, $X_0 = 0$ a.e., be a martingale with integrable p-th power. Consider the Doob decomposition of the non-negative submartingale $(|X_n|^p, \mathcal{F}_n)$, $n \geqq 0$, i.e. let

(1) $$|X_n|^p = M_n^{(p)} + A_n^{(p)}, \quad n \geqq 0,$$

where $(M_n^{(p)}, \mathcal{F}_n)$ is the martingale and $(A_n^{(p)}, n \geqq 0)$ is the natural increasing process corresponding to the submartingale $(|X_n|^p, \mathcal{F})$. NEVEU has shown that

$$X_\infty = \lim_{n \to +\infty} X_n$$

exists and is finite a.e. on the event $\{A_\infty^{(p)} < +\infty\}$, where

$$A_\infty^{(p)} = \lim_{n \to +\infty} A_n^{(p)}.$$

Further, if

(2) $$E\left(\sup_{n \geqq 0} |X_{n+1} - X_n|^p\right) < +\infty,$$

then on the event $\{A_\infty^{(p)} = +\infty\}$ we have

(3) $$\limsup_{n \to +\infty} X_n = +\infty \quad \text{and} \quad \liminf_{n \to +\infty} X_n = -\infty.$$

The aim of the present note is to generalize this result by taking instead of the power functions x^p, $x \geqq 0$, $p \geqq 1$, the so called Young-functions.

The definition and the properties of the Young-functions are given e.g. in NEVEU [1] or in KRASNOSEL'SKII and RUTICKII [2]. One of the properties used in this paper is the so called Δ_2-property of some Young-functions Φ. This means that the inequality

(4) $$\Phi(2x) \leqq K\Phi(x)$$

holds for all $x > 0$, where $K > 0$ is a constant.

2. In this note we prove the following

THEOREM. *Let $\Phi(x)$ be a Young-function and let $(X_n, \mathcal{F}_n)$, $n \geqq 0$, $X_0 = 0$, be a martingale with $E(\Phi(|X_n|)) < +\infty$, $n \geqq 0$. Then the random variable*

$$X_\infty = \lim_{n\to+\infty} X_n$$

exists and is a.s. finite on the event $\{A_\infty^{(\Phi)} = +\infty\}$, where $(A_n^{(\Phi)}, n \geqq 0)$ denotes the natural increasing process in the Doob decomposition of the submartingale $(\Phi(|X_n|), \mathcal{F}_n)$.

Further, if $\Phi(x)$ satisfies the Δ_2-condition and

$$E\left(\sup_{n\geqq 0} \Phi(|X_{n+1} - X_n|)\right) < +\infty \tag{5}$$

holds then on the event $\{A_\infty^{(\Phi)} < +\infty\}$ we have

$$\limsup_{n\to+\infty} X_n = +\infty \quad \text{and} \quad \liminf_{n\to+\infty} X_n = -\infty\,. \tag{6}$$

PROOF. Since

$$E(\Phi(|X_n|)) < +\infty, \quad n \geqq 0,$$

the non-negative submartingale $(\Phi(|X_n|), \mathcal{F}_n)$ can be decomposed according to the method of Doob to have

$$\Phi(|X_n|) = M_n^{(\Phi)} + A_n^{(\Phi)}, \quad n \geqq 0,$$

where the increasing process $(A_n^{(\Phi)}, n \geqq 0)$ is defined by the formula

$$A_{n+1}^{(\Phi)} - A_n^{(\Phi)} = E(\Phi|X_{n+1}|)|\mathcal{F}_n) - \Phi(|X_n|), \quad n \geqq 0, \quad A_0^{(\Phi)} = 0, \tag{7}$$

while $(M_n^{(\Phi)}, \mathcal{F}_n)$ is the corresponding martingale. Consequently, both terms of the decomposition are finitely integrable. From the supposition it also follows that $(\Phi(X_n^+), \mathcal{F}_n)$, $n \geqq 0$, is also a non-negative and integrable submartingale.

Let us denote by $(A_n', n \geqq 0)$ the increasing process in the Doob decomposition of this submartingale. We shall show that

(a′) $\lim\limits_{n\to+\infty} X_n$ exists and is finite a.s. on $\{A_\infty' < +\infty\}$,

(b′) $\limsup\limits_{n\to+\infty} X_n = +\infty$ a.s. on $\{A_\infty' = +\infty\}$.

This will be sufficient to prove the theorem, because using this result to the martingale $(-X_n, \mathcal{F}_n)$, $n \geqq 0$, we get

(a″) $\lim\limits_{n\to+\infty} X_n$ exists and is finite a.s. on $\{A_\infty'' < +\infty\}$,

(b″) $\liminf\limits_{n\to-\infty} X_n = -\infty$ a.s. on $\{A_\infty'' = +\infty\}$.

where $(A''_n, n \geqq 0)$ denotes the increasing process in the Doob decomposition of the submartingale $(\Phi(X_n^-), \mathcal{F}_n)$, $n \geqq 0$. Collecting the two results (a′), (b′) and (a″), (b″) gives

$$\{A'_\infty < +\infty\} = \{A''_\infty < +\infty\}$$

(in fact, when the limit $\lim\limits_{n \to +\infty} X_n$ exists on R then $\limsup\limits_{n \to +\infty} X_n$ and $\liminf\limits_{n \to +\infty} X_n$ cannot be infinite) and it also follows that

$$A_n^{(\Phi)} = A'_n + A''_n, \quad n \geqq 0. \tag{8}$$

Thus we see that $A_\infty^{(\Phi)}$ is finite a.s. at the same time as A'_∞ and A''_∞. Therefore, in this way the theorem will be proved.

To prove the validity of (8) we can proceed as follows: write $|X_n|$ in the form

$$|X_n| = X_n I(X_n \geqq 0) + (-X_n I(X_n < 0)) = X_n^+ + X_n^- .$$

Then we have

$$\Phi(|X_n|) = \Phi(X_n^+) + \Phi(X_n^-),$$

since

$$\{X_n \geqq 0\} \cap \{X_n < 0\} = \emptyset$$

and

$$\Phi(0) = 0 .$$

It follows from the definition of the increasing process that

$$A_{n+1}^{(\Phi)} - A_n^{(\Phi)} = E(\Phi(|X_{n+1}|)|\mathcal{F}_n) - \Phi(|X_n|)$$
$$A'_{n+1} - A'_n = E(\Phi(X_{n+1}^+)|\mathcal{F}_n) - \Phi(X_n^+)$$
$$A''_{n+1} - A''_n = E(\Phi(X_n^-)|\mathcal{F}_n) - \Phi(X_n^-)$$

for every $n \geqq 0$. Summing up the last two formulas gives

$$(A'_{n+1} + A''_{n+1}) - (A'_n + A''_n) = \tag{9}$$
$$= E([\Phi(X_{n+1}^+) + \Phi(X_{n+1}^-)]|\mathcal{F}_n) -]\Phi(X_n^+) + \Phi(X_n^-)] =$$
$$= E(\Phi(|X_{n+1}|)|\mathcal{F}_n) - \Phi(|X_n|) = A_{n+1}^{(\Phi)} - A_n^{(\Phi)}$$

for every $n \geqq 0$, which shows the validity of (8), since $A_0^{(\Phi)} = A'_0 = A''_0 = 0$.

Now let us turn to prove (a′) and (b′).

(a′) For every real $a > 0$ let ν_a be the stopping time defined by the formula

$$\nu_a = \begin{cases} \min(n : A'_{n+1} > a), & \text{if} \quad A'_\infty > a \\ +\infty, & \text{if} \quad A'_\infty \leqq a . \end{cases}$$

Then trivially $A'_{\nu_a} \leqq a$, and consequently,

$$\lim_{n \to +\infty} E(\Phi(X_{\nu_a \wedge n}^+)) = \lim_{n \to +\infty} E(A'_{\nu_a \wedge n}) = E(A'_{\nu_a}) \leqq a \tag{10}$$

since by supposition $X_0 = 0$ a.e. and it is not difficult to show that $A'_{\nu_a \wedge n}$ is the increasing process which correponds to the submartingale $(\Phi(X^+_{\nu_a \wedge n}), \mathcal{F}_n)$ in the decomposition of Doob. The Young-inequality

$$u\,v \leqq \Phi(u) + \Psi(v),$$

which holds for every $u \geqq 0$, $v \geqq 0$ and for any pair (Φ, Ψ) of conjugate Young-functions applied to (10) with $v = 1$ gives

$$\lim_{n\to+\infty} E(X^+_{\nu_a \wedge n}) \leqq \lim_{n\to+\infty} E(\Phi(X^+_{\nu_a \wedge n})) + \Psi(1) \leqq a + \Psi(1) < +\infty .$$

This implies that the martingale $(X_{\nu_a \wedge n}, \mathcal{F}_n)$, $n \geqq 0$, converges a.e. to a finite and integrable limit (see, NEVEU [1], Theorem IV$-$1$-$2). Thus the limit $\lim\limits_{n\to+\infty} X_n = X_\infty$ exists and is finite a.s. on the event $\{\nu_a = +\infty\} = \{A'_\infty \leqq a\}$. Letting $a \uparrow +\infty$ we obtain that $X_\infty = \lim\limits_{n\to+\infty} X_n$ exists a.s. on the event $\{A'_\infty < +\infty\}$.

(b′) For arbitrary real number $a > 0$ let ν'_a be the stopping time defined by the formula

$$\nu'_a = \begin{cases} \min(n : X_n > a), & \text{if } \sup\limits_{n\geqq 0} X_n > a , \\ +\infty, & \text{if } \sup\limits_{n\geqq 0} X_n \leqq a . \end{cases}$$

Then the condition

$$E(\Phi(\sup_{n\geqq 0}(X_{n+1} - X_n)^+)) < +\infty$$

implies that

$$E(A'_{\nu'_a}) = \lim_{m\to+\infty} E(\Phi(X^+_{\nu'_a \wedge m}))$$

is finite. Indeed, $X_{\nu'_a \wedge m} \leqq a$ on the event $\{\nu'_a > m\}$ and on the complementary event $\{\nu'_a \leqq m\}$ we have

$$X_{\nu'_a \wedge m} \leqq a + (X_{\nu'_a} - X_{\nu'_a - 1})^+ \leqq a + \sup_{n\geqq 0}(X_{n+1} - X_n)^+ \leqq$$

$$\leqq 2 \max(a, \sup_{n\geqq 0}(X_{n+1} - X_n)^+) .$$

Consequently,

$$\Phi(X^+_{\nu'_a \wedge m}) \leqq \max\{\Phi(2a), \Phi(2\sup_{n\geqq 0}(X_{n+1} - X_n)^+)\} \leqq$$

$$\leqq K \max\{\Phi(a), \Phi(\sup_{n\geqq 0}(X_{n+1} - X_n)^+)\}$$

since by supposition Φ satisfies the Δ_2-condition. From this we get that

$$\lim_{m\to+\infty} E(\Phi(X^+_{\nu'_a \wedge m})) = E(A'_{\nu'_a}) \leqq$$

$$\leqq K\{\Phi(a) + E(\Phi(\sup_{n\geqq 0}(X_{n+1} - X_n)^+))\} < +\infty .$$

Thus the positive random variable $A'_{\nu'_a}$ belongs to L^1 and so is finite a.e. Consequently, $A'_\infty < +\infty$ a.s. on the event

$$\{\nu'_a = +\infty\} = \{\sup_{n \geqq 0} X_n \leqq a\}.$$

If $a \uparrow +\infty$ then we have $A'_\infty < +\infty$ on the event

$$\{\sup_{n \geqq 0} X_n < +\infty\}$$

or, equivalently,

$$\sup_{n \geqq 0} X_n = +\infty \quad \text{on} \quad \{A'_\infty = +\infty\}.$$

To complete the proof it remains to observe that

$$\{\sup_{n \geqq 0} X_n = +\infty\} = \{\limsup_{n \to +\infty} X_n = +\infty\}.$$

This follows from the fact that for all $n \geqq 0$ we have $X_n < +\infty$ a.e.

REMARK. If we would have contented ourselves with applying the line of reasoning which gave (a′) and (b′) above to the submartingale $(\Phi(|X_n|), \mathcal{F}_n)$, $n \geqq 0$, and its associated increasing process $(A_n^{(\Phi)}, n \geqq 0)$, we would only have obtained

$$\limsup_{n \to +\infty} |X_n| = +\infty \quad \text{on} \quad \{A_\infty^{(\Phi)} = +\infty\},$$

which is weaker than the result given in the theorem.

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UNIFORMLY COMPOSITION-CLOSED FUNCTION CLASSES

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0. Introduction. J. R. ISBELL [6] and Á. CSÁSZÁR [1] have introduced the concept of various kinds of composition-closed function classes; the second author examined them thoroughly (cf. also [2]). These classes are defined as follows.

Let X be a set, Φ a function class on X (i.e. a class of real-valued functions defined on X), I a set of indices, $f_i \in \Phi$ for $i \in I$. Consider the product set $E = \mathbf{R}^I = \bigtimes_{i \in I} E_i$, $E_i = \mathbf{R}$ for $i \in I$, and define $h: X \to E$ by $\pi_i \circ h = f_i$ where $\pi_i: E \to E_i$ is the projection. Let $\overline{h(X)}$ denote the closure of $h(X)$ with respect to the product topology of E arising from the euclidean topology of E_i.

Now Φ is said to be *strongly composition-closed* (scc) if $k \circ h \in \Phi$ whenever the system $\{f_i : i \in I\} \subset \Phi$ is arbitrarily chosen and $k \in C(h(X))$ (where $C(T)$ denotes, for an arbitrary topological space T, the set of all continuous real-valued functions on T, and $h(X)$ is equipped with the subspace topology obtained from the product topology of E). Φ is said to be *composition-closed* (cc) if $k \circ h \in \Phi$ for an arbitrary system $\{f_i\}$ and $k \in C(\overline{h(X)})$, and *weakly composition-closed* (wcc) if $k \circ h \in \Phi$ for an arbitrary system $\{f_i\}$ and $k \in C(E)$.

Similar definitions lead, with the restriction to countable or finite index sets I, to the *countably strongly composition-closed* (cscc), *countably composition-closed* (ccc), *countably weakly composition-closed* (cwcc), *finitely strongly composition-closed* (fscc), *finitely composition-closed* (fcc), *finitely weakly composition-closed* (fwcc) classes. Their interrelations are the following ones ([2], p. 44):

$$\begin{array}{ccccc}
\text{scc} & \Rightarrow & \text{cc} & \Rightarrow & \text{wcc} \\
\Downarrow & & \Downarrow & & \Updownarrow \\
\text{cscc} & \Rightarrow & \text{ccc} & \Leftrightarrow & \text{cwcc} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{fscc} & \Rightarrow & \text{fcc} & \Leftrightarrow & \text{fwcc}
\end{array}$$

The purpose of this paper is to investigate function classes defined similarly to the above definitions, first of all those obtained with the only modification that the function k has to be uniformly continuous with respect to the product uniformity of E arising from the euclidean uniformity of E_i, or to the subspace uniformity induced on $h(X)$ or $\overline{h}(X)$ by this product uniformity.

1. Fundamental definitions. We give a precise formulation to the above definition. Let us first introduce the following notation. If $(X, \mathfrak{U})$ is a uniform space, let $C(\mathfrak{U})$ denote the set of all real-valued functions $f: X \to \mathbf{R}$ that are uniformly continuous with respect to $\mathfrak{U}$ (and the euclidean uniformity of $\mathbf{R}$). Let $C^*(\mathfrak{U})$ be the set of all bounded elements of $C(\mathfrak{U})$. If $A \subset X$, let $\mathfrak{U}|A$ denote the subspace uniformity induced by $\mathfrak{U}$ on A.

Definition 1.1. Let X, Φ, I, $\{f_i : i \in I\} \subset \Phi$, E, h, π_i, $\overline{h(X)}$ have the same meaning as above. The class $\Phi \neq \emptyset$ is said to be *uniformly strongly composition-closed* (uscc) if $k \circ h \in \Phi$ whenever $k \in C(\mathfrak{U}_I | h(X))$, *uniformly composition-closed* (ucc) if $k \circ h \in \Phi$ whenever $k \in C(\mathfrak{U}_I | \overline{h(X)})$, and *uniformly weakly composition-closed* (uwcc) if $k \circ h \in \Phi$ whenever $k \in C(\mathfrak{U}_I)$, where $\mathfrak{U}_I$ denotes the product uniformity on E arising from the euclidean uniformity of E_i.

As an easy example, we see that $C(\mathfrak{U})$ is uscc for every uniform space $(X, \mathfrak{U})$; indeed, $f_i \in C(\mathfrak{U})$ implies that h is $(\mathfrak{U}, \mathfrak{U}_I)$-uniformly continuous, hence $k \circ h \in C(\mathfrak{U})$ whenever $k \in C(\mathfrak{U}_I | h(X))$.

Definition 1.2. If we restrict ourselves, in Definition 1.1, to countable or finite index sets I, then we obtain *uniformly countably strongly composition-closed* (ucscc), *uniformly countably composition-closed* (uccc), *uniformly countably weakly composition-closed* (ucwcc) and *uniformly finitely strongly composition-closed* (ufscc), *uniformly finitely composition-closed* (ufcc), *uniformly finitely weakly composition-closed* (ufwcc) classes respectively.

The number of the above defined classes can be immediately reduced to 6:

Lemma 1.3. *Any uniformly (countably, finitely) composition-closed class is uniformly (countably, finitely) strongly composition-closed.*

Proof. Every $\mathfrak{U}_I | h(X)$-uniformly continuous function k has a $\mathfrak{U}_I | \overline{h(X)}$-uniformly continuous extension ([3], (6.2.7)). ▌

Thus, in the sequel, we can restrict ourselves to the examination of ucc, uwcc, uccc, ucwcc, ufcc, and ufwcc classes.

Lemma 1.4. *If Φ is (countably, finitely) composition-closed or (countably, finitely) weakly composition-closed, then it is uniformly (countably, finitely) composition-closed or uniformly (countably, finitely) weakly composition-closed respectively.*

Proof. If $k \in C(\mathfrak{U} | \overline{h(X)})$ or $k \in C(\mathfrak{U})$ then $k \in C(\overline{h(X)})$ or $k \in C(E)$ respectively. ▌

The number of the distinct uniformly composition-closed classes is still reduced by the following

LEMMA 1.5. *Every uniformly countably weakly composition-closed class is uniformly weakly composition-closed.*

PROOF. Let $k \in C(\mathfrak{U}_I)$. It is well-known (cf. [1], pp. 148–149) that there exists a countable subset $I' \subset I$ such that $k(u) = k(v)$ whenever $u, v \in E$ and $\pi_i(u) = \pi_i(v)$ for $i \in I'$. Hence if we set

$$\Phi' = \{f_i : i \in I'\}, \quad E' = \underset{i \in I'}{\times} E_i, \quad \pi_i' : E' \to E_i,$$

$$h' : X \to E', \quad \pi_i' \circ h' = f_i \quad (i \in I')$$

where π_i' is the projection onto E_i, and if $\pi : E \to E'$ is the projection defined by $\pi_i = \pi_i' \circ \pi$ for $i \in I'$, finally if $\mathfrak{U}'$ is the product uniformity on E', then a function $k \in C(\mathfrak{U}_I)$ can be written in the form $k = k' \circ \pi$ where $k' \in C(\mathfrak{U}')$ (observe that π is a $(\mathfrak{U}_I, \mathfrak{U}')$-quotient map by [3], (7.3.26) and (7.3.27), and apply the uniform analogon of [3], (7.4.5)). Therefore $k \circ h = k' \circ h'$ implies $k \circ h \in \Phi$ provided that Φ is ucwcc. ∎

A similar statement holds for uccc and ucc classes but the proof is based on completely different tools.

LEMMA 1.6. *Every uniformly countably composition-closed class Φ is uniformly closed (i.e. $f_n \in \Phi$ implies $f \in \Phi$ whenever $f_n \to f$ uniformly).*

PROOF. Suppose $f_n \in \Phi$, $f_n \to f$ uniformly, and set $I = \mathbf{N}$. Define $k : h(X) \to \mathbf{R}$ by

$$k(h(x)) = f(x) \quad \text{for} \quad x \in X.$$

This is possible because $h(x) = h(y)$ implies $f_n(x) = f_n(y)$ for $n \in \mathbf{N}$, hence $f(x) = f(y)$. For a given $\varepsilon > 0$, choose $n_0 \in \mathbf{N}$ such that $|f_{n_0}(x) - f(x)| < \frac{\varepsilon}{3}$ for $x \in X$. Then $u, v \in h(X)$, $|\pi_{n_0}(u) - \pi_{n_0}(v)| < \frac{\varepsilon}{3}$ implies $u = h(x)$, $v = h(y)$, $|f_{n_0}(x) - f_{n_0}(y)| < \frac{\varepsilon}{3}$ and

$$|f(x) - f(y)| = |k(u) - k(v)| < \varepsilon.$$

Thus $k \in \mathfrak{U}_I | h(X)$. Now if Φ is uccc, then $f = k \circ h$ by 1.1. ∎

A similar statement can be proved for ufcc classes. For this purpose, let us call, according to [4], a function class Φ *coherently closed* if $f_n \in \Phi$ implies $f \in \Phi$ whenever $f_n \to f$ pointwise and there exist functions $g_1, \ldots, g_m \in \Phi$ and $\delta > 0$ such that $0 < \varepsilon \leqq \delta$, $x, y \in X$ and $|g_i(x) - g_i(y)| \leqq \varepsilon$ for $i = 1, \ldots, m$ imply $|f_n(x) - f_n(y)| \leqq \varepsilon$ for $n \in \mathbf{N}$.

Now we can prove:

LEMMA 1.7. *Every uniformly finitely composition-closed class is coherently closed.*

PROOF. Let Φ be ufcc, $f_n \in \Phi$ for $n \in \mathbf{N}$, $f_n \to f$ pointwise, and $g_1, \ldots, g_m \in \Phi$ be functions with the above properties. Define $h^* : X \to \mathbf{R}^m$ by

$$h^*(x) = (g_1(x), \ldots, g_m(x))$$

and $k : h^*(X) \to \mathbf{R}$ by

$$k(h^*(x)) = f(x).$$

This is possible because $h^*(x) = h^*(y)$ implies $g_i(x) = g_i(y)$ for $i = 1, \ldots, m$, hence $f_n(x) = f_n(y)$ for $n \in \mathbf{N}$ and $f(x) = f(y)$.

Clearly $k \in C(\mathfrak{U}_m | h^*(X))$ where $\mathfrak{U}_m$ denotes the euclidean uniformity of $\mathbf{R}^m$. In fact, if $u, v \in h^*(X)$ and

$$|\pi_i(u) - \pi_i(v)| \leqq \varepsilon < \delta \quad \text{for} \quad i = 1, \ldots, m,$$

then choose $x, y \in X$ satisfying $u = h^*(x)$, $v = h^*(y)$ so that $|g_i(x) - g_i(y)| \leqq \varepsilon$ for each i, whence

$$|f_n(x) - f_n(y)| \leqq \varepsilon \quad \text{for} \quad n \in \mathbf{N}$$

and

$$|k(u) - k(v)| = |f(x) - f(y)| \leqq \varepsilon.$$

Therefore $f = k \circ h^* \in \Phi$. ∎

According to [5], a function class Φ is said to be a subtractive lattice if $f, g \in \Phi$ implies $f - g \in \Phi$, $\max(f, g) \in \Phi$, $\min(f, g) \in \Phi$. A ufwcc class is obviously a subtractive lattice.

For an arbitrary class Φ of real-valued functions on X, we denote by $\mathfrak{U}(\Phi)$ the weak uniformity of Φ, i.e. the coarsest uniformity with respect to which every $f \in \Phi$ is uniformly continuous.

We shall need the following

LEMMA 1.8 ([4], Satz 3). *If Φ is a coherently closed subtractive lattice that contains all constants, then every $\mathfrak{U}(\Phi)$-uniformly continuous function is uniform limit of a sequence taken from Φ.* ∎

Now we can prove:

LEMMA 1.9. *A class Φ is uniformly composition-closed iff it is a uniformly closed and coherently closed subtractive lattice that contains all constants.*

PROOF. Let Φ be ucc. Then it is obviously ufwcc, hence a subtractive lattice that contains all constants because a constant function $k : \mathbf{R} \to \mathbf{R}$ belongs to $C(\mathfrak{U}_1)$. Φ is clearly uccc, hence uniformly closed by 1.6, and ufcc, hence coherently closed by 1.7.

Now suppose that Φ satisfies the conditions in the statement. By 1.8, $C(\mathfrak{U}(\Phi)) \subset \Phi$, and obviously $\Phi \subset C(\mathfrak{U}(\Phi))$ so that $\Phi = C(\mathfrak{U}(\Phi))$ and Φ is ucc. ∎

Now we can formulate the following analogon of 1.5:

COROLLARY 1.10. *Every uniformly countably composition-closed class is uniformly composition-closed.*

Proof. A uccc class is ufcc and ufwcc, hence a uniformly closed and coherently closed subtractive lattice containing all constants. ▌

Corollary 1.11. *A class is uniformly composition-closed iff it is uniformly closed and unifohmly finitely composition-closed.* ▌

Corollary 1.12. *A class Φ is uniformly composition-closed iff there exists a uniformity $\mathfrak{U}$ such that $\Phi = C(\mathfrak{U})$.*

Proof. We know that the condition is sufficient. It is necessary because the proof of 1.9 shows that a ucc class coincides with $C(\mathfrak{U}(\Phi))$. ▌

Theorem 1.13. *We have the following implications:*

$$\begin{array}{ccccc}
\text{uscc} & \Leftrightarrow & \text{ucc} & \Rightarrow & \text{uwcc} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{ucscc} & \Leftrightarrow & \text{uccc} & \Rightarrow & \text{ucwcc} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{ufscc} & \Leftrightarrow & \text{ufcc} & \Rightarrow & \text{ufwcc}
\end{array}$$

Proof. The arrows from the left to the right and those from above to below are obvious. 1.3 yields ucc⇒uscc, uccc⇒ucscc, and ufcc⇒ufscc. 1.5 furnishes ucwcc⇒uwcc, and uccc⇒ucc is obtained from 1.10. ▌

As a consequence of 1.13 we can restrict ourselves to the simpler diagram

$$\begin{array}{ccc}
\text{ucc} & \Rightarrow & \text{uwcc} \\
\Downarrow & & \Downarrow \\
\text{ufcc} & \Rightarrow & \text{ufwcc}
\end{array}$$

These implications cannot be reversed. In fact, we shall see in 2.6 that ufcc⇏uwcc, hence ufcc⇏ucc and ufwcc⇏uwcc. In order to prove uwcc⇏ufcc, let us first introduce the following notation: if Φ is a function class on Y and $X \subset Y$, let $\Phi|X$ denote the set of all restrictions $f|X$ for $f \in \Phi$.

Now it is easy to prove:

Lemma 1.14. *If $(Y, \mathfrak{U})$ is a uniform space and $X \subset Y$, then $C(\mathfrak{U})|X$ is uniformly weakly composition-closed.*

Proof. For $f_i : X \to \mathbf{R}$, $f_i = g_i|X$, $g_i \in C(\mathfrak{U})$, define $h : X \to E = \mathbf{R}^I$ as always, and similarly $h' : Y \to \mathbf{R}^I$ by $\pi_i \circ h' = g_i$. If $k \in C(\mathfrak{U}_I)$, then $k \circ h' \in C(\mathfrak{U})$, consequently $k \circ h = (k \circ h')|X \in C(\mathfrak{U})|X$. ▌

Now consider $\Phi = C(\mathfrak{U}_1|)\mathbf{N}$. By 1.14, Φ is uwcc, but it fails to be ufcc: indeed, $f(x) = x$ defines, for $x \in \mathbf{N}$, a function belonging to Φ and, for the family $\{f\}$, we have $h = f$, $h(\mathbf{N}) = \mathbf{N}$, and $k(x) = x^2$ $(x \in \mathbf{N})$ furnishes a function $k \in \mathfrak{U}_1|\mathbf{N}$ because $\mathfrak{U}_1|\mathbf{N}$ is the discrete uniformity on $\mathbf{N}$. However, $k \circ h = k$ does not belong to Φ. This example shows that uwcc⇏ufcc, hence uwcc⇏ucc and ufwcc⇏ufcc.

Concerning the relation between uniform and ordinary composition-closed classes, it is enough to consider $\Phi = C(\mathfrak{U}_1)$; it is ucc by 1.12, but it is not fwcc since $f \in \Phi$ for $f(x) = x$ $(x \in \mathbf{R})$, while $k \circ f \notin \Phi$ if $k(x) = x^2$ $(x \in \mathbf{R})$, $k \in C(\mathbf{R})$.

2. Classes of the form $C(\mathfrak{U})$, $C(\mathfrak{U})|X$, $C^*(\mathfrak{U})$, $C^*(\mathfrak{U})|X$. We begin with a slightly more precise formulation of 1.12. For this purpose, let us recall that a uniformity $\mathfrak{U}$ is said to be a *c-uniformity* if it is of the form $\mathfrak{U}(\Phi)$, i.e. if there exists a function class Φ such that $\mathfrak{U}$ is the weak uniformity of Φ. Now we can prove:

THEOREM 2.1 *For a function class Φ on X, the following statements are equivalent:*

(a) *Φ is uniformly composition-closed,*

(b) *there exists a c-uniformity $\mathfrak{U}$ on X such that $\Phi = C(\mathfrak{U})$,*

(c) *there exists a uniformity $\mathfrak{U}$ on X such that $\Phi = C(\mathfrak{U})$,*

(d) *there exist a set $Y \supset X$ and a c-uniformity $\mathfrak{U}$ on Y such that X is dense in Y and $\Phi = C(\mathfrak{U})|X$,*

(e) *there exist a set $Y \supset X$ and a uniformity $\mathfrak{U}$ on Y such that X is dense in Y and $\Phi = C(\mathfrak{U})|X$.*

PROOF. (a)$\Rightarrow$(b) is contained in the proof of 1.12.

(b) $\left\{\begin{matrix}\Rightarrow(\text{c}) \Rightarrow \\ \Rightarrow(\text{d}) \Rightarrow\end{matrix}\right\}$ (e) is obvious.

(e)$\Rightarrow$(a): If X is dense in $(Y, \mathfrak{U})$, then the extension theorem for uniformly continuous maps implies $C(\mathfrak{U})|X = C(\mathfrak{U}|X)$. ∎

2.1 corresponds to [2], Theorem 2 and Theorem 4. The following theorem is the uniform analogon of [2], Theorem 3:

THEOREM 2.2. *For a function class Φ on X, the following statements are equivalent:*

(a) *Φ is uniformly weakly composition-closed,*

(b) *there exist a set $Y \supset X$ and a c-uniformity $\mathfrak{U}$ on Y such that $\Phi = C(\mathfrak{U})|X$,*

(c) *there exist a set $Y \supset X$ and a uniformity $\mathfrak{U}$ on Y such that $\Phi = C(\mathfrak{U})|X$.*

PROOF. (a)$\Rightarrow$(b): Choose an index set I such that $\Phi = \{f_i : i \in I\}$, construct E, h as usually, and define $Y \supset X$ such that there exist a bijection $h' : Y - X \to E - h(X)$, and a map $g : Y \to E$ such that

$$g(x) = \begin{cases} h(x) & \text{if } \quad x \in X, \\ h'(x) & \text{if } \quad x \in Y - X. \end{cases}$$

Let $\mathfrak{U}$ be the initial uniformity on X corresponding to g and the uniformity $\mathfrak{U}_I$ on E.

$\mathfrak{U}$ is a c-uniformity because it is the weak uniformity of $\{\pi_i \circ g : i \in I\}$.

The equality $f_i = (\pi_i \circ g)|X$ yields $\Phi \subset C(\mathfrak{U})|X$. Conversely if $f \in C(\mathfrak{U})$ then it can be factorized in the form $f = k \circ g$ where $k \in C(\mathfrak{U}_I)$. Since Φ is uwcc, $f|X = k \circ h \in \Phi$, consequently $C(\mathfrak{U})|X \subset \Phi$.

(b)$\Rightarrow$(c) is obvious, and (c)$\Rightarrow$(a) is obtained from 1.14. ∎

Let us now consider classes of the form $C^*(\mathfrak{U})$ and $C^*(\mathfrak{U})|X$. For this purpose, we shall say that a class Φ of real-valued functions on X is *bounded* if every $f\in\Phi$ is bounded. Classes of the form $C^*(\mathfrak{U})$ or $C^*(\mathfrak{U})|X$ are clearly bounded.

We also need the following analogon of the Tietze—Urysohn extension theorem (cf. [3], 4.2.f.9):

LEMMA 2.3. *Let* $(Y, \mathfrak{U})$ *be a uniform space,* $X\subset Y$, *and* $f\in C^*(\mathfrak{U}|X)$. *Then there exists a* $g\in C^*(\mathfrak{U})$ *such that* $g|X=f$. ∎

Now we can formulate the following characterization theorem:

THEOREM 2.4. *Let* Φ *be a bounded function class on* X. *The following statements are equivalent:*

(a) Φ *is uniformly composition-closed,*
(b) *there is a c-uniformity* $\mathfrak{U}$ *on* X *such that* $\Phi = C^*(\mathfrak{U})$,
(c) *there is a uniformity* $\mathfrak{U}$ *on* X *such that* $\Phi = C^*(\mathfrak{U})$,
(d) *there exist a set* $Y\supset X$ *and a c-uniformity* $\mathfrak{U}$ *on* Y *such that* $\Phi = C^*(\mathfrak{U})|X$,
(e) *there exist a set* $Y\supset X$ *and a uniformity* $\mathfrak{U}$ *on* Y *such that* $\Phi = C^*(\mathfrak{U})|X$.

PROOF. (a)→(b)→(c) follows from 2.1. (b)⇒(d)⇒(e) and (c)→(e) are obvious.

(e)⇒(a): By 2.3 $C^*(\mathfrak{U}|X)\subset C^*(\mathfrak{U})|X$, and the opposite inclusion is obvious. Hence $\Phi = C^*(\mathfrak{U}|X)$. Assume $f_i\in\Phi$ for $i\in I$, and define E, h as usually. The set $\pi_i(h(X))=f_i(X)$ is bounded for each i, hence there are compact intervals $[a_i, b_i]\subset\mathbf{R}$ such that

$$h(X)\subset \bigtimes_{i\in I}[a_i, b_i].$$

Thus $\overline{h(X)}$ is compact. If now $k\in C(\mathfrak{U}_I|\overline{h(X)})$ then k is bounded and $k\circ h\in C^*(\mathfrak{U}|X)=\Phi$. By 1.3 Φ is ucc. ∎

COROLLARY 2.5. *A bounded and uniformly weakly composition-closed class is uniformly composition-closed.*

PROOF. If Φ is bounded and uwcc, then $\Phi = C(\mathfrak{U})|X$ by 2.2 for a suitable uniform space $(Y, \mathfrak{U})$ such that $Y\supset X$. Hence obviously $\Phi = C^*(\mathfrak{U})|X$ and Φ is ucc by 2.4. ∎

Now we are able to produce a class that is ufcc without being uwcc:

EXAMPLE 2.6. *Let* $X\subset\mathbf{R}^2$ be the union of all straight lines $L_n=\{(x, n): x\in\mathbf{R}\}$ for $n\in\mathbf{N}$, equipped with the uniformity $\mathfrak{U}_2|X$. Let Φ be the class of all those $f\in C^*(\mathfrak{U})$ for which there is an n_0 such that $f(x, n)=0$ if $n\geqq n_0$. By 2.4 Φ is ufcc. If Φ were uwcc, then it had to be ucc by 2.5, hence uniformly closed by 1.11. However, this fails to be true: the function

$$f(x, n) = \frac{1}{n}\sin nx$$

is the uniform limit of a sequence taken from Φ but does not belong to Φ (cf. [6], p. 103 and [1], pp. 149–150). ∎

In the following theorems we assume that a uniformity is given on X. They correspond to [2], Theorems 5–7, and the role of completely regular topologies is played in them by c-uniformities.

THEOREM 2.7. *Let $\mathfrak{U}$ be a c-uniformity on X and Φ be a function class on X. The equality $\Phi = C(\mathfrak{U})$ holds iff $\mathfrak{U}$ is uniformly composition-closed and $\mathfrak{U} = \mathfrak{U}(\Phi)$.*

PROOF. If $\Phi = C(\mathfrak{U})$ then Φ is ucc. Let Φ' be a function class such that $\mathfrak{U} = \mathfrak{U}(\Phi')$. Then $\Phi' \subset C(\mathfrak{U}) = \Phi$, hence $\mathfrak{U} = \mathfrak{U}(\Phi') \subset \mathfrak{U}(\Phi)$, and obviously $\mathfrak{U}(\Phi) \subset \mathfrak{U}$, consequently $\mathfrak{U} = \mathfrak{U}(\Phi)$.

Conversely if $\mathfrak{U}$ is ucc and $\mathfrak{U} = \mathfrak{U}(\Phi)$, then $\Phi = C(\mathfrak{U})$ by the proof of 1.12. ∎

THEOREM 2.8. *Let $(X, \mathfrak{U})$ be a uniform space and Φ be a function class on X. There exist a set $Y \supset X$ and a c-uniformity $\mathfrak{U}'$ on Y such that $\mathfrak{U} = \mathfrak{U}'|X$ and $\Phi = C(\mathfrak{U}')|X$ iff Φ is uniformly weakly composition-closed and $\mathfrak{U} = \mathfrak{U}(\Phi)$.*

PROOF. If Φ is uwcc, then by 2.2 $\Phi = C(\mathfrak{U}')|X$ for a suitable set $Y \supset X$ and a c-uniformity $\mathfrak{U}'$ on Y. For $\Phi' = C(\mathfrak{U}')$ we have $\mathfrak{U}' = \mathfrak{U}(\Phi')$ by 2.7. By [3], (4.2.10), we obtain $\mathfrak{U}'|X = \mathfrak{U}(\Phi)$, hence $\mathfrak{U}'|X = \mathfrak{U}$ if $\mathfrak{U} = \mathfrak{U}(\Phi)$.

Conversely assume $\mathfrak{U} = \mathfrak{U}'|X$ and $\Phi = C(\mathfrak{U}')|X$ for a c-uniformity $\mathfrak{U}'$. Then Φ is uwcc by 2.2, and the above argument yields $\mathfrak{U}'|X = \mathfrak{U}(\Phi)$, hence $\mathfrak{U} = \mathfrak{U}(\Phi)$. ∎

THEOREM 2.9. *Let $\mathfrak{U}$ be a precompact c-uniformity on X, and Φ be a bounded function class on X. We have $\Phi = C^*(\mathfrak{U})$ iff Φ is uniformly composition-closed and $\mathfrak{U} = \mathfrak{U}(\Phi)$.*

PROOF. If $\Phi = C^*(\mathfrak{U})$ then Φ is ucc by 2.4. Since $\mathfrak{U}$ is precompact, we have $C^*(\mathfrak{U}) = C(\mathfrak{U})$ and $\mathfrak{U} = \mathfrak{U}(\Phi)$ by 2.7.

Conversely let Φ be ucc and $\mathfrak{U} = \mathfrak{U}(\Phi)$. Then $\Phi = C(\mathfrak{U})$ by 2.7, hence $\Phi = C^*(\mathfrak{U})$ by the precompactness of $\mathfrak{U}$. ∎

3. Classes of the form $C^*(X)$ and $C^*(Y)|X$. We denote, of course, by $C^*(X)$ the class of all bounded, continuous real-valued functions on a topological space X. Similarly, if $\mathfrak{T}$ is the topology of X, we write $C(\mathfrak{T})$ and $C^*(\mathfrak{T})$ for $C(X)$ and $C^*(X)$ respectively.

LEMMA 3.1. *Every class $C^*(Y)|X$ is weakly composition-closed; it is composition-closed if X is dense in Y, in particular if $X = Y$.*

PROOF. Let $f_i = f_i'|X$, $f_i' \in C^*(Y)$. Define E, h as usually, and $h' : Y \to E$ by $\pi_i \circ h' = f_i'$. Then $f_i'(Y) \subset [a_i, b_i] \subset \mathbf{R}$ for each i, hence $h'(Y)$ is contained in the compact set

$$K = \underset{i \in I}{\times} [a_i, b_i] \subset E .$$

Thus if $k \in C(E)$, then $k|h'(Y)$ is bounded, and $k \circ h' \in C^*(Y)$, $k \circ h = (k \circ h')|X \in C^*(Y)|X$.

If X is dense in Y, then $h'(Y)\subset\overline{h(X)}\subset K$, hence $k\in C(\overline{h(X)})$ implies that k is bounded, and $k\circ h=(k\circ h')\,|X\in C^*(Y)|\,X$. ▮

Observe that $C^*(X)$ need not be fscc; indeed, consider $X=(0,\ 1)$, $f(x)=x$ for $x\in X$, and $k(u)=\dfrac{1}{u}$ for $h(X)=f(X)=(0,1)$.

COROLLARY 3.2. *Every class $C^*(Y)|X$ is uniformly composition-closed.*

PROOF. A wcc class is ccc, hence uccc by 1.4 and ucc by 1.13. ▮

3.1 admits the following converses:

THEOREM 3.3. *Let Φ be a bounded function class on X. Φ is weakly composition-closed iff there exist a set $Y\supset X$ and a topology on Y such that $\Phi=C^*(Y)|X$.*

PROOF. The sufficiency is contained in 3.1. Now if Φ is wcc, then, by [2], Theorem 3, $\Phi=C(Y)|X$ for a suitable topological space Y. Then $\Phi=C^*(Y)|X$ because every f has a bounded extension in $C(Y)$. ▮

A similar argument furnishes by [2], Theorem 4:

THEOREM 3.4 *Let Φ be a bounded function class on X. Φ is composition-closed iff there exist a set $Y\supset X$ and a topology on Y such that X is dense in Y and $\Phi=C^*(Y)|X$.* ▮

In order to characterize the classes $C^*(X)$, we need a concept introduced in [5]: a function class Φ will be said to be *envelope-closed* if $f\in\Phi$ whenever there are $\emptyset\neq\Phi_1\subset\Phi$ and $\emptyset\neq\Phi_2\subset\Phi$ such that $f=\sup\Phi_1=\inf\Phi_2$. This concept occurs in the following

LEMMA 3.5 ([5], Theorem 8). *Let Φ be a bounded function class on X. There exists a topology $\mathfrak{T}$ on X such that $\Phi=C^*(\mathfrak{T})$ iff Φ is an envelope-closed subtractive lattice containing all constants.* ▮

Now we can prove:

THEOREM 3.6. *Let Φ be a bounded function class on X. There exists a topology on X such that $\Phi=C^*(X)$ iff Φ is finitely composition-closed and envelope-closed.*

PROOF. The necessity results from 3.1 and 3.5. Conversely, if Φ is fcc, then it is a subtractive lattice and contains all constants so that, by 3.5 again, $\Phi=C^*(X)$ for a suitable topology on X. ▮

THEOREM 3.7. *Let X be a completely regular space and Φ be a bounded function class on X. Then $\Phi=C^*(X)$ iff Φ is finitely composition-closed, envelope-closed, and the cozero-sets of the elements of Φ constitute a base for X.*

PROOF. If $\Phi=C^*(X)$ then Φ is fcc and envelope-closed by 3.6 and the last condition is obvious. Conversely if all conditions are fulfilled then 3.6 implies that there is a topology $\mathfrak{T}'$ on X such that $\Phi=C^*(\mathfrak{T}')$; it is well-known that $C^*(\mathfrak{T}')=C^*(\mathfrak{T})$ if $\mathfrak{T}$ is the topology whose base is composed of the cozero-sets of the elements of $C^*(\mathfrak{T}')$, i.e. $\Phi=C^*(\mathfrak{T}')=C^*(\mathfrak{T})=$ $=C^*(X)$. ▮

THEOREM 3.8. *Let* $(X, \mathfrak{U})$ *be a uniform space. We have* $C^*(\mathfrak{U}) = C^*(\mathfrak{T})$ *where* $\mathfrak{T}$ *is the topology induced by* $\mathfrak{U}$ *iff* $C^*(\mathfrak{U}) = \Phi$ *is envelope-closed.*

PROOF. The necessity follows from 3.6. Conversely if $\Phi = C^*(\mathfrak{U})$ is envelope-closed, then Φ is a bounded subtractive lattice containing all constants, hence, by 3.5, it is of the form $\Phi = C^*(\mathfrak{T}')$ where $\mathfrak{T}'$ is a suitable topology on X. By 3.6 Φ is fcc so that 3.7 can be applied because the cozero-sets of $C^*(\mathfrak{U})$ constitute a base for $\mathfrak{T}$ by [3], 4.2.f.2. ▌

4. Boundedly composition-closed classes. Another characterization of the classes $C^*(X)$ can be given with the help of the following

DEFINITION 4.1. A function class Φ on X is said to be *boundedly strongly composition-closed* (sc*c), *boundedly composition-closed* (c*c), or *boundedly weakly composition-closed* (wc*c) if $f_i \in \Phi$ $(i \in I)$ implies, with the usual meaning of E, h, that we have $k \circ h \in \Phi$ whenever $k \in C^*(h(X))$, or $k \in C^*(\bar{h}(X))$, or $k \in C^*(E)$, respectively.

LEMMA 4.2. *If* Φ *is boundedly strongly composition-closed and bounded, then it is envelope-closed.*

PROOF. Suppose $f = \sup \Phi_1 = \inf \Phi_2$ where

$$\emptyset \neq \Phi_1 = \{f_i : i \in I_1\}, \quad \emptyset \neq \Phi_2 = \{f_i : i \in I_2\},$$

and $f_i \in \Phi$ for $i \in I = I_1 \cup I_2$. For these f_i and I, define E and h in the usual way.

We shall define $k : h(X) \to \mathbf{R}$ by

$$k(h(x)) = f(x) \quad (x \in X).$$

This is possible since $x, y \in X$, $h(x) = h(y)$ implies $f_i(x) = f_i(y)$ for every $i \in I$, hence $f(x) = f(y)$. The inequality

$$f_{i_1}(x) \leq k(h(x)) \leq f_{i_2}(x) \quad (i_1 \in I_1,\ i_2 \in I_2)$$

shows that k is bounded. We show that $k \in C^*(h(X))$.

In fact, for $z = h(x) \in h(X)$, $x \in X$, and $\varepsilon > 0$, choose $i_1 \in I_1$ and $i_2 \in I_2$ such that

$$k(z) - \frac{\varepsilon}{2} < f_{i_1}(x) \leq f_{i_2}(x) < k(z) + \frac{\varepsilon}{2}.$$

If now $u = h(y) \in h(X)$ satisfies

$$|\pi_{i_1}(u) - \pi_{i_1}(z)| < \frac{\varepsilon}{2}, \quad |\pi_{i_2}(u) - \pi_{i_2}(z)| < \frac{\varepsilon}{2}$$

then

$$k(z) - \varepsilon < f_{i_1}(y) \leq f(y) = k(u) \leq f_{i_2}(y) < k(z) + \varepsilon,$$

according to the statement.

By hypothesis $k \circ h \in \Phi$, i.e. $f \in \Phi$. ▌

THEOREM 4.3. *A bounded function class Φ on X is boundedly strongly composition-closed iff it is of the form $\Phi = C^*(X)$ for a suitable topology on X.*

PROOF. If Φ is bounded and sc*c then by 4.2 it is envelope-closed and a subtractive lattice containing all constants (because $k_1(u, v) = u - v$, $k_2(u, v) = \max(u, v)$, $k_3(u, v) = \min(u, v)$ are bounded on bounded subsets of $\mathbf{R}^2$), hence $\Phi = C^*(X)$ by 3.5. The converse is obvious. ▌

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THE MAXIMAL SQUARE-FREE, BI-UNITARY DIVISOR OF m WHICH IS PRIME TO n, I.

By

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§ 1. Introduction. It is well-known that a divisor $d>0$ of a positive integer n is called unitary if $d\delta=n$ and $(d, \delta)=1$. We write $d\|n$ to say that d is a unitary divisor of n. Analogous to this the second named author (cf. [5]) introduced the notion of a bi-unitary divisor. A divisor $d>0$ of the positive integer n is called bi-unitary if $d\delta=n$ and $(d, \delta)^{**}=1$, where $(d, \delta)^{**}$ is the greatest common unitary divisor of both d and δ. Recently the second named author and M. V. Subba Rao (cf. [6]) generalized the concept of bi-unitary convolution and obtained several interesting asymptotic formulae. They considered the bi-unitary k-ary product $F_k^{**}(n)$ of arithmetical functions $f(n)$ and $g(n)$ defined by

$$F_k^{**}(n)=\sum_{\substack{d\delta=n \\ (d, \delta)_k^{**}=1}} f(d)g(\delta),$$

where k is a fixed positive integer and $(a, b)_k^{**}$ the greatest among the common k-th power unitary divisors of a and b. In case $k=1$, this reduces to the bi-unitary convolution.

As usual, by a k-free integer, we mean a positive integer which is not divisible by the k-th power of any prime. Recently, an estimate for the summatory function $\gamma_k(m; n)$ has been established by the authors (cf. [7]) where $\gamma_k(m; n)$ denotes the maximal k-free divisor of m which is prime to n. In particular, when $k=2$, $\gamma_2(m; n)=\gamma(m; n)$; the maximal square-free divisor of m which is prime to n. Let $\gamma(m)$ and $\delta(m)$ respectively denote the maximal square-free divisor of m and the maximal square-free odd divisor of m. It is clear that $\gamma(m; 1)=\gamma(m)$ and $\gamma(m; 2)=\delta(m)$. Let $\gamma^{**}(m; n)$ denote the maximal square-free bi-unitary divisor of m which is prime to n and let $\gamma^{**}(m)$ and $\delta^{**}(m)$ respectively denote the maximal square-free, bi-unitary divisor of m and the maximal square-free, bi-unitary odd divisor of m.

In this paper we establish asymptotic formulae for

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \gamma(m;n) \quad \text{and} \quad \sum_{\substack{m \leq x \\ (m,n)=1}} \gamma^{**}(m;n)$$

with uniform O-estimates for the error terms (see **§ 3**) and deduce several corollaries. Further on the assumption of the Riemann hypothesis, we improve the order estimates of the error terms in the asymptotic formulae. In **§ 2** we prepare the necessary background that is needed in establishing the asymptotic formula of **§ 3**.

§ 2. Preliminaries. Let $\mu(n)$ denote the Möbius function and let $\varphi(n)$ denote the Euler totient function, $\varphi^*(n)$ denote the unitary analogue of $\varphi(n)$ (cf. [1]. § 1), $\psi(n)$ denote Dedekind's ψ-function (cf. [3] p. 123) and $J(n)$ denote the Jordan totient function of second order (cf. [3], p. 147). These functions have the following arithmetical forms.

$$\varphi(n) = \sum_{d\delta=n} \mu(d)\,\delta = n \prod_{p|n}\left(1-\frac{1}{p}\right) \tag{2.1}$$

$$\varphi^*(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} \mu^*(d)\,\delta = n \prod_{p^\alpha \| n}\left(1+\frac{1}{p^\alpha}\right) \tag{2.2}$$

$$\psi(n) = \sum_{d\delta=n} \mu^2(d)\,\delta = n \prod_{p|n}\left(1+\frac{1}{p}\right) \tag{2.3}$$

$$J(n) = \sum_{d\delta=n} \mu(d)\,\delta^2 = n^2 \prod_{p|n}\left(1-\frac{1}{p^2}\right) \tag{2.4}$$

where $\mu^*(n)$ is the unitary analogue of the Möbius μ-function defined by $\mu^*(n)=(-1)^{\omega(n)}$, $\omega(n)$ being the number of distinct prime factors of $n>1$ and $\omega(1)=0$. We note that

$$\varphi^*(n) = \varphi(n), \quad \text{if } n \text{ is square-free.} \tag{2.5}$$

It is clear from (2.1), (2.3) and (2.4) that

$$\psi(n) = \frac{J(n)}{\varphi(n)}. \tag{2.6}$$

Let $H(n)$ be the function defined by $H(1)=1$ and

$$H(n) = n^2 \prod_{p|n}\left(1-\frac{1}{p(p+1)}\right) \quad \text{for} \quad n>1. \tag{2.7}$$

REMARK 2.1. It is clear that $\varphi(n) \leq n$, $\psi(n) \geq n$, $\dfrac{1}{J(n)} = O\left(\dfrac{1}{n^2}\right)$, since $J(n) = n^2 \prod_{p|n}\left(1-\dfrac{1}{p^2}\right) > n^2 \prod_{p}\left(1-\dfrac{1}{p^2}\right) = \dfrac{n^2}{\zeta(2)}$ (cf. [4]. Theorem 280), where

$\zeta(s)$ is Riemann zeta function defined by $\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}$ for $s>1$. Further, $\frac{1}{H(n)}=O\left(\frac{1}{n^2}\right)$, since

$$H(n)>n^2\prod_p\left(1-\frac{1}{p(p+1)}\right)>n^2\prod_p\left(1-\frac{1}{p^2}\right)=\frac{n^2}{\zeta(2)}.$$

Let $\Theta(n)$ denote the number of square-free divisors of n. Let α and β be the constants given by

$$\alpha=\prod_p\left(1-\frac{1}{p(p+1)}\right) \tag{2.8}$$

$$\beta=\prod_p\left(1-\frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right). \tag{2.9}$$

Let $G(n)$ be the function defined by

$$G(n)=n\prod_{p\mid n}\left(1-\frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right)\quad\text{for}\quad n>1\quad\text{and}\quad G(1)=1. \tag{2.10}$$

Let $l(n)=1$ or 0 according as $n=1$ or $n>1$ and note that $l(n)$ is multiplicative.

Throughout the discussion x denotes a real variable >3, ε denotes a preassigned positive real number, u and n are fixed positive integers. All the O-estimates that appear in this paper are independent of x, u and n but may depend on ε. We describe this situation by mentioning the word "uniformly" at the end of each formula.

We need the following Lemmas to prove our main theorems.

LEMMA 2.1. (cf. [2] Lemma 3.1, $s=2$)

$$\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}=\frac{\zeta(2)J(u)}{u^2}+O\left(\frac{1}{x}\right) \tag{2.11}$$

uniformly.

LEMMA 2.2. (cf. [4], Theorem 63)

$$\sum_{d\mid n}\varphi(d)=n. \tag{2.12}$$

LEMMA 2.3.

$$\gamma(m;n)=\sum_{d\mid m}\mu^2(d)\varphi(d)\,l\big((d,n)\big). \tag{2.13}$$

PROOF. By Lemma 2.2, we have

$$\gamma(m;n)=\sum_{d|\gamma(m;n)}\varphi(d)=\sum_{\substack{d|m\\ d\text{ square-free}\\ (d,n)=1}}\varphi(d)=\sum_{d|m}\mu^2(d)\varphi(d)\,l\big((d,n)\big).$$

LEMMA 2.4. (cf. [6], Lemma 2.1, $k=1$)

$$\sum_{\substack{d\|m\\ d\|n}}\mu^*(d)=\begin{cases}1 & \text{if}\quad (m,n)^{**}=1\,.\\ 0 & \text{otherwise}\,.\end{cases}\tag{2.14}$$

LEMMA 2.5. (cf. [1], Corollary 2.1.1)

$$\sum_{d\|n}\varphi^*(d)=n\,.\tag{2.15}$$

LEMMA 2.6.

$$\gamma^{**}(m;n)=\sum_{\substack{d^2\delta=m\\ (d,\delta)=1}}\mu(d)\varphi(d)\,l\big((d,n)\big)\,\gamma(\delta;n)\,.\tag{2.16}$$

PROOF. We have $d\|\gamma^{**}(m,n)$ if and only if d is a square-free, bi-unitary divisor of m and $(d,n)=1$. Hence by Lemma 2.5 and (2.5),

$$\gamma^{**}(m;n)=\sum_{d\|\gamma^{**}(m;n)}\varphi^*(d)=\sum_{\substack{d\delta=m\\ (d,\delta)^{**}=1\\ (d,n)=1\\ d\text{ square-free}}}\varphi^*(d)=$$

$$=\sum_{\substack{d\delta=m\\ (d,\delta)^{**}=1}}\mu^2(d)\varphi(d)\,l\big((d,n)\big)=\sum_{d\delta=m}\mu^2(d)\varphi(d)\,l\big((d,n)\big)\,l\big((d,\delta)^{**}\big)\,.$$

Now, by Lemma 2.4, if follows that

$$\gamma^{**}(m;n)=\sum_{d\delta=m}\mu^2(d)\varphi(d)\,l\big((d,n)\big)\sum_{\substack{t\|d\\ t\|\delta}}\mu^*(t)=$$

$$=\sum_{d\delta=m}\mu^2(d)\varphi(d)\,l\big((d,n)\big)\sum_{\substack{tt_1=d\\ tt_2=\delta\\ (t,t_1)=(t,t_2)=1}}\mu^*(t)=$$

$$=\sum_{\substack{t^2t_1t_2=m\\ (t,t_1t_2)=1}}\mu^2(tt_1)\varphi(tt_1)\,l\big((tt_1,n)\big)\mu^*(t)=$$

$$=\sum_{\substack{t^2t_1t_2=m\\ (t,t_1t_2)=1}}\mu^2(t)\mu^2(t_1)\varphi(t)\varphi(t_1)\,l\big((t_1,n)\big)\,l\big((t,n)\big)\mu^*(t)=$$

$$=\sum_{\substack{t^2u=m\\ (t,u)=1}}\mu(t)\varphi(t)\,l\big((t,n)\big)\sum_{t_1t_2=u}\mu^2(t_1)\varphi(t_1)\,l\big((t_1,n)\big)=$$

$$=\sum_{\substack{t^2u=m\\ (t,u)=1}}\mu(t)\varphi(t)\,l\big((t,n)\big)\,\gamma(u,n)\,.$$

by Lemma 2.3. Hence Lemma 2.6 follows.

LEMMA 2.7. (cf. [7], Corollary 4.1.3). For $x \geq 3$ and $n \geq 1$

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu^2(m)\varphi(m) = \frac{3\alpha n^3 x^2}{\pi^2 \psi(n) H(n)} + O\left(\Theta(n) x^{\frac{3}{2}} \delta(x)\right); \tag{2.17}$$

uniformly, where $\delta(x)$ is given by

$$\delta(x) = \begin{cases} \exp\left\{-A \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right\}, & \text{for} \quad x \geq 3, \\ 1, \quad \text{for} \quad 0 < x < 3, \end{cases} \tag{2.18}$$

A being a positive absolute constant.

LEMMA 2.8. (cf. [7], Corollary 4.2.3). If the Riemann hypothesis is true, then for $x \geq 3$ and $n \geq 1$

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu^2(m)\varphi(m) = \frac{3\alpha n^3 x^2}{\pi^2 \psi(n) H(n)} + O\left(\Theta(n) x^{\frac{7}{5}} \omega(x)\right); \tag{2.19}$$

uniformly, where $\omega(x)$ is defined by

$$\omega(x) = \begin{cases} \exp\{A \log x (\log\log x)^{-1}\}, & \text{for} \quad x \geq 3, \\ 1, \quad \text{for} \quad 0 < x < 3, \end{cases} \tag{2.20}$$

A being a positive absolute constant.

LEMMA 2.9. For $u \geq 1$

$$\sum_{\substack{m=1 \\ (m,u)=1}}^{\infty} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = \frac{\beta u}{G(u)}, \tag{2.21}$$

where β and $G(u)$ are defined by (2.9) and (2.10) respectively.

PROOF. The series is absolutely convergent, since

$$\left|\frac{\mu(m)\varphi^2(m)}{m^3 H(m)}\right| = O\left(\frac{1}{m^3}\right),$$

by Remark 2.1, and the general term of the series is a multiplicative function of m, so that the Lemma follows, expanding the series on the left into an infinite product of Euler type (cf. [4], theorem 286).

LEMMA 2.10. For $u \geq 1$

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = \frac{\beta u}{G(u)} + O\left(\frac{1}{x^2}\right). \tag{2.22}$$

PROOF. We have by Remark 2.1,

$$\sum_{\substack{m > x \\ (m,u)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = O\left(\sum_{\substack{m > x \\ (m,u)=1}} \frac{1}{m^3}\right) = O\left(\sum_{m > x} \frac{1}{m^3}\right) = O\left(\frac{1}{x^2}\right).$$

Since

$$\sum_{\substack{m\leq x\\(m,u)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = \sum_{\substack{m=1\\(m,u)=1}}^{\infty} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} - \sum_{\substack{m>x\\(m,u)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)}.$$

Now, Lemma 2.10 follows by Lemma 2.9.

§ 3. Main Results. First we prove the following.

THEOREM 3.1. *For $x\geq 3$ and for fixed positive integers n and u*

$$\sum_{\substack{m\leq x\\(m,u)=1}} \gamma(m;n) = \frac{\alpha\, un^3 J(u)\, x^2}{2\psi(un) H(un)} + O\left(\Theta(un)\, x^{\frac{3}{2}}\,\delta(x)\right); \tag{3.1}$$

uniformly where $\delta(x)$ is given by (2.18).

PROOF. By Lemmas 2.3 and 2.7, we have

$$\sum_{\substack{m\leq x\\(m,u)=1}} \gamma(m;n) = \sum_{\substack{m\leq x\\(m,u)=1}} \sum_{d\delta=n} \mu^2(d)\varphi(d)\, l((d,n)) = \tag{3.2}$$

$$= \sum_{\substack{d\delta\leq x\\(d\delta,u)=1\\(d,n)=1}} \mu^2(d)\varphi(d) = \sum_{\substack{\delta\leq x\\(\delta,u)=1}} \sum_{\substack{d\leq x/\delta\\(d,un)=1}} \mu^2(d)\varphi(d) =$$

$$= \sum_{\substack{\delta\leq x\\(\delta,u)=1}} \left\{\frac{3\alpha(un)^3}{\pi^2\psi(un)H(un)}\cdot\frac{x^2}{\delta^2} + O\left(\Theta(un)\frac{x^{3/2}}{\delta^{3/2}}\,\delta\left(\frac{x}{\delta}\right)\right)\right\} =$$

$$= \frac{3\alpha(un)^3 x^2}{\pi^2\psi(un)H(un)} \sum_{\substack{m\leq x\\(m,u)=1}} \frac{1}{m^2} + O\left(\Theta(un)\sum_{\substack{m\leq x\\(m,u)=1}} \left(\frac{x}{m}\right)^{3/2}\delta\left(\frac{x}{m}\right)\right).$$

By (2.18) it is clear that $x^{\varepsilon}\delta(x)$ is monotonic increasing for every $\varepsilon>0$, so that we have

$$\sum_{\substack{m\leq x\\(m,u)=1}} \left(\frac{x}{m}\right)^{3/2}\delta\left(\frac{x}{m}\right) = \sum_{\substack{m\leq x\\(m,u)=1}} \left(\frac{x}{m}\right)^{(3/2)-\varepsilon}\left(\frac{x}{m}\right)^{\varepsilon}\delta\left(\frac{x}{m}\right) =$$

$$= O\left(\sum_{\substack{m\leq x\\(m,u)=1}} \left(\frac{x}{m}\right)^{(3/2)-\varepsilon} x^{\varepsilon}\,\delta(x)\right) = \left(x^{3/2\varepsilon}\,\delta(x)\sum_{\substack{m\leq x\\(m,u)=1}} \frac{1}{m^{(3/2)-\varepsilon}}\right) =$$

$$= O\left(x^{3/2}\,\delta(x)\right).$$

Hence by Lemma 2.1, Remark 2.1, and the above disuccion, (3.2) becomes

$$\sum_{\substack{m\leq x\\(m,u)=1}} \gamma(m;n) = \frac{3\alpha(un)^3 x^2}{\pi^2\psi(un)H(un)}\left\{\frac{\zeta(2)J(u)}{u^2} + O\left(\frac{\Theta(u)}{x}\right)\right\} +$$

$$+O\left(\Theta(un)\,x^{3/2}\,\delta(x)\right) = \frac{\alpha\, un^3\, J(u)\, x^2}{2\psi(un)\, H(un)} + O(x) + O\left(\Theta(un)\,x^{3/2}\,\delta(x)\right) =$$

$$= \frac{\alpha\, un^3\, J(u)\, x^2}{2\psi(un)\, H(un)} + O\left(\Theta(un)\,x^{3/2}\,\delta(x)\right).$$

Hence Theorem 3.1 follows.

COROLLARY 3.1.1. ($u = 1$). For $x \geq 3$ and for any fixed positive integer n,

$$\sum_{m \leq x} \gamma(m; n) = \frac{\alpha\, n^3\, x^2}{2\psi(n)\, H(n)} + O\left(\Theta(n)\, x^{3/2}\, \delta(x)\right); \tag{3.3}$$

uniformly.

COROLLARY 3.1.2. ($n = 1$). For $x \geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \gamma(m) = \frac{\alpha\, u\, \varphi(u)\, x^2}{2H(u)} + O\left(\Theta(u)\, x^{3/2}\, \delta(x)\right), \tag{3.4}$$

uniformly.

COROLLARY 3.1.3. ($n = 2$). For $x \geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \delta(m) = \frac{2\alpha\, u\, \varphi(u)\, x^2}{5H(u)} + O\left(\Theta(u)\, x^{3/2}\, \delta(x)\right); \tag{3.5}$$

uniformly.

REMARK 3.1. Corollary 3.1.1 has already been established by the authors (cf. [7], Corollary 4.3.4). Formulas (3.3) and (3.4) in the case $u = 1$ reduce to those results already established by the authors (cf. [7], Corollary 4.3.5 and 4.3.6)

THEOREM 3.2. *If the Riemann hypothesis is true then for $x \geq 3$ and for fixed positive integers u and n*

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \gamma(m; n) = \frac{\alpha\, un^3\, J(u)\, x^2}{2\psi(un)\, H(un)} + O\left(\Theta(un)\, x^{7/5}\, \omega(x)\right); \tag{3.6}$$

uniformly, where $\omega(x)$ is given by (2.20).

PROOF. Following the same procedure adopted in the proof of Theorem 3.1 and making use of Lemma 2.8 instead of Lemma 2.7, we get the following instead of (3.2).

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \gamma(m; n) = \frac{3\alpha\, (un)^3\, x^2}{\pi^2\, \psi(un)\, H(un)} \sum_{\substack{m \leq x \\ (m,u)=1}} \frac{1}{m^2} + \tag{3.7}$$

$$+ O\left(\Theta(un)\, x^{7/5} \sum_{\substack{m \leq x \\ (m,u)=1}} \frac{\omega\left(\dfrac{x}{m}\right)}{m^{7/5}}\right).$$

Since $\omega(x)$ is increasing, we have

$$\sum_{\substack{m\le x\\(m,u)=1}} \frac{\omega\left(\dfrac{x}{m}\right)}{m^{7/5}} = O\left(\omega(x)\sum_{\substack{m\le x\\(m,u)=1}} \frac{1}{m^{7/5}}\right) = O(\omega(x)).$$

Hence (3.7) reduces to

$$\sum_{\substack{m\le x\\(m,u)=1}} \gamma(m;n) = \frac{3\alpha\,(un)^3\,x^2}{\pi^2\,\varphi(un)\,H(un)} \sum_{\substack{m\le x\\(m,u)=1}} \frac{1}{m^2} + O(\Theta(un)\,x^{7/5}\,\omega(x)).$$

Now, Theorem 3.2 follows by Lemma 2.1 and Remark 2.1.

COROLLARY 3.2.1. ($u=1$). If the Riemann hypothesis is true then for $x\geq 3$ and for any fixed positive integer n,

$$\sum_{m\le x}\gamma(m;n) = \frac{\alpha\,n^3\,x^2}{2\psi(n)\,H(n)} + O(\Theta(n)\,x^{7/5}\,\omega(x)); \tag{3.8}$$

uniformly.

COROLLARY 3.2.2. ($n=1$). If the Riemann hypothesis is true then for $x\geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m\le x\\(m,u)=1}}\gamma(m) = \frac{\alpha\,u\,\varphi(u)\,x^2}{2H(u)} + O(\Theta(u)\,x^{7/5}\,\omega(x)); \tag{3.9}$$

uniformly.

COROLLARY 3.2.3. ($n=2$). If the Riemann hypothesis is true then for $x\geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m\le x\\(m,u)=1}}\delta(m) = \frac{2\alpha\,u\,\varphi(u)\,x^2}{5H(u)} + O(\Theta(u)\,x^{7/5}\,\omega(x)); \tag{3.10}$$

uniformly.

REMARK 3.2. Corollary 3.2.1 has already been established by the authors (cf. [7], corollary 4.4.4). Formulas (3.9) and (3.10) in the case $u=1$ reduce to those results already established by the authors (cf. [7], corollaries 4.4.5 and 4.4.6)

THEOREM 3.3. *For $x\geq 3$ and fixed positive integers n and u,*

$$\sum_{\substack{m\le x\\(m,u)=1}}\gamma^{**}(m;n) = \frac{\alpha\,\beta\,u^2\,n^4\,J(u)\,x^2}{2\psi(un)\,H(un)\,G(un)} + O(\Theta(un)\,x^{3/2}\,\delta(x)); \tag{3.11}$$

uniformly where β, $G(n)$ are respectively given by (2.9) *and* (2.10).

PROOF. By Lemma 2.6 and theorem 3.1, we have

$$(3.12)\quad \sum_{\substack{m\le x\\(m,u)=1}} \gamma^{**}(m;n) = \sum_{\substack{m\le x\\(m,u)=1}} \sum_{\substack{d^2\delta=m\\(d,\delta)=1}} \mu(d)\varphi(d)\, l((d,n))\,\gamma(\delta,n) =$$

$$= \sum_{\substack{d^2\delta\le x\\(d\delta,u)=(d,\delta)=(d,n)=1}} \mu(d)\varphi(d)\gamma(\delta;n) = \sum_{\substack{d\le\sqrt{x}\\(d,un)=1}} \mu(d)\varphi(d) \sum_{\substack{\delta\le x/d^2\\(\delta,ud)=1}} \gamma(\delta;n) =$$

$$= \sum_{\substack{d\le\sqrt{x}\\(d,un)=1}} \mu(d)\varphi(d)\left\{\frac{\alpha u d n^3 J(ud)x^2}{2\psi(und)H(und)d^4} + O\left[\Theta(und)\left(\frac{x}{d^2}\right)^{3/2}\delta\left(\frac{x}{d^2}\right)\right]\right\} =$$

$$= \frac{\alpha u n^3 J(u)x^2}{2\psi(un)H(un)} \sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} +$$

$$+ O\left(\Theta(un)\sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \Theta(m)\left(\frac{x}{m^2}\right)^{3/2}\delta\left(\frac{x}{m^2}\right)\right).$$

Since $x^{\varepsilon}\delta(x)$ is monotonic increasing for every $\varepsilon>0$, and $\Theta(m)=O(m^{\varepsilon})$, since $\Theta(m)\le\tau(m)$ where $\tau(m)$ is the number of all divisors of m and $\tau(m)= = O(m^{\varepsilon})$ (cf. [4], Theorem 315), the sum in the O-term of (3.12) becomes

$$\sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \Theta(m)\left(\frac{x}{m^2}\right)^{3/2}\delta\left(\frac{x}{m^2}\right) = \sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \Theta(m)\left(\frac{x}{m^2}\right)^{(3/2)-\varepsilon}\left(\frac{x}{m^2}\right)^{\varepsilon}\delta\left(\frac{x}{m^2}\right) \le$$

$$= O\left(\sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \Theta(m)\left(\frac{x}{m^2}\right)^{3/2-\varepsilon} x^{\varepsilon}\delta(x)\right) = O\left(x^{3/2}\delta(x)\sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \frac{\Theta(m)}{m^{3-2\varepsilon}}\right) =$$

$$= O\left(x^{3/2}\delta(x)\right).$$

Now, (3.12) reduces to

$$\sum_{\substack{m\le x\\(m,u)=1}} \gamma^{**}(m;n) = \frac{\alpha u n^3 J(u)x^2}{2\psi(un)H(un)} \sum_{\substack{m\le\sqrt{x}\\(m,un)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} +$$

$$+ O\left(\Theta(un)x^{3/2}\delta(x)\right).$$

By Lemma 2.10 and Remark 2.1, it follows that

$$\sum_{\substack{m\le x\\(m,un)=1}} \gamma^{**}(m;n) = \frac{\alpha u n^3 J(u)x^2}{2\psi(un)H(un)}\left\{\frac{\beta un}{G(un)} + O\left(\frac{1}{x}\right)\right\} +$$

$$+ O\left(\Theta(un)x^{3/2}\delta(x)\right) = \frac{\alpha\beta u^2 n^4 J(u)x^2}{2\psi(un)H(un)G(un)} + O\left(\Theta(un)x^{3/2}\delta(x)\right).$$

Hence Theorem 3.3 follows.

COROLLARY 3.3.1. ($u = 1$). For $x \geq 3$ and for any fixed positive integer n,

$$\sum_{m \leq x} \gamma^{**}(m; n) = \frac{\alpha \beta n^4 x^2}{2\psi(n) H(n) G(n)} + O\left(\Theta(n) x^{3/2} \delta(x)\right); \tag{3.13}$$

uniformly.

COROLLARY 3.3.2. ($n = 1$). For $x \geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m \leq x \\ (m, u) = 1}} \gamma^{**}(m) = \frac{\alpha \beta u^2 \varphi(u) x^2}{2H(u) G(u)} + O\left(\Theta(u) x^{3/2} \delta(x)\right); \tag{3.14}$$

uniformly.

COROLLARY 3.3.3. ($u = 1$, $n = 1$). For $x \geq 3$

$$\sum_{m \leq x} \gamma^{**}(m) = \frac{1}{2} \alpha \beta x^2 + O\left(x^{3/2} \delta(x)\right); \tag{3.15}$$

uniformly.

COROLLARY 3.3.4. ($u = 1$, $n = 2$). For $x \geq 3$

$$\sum_{m \leq x} \delta^{**}(m) = \frac{32}{75} \alpha \beta x^2 + O\left(x^{3/2} \delta(x)\right). \tag{3.16}$$

uniformly.

THEOREM 3.4. *If the Riemann hypothesis is true then for $x \geq 3$ and fixed positive integers n and u*

$$\sum_{\substack{m \leq x \\ (m, u) = 1}} \gamma^{**}(m; n) = \frac{\alpha \beta u^2 n^4 J(u) x^2}{2\psi(un) H(un) G(un)} + O\left(\Theta(un) x^{7/5} \omega(x)\right);$$

(3.17)

uniformly where $\omega(x)$ is given by (2.20).

PROOF. Following the same procedure adopted in the proof of Theorem 3.3 and making use of Theorem 3.2 instead of Theorem 3.1 and by observing that $\omega(x)$ is monotonic increasing, we get Theorem 3.9.

COROLLARY 3.4.1. ($u = 1$). If the Riemann hypothesis is true then for $x \geq 3$ and for any fixed positive integer n

$$\sum_{m \leq x} \gamma^{**}(m; n) = \frac{\alpha \beta n^4 x^2}{2\psi(n) H(n) G(n)} + O\left(\Theta(n) x^{7/5} \omega(x)\right); \tag{3.18}$$

uniformly.

COROLLARY 3.4.2. ($n = 1$). If the Riemann hypothesis is true then for $x \geq 3$ and for any fixed positive integer u,

$$\sum_{\substack{m \leq x \\ (m, u) = 1}} \gamma^{**}(m) = \frac{\alpha \beta u^2 \varphi(u) x^2}{2H(u) G(u)} + O\left(\Theta(u) x^{7/5} \omega(x)\right); \tag{3.19}$$

uniformly.

COROLLARY 3.4.3. ($u=1$, $n=1$). If the Riemann hypothesis is true then for $x\geq 3$

$$\sum_{m\leq x}\gamma^{**}(m)=\frac{1}{2}\alpha\beta x^2+O\left(x^{7/5}\omega(x)\right); \tag{3.20}$$

uniformly.

COROLLARY 3.4.4. ($u=1$, $n=2$). If the Riemann hypothesis is true then for $x\geq 3$,

$$\sum_{m\leq x}\delta^{**}(m)=\frac{32}{75}\alpha\beta x^2+O\left(x^{7/5}\omega(x)\right); \tag{3.21}$$

uniformly.

THEOREM 3.5. *For $x\geq 3$ and fixed positive integers n and u,*

$$\sum_{\substack{m\leq x\\(m,u)=1}}\frac{\gamma(m;n)}{m}=\frac{\alpha u n^3 J(u)x}{\psi(un)H(un)}+O\left(\Theta(un)x^{1/2}\delta(x)\right); \tag{3.22}$$

uniformly.

PROOF. The proof follows by Theorem 3.1 and by partial summation (cf. [4], Theorem 421).

THEOREM 3.6. *If the Riemann hypothesis is true then for $x\geq 3$ and fixed positive integers n and u*

$$\sum_{\substack{m\leq x\\(m,u)=1}}\frac{\gamma(m;n)}{m}=\frac{\alpha u n^3 J(u)x}{\psi(un)H(un)}+O\left(\Theta(un)x^{2/5}\omega(x)\right); \tag{3.23}$$

uniformly.

PROOF. The proof follows by Theorem 3.2 and by partial summation (cf. [4], Theorem 421).

THEOREM 3.7. *For $x\geq 3$ and fixed positive integers n and u,*

$$\sum_{\substack{m\leq x\\(m,u)=1}}\frac{\gamma^{**}(m;n)}{m}=\frac{\alpha\beta u^2 n^4 J(u)x}{\psi(un)H(un)G(un)}+O\left(\Theta(un)x^{1/2}\delta(x)\right); \tag{3.24}$$

uniformly.

PROOF. The proof follows by Theorem 3.3 and by partial summation (cf. [4], Theorem 421)

THEOREM 3.8. *If the Riemann hypothesis is true then for $x \geq 3$ and fixed positive integers n and u,*

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \frac{\gamma^{**}(m;n)}{m} = \frac{\alpha\beta u^2 n^4 J(u)x}{\psi(un) H(un) G(un)} + O(\Theta(un) x^{2/5} \omega(x)); \tag{3.25}$$

uniformly.

PROOF. The proof follows by Theorem 3.2 and by partial summation (cf. [4], Theorem 421)

In conclusion, we would like to mention that an asymptotic formula for the sum

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \frac{\gamma^{**}(m;n)}{m^2}$$

will be established in a separate paper.

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THE MAXIMAL SQUARE-FREE, BI-UNITARY DIVISOR OF m WHICH IS PRIME TO n, II.

By

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§ 1. Introduction. It is well-known that a divisor $d>0$ of a positive integer n is called unitary if $d\delta = n$ and $(d, \delta) = 1$. We write $d\|n$ to say that d is a unitary divisor of n. Analogous to this notion the second named author (cf. [8]) introduced the notion of a bi-unitary divisor. A divisor $d>0$ of the positive integer n is called bi-unitary if $d\delta = n$ and $(d, \delta)^{**} = 1$, where $(d, \delta)^{**}$ denotes the greatest common unitary divisor of both d and δ.

As usual, by a k-free integer, we mean a positive integer which is not divisible by the k-th power of any prime. Recently, asymptotic formulae for the sums

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{q_k(m)\,\varphi(m)}{m^2} \quad \text{and} \quad \sum_{m \leq x} \frac{\gamma_k(m;n)}{m^2}$$

with uniform O-estimates for the error terms have been obtained (cf. [9] § 4.) by the authors, where $\varphi(n)$ is the Euler totient function and $q_k(m) = 1$ or 0 according as m is k-free or not k-free and $\gamma_k(m; n)$ denotes the maximal k-free divisor of m which is prime to n. In particular, we note that $\gamma_2(m; n) = \gamma(m; n)$, the maximal square-free divisor of m which is prime to n. It is clear that $\gamma(m; 1) = \gamma(m)$, the maximal square-free divisor of m and $\gamma(m; 2) = \delta(m)$, the maximal odd square-free divisor of m. Let $\gamma^{**}(m; n)$ denote the maximal square-free, bi-unitary divisor of m which is prime to n and let $\gamma^{**}(m)$ and $\delta^{**}(m)$ respectively denote the maximal square-free, bi-unitary divisor of m and the maximal square-free, bi-unitary odd divisor of m.

In this paper, we establish an asymptotic formula for the sum

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\gamma^{**}(m;n)}{m^2}$$

with uniform O-estimates for the error terms (see **§ 3**) and deduce several results as particular cases. Also, we improve the O-estimates of the error terms on the assumption of the Riemann hypothesis.

§ 2. Preliminaries. Let $\mu(n)$ denote the Möbius function and let $\varphi(n)$ denote the Euler totient function, $\psi(n)$ denote the Dedekind ψ-function (cf. [4] p. 123) and $J(n)$ denote the Jordan totient function of order two (cf. [4], p. 147). Let $H(n)$ be the arithmetical function given by

$$H(n) = n^2 \sum_{d|n} \frac{\mu(d)}{\psi(d)\, d} . \tag{2.1}$$

These functions have the following arithmetical forms:

$$\varphi(n) = \sum_{d\delta=n} \mu(d)\, \delta = n \prod_{p|n} \left(1-\frac{1}{p}\right), \tag{2.2}$$

$$\psi(n) = \sum_{d\delta=n} \mu^2(d)\, \delta = n \prod_{p|n} \left(1+\frac{1}{p}\right), \tag{2.3}$$

$$J(n) = \sum_{d\delta=n} \mu(d)\, \delta^2 = n^2 \prod_{p|n} \left(1-\frac{1}{p^2}\right), \tag{2.4}$$

$$H(n) = n^2 \sum_{d\delta=n} \frac{\mu(d)}{\psi(d)\, d} = n^2 \prod_{p|n} \left(1-\frac{1}{p(p+1)}\right). \tag{2.5}$$

Remark 2.1. It is clear that $\varphi(n) \leq n$, $\psi(n) \geq n$ and $\dfrac{1}{J(n)} = O\left(\dfrac{1}{n^2}\right)$, since

$$J(n) = n^2 \prod_{p|n} \left(1-\frac{1}{p^2}\right) > n^2 \prod_{p} \left(1-\frac{1}{p^2}\right) = \frac{n^2}{\zeta(2)}$$

(cf. [5], Theorem 280) where $\zeta(s)$ is the Riemann Zeta function defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad \text{for} \quad s > 1 .$$

Further

$$\frac{1}{H(n)} = O\left(\frac{1}{n^2}\right),$$

since

$$H(n) > n^2 \prod_{p} \left(1-\frac{1}{p(p+1)}\right) > n^2 \prod_{p} \left(1-\frac{1}{p^2}\right).$$

Let $\zeta'(s)$ denote the derivative of $\zeta(s)$ with respect to s and $\Theta(n)$ denote the number of square-free divisors of n. Let $G(n)$ be the function defined by

$$G(n) = n \prod_{p|n} \left(1-\frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right) \quad \text{for} \quad n > 1 \tag{2.6}$$

and $G(1) = 1$.

Let α and β be the constants given by

$$\alpha = \prod_p \left(1 - \frac{1}{p(p+1)}\right), \tag{2.7}$$

$$\beta = \prod_p \left(1 - \frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right). \tag{2.8}$$

Let $I(n) = 1$ or 0 according as $n = 1$ or $n > 1$ and note that $I(n)$ is multiplicative.

Throughout the discussion x denotes a real variable ≥ 3, ε denotes a pre-assigned positive real number, u and n are fixed positive integers. All the O-estimates that appear in this paper are independent of x, u and n but may depend on ε. We describe this situation by mentioning the word "uniformly" at the end of each formula.

We need the following Lemmas to prove our main Theorems.

LEMMA 2.1. (cf. [7], Lemma 2.3)

$$\gamma(m; n) = \sum_{d\delta = m} \mu^2(d)\, \varphi(d)\, I((d, n)). \tag{2.9}$$

LEMMA 2.2. (cf. [7], Lemma 2.6)

$$\gamma^{**}(m; n) = \sum_{\substack{d^2\delta = m \\ (d,\delta)=1}} \mu(d)\, \varphi(d)\, I((d, n))\, \gamma(\delta; n). \tag{2.10}$$

LEMMA 2.3. (cf. [2], Lemma 3.1, $s = 2$). For $x \geq 3$ and $u \geq 1$,

$$\sum_{\substack{m \leq x \\ (m,u)=1}} \frac{1}{m^2} = \frac{\zeta(2)\, J(u)}{u^2} + O\left(\frac{1}{x}\right). \tag{2.11}$$

uniformly.

LEMMA 2.4.

$$\beta(n) \equiv -\frac{n^2}{J(n)} \sum_{d|n} \frac{\mu(d) \log d}{d^2} = \begin{cases} \sum_{p|n} \frac{\log p}{p^2-1}, & \text{if} \quad n > 1, \\ 0, & \text{if} \quad n = 1. \end{cases} \tag{2.12}$$

$$A(n) \equiv -\frac{\psi(n)}{n} \sum_{d|n} \frac{\mu(d) \log d}{\psi(d)} = \begin{cases} \sum_{p|n} \frac{\log p}{p}, & \text{if} \quad n > 1, \\ 0, & \text{if} \quad n = 1. \end{cases} \tag{2.13}$$

$$B(n) \equiv -\frac{\psi(n)}{n} \sum_{d|n} \frac{\mu(d)\, \beta(d)}{\psi(d)} = \begin{cases} \sum_{p|n} \frac{\log p}{p(p^2-1)}, & \text{if} \quad n > 1, \\ 0, & \text{if} \quad n = 1. \end{cases} \tag{2.14}$$

$$(2.15)\quad C(n) \equiv -\frac{n^2}{H(n)} \sum_{d|n} \frac{\mu(d)\log d}{\psi(d)\,d} = \begin{cases} \sum_{p|n} \dfrac{\log p}{p^2+p-1}, & \text{if } n>1, \\ 0, & \text{if } n=1. \end{cases}$$

$$(2.16)\quad D(n) \equiv -\frac{n^2}{H(n)} \sum_{d|n} \frac{\mu(d)\beta(d)}{\psi(d)\,d} = \begin{cases} \sum_{p|n} \dfrac{\log p}{(p^2-1)(p^2+p-1)}, & \text{if } n>1, \\ 0, & \text{if } n=1. \end{cases}$$

REMARK 2.2. Proofs of equalities (2.12) to (2.16) can be given in the same way as the proof of the following:

$$(2.17)\quad \alpha(n) \equiv -\frac{n}{\varphi(n)} \sum_{d|n} \frac{\mu(d)\log d}{d} = \begin{cases} \sum_{p|n} \dfrac{\log p}{(p-1)}, & \text{if } n>1, \\ 0, & \text{if } n=1 \end{cases}$$

which was given originally by DAVENPORT (cf. [3], pp. 293–294). For alternative proofs of (2.17) we refer to LANDAU (cf. [6], p–245) and BERGMANN [1]. In fact, DAVENPORT used the notation $\omega(n)$ for $\alpha(n)$ defined above.

Set

$$(2.18)\quad F(n) = \sum_{p|n} \frac{(p-1)\log p}{(p^2+p-1)}.$$

Then we observe that

$$(2.19)\quad A(n)+B(n)-2C(n)-D(n) = F(n).$$

REMARK 2.3. It is clear that $F(n)=O(\Theta(n))$ and $\beta(n)=O(\Theta(n))$, since

$$F(n) = \sum_{p|n} \frac{(p-1)\log p}{(p^2+p-1)} < \sum_{p|n} \frac{\log p}{(p-1)} \le \sum_{p|n} 1 \le \Theta(n)$$

and

$$\beta(n) = \sum_{p|n} \frac{\log p}{(p^2-1)} < \sum_{p|n} \frac{\log p}{(p-1)} \le \Theta(n).$$

LEMMA 2.5. (cf. [9], Corollary 4.1.3). For $x \ge 3$ and $n \ge 1$,

$$(2.20)\quad \sum_{\substack{m \le x \\ (m,n)=1}} \frac{\mu^2(m)\varphi(m)}{m^2} =$$

$$= \frac{\alpha n^3}{\zeta(2)\psi(n)H(n)} \left[\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + F(n) + \sum_{p} \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] + \\ + O(\Theta(n)\, x^{-1/2}\, \delta(x));$$

uniformly, where $F(n)$ is given by (2.18) and $\delta(x)$ is defined by

$$(2.21)\qquad \delta(x)=\begin{cases}\exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}, & \text{for}\quad x\geq 3,\\ 1 \quad\text{for}\quad 0<x<3, & \end{cases}$$

A being a positive absolute constant.

LEMMA 2.6. (cf. [9], Corollary 4.2.3). If the Riemann hypothesis is true then for $x\geq 3$ and $n\geq 1$,

$$(2.22)\qquad \sum_{\substack{m\leq x\\(m,n)=1}}\frac{\mu^2(m)\varphi(m)}{m^2}=$$

$$=\frac{\alpha n^3}{\zeta(2)\,\psi(n)\,H(n)}\left[\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+F(n)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]+$$
$$+O\left(\Theta(n)\,x^{-3/5}\,\omega(x)\right);$$

uniformly, where $\omega(x)$ is the function given by

$$(2.23)\qquad \omega(x)=\begin{cases}\exp\{A\log x(\log\log x)^{-1}\}. & \text{for}\quad x\geq 3,\\ 1\quad\text{for}\quad 0<x<3. & \end{cases}$$

LEMMA 2.7. For $s>1$ and $n\geq 1$

$$(2.24)\qquad \sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu(m)\varphi^2(m)}{m^s H(m)}=\prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^{s+1}(p^2+p-1)}\right).$$

PROOF. By Remark 2.1, we have

$$\left|\frac{\mu(m)\varphi^2(m)\,l((m,n))}{m^s H(m)}\right|=O\left(\frac{1}{m^s}\right)$$

and so the above series is absolutely convergent for $s>1$. Further, the general term of the series is a multiplicative function of m and hence the series can be expanded into an infinite product of Euler type (cf. [5], Theorem 286). Thus we have,

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu(m)\varphi^2(m)}{m^s H(m)}=\sum_{m=1}^{\infty}\frac{\mu(m)\varphi^2(m)\,l((m;n))}{m^s H(m)}=$$

$$=\prod_p\left(\sum_{i=0}^{\infty}\frac{\mu(p^i)\varphi^2(p^i)\,l((p^i,n))}{p^{is}H(p^i)}\right)=\prod_p\left(1-\frac{(p-1)^2\,l((p,n))}{p^s\cdot p^2\left(1-\dfrac{1}{p(p+1)}\right)}\right)=$$

$$=\prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p-1)^2(p+1)}{p^{s+1}(p^2+p-1)}\right).$$

Hence Lemma 2.7 follows.

LEMMA 2.8. For $s>1$ and $n\geq 1$

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty} \frac{\mu(m)\varphi^2(m)\log m}{m^s H(m)} = \tag{2.26}$$

$$= -\prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^{s+1}(p^2+p-1)}\right)\sum_{\substack{p\\p\nmid n}}\frac{(p^2-1)(p-1)\log p}{p^{s+1}(p^2+p-1)-(p^2-1)(p-1)}.$$

PROOF. This series is uniformly convergent for $s\geq 1+\varepsilon>1$ and so by termwise differentiation of the series in (2.24) with respect to s we get the series in (2.25) with a minus sign before the sum. For finding the derivative of the right hand side expression of (2.24) with respect to s, we write

$$f(s) = \prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^{s+1}(p^2+p-1)}\right).$$

Then

$$\log f(s) = \sum_{\substack{p\\p\nmid n}}\log\left(1-\frac{(p^2-1)(p-1)}{p^{s+1}(p^2+p-1)}\right)$$

so that

$$\frac{f'(s)}{f(s)} = \sum_{\substack{p\\p\nmid n}}\frac{(p^2-1)(p-1)\log p}{p^{s+1}(p^2+p-1)-(p^2-1)(p-1)}$$

and this gives

$$f'(s) = \prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^{s+1}(p^2+p-1)}\right)\sum_{\substack{p\\p\nmid n}}\frac{(p^2-1)(p-1)\log p}{p^{s+1}(p^2+p-1)-(p^2-1)(p-1)}.$$

Hence Lemma 2.8 follows.

From (2.24) and (2.25) for $s=3$, we have the following

$$\sum_{\substack{m\leqq 1\\(m,n)=1}}^{\infty}\frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = \prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right)\equiv\frac{\beta\cdot n}{G(n)}, \tag{2.26}$$

$$\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu(m)\varphi^2(m)\log m}{m^3 H(m)} = \tag{2.27}$$

$$= -\prod_{\substack{p\\p\nmid n}}\left(1-\frac{(p^2-1)(p-1)}{p^4(p^2+p-1)}\right)\sum_{\substack{p\\p\nmid}}\frac{(p^2-1)(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}\equiv$$

$$\equiv -\frac{\beta n}{G(n)}\sum_{\substack{p\\p\nmid n}}\frac{(p^2-1)(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}$$

where β and $G(n)$ are given by (2.8) and (2.6) respectively.

LEMMA 2.9. For $x \geq 3$ and $n \geq 1$,

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = \frac{\beta n}{G(n)} + O\left(\frac{1}{x^2}\right), \tag{2.28}$$

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)\log m}{m^3 H(m)} = \tag{2.29}$$

$$= -\frac{\beta n}{G(n)} \sum_{\substack{p \\ p \nmid n}} \frac{(p^2-1)(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} + O\left(\frac{\log x}{x^2}\right).$$

PROOF. By Remark 2.1, we have

$$\sum_{\substack{m > x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} = O\left(\sum_{\substack{m > x \\ (m,n)=1}} \frac{1}{m^3}\right) = O\left(\sum_{m > x} \frac{1}{m^3}\right) = O\left(\frac{1}{x^2}\right),$$

$$\sum_{\substack{m > x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)\log m}{m^3 H(m)} = O\left(\sum_{\substack{m > x \\ (m,n)=1}} \frac{\log m}{m^3}\right) =$$

$$= O\left(\sum_{m > x} \frac{\log m}{m^3}\right) = O\left(\frac{\log x}{x^2}\right).$$

Hence (2.28) and (2.29) follow by (2.26) and (2.27) and the above two O-estimates.

LEMMA 2.10. For $x \geq 3$ and $n \geq 1$,

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)F(m)}{m^3 H(m)} = \tag{2.30}$$

$$= -\frac{\beta n}{G(n)} \sum_{\substack{p \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} + O\left(\frac{\log x}{x^2}\right).$$

PROOF. By (2.18) and (2.28), we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)F(m)}{m^3 H(m)} = \tag{2.31}$$

$$= \sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu(m)\varphi^2(m)}{m^3 H(m)} \sum_{p\delta = m} \frac{(p-1)\log p}{(p^2+p-1)} =$$

$$= -\sum_{\substack{p\delta \leq x \\ (p,\delta)=1}} \frac{\mu(\delta)(p-1)^2\varphi^2(\delta)(p-1)\log p}{p^3\delta^3 H(\delta)p^2\left(1-\frac{1}{p(p+1)}\right)(p^2+p-1)} =$$

$$= - \sum_{\substack{p\delta \leq x \\ (p\delta, n)=1 \\ (p,\delta)=1}} \frac{\mu(\delta)(p-1)^2(p^2-1)\varphi^2(\delta)\log p}{p^4(p^2+p-1)^2\delta^3 H(\delta)} =$$

$$= - \sum_{\substack{p \leq x \\ (p,n)=1}} \frac{(p-1)^2(p^2-1)\log p}{p^4(p^2+p-1)^2} \sum_{\substack{\delta \leq x/p \\ (\delta, pn)=1}} \frac{\mu(\delta)\varphi^2(\delta)}{\delta^3 H(\delta)} =$$

$$= - \sum_{\substack{p \leq n \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{p^4(p^2+p-1)^2} \cdot$$

$$\cdot \left\{ \frac{\beta\, n\, p^4(p^2+p-1)}{G(n)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} + O\left(\frac{p^2}{x^2}\right) \right\} =$$

$$= -\frac{\beta n}{G(n)} \sum_{\substack{p \leq x \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{(p^2+p-1)p^4\{(p^2+p-1)-(p^2-1)(p-1)\}} +$$

$$+ O\left(\frac{1}{x^2} \sum_{\substack{p \leq x \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{p^2(p^2+p-1)^2} \right) =$$

$$= -\frac{\beta n}{G(n)} \sum_{\substack{p \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} +$$

$$+ \frac{\beta n}{G(n)} \sum_{\substack{p > x \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} +$$

$$+ O\left(\frac{1}{x^2} \sum_{\substack{p \leq x \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{p^2(p^2+p-1)^2} \right).$$

By (2.6) and (2.8) we have $\dfrac{\beta n}{G(n)} < 1$ and note that

$$(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\} > p^7,$$

so that the second term in (2.31) is

$$O\left(\sum_{\substack{p > x \\ p \nmid n}} \frac{(p-1)^2(p^2-1)\log p}{p^7} \right) = O\left(\sum_{\substack{p > x \\ p \nmid n}} \frac{\log p}{p^3} \right) =$$

$$= O\left(\sum_{p > x} \frac{\log p}{p^3} \right) = O\left(\sum_{m > x} \frac{\log m}{m^3} \right) = O\left(\frac{\log x}{x^2} \right)$$

and further note that $(p-1)^2(p^2-1)<p(p^2+p-1)^2$, so that the O-term in (2.31) becomes

$$O\left(\frac{1}{x^2}\sum_{\substack{p\le x\\ p\nmid n}}\frac{\log p}{p}\right)=O\left(\frac{1}{x^2}\sum_{p\le x}\frac{\log p}{p}\right)=O\left(\frac{\log x}{x^2}\right),$$

since

$$\sum_{p\le x}\frac{\log p}{p}=O(\log x)$$

(cf. [5], Theorem 425). Hence Lemma 2.10 follows.

LEMMA 2.11. For $x\ge 3$ and $n\ge 1$,

$$\sum_{\substack{m\le x\\ (m,n)=1}}\frac{\mu(m)\varphi^2(m)\beta(m)}{m^3H(m)}= \tag{2.33}$$

$$=-\frac{\beta n}{G(n)}\sum_{\substack{p\\ p\nmid n}}\frac{(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}+O\left(\frac{\log x}{x^2}\right).$$

PROOF. By (2.12) and (2.28), we have

$$\sum_{\substack{m\le x\\ (m,n)=1}}\frac{\mu(m)\varphi^2(m)\beta(m)}{m^3H(m)}=\sum_{\substack{m\le x\\ (m,n)=1}}\frac{\mu(m)\varphi^2(m)}{m^3H(m)}\sum_{p\delta=m}\frac{\log p}{p^2-1}= \tag{2.33}$$

$$=-\sum_{\substack{p\delta\le x\\ (p\delta,n)=1\\ (p,\delta)=1}}\frac{\mu(\delta)\varphi^2(\delta)(p-1)^2\log p}{p^3\delta^3p^2\left(1-\dfrac{1}{p(p+1)}\right)H(\delta)(p^2-1)}=$$

$$=-\sum_{\substack{p\delta\le x\\ (p\delta,n)=1\\ (p,\delta)=1}}\frac{\mu(\delta)\varphi^2(\delta)(p-1)\log p}{\delta^3p^4(p^2+p-1)H(\delta)}=$$

$$=-\sum_{\substack{p\le x\\ p\nmid n}}\frac{(p-1)\log p}{p^4(p^2+p-1)}\sum_{\substack{\delta\le x/p\\ (\delta,pn)=1}}\frac{\mu(\delta)\varphi^2(\delta)}{\delta^3H(\delta)}=$$

$$=-\sum_{\substack{p\le x\\ p\nmid n}}\frac{(p-1)\log p}{p^4(p^2+p-1)}\left\{\frac{\beta n\,p^4(p^2+p-1)}{G(n)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}+O\left(\frac{p^2}{x^2}\right)\right\}=$$

$$=-\frac{\beta n}{G(n)}\sum_{\substack{p\le x\\ p\nmid n}}\frac{(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}+$$

$$+O\left(\frac{1}{x^2}\sum_{\substack{p\le x\\ p\nmid n}}\frac{(p-1)\log p}{p^2(p^2+p-1)}\right)=$$

$$= -\frac{\beta n}{G(n)} \sum_{\substack{p \\ p \nmid n}} \frac{(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} +$$

$$+\frac{\beta n}{G(n)} \sum_{\substack{p > x \\ p \nmid n}} \frac{(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} +$$

$$+O\left(\frac{1}{x^2}\sum_{\substack{p \leqq x \\ p \nmid n}} \frac{(p-1)\log p}{p^2(p^2+p-1)}\right)$$

we have $\frac{\beta n}{G(n)} < 1$ and $p^4(p^2+p-1)-(p^2-1)(p-1) > p^3(p-1)$, so that the second term in (2.33) is

$$O\left(\sum_{\substack{p > x \\ p \nmid n}} \frac{\log p}{p^3}\right) = O\left(\sum_{p > x} \frac{\log p}{p^3}\right) = O\left(\sum_{m > x} \frac{\log m}{m^3}\right) = O\left(\frac{\log x}{x^2}\right)$$

and further $p^2(p^2+p-1) > p-1$, so that the O-term in (2.33) becomes

$$O\left(\frac{1}{x^2}\sum_{\substack{p \leqq x \\ p \nmid n}} \frac{\log p}{p}\right) = O\left(\frac{1}{x^2}\sum_{p \leqq x} \frac{\log p}{p}\right) = O\left(\frac{\log x}{x^2}\right),$$

since

$$\sum_{p \leqq x} \frac{\log p}{p} = O(\log x)$$

(cf. [5], Theorem 425). Hence Lemma 2.11 follows.

LEMMA 2.12. (cf. [10], Lemma 2.2, $s = 2$). For $x \geqq 3$ and $u \geqq 1$,

$$\sum_{\substack{m \leqq x \\ (m,u)=1}} \frac{\log m}{m^2} = -\frac{\zeta(2)J(u)}{u^2}\left(\beta(u)+\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log x}{x}\right); \tag{2.34}$$

uniformly.

§ 3. Main Results. First we prove the following

THEOREM 3.1. *For $x \geqq 3$ and for fixed positive integers u and n.*

$$\sum_{\substack{m \leqq x \\ (m,u)=1}} \frac{\gamma(m;n)}{m^2} = \tag{3.1}$$

$$= \frac{\alpha u n^3 J(u)}{\psi(un)H(un)}\left[\log x+\gamma+\beta(u)+F(un)+\sum_{p}\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]+$$

$$+O\left(\Theta(un)x^{-1/2}\delta(x)\right);$$

uniformly.

PROOF. By Lemmas 2.1 and 2.5, we have

$$\sum_{\substack{m\le x\\(m,u)=1}}\frac{\gamma(m;n)}{m^2}=\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}\sum_{\substack{d\,\delta=m\\(d,n)=1}}\mu^2(d)\,\varphi^2(d)=$$

$$=\sum_{\substack{d\,\delta\le x\\(d\,\delta,\,u)=(d,n)=1}}\frac{\mu^2(d)\,\varphi(d)}{d^2\,\delta^2}=\sum_{\substack{\delta\le x\\(\delta,u)=1}}\frac{1}{\delta^2}\sum_{\substack{d\le x/\delta\\(d,un)=1}}\frac{\mu^2(d)\,\varphi(d)}{d^2}=$$

$$=\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}\left\{\frac{\alpha\,u^3\,n^3}{\zeta(2)\,\psi(un)\,H(un)}\left[\log x-\log m+\gamma-\frac{\zeta'(2)}{\zeta(2)}+F(un)+\right.\right.$$

$$\left.\left.+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]+O\left(\Theta(un)\,(x/m)^{-1/2}\,\delta(x/m)\right)\right\}=$$

$$=\frac{\alpha\,u^3\,n^3}{\zeta(2)\,\psi(un)\,H(un)}\left\{\left[\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+F(un)+\right.\right.$$

$$\left.\left.+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}-\sum_{\substack{m\le x\\(m,u)=1}}\frac{\log m}{m^2}\right\}+$$

$$+O\left(\Theta(un)\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-1/2}\delta\left(\frac{x}{m}\right)\right).$$

Now, applying Lemmas 2.3 and 2.12, we obtain

$$\sum_{\substack{m\le x\\(m,u)=1}}\frac{\gamma(m;n)}{m^2}=\frac{\alpha\,u^3\,n^3}{\zeta(2)\,\psi(un)\,H(un)}\cdot\tag{3.2}$$

$$\cdot\left\{\left[\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+F(un)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]\cdot\right.$$

$$\left.\cdot\left[\frac{\zeta(2)\,J(u)}{u^2}+O\left(\frac{1}{x}\right)\right]-\left[-\frac{\zeta(2)\,J(u)}{u^2}\left(\beta(u)+\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log x}{x}\right)\right]\right\}+$$

$$+O\left(\Theta(un)\sum_{m\le x}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-1/2}\delta\left(\frac{x}{m}\right)\right).$$

By (2.21) it is clear that $x^\varepsilon\,\delta(x)$ is monotonic increasing for every $\varepsilon>0$, and so for $0<\varepsilon<1/2$, the last O-term in (3.2) is

$$O\left(\Theta(un)\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}\left(\frac{x}{m}\right)^{(-1/2)-\varepsilon}\left(\frac{x}{m}\right)^{\varepsilon}\delta\left(\frac{x}{m}\right)\right)=$$

$$=\left(\Theta(un)\,x^\varepsilon\,\delta(x)\,x^{(-1/2)-\varepsilon}\sum_{m\le x}m^{(-3/2)+\varepsilon}\right)=O\left(\Theta(un)\,x^{-1/2}\,\delta(x)\right).$$

Further, by Remarks 2.1 and 2.3, it is clear that each of the first and second O-terms that arise in (3.2) are each

$$= O\left(\frac{\Theta(un)\log x}{x}\right) = O\left(\Theta(un)\, x^{-1/2}\, \delta(x)\right).$$

Now Theorem 3.1 follows from the above discussion and (3.2).

COROLLARY 3.1.1. ($u = 1$). For $x \geq 3$ and $n \geq 1$,

$$\sum_{m\leq x}' \frac{\gamma(m;n)}{m^2} = \tag{3.3}$$

$$= \frac{\alpha\, n^3}{\psi(n)\, H(n)}\left[\log x + \gamma + F(n) + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] +$$
$$+ O\left(\Theta(n)\, x^{-1/2}\, \delta(x)\right);$$

uniformly.

COROLLARY 3.1.2. ($n = 1$). For $x \geq 3$ and $u \geq 1$,

$$\sum_{\substack{m\leq x\\(m,u)=1}}' \frac{\gamma(m)}{m^2} = \tag{3.4}$$

$$= \frac{\alpha(u)\, J(u)}{\psi(u)\, H(u)}\left[\log x + \gamma + \beta(u) + F(u) + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] +$$
$$+ O\left(\Theta(u)\, x^{-1/2}\, \delta(x)\right);$$

uniformly.

COROLLARY 3.1.3. ($n = 2$). For $x \geq 3$ and $u \geq 1$,

$$\sum_{\substack{m\leq x\\(m,u)=1}}' \frac{\delta(m)}{m^2} = \tag{3.5}$$

$$= \frac{8\alpha\, u\, J(u)}{\psi(2u)\, H(2u)}\left[\log x + \gamma + \beta(u) + F(2u) + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] +$$
$$+ O\left(\Theta(2u)\, x^{-1/2}\, \delta(x)\right);$$

uniformly.

REMARK 3.1. Corollary 3.1.1. has already been established by the authors (cf. [9], Corollary 4.3.3). Formulas (3.4) and (3.5) in the case $u = 1$ reduce to those results already established by the authors (cf. [9], Corollaries 4.3.4 and 4.3.5).

THEOREM 3.2. *If the Riemann hypothesis is true then for $x \geq 3$ and for any fixed positive integers u and n, the sum of the error terms in* (3.1) *can be replaced by* $O\left(\Theta(un)\, x^{-3/5}\, \omega(x)\right)$ *where* $\omega(x)$ *is given by* (2.23).

PROOF. Following the same procedure adopted in the proof of Theorem 3.1 and making use of Lemma 2.6 instead of Lemma 2.5, we get (3.2) with

$$O\left(\Theta(un)\sum_{\substack{m\leq x\\(m,u)=1}}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-1/2}\delta\left(\frac{x}{m}\right)\right)$$

replaced by

$$O\left(\Theta(un)\sum_{\substack{m\leq x\\(m,u)=1}}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-3/5}\omega\left(\frac{x}{m}\right)\right).$$

since $\omega(x)$ is monotonic increasing this O-term is

$$O\left(\Theta(un)\,x^{-3/5}\,\omega(x)\sum_{\substack{m\leq x\\(m,u)=1}}m^{-7/5}\right)=O\left(\Theta(un)\,x^{-3/5}\,\omega(x)\right).$$

The rest of the argument is the same as in the proof of Theorem 3.1.

COROLLARY 3.2.1. ($u=1$). If the Riemann hypothesis is true then for $x\geq 3$ and $n\geq 1$, the error terms in (3.3) can be replaced by

$$O\left(\Theta(n)\,x^{-3/5}\,\omega(x)\right).$$

COROLLARY 3.2.2. ($n=1$). If the Riemann hypothesis is true then for $x\geq 3$ and $u\geq 1$, the error terms in (3.4) can be replaced by

$$O\left(\Theta(u)\,x^{-3/5}\,\omega(x)\right).$$

COROLLARY 3.2.3. ($n=2$). If the Riemann hypothesis is true then for $x\geq 3$ and $u\geq 1$, the error terms in (3.5) can be replaced by

$$O\left(\Theta(2u)\,x^{-3/5}\,\omega(x)\right).$$

REMARK 3.2. Corollary 3.2.1 has already been established by the authors (cf. [9], corollary 4.4.2)

THEOREM 3.3. *For $x\geq 3$ and for fixed positive integers u and n,*

$$\sum_{\substack{m\leq x\\(m,u)=1}}\frac{\gamma^{**}(m;n)}{m^2}=\frac{\alpha\,\beta\,u^2\,n^4\,J(u)}{\psi(un)\,H(un)\,G(un)}\cdot \tag{3.6}$$

$$\cdot\left[\log x+\gamma+\beta(u)+F(un)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}+\right.$$

$$\left.+\sum_{\substack{p\\p\nmid un}}\frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\left(p^4(p^2+p-1)-(p^2-1)(p-1)\right)}\right]+$$

$$+O\left(\Theta(un)\,x^{-1/2}\,\delta(x)\right);$$

uniformly.

PROOF. By Lemma 2.2 and Theorem 3.1, we have

$$\sum_{\substack{m\le x\\(m,u)=1}}\frac{\gamma^{**}(m;n)}{m^2}=\sum_{\substack{m\le x\\(m,u)=1}}\frac{1}{m^2}\sum_{\substack{d^2\delta=m\\(d,\delta)=1}}\mu(d)\,\varphi(d)\,I((d;n))\,\gamma(\delta;n)=$$

$$=\sum_{\substack{d^2\delta\le x\\(d\delta,u)=(d,\delta)=(d,u)=1}}\frac{\mu(d)\,\varphi(d)\,\gamma(\delta;n)}{d^4\,\delta^2}=\sum_{\substack{d\le\sqrt{x}\\(d,du)=1}}\frac{\mu(d)\,\varphi(d)}{d^4}\sum_{\substack{\delta\le x/d^2\\(\delta,du)=1}}\frac{\gamma(\delta;n)}{\delta^2}=$$

$$=\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi(d)}{d^4}\left\{\frac{\alpha\,dun^3\,J(du)}{\psi(dun)\,H(dun)}\cdot\right.$$

$$\cdot\left[\log\frac{x}{d^2}+\gamma+\beta(du)+F(dun)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]+$$

$$\left.+O\left(\Theta(dun)\,(x/d^2)^{-1/2}\,\delta(x/d^2)\right)\right\}=\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi(d)}{d^4}\cdot$$

$$\cdot\left\{\frac{\alpha\,un^3\,J(u)\,J(d)\,d}{\psi(un)\,H(un)\,\psi(d)\,H(d)}\left[\log x-2\log d+\gamma+\beta(d)+\beta(u)+F(d)+\right.\right.$$

$$\left.\left.+F(un)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]+O\left(\Theta(un)\,\Theta(d)\,(x/d^2)^{-1/2}\,\delta(x/d^2)\right)\right\}=$$

$$=\frac{\alpha\,un^3\,J(u)}{\psi(un)\,H(un)}\left\{\left[\log x+\gamma+\beta(u)+F(un)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]\cdot\right.$$

$$\cdot\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi^2(d)}{d^3\,H(d)}-2\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi^2(d)\log d}{d^3\,H(d)}+$$

$$\left.+\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi^2(d)\,\beta(d)}{d^3\,H(d)}+\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\mu(d)\,\varphi^2(d)\,F(d)}{d^3\,H(d)}\right\}+$$

$$+O\left(\Theta(un)\sum_{\substack{d\le\sqrt{x}\\(d,un)=1}}\frac{\Theta(d)}{d^3}\left(\frac{x}{d^2}\right)^{-1/2}\delta\left(\frac{x}{d^2}\right)\right).$$

Now, by Lemma 2.9, 2.11 and 2.10, and by Remarks (2.1, 2.3) we have

$$=\frac{\alpha\,un^3\,J(u)}{\psi(un)\,H(un)}\left\{\left[\log x+\gamma+\beta(u)+F(un)+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right]\cdot\right.$$

$$\cdot\left[\frac{\beta\,un}{G(un)}+O\left(\frac{1}{x}\right)\right]-$$

$$-2\left[-\frac{\beta un}{G(un)}\sum_{\substack{p\\ p\nmid un}}\frac{(p^2-1)(p-1)\log p}{p^4(p^2+p-1)-(p^2-1)(p-1)}+O\left(\frac{\log x}{x}\right)\right]+$$

$$+\left[-\frac{\beta un}{G(un)}\sum_{\substack{p\\ p\nmid un}}\frac{(p-1)\log p}{\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}+O\left(\frac{\log x}{x}\right)\right]+$$

$$+\left[-\frac{\beta un}{G(un)}\sum_{\substack{p\\ p\nmid un}}\frac{(p-1)^2(p^2-1)\log p}{(p^2-p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}+\right.$$

$$\left.\left.+O\left(\frac{\log x}{x}\right)\right]\right\}+O\left(\Theta(un)\sum_{\substack{d\leq\sqrt{x}\\ (d,un)=1}}\frac{\Theta(d)}{d^3}\left(\frac{x}{d^2}\right)^{-1/2}\delta\left(\frac{x}{d^2}\right)\right).$$

$$\sum_{\substack{m\leq x\\ (m,u)=1}}\frac{\gamma^{**}(m;n)}{m^2}=\frac{\alpha\beta u^2n^4J(u)}{\varphi(un)H(un)G(un)}\cdot \tag{3.7}$$

$$\cdot\left[\log x+\gamma+\beta(u)+F(un)+\sum_{p}\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}+\right.$$

$$\left.+\sum_{\substack{p\\ p\nmid un}}\frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}\right]+\left(\Theta(un)\frac{\log x}{x}\right)+$$

$$+O\left(\Theta(un)\sum_{\substack{d\leq\sqrt{x}\\ (d,un)=1}}\frac{\Theta(d)}{d^2}\left(\frac{x}{d^2}\right)^{-1/2}\delta\left(\frac{x}{d^2}\right)\right),$$

since $x^{\varepsilon}\delta(x)$ is monotonic increasing for every $\varepsilon>0$, and $\Theta(m)\leq\tau(m)=$ $=O(m^{\varepsilon})$ (cf. [5]. Theorem 315) where $\tau(m)$ is the number of all divisors of m, the sum in the second O-term of (3.7) is

$$\sum_{\substack{m\leq\sqrt{x}\\ (m,un)=1}}\frac{\Theta(m)}{m^3}\left(\frac{x}{m^2}\right)^{-1/2}\delta\left(\frac{x}{m^2}\right)=$$

$$=\sum_{\substack{m\leq\sqrt{x}\\ (m,un)=1}}\frac{\Theta(m)}{m^3}\left(\frac{x}{m^2}\right)^{(-1/2)-\varepsilon}\left(\frac{x}{m^2}\right)^{\varepsilon}\delta\left(\frac{x}{m^2}\right)=$$

$$=O\left(x^{\varepsilon}\delta(x)\sum_{\substack{m\leq\sqrt{x}\\ (m,un)=1}}\frac{\Theta(m)}{m^3}\frac{x^{(-1/2)-\varepsilon}}{m^{(-1/2)-\varepsilon}}\right)=O\left(x^{-1/2}\delta(x)\sum_{\substack{m\leq\sqrt{x}\\ (m,un)=1}}\frac{\Theta(m)}{m^{2-2\varepsilon}}\right)=$$

$$=O\left(x^{-1/2}\delta(x)\frac{1}{m^{2-3\varepsilon}}\right)=O\left(x^{-1/2}\delta(x)\right).$$

Hence Theorem 3.3. follows.

COROLLARY 3.3.1. ($u = 1$). For $x \geqq 3$ and $n \geqq 1$,

$$(3.8)\qquad \sum_{m \leqq x} \frac{\gamma^{**}(m;n)}{m^2} = \frac{\alpha\beta n^2}{\psi(n) H(n) G(n)} \Bigg[\log x + \gamma + F(n) + + \sum_{p} \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)} + + \sum_{\substack{p \\ p \nmid n}} \frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} \Bigg] + + O\left(\Theta(n)\, x^{-1/2}\, \delta(x)\right).$$

uniformly.

COROLLARY 3.3.2, ($n = 1$). For $x \geqq 3$ and $u \geqq 1$.

$$(3.9)\qquad \sum_{\substack{m \leqq x \\ (m,u)=1}} \frac{\gamma^{**}(m)}{m^2} = \frac{\alpha\beta u^2 \varphi(u)}{H(u) G(u)} \Bigg[\log x + \gamma + \beta(u) + F(u) + + \sum_{p} \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)} + + \sum_{\substack{p \\ p \nmid u}} \frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} \Bigg] + + O\left(\Theta(u)\, x^{-1/2}\, \delta(x)\right);$$

uniformly.

COROLLARY 3.3.3. ($n = 2$). For $x \geqq 3$ and $u \geqq 1$:

$$(3.10)\qquad \sum_{\substack{m \leqq x \\ (m,u)=1}} \frac{\delta^{**}(m)}{m^2} = \frac{16\,\alpha\beta u^2 J(u)}{\psi(2u) H(2u) G(2u)} \Bigg[\log x + \gamma + \beta(u) + F(2u) + + \sum_{p} \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)} + + \sum_{\substack{p \\ p \nmid 2u}} \frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} \Bigg] + + O\left(\Theta(2u)\, x^{-1/2}\, \delta(x)\right);$$

uniformly.

COROLLARY 3.3.4. ($u = 1$, $n = 1$). For $x \geqq 3$,

$$(3.11)\ \sum_{m \leqq x} \frac{\gamma^{**}(m)}{m^2} = \alpha\beta \Bigg[\log x + \gamma + \sum_{p} \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)} + + \sum_{p} \frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}} \Bigg] + O\left(x^{-1/2}\, \delta(x)\right);$$

uniformly.

COROLLARY 3.3.5. ($u=1$, $n=2$). For $x\geq 3$,

$$\sum_{m\leq x}\frac{\delta^{**}(m)}{m^2}=\frac{64}{77}\alpha\beta\left[\log x+\gamma+\sum_p\frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}+\right.$$
$$\left.+\sum_{\substack{p\\ p\text{ odd}}}\frac{(p-1)(2p^4+p^3-4p^2-2p+2)\log p}{(p^2+p-1)\{p^4(p^2+p-1)-(p^2-1)(p-1)\}}\right]+$$
$$+O\left(x^{-1/2}\delta(x)\right), \tag{3.12}$$

uniformly.

THEOREM 3.4. *If the Riemann hypothesis is true then for $x\geq 3$ and for fixed positive integers u and n, the sum of the error terms in* (3.6) *can be replaced by*

$$O\left(\Theta(un)\,x^{-3/5}\,\omega(x)\right).$$

PROOF. Following the same procedure adopted in the proof of Theorem 3.3 and making use of Theorem 3.2 instead of Theorem 3.1, we get (3.7) with

$$O\left(\Theta(un)\sum_{\substack{m\leq\sqrt{x}\\(m,un)=1}}\frac{\Theta(m)}{m^3}\left(\frac{x}{m^2}\right)^{-1/2}\delta\left(\frac{x}{m^2}\right)\right)$$

replaced by

$$O\left(\Theta(un)\sum_{\substack{m\leq\sqrt{x}\\(m,un)=1}}\frac{\Theta(m)}{m^3}\left(\frac{x}{m^2}\right)^{-3/5}\omega\left(\frac{x}{m^2}\right)\right).$$

Since $\omega(x)$ is monotonic increasing and $\Theta(m)\leq\tau(m)=O(m^{\varepsilon})$, (cf. [5], Theorem 315), this O-term is

$$O\left(\Theta(un)\,x^{-3/5}\,\omega(x)\sum_{\substack{m\leq\sqrt{x}\\(m,un)=1}}\frac{\Theta(m)}{m^{9/5}}\right)=O\left(\Theta(un)\,x^{-3/5}\,\omega(x)\right).$$

Hence Theorem 3.4 follows.

COROLLARY 3.4.1. ($u=1$). If the Riemann hypothesis is true then for $x\geq 3$ and $n\geq 1$, the error terms in (3.8) can be replaced by

$$O\left(\Theta(n)\,x^{-3/5}\,\omega(x)\right).$$

COROLLARY 3.4.2. ($n=1$). If the Riemann hypothesis is true then for $x\geq 3$ and $u\geq 1$, the error terms in (3.9) can be replaced by

$$O\left(\Theta(u)\,x^{-3/5}\,\omega(x)\right).$$

COROLLARY 3.4.3. ($u = 2$). If the Riemann hypothesis is true then for $x \geq 3$ and $n \geq 1$, the error terms in (3.10) can be replaced by

$$O\left(\Theta(2u)\, x^{-3/5}\, \omega(x)\right).$$

COROLLARY 3.4.4. ($u = 1$; $n = 1$). If the Riemann hypothesis is true then for $x \geq 3$ the error term in (3.11) and that in (3.12) can be replaced by

$$O\left(x^{-3/5}\, \omega(x)\right).$$

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VARIETIES WITH DIRECTLY DECOMPOSABLE DIAGONAL SUBALGEBRAS

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The concept of diagonal subalgebras is frequently used in characterizations of polynomially complete algebras and polynomial interpolation, see [6], [7] and references there. Since they are subalgebras of a direct product, it is important to know under which conditions they are determined by their projections. In other words, it is a generalization of the problem solved in [5] for congruences, if instead of congruences other relations (not necessarily symmetric or transitive) are considered. The first attempt in this direction was done in [3] and for lattices and similar algebras this problem is solved in [2], provided these diagonal subalgebras are symmetric (so called tolerances). One characterization of direct decomposable diagonal subalgebras of a given algebra is contained in [4]; it is based on the notion of nonindexed products of algebras. The objective of this paper is to give a polynomial characterization of varieties having directly decomposable diagonal subalgebras. For congruences, [5] contains such characterization in the form of Mal'cev conditions. For general diagonal subalgebras, these conditions are derived in the form of $\forall\,\exists$-conditions similarly as for lattice identities in [1].

1. General polynomial conditions for direct decomposability

Let $\mathfrak{A}=(A, F)$ be an algebra. The set $\Delta=\{\langle a, a\rangle;\ a\in A\}$ is called a *diagonal* of the direct product $\mathfrak{A}\times\mathfrak{A}$. Each subalgebra $\mathfrak{B}$ of $\mathfrak{A}\times\mathfrak{A}$ containing Δ is called a *diagonal subalgebra*. Clearly Δ and $\mathfrak{A}\times\mathfrak{A}$ are diagonal subalgebras. In other terminology used by the author e.g. in [1], diagonal subalgebras are reflexive binary relations on $\mathfrak{A}$ with the Substitution Property with respect to F.

Denote by $\mathcal{R}(\mathfrak{A})$ the set of all diagonal subalgebras of $\mathfrak{A}\times\mathfrak{A}$. Clearly $\mathcal{R}(\mathfrak{A})$ is a complete lattice with respect to set inclusion, where the lattice meet coincides with set intersection. Denote by $\vee_A$ the join in $\mathcal{R}(\mathfrak{A})$. If $a, b\in A$, denote by $R_A(a, b)$ the least diagonal subalgebra of $\mathfrak{A}\times\mathfrak{A}$ containing the pair $\langle a, b\rangle$. The following lemma is obvious

LEMMA 1. Let $\mathfrak{A} = (A, F)$, $a, b \in A$, $R_\gamma \in \mathcal{R}(\mathfrak{A})$ for $\gamma \in \Gamma$.

(a) $\langle x, y \rangle \in R_A(a, b)$ if and only if there exists a unary algebraic function φ over $\mathfrak{A}$ such that $x = \varphi(a)$, $y = \varphi(b)$.

(b) $\langle x, y \rangle \in \bigvee_A \{R_\gamma;\ \gamma \in \Gamma\}$ if and only if there exists an m-ary polynomial p over $\mathfrak{A}$, indices $\gamma_1, \ldots, \gamma_m \in \Gamma$ and elements $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ A such that $\langle a_i, b_i \rangle \in R_{\gamma_i}$ for $i = 1, \ldots, m$ and $x = p(a_1, \ldots, a_m)$, $y = p(b_1, \ldots, b_m)$.

DEFINITION 1. A variety $\mathcal{V}$ of algebras has *directly decomposable diagonal subalgebras* if for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ and every $R \in \mathcal{R}(\mathfrak{A} \times \mathfrak{B})$ there exist $S \in \mathcal{R}(\mathfrak{A})$, $T \in \mathcal{R}(\mathfrak{B})$ such that $R = S \times T$.

DEFINITION 2. A variety $\mathcal{V}$ of algebras *has the property* (P) if for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ and every $R_1, R_2 \in \mathcal{R}(\mathfrak{A})$, $S_1, S_2 \in \mathcal{R}(\mathfrak{B})$ the following identity is valid

$$(R_1 \vee_A R_2) \times (S_1 \vee_B S_2) = (R_1 \times S_1) \vee_{A \times B} (R_2 \times S_2).$$

REMARK. If instead of diagonal subalgebras only congruences are considered, the foregoing identity is true in every variety, [5]. However, in the case of general diagonal subalgebras such assertion is not true.

THEOREM 1. *For a variety of algebras $\mathcal{V}$, the following conditions are equivalent:*

(1) *$\mathcal{V}$ has the property (P).*

(2) *For every n-ary polynomial p and every m-ary polynomial q and each $0 \leq k \leq n$, $0 \leq h \leq m$ there exist an $(c+d)$-ary polynomial s and polynomials: k-ary t_i, h-ary u_i $(i \leq c)$ $(n-k)$-ary v_j, $(m-h)$-ary w_j $(j \leq d)$ such that*

$$p(x_1, \ldots, x_n) = s(\mathbf{t}_i(x_1, \ldots, x_k),\ \mathbf{v}_j(x_{k+1}, \ldots, x_n)),$$
$$q(x_1, \ldots, x_m) = s(\mathbf{u}_i(x_1, \ldots, x_h),\ \mathbf{w}_j(x_{h+1}, \ldots, x_m)).$$

PROOF. Clearly

$$(R_1 \vee_A R_2) \times (S_1 \vee_B S_2) \supseteq (R_1 \times S_1) \vee_{A \times B} (R_2 \times S_2)$$

for each $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ and every $R_1, R_2 \in \mathcal{R}(\mathfrak{A})$, $S_1, S_2 \in \mathcal{R}(\mathfrak{B})$. Prove the converse inclusion only.

(1)$\Rightarrow$(2): Let p be an n-ary and q an m-ary polynomials over $\mathcal{V}$ and $k \in \{0, \ldots, n\}$, $h \in \{0, \ldots, m\}$. Let

$$\mathfrak{A} = F_{2n}(x_1, \ldots, x_n,\ y_1, \ldots, y_n), \quad \mathfrak{B} = F_{2m}(x_1, \ldots, x_m,\ y_1, \ldots, y_m)$$

be free algebras in $\mathcal{V}$ with generating sets

$$\{x_1, \ldots, x_n,\ y_1, \ldots, y_n\}$$

and

$$\{x_1, \ldots, x_m,\ y_1, \ldots, y_m\},$$

respectively. Put

$$R_1 = \vee_A\{R_A(x_i, y_i);\ i \le k\},\quad R_2 = \vee_A\{R_A(x_i, y_i);\ k < i \le n\},$$
$$S_1 = \vee_B\{R_B(x_j, y_j);\ j \le h\},\quad S_2 = \vee_B\{R_B(x_j, y_j);\ h < j \le m\}.$$

Hence $R_1, R_2 \in \mathcal{R}(\mathfrak{A})$, $S_1, S_2 \in \mathcal{R}(\mathfrak{B})$ and, by Lemma 1,

$$\langle p(\mathbf{x}_i), p(\mathbf{y}_i)\rangle \in \vee_A\{R_A(x_i, y_i);\ i = 1, \dots, n\} = R_1 \vee_A R_2,$$
$$\langle q(\mathbf{x}_j), q(\mathbf{y}_j)\rangle \subset \vee_B\{R_B(x_j, y_j);\ j = 1, \dots, m\} = S_1 \vee_B S_2.$$

By (1), we obtain

$$\langle [p(\mathbf{x}_i), q(\mathbf{x}_j)], [p(\mathbf{y}_i) q(\mathbf{y}_j)]\rangle \in (R_1 \times S_1) \vee_{A\times B} (R_2 \times S_2)$$

and, by Lemma 1, there exists an $(c+d)$-ary polynomial s over $\mathcal{V}$ such that

$$\text{(I)}\quad \begin{aligned} [p(\mathbf{x}_i), q(\mathbf{x}_j)] &= s([a_1, b_1], \dots, [a_{c+d}, b_{c+d}]),\\ [p(\mathbf{y}_i), q(\mathbf{y}_j)] &= s([a_1', b_1'], \dots, [a_{c+d}', b_{c+d}']) \end{aligned}$$

for some

$$\langle [a_i, b_i], [a_i', b_i']\rangle \in R_1 \times S_1 \quad \text{for} \quad i \le c,$$
$$\langle [a_i, b_i], [a_i', b_i']\rangle \in R_2 \times S_2 \quad \text{for} \quad c < i \le c+d.$$

Suppose $i \le c$. Since $R_1 = \vee_A\{R_A(x_i, y_i);\ i = 1, \dots, k\}$ and $S_1 = \vee_B\{R_B(x_j, y_j);\ j = 1, \dots, h\}$, there exist (by Lemma 1) algebraic functions φ_i (k-ary) and ψ_i (h-ary) such that

$$a_i = \varphi_i(x_1, \dots, x_k) \quad \text{and} \quad b_i = \psi_i(x_1, \dots, x_h),$$
$$a_i' = \varphi_i(y_1, \dots, y_k) \quad \text{and} \quad b_i' = \psi_i(y_1, \dots, y_h).$$

Since φ_i, ψ_i are algebraic functions over free algebras $\mathfrak{A}$, $\mathfrak{B}$, there exist polynomaials t_i^*, u_i^* such that

$$\varphi_i(\xi_1, \dots, \xi_k) = t_i^*(\xi_1, \dots, \xi_k, x_1, \dots, x_n, y_1, \dots, y_n),$$
$$\psi_i(\xi_1, \dots, \xi_h) = u_i^*(\xi_1, \dots, \xi_h, x_1, \dots, x_m, y_1, \dots, y_m).$$

Analogously it can be proved the existence of suitable v_i^*, w_i^* (for $c < i \le c+d$) such that (I) implies

$$p(\mathbf{x}_i) = s\big(\mathbf{t}_i^*(x_1, \dots, x_k, \mathbf{x}_i, \mathbf{y}_i),\ \mathbf{v}_j^*(x_{k+1}, \dots, x_n, \mathbf{x}_i, \mathbf{y}_i)\big),$$
$$p(\mathbf{y}_i) = s\big(\mathbf{t}_i^*(y_1, \dots, y_k, \mathbf{x}_i, \mathbf{y}_i),\ \mathbf{v}_j^*(y_{k+1}, \dots, y_n, \mathbf{x}_i, \mathbf{y}_i)\big).$$

Hence it is clear that t_i^* does not depend on the j-th variable for $j > k$, thus

$$t_i^*(\xi_1, \dots, \xi_k, \mathbf{x}_i, \mathbf{y}_i) = t_i(\xi_1, \dots, \xi_k)$$

for some k-ary polynomial t_i over $\mathcal{V}$. Analogously it can be shown for v_j^*. If q is used instead of p, it can be shown also for u_i^* and w_j^* whence (2) is evident.

(2)⇒(1). Suppose $\langle a, b\rangle \in (R_1 \vee_A R_2)\times(S_1 \vee_B S_2)$, where $a=[a_1, a_2]$, $b=[b_1, b_2]$. Since $\langle a_1, a_2\rangle \in R_1 \vee_A R_2$ and $\langle b_1, b_2\rangle \in S_1 \vee_B S_2$, by Lemma 1 there exist an n-ary polynomial p and an m-ary polynomial q and elements $x_i, y_i \in A$ and $x'_j, y'_j \in B$ such that

$$\begin{aligned} a_1 &= p(x_1, \dots, x_n) \\ b_1 &= p(y_1, \dots, y_n), \quad \text{where} \quad \langle x_i, y_i\rangle \in R_1 \quad \text{for} \quad i \le k \\ & \qquad\qquad\qquad\qquad\quad\;\; \langle x_i, y_i\rangle \in R_2 \quad \text{for} \quad k < i \le n \end{aligned}$$

and

$$\begin{aligned} a'_1 &= q(x'_1, \dots, x'_m) \\ b'_1 &= q(y'_1, \dots, y'_m), \quad \text{where} \quad \langle x'_j, y'_j\rangle \in S_1 \quad \text{for} \quad j \le h \\ & \qquad\qquad\qquad\qquad\quad\;\; \langle x'_j, y'_j\rangle \in S_2 \quad \text{for} \quad h < j \le m \end{aligned}$$

for some $k \in \{0, \dots, n\}$, $h \in \{0, \dots, m\}$. Hence

$$\langle [t_i(x_1, \dots, x_k), u_i(x'_1, \dots, x'_h)],$$
$$[t_i(y_1, \dots, y_k), u_i(y'_1, \dots, y'_h)]\rangle \in R_1 \times S_1 .$$
$$\langle [v_j(x_{k+1}, \dots, x_n), w_j(x'_{h+1}, \dots, x'_m)],$$
$$[v_j(y_{k+1}, \dots, y_n), w_j(y'_{h+1}, \dots, y'_m)]\rangle \in R_2 \times S_2$$

and, by (2), also $\langle a, b\rangle \in (R_1\times S_1) \vee_{A\times B} (R_2 \times S_2)$.

Corollary. Let $\mathcal{V}$ be a variety of algberas. The following conditions are equivalent:

(1′) For each $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ and arbitrary $R_\gamma \in \mathcal{R}(\mathfrak{A})$, $S_\gamma \in \mathcal{R}(\mathfrak{B})$, $\gamma \in \Gamma$ (Γ has an arbitrary cardinality), we have

$$\vee_A \{R_\gamma; \gamma \in \Gamma\} \times \vee_B \{S_\gamma; \gamma\in\Gamma\} = \vee_{A\times B}\{(R_\gamma \times S_\gamma); \gamma\in\Gamma\}.$$

(2′) $\mathcal{V}$ fulfils the condition (2) of Theorem 1.

Proof. (1′)⇒(2′) is trivial (by Theorem 1). Prove (2′)⇒(1′). The inclusion $\supseteq$ in (1′) is evident. Prove the converse inclusion. It can be easily done for $\Gamma = \{1, \dots, n\}$ by induction. Suppose

$$\langle a, b\rangle = \langle [a_1, a_2], [b_1, b_2]\rangle \in \vee_A \{R_\gamma; \gamma\in\Gamma\} \times \vee_B \{S_\gamma; \gamma \in \Gamma\}.$$

By Lemma 1, there exist an n-ary polynomial p and m-ary polynomial q with

$$\begin{aligned} a_1 &= p(x_1, \dots, x_n), \quad b_1 = p(y_1, \dots, y_n) \quad \text{for some} \quad \langle x_i, y_i\rangle \in R_{\gamma_i}, \\ a_2 &= q(x'_1, \dots, x'_m), \quad b_2 = q(y'_1, \dots, y'_m) \quad \text{for some} \quad \langle x'_j, y'_j\rangle \in S_{\delta_j} \end{aligned}$$

for $\gamma_i, \delta_j \in \Gamma$. Accordingly

$$\langle a, b\rangle \in \vee_A \{T_{\gamma_i}; i = 1, \dots, n) \times \vee_B \{S_{\delta_j}; j=1, \dots, m\}.$$

However, for finite index sets the inclusion is fulfiled by the assumption, thus $\langle a, b\rangle \in \vee_{A\times B}\{(R_\gamma \times S_\gamma); \gamma\in\Gamma\}$.

LEMMA 2. Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras of the same type satisfying the identity

$$\vee_{A\times B}\{(R_\gamma\times S_\gamma);\ \gamma\in\Gamma\} = \vee_A\{R_\gamma;\ \gamma\in\Gamma\}\times\vee_B\{S_\gamma;\ \gamma\in\Gamma\}$$

for an arbitrary index set Γ and each diagonal subalgebras $R_\gamma\in\mathcal{R}(\mathfrak{A})$, $S_\gamma\in\mathcal{R}(\mathfrak{B})$. The following conditions are equivalent:

(a) If $R\in\mathcal{R}(\mathfrak{A}\times\mathfrak{B})$, there exist $R_1\in\mathcal{R}(\mathfrak{A})$ and $R_2\in\mathcal{R}(\mathfrak{B})$ such that $R=R_1\times R_2$;

(b) $\langle a, b\rangle\in R$ implies $R_A(a_1, b_1)\times R_B(a_2, b_2)\subseteq R$, where $a=[a_1, a_2]$, $b=[b_1, b_2]$.

PROOF. (a)$\Rightarrow$(b) is evident since $R_A(a_1, b_1)\subseteq R_1$, $R_B(a_2, b_2)\subseteq R_2$. Prove (b)$\Rightarrow$(a). Put

$$R_1=\{\langle a_1, b_1\rangle;\ \langle[a_1, x_2], [b_1, y_2]\rangle\in R \quad \text{for some} \quad x_2, y_2\in B\},$$
$$R_2=\{\langle a_2, b_2\rangle;\ \langle[x_1, a_2], [y_1, b_2]\rangle\in R \quad \text{for some} \quad x_1, y_1\in A\}.$$

Clearly $R_1\in\mathcal{R}(\mathfrak{A})$, $R_2\in\mathcal{R}(\mathfrak{B})$ and

$$R_1=\vee_A\{R_A(a_1, b_1);\ \langle a, b\rangle\in R\},$$
$$R_2=\vee_B\{R_B(a_2, b_2);\ \langle a, b\rangle\in R\}.$$

By (b) and the assumption of Lemma 2,

$$R_1\times R_2=\vee_{A\times B}\{R_A(a_1, b_1)\times R_B(a_2, b_2);\ \langle a, b\rangle\in R\}\subseteq R.$$

The converse inclusion is evident.

THEOREM 2. *Let $\mathcal{V}$ be a variety of algebras which has the property (P). The following conditions are equivalent:*

(3) *$\mathcal{V}$ has directly decomposable diagonal subalgebras*

(4) *For any two $(n+1)$-ary polynomials p, q there exist $k\geq 0$ and a $(k+1)$-ary polynomial r and a $(n+2)$-ary polynomials $t_1, \ldots, t_k$, $w_1, \ldots, w_k$ such that*

$$p(x, z_1, \ldots, z_n)=r(x, \mathbf{t}_j(x, y, z_1, \ldots, z_n)),$$
$$p(y, z_1, \ldots, z_n)=r(y, \mathbf{t}_j(x, y, z_1, \ldots, z_n)),$$
$$q(x, z_1, \ldots, z_n)=r(x, \mathbf{w}_j(x, y, z_1, \ldots, z_n)),$$
$$q(y, z_1, \ldots, z_n)=r(y, \mathbf{w}_j(x, y, z_1, \ldots, z_n)).$$

PROOF. (3)$\Rightarrow$(4): Let p and q be $(n+1)$-ary polynomials over $\mathcal{V}$ and $F_{n+2}(x, y, z_1, \ldots, z_n)$ a free algebra in $\mathcal{V}$ with the generatig set $\{x, y, z_1, \ldots, z_n\}$. For the sake of brevity, denote

$$c_1=p(x, z_1, \ldots, z_n),\quad c_2=q(x, z_1, \ldots, z_n),$$
$$d_1=p(y, z_1, \ldots, z_n),\quad d_2=q(y, z_1, \ldots, z_n).$$

By Lemma 1,

$$\langle[c_1, c_2], [d_1, d_2]\rangle\in R_A(x, y)\times R_A(x, y).$$

Since $\mathcal{V}$ has the property (P), the assumption of Lemma 2 is satisfied (by the Corollary) and it implies

$$\langle [c_1, c_2], [d_1, d_2] \rangle \in R_{A\times A}([x, x], [y, y]).$$

By Lemma 1, there exists a unary algebraic function φ over $\mathfrak{A}$ such that

$$[c_1, c_2] = \varphi([x, x]), \quad [d_1, d_2] = \varphi([y, y]),$$

i.e. there exist a $(k+1)$-ary polynomial r and elements $[\alpha_i, \beta_i] \in A\times A$ with

$$[c_1, c_2] = r([x, x], [\alpha_1, \beta_1], \ldots, [\alpha_k, \beta_k]),$$
$$[d_1, d_2] = r([y, y], [\alpha_1, \beta_1], \ldots, [\alpha_k, \beta_k]).$$

Since α_i, β_i are elements of $F_{n+2}(x, y, z_1, \ldots, z_n)$, there exist $(n+2)$-ary polynomials t_i, w_i such that

$$\alpha_i = t_i(x, y, z_1, \ldots, z_n), \quad \beta_i = w_i(x, y, z_1, \ldots, z_n).$$

Whence (4) is evident.

(4)$\Rightarrow$(3): Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$ and $R \in \mathcal{R}(\mathfrak{A}\times\mathfrak{B})$. Suppose $\langle a, b\rangle \in R$, i.e. $R_{A\times B}(a, b) \subseteq R$ for $a = [a_1, a_2]$, $b = [b_1, b_2]$. By Lemma 2, it remains to prove the inclusion

$$R_A(a_1, b_1) \times R_B(a_2, b_2) \subseteq R.$$

Let $\langle x, y\rangle \in R_A(a_1, b_1) \times R_B(a_2, b_2)$. Then $\langle x_1, y_1\rangle \in R_A(a_1, b_1)$, $\langle x_2, y_2\rangle \in R_B(a_2, b_2)$ and, by Lemma 1, there exist an $(n+1)$-ary polynomial p and an $(m+1)$-ary polynomial q such that

$$x_1 = p(a_1, c_1, \ldots, c_n), \quad y_1 = p(b_1, c_1, \ldots, c_n),$$
$$x_2 = q(a_2, d_1, \ldots, d_m), \quad y_2 = q(b_2, d_1, \ldots, d_m)$$

for some c_i of $\mathfrak{A}$ and d_j of $\mathfrak{B}$. Without loss of generality, suppose $m = n$. By (4)

$$x_1 = r(a_1, \mathbf{t}_j(a_1, b_1, c_1, \ldots, c_n)),$$
$$y_1 = r(b_1, \mathbf{t}_j(a_1, b_1, c_1, \ldots, c_n))$$

and

$$x_2 = r(a_2, \mathbf{w}_j(a_2, b_2, d_1, \ldots, d_n)),$$
$$y_2 = r(b_2, \mathbf{w}_j(a_2, b_2, d_1, \ldots, d_n)).$$

By Lemma 1,

$$\langle x, x\rangle = \langle [x_1, x_2], [y_1, y_2]\rangle \in R_{A\times B}([a_1, a_2], [b_1, b_2]) \subseteq R$$

and (3) is proved.

THEOREM 3. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(A) *$\mathcal{V}$ has directly decomposable diagonal subalgebras;*
(B) *$\mathcal{V}$ satisfies the conditions* (2) *and* (4).

PROOF. By Theorems 1 and 2, (B) implies (A). Prove (A)$\Rightarrow$(B). Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$, $R_1, R_2 \in \mathcal{R}(\mathfrak{A})$, $S_1, S_2 \in \mathcal{R}(\mathfrak{B})$. Clearly

$$(R_1\times S_1) \vee_{A\times B} (R_2\times S_2) \subseteq (R_1 \vee_A R_2)\times(S_1 \vee_B S_2).$$

Since $\mathcal{V}$ has directly decomposable diagonal subalgebras, there exist $T_1 \in \mathcal{R}(\mathfrak{A})$ and $T_2 \in \mathcal{R}(\mathfrak{B})$ such that

$$(R_1 \times S_1) \vee_{A\times B} (R_2 \times S_2) = T_1 \times T_2 \subseteq (R_1 \vee_A R_2) \times (S_1 \vee_B S_2),$$

whence

$$T_1 \subseteq R_1 \vee_A R_2, \quad T_2 \subseteq S_1 \vee_B S_2 .$$

However,

$$R_1 = pr_1 (R_1 \times S_1) \subseteq pr_1 [(R_1 \times S_1) \vee_{A\times B} (R_2 \times S_2)],$$
$$R_2 = pr_1 (R_2 \times S_2) \subseteq pr_1 [(R_1 \times S_1) \vee_{A\times B} (R_2 \times S_2)],$$

whence

$$R_1 \vee_A R_2 \subseteq pr_1 [(R_1 \times S_1) \vee_{A\times B} (R_2 \times S_2)] = T_1 ,$$

therefore

$$T_1 = R_1 \vee_A R_2 .$$

Analogously we can prove $T_2 = S_1 \vee_B S_2$, thus $\mathcal{V}$ has the property (P). By Theorem 1 and Theorem 2, also (B) is satisfied.

2. Applications in some varieties of algebras

Theorem 4. *The variety of all lattices has directly decomposable diagonal subalgebras.*

Proof. Let p or q be an n-ary or an m-ary lattice polynomial, respectively, and $0 \le k \le n$, $0 \le h \le m$. Put $c = k + h + 1$, $d = n + m - k - h + 1$. Put

$$s(z_1, \ldots, z_{c+d}) = \big(z_c \wedge p(z_1, \ldots, z_k, z_{c+1}, \ldots, z_{c+n-k})\big) \vee$$
$$\vee \big(z_{c+d} \wedge q(z_{k+1}, \ldots, z_{k+h}, z_{n+h+2}, \ldots, z_{c+d-1})\big),$$

$$t_i(x_1, \ldots, x_k) = x_i, \quad u_i(x_1, \ldots, x_h) = G = \bigvee_{f=1}^{n+m} x_f \quad \text{for} \quad i \le k,$$

$$t_i(x_1, \ldots, x_k) = G, \quad u_i(x_1, \ldots, x_h) = x_{i-k} \quad \text{for} \quad k < i \le k+h,$$

$$t_c(x_1, \ldots, x_k) = G, \quad u_c(x_1, \ldots, x_h) = L = \bigwedge_{f=1}^{n+m} x_f,$$

$$v_j(x_1, \ldots, x_{n-k}) = x_j, \quad w_j(x_1, \ldots, x_{m-h}) = G \quad \text{for} \quad j \le n-k,$$
$$v_j(x_1, \ldots, x_{n-k}) = G, \quad w_j(x_1, \ldots, x_{m-h}) = x_{j-n+k} \quad \text{for} \quad n-k < j < d-1,$$
$$v_d(x_1, \ldots, x_{n-k}) = L, \quad w_d(x_1, \ldots, x_{m-h}) = G .$$

Then

$$s(t_1, \ldots, t_c, v_1, \ldots, v_d) = p(x_1, \ldots, x_n),$$
$$s(u_1, \ldots, u_c, w_1, \ldots, w_d) = q(x_1, \ldots, x_m)$$

thus (2) is satisfied. It remains to prove the condition (4). Let p and q be $(n+1)$-ary lattice polynomials. Clearly

$$p(x, z_1, \ldots, z_n) = \big(G_0 \wedge p(x, z_1, \ldots, z_n)\big) \vee \big(L_0 \wedge q(x, z_1, \ldots, z_n)\big),$$
$$p(y, z_1, \ldots, z_n) = \big(G_0 \wedge p(y, z_1, \ldots, z_n)\big) \vee \big(L_0 \wedge q(y, z_1, \ldots, z_n)\big),$$

$$q(x, z_1, \ldots, z_n) = (L_0 \wedge p(x, z_1, \ldots, z_n)) \vee (G_0 \wedge q(x, z_1, \ldots, z_n)),$$
$$q(y, z_1, \ldots, z_n) = (L_0 \wedge p(y, z_1, \ldots, z_n)) \vee (G_0 \wedge q(y, z_1, \ldots, z_n)),$$

for

$$G_0 = x \vee y \vee \left(\bigvee_{i=1}^{n} z_i\right), \quad L_0 = x \wedge y \wedge \left(\bigwedge_{i=1}^{n} z_i\right),$$

whence r, t_j and v_j are evident. Thus also (4) is satisfied.

THEOREM 5. *Let $\mathcal{V}$ be a variety with two binary and two nullary operations: $+$, $\cdot$, 0, 1, satisfying the identities*

$$x+0 = 0+x = 1\cdot x = x\cdot 1 = x$$
$$0\cdot x = 0 = x\cdot 0.$$

Then $\mathcal{V}$ has directly decomposable diagonal subalgebras.

The proof is similar to that of Theorem 4, see e.g. [4].

3. Directly decomposable tolerances

A *tolerance* T on an algebra $\mathfrak{A}$ is a symmetric diagonal subalgebra of $\mathfrak{A}\times\mathfrak{A}$, i.e. $\langle a, b\rangle\in T$ if and only if $\langle b, a\rangle\in T$, see [1], [2], [3] and numerous references there. Denote by $LT(\mathfrak{A})$ the set of all tolerances on an algebra $\mathfrak{A}$. By Lemma 1 in [1], $LT(\mathfrak{A})$ is an algebraic lattice (with respect to the set inclusion). Denote by $T_A(a, b)$ the least tolerance on $\mathfrak{A}$ containing a pair $\langle a, b\rangle$.

LEMMA 3. Let $\mathfrak{A} = (A, F)$ and $a, b\in A$.

(a) $\langle x, y\rangle\in T_A(a, b)$ if and only if there exists a binary algebraic function φ over $\mathfrak{A}$ such that $x = \varphi(a, b)$, $y = \varphi(b, a)$.

(b) The lattice $LT(\mathfrak{A})$ is a sublattice of $\mathcal{R}(\mathfrak{A})$.

For the proof, see Lemma 1 and Lemma 2 in [1].

Now, we can define direct decomposability of tolerances:

DEFINITION 3. A variety $\mathcal{V}$ of algebras has *directly decomposable tolerances* if for each $\mathfrak{A}$, $\mathfrak{B}\in\mathcal{V}$ and every $T\in LT(\mathfrak{A}\times\mathfrak{B})$ there exist $T_1\in LT(\mathfrak{A})$ and $T_2\in LT(\mathfrak{B})$ such that $T = T_1\times T_2$.

Evidently, if $\mathcal{V}$ has directly decomposable diagonal subalgebras, it has also directly decomposable tolerances. Also the condition (P) can be easily formulated for tolerances and Theorem 1 and Lemma 2 remain valid in this case. If we use Lemma 3 instead of Lemma 1 in the proof of Theorem 2, we obtain the following characterization:

THEOREM 6. *For a variety $\mathcal{V}$, the folloving conditions are equivalent:*

(A_0) *$\mathcal{V}$ has directly decomposable tolerances;*

(B_0) *$\mathcal{V}$ satisfies (2) and for any two $(n+2)$-ary polynomials p, q there exist a $(k+2)$-ary polynomial r and $(n+2)$-ary polynomials $t_1, \ldots, t_k, w_1, \ldots, w_k$ such that*

$$p(x, y, z_1, \ldots, z_n) = r(x, y, \mathbf{t}_j(x, y, z_1, \ldots, z_n)),$$
$$p(y, x, z_1, \ldots, z_n) = r(y, x, \mathbf{t}_j(x, y, z_1, \ldots, z_n)),$$
$$q(x, y, z_1, \ldots, z_n) = r(x, y, \mathbf{w}_j(x, y, z_1, \ldots, z_n)),$$
$$q(y, x, z_1, \ldots, z_n) = r(y, x, \mathbf{w}_j(x, y, z_1, \ldots, z_n)).$$

This result is an essantial generalization of results derived in [2] and [3]. By Theorem 4 and Theorem 5, also the variety of all lattices and each variety with $+$, $\cdot$, 0, 1, satisfying the prescribed identities have directly decomposable tolerances.

REMARK. Although the direct decomposability of diagonal subalgebras implies direct decomposability of tolerances, Theorems 3 and 6 show that the converse statement is not true in a general case of a variety.

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ON THE CONVERGENCE WITH RESPECT TO THE σ-IDEAL

By

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S. HARTMAN and E. MARCZEWSKI in [1] have considered some aspects of convergence in measure. This paper deals with similar kind of problems related with more general type of convergence.

Let $(X, \mathcal{S})$ be a measurable space Let $\mathcal{J}\subset\mathcal{S}$ be a proper σ-ideal of sets such that every family of disjoint sets in $\mathcal{S}-\mathcal{J}$ is at most denumerable (so called countable chain condition, in abbr. C.C.C.). Let (Y, ϱ) be a metric space. For a set $A\subset Y$ we shall denote by Fr A the boundary of A, i.e. $\bar{A}\cap\overline{(Y-A)}$, and by Int A the interior of A. A mapping $f: X\to Y$ is $\mathcal{S}$-measurable if and only if $f^{-1}(B)\in\mathcal{S}$ for every Borel subset B of Y.

We shall say that $\mathcal{J}$-almost every point of $A\subset X$ has some property W if and only if the set of points in A, which have not this property belongs to the σ-ideal $\mathcal{J}$. In that case we shall say also that the property W holds $\mathcal{J}$-almost everywhere (abbr. $\mathcal{J}$-a.e.) on A.

DEFINITION (see [2]). We shall say that a sequence $\{f_n\}$, $n\in\mathbf{N}$ of $\mathcal{S}$-measurable functions transforming X into Y converges with respect to (abbr. wrt) the σ-ideal $\mathcal{J}$ to the $\mathcal{S}$-measurable function f transforming X into Y (abbr. $f_n\to f$ wrt $\mathcal{J}$) if and only if every subsequence $\{f_{m_n}\}$ of $\{f_n\}$ contains a subsequence $\{f_{p_{m_n}}\}$ convergent $\mathcal{J}$-a.e. to f.

DEFINITION (compare [1]). We shall say that a sequence of sets $E_n\in\mathcal{S}$, $n\in\mathbf{N}$ converges with respect to the σ-ideal $\mathcal{J}$ to the set $E\in\mathcal{S}$ (abbr. $E_n\to E$ wrt $\mathcal{J}$) if and only if the sequence $\{\chi_{E_n}\}$, $n\in\mathbf{N}$ of characteristic functions of E_n is convergent with respect to the σ-ideal $\mathcal{J}$ to the characteristic function of E.

It is easy to see that the sequence $\{\chi_{E_n}\}$ of characteristic functions of E_n is convergent $\mathcal{J}$-a.e. if and only if $\limsup_n E_n-\liminf_n E_n\in\mathcal{J}$ and $\chi_{E_n}\to\chi_E$, $n\to\infty$ $\mathcal{J}$-a.e. if and only if $\left(E\setminus\liminf_n E_n\right)\cup\left(\limsup_n E_n\setminus E\right)\in\mathcal{J}$. Obviously

(i) If $E_n\to E$ wrt $\mathcal{J}$ and $F_n\to F$ wrt $\mathcal{J}$, then $(E_n\cup F_n)\to E\cup F$ wrt $\mathcal{J}$, $(E_n\cap F_n)\to E\cup F$ wrt $\mathcal{J}$, $(E_n-F_n)\to E-F$ wrt $\mathcal{J}$.

A class $\mathcal{V}$ of Borel subsets of Y is called a basis of Y, if every open set $G \subset Y$ is the union of a denumerable subclass of $\mathcal{V}$. A basis $\mathcal{V}$ is called an additive basis of Y if union of any two sets of $\mathcal{V}$ belongs to $\mathcal{V}$. Obviously

(ii) If $\mathcal{V}$ is a denumerable basis of Y, then for every $\varepsilon > 0$ the class of all sets $V \in \mathcal{V}$ such that $\delta(V) < \varepsilon$ is again a basis of Y.

(iii) If Y is a separable metric space (i.e. if there is a denumerable basis of Y) then every basis of Y contains a denumerable basis.

Now we shall prove that

(iv) For every $\mathcal{S}$-measurable function $f: X \to Y$ the class $\mathcal{G}$ of all open subsets $G \subset Y$ such that $f^{-1}(\operatorname{Fr} G) \in \mathcal{J}$ is an additive basis of Y.

The relation $\operatorname{Fr} E \cup \operatorname{Fr} F \supset \operatorname{Fr}(E \cup F)$ implies the additivity of $\mathcal{G}$. For every open set $G \subset Y$ the set R of all $r > 0$ such that $f^{-1}(\{y: \varrho(y, Y-G) = r\}) \notin \notin \mathcal{J}$ is denumerable (it follows from C.C.C.), so there exists a sequence of positive numbers $r_n \notin \mathbf{R}$, $n \in \mathbf{N}$ convergent to 0. Put $V_n = \{y: \varrho(y, Y-G) > r_n\}$. The set V_n is open and $f^{-1}(\operatorname{Fr} V_n) \subset f^{-1}(\{y: \varrho(y, Y-G) = r_n\}) \in \mathcal{J}$ for every $n \in \mathbf{N}$. Then $V_n \in \mathcal{G}$ for every n, $G = \bigcup_{n=1}^{\infty} V_n$, so $\mathcal{G}$ is a basis of Y.

A class $\mathcal{V}$ of Borel subsets of Y is called a generalized basis of Y if the smallest field $\mathcal{V}_0$ containing $\mathcal{V}$ is a basis of Y. Consequently

(v) Every basis of Y is a generalized basis of Y.

THEOREM 1. *If $f_n \to f$ wrt $\mathcal{J}$ and E is a Borel subset of Y such that $f^{-1}(\operatorname{Fr} E) \in \mathcal{J}$ then $f_n^{-1}(E) \to f^{-1}(E)$ wrt $\mathcal{J}$.*

PROOF. Let $\{\chi_{f_{m_n}^{-1}(E)}\}$ be an arbitrary subsequence of $\{\chi_{f_n^{-1}(E)}\}$. From the assumption it follows that there exists a subsequence $\{f_{p_{m_n}}\}$ of $\{f_{m_n}\}$ convergent to f except on a set $A \in \mathcal{J}$. We have

$$X = [(f^{-1}(E) - A) \cap f^{-1}(\operatorname{Int} E)] \cup [(f^{-1}(E) - A) \cap f^{-1}(\operatorname{Fr} E)] \cup$$
$$\cup \Big[\big(X - (f^{-1}(E) \cup A)\big) \cap f^{-1}(\operatorname{Int}(Y - E))\Big] \cup$$
$$\cup \Big[\big(X - (f^{-1}(E) \cup A)\big) \cap f^{-1}(\operatorname{Fr} E)\Big] \cup A\,.$$

If $x \in f^{-1}(E) - A$ and $f(x) \in \operatorname{Int} E$ then $f_{p_{m_n}}(x) \to f(x)$, $n \to \infty$, so there exists N_0 such that $f_{p_{m_n}}(x) \in E$ for $n > N_0$. Therefore $x \in f_{p_{m_n}}^{-1}(E)$ for $n \geq N_0$ and consequently $\chi_{f_{p_{m_n}}^{-1}(E)}(x) \to \chi_{f^{-1}(E)}(x)$, $n \to \infty$. Similarly for

$$x \in \big(X - (f^{-1}(E) \cup A)\big) \cap f^{-1}(\operatorname{Int}(Y - E))\,.$$

Since the remaining sets belong to $\mathcal{J}$, so the sequence $\{\chi_{f_{p_{m_n}}^{-1}(E)}\}$ converges to $\chi_{f^{-1}(E)}$ $\mathcal{J}$-a.e. on X. From the arbitrariness of the subsequence $\{\chi_{f_{m_n}^{-1}(E)}\}$ it follows that $f_n^{-1}(E) \to f^{-1}(E)$ wrt $\mathcal{J}$.

THEOREM 2. *Let $\mathcal{V}$ be a denumerable generalized basis of Y. If $f_n^{-1}(V) \to f^{-1}(V)$ wrt $\mathcal{J}$ for every $V \in \mathcal{V}$, then $f_n \to f$ wrt $\mathcal{J}$.*

PROOF. From proposition (i) it follows that the convergence $f_n^{-1}(V) \to f^{-1}(V)$ wrt $\mathcal{J}$ holds for every set V belonging to the smallest field $\mathcal{V}_0$ containing $\mathcal{V}$.

Suppose that the sequence $\{f_n\}$ is not convergent with respect to the σ-ideal $\mathcal{J}$ to the function f. Put $g_n(x) = \varrho(f_n(x), f(x))$ for $x \in X$. Obviously $g_n(x) \geq 0$ for $x \in X$ and the sequence $\{g_n\}$ does not converge to zero with respect to the σ-ideal $\mathcal{J}$. By lemma 4 in [2] there exist a subsequence $\{f_{m_n}\}$ of $\{f_n\}$, a set $A_0 \in \mathcal{S} - \mathcal{J}$ and a natural number k_0 such that for every subsequence $\{f_{p_{m_n}}\}$ of $\{f_{m_n}\}$ we have $\limsup_n \varrho(f_{p_{m_n}}(x), f(x)) \geq \frac{1}{k_0}$ $\mathcal{J}$-a.e. on A_0.

Let ε be a positive number such that $\frac{1}{k_0} > \varepsilon$. By the definition of generalized basis and proposition (ii) there exists a sequence $\{V_n\}$ of sets belonging to $\mathcal{V}_0$ such that $Y = \bigcup_{n=1}^{\infty} V_n$ and $\delta(V_n) < \varepsilon$. Since $A_0 = \bigcup_{n=1}^{\infty} [A_0 \cap f^{-1}(V_n)]$, there exists n_0 such that $A_0 \cap f^{-1}(V_{n_0}) \notin \mathcal{J}$. Put $V = V_{n_0}$, $E = A_0 \cap f^{-1}(V)$ and let $\{f_{p_{m_n}}\}$ be an arbitrary subsequence of $\{f_{m_n}\}$. Then $\limsup_n \varrho(f_{p_{m_n}}(x), f(x)) \geq \frac{1}{k_0} > \varepsilon$ $\mathcal{J}$-a.e. on E. For infinitely many n's $f_{p_{m_n}}(x) \notin V$, hence $\liminf_n \chi_{f^{-1}_{p_{m_n}}(V)}(x) = 0$ $\mathcal{J}$-a.e. on E and $\chi_{f^{-1}(V)}(x) = 1$ for $x \in E$. From the arbitrariness of $\{f_{p_{m_n}}\}$ it follows that the sequence $\{f_{m_n}^{-1}(V)\}$ does not contain a subsequence convergent to $f^{-1}(V)$ $\mathcal{J}$-a.e. on X — a contradiction.

THEOREM 3. *If Y is a separable metric space, and $f_n, f: X \to Y$ are $\mathcal{S}$-measurable functions, then the following conditions are equivalent:*

I. $f_n \to f$ wrt $\mathcal{J}$.
II. *There exists a denumerable generalized basis $\mathcal{V}$ of Y such that $f_n^{-1}(V) \to f^{-1}(V)$ wrt $\mathcal{J}$ for every $V \in \mathcal{V}$.*
III. *There exists a denumerable basis $\mathcal{V}$ of Y such that $f_n^{-1}(V) \to f^{-1}(V)$ wrt $\mathcal{J}$ for every $V \in \mathcal{V}$.*
IV. $f_n^{-1}(B) \to f^{-1}(B)$ wrt $\mathcal{J}$ *for every Borel set $B \subset Y$ such that $f^{-1}(\mathrm{Fr}\, B) \in \mathcal{J}$.*

The proof of the above theorem is essentialy the same as the proof of theorem 3 in [1], p. 129, so we shall omit it.

Denote by $\mathcal{H}$ the class of pairs of sets $(E, F) \in \mathcal{S} \times \mathcal{S}$ *fulfilling the following conditions:*

(1) If $E \in \mathcal{J}$ or $F \in \mathcal{J}$, then $(E, F) \in \mathcal{H}$;
(2) If $(E, F) \in \mathcal{H}$, then $(X - E, X - F) \in \mathcal{H}$;
(3) If $E \cap F \in \mathcal{J}$, then $E \in \mathcal{J}$ or $F \in \mathcal{J}$ for every $(E, F) \in \mathcal{H}$;
(4) For every sequence $\{(E_n, F_n)\}$ if $(E_n, F_n) \in \mathcal{H}$ for each $n \in \mathbf{N}$, $E_n \to E$ wrt $\mathcal{J}$ and $F_n \to F$ wrt $\mathcal{J}$, then $(E, F) \in \mathcal{H}$.

If $(E, F)\in\mathcal{N}$, then we shall say that the sets E and F are independent. Two classes $\mathcal{E}$ and $\mathcal{F}$ contained in $\mathcal{S}$ are called independent, if any two sets $E\in\mathcal{E}$ and $F\in\mathcal{F}$ are independent. Let $\mathcal{B}$ denote the family of the Borel subsets of Y. Two $\mathcal{S}$-measurable functions f and g are called independent if the classes $\mathcal{B}_f=\{f^{-1}(B): B\in\mathcal{B}\}$ and $\mathcal{B}_g=\{g^{-1}(B): B\in\mathcal{B}\}$ are independent.

If $(X, \mathcal{S})$ is a measurable space and $\mathcal{J}\subset\mathcal{S}$ is a proper σ-ideal of subsets of X, then there always exists a family $\mathcal{N}\subset\mathcal{S}\times\mathcal{S}$ fulfilling the conditions (1)–(4). Indeed, put $\mathcal{N}=(\mathcal{J}\times\mathcal{S})\cup(\mathcal{S}\times\mathcal{J})\cup(\mathcal{J}'\times\mathcal{S})\cup(\mathcal{S}\times\mathcal{J}')$ where $\mathcal{J}'=\{A\in\mathcal{S}: X-A\in\mathcal{J}\}$. This is, in fact, the smallest family having all required properties.

Observe that in the case of probability space the family of all pairs of independent sets fulfills (1)–(4).

Let Q and $\mathcal{K}$ be two subclasses of $\mathcal{S}$.

Definition. We shall say that Q is $\mathcal{J}$-dense on $\mathcal{K}$ if and only if for every set $E\in\mathcal{K}$ there exists a sequence of sets $\{E_n\}$, $E_n\in Q$ for $n\in\mathbf{N}$, convergent with respect to the σ-ideal $\mathcal{J}$ to E.

Lemma 1. Let the family of all $\mathcal{S}$-measurable functions endowed with the convergence with respect to the σ-ideal $\mathcal{J}$ be a topological space (see [2], [3]). If $f: X\to Y$ is an $\mathcal{S}$-measurable function and $\mathcal{V}$ is an additive basis of Y, then the class $\mathcal{V}_f=\{f^{-1}(V): V\in\mathcal{V}\}$ is $\mathcal{J}$-dense on $\mathcal{B}_f$.

Proof. It is well-know that $\mathcal{B}=\bigcup_{\alpha<\Omega}\mathcal{G}_\alpha$, where $\mathcal{G}_0$ is a class of open sets in Y, $\mathcal{G}_\alpha=\left(\bigcup_{\gamma<\alpha}\mathcal{G}_\gamma\right)_\sigma$ if α is an even ordinal number, $\mathcal{G}_\alpha=\left(\bigcup_{\gamma<\alpha}\mathcal{G}_\gamma\right)_\delta$ if α is an odd ordinal number and Ω stands for the smallest non-denumerable ordinal number. It suffices to prove that the class $\mathcal{V}_f$ is $\mathcal{J}$-dense on $(\mathcal{G}_\alpha)_f$ for $\alpha<\Omega$. We shall proceed by transfinite induction. If $G\in\mathcal{G}_0$, then $G=\bigcup_{i=1}^{\infty}V_i$ where $V_i\in\mathcal{V}$ for every $i\in\mathbf{N}$. Hence

$$f^{-1}(G)=\bigcup_{i=1}^{\infty}f^{-1}(V_i).$$

Put

$$A_n=\bigcup_{i=1}^{n}f^{-1}(V_i)=f^{-1}\left(\bigcup_{i=1}^{n}V_i\right).$$

We have $\bigcup_{i=1}^{n}V_i\in\mathcal{V}$, so $A_n\in\mathcal{V}_f$, $A_n\subset A_{n+1}$ for $n\in\mathbf{N}$ and $\bigcup_{n=1}^{\infty}A_n=f^{-1}(G)$. Therefore $\chi_{A_n}(x)\to\chi_{f^{-1}(G)}(x)$, $n\to\infty$ for every $x\in X$. Consequently $A_n\to f^{-1}(G)$ wrt $\mathcal{J}$. Suppose that the class $\mathcal{V}_f$ is $\mathcal{J}$-dense on $(\mathcal{G}_\gamma)_f$ for every $\gamma<\alpha$, where α is a countable ordinal. We shall prove that $\mathcal{V}_f$ is $\mathcal{J}$-dense on $(\mathcal{G}_\alpha)_f$. If α is an even ordinal number, then $\mathcal{G}_\alpha$ is an α-additive class, i.e. it consists of the denumerable unions of increasing sets of classes less than α. If $G\in\mathcal{G}_\alpha$, then $G=\bigcup_{n=1}^{\infty}G_n$ where $G_n\subset G_{n+1}$, $G_n\in\mathcal{G}_{\gamma_n}$, $\gamma_n<\alpha$ for every $n\in\mathbf{N}$.

We have $f^{-1}(G)=\bigcup_{n=1}^{\infty} f^{-1}(G_n)$, so $f^{-1}(G_n)\to f^{-1}(G)$ wrt $\mathcal{J}$ and $f^{-1}(G_n)\in \in(\mathcal{G}_{\gamma_n})_f$. From the assumption it follows that for every n there exists a sequence of sets $\{A_{n,m}\}$ in $\mathcal{V}_f$ convergent with respect to the σ-ideal $\mathcal{J}$ to $f^{-1}(G_n)$. Since the family of $\mathcal{S}$-measurable functions with convergence with respect to the σ-ideal $\mathcal{J}$ is a topological space, so there exists a sequence of sets $\{A_{n_k,m_k}\}$, $k\in\mathbf{N}$ in $\mathcal{V}_f$ convergent with respect to the σ-ideal $\mathcal{J}$ to $f^{-1}(G)$. Consequently the class $\mathcal{V}_f$ is $\mathcal{J}$-dense on $(\mathcal{G}_\alpha)_f$. If α is an odd ordinal number, then $\mathcal{G}_\alpha$ is an α-multiplicative class, i.e. for $G\in\mathcal{G}_\alpha$ we have $G=\bigcap_{n=1}^{\infty} G_n$, where $G_n\supset G_{n+1}$, $G_n\in\mathcal{G}_{\gamma_n}$, $\gamma_n<\alpha$ for every $n\in\mathbf{N}$. Obviously $f^{-1}(G)= =\bigcap_{n=1}^{\infty} f^{-1}(G_n)$, so $f^{-1}(G_n)\to f^{-1}(G)$ wrt $\mathcal{J}$ and $f^{-1}(G_n)\in(\mathcal{G}_{\gamma_n})_f$. In the same way as earlier from the assumption it follows that there exists a sequence of sets in $\mathcal{V}_f$ convergent with respect to the σ-ideal $\mathcal{J}$ to $f^{-1}(G)$. Consequently the class $\mathcal{V}_f$ is $\mathcal{J}$-dense on $(\mathcal{G}_\alpha)_f$.

Observe that in Lemma 1 we cannot omit the assumption that the family of $\mathcal{S}$-measurable funtions with the convergence with respect to the σ-ideal $\mathcal{J}$ is a topological space. Indeed, put $X=Y=R$, $f(x)=x$ for $x\in X$. Denote by $\mathcal{S}$ the σ-field of Borel sets and by $\mathcal{J}$ the σ-ideal of sets of measure zero and of the first category. It is easy to see from the third example in [2] that the family of $\mathcal{S}$-measurable functions in that case is not a topological space. Let $\mathcal{V}$ denote the family of the finite unions of open intervals and let $F\subset \subset[0,1]$ be a set of type F_σ of the first category which has a positive measure on every open interval $(a,b)\subset[0,1]$. We shall prove that there is no sequence $\{V_n\}\subset\mathcal{V}$, convergent to F with respect to the σ-ideal $\mathcal{J}$. Indeed, suppose that $V_n\to F$, $n\to\infty$ in measure and let $\{V_{m_n}\}$ be an arbitrary subsequence of $\{V_n\}$. Put $V=\limsup_n V_{m_n}$. Then V is a set of type G_δ dense in $[0,1]$, because V has a positive measure on every open interval $(a,b)\subset[0,1]$. Hence V is the set of the second category. The set $V\triangle F$ is of the second category, consequently the sequence $\left\{\chi_{V_{m_n}}\right\}$ is not convergent to χ_F except on a set of the first category. From the arbitrariness of the subsequence $\{V_{m_n}\}$ it follows that the sequence $\{V_n\}$ is not convergent with respect to the σ-ideal of the sets of measure zero and of the first category to the set F.

In the sequel we shall suppose that the family of $\mathcal{S}$-measurable functions with the convergence with respect to the σ-ideal $\mathcal{J}$ is a topological space.

LEMMA 2. Let $\mathcal{V}$ be an additive basis of Y. Two functions f and g are independent if and only if the classes $\mathcal{V}_f$ and $\mathcal{V}_g$ are independent.

PROOF. The necessity is obvious. To prove the sufficiency it is enough to observe that if $\mathcal{V}_f$ and $\mathcal{V}_g$ are independent then, by Lemma 1 and condition (4), the classes $\mathcal{B}_f$ and $\mathcal{B}_g$ are also independent. Consequently the functions f and g are independent.

THEOREM 4. *If the functions f_n and g_n are independent for every $n\in\mathbf{N}$, $_n\to f$ wrt $\mathcal{J}$ and $g_n\to g$ wrt $\mathcal{J}$ then f and g are independent.*

PROOF. Similarly as in (iv) it is easy to prove that the class $\mathcal{V}$ of all open sets $G\subset Y$ such that $f^{-1}(\mathrm{Fr}\,G)\in\mathcal{J}$ and $g^{-1}(\mathrm{Fr}\,G)\in\mathcal{J}$ is an additive basis of Y. In view of Theorem 1 we have $f_n^{-1}(G_1)\to f^{-1}(G_1)$ wrt $\mathcal{J}$ and $g_n^{-1}(G_2)\to g^{-1}(G_2)$ wrt $\mathcal{J}$ for all $G_1, G_2\in\mathcal{V}$. The functions f_n and g_n are independent for every $n\in\mathbf{N}$, so by condition (4) and Lemma 2 f and g are independent.

LEMMA 3. If Y is a separable metric space, then f is independent with respect to itself if and only if f is constant $\mathcal{J}$-a.e. (i.e. if there exists $y_0\in Y$ such that $\{x: f(x)\neq y_0\}\in\mathcal{J}$).

PROOF. Suppose that f is constant $\mathcal{J}$-a.e. Then for every set $A\in\mathcal{B}_f$ we have $A\in\mathcal{J}$ or $X-A\in\mathcal{J}$. Hence for every $A_1, A_2\in\mathcal{B}_f$ the pair $(A_1, A_2)\in\mathcal{M}$. Consequently f is independent with respect to itself.

Suppose now that f is independent with respect to itself and let $\mathcal{V}$ be a denumerable basis of Y. Put $\mathcal{V}_n=\left\{V\in\mathcal{V}:\delta(V)<\frac{1}{n}\right\}$ for every $n\in\mathbf{N}$. Obviously for every n the class $\mathcal{V}_n$ is also a denumerable basis of Y. Therefore for every n there exists a set $V_n\in\mathcal{V}_n$ such that $f^{-1}(V_n)\notin\mathcal{J}$. We have $f^{-1}(V_n)\cap f^{-1}(Y-V_n)=0\in\mathcal{J}$, so in virtue of (3) $f^{-1}(Y-V_n)\in\mathcal{J}$. Observe that

$$f^{-1}\left(\bigcap_{n=1}^{\infty} V_n\right)\notin\mathcal{J}$$

because

$$X-f^{-1}\left(\bigcap_{n=1}^{\infty} V_n\right)=\bigcup_{n=1}^{\infty} f^{-1}(Y-V_n)\in\mathcal{J}.$$

Hence $\bigcap_{n=1}^{\infty} V_n\neq 0$. Since $\delta(V_n)<\frac{1}{n}$ then there exists $y_0\in Y$ such that $\bigcap_{n=1}^{\infty} V_n=\{y_0\}$ and $\{x: f(x)\neq y_0\}\in\mathcal{J}$. Consequently f is constant $\mathcal{J}$-a.e.

From Theorem 4 and Lemma 3 we obtain

THEOREM 5. *If Y is a separable metric space, $f_n\to f$ wrt $\mathcal{J}$ and for every K there exist $n>m>K$ such that f_n and f_m are independent, then f is constant* $\mathcal{J}$-**a.e.**

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REMARKABLE DECOMPOSITIONS OF L^{Φ}-RANDOM VARIABLES

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1. Introduction. In this note we give some remarkable Davis-type [1] decompositions for the random variables belonging to L^{Φ}-spaces. These decompositions will be given under various conditions imposed on the power of the Young-function Φ and on the power q of its conjugate Young-function Ψ. The methods of these decompositions are interesting in themselves. At the same time they lead to some useful applications.

2. Basic notions and definitions. We refer to our work [2] and [3] and we use the notions of them. We work on a fixed probability space $(\Omega, \mathcal{A}, P)$. Consider the random variable $X \in L^1$ and an increasing sequence of σ-fields $\{\mathcal{F}_n \uparrow\}$, $n \geq 0$. We work with the regular martingale

$$X_n = E(X|\mathcal{F}_n), \quad n \geq 0,$$

where we suppose that $X_0 = 0$ a.e.

We also consider the pair (Φ, Ψ) of conjugate Young-functions.

2.1. Definition. We say that a random variable X is L^{Φ}-predictable if there is a sequence $\{\lambda_n\}$ such that the conditions

a) $|X_n| \leq \lambda_{n-1}$ a.e., $n \geq 1$,
b) $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$,
c) λ_n is $\mathcal{F}_n$-measurable,
d) $\lim\limits_{n \to +\infty} \lambda_n = \lambda_\infty \in L^{\Phi}$,

hold.

We define $\mathcal{D}_\Phi$ as the class of L^{Φ}-predictable functions X and we put

$$\|X\|_{\mathcal{D}_\Phi} = \inf_{\{\lambda_n\}} \|\lambda_\infty\|_\Phi ,$$

where the "inf" is taken over all the sequences $\{\lambda_n\}$ satisfying the preceding conditions. The sequence $\{\lambda_n\}$ will be called an L^{Φ}-predicting sequence of X.

2.2. Definition. Consider the set of random variables γ defined as follows:

$$\Lambda_X^{(\Phi)} = \{\gamma : \gamma \in L^\Phi, \quad E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \quad \text{a.e.} \quad \forall\, n \geq 1\}.$$

We say that $X \in \mathcal{X}_\Phi$ if $\Lambda_X^{(\Phi)}$ is not empty and we put

$$\|X\|_{\mathcal{X}_\Phi} = \inf_{\gamma \in \Lambda_X^{(\Phi)}} \|\gamma\|_\Phi .$$

2.3. Definition. Consider the set of random variables γ defined as follows:

$$\Gamma_X^{(\Phi)} = \{\gamma : \gamma \in L^\Phi, \quad E(|X - X_n| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \quad \text{a.e.} \quad \forall\, n \geq 0\}.$$

We say that $X \in \mathcal{X}_\Phi^+$ if $\Gamma_X^{(\Phi)}$ is not empty and we set

$$\|X\|_{\mathcal{X}_\Phi^+} = \inf_{\gamma \in \Gamma_X^{(\Phi)}} \|\gamma\|_\Phi .$$

2.1. Remars. It is easily seen that

1. $\|\cdot\|_{\mathcal{D}_\Phi}$, $\|\cdot\|_{\mathcal{X}_\Phi}$, and $\|\cdot\|_{\mathcal{X}_\Phi^+}$ are norms on the spaces $\mathcal{D}_\Phi$, $\mathcal{X}_\Phi$, and $\mathcal{X}_\Phi^+$ respectively.
2. $X \in \mathcal{X}_\Phi$ iff $E(|X_n - X_{k-1}| \mid \mathcal{F}_k) \leq E(\gamma \mid \mathcal{F}_k)$ a.e. for all $n \geq k \geq 1$ and all $\gamma \in \Lambda_X^{(\Phi)}$.
3. $\mathcal{X}_\Phi \subset \mathcal{X}_\Phi^+$, moreover we have

$$\|X\|_{\mathcal{X}_\Phi^+} \leq 2\, \|X\|_{\mathcal{X}_\Phi} .$$

3. The Davis-type decompositions of the L^Φ-random variables. These decompositions will be given under various conditions imposed on the power p of the Young-function Φ and on the power q of its conjugate Young-function Ψ.

3.1. Theorem. *Let $X \in L^\Phi$ and suppose that both Φ and its conjugate Ψ have finite power p and q, respectively. Then X can be written in the form*

$$X = Y + Z$$

where $Y \in L^\Phi$ and $E(Y \mid \mathcal{F}_n)$, $n \geq 0$ is a regular martingale for which $E(Y \mid \mathcal{F}_0) = 0$ a.e. *and*

$$\sum_{i=1}^{\infty} |E(Y \mid \mathcal{F}_i) - E(Y \mid \mathcal{F}_{i-1})| \in L^\Phi .$$

while

$$Z \in \mathcal{D}_\Phi .$$

Moreover, we have

$$\left\| \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \right\|_\Phi \leq (4 + 4p)\, q\, \|X\|_\Phi$$

and

$$\|Z\|_{\mathcal{D}_\Phi} \leq (13+4p)\, q\, \|X\|_\Phi .$$

PROOF. Let $X_n^* = \max_{1\leq k\leq n} |X_k|$ and let $X^* = \sup_{k\geq 1} |X_k|$. By using the maximal inequality of Doob (cf. [6], Lemma 1.) we have

$$\|X_n^*\|_\Phi \leq q\, \|X_n\|_\Phi \leq q\, \|X\|_\Phi ,$$

so, we get

$$\|X^*\|_\Phi \leq q\, \|X\|_\Phi .$$

Consequently, $X^* \in L^\Phi$ and so there exists a random variable γ belonging to L^Φ such that the inequality

$$E(|X-X_{n-1}|\,|\mathcal{F}_n) \leq E(\gamma|\mathcal{F}_n) = \gamma_n$$

holds a.e. and for all $n\geq 1$. For example, we can choose $\gamma = 2X^*$. Let

$$\gamma_0^* = 0, \quad \gamma_n^* = \max_{1\leq k\leq n} \gamma_k \quad \text{and} \quad \gamma^* = \sup_{k\geq 1} \gamma_k .$$

Let us define $Y_0 = 0$ and for $n\geq 1$ let

$$Y_n = \sum_{i=1}^n d_i' = \sum_{i=1}^n \{d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*) - E(d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*)|\mathcal{F}_{i-1})\},$$

further, put $Z_0 = 0$ and, again, for $n\geq 1$ let

$$Z_n = \sum_{i=1}^n d_i'' = \sum_{i=1}^n \{d_i I(\gamma_i^* < 2\gamma_{i-1}^*) - E(d_i I(\gamma_i^* < 2\gamma_{i-1}^*)|\mathcal{F}_{i-1})\}.$$

It is easily seen that $X_n = Y_n + Z_n$, $n\geq 0$, further that $(Y_n, \mathcal{F}_n)$ and $(Z_n, \mathcal{F}_n)$ are martingales. Here $\{d_i\}$, $\{d_i'\}$ and $\{d_i''\}$ denote the martingale difference sequences corresponding to $(X_n, \mathcal{F}_n)$, $(Y_n, \mathcal{F}_n)$ and $(Z_n, \mathcal{F}_n)$, respectively.

On the event $\{\gamma_i^* \geq 2\gamma_{i-1}^*\}$ the inequality

$$\gamma_i^* \leq 2(\gamma_i^* - \gamma_{i-1}^*)$$

trivially holds. Note that

$$|d_i| = E(|X_i - X_{i-1}|\,|\mathcal{F}_i) \leq E(\gamma|\mathcal{F}_i) \leq \gamma_i^* .$$

Consequently,

$$\begin{aligned}|d_i'| = |Y_i - Y_{i-1}| &= |d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*) - E(d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*)|\mathcal{F}_{i-1})| \leq \\ &\leq |d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*)| + E(|d_i I(\gamma_i^* \geq 2\gamma_{i-1}^*)|\,|\mathcal{F}_{i-1}) \leq \\ &\leq 2(\gamma_i^* - \gamma_{i-1}^*) + 2E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1}).\end{aligned}$$

Therefore, for arbitrary $n\geq 1$ we have

$$\sum_{i=1}^n |Y_i - Y_{i-1}| \leq 2\gamma_n^* + 2\sum_{i=1}^n E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1})$$

and so

$$\sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \le 2\gamma^* + 2 \sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^* | \mathcal{F}_{i-1}).$$

It follows that

$$\sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \in L^{\Phi}.$$

To see this, by the maximal inequality of Doob and by the Jensen inequality we have

$$\|\gamma^*\|_{\Phi} \le q \sup_{n\ge 1} \|\gamma_n\|_{\Phi} \le q\, \|\gamma\|_{\Phi}.$$

So, $\gamma^* \in L^{\Phi}$. On the other hand, by the convexity lemma (cf. NEVEU [7], p. 219.) we can also prove that

$$\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^* | \mathcal{F}_{i-1}) \in L^{\Phi}.$$

In fact, we have

$$E\left(\Phi\left(\frac{\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^* | \mathcal{F}_{i-1})}{p\,\|\gamma^*\|_{\Phi}}\right)\right) \le E\left(\Phi\left(\frac{\sum_{i=1}^{\infty} (\gamma_i^* - \gamma_{i-1}^*)}{\|\gamma^*\|_{\Phi}}\right)\right) =$$

$$= E\left(\Phi\left(\frac{\gamma^*}{\|\gamma^*\|_{\Phi}}\right)\right) \le 1.$$

So this implies that

$$\left\|\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^* | \mathcal{F}_{i-1})\right\|_{\Phi} \le p\,\|\gamma^*\|_{\Phi}.$$

Here we have supposed that $\|\gamma^*\|_{\Phi} > 0$. This supposition can be made without any loss of generality since otherwise the inequality would be trivial.

Hence

$$\sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \in L^{\Phi}.$$

Therefore, we deduce that the martingale $(Y_n, \mathcal{F}_n)$ is regular. Denoting by Y its a.e. limit, it is easily seen that

$$|Y| \le \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \in L^{\Phi}$$

and

$$Y_n = E(Y | \mathcal{F}_n), \quad n \ge 0.$$

Now we turn to the martingale $(Z_n, \mathcal{F}_n)$. Since

$$Z_n = X_n - Y_n$$

we deduce that it is also a regular martingale and with some $Z \in L^{\Phi}$ we have

$$Z_n = E(Z|\mathcal{F}_n), \quad n \geq 0 .$$

We show now that

$$Z \in \mathcal{D}_{\Phi} .$$

From the definition of Z_n it follows that

$$|d_n''| = |Z_n - Z_{n-1}| \leq |d_n| \, I(\gamma_n^* < 2\gamma_{n-1}^*) + E\left(|d_n| \, I(\gamma_n^* < 2\gamma_{n-1}^*)|\mathcal{F}_{n-1}\right) \leq$$
$$\leq \gamma_n^* \, I(\gamma_n^* < 2\gamma_{n-1}^*) + E\left(\gamma_n^* \, I(\gamma_n^* < 2\gamma_{n-1}^*)|\mathcal{F}_{n-1}\right) \leq$$
$$\leq 2\gamma_{n-1}^* + 2\gamma_{n-1}^* = 4\gamma_{n-1}^* .$$

Consequently, the relation

$$X_{n-1} = Y_{n-1} + Z_{n-1}$$

implies that

$$|Z_n| = |Z_{n-1}| + |Z_n - Z_{n-1}| \leq |X_{n-1}| + |Y_{n-1}| + 4\gamma_{n-1}^* \leq$$
$$\leq 4\gamma_{n-1}^* + X_{n-1}^* + \sum_{i=1}^{n-1} |Y_i - Y_{i-1}| .$$

Let

$$\lambda_n = 4\gamma_n^* + X_n^* + \sum_{i=1}^{n} |Y_i - Y_{i-1}|, \quad n \geq 1 ,$$

and put $\lambda_0 = 0$ a.e. It is easily seen that λ_n is $\mathcal{F}_n$-measurable and that $\lambda_n \in L^{\Phi}$. Also we have $\lambda_n \uparrow \lambda_\infty$ as $n \to +\infty$, where

$$\lambda_\infty = 4\gamma^* + \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| + X^* \in L^{\Phi} .$$

Finally, from the above estimations it follows that

$$\left\| \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \right\|_{\Phi} \leq \left\| 2\gamma^* + 2\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1}) \right\|_{\Phi} \leq$$

$$\leq 2\,\|\gamma^*\|_{\Phi} + 2\left\| \sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1}) \right\|_{\Phi} \leq 2\,\|\gamma^*\|_{\Phi} + 2p\,\|\gamma^*\|_{\Phi} = (2+2p)\,\|\gamma^*\|_{\Phi}$$

and that

$$\|Z\|_{\mathcal{D}_{\Phi}} \leq 4\,\|\gamma^*\|_{\Phi} + \left\| \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \right\|_{\Phi} + \|X^*\|_{\Phi} \leq$$

$$\leq 4\,\|\gamma^*\|_{\Phi} + (2+2p)\,\|\gamma^*\|_{\Phi} + \|X^*\|_{\Phi} = (6+2p)\,\|\gamma^*\|_{\Phi} + \|X^*\|_{\Phi} .$$

Especially, if $\gamma_n = 2X_n^*$, we have

$$\left\| \sum_{i=1}^{\infty} |Y_i - Y_{i-1}| \right\|_{\Phi} \leq (4+4p)\,\|X^*\|_{\Phi} \leq (4+4p)\,q\,\|X\|_{\Phi}$$

and

$$\|Z\|_{\mathcal{D}_\Phi} \leq (13+4p)\|X^*\|_\Phi \leq (13+4p)\, q\, \|X\|_\Phi .$$

This proves the assertion.

We can formulate the preceding assertion under other conditions, too. Namely, we have

3.2. THEOREM. *Let $X \in L^\Phi$ where Φ is a Young-function. We suppose that Φ has finite power p and also that*

$$\sup_{x>0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t} \cdot dt = c < +\infty$$

holds, where $\varphi(t)$ denotes the right-hand side derivative of Φ. Then X can be written in the form

$$X = Y + Z$$

where $Y \in L_\Phi$ and $E(Y|\mathcal{F}_n)$, $n \geq 0$ is a regular martingale for which $E(Y|\mathcal{F}_0) = 0$ a.e. and

$$\sum_{i=1}^{\infty} |E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})| \in L^\Phi .$$

while $Z \in \mathcal{D}_\Phi$.

Moreover, we have

$$\left\| \sum_{i=1}^{\infty} |E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})| \right\|_\Phi \leq (4+4p)\, pc\, \|X\|_\Phi$$

and

$$\|Z\|_{\mathcal{D}_\Phi} \leq (13+4p)\, pc\, \|X\|_\Phi .$$

PROOF. We can follow the same method of proof as in the preceding theorem. Since p, the power of Φ, and

$$\sup_{x>0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\, dt$$

are finite it follows that $X^* \in L^\Phi$ (cf. [4] Theorem 1.) Taking for example $\gamma = 2X^*$ we can in a completely analogous manner construct the martingales $(Y_n, \mathcal{F}_n)$ and $(Z_n, \mathcal{F}_n)$ as in the proof of preceding assertion.

Under the present conditions we deduce that

$$\gamma^* = \sup_{n \geq 0} E(\gamma|\mathcal{F}_n) \in L^\Phi$$

and since p is finite the convexity lemma can also be applied to prove that

$$\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^* | \mathcal{F}_{i-1}) \in L^\Phi .$$

The other steps of the proof are the same as in the preceding theorem. This proves the assertion.

A third condition under which the same Davis-type decomposition is given can be formulated in the following:

3.3. THEOREM. *Let $X \in L^\Phi$ and suppose that the conjugate Young-function Ψ has finite power q. Also, suppose that*

$$\sup_{n>0} \frac{1}{\psi(x)} \int_0^x \frac{\psi(t)}{t}\,dt = c'$$

is finite, where $\psi(x)$ denotes the right-hand side derivative of $\Psi(x)$. Then X can be written in the form

$$X = Y + Z$$

where $Y \in L^\Phi$ and $E(Y|\mathcal{F}_n)$, $n \geq 0$ is a regular martingale for which $E(Y|\mathcal{F}_0) = 0$ a.e. *and*

$$\sum_{i=1}^{\infty} |E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})| \in L^\Phi,$$

while $Z \in \mathcal{D}_\Phi$.

Moreover, we have

$$\left\| \sum_{i=1}^{\infty} |E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})| \right\|_\Phi \leq (4+4qc')\,q\,\|X\|_\Phi,$$

and

$$\|Z\|_{\mathcal{D}_\Phi} \leq (13+4qc')\,q\,\|X\|_\Phi.$$

PROOF. We can follow the main lines of the proof of the preceding two theorems. Since q is finite by the Doob inequality we deduce that $X^* \in L^\Phi$. Thus, taking for example $\gamma = 2X^*$, we can construct the martingales $(Y_n, \mathcal{F}_n)$ and $(Z_n, \mathcal{F}_n)$ as in Theorem 3.1. The finiteness of q also implies that

$$\gamma^* = \sup_{n \geq 0} E(Y|\mathcal{F}_n) \in L^\Phi.$$

Further, the finiteness of q together with the condition that

$$\sup_{x>0} \frac{1}{\psi(x)} \int_0^x \frac{\psi(t)}{t}\,dt = c' < +\infty$$

imply that for the Young-function Φ and for the random variable

$$\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1})$$

the convexity inequality (cf. [5], Theorem 3.), can be applied to show that

$$\sum_{i=1}^{\infty} E(\gamma_i^* - \gamma_{i-1}^*|\mathcal{F}_{i-1})$$

also belongs to L^Φ.

The other steps of the proof are the same as in the preceding two theorems. This proves the assertion.

4. Two inequalities and an application of them to the preceding decompositions. Now we shall deduce two interesting principles and we use them to the decompositions of the preceding section.

4.1. THEOREM. *Let* $(X_n, \mathcal{F}_n)$, $n \geq 0$, *be a martingale and suppose that* $X_0 = 0$ a.e. *Let us denote by* $\{d_i\}$ *the difference sequence of this martingale and suppose that for all* $i \geq 1$ *we have* a.e.

$$|d_i| \leq \delta_i ,$$

where the random variables δ_i *are* $\mathcal{F}_i$*-measurable and such that*

$$\sum_{i=1}^{\infty} \delta_i \in L^{\Phi} .$$

If $Y \in \mathcal{K}_{\Psi}$ *then the expectation* $E(X_n Y_n)$ *exists and is finite. Further, we have*

$$|E(X_n Y_n)| = \left|\sum_{i=1}^{n} E(d_i d_i')\right| \leq 2 \left\|\sum_{i=1}^{n} \delta_i\right\|_{\Phi} \|Y_n\|_{\mathcal{K}_{\Psi}} ,$$

where $\{d_i'\}$ *denotes the difference sequence corresponding to the martingale* $(Y_n, \mathcal{F}_n)$. *Moreover*

$$\lim_{n \to +\infty} E(X_n Y_n)$$

exists and is finite. We have

$$\left|\lim_{n \to +\infty} E(X_n Y_n)\right| \leq 2 \left\|\sum_{i=1}^{\infty} \delta_i\right\|_{\Phi} \|Y\|_{\mathcal{K}_{\Psi}} .$$

PROOF. Since $Y \in \mathcal{K}_{\Psi}$, then by Remark 2.1.2. we deduce that with any $\gamma \in \Lambda_Y^{(\Psi)}$

$$|d_i'| = E(|Y_i - Y_{i-1}| \mid \mathcal{F}_i) \leq E(\gamma \mid \mathcal{F}_i), \quad i \geq 1 .$$

It follows that

$$E(|d_i d_j'|) \leq E\left(\delta_i E(\gamma \mid \mathcal{F}_j)\right) \leq 2 \|\delta_i\|_{\Phi} \|E(\gamma \mid \mathcal{F}_j)\|_{\Psi} \leq 2 \|\delta_i\|_{\Phi} \|\gamma\|_{\Psi} < +\infty .$$

Here we have used the generalized Hölder inequality. Consequently, $E(X_n Y_n)$ is finite and

$$|E(X_n Y_n)| = \left|\sum_{i=1}^{n} E(d_i d_i')\right| \leq \sum_{i=1}^{n} E\left(\delta_i E(\gamma \mid \mathcal{F}_i)\right) =$$

$$= E\left(\left(\sum_{i=1}^{n} \delta_i\right) \gamma\right) \leq 2 \left\|\sum_{i=1}^{n} \delta_i\right\|_{\Phi} \|\gamma\|_{\Psi} .$$

Further, by the generalized Hölder inequality with arbitrary indices $m \leq n$ we have

$$|E(X_n Y_n) - E(X_m Y_m)| = \left|\sum_{i=m+1}^{n} E(d_i d_i')\right| \le \sum_{i=m+1}^{n} E(\delta_i E(\gamma|\mathcal{F}_i)) =$$

$$= \sum_{i=m+1}^{n} E(\delta_i \gamma) = E\left(\left(\sum_{i=m+1}^{n} \delta_i\right)\gamma\right) \le 2\left\|\sum_{i=m+1}^{n} \delta_i\right\|_{\Phi} \|\gamma\|_{\Psi}.$$

We deduce that $\{E(X_n Y_n)\}$ forms a Cauchy sequence. Consequently,

$$\lim_{n\to+\infty} E(X_n Y_n)$$

exists. Finally, we trivially have

$$\left|\lim_{n\to+\infty} E(X_n Y_n)\right| \le \lim_{n\to+\infty} \sum_{i=1}^{n} E(|d_i d_i'|) \le \lim_{n\to+\infty} \sum_{i=1}^{n} E(\delta_i E(\gamma|\mathcal{F}_i)) =$$

$$= \sum_{i=1}^{\infty} E(\delta_i \gamma) = E\left(\left(\sum_{i=1}^{\infty} \delta_i\right)\gamma\right) \le 2\left\|\sum_{i=1}^{\infty} \delta_i\right\|_{\Phi} \|\gamma\|_{\Psi}.$$

This proves the assertion.

We recall from [3] the following notion. To each random variable $X \in \mathcal{D}_{\Phi}$ we can order a random variable X_C where $C > 0$ is arbitrary number. X_C converges a.e. and in $\mathcal{D}_{\Phi}$ to X as $C \to +\infty$. We have

$$|X_C| \le (2 + 2\log 2)\, C.$$

Now, we can prove the following:

4.2. THEOREM. *Let (Φ, Ψ) be a pair of conjugate Young-functions and suppose that $X \in \mathcal{D}_{\Phi}$ and $Y \in \mathcal{K}_{\Psi}$. Then the expectation*

$$E(X_n Y_n)$$

exists and is finite for arbitrary fixed $n \ge 1$, and we have

$$E(X_n Y_n) = \sum_{i=1}^{n} E(d_i d_i'),$$

where $\{d_i\}$ and $\{d_i'\}$ denote the difference sequences corresponding to the martingales $(X_n, \mathcal{F}_n)$ and $(Y_n, \mathcal{F}_n)$, erspectively. We also have that

$$|E(X_n Y_n)| = \left|\lim_{C\to+\infty} E(X_{n_C} Y_n)\right| \le (16 + 16\log 2)\, \|X_n\|_{\mathcal{D}_{\Phi}} \|Y_n\|_{\mathcal{K}_{\Psi}}.$$

PROOF. $|X_n|$ is bounded by $\lambda_{n-1} \le \lambda_{\infty} \in L^{\Phi}$. Consequently,

$$|d_n| = |X_n - X_{n-1}| \le 2\lambda_{n-1} \in L^{\Phi}.$$

Further, if $Y \in \mathcal{K}_{\Psi}$ then with any $\gamma \in \Delta_Y^{(\Psi)}$, we have from Remark 2.1.2. that

$$|d_i'| = |Y_i - Y_{i-1}| = E(|Y_i - Y_{i-1}|\,|\mathcal{F}_i) \le E(\gamma|\mathcal{F}_i).$$

Also, together with $\gamma \in L^{\Psi}$ it is also true that $E(\gamma|\mathcal{F}_i) \in L^{\Psi}$ for arbitrary $i \ge 1$. Consequently, $d_i' \in L^{\Psi}$ and by the generalized Hölder inequality we have

$$E(|d_i d_j'|) \le 2E(\lambda_{i-1} E(\gamma|\mathcal{F}_j)) \le 4\|\lambda_{i-1}\|_{\Phi} \|E(\gamma|\mathcal{F}_j)\|_{\Psi} < +\infty.$$

This means that

$$|E(X_n Y_n)| < +\infty \quad \text{and} \quad E(X_n Y_n) = \sum_{i=1}^{n} E(d_i d_i').$$

Now we prove that

$$E(X_n Y_n) = \lim_{C \to +\infty} E(X_{n_C} Y_n),$$

where X_{n_C} is the random variable defined above for arbitrary $C > 0$. In [3] we proved that for arbitrary $C > 0$ we have the inequality

$$|X_n - X_{n_C}| \le \sum_{i=1}^{n} d_i \Theta_C(\lambda_{i-1}) \le 2\lambda_{n-1} \Theta_C(\lambda_{n-1}),$$

where

$$0 \le \Theta_C(\lambda_{n-1}) = 1 - \left(\frac{2C}{\lambda_{n-1}} - 1\right) \wedge 1 \le 1.$$

The limit of $\Theta_C(\lambda_{n-1})$ is 0 as $C \to +\infty$. So, $X_{n_C} - X_n \to 0$ a.e. and

$$|X_n - X_{n_C}|\,|Y_n| \le 2\lambda_{n-1}\,|Y_n|.$$

Here the right-hand side is integrable since again by the generalized Hölder inequality

$$E(\lambda_{n-1}|Y_n|) \le 2\,\|\lambda_{n-1}\|_{\Phi}\,\|Y_n\|_{\Psi} < +\infty.$$

Consequently, by the Lebesgue dominated convergence theorem we have

$$E(|X_n - X_{n_C}|\,|Y_n|) \to 0$$

as $C \to +\infty$.

Finally, we prove that

$$|E(X_n Y_n)| \le (16 + 16 \log 2)\,\|X_n\|_{\mathcal{D}_\Phi}\,\|Y_n\|_{\mathcal{K}_\Psi}.$$

This follows from the following remarks: together with X the random variable X_n also belongs to $\mathcal{D}_\Phi$ and trivially $\|X_n\|_{\mathcal{D}_\Phi} \le \|X\|_{\mathcal{D}_\Phi}$. Also, if $Y \in \mathcal{K}_\Psi$ then trivially $Y_n \in \mathcal{K}_\Psi \subset \mathcal{K}_\Psi^*$ and by Remark 2.1.3.

$$\|Y_n\|_{\mathcal{K}_\Psi^*} \le 2\,\|Y_n\|_{\mathcal{K}_\Psi} \le 2\,\|Y\|_{\mathcal{K}_\Psi}.$$

Consequently, by our generalization of Herz inequality (cf. [3]) we get that

$$|E(X_n Y_n)| \le (8 + 8 \log 2)\,\|X\|_{\mathcal{D}_\Phi}\,\|Y_n\|_{\mathcal{K}_\Psi^*} \le (16 + 16 \log 2)\,\|X_n\|_{\mathcal{D}_\Phi}\,\|Y_n\|_{\mathcal{K}_\Psi}.$$

This proves the assertion.

The first assertion of the preceding section enables us to formulate the following:

4.1. Corollary. Let (Φ, Ψ) be a pair of conjugate Young-functions and suppose that the power p of Φ as well as the power q of Ψ are finite. If $X \in L^\Phi$ and $Y \in \mathcal{K}_\Psi$ then for all $n \ge 1$ the expectation

$$E(X_n Y_n)$$

is finite. Namely, we have for all $n \geq 1$

$$|E(X_n Y_n)| \leq C_{\Phi} \|X\|_{\Phi} \|Y\|_{\mathcal{X}_{\Psi}},$$

where

$$C_{\Phi} = q\left(8+8p+(16+16\log 2)(13+4p)\right).$$

Moreover, $\lim\limits_{n\to+\infty} E(X_n Y_n)$ exists and

$$\left|\lim_{n\to+\infty} E(X_n Y_n)\right| \leq C_{\Phi} \|X\|_{\Phi} \|Y\|_{\mathcal{X}_{\Psi}}.$$

Proof. Theorem 3.1. implies that

$$X = X' + X'',$$

where $X' \in L^{\Phi}$ is such that the corresponding martingale

$$X_n' = E(X'|\mathcal{F}_n), \quad n \geq 0, \quad X_0' = 0 \quad \text{a.e.}$$

satisfies the inequality

$$\left\|\sum_{i=1}^{\infty} |X_i' - X_{i-1}'|\right\|_{\Phi} \leq (4+4p)\, q \|X\|_{\Phi} < +\infty.$$

Consequently, by Theorem 4.1. we have that $E(X_n' Y_n)$ is finite for all $n \geq 1$ and

$$E(X_n' Y_n) \leq 2(4+4p)\, q \|X\|_{\Phi} \|Y\|_{\mathcal{X}_{\Psi}}.$$

Also, we have that $X'' \in \mathcal{D}_{\Phi}$ and so by the preceding theorem

$$E(X_n'' Y_n)$$

is finite for arbitrary $n \geq 1$ and

$$E(X_n'' Y_n) \leq (16+16\log 2)(13+4p)\, q \|X\|_{\Phi} \|Y\|_{\mathcal{X}_{\Psi}}.$$

This proves the first part of the corollary. To prove the second one we have only to show the validity of the Cauchy property for the sequence $E(X_n Y_n)$. By Theorem 4.1. and by Theorem 4.2. we have that

$$E(X_n Y_n) = \sum_{i=1}^{n} E(d_i d_i')$$

where $\{d_i\}$ and $\{d_i'\}$ are the difference sequence of $(X_n, \mathcal{F}_n)$ and $(Y_n, \mathcal{F}_n)$ respectively. From this for $n \geq m$ we deduce that

$$E(X_n Y_n) - E(X_m Y_m) = \sum_{i=m+1}^{n} E(d_i d_i') = E\left((X_n - X_m) Y_n\right).$$

Consequently, by the preceding inequality

$$|E(X_n Y_n) - E(X_m Y_m)| \leq C_\Phi \|X_n - X_m\|_\Phi \|Y_n\|_{\mathcal{K}_\Psi}.$$

Since Φ has finite power and

$$\sup_{n \geq 0} \|X_n\|_\Phi \leq \|X\|_\Phi,$$

further $X_n \to X$ a.e. we get that the right-hand side tends to 0 as $n, m \to +\infty$, since $\|Y_n\|_{\mathcal{K}_\Psi} \leq \|Y\|_{\mathcal{K}_\Psi}$ (cf. [8].). This proves the assertion.

In the same manner by using Theorem 3.4. and Theorem 3.3. we prove the following two corollaries:

4.2. Corollary. Let (Φ, Ψ) be a pair of conjugate Young-functions. We suppose that Φ has finite power p and that

$$c = \sup_{x > 0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\, dt < +\infty$$

holds, where φ denotes the right-hand side derivative of Φ. If $X \in L^\Phi$ and $Y \in \mathcal{K}_\Psi$ then for arbitrary $n \geq 1$ the expectation

$$E(X_n Y_n)$$

is finite. Further, for all $n \geq 1$ we have

$$|E(X_n Y_n)| \leq C'_\Phi \|X\|_\Phi \|Y\|_{\mathcal{K}_\Psi},$$

where C'_Φ is the constant

$$C'_\Phi = pc\,(8 + 8p + (16 + 16 \log 2)(13 + 4p)).$$

Moreover, $\lim_{n \to +\infty} E(X_n Y_n)$ exists, is finite and

$$\left|\lim_{n \to +\infty} E(X_n Y_n)\right| \leq C'_\Phi \|X\|_\Phi \|Y\|_{\mathcal{K}_\Psi}.$$

4.3. Corollary. Let (Φ, Ψ) be a pair of conjugate Young-functions and suppose that Ψ has finite power q. Also, suppose that

$$c' = \sup_{x > 0} \frac{1}{\psi(x)} \int_0^x \frac{\psi(t)}{t}\, dt$$

is finite, where $\psi(x)$ denotes the right-hand side derivative of Ψ. If $X \in L^\Phi$ and $Y \in \mathcal{K}_\Psi$ then for all $n \geq 1$ the expectation

$$E(X_n Y_n)$$

is finite. Further for all $n \geq 1$ we have

$$|E(X_n Y_n)| \leq c''_{\Phi} \|X\|_{\Phi} \|Y\|_{\mathcal{X}_{\Psi}},$$

where c''_{Φ} is the constant

$$c''_{\Phi} = q\left(8+8qc'+(16+16\log 2)(13+4qc')\right).$$

The inequalitites of these corollaries are in — between the generalized Hölder and the Fefferman—Garsia inequalities.

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THE AVERAGE NUMBER OF GAMES IN THE "RED-AND-BLACK" CASINO

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In this paper we are dealing mainly with a special kind of casino, namely with the so called red-and-black casino. In this gamble the gambler cannot stake more than he possesses. Denote his initial amount of money by x (where $0 < x < 1$). If he wins — this occurs with probability p —, then he gets back his stake and as much more again, he loses his stake with probability $q = 1-p$. His goal is to obtain at least 1 unit of money. DUBINS and SAVAGE [1] (Theorem 5.52.) characterize — in the case if $p < \frac{1}{2}$ — those strategies for which the probability of attaining the goal 1 is maximal. A useful common property of these strategies is that if the gambler uses some of them then at the end he has just 1 or 0 unit of money. The "bold" strategy belongs to this family. This is the following: the gambler bets his entire money if this is smaller than $\frac{1}{2}$, otherwise he bets a smaller amount of money which is just enough to reach his goal (if he wins). We shall prove that in a large class of strategies the "bold" one minimizes the average number of games until the gambler reaches his goal or becomes ruined.

We shall use the following notations:

X_i denotes the outcome of the i-th game. It is equal to $+1$ if the gambler wins, and its value is -1 if he loses, $i = 1, 2, \ldots$. They are independent random variables. $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ $i = 1, 2, \ldots$. $\mathcal{F}_0 = \{0, \Omega\}$. The strategy of the gambler means to choose random variables $(S_n)_{n=1,2,\ldots}$ such that $0 < S_n < 1$ — the gambler's decision before the n-th game depends only on the outcomes of the first $(n-1)$ games and his initial amount of money. So, we require that S_n would be $\mathcal{F}_{n-1}$ measurable $n = 1, 2, \ldots$. Let Y_n be the total amount of money of the gambler after the n-th game ($Y_0 = x$). They are defined by the formula

$$Y_n = Y_{n-1} + Y_{n-1} S_n X_n \quad n = 1, 2, \ldots .$$

Finally, let

$$\nu = \inf(n : Y_n = 0 \quad \text{or} \quad Y_n \geq 1)$$

where, as usual, the inf of the empty set is equal to $+\infty$.

First we characterize the average number of games using a strategy which minimizes this. We shall refer to such strategies as "optimal" ones.

PROPOSITION 1. Suppose that using a given strategy the expected number of games — denoted by $M(x)$ — is a bounded function of x and satisfies the following inequality

$$\text{(1)} \qquad M(x) \leq 1 + pM(x+y) + qM(x-y), \quad 0 \leq x-y < x < 1,$$

where $M(0) = 0$, $M(z) = 0$ if $z \geq 1$.

In this case there isn't any other strategy for which the average number of games is less than $M(x)$ (at any $x \in (0, 1)$).

PROOF. First we prove that the sequence of the following random variables forms a submartingale for any strategy: $(M(Y_n) + n)_{n=1,2,\dots}$.

Observe that the inequality (1) trivially holds if $x \geq 1$. Let A be an atom in the σ-field $\mathcal{F}_n$. Then the random variables Y_n and S_{n+1} are constants, the random variable X_{n+1} is $+1$ — with conditional probability p — or -1 — with conditional probability q — on this event. Hence we can argue as follows (using the independence of X_{n+1} and χ_A):

$$\int_A [M(Y_{n+1}) + n + 1]\, dP = \int_A [M(Y_n + Y_n S_{n+1} X_{n+1}) + n + 1]\, dP =$$

$$= \int_A [1 + pM(Y_n + Y_n S_{n+1}) + qM(Y_n - Y_n S_{n+1}) + n]\, dP \geq$$

$$\geq \int_A [M(Y_n) + n]\, dP.$$

Thus

$$E(M(Y_n) + n) \geq M(Y_0) = M(x)$$

for every n.

Since M is bounded and $\nu \wedge n$ increases as $n \to \infty$, we get

$$E(M(Y_\nu)) + E(\nu) \geq M(x).$$

If $E(\nu) = \infty$ then obviously $E(\nu) \geq M(x)$.

If $E(\nu) < \infty$ then $P(Y_\nu = 0 \text{ or } Y_\nu \geq 1) = 1$, so $P(M(Y_\nu) = 0) = 1$, therefore $E(\nu) \geq M(x)$.

COROLLARY. If we restrict ourselves to the strategies for which $P(Y_\nu = 0$ or $Y_\nu = 1) = 1$, then it is enough to show inequality (1) in the case $0 \leq x - y < x < x + y \leq 1$.

Now we show that the well-known strategy "double or quits" assures the minimal expectation of the number of games.

PROPOSITION 2. Suppose that the gambler uses the following strategy: in every game he stakes all of his money. This strategy is "optimal".

PROOF. Suppose that $\frac{1}{2^n} \le x < \frac{1}{2^{n-1}}$ for some n. Then the number of games is at least k with probability p^k if $0 \le k < n$, if $k \ge n$ then the probability in question is 0. Denoting the expected number of games using the "double or quits" strategy by $M_d(x)$ we have got

$$M_d(x) = \sum_{k=0}^{n-1} p^k \quad \text{if} \quad \frac{1}{2^n} \le x < \frac{1}{2^{n-1}}.$$

Obviously $M_d(x) = 0$ if $x = 0$ or $x \ge 1$ and

$$M_d(x) \le \frac{1}{1-p} = \frac{1}{q}.$$

So it is enough to prove that M_d satisfies the inequality (1).

If $x \geqq \frac{1}{2}$ then $M_d(x) = 1$ so (1) holds for every $0 \leqq y \leqq x$.

If $\frac{1}{2^{n+1}} \le x < \frac{1}{2^n}$ for some n, then $x+y < \frac{1}{2^{n-1}}$. So $pM_d(x+y) \ge \ge M_d(x)$. Thus (1) holds.

Now let us restrict ourselves to such strategies for which $P(Y_r = 0$ or $Y_r = 1) = 1$. The following theorem holds.

THEOREM. *There exists a unique "optimal" strategy in this family, namely the "bold" one.*

PROOF. Denote the expected number of games using the "bold" strategy by $M_b(x)$. First we give an explicit expression of the function $M_b(x)$. (For the sake of simplicity we introduce a function f in the following way: $f(0) = = p$, $f(1) = q$.)

If x is binary rational, and

$$x = \sum_{i=1}^{k} \frac{x_i}{2^i} \quad (x_k = 1)$$

then

$$M_b(x) = 1 + \sum_{i=1}^{k} \prod_{j=1}^{i} f(x_j). \tag{2}$$

If x is binary irrational and

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

then

$$M_b(x) = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} f(x_j). \tag{3}$$

From these expressions it follows that the function M_b is continuous at every binary irrational number, and M_b is bounded by $[1-\max(p, q)]^{-1}$. So for the optimality of "bold" strategy it is enough to show the inequality (1) in the case $0 \leq x-y \leq x \leq x+y \leq 1$.

By the formula of total expectations

$$M_b(x) = \begin{cases} 1+pM_b(2x) & \text{if} \quad 0<x<\frac{1}{2} \\ 1+qM_b(2x-1) & \text{if} \quad \frac{1}{2} \leq x<1 \end{cases} \tag{4}$$

and $M_b(0)=M_b(1)=0$.

Denote $M(x, y)=1+pM_b(x+y)+qM_b(x-y)-M_b(x)$.

If at least one of the x, y, $x+y$, $x-y$, $2x$, $2x-\frac{1}{2}$ is 0 or 1 then easy computation shows that $M(x, y) \geq 0$. So we can suppose that $0<x-y$, $x+y<1$, $x \neq \frac{1}{4}$, $x \neq \frac{1}{2}$, $x \neq \frac{3}{4}$. First we show the validity of (1) for binary rationals x and y, using induction on the binary order of x and y. This and the continuity of M_b at binary irrationals implies that (1) holds for every x, y satisfying the constraints $0 \leq x-y \leq x \leq x+y \leq 1$.

The validity of the forthcoming equalities can be justifield using the equality (4).

Case 1. $x+y<\frac{1}{2}$. In this case $M(x, y)=q+pM(2x, 2y)$. The induction hypothesis justifies that $M(x, y) \geq 0$.

Case 2. $x-y \geq \frac{1}{2}$ then $M(x, y)=p+qM(2x-1, 2y)$. As in the case 1. from this follows that $M(x, y) \geq 0$.

Case 3. $x<\frac{1}{2}<x+y$. Consider two subcases.

(i) $p<q: M(x, y) \geq 1+p+q-M_b(x)=1-pM_b(2x) \geq 1-p\frac{1}{1-q}=0$.

(ii) $p \geq q$: since $x>y$, x is necessarily greater than $\frac{1}{4}$, so

$$\frac{1}{2}>2x-\frac{1}{2}>0 .$$

$$M(x, y)=q+qM\left(2x-\frac{1}{2}, 2y-\frac{1}{2}\right)+q(p-q)M_b\left(2x-\frac{1}{2}-2y-\frac{1}{2}\right).$$

The induction hypothesis justifies that $M(x, y) \geq 0$.

Case 4. $x-y<\frac{1}{2}<x$. The proof is similar to that of the preceding one.

From these it immediately follows that the "bold" strategy is optimal.

It remains to prove that it is unique. We have known already that if the gambler uses an arbitrary strategy then $M_b(Y_{\nu\wedge n})+\nu\wedge n$ is a submartingale, so $E(\nu)\geq M_b(x)$. The equality holds if and only if this submartingale is a martingale. This means that $P(M(Y_n, S_{n+1}Y_n)=0)=1$ for every n. But $M(x, y)=0$ if and only if $x=y$ or $x+y=1$. This yields the "bold" strategy. ▌

It is interesting to note that this strategy assures the maximal probability of attaining the gambler's goal, the 1 unit of money, in the case $p<q$.

As concerns the question what strategy assures the maximal average number of games it is clear that for the strategy $S_n=0$ for every n we have $E(\nu)=\infty$. Moreover, if the gambler can stake arbitrary small amount of money then we cannot expect "reasonable" answer to this question. Let us modify a little the rule of gamble. Suppose that x, the target sum k and the bets in every game are natural numbers (particularly they are ≥ 1). If we restrict ourselves to such strategies for which $P(Y_\nu=0 \text{ or } Y_\nu=k)=1$ then the following strategy assures the maximal average number of games: in every game the bets are just 1 unit of money. Denote this average by $M_r(x)$. We know that

$$M_r(x)=\begin{cases} -\dfrac{x}{p+q}+\dfrac{k}{p-q}\cdot\dfrac{1-\left(\dfrac{q}{p}\right)^x}{1-\left(\dfrac{q}{p}\right)^k} & \text{if } \quad p\neq q \\ x(k-x) & \text{if } \quad p=q\,. \end{cases}$$

$(0<x<k)$.

Since obviously an assertion similar to Proposition 1. holds in this case, too, it is enough to prove that

$$(5) \qquad 1+pM_r(x+y)+qM_r(x-y)-M_r(x)<0 \quad 0<x-y<x<x+y\leq k\,.$$

An easy computation shows that this holds.

Remark. If we omit the restriction $P(Y_\nu=0 \text{ or } Y_\nu=k)=1$ then the above strategy ceases to assure the maximal average number of games, since inequality (5) doesn't hold without the constraint $x+y\leq k$. On the other hand in the case $p=q=\dfrac{1}{2}$ the required strategy can be determined easily. This is the following: the gambler stakes in every game 1 unit of money except if he has just $(k-1)$ units of money when he stakes more than 1 unit of money, say l. l is approximately $\dfrac{2}{5}k$ for large k. The exact value of l can be determined by solving an equation of order two.

We hope to return to these questions in a forthcoming paper.

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GENERALIZED ČECH-COMPLETE SPACES

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1. Introduction. There are two important classes of topological spaces whose definition, in its original form, is an external one, it uses the properties of the space relative to other spaces that contain it. Hence a space X is said to be *H-closed* if it is a Hausdorff space and it is closed in every Hausdorff space Y that contains X as a subspace; and X is said to be *Čech-complete* if it is a Tychonoff space and is a G_δ in every Hausdorff space that contains it as a dense subspace.

Besides these external definitions, there exist internal characterizations for both classes of spaces. That one for H-closed spaces was known for P. S. ALEXANDROFF and P. S. URYSOHN who first defined H-closed spaces and says that X is H-closed iff it is a Hausdorff space and is *almost compact;* the last property means that, in each open cover of X, there is a finite number of members the union of which is dense in X.

The internal characterization of Čech-complete spaces was discovered much later ([1], [2]). It says that a Tychonoff space X is Čech-complete iff there exists in X a sequence $(\mathfrak{C}_n)$ of open covers such that if, in a centred system $\mathfrak{A}$ of closed sets, there is for every $n\in\mathbf{N}$ a set $A_n\in\mathfrak{A}$ contained in a member $C_n\in\mathfrak{C}_n$, then $\bigcap\mathfrak{A}\neq\emptyset$ (a system is said to be *centred* iff it has the finite intersection property).

It is easy to observe that, in contrast to the original external definitions in which it is essential to assume that X is Hausdorff or Tychonoff, respectively, the internal characterizations postulate properties that remain meaningful for arbitrary topological spaces. Hence the question arises quite naturally whether the properties contained in the internal characterizations are equivalent or not to suitable external properties without any *a priori* restriction concerning the class of topological spaces in question.

A positive answer to this question is known for the case of H-closed spaces ([3], [4], [5], [6]). The purpose of the present paper is to show that the situation is similar for the case of Čech-complete spaces, the methods applied in connection with H-closed spaces and the results obtained with the help of them being also useful in dealing with the problem of Čech-complete spaces.

2. Results on generalized H-closed spaces. Let Y be a topological space, X a subspace of Y. We say ([7], [3], [4]) that Y is *relatively* T_2 with respect to X iff $x \in Y$, $y \in Y-X$, $x \neq y$ implies that x and y have disjoint neighbourhoods. Y is said to be an *extension* of X iff X is dense in Y. Y is said ([4]) to be an *ordinary extension* of X iff it is an extension and is relatively T_2 with respect to X.

A *filter* $\mathfrak{s}$ in a topological space X is said to be *open* iff it is generated by a filter base composed of open sets. A *maximal open filter* $\mathfrak{s}$ is an open filter such that, for every open filter $\mathfrak{s}_1 \supset \mathfrak{s}$, we have $\mathfrak{s}_1 = \mathfrak{s}$. A point $x \in X$ is a *cluster point* of an arbitrary system $\mathfrak{A}$ of subsets of X iff $x \in \bigcap\{\bar{A} : A \in \mathfrak{A}\}$.

Now the theorem on generalized H-closed spaces is the following ([4], Theorem (1.5); [5], Lemma 1.2; [6], p. 132):

(2.1) *For an arbitrary topological space X, the following statements are quivalent:*

(a) *X is almost compact,*
(b) *In X every open filter has a cluster point,*
(c) *In X every maximal open filter is convergent,*
(d) *If X is a subspace of a space Y relatively T_2 with respect to X, then X is closed in Y.*

One of the main purposes of [4] was to generalize, for arbitrary spaces, some constructions, known for Hausdorff spaces, of almost compact extensions. For our present purposes, one is needed of them, the *Fomin extension* of X. This is obtained in the following way.

In an arbitrary topological space X, consider all non-convergent maximal open filters and define a set $Y \supset X$ such that there exists a one-to-one map $\mathfrak{s}$ from $Y-X$ onto the set of these filters. Thus $\mathfrak{s}(p)$ is, for $p \in Y-X$, a non-convergent maximal open filter in X. For $x \in X$, define $\mathfrak{s}(x)$ to be the neighbourhood filter of x. For an open set $G \subset X$, set

$$s(G) = \{y \in Y : G \in \mathfrak{s}(y)\},$$

and equip Y with the topology having

$$\{s(G) : G \text{ is open in } X\}$$

as a base. Then we have ([4], (2.13)):

(2.2) *The space Y described above is called the Fomin extension of X. Y is an ordinary, almost compact extension of X, T_2 if X is T_2.*

From the definition of a maximal open filter it is easy to deduce the following proposition that will be useful in the sequel ([4], (2.4)):

(2.3) *If $\mathfrak{s}$ is a maximal open filter and G is an open set such that $G \cap S \neq \emptyset$ or $S \in \mathfrak{s}$, then $G \in \mathfrak{s}$.*

An easy application of the Kuratowski-Zorn lemma yields ([4], (2.5)):

(2.4) *Every open filter is contained in a maximal open filter.*

3. Regularly embedded subsets. We shall need the following concept:

Definition 3.1. A subset A of a topological space X is said to be *regularly embedded* in X if, whenever $x \in A \subset G$ and G is open, there exists an open set V such that $X \in V \subset \bar{V} \subset G$.

Proposition 3.2. *X is regular iff every one-element subset of X is regularly embedded.* ▌

Proposition 3.3. *Every open-closed subset of X is regularly embedded.* ▌

Proposition 3.4. *If A_i is regularly embedded in X for $i \in I$, then $A = \bigcup_{i \in I} A_i$ is regularly embedded.* ▌

Proposition 3.5. *Let X be a subspace of Y, $A \subset X \subset Y$. If A is regularly embedded in Y, then it is regularly embedded in X too.*

Proof. Let G be open (in Y) and $A \subset G \cap X$, $x \in A$. Then there exist an open set H and a closed set F (always in Y) such that $x \in H \subset F \subset G$, hence

$$x \in H \cap X \subset F \cap X \subset G \cap X ,$$

and $H \cap X$ is relatively open, $F \cap X$ relatively closed in X. ▌

The following property of the Fomin extension will be useful:

Theorem 3.6. *Every space X is regularly embedded in its Fomin extension Y.*

Proof. Let H be open in Y, $X \subset H \subset Y$, $x \in X$. There is a set G, open in X, such that $x \in s(G) \subset H$; we show $\overline{s(G)} \subset H$ for the closure in Y. By $X \subset H$ it suffices to check $\overline{s(G)} - X \subset H$. But this is a consequence of the equality

$$\overline{s(G)} - X = s(G) - X . \tag{3.6.1}$$

In order to see (3.6.1), let $y \in Y - s(G)$, $y \in Y - X$; then $G \notin \tilde{s}(y)$, hence by (2.3) there is a G_1, open in X, such that $G_1 \in \tilde{s}(y)$, $G \cap G_1 = \emptyset$. Therefore $s(G) \cap s(G_1) = \emptyset$ and $s(G_1)$ is a neighbourhood of y which does not intersect $s(G)$; by this (3.6.1) is established. ▌

4. Čech spaces. For obtaining an easier formulation of the internal characterization of Čech-complete spaces and of related properties, let us introduce the following terminology.

Definition 4.1. Let $(\mathfrak{C}_n)$ be a sequence of systems of sets, and $\mathfrak{A}$ a system of sets. $\mathfrak{A}$ is said to be *subordinate* to the sequence $(\mathfrak{C}_n)$ if, for every $n \in \mathbf{N}$, there are a set $A_n \in \mathfrak{A}$ and a set $C_n \in \mathfrak{C}_n$ such that $A_n \subset C_n$.

Definition 4.2. Let X be a topological space. A *Čech sequence (Čech f-sequence, Čech g-sequence)* in X is a sequence $(\mathfrak{C}_n)$ of open covers of X such that every centred system $\mathfrak{A}$ (composed of closed sets, composed of open sets) subordinate to $(\mathfrak{C}_n)$ has a cluster point.

PROPOSITION 4.3. *Every Čech sequence is a Čech f-sequence and a Čech g-sequence* ▌

Now the internal characterization of Čech-complete spaces can be formulated as follows:

(4.4) *A Tychonoff space is Čech-complete iff there exists a Čech f-sequence in X.*

It is known ([8]) that Čech f-sequences can be replaced here by Čech sequences:

(4.5) *A Tychonoff space is Čech-complete iff there exists a Čech sequence in X.*

Moreover, it is not difficult to show that we could replace Čech f-sequences by Čech g-sequences also.

LEMMA 4.6. *In a regular space every Čech g-sequence is a Čech f-sequence.*

PROOF. Let $(\mathfrak{C}_n)$ be a Čech g-sequence in the regular space X and $\mathfrak{A}$ a centred system of closed sets subordinate to $(\mathfrak{C}_n)$. Denote by $\mathfrak{B}$ the system of all open sets that contain at least one element of $\mathfrak{A}$. Clearly $\mathfrak{B}$ is a centred system of open sets subordinate to $(\mathfrak{C}_n)$ because $A_n\in\mathfrak{A}$, $C_n\in\mathfrak{C}_n$, $A_n\subset C_n$ implies $C_n\in\mathfrak{B}$. By hypothesis there exists an $x\in\bigcap\{\bar{B}:B\in\mathfrak{B}\}$. Suppose $x\notin A$ for a set $A\in\mathfrak{A}$. By the regularity of X there are open sets U and V such that $x\in V$, $A\subset U$, $U\cap V=\emptyset$. Then $U\in\mathfrak{B}$ does not intersect the neighbourhood V of x; this contradiction shows that $x\in\bigcap\mathfrak{A}$. ▌

LEMMA 4.7. *If, in a regular space, there exists a Čech f-sequence, then there exists a Čech sequence also.*

PROOF. Let X be regular and $(\mathfrak{C}_n)$ a Čech f-sequence in X. Let $\mathfrak{B}_n$ consist of all open sets B such that $\bar{B}\subset C\in\mathfrak{C}_n$ for a suitable C. By the regularity of X this is a cover. Let $\mathfrak{A}$ be an arbitrary centred system subordinate to $(\mathfrak{B}_n)$, and denote $\bar{\mathfrak{A}}=\{\bar{A}:A\in\mathfrak{A}\}$. Clearly $\bar{\mathfrak{A}}$ is a centred system of closed sets, subordinate to $(\mathfrak{C}_n)$, hence there exists an $x\in\bigcap\bar{\mathfrak{A}}$. ▌

THEOREM 4.8. *For a regular space X, the following statements are equivalent:*

(a) *There is a Čech g-sequence in X,*
(b) *There is a Čech f-sequence in X,*
(c) *There is a Čech sequence in X.*

PROOF. (a)⇒(b) : 4.6. (b)⇒(c) : 4.7. (c)⇒(a) : 4.3. ▌

Now let us introduce the following terminology (cf. [8], p. 398):

DEFINITION 4.9. A space X is said to be a *Čech space (Čech f-space, Čech g-space)* iff there exists a Čech sequence (Čech f-sequence, Čech g-sequence) in X.

By 4.8. the three concepts coincide in the class of regular spaces, and by (4.4) or (4.5) they coincide with Čech-complete spaces in the class of Tychonoff spaces. We shall show that, in a certain sense, Čech g-spaces can be considered as a natural generalization of Čech-complete spaces; however, Čech spaces have many similar properties too (see [8]).

PROPOSITION 4.10. *Every Čech space is a Čech g-space and a Čech f-space. If X is regular and a Čech g-space or a Čech f-space, then it is a Čech space.*

PROOF. 4.3, 4.8. ▌

5. Properties of Čech g-spaces. By (2.1), we have obviously:

THEOREM 5.1. *Every almost compact space is a Čech g-space.* ▌

Concerning heredity properties, we can prove two statements:

THEOREM 5.2. *A regularly embedded open subspace of a Čech g-space is a Čech g-space.*

PROOF. Let Y be a Čech g-space, $X \subset Y$ regularly embedded and open, $(\mathfrak{C}_n)$ a Čech g-sequence in Y. Define $\mathfrak{B}_n$ to be the system of those open subsets B of X for which $\overline{B} \subset X$ (with the closure in Y) and $B \subset C_n$ for a suitable $C_n \in \mathfrak{C}_n$. Then $\mathfrak{B}_n$ is a cover of X; in fact, if $x \in X$, there is a $C_n \in \mathfrak{C}_n$ such that $x \in C_n$, and an open V such that $x \in V \subset \overline{V} \subset X$, then $x \in C_n \cap V \in \mathfrak{B}_n$. We show that $(\mathfrak{B}_n)$ is a Čech g-sequence in X.

In fact, if $\mathfrak{A}$ is a centred system of open subsets of X, subordinate to $(\mathfrak{B}_n)$, then it is also a centred system of open subsets of Y subordinate to $(\mathfrak{C}_n)$, hence there exists $x \in \bigcap \{\overline{A} : A \in \mathfrak{A}\}$. Choosing $A_n \in \mathfrak{A}$ such that $A_n \subset B_n \in \in \mathfrak{B}_n$, we see that $\overline{A}_n \subset X$ so that $x \in X$ and it is a cluster point of $\mathfrak{A}$ in the subspace X. ▌

THEOREM 5.3. *A regularly embedded, dense G_δ subspace of a Čech g-space is a Čech g-space.*

PROOF. Let $(\mathfrak{C}_n)$ be a Čech g-sequence in Y, $X \subset Y$ dense and regularly embedded, further $X = \bigcap_1^\infty G_n$, G_n open (in Y). Define $\mathfrak{B}_n$ to be the system of those relatively open subsets B of X for which $B = H \cap X$, H open in Y, $\overline{H} \subset G_n$ (closure in Y) and $H \subset C_n \in \mathfrak{C}_n$ with a suitable C_n. Then $\mathfrak{B}_n$ is a cover of X; in fact, if $x \in X$, there is a $C_n \in \mathfrak{C}_n$ such that $x \in C_n$, and an open (in Y) V such that $x \in V \subset \overline{V} \subset G_n$ and $x \in V \subset C_n \cap X \in \mathfrak{B}_n$. We show that $(\mathfrak{B}_n)$ is a Čech g-sequence in X.

Let $\mathfrak{A}$ be a centred system of relatively open subsets of X, subordinate to $(\mathfrak{B}_n)$. Define $\mathfrak{A}'$ as the system of those open (in Y) sets A' for which $A' \cap X \in \mathfrak{A}$. Then $\mathfrak{A}'$ is centred and subordinate to $(\mathfrak{C}_n)$; in fact, for every $n \in \mathbf{N}$, we find a set $A_n \in \mathfrak{A}$ such that $A_n \subset B_n \in \mathfrak{B}_n$, hence sets D_n, H_n, open in Y, such that

$$A_n = D_n \cap X \subset B_n = H_n \cap X \subset C_n \in \mathfrak{C}_n, \quad \overline{H}_n \subset G_n,$$

and

$$A_n = D_n \cap H_n \cap C_n \cap X, \quad D_n \cap H_n \cap C_n \in \mathfrak{A}'.$$

Therefore there exists $x\in\bigcap\{\bar{A}' : A'\in\mathfrak{A}'\}$. Since, with the above notation, $x\in\overline{D_n\cap H_n\cap C_n}\subset H_n\subset G_n$ for every n, we have $x\in X$. Finally if $A = D\cap X$, $D\in\mathfrak{A}'$, then $x\in\bar{D}\subset\overline{D\cap X} = \bar{A}$ since D is open and X is dense in Y. Hence $x\in\bar{A}\cap X$ and x is a cluster point of $\mathfrak{A}$ in the subspace X. ∎

We shall prove a product theorem for Čech g-spaces. For this purpose, let us first define:

DEFINITION 5.4. A Čech sequence (g-sequence, f-sequence) $(\mathfrak{C}_n)$ is said to be *monotone* iff $\mathfrak{C}_{n+1}\subset\mathfrak{C}_n$ for $n\in\mathbf{N}$.

LEMMA 5.5. *In a Čech space (g-space, f-space) there exists a monotone Čech sequence (g-sequence, f-sequence).*

PROOF. Let $(\mathfrak{C}_n)$ be an arbitrary Čech sequence (g-sequence, f-sequence) and let $\mathfrak{B}_n$ denote the system of all open sets B such that $B\subset\bigcap_1^n C_i$ for suitable sets $C_i\in\mathfrak{C}_i$ $(i = 1, \dots, n)$. Obviously $\mathfrak{B}_{n+1}\subset\mathfrak{B}_n$, $\mathfrak{B}_n$ is an open cover, and a centred system $\mathfrak{A}$ subordinate to $(\mathfrak{B}_n)$ is subordinate to $(\mathfrak{C}_n)$; hence $(\mathfrak{B}_n)$ is the sequence we are looking for. ∎

THEOREM 5.6. *The product of countably many Čech g-spaces is a Čech g-space.*

PROOF. Let $(\mathfrak{C}^i_n)$ $(n\in\mathbf{N})$ be a Čech g-sequence in X_i for $i\in\mathbf{N}$. By 5.5. we can assume that every sequence $(\mathfrak{C}^i_n)$ is monotone. In the space $X = \bigtimes_{i=1}^{\infty} X_i$, define $\mathfrak{C}_n$ to be the system of all products $\bigtimes_{i=1}^{\infty} C_i$ such that $C_i\in\mathfrak{C}^i_n$ for $i\leqq n$ and $C_i = X_i$ for $i>n$. Clearly $\mathfrak{C}_n$ is an open cover of X.

We show that $(\mathfrak{C}_n)$ is a Čech g-sequence in X. Let $\mathfrak{A}$ be a centred system of open sets in X, subordinate to $(\mathfrak{C}_n)$. The finite intersections of the members of $\mathfrak{A}$ constitute an open filter base that generates an open filter; by (2.4) the latter is contained in a maximal open filter $\mathfrak{s}\supset\mathfrak{A}$. For a given $i\in\mathbf{N}$ consider the system

$$\mathfrak{A}_i = \{p_i(S) : S\in\mathfrak{s},\quad S \text{ is open}\},$$

where $p_i : X\to X_i$ denotes the projection. Clearly $\mathfrak{A}_i$ is a centred system of open subsets of X_i. Moreover, $\mathfrak{A}_i$ is subordinate to $(\mathfrak{C}^i_n)$. In fact, if $n\geqq i$, then there are an $A\in\mathfrak{A}$ and a $C\in\mathfrak{C}_n$ such that $A\subset C$. Now $C = \bigtimes_{i=1}^{\infty} C_i$ and $C_i\in\mathfrak{C}^i_n$ for $i\leqq n$, hence $p_i(A)\subset C_i\in\mathfrak{C}^i_n$. If $n<i$, we first select a set $C_i\in\mathfrak{C}^i_i$ and an $A\in\mathfrak{A}$ such that $p_i(A)\subset C_i$ and observe that, by the monotonity of the sequence $(\mathfrak{C}^i_n)$, $C_i\in\mathfrak{C}^i_n$.

Therefore there is $x_i\in\bigcap\{\bar{A}_i : A_i\in\mathfrak{A}_i\}$ for each $i\in\mathbf{N}$. Define $x = (x_i)$. We show that x is a cluster point of $\mathfrak{A}$. In fact, if $V = \bigtimes_{i=1}^{\infty} V_i$ is a neighbourhood of x such that $x_i \in V_i$ and V_i is open in X_i for $i \leq n$, $V_i = X_i$ for $i > n$, then $V_i \cap p_i(S) \neq \emptyset$ for any open set $S \in \mathfrak{s}$, i.e. $S \cap p_i^{-1}(V_i) \neq \emptyset$ for the same S, and by (2.3) $p_i^{-1}(V_i) \in \mathfrak{s}$ for $i \leq n$ so that

$$V = \mathop{\times}_{i=1}^{\infty} V_i = \bigcap_{i=1}^{n} p_i^{-1}(V_i) \in \mathfrak{s}$$

and $V \cap A \neq \emptyset$ for $A \in \mathfrak{A}$. ▌

For the sum of topological spaces, it is easy to show:

THEOREM 5.7. *If $X = \bigcup_{i \in I} X_i$, the subspaces X_i are disjoint, open and each of them is a Čech g-space, then so is X.*

PROOF. Let $(\mathfrak{C}_n^i)$ be a Čech g-sequence in X_i and define $\mathfrak{C}_n = \bigcup_{i \in I} \mathfrak{C}_n^i$. Clearly $\mathfrak{C}_n$ is an open cover of X. If $\mathfrak{A}$ is an open, centred system in X subordinate to $(\mathfrak{C}_n)$, then there is an i such that $\mathfrak{A}$ is subordinate to $(\mathfrak{C}_n^i)$ because $\mathfrak{A}$ is centred and the subspaces X_i are disjoint so that $A_n \subset C_n \in \mathfrak{C}_n^i$, $A_m \subset C_m \in \in \mathfrak{C}_m^j$, A_n, $A_m \in \mathfrak{A}$ is possible only if $i = j$. Select an i of this kind and define

$$\mathfrak{B} = \{A \cap X_i : A \in \mathfrak{A}\}.$$

$\mathfrak{B}$ is a centred system because the intersection of finitely many members of $\mathfrak{A}$ has a still non-empty intersection with an $A_n \subset C_n \in \mathfrak{C}_n^i$, $A_n \in \mathfrak{A}$, that is contained in X_i. Hence $\mathfrak{B}$ has a cluster point in X_i which is a cluster point of $\mathfrak{A}$ in X. ▌

Let us note that conversely:

THEOREM 5.7. *If $X = \bigcup_{i \in I} X_i$ is a Čech g-space and the subspaces X_i are open and disjoint then they are Čech g-spaces as well.*

PROOF. 3.3. and 5.2. ▌

6. Generalization of Čech-completeness. We show that, for Čech g-spaces, there is an external characterization without any separation axiom.

THEOREM 6.1. *For an arbitrary topological space X, the following statements are equivalent:*

(a) *X is G_δ in every ordinary extension Y.*
(b) *X is G_δ in its Fomin extension Y,*
(c) *X is a Čech g-space.*

PROOF. (a)$\Rightarrow$(b) : (2.2).

(b)$\Rightarrow$(c): The Fomin extension Y is almost compact by (2.2) and X is regularly embedded in Y by 3.6. Hence Y is a Čech g-space by 5.1 and the same holds for X by 5.3.

(c)$\Rightarrow$(a): Let Y be an ordinary extension of X and $(\mathfrak{C}_n)$ a Čech g-sequence in X. Represent, for a given n, each $C \in \mathfrak{C}_n$ in the form $C = D \cap X$ where D is open in Y, and denote by G_n the union of these sets D. Then $X \subset G_n$ and G_n is open in Y.

We show $X=\bigcap_1^\infty G_n$. In fact, assume $y\in\bigcap_1^\infty G_n$, $y\in Y-X$. Let $\mathfrak{B}$ denote the system of all open neighbourhoods of y and define

$$\mathfrak{A}=\{B\cap X : B\in\mathfrak{B}\}.$$

Then $\mathfrak{A}$ is a centred system of open subsets of X and it is subordinate to $(\mathfrak{C}_n)$ because, for $n\in\mathbf{N}$, there is an open D such that $C=D\cap X\in\mathfrak{C}_n$, $y\in D\subset G_n$ and then $D\in\mathfrak{B}$, $C\in\mathfrak{A}\cap\mathfrak{C}_n$. Hence there is a cluster point $x\in X$ of $\mathfrak{A}$. Since Y is an ordinary extension of X, the points x and y must have disjoint open neighbourhoods U and V. But then $V\in\mathfrak{B}$, $V\cap X\in\mathfrak{A}$ so that $U\cap V\cap X=\emptyset$ contradicts the fact that x is a cluster point of $\mathfrak{A}$. By this, we have established the equality $X=\bigcap_1^\infty G_n$. ∎

Observe that the fundamental theorem (4.4) is a corollary of 6.1. In fact, if X is a Tychonoff space that is G_δ in every Hausdorff extension Y then, in particular, X is a G_δ in its Fomin extension by (2.2). Hence X is a Čech g-space and, by 4.10, a Čech f-space. Conversely if X (still Tychonoff) is a Čech f-space, then it is a Čech g-space by 4.10, hence a G_δ in every Hausdorff extension (which is ordinary of course).

We obtain in the same way a proof of (4.5).

7. Properties of Čech (f-) spaces. We add some results on Čech f-spaces and some improvements of known results on Čech spaces.

We first show that the concept of a Čech g-space is distinct from those of Čech space and Čech f-space:

THEOREM 7.1. *There exists a Hausdorff space that is a Čech g-space without being a Čech f-space.*

PROOF. Let X be the real line equipped with the topology for which the sets $I-M$ constitute a base, where I is an open interval and M is countable. X is a T_2-space since its topology is finer than the usual topology of the real line.

Let Y be the Fomin extension of X; it is T_2 and almost compact by (2.2), hence a Čech g-space by 5.1.

Let $(\mathfrak{C}_n)$ be an arbitrary sequence of open covers of Y. Select $C_n\in\mathfrak{C}_n$ such that $0\in C_n$, and then an open interval I_n and a countable set M_n such that $0\in I_n-M_n\subset C_n$. Define

$$M_0=\bigcup_1^\infty M_n\cup\{0\}$$

and

$$x_k\in\left(\left(\bigcap_1^k I_n\right)\cap\left(-\frac{1}{k},\ \frac{1}{k}\right)\right)-M_0,$$

$$A_n=\{x_k : k\geqq n\}.$$

By

$$A_n\subset I_n-M_0\subset I_n-M_n\subset C_n$$

the system

$$\mathfrak{A} = \{A_n : n \in \mathbf{N}\}$$

is a centred system subordinate to $(\mathfrak{C}_n)$. We show that each A_n is closed in Y; this will prove that Y is not a Čech f-space because clearly $\bigcap_1^\infty A_n = \emptyset$.

Let $y \in Y - A_n$. If $y \in X$ and $y \neq 0$ then $y \notin \bar{A}_n$ since the topology of X is finer than the usual topology of $\mathbf{R}$. We have also $0 \notin \bar{A}_n$ since $(-1, 1) - A_n$ is a neighbourhood of 0 in X. Now let $y \in Y - X$. Choose open, disjoint neighbourhoods U and V of 0 and y respectively, further an open interval I' and a countable set M' such that $0 \in I' - M' \subset U$. Observe that $V \cap I' \neq \emptyset$ would imply the existence of an open interval I and a countable set M such that $I - M \subset V$, $I' \cap (I - M) \neq \emptyset$, whence $(I' - M') \cap (I - M) = (I' \cap I) - (M' \cup M) \neq \emptyset$ because $I' \cap I \neq \emptyset$ is uncountable; this is impossible by $U \cap V = \emptyset$. Hence $V \cap I' = \emptyset$ so that $V \cap A_n$ is finite and $y \notin \bar{A}_n$ again. ▌

Corollary 7.2. *There exists a Hausdorff space that is a Čech g-space without being a Čech space.*

Proof. 7.1 and 4.10. ▌

Problem 7.3. Does there exist a Čech f-space that is not a Čech g-space?

Problem 7.4. Does there exist a Čech f-space that is not a Čech space?

For the heredity properties of Čech f-spaces we can prove:

Theorem 7.5. *A closed subspace of a Čech f-space is a Čech f-space.*

Proof. Let $(\mathfrak{C}_n)$ be a Čech f-sequence in Y and $X \subset Y$ a closed subspace. If

$$\mathfrak{B}_n = \{C \cap X : C \in \mathfrak{C}_n\}$$

and $\mathfrak{A}$ is a centred system of closed subsets of X, subordinate to $(\mathfrak{B}_n)$, then $\mathfrak{A}$ has a cluster point in Y which necessarily belongs to X. ▌

The corresponding property of Čech spaces (which can be proved in the same manner) is known ([8], (9.2.22)). For Čech spaces we can also prove:

Theorem 7.6. *A regularly embedded G_δ subspace of a Čech space is a Čech space.*

Proof. Let $(\mathfrak{C}_n)$ be a Čech sequence in Y and $X = \bigcap_1^\infty G_n$, G_n open in Y, X regularly embedded in Y. Define $\mathfrak{B}_n$ to be the system of those relatively open subsets B of X for which $B = H \cap X$, H open in Y, $\bar{H} \subset G_n$ and $H \subset C_n \in \mathfrak{C}_n$ for a suitable C_n. Similarly as in the proof of 5.3, we see that $\mathfrak{B}_n$ is a cover of X. Now if $\mathfrak{A}$ is a centred system of subsets of X, subordinate to $(\mathfrak{B}_n)$, then it is obviously subordinate to $(\mathfrak{C}_n)$, hence has a cluster point in Y. Since $A_n \in \mathfrak{A}$, $B_n \in \mathfrak{B}_n$, $B_n = H_n \cap X$, $\bar{H}_n \subset G_n$, $A_n \subset B_n$ imply $\bar{A}_n \subset G_n$, this cluster point is in X. ▌

This is a slight generalization of the known fact ([8], (9.2.23)) that a G_δ subspace of a regular Čech space is a Čech space; in fact, by 3.2 and 3.4 every subspace of a regular space is regularly embedded.

A product theorem corresponding to 5.6 is known for Čech spaces ([8], (9.2.24)) and can be proved by the same method, using ultrafilters instead of maximal open filters. For Čech f-spaces, the method does not work because the projections are not closed maps.

The proof of 5.7, with obvious modifications, furnishes:

THEOREM 7.7. *If $X = \bigcup_{i \in I} X_i$, the subspaces X_i are disjoint, open and each of them is a Čech (f-) space, then so is X.* ∎

The following theorem is known for Čech spaces ([8], (9.2.21)):

THEOREM 7.8. *In every Čech g-space there exists a sequence $(\mathfrak{B}_n)$ of bases such that if the system $\mathfrak{B} = \{B_n : n \in \mathbf{N}\}$ is centred and $B_n \in \mathfrak{B}_n$ then $\mathfrak{B}$ has a cluster point.*

PROOF. If $(\mathfrak{C}_n)$ is a Čech g-sequence, it suffices to define $\mathfrak{B}_n$ to be the system of all open sets B such that $B \subset C_n \in \mathfrak{C}_n$ for a suitable C_n. ∎

COROLLARY 7.9. *If X is a Čech g-space and every non-empty open set in X contains the closure of a non-empty open subset, then X is a Baire space.*

PROOF. [9], (5.1). ∎

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A REPRESENTATION OF R^+ WITH AN APPLICATION BY SOBOLEV SPACE OPERATORS OF PARABOLIC NON-SMOOTH BOUNDARY VALUE PROBLEMS

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Introduction

In this paper, using a special domain transformation for domains with non-smooth boundaries of parabolic mixed type boundary value problems defined as in [12], we construct a representation of the one-parameter translation semigroup R^+ by continuous linear operators of Sobolev spaces of functions satisfying the mixed type boundary value problem

$$\frac{\partial U}{\partial \tau}+q\cdot U=0$$

for some fixed q and an orientation function τ on the boundary.

The class of domains considered here (called generalized cylinders) consists of open subsets of R^{n+1} with orientation function τ on the boundary satisfying some restrictions necessary to formulate the boundary value problem and to obtain the tools for applying the sophisticated methods used in [6], [8], [9], [10]. These conditions and results obtained in [11] and [12] are summarized in **§ 1.**

In **§ 2.** (theorem 2.1) the representation is constructed.

In **§ 3.** using the methods of [9], [10], we outline the proof of an existence theorem for a mixed type boundary value problem as an illustration of the results. The conditions in the existence theorem show that the non-smoothness of the boundary makes some restrictions on the parabolic equation necessary too which do not arise in the case of smooth boundaries. We think worth mentioning that this existence theorem presents an approach to an unsolved problem exposed in [9] (problem 5, Problems 18., chapter I.).

§ 1. Preliminaries

In this § we give the definition of the generalized cylinder and summarize some results concerning it gained in [11] and [12].

Let the set $\Omega\subset(t_0, \infty)\times R^n\subset R^{n+1}$ be open for a fixed $t_0\in R$. Let us denote the boundary of Ω in $\Omega_1:=(t_0, \infty)\times R^n$ by $\partial_{\Omega_1}\Omega$.

DEFINITION 1.1: The unit vector $\tau\in R^{n+1}$ points into Ω at $x\in\partial_{\Omega_1}\Omega$ if the inclusions

$$(1.1)\qquad x+t\cdot\tau'\in\Omega \qquad \text{with} \quad t>0$$

and

$$x+t\cdot\tau'\in\Omega_1\setminus\overline{\Omega}^{\Omega_1} \quad \text{with} \quad t<0$$

hold with $t\in R$, $|t|<\delta$ and $\tau'\in R^{n+1}$, $\|\tau'-\tau\|<\varepsilon$ for a suitable fixed pair of positive numbers ε and δ.

DEFINITION 1.2: The boundary $\partial_{\Omega_1}\Omega$ is called orientable if a Lipschitzian function $\tau:\partial_{\Omega_1}\Omega\to R^{n+1}$, $\|\tau\|=1$ exists with value $\tau(x)$ pointing into Ω for each $x\in\partial_{\Omega_1}\Omega$.

DEFINITION 1.3: The boundary $\partial_{\Omega_1}\Omega$ is uniformly orientable if it is orientable and a neighbourhood $w(p)$ and positive numbers $\varepsilon(p)>0$ and $\delta(p)>0$ exist for each $p\in\partial_{\Omega_1}\Omega$ fulfilling the condition of definition 1.1 with $\tau:=\tau(x)$, $\delta:=\delta(p)$ and $\varepsilon:=\varepsilon(p)$ for each $x\in w(p)\cap\partial_{\Omega_1}\Omega$.

Now we are in the position to define the concept of the generalized cylinder.

DEFINITION 1.4: An open set $\Omega\subset\Omega_1\subset R^{n+1}$ with uniformly orientable boundary $\partial_{\Omega_1}\Omega$ is a generalized cylinder if it satisfies the following restrictions:

1. For arbitrary $[t_1, t_2]\subset[t_0, \infty)$ the closed set

$$(1.2)\qquad \bar{\Omega}\subset[t_1, t_2]\times R^n$$

is compact.

2. The set

$$(1.3)\qquad \operatorname{int}_{t\times R^n}(\{t\}\times R^n\cap\bar{\Omega}),\quad t\in[t_0, \infty)$$

is non-empty, where $\operatorname{int}_{t\times R^n}$ denotes interior in $t\times R^n$.

3. The orientation vector $\tau(t, x)$ is not parallel with $e_t:=(1, 0)\in R\times R^n$ for any $(t, x)\in\partial_{\Omega_1}\Omega$.

Now we summarize the results about generalized cylinders gained in papers [11] and [12].

Let us suppose that $\Omega\subset\Omega_1\subset R^n$ is a generalized cylinder with orientation function τ on its boundary $\partial_{\Omega_1}\Omega$. By theorem A1.2 in [11] the orientation function τ has a Lipschitzian extension f which is infinitely differentiable in the set $\Omega_1\setminus\partial_{\Omega_1}\Omega$. Considering the solutions of the differential equation

$$(1.4)\qquad \dot{y}=f(y)$$

we gain the following theorem (theorem 2.1 in [11]):

THEOREM 1.1: *If Ω is a generalized cylinder then a C^∞ manifold can be given on Ω_1 with the help of a family of triplets $\{U_\alpha, V_\alpha, \Psi_\alpha\}_{\alpha\in A}$ satisfying the following properties:*

1. *The family* $\{U_\alpha\}_{\alpha\in A}$ *of open sets in* Ω_1 *covers* $\partial_{\Omega_1}\Omega$ *and* $V\subset R^{n-1}$ *is open for each* $\alpha\in A$.

2. *The mapping*

$$\Psi_\alpha:(0,2)\times V_\alpha\to U_\alpha \tag{1.5}$$

is co-ordinate function (this means that

$$\Psi_{\alpha_1}^{-1}\circ\Psi_{\alpha_2}:(0,2)\times V_{\alpha_2}\to(0,2)\times V_{\alpha_1} \tag{1.6}$$

is in C^∞ *for each* $\alpha_1,\alpha_2\in A$, $U_{\alpha_1}\cap U_{\alpha_2}\neq\emptyset$).

3. $\Psi_\alpha(1,V_\alpha)=U_\alpha\cap\partial_{\Omega_1}\Omega$ *for each* $\alpha\in A$.

4. *The restriction*

$$\Psi_\alpha:\big((0,1)\cup(1,2)\big)\times V_\alpha\to U_\alpha \tag{1.7}$$

is a C^∞ *mapping for each* $\alpha\in A$.

Theorem 1.1 is valid for arbitrary open set $\Omega\subset\Omega_1\subset R^n$ with orientable boundary $\partial_{\Omega_1}\Omega$ and open set Ω_1 ([11]). The following theorems play basic role in the present paper.

THEOREM 1.2: *If the boundary* $\partial_{\Omega_1}\Omega$ *of the generalized cylinder* Ω *satisfies the orientability conditions with universal* $\varepsilon(t_1)>0$, $\delta(t_1)>0$ *in the interval* $[t_0,t_1]\times R^n$ *for arbitrary* $t_0<t_1<\infty$ *and*

$$1>|f_0(p)|/\varepsilon(p) \tag{1.8}$$

holds for each $p\in\partial_{\Omega_1}\Omega$ *with* $\varepsilon(p)$ *given in definition 1.3, then an n-dimensional* C^∞ *manifold structure on the set* $\Omega_{t_0}:=\overline{\Omega}\cap\{t_0\}\times R^n$ *and a* C^∞ *homeomorphism* $\mathcal{H}:[t_0,\infty)\times\Omega_{t_0}\to\overline{\Omega}$ *exist with the following properties* ($\overline{\Omega}$ *is considered as a* C^∞ *manifold with boundary by theorem 1.1*):

1. *The boundary of* Ω_{t_0} *in* $\{t_0\}\times R^n$ *is an* $n-1$ *dimensional* C^∞ *submanifold.*

2. *The boundary* $\partial_{\Omega_1}\Omega$ *as an n-dimensional* C^∞ *submanifold is the image of the product manifold* $[t_0,\infty)\times\partial\Omega_{t_0}$ *with the map* $\mathcal{H}$.

3. *If* g *and* s_h *denote a smooth curve in* $\overline{\Omega}$ *with properties* $g(s_0)\in\partial_{\Omega_1}\Omega$, $g'(s_0)=\tau\big(g(s_0)\big)$ *and a* C^∞ *transformation of* $\overline{\Omega}$ *into itself given by*

$$s_h(u):=\mathcal{H}\big(e_t\cdot h+\mathcal{H}^{-1}(u)\big),\quad u\in\overline{\Omega} \tag{1.9}$$

for arbitrary $h\in R$, $h\geq 0$ *respectively, then*

$$\big[s_h\big(g(s_0)\big)\big]'=\tau\Big(s_h\big(g(s_0)\big)\Big) \tag{1.10}$$

holds.

THEOREM 1.3: *If a generalized cylinder* Ω *satisfying the conditions of theorem 1.2 has an orientation function* τ *orthogonal to* $e_t=(1,0)$ *at each point of* $\partial_{\Omega_1}\overline{\Omega}$ *then theorem 1.2 holds with a homeomorphism*

$$\mathcal{H}=:(\mathcal{H}_0,\mathcal{H}_1):[t_0,\infty)\times\Omega_{t_0}\to\Omega\subset R\times R^n$$

that satisfies

$$\mathcal{H}_0(t, p) = t, \quad (t, p) \in [t_0, \infty) \times \Omega_{t_0}. \tag{1.12}$$

Using the manifold structure given by theorem 1.1 on $\overline{\Omega}$ and $\partial_{\Omega_1}\Omega$ we can define spaces of differentiable functions as follows:

DEFINITION 1.5:

1. $C^k(\partial_{\Omega_1}\Omega, \tau) := \{u | u : \partial_{\Omega_1}\Omega \to R$ and u is continuously differentiable on the manifold k times.$\}$
2. $C^k(\overline{\Omega}, \tau) := \{u | u : \overline{\Omega} \to R$ and u is continuously differentiable on the manifold k times.$\}$
3. $C_0^\infty(\overline{\Omega}, \tau) := \left\{u | u \in \bigcap_{k=1}^{\infty} C^k(\dot{\Omega}, \tau) \text{ and } u \text{ has compact support.}\right\}$

In the definition 3.1 τ appears to emphasize that the differentiation is meant on the C^∞ manifold generated by τ.

With the help of a special partition of unity and the local co-ordinates an $n+1$ dimensional measure ν and an n-dimensional measure ϱ can be defined on $\overline{\Omega}$ and on the submanifold $\partial_{\Omega_1}\Omega$ resp. preserving the rule of partial integration for the derivatives in the sense of the manifold structure (see [11], **§ 3.**).

Using the measures ν and ϱ, families of Sobolev spaces $\{H^s(\Omega, \tau)\}_{s\in R}$ $\{H^s(\partial_{\Omega_1}\Omega, \tau)\}_{s\in R}$ can be defined on Ω and on the boundary $\partial_{\Omega_1}\Omega$ in a way following the one in chapter I. of [9] and **§ 3.** of [11]. The trace theorems are also valid for these Sobolev spaces ([11], **§ 3.**, section 2.).

Let $\varphi \in C^l(\partial_{\Omega_1}\Omega, \tau)$. Our aim is to give a representation of the semigroup $\{S^h\}_{h\in R^+}$ given in (1.9) by continuous linear injections of subspaces of Sobolev spaces defined by the boundary condition

$$\frac{\partial U}{\partial \tau} + \varphi \cdot u = 0. \tag{1.13}$$

In **§ 2.** we formulate the problem exactly and give the formulae of the infinitesimal generators of the representations.

In **§ 3.** we prove an existence theorem with the help of infinitesimal generators for mixed type boundary value problem.

§ 2. A representation of the semigroup $\{S_h\}_{h\in R^+}$

In this **§** we use the notations introduced in **§ 1.**

Let $\varphi \in C^l(\partial_{\Omega_1}\Omega, \tau)$ be a fixed bounded function with bounded first derivatives. $C^l_{0,\varphi}(\overline{\Omega}, \tau)$ denotes the function space given by

$$C^l_{0,\varphi}(\overline{\Omega}, \tau) := \left\{u \,\middle|\, \Phi(u) := \frac{\partial u}{\partial \tau} + \varphi \cdot u \Big|_{\partial_{\Omega_1}\Omega} = 0, \right. \tag{2.1}$$

$$\left. u|_{\Omega_{t_0}} = 0, \quad \frac{\partial u}{\partial t}\Big|_{\Omega_{t_0}} = 0, \quad u \in C^l(\Omega, \tau) \cap H^l(\Omega, \tau)\right\}.$$

The space $C^l_{0,\varphi}(\Omega, \tau)$ contains the functions of $C_0^\infty(\Omega)$ therefore it is non-empty and wide enough.

By the trace theorem (see [11], § 3. section 2.) a continuous linear mapping

$$(2.2)\qquad H^s(\Omega, \tau) \to \bigtimes_{i=0}^{k} H^{s-i-1/2}(\partial_{\Omega_1}\Omega, \tau)$$

$$(2.3)\qquad u \to \left\{\frac{\partial^i u}{\partial \tau'}\Big| i = 0, 1, \ldots, k\right\}, \quad u \in H^s(\Omega, \tau)$$

exists.

It is easy to prove using the mapping given in (2.3), that the boundary operator $\Phi: C^l(\overline{\Omega}, \tau) \to C^{l-1}(\partial_{\Omega_1}\Omega, \tau)$ defined in (2.1) can be extended to a continuous linear mapping from $H^1(\Omega, \tau)$ into $H^{-1/2}(\partial_{\Omega_1}\Omega, \tau)$. Denoting this extension also by Φ, the subspace $\Phi^{-1}(O) \subset H^1(\Omega, \tau)$ contains $C^1_{0,\varphi}(\partial_{\Omega_1}\Omega, \tau)$ and by the continuity of Φ it is closed.

Denoting by $H^1_{0,\varphi}(\Omega, \tau)$ the closure of $C^1_{0,\varphi}(\Omega, \tau)$ in $H^1(\Omega, \tau)$, the relation

$$(2.4)\qquad H^1_{0,\varphi}(\Omega, \tau) \subset \Phi^{-1}(O)$$

follows from the preceding considerations.

Our aim is to give a continuous bounded one-parameter semigroup $\{Q_h\}_{h\in R^+}$ of the continuous linear operators $Q_h: H^1_{0,\varphi}(\Omega, \tau) \to H^1_{0,\varphi}(\Omega, \tau)$ representing the semigroup $\{S_h\}_{h\in R^+}$.

First we define a family of continuous mappings

$$\{Q_h\}_{h\in R^+}, \quad Q_{0,h}: C^1_{0,\varphi}(\Omega, \tau) \to C^1(\Omega, \tau)$$

as follows:

$$(2.5)\quad Q_{0,h}(u)(t, z) := \begin{cases} u(t', z') & \text{if} \quad \exists (t', z') \in \overline{\Omega}, \quad (t, z) = S_h(t', z') \\ 0, & \text{if} \quad \nexists (t', z') \in \overline{\Omega} \quad \text{with} \quad (t, z) = S_h(t', z') \end{cases}$$

for $u \in C^0_{1,\varphi}(\Omega, \tau)$ and $(t, z) \in \Omega$.

As the function φ is not invariant with respect to the translations S_h, $h \in R^+$ it is evident that in general $Q_{0,h}(u) \notin C^1_{0,\varphi}(\Omega, \tau)$.

Now we shall prove the following lemma.

LEMMA 2.1: To the family of mappings $\{Q_{0,h}\}_{h\in R^+}$ given by (2.5) a family of functions $\{Q_{1,h}\}_{h\in R^+} \subset C^l(\Omega, \tau)$ can be found such that the linear operators $\{Q_h\}_{h\in R^+}$ given by

$$(2.6)\quad Q_h(u) := Q_{1,h}(t, x) \cdot Q_{0,h}(u)(t, x), \quad u \in C^1_{0,\varphi}(\Omega, \tau), \quad (t, x) \in \Omega$$

are continuous mappings of $C^1_{0,\varphi}(\Omega, \tau)$ into $C^1_{0,\varphi}(\Omega, \tau)$ for $h \in R^+$ and they form a continuous bounded one-parameter semigroup.

PROOF: The function φ can be extended as a constant function along the solutions of (1.4) in each co-ordinate neighbourhood U_α, $\alpha \in A$ given in theorem 1.1. Taking these extensions we have a well-defined extension $\tilde{\varphi} \in C^l$ of φ on the set $\bigcup_{\alpha\in A} U_\alpha$.

Let us formulate now the condition of the inclusion $Q_h(u)\in C^1_{0,\varphi}(\Omega, \tau)$ for arbitrary $u\in C^1_{0,\varphi}(\Omega, \tau)$. With the help of the solution $y(\cdot, s_0, z)$, $z\in\partial_{\Omega_1}\Omega$ of (1.4) we have the condition

$$(2.7)\quad \frac{\partial}{\partial s}\Big(Q_h(u)\left(y(s, s_0, z)\right)\Big)+\varphi\left(y(s, s_0, z)\right)\cdot Q_h(u)\left(y(s, s_0, z)\right)|_{s=s_0} = 0 .$$

Taking into account the property of S_h expressed by

$$(2.8)\qquad y(s_0, s_0, S_h(z)) = S_h\left(y(s_0, s_0, z)\right)$$

proved in [12] (proof of lemma 2.5) and the definitions (2.5) and (2.6), the condition (2.7) can be written into the form:

$$(2.9)\qquad \frac{\partial}{\partial s}\left\{Q_{1,h}\Big(S_h\left(y(s, s_0, z)\right)\Big)\cdot Q_{0,h}(u)\Big(S_h\left(y(s, s_0, z)\right)\Big)\right\}+$$

$$+\varphi\Big(S_h\left(y(s, s_0, z)\right)\Big)\cdot Q_{1,h}\Big(S_h\left(y(s, s_0, z)\right)\Big)\cdot Q_{0,h}(u)\Big(S_h\left(y(s, s_0, z)\right)\Big)\Big|_{s=s_0} = 0.$$

By the definitions of $Q_{0,h}$ and the function space $C^1_{0,\varphi}(\Omega, \tau)$ we gain the following relations:

$$(2.10)\qquad Q_{0,h}(0)\left(S_h(y)\right) = u(y), \quad y\in\overline{\Omega}$$

and

$$(2.11)\qquad \frac{\partial u}{\partial s}\left(y(s, s_0, z)\right)+\varphi\left(y(s, s_0, z)\right)\cdot u\left(y(s, s_0, z)\right)|_{s=s_0} = 0$$

for any $u\in C^1_{0,\varphi}(\Omega, \tau)$.

Writing (2.10) and (2.11) and the evident consequence $\varphi\left(y(s, s_0, z)\right) = Q_{0,h}(\overline{\varphi})\Big(S_h\left(y(s, s_0, z)\right)\Big)$ of (2.5) into (2.9) we have the condition

$$(2.12)\qquad \left\{\frac{\partial}{\partial s}\left[Q_{1,h}\Big(S_h\left(y(s, s_0, z)\right)\Big)\right]+Q_{1,h}\Big(S_h\left(y(s, s_0, z)\right)\Big)\times\right.$$

$$\left.\times\left[\overline{\varphi}\Big(S_h\left(y(s, s_0, z)\right)\Big)-Q_{0,h}(\varphi)\Big(S_h\left(y(s, s_0, z)\right)\Big)\right]\right\}\times u\left(y(s, s_0, z)\right)|_{s=s_0} = 0$$

equivalent to (2.9). As (2.12) is valid for arbitrary $u\in C^1_{0,\varphi}(\Omega, \tau)$ we gain the sufficient condition

$$(2.13)\qquad \frac{\partial}{\partial s}\, Q_{1,h}\Big(S_h\left(y(s, s_0, z)\right)\Big)+Q_{1,h}\left(y(s, s_0, z)\right)\times$$

$$\times\left[\overline{\varphi}\Big(S_h\left(y(s, s_0, z)\right)\Big)-Q_{0,h}(\varphi)\Big(S_h\left(y(s, s_0, z)\right)\Big)\right] = 0$$

for satisfying (2.7).

Considering the condition (2.13) as a parametric differential equation with the parameter $z\in\partial_{\Omega_1}\Omega$ its parametric solution has the form

$$(2.14)\quad Q_{1,h}\left(y(s, s_0, z)\right) = C(z, h)\cdot e^{-\int\limits_0^s \left[\overline{\varphi}\left(y(u, s_0, z)\right)-Q_{0,h}(\overline{\varphi})\left(y(u, s_0, z)\right)\right] du}$$

Let us define a function $\overline{\Phi}: \bigcup_{\alpha \in A} U_\alpha \to R$ by:

$$(2.15) \qquad \overline{\Phi}(w) := \int_0^s \overline{\varphi}(y(u, s_0, z))\, du, \quad w = y(s, s_0, z), \quad z \in \partial_{\Omega_1} \Omega$$

for $w \in \bigcap_{\alpha \in A} u_\alpha$.

From the boundedness of the function φ and that of its first derivatives it is easy to give a bounded function $\hat{\Phi} \in C^1(\Omega, \tau)$ with bounded first derivatives coinciding with $\overline{\Phi}$ on an open subset of Ω_1 containing $\partial_{\Omega_1} \Omega$. Denoting by L the C^1-norm of $\hat{\Phi}$, let us define the required function $Q_{1,h}$ as follows:

$$(2.16) \qquad Q_{1,h}(t, x) := e^{-L \cdot h} \cdot e^{-(\hat{\Phi}(t,x) - Q_{0,h}(\hat{\Phi})(t,x))}$$

if $(t, x) \in \Omega$, $h \in R^+$.

It is easy to check that

$$(2.17) \qquad Q_h(u) := Q_{1,h} \cdot Q_{0,h}(u) \in C^1_{0,\varphi}(\Omega, \tau)$$

for arbitrary $u \in C^1_{0,\varphi}(\Omega, \tau)$. Moreover the family $\{Q_h\}_{h \in R^+}$ is a continuous bounded semigroup of operators of $C^1_{0,\varphi}(\Omega, \tau)$. This proves our lemma.

It is easy to prove that the operators of the semigroup $\{Q_h\}_{h \in R^+}$ as $H^1_{0,\varphi}(\Omega, \tau) \to H^1_{0,\varphi}(\Omega, \tau)$ mappings are also continuous.

If we take the measure defined by means of the transformation $\mathcal{H}: [t_0, \infty) \times \Omega_{t_0} \to \Omega$ on Ω and equivalent to the Lebesgue measure then using the norm on $H_{0,\varphi}(\Omega, \tau)$ generated by this measure leads to the boundedness of the semigroup $\{Q_h\}_{h \in R^+}$.

The summary of the above considerations is the following theorem:

Theorem 2.1: *The family $\{Q_h\}_{h \in R^+}$ of continuous operators of $H^1_{0,\varphi}(\Omega, \tau)$ given by* (2.17) *is a continuous bounded one-parameter semigroup with the parameter $h \in R^+$.*

The infinitesimal generator of the semigroup has the form

$$(2.18) \qquad A u = \frac{\partial \mathcal{H}}{\partial t} \cdot \frac{\partial u}{\partial \mathcal{H}} + u\left(-L - \frac{\partial \hat{\Phi}}{\partial \mathcal{H}} \cdot \frac{\partial \mathcal{H}}{\partial t}\right), \quad u \in H^1_{0,\varphi}(\Omega, \tau)$$

in the case of theorem 1.2 and

$$(2.19) \qquad A u = \frac{\partial u}{\partial t} + u\left(-L - \frac{\partial \hat{\Phi}}{\partial t}\right) + \frac{\partial \mathcal{H}_1}{\partial t} \cdot \frac{\partial u}{\partial \mathcal{H}_1}, \quad u \in H^1_{0,\varphi}(\Omega, \tau)$$

in the case of theorem 1.3 with $\mathcal{H}_1$ given by $\mathcal{H} =: (\mathcal{H}_0, \mathcal{H}_1) \in R \times R^n$.

Proof: The formulae (2.18) and (2.19) are simple consequences of the definition of the semigroup $\{Q_h\}_{h \in R^+}$.

In the next § we prove existence theorem for a mixed type parabolic boundary value problem as applications for theorem 2.1.

§ 3. Existence theorem for a parabolic mixed type boundary value problem

In this § we follow the way given in chapter I. of [10]. As this § is intended to illustrate the use of the preceding considerations, we shall outline only the proofs with referring to the sources.

From now we suppose that the orientation function τ of the generalized cylinder Ω has values orthogonal to $(1, 0)\in R\times R^n$. This property implies the validity of (1.12) in theorem 1.3 and the infinitesimal generator of the semigroup $\{Q_h\}_{h\in R^+}$ given in theorem 2.1 has the form of (2.19).

Now we have to define some function spaces.

First of all let us decompose Ω into the disjoint union of the sets $\Omega_t = \Omega\cap\{t\}\times R^n$, $t\in(t_0, \infty) =: I$ as C^∞ manifolds generated by τ. $H^1_{0,\varphi}(\Omega_t, \tau)$ denotes the closed subspace of the Sobolev space $H^1(\Omega_t, \tau)$ defined the same way as $H^1_{0,\varphi}(\Omega, \tau)$ is given in § 2.

An element u of the product space $\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)$ is called square integrable if the positive function $\|u\|_{t,\varphi,\tau}:(t_0, \infty)\to R^+$ is measurable and square integrable with norm $\|\cdot\|_{t,\varphi,\tau}$ in $H^1_{0,\varphi}(\Omega_., \tau)$.

Now we shall choose a class from the elements of the product space $\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)$ as follows:

$$L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big) := \Big\{u \,|\, u\in \underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau), \text{ the mapping } t\to Q_t^{-1}u_t, \ (t_0, \infty)\to H^1_{0,\varphi}(\Omega_t, \tau) \text{ is measurable in strong sense and } u \text{ is square integrable}\Big\}. \tag{3.1}$$

(From the definition of Q_h in theorem 2.1 the existence of Q_h^{-1}, $h\in(t_0, \infty)$ follows obviously). It is easy to check that $L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big)$ is a Hilbert space with the scalar product

$$\int_{t_0}^{\infty} (u_t, v_t)_t\,dt =: [u, v] \tag{3.2}$$

with $(u_t, v_t)_t$ being scalar product in $H^1_{0,\varphi}(\Omega_t, \tau)$, $t\in(t_0, \infty)$ for $u, v\in L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big)$.

The simplest way for the definition of the Sobolev derivatives by parameter t in $L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big)$ is to consider the elements of $L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big)$ as functions $\overline{\Omega}\to R$ being in $L_2(\overline{\Omega}, \tau)$ by virtrue of definition (3.1) and to take the differential operator $\frac{d}{dt}$ in the sense of distributions on them. Selecting a class of elements from $L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau)\Big)$ fulfilling the condition

$$H^1\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t,\tau)\Big):=\Big\{u \,\Big|\, \frac{du}{dt}\circ\mathcal{H}\in L_2(\Omega,\tau) \quad u\in L_2\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t,\tau)\Big)\Big\}$$
(3.3)

it is easy to see with the help of Fubini's theorem that the space

$$H^1\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t,\tau)\Big)$$

coincides with $H^1_{0,\varphi}(\Omega,\tau)$ (Details can be found at theorem 1.3 in chapter I. of [9]).

Now we may turn our attention to the parabolic differential operators.

First we define a family of strictly positive definite bilinear forms giving a measurable function $t\to a(t;u(t),v(t))$, $t\in[t_0,\infty)$ for each pair $u,v\in H^1\Big(\underset{t\in I}{\times} H^1_{0,\varphi}(\Omega_t,\tau)\Big)$.

If A denotes a bounded measurable $n\times n$ matrix-function $\bar\Omega\to R^{n\times n}$ with the properties

$$(3.4)\qquad \begin{aligned}&(A(t,x)\xi,\xi)\geq\alpha\cdot\|\xi\|^2\\ &|(A(t,x)\xi,\eta)|\leq\bar\alpha\cdot\|\xi\|\cdot\|\eta\|\end{aligned}$$

for $\alpha,\bar\alpha>0$, $\xi,\eta\in R^n$ and C is a bounded measurable function $\Omega\to R$ then let the bilinear form $a(t;\cdot,\cdot)$ be given by

$$(3.5)\qquad a(t;u,v):=\int_{\Omega_t}\Big\{(A(t;\cdot)D_xu,D_xv)-\Big(\frac{\partial\mathcal{H}_1}{\partial t},D_xu\Big)v+$$
$$+\Big[C(t,\cdot)+L+\frac{\partial\hat\Phi}{\partial\mathcal{H}}\frac{\partial\mathcal{H}}{\partial t}\Big]u\cdot v\Big\}d\nu+\int_{\partial\Omega_t}\varphi(t,\cdot)u\cdot v\,d\varrho,\quad u\in H^1_{0,\varphi}(\Omega_t\tau)$$

with the functions $\mathcal{H}$, φ, Φ and the constant L given in the preceding **§**, the measures ν, ϱ given in **§ 1.** and the Sobolev derivation in $H^1(\Omega_t,\tau)$ denoted by D_x.

By virtue of the trace theorem the restriction $u|_{\partial\Omega_t}$ of any function $u\in H^1(\Omega_t,\tau)$ is in $L_2(\partial\Omega_t,\tau)$ and this mapping is continuous. Therefore considering the last term of (3.5), we obtain the trivial upper bound

$$(3.6)\qquad \int_{\partial\Omega_t}\varphi\cdot u\cdot v\,d\varrho\leq\|\varphi\|_{L_\infty}\cdot c_0\cdot\|u\|_{L_2(\partial\Omega_t,\tau)}\cdot\|v\|_{L_2(\partial\Omega_t,\tau)}\leq$$
$$\leq\|\varphi\|_{L_\infty}\cdot c\cdot\|u\|_{H^1(\Omega_t,\tau)}\cdot\|v\|_{H^1(\Omega_t,\tau)}$$

for $u,v\in H^1_{0,\varphi}(\Omega_t,\tau)$ and any $t\in[t_0,\infty)$.

On the other hand, once an interval $[t_0,T]$ is fixed the constant L given in the preceding **§** can be chosen so large that the inequality

$$(3.7)\qquad c(t,x)+L-\Big(\frac{\partial\hat\Phi}{\partial\mathcal{H}},\frac{\partial\mathcal{H}}{\partial t}\Big)(t,x)\geq(\varepsilon+c)\cdot\|\varphi\|_{L_\infty}\cdot\Big\|\frac{\partial\mathcal{H}}{\partial t}\Big\|_{L_\infty}$$

is satisfied for each $(t,x)\in\overline{\Omega}\cap[t_0,T]\times R^n$.

Now we formulate a condition for the positive definiteness of $a(t; \cdot, \cdot)$, $t \in [t_0, T]$ in the following lemma:

LEMMA 3.1: If the lower bound α of the matrix function A satisfies the inequality

$$\alpha > \frac{1}{2} \left\| \frac{\partial \mathcal{H}}{\partial t} \right\|_{L_\infty} + \|\varphi\|_{L_\infty} \cdot c \tag{3.8}$$

then the continuous bilinear form $a(t; \cdot, \cdot): H^1_{0,\varphi}(\Omega_t, \tau) \times H^1_{0,\varphi}(\Omega_t, \tau) \to R$ given in (3.5) is strictly positive definite, i.e.

$$a(t; u, u) \geq \alpha_0 \|u\|^2_{H^1_{0,\varphi}(\Omega_t, \tau)} \tag{3.9}$$

holds with a lower bound $\alpha_0 > 0$ for $u \in H^1_{0,\varphi}(\Omega_t, \tau)$ and $t \in [t_0, T]$.

Using the inequalities (3.6) and (3.7) the proof of the lemma follows evidently.

As the semigroup $\{Q_h\}_{h \in R^+}$ is bounded, the spectrum of Λ given in (2.19) is in R^+. Therefore Λ is a positive semidefinite operator. It is easy to see that the operator Λ can be considered as a continuous mapping

$$H^1 \Big(\underset{t \in I}{\times} H^1_{0,\varphi}(\Omega_t, \tau) \Big) \to L_2(\Omega, \tau).$$

Summarizing the above considerations, we obtain a strictly positive definite bilinear form

$$B: H^1 \Big(\underset{t \in (t_0, T)}{\times} H^1_{0,\varphi}(\Omega_t, \tau) \Big) \times H^1 \Big(\underset{t \in (t_0, T)}{\times} H^1_{0,\varphi}(\Omega_t, \tau) \Big) \to R$$

in the form

$$B(u, v) := \int_{t_0}^{T} a(t; u, v)\, dt + (\Lambda u, u) \tag{3.10}$$

for

$$u, v \in H^1 \Big(\underset{t \in (t_0, T)}{\times} H^1_{0,\varphi}(\Omega_t, \tau) \Big).$$

By virtue of the proof of theorem 1.1 in [9] the equation

$$B(u, v) = (f, v) \tag{3.11}$$

has a unique solution $u \in H^1 \Big(\underset{t \in (t_0, T)}{\times} H^1_{0,\varphi}(\Omega_t, \tau) \Big)$ for arbitrary fixed $f \in L_2(\Omega_{(t_0, T)}, \tau)$.

Let us suppose now that the matrix-function A and the solution u of (3.11) are continuously differentiable in the sense of the manifold defined on Ω. Then using the formulae of partial integration, we obtain the following form equivalent to (3.11):

$$\int_{\Omega_{(t_0, T)}} \left(\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \partial_i (A\, D_x u)_i + c \cdot u \right) v\, d\nu + \tag{3.12}$$

$$+ \int_{\partial \Omega_{(t_0, T)}} \{(A\, D_x u, \tau) + \varphi \cdot u\} v\, d\varrho = \int_{\Omega_{(t_0, T)}} f \cdot v\, d\nu.$$

If the function u satisfies the condition

$$(3.13) \qquad (D_x u, A^* \tau) + \varphi \cdot u = 0$$

then the solution u of (3.11) solves also the second order parabolic differential equation

$$(3.14) \qquad \frac{\partial u}{\partial \tau} - \sum_{i=1}^{n} \partial_i (A D_x u)_i + c \cdot u = f$$

with the boundary condition (3.13).

Therefore if both vectors $A^* \tau$ and τ point into Ω then taking the manifold generated by τ and the Sobolev space $H^1 \left(\underset{t \in I}{\times} H^1_{0,\varphi} (\Omega_t, A^* \tau) \right)$ generated by $A^* \tau$ and applying the existence theorem outlined above on

$$H^1 \left(\underset{t \in I}{\times} H^1_{0,\varphi} (\Omega_t, A^* \tau) \right)$$

instead of

$$H^1 \left(\underset{t \in I}{\times} H^1_{0,\varphi} (\Omega_t, \tau) \right)$$

we get the following theorem:

THEOREM 3.1: *If*

(a) *the mapping A is bounded continuously differentiable strictly positive definite matrix function $A : \overline{\Omega} \to R^{n \times n}$,*

(b) *the function $C : \overline{\Omega} \to R$ satisfies the conditions of lemma* 3.1,

(c) *both vector functions τ and $A^* \tau$ point into Ω and*

(d) *$\varphi \in C^I (\partial_\Omega, \Omega, \tau)$ and its first derivatives are both bounded*

then the parabolic equation

$$(3.15) \qquad \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \partial_i (A D_x u)_i + c \cdot u = f$$

has a unique solution $u \in H^1 \left(\underset{t \in (t_0, T)}{\times} H^1_{0,\varphi} (\Omega_t, A^* \tau) \right)$ in the sense expressed by (3.11) *for arbitrary $f \in L_2 (\overline{\Omega}, \tau)$ and $[t_0\ T] \subset [t_0, \infty)$.*

This solution satisfies the boundary condition (3.13) also in the generalized sense.

The condition imposed on τ and $A^* \tau$ holds evidently in the case of smooth boundaries for any positive definite matrix function $A : \overline{\Omega} \to R^{n \times n}$.

Our aim with the choice of this example to illustrate our results was to show the effect of the non-smoothness of the boundary on the solvability of the parabolic boundary value problem stated in (3.15) and (3.13).

The methods used in [9], [10] for the investigation of the smoothness of the solutions can also be applied as it is shown in [11].

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PSEUDOCOMPACT EXTENSIONS

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0. Introduction

The theory of extensions is one of the largest and richest part of general topology. The theory of compact and realcompact extensions has been examined especially minutiously.

Our paper deals with pseudocompact extensions. We introduce the notion of pseudocompactification and will look for spaces which have a pseudocompactification. A type of pseudocompactification will be constructed to each completely regular T_1-space. The notion of pseudocompactificational classes of functions will be introduced and we shall establish a necessary and sufficient condition for a class of functions to be pseudocompactificational. Using these tools, each pseudocompactification of a completely regular T_1-space will be constructed. Finally, we shall examine the following problem: what kinds of spaces have a maximal pseudocompactification?

Throughout this paper "space" will mean a completely regular T_1-space. The classes of the continuous and the bounded continuous functions of a space X will be denoted by $C(X)$ and $C^*(X)$, respectively.

1. Pseudocompactifications

1.1 Definition. A pseudocompact extension Y of the space X is called a *pseudocompactification* of X.

Since the Čech-Stone compactification is pseudocompact, it is clear that every topological space has at least one pseudocompactification.

1.2 Definition. Let X be a topological space, $f\in C(X)$. f can be extended to a function $f^*:\beta X\to\mathbf{R}^*$, where $\mathbf{R}^*=\mathbf{R}\cup\{\infty\}$ is the Alexandroff compactification of $\mathbf{R}$. The point $x\in\beta X$ is *infinite* if there is an $f\in C(X)$ such that $f^*(x)=\infty$. If $x\in\beta X$ is not infinite, it is *finite*.

It is clear that every $x\in X$ is finite in βX.

The following characterizations of realcompact and pseudocompact spaces are well-known:

1.3 THEOREM. *X is pseudocompact iff every $x \in \beta X$ is finite.*
X is realcompact iff every $x \in \beta X - X$ is infinite.

Now we can easily construct a pseudocompactification which is not compact.

1.4 THEOREM. *Let X be a topological space and*

$$\pi X = \{x \in \beta X : x \text{ is infinite}\} \cup X .$$

πX is a pseudocompactification of X and πX is compact iff X is realcompact.

PROOF. $X \subset \pi X \subset \beta X$, so X is dense in πX and $\beta \pi X = \beta X$. Consider $f \in C(\pi X)$, then $f|X \in C(X)$. $f|X$ can be extended to βX, let this function be f^*. Since the extension is unique so $f^*|\pi X = f$, i.e. f^* is an extension of f, too. For a point $x \in \beta X - \pi X$, we have $f^*(x) \in \mathbf{R}$, since $f^*(x) = \infty$ implies $x \in \pi X$. This means that every point of $\beta X - \pi X = \beta \pi X - \pi X$ is finite, thus, πX is pseudocompact.

Let πX be compact. Then, of course, $\pi X = \beta X$, so every point $x \in \beta X - X$ is infinite, that is, X is realcompact. On the other hand, if X is realcompact, then, of course, $\pi X = \beta X$, i.e. πX is compact.

1.5 COROLLARY. Every non-realcompact space has a non-compact pseudocompactification, e.g. πX.

1.6. DEFINITION. The class Φ of functions on X is *(pseudo)-compactificational* if there is a (pseudo)-compactification Y of X such that $C(Y)|X = \Phi$.

The following theorem, characterizing the compactificational classes of functions is well-known. (See [2], 1.4, 1.10)

1.7 THEOREM. *The class Φ of functions on X is compactificational iff*

(a) $\Phi \subset C^*(X)$,

(b) *Φ is closed under pointwise addition, multiplication and contains the constant functions,*

(c) *Φ is closed under uniform convergence,*

(d) *$\mathfrak{Z}(\Phi) = \{f^{-1}(\{0\}) : f \in \Phi\}$ forms a closed base in X.*

1.8 THEOREM. *Φ is a compactificational class of functions iff it is a pseudocompactificational one.*

PROOF. If Φ is a compactificational class then it is pseudocompactificational too, since a compactification is also a pseudocompactification.

If Φ is a pseudocompactificational class then there exists a pseudocompactification Y such that $\Phi = C(Y)|X$. In this case, conditions (a), (b), (c) and (d) of Theorem 1.7 are satisfied, consequently Φ is a compactificational class of functions, too.

1.9 Definition. Let X be a topological space, Y and Y' two pseudocompactifications of X. Y is *finer* than Y' if there is a map $q : Y \to Y'$ which is onto and keeps the points of X fixed.

1.10 Theorem. *Let Y and Y' be two pseudocompactifications of the topological space X, Φ and Φ' the corresponding pseudocompactificational classes of functions and suppose that Y is finer than Y'. Then $\Phi \supset \Phi'$.*

Proof. If $q : Y \to Y'$ is an onto map which keeps the points of X fixed and $f \in \Phi'$, then there is a $g \in C(Y')$ such that $g|X = f$. Then, clearly $g \circ q \in C(Y)$ and $g \circ q|X = f$ since q fixes the points of X.

1.11 Remark. It is easy to see that if Y and Y' are compactifications, the converse of the theorem is also true, i.e., if $\Phi \supset \Phi'$ then Y is finer than Y'. This means that two compactifications of X are equivalent (in the usual sense, i.e., between them there is a homeomorphism which keeps the points of X fixed), if and only if the two compactificational classes of functions are the same. The fact that this converse fails to hold for pseudocompactifications is shown by βX and πX; both of them are pseudocompactifications of X and $\Phi = C^*(X)$ for each case, but, of course, for a non-realcompact X, neither of them is finer than the other one.

1.12 Theorem. *Let X be a topological space, Φ a pseudocompactificational class of functions, Y a pseudocompactification belonging to Φ. Then every $Y \subset Z \subset \beta Y$ is a pseudocompactification of X belonging to Φ and βY is the compactification belonging to Φ. Hence every pseudocompactification belonging to Φ is a subspace of the compactification belonging to Φ.*

Proof. If Y is a pseudocompactification of X belonging to Φ, then $\Phi = C(Y)|X$. Every member of $C(Y)$ is bounded, hence it can be extended to βY, so every member of Φ can be extended to βY.

Thus, if $Y \subset Z \subset \beta Y$, then every element of Φ can be extended to Z (in fact, they can be extended even to βY), but no function of $C(X) - \Phi$ can be extended to Z (not even to Y). On the other hand Z is an extension of Y and Z is pseudocompact since the pseudocompact subspace Y is dense in it.

Applying this result to $Z = \beta Y$, we can see that βY is a pseudocompactification belonging to Φ too. On the other hand βY is also a compactification, thus, according to the Remark above, βY is the compactification of X belonging to Φ. So the proof is complete.

Applying 1.8 and 1.12, we can classify the pseudocompactifications of a space X according to the pseudocompactificational classes of functions which are characterized by 1.7 and 1.8. We have seen that if qX is the compactification belonging to the class Φ, then, for every pseudocompactification Y belonging to Φ, we have $X \subset Y \subset qX$. To find every pseudocompactification of X it is enough to prove:

1.13 Theorem. *Let X be a topological space, Φ a compactificational class of functions on X, qX the compactification belonging to Φ. For arbitrary $f \in C(X)$, let us consider*

$$A_f = \{x \in \varphi X : f(\mathfrak{W}(x)(\cap)\{X\}) \text{ does not converge in } \mathbf{R}\}$$

where $\mathfrak{W}(x)$ *is the neighbourhood filter of* x *in* φX. *It is clear that* $A_f = \emptyset$ *iff* $f \in \Phi$. *Let us denote*

$$\sigma = \{S : S \cap A_f \neq \emptyset, \quad \forall f \in C(x) - \Phi\}.$$

Y *is a pseudocompactification of* X *belonging to* Φ *iff there is an* $S \in \sigma$ *such that*

$$X \cup S = Y \subset \varphi X.$$

PROOF. Necessity: Let Y be a pseudocompactification of X belonging to Φ and $f \in C(X)$ arbitrary. If $A_f \cap (Y - X) = \emptyset$, then $f(\mathfrak{W}(x)(\cap)\{X\})$ is convergent in $\mathbf{R}$ for every $x \in Y$, thus f can be extended to Y because of the regularity of $\mathbf{R}$. Hence $f \in \Phi$. This shows that $Y - X \in \sigma$.

Sufficiency: If $S \in \sigma$ then $X \subset S \cup X = Y \subset \varphi X$, i.e., Y is an extension of X. Every function of Φ can be extended to Y (in fact, to φX). If $f \in C(X) - \Phi$, then $A_f \cap S \neq \emptyset$, that is, there exists a point $x \in S$ such that $f(\mathfrak{W}(x)(\cap)\{X\})$ is not convergent. Hence, f cannot be extended to Y. So the functions that can be extended to Y are exactly those which are in Φ. On the other hand, the members of Φ are bounded, i.e., every continuous function on Y is bounded. Hence Y is pseudocompact.

2. Maximal pseudocompactifications

As well-known, βX is the finest compactification of X (in the sense of 1.9). Evidently the question arises whether there is always a finest pseudocompactification or not.

It is easy to see, the answer is no. In fact none of the pseudocompactifications of X can be finer than βX, except βX itself. Suppose that the pseudocompactification Y of X is finer than βX, i.e., there is a mapping $\varphi : Y \to \beta X$ onto and fixing the points of X. βX is the finest compactification of X and βY is also a compactification of X, hence there is a continuous map $\psi : \beta X \to \beta Y$ which keeps the points of X fixed. $\chi = \psi \circ \varphi : Y \to \beta Y$ is a map onto βY which fixes the points of X. This means that χ is an extension of $\mathrm{id}_X : X \to \beta Y$ to Y, but then, because of the uniqueness of the extension of an arbitrary function into βY, $\chi = \mathrm{id}_Y$, thus it cannot be onto unless $Y = \beta Y$ and $Y = \beta X$. Consequently there is no pseudocompactification finer than βX, except βX itself.

On the other hand, it is clear that βX itself is not a maximal pseudocompactification, unless if every pseudocompactification of X is compact.

Let us then modify the question to read: For every pseudocompactification Y of X is there a pseudocompactification Y' which is finer than Y and $X \subset Y' \subset \beta X$?

Let X be a topological space and $X \subset Y$ a pseudocompactification of X. βY is a compactification of X so $\mathrm{id}_X : X \to \beta Y$ can be extended to βX. Let us denote the extension by $\varphi : \beta X \to \beta Y$. φ is unique, since βY is a T_2-

space. Let us consider $\varphi^{-1}(Y)$. If there is a pseudocompactification $X\subset Y'\subset\subset\beta X$ such that there exists an onto map $\psi: Y'\to Y$ fixing the points of X then, because of the uniqueness of extensions, $\psi=\varphi|Y'$. Hence $Y'\subset\varphi^{-1}(Y)$ i.e., $\varphi^{-1}(Y)$ is a pseudocompactification of X, $X\subset\varphi^{-1}(Y)\subset\beta X$ and $\varphi(\varphi^{-1}(Y))=Y$. Thus, we have proved the following

2.1 THEOREM. *Let Y be a pseudocompactification of X, $\varphi:\beta X\to\beta Y$ the extension of $\mathrm{id}_X: X\to\beta Y$. There is a pseudocompactification Y' of X such that Y' is finer than Y and $X\subset Y'\subset\beta X$ iff $\varphi^{-1}(Y)$ is pseudocompact. In this case, $\varphi^{-1}(Y)$ contains every Y' which satisfies the condition above.*

The following two lemmas are well-known:

2.2 LEMMA. The four statements below are equivalent for an arbitrary topological space X:

(a) X is open in every T_2-extension;

(b) X is open in every compactification;

(c) X is open in βX;

(d) X is locally compact.

(a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) is obvious, for (d)$\Rightarrow$(a), see e.g. [3], 3.3.9.

2.3 LEMMA. In order that a space X be pseudocompact, it is necessary and sufficient that, for any decreasing sequence $\{V_n\}_{n\in\mathbf{N}}$ of non-empty open sets, $\bigcap_{n\in\mathbf{N}}\mathrm{cl}\,V_n$ be non-empty. See [4], 9.13.

2.4 THEOREM. *Let X be a locally compact space, Y a pseudocompactification of X, $\varphi:\beta X\to\beta Y$ the extension of $\mathrm{id}_X: X\to\beta Y$ to βX. Then $\varphi^{-1}(Y)$ is pseudocompact.*

PROOF. Let $\emptyset\neq V\subset\varphi^{-1}(Y)$ be open in $\varphi^{-1}(Y)$. X is dense in βX, hence in $\varphi^{-1}(Y)$ too, that is $V'=V\cap X\neq\emptyset$ and V' is open in X. βY is a compactification of the locally compact space X, thus $X\subset\beta Y$ is open, i.e., $V'\subset X$ is open in βY and also in Y. Consequently $\emptyset\neq V'\subset\varphi(V)$, since φ is the extension of id_X. Hence we have proved that, for every open set $\emptyset\neq V\subset\varphi^{-1}(Y)$, $\mathrm{int}_Y\,\varphi(V)\neq\emptyset$.

Let $\emptyset\neq V_i\subset\varphi^{-1}(Y)$ be open, $V_{i+1}\subset V_i$ $(i\in\mathbf{N})$. This is a filter base. Let us consider the filter σ generated by it. Let us denote $V_i^*=\mathrm{int}_Y(\varphi(V_i))$. Then $\emptyset\neq V_i^*\subset Y$ are open, and $V_{i+1}^*\subset V_i^*$ $(i\in\mathbf{N})$. Y is pseudocompact, hence there exists $y\in Y$ such that $y\in\bigcap_{i\in\mathbf{N}}\mathrm{cl}_Y\,V_i^*$.

If $S^*\in\varphi(\sigma)$, there is an $S\in\sigma$ such that $\varphi(S)=S^*$. By $S\in\sigma$ there is an $i\in\mathbf{N}$ such that $V_i\subset S$. Hence $V_i^*\subset\varphi(V_i)\subset\varphi(S)=S^*$, i.e. $y\in\mathrm{cl}\,S^*$. This means that y is an accumulation point of the filter base $\varphi(\sigma)$. φ is a perfect map (since it is a map from a compact space to a Hausdorff space), thus there exists an accumulation point $x\in\varphi^{-1}(y)\subset\varphi^{-1}(Y)$ of σ, i.e. of $\{V_i\}_{i\in\mathbf{N}}$ (see e.g. [1] I. 10.2. 1 Theorem.) Hence $\varphi^{-1}(Y)$ is pseudocompact.

2.5 COROLLARY. If X is a locally compact space, then for every pseudocompactification Y of X there exists a pseudocompactification Y' such that $X \subset Y' \subset \beta X$ and Y' is finer than Y.

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ON THE CHARACTERIZATION OF ADDITIVE FUNCTIONS

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Throughout this paper f denotes an additive arithmetical function and let

$$g(n) = \max\{f(n), \ldots, f(n+k-1)\}$$

with an arbitrary but fixed k.

P. ERDŐS proved in [2], that if f is monotone, then $f = c \cdot \log n$.

J. BIRCH [1] and I. KÁTAI [4] showed that the same holds, if f is monotone on a set having upper density one.

A. IVÁNYI proved in [3] that if g is strictly monotone (increasing) with $k = 2$, then $f = c \cdot \log n$, but if g is only monotone, then this is not true.

The purpose of this paper is to generalize these results in two directions. We can summarize our results as follows:

1. If g is a strictly monotone increasing function with an arbitrary fixed k on a set having upper density one, then $f = c \cdot \log n$.

2. If g is convergent, then g is a constant-function, which is non-negative.

We examine the functions f and g — if g is monotone — in general too.

1. I. KÁTAI raised the following problem in [4]: Let the function

$$g(n) := \max\{f(n),\ f(n+1)\}$$

be a monotone increasing function. What can we say about the function f?

A. IVÁNYI showed an example, when g is monotone increasing, while f fluctuates:

$$f(p^\alpha) = \begin{cases} 1 & \text{if} \quad p = 2 \\ 0 & \text{if} \quad p \neq 2. \end{cases} \tag{1.1}$$

If we further require f to be completely additive, we can give the following example:

$$f(p)=\begin{cases}-1 & \text{if} \quad p=2\\ 0 & \text{if} \quad p\neq 2.\end{cases} \tag{1.2}$$

A. IVÁNYI proved in [3], that if g is strictly monotone increasing with $k=2$, then $f=c\cdot\log n$. This theorem is true, even if the assumption holds only on a set having upper density one and k is an arbitrary natural number:

THEOREM 1. *If g — with an arbitrary fixed k — is a strictly monotone increasing function on a set having upper density one, then $f=c\cdot\log n$.*

PROOF. Let $k=2$. We denote by A our set having upper density one. First we show, that if

$$a\in A \quad \text{and} \quad a-1\in A, \quad \text{then} \quad g(a)=f(a+1).$$

Indeed, $g(a-1)<g(a)$ means

$$f(a-1)<f(a+1) \quad \text{or} \quad f(a)<f(a+1),$$

consequently we have $g(a)=f(a+1)$.

Consider the set $A'=\{a\in A\,|\,a-1\in A \text{ and } a-2\in A\}$. For $a\in A'$ we have

$$f(a)=g(a-1)<g(a)=f(a+1). \tag{1.3}$$

Now we prove, that f is monotone on A'. Let a and a_1 denote two elements of A' and $a<a_1$. Since g is strictly monotone increasing on A, and (1.3) is valid, thus

$$f(a)<g(a)\leq g(a_1-1)=f(a_1),$$

consequently $f(a)<f(a_1)$.

It is easy to see, that if A is a set having upper density one, then so is A' too, and so we can use the above-mentioned result of KÁTAI [4] and BIRCH [1]:

If an additive arithmetical function is monotone on a set having upper density one, then $f=c\cdot\log n$.

For $k<2$, the proof is similar. ∎

2. Instead of the monotonity of the function g, let us examine the convergence of g.

We prove the following theorem:

THEOREM 2. *If $\lim\limits_{n\to\infty} g(n)=c$, then $c\geq 0$ and $g=c$.*

PROOF. (a) g cannot converge to a negative number c.

If g converges, then g and f have an upper bound, say $d>0$.

Moreover, to an arbitrary $\varepsilon>0$, we can choose an n_0 such that for $n\geq n_0$

$$|g(n)-c|<\varepsilon$$

and

$$f(n) < c + \varepsilon .$$

Consider the following system of congruences:

$$(2.1)\quad x+i \equiv p_{(i-1)r+1} \dots p_{ir} \bmod (p_{(i-1)r+1} \dots p_{ir})^2 \quad (i = 1, \dots, k),$$

where the p_j are different primes, $f(p_j) < c+\varepsilon$, r is a fixed integer and $\varepsilon < -\frac{c}{2}$.

Then

$$(2.2)\qquad f(x+i) = f[p_{(i-1)r+1} \dots p_{ir}(1+sp_{(i-1)r+1} \dots p_{ir})] =$$
$$= f(p_{(i-1)r+1}) + \dots + f(p_{ir}) + f(1+s \dots) < rc + r\varepsilon + d <$$
$$< \frac{rc}{2} + d < 2c \quad (i = 1, \dots, k)$$

if r is large enough.

So $g(x+1) < 2c$, consequently g cannot converge to $c < 0$.

(b) g cannot converge to 0 unless $g = 0$.

If there exists a t for which $f(t) = a > 0$, then f cannot be positive or 0 on an infinite set of numbers coprime with t, otherwise $f(n) \geq a$ would be valid on an infinite set. Consequently, f can be positive only either (i) on infinitely many powers of a fixed prime, or (ii) on finitely many powers of finitely many primes. Moreover f can be 0 only on finitely many powers of finitely many primes, and in the case (i) on infinitely many powers of the given prime too.

Consequently, f must be negative on all other prime-powers.

If $f(q_i) < -\varepsilon$ for infinitely many prime-powers q_i of different primes with a positive ε, then we can find an infinite set, where g is less than $-d$ (d was an upper bound of f).

Indeed if we look at the (2.1) system of congruences, and instead of the primes p_i we put our prime-powers q_i, then instead of (2.2) we can assert

$$f(x+i) < -r\varepsilon + d < -d \quad (i = 1, \dots, k),$$

if r is large enough. So $g(x+1) < -d$ on an infinite set, which is impossible, if $\lim g = 0$.

Consequently, to an arbitrary $\varepsilon > 0$

$$-\varepsilon < f(q_i) < 0$$

except the powers of finitely many primes.

Now if we assume $f(t) > 0$ for some t, then — taking $\varepsilon = \frac{f(t)}{2}$ — we have for almost all prime-powers q_i $-\frac{f(t)}{2} < f(q_i) < 0$, and consequently $f(tq_i) > \frac{f(t)}{2}$. This contradicts $g(n) \to 0$. Thus $f \leq 0$, and so $g \leq 0$ too.

But g cannot be negative either. If $g(x_0)<0$ for some x_0, then we can construct an infinite set A, on which $g(x)\le g(x_0)$ $(x\in A)$.

Consider the solutions of the congruence

$$x\equiv x_0 \mod (x_0\dots(x_0+k-1))^2. \tag{2.3}$$

Then

$$x+i=(x_0+i)\bigl(1+tx_0^2\dots(x_0+i)\dots(x_0+k-1)^2\bigr)\quad (i=0,\dots,k-1)$$

and so

$$f(x+i)=f(x_0+i)+f(1+t\dots)\le f(x_0+i), \tag{2.4}$$

because $f\le 0$. So $g(x)\le g(x_0)$ too, which is impossible, if $\lim g=0$. We proved, that $g=0$.

We also obtained, that f cannot be negative on the powers of more than $k-1$ primes, and on all other prime-powers f must be 0. f can be negative on the powers of $k-1$ primes, if the primes are larger than k: we can always find a number z among k adjacent numbers, which is relatively prime to the given primes on which f is negative, so $f(z)=0$ gives the maximum on all k adjacent numbers. We can say more too: If $q_1<\dots<q_r<k<q_{r+1}<$ $<\dots<q_s$, where q_i are the powers of different primes and $f(q_i)<0$, then $s<k-r$.

To prove this we look at the system of congruences

$$x+j\equiv q_{r-j} \mod q_{r-j}^2 \quad (j=0,\dots,r-1)$$

and let $q_1>2$.

Then

$$q_{r-j}\|x+j \quad\text{and}\quad q_{r-j}\|x+j+q_{r-j} \quad (j=0,\dots,r-1).$$

Because of $q_{j+1}-q_j\ge 2$ we have

$$x<x+1<\dots<x+r-1<x+r-1+q_1<$$
$$<x+r-2+q_2<\dots<x+1+q_{r-1}<x+q_r<x+k.$$

So

$$f(x+j)=f(q_{r-j}\,y)=f(q_{r-j})+f(y)<0$$

and

$$f(x+j+q_{r-j})<0 \quad\text{too}\quad (j=0,\dots,r-1),$$

i.e. f is negative on these $2r$ numbers.

Consider further $x+u_t\equiv q_t \mod q_t^2$ $(t=r+1,\dots,s)$, where $u_t=r$, $r+1,\dots$, but $u_t\ne j+q_{r-j}$ for any j. If $s\ge k-r$, then $g(x)=\max\{f(x+i)\mid i=$ $=0,\dots,k-1\}$ is negative, which contradicts $g=0$.

If $q_1=2$ let $x\equiv 2 \mod 4$. Then $2\|x+4\nu$ $\nu=0,\dots,\left[\dfrac{k}{4}\right]$, and so $f(x+4\nu)<0$. Thus f cannot be negative on more than $k-1-\left(\left[\dfrac{k}{4}\right]+1\right)$ powers of other different primes, i.e. $s<k-\left[\dfrac{k}{4}\right]$ is true too.

(c) g cannot converge to a positive number c unless $g = c$.

First we show, that c is an upper bound of the function f. If there is a number b for which

$$f(b) = c+\delta$$

with a positive number δ, then f cannot be positive or 0 on an infinite set of numbers q_i, which are relatively prime to b, since

$$f(bq_i) \geq c+\delta$$

would hold, which is impossible, if $\lim\limits_{n\to+\infty} g(n) = c$.

So f is negative on the powers of infinitely many primes. Similarly to (b), we can prove, that to an arbitrary $\varepsilon > 0$, there are infinitely many prime-powers q_i (of different primes), for which

$$-\varepsilon < f(q_i) < 0 .$$

If we choose $\varepsilon := \dfrac{\delta}{2}$, then

$$f(bq_i) = f(b)+f(q_i) > c+\delta-\frac{\delta}{2} > c+\frac{\delta}{2} > c$$

with infinitely many q_i, which is impossible. Thus we proved, that c is an upper bound of f.

Fix an arbitrary $\varepsilon < \dfrac{c}{2}$. Look at the numbers z_i, which give the maximum (i.e. the values of g). Because of $g(n) \to c$, there is an n_0 to an arbitrary $\varepsilon > 0$ such, that if $z_i > n_0$, then

$$f(z_i) > c-\varepsilon .$$

Look at the possible values of (z_i, z_{i+1}). Because of $z_{i+1}-z_i < k$, there is a natural number u, such that

$$(z_i, z_{i+1}) = u$$

occurs infinitely often. So

$$c \geq f([z_i, z_{i+1}]) = f(z_i)+f(z_{i+1})-f(u) \geq 2c-2\varepsilon-f(u) ,$$

i.e. $f(u) \geq c-2\varepsilon$ with an arbitrary $\varepsilon > 0$. Consequently $f(u) = c$.

For $(w, u) = 1$

$$f(wu) = f(w)+f(u) = f(w)+c \leq c ,$$

i.e. $f(w) \leq 0$.

Now we are ready to prove $g = c$. Assume indirectly, that there exists a number x_0, with $g(x_0) = c_0 < c$, and look at the congruence (2.3). If we choose $t = ut'$, then because of $f(1+ut') \leq 0$, (2.4) is true. So $g(x) \leq g(x_0) = c_0 < c$ on an infinite set, which contradicts $\lim\limits_{n\to+\infty} g(n) = c$.

Let us examine what can we say about f in this case. Let ν denote the least number, where $f(\nu)=c$. Let

$$\nu = p_1^{\alpha_1} \dots p_s^{\alpha_s}.$$

Clearly $f(p^\alpha) \leq 0$, if $p \neq p_i$ $(i=1, \dots, s)$.

For $p = p_i$

$$f\left(\frac{\nu}{p_i^{\alpha_i}} \cdot p_i^{\beta_i}\right) \leq f(\nu) = c$$

implies

$$f(p_i^{\beta_i}) \leq f(p_i^{\alpha_i}) \quad \text{for all} \quad \beta_i,$$

moreover

$$f(p_i^{\beta_i}) < f(p_i^{\alpha_i}) \quad \text{if} \quad \beta_i < \alpha_i$$

(ν is the least number, where $f(\nu)=c$). So the maximums come only from the multiples of ν.

We can also prove, that f cannot be negative on the powers of more than $\left[\frac{k}{\nu}\right]-1$ primes different from p_i.

Since the multiples of ν give the maximums, so f must be c on least one of the multiples of ν among any adjacent k numbers. If $x = 1 + j\nu$, then

$$g(x) = f(x + m\nu - 1) = f(j'\nu)$$

with $j' = j+1$ or $j+2 \dots$ or $j+\left[\frac{k}{\nu}\right]$.

If $f(q_t) < 0$ with powers of different primes different from p_i,

$$\left(t = 1, \dots, \left[\frac{k}{\nu}\right]\right)$$

then we can choose

$$j + t \equiv q_t \bmod q_t^2 \quad \left(t = 1, \dots, \left[\frac{k}{\nu}\right]\right),$$

and so $f[(j+t)\nu] = f(q_t s_t) = f(q_t) + f(s_t) \leq f(q_t) + c < c$. So $g(x) < c$, and this contradicts $g = c$. So f cannot be negative on the powers of more than $\left[\frac{k}{\nu}\right]-1$ primes different from p_i.

3. Let us examine now, what can we say about the original problem of KÁTAI: What is f like, when g is monotone?

(A) If g is strictly monotone increasing, then according to Theorem 1 $f = c \cdot \log n$.

(B) If g is monotone increasing and bounded, then it is convergent too, and so according to Theorem 2 $g = c$. During the proof of Theorem 2, we received several informations on f. These can be summarized as follows:

If $c = 0$, then $f(p^\alpha) = 0$ except the powers of at most $k-r-1$ primes, on which f can be negative (r denotes the number of the prime-powers smaller than k of different primes, on which f is negative).

If $c > 0$ and ν is the least number for which $g(\nu) = c$, then f cannot be negative on the powers of more than $\left[\frac{k}{\nu}\right] - 1$ primes coprime with ν.

Examples were given by (1.1) and (1.2) for $c = 1$ and $c = 0$ respectively. These can be generalized as follows

$$f(p^\alpha) = \begin{cases} c > 0 & \text{if} \quad p = p_0 < k \\ 0 & \text{if} \quad p \neq p_0 , \end{cases} \tag{3.1}$$

$$f(p) = \begin{cases} c_1 < 0 & \text{if} \quad p = p_0 \\ 0 & \text{if} \quad p \neq p_0 . \end{cases} \tag{3.2}$$

(This example is completely additive.)

We can show such an example too, where f is positive on an arbitrary finite set of primes, if k is large enough:

$$f(p^\alpha) = \begin{cases} c_t > 0 & \text{if} \quad p = p_t \quad (t = 1, \ldots, s) \\ 0 & \text{if} \quad p \neq p_t . \end{cases} \tag{3.3}$$

and $k \geq p_0 \ldots p_s$.

(C) Finally, let us examine the case, when g is monotone increasing (not strictly) and g is unbounded. We can show examples, where $f \neq c \cdot \log n$.

$$f(p) = \begin{cases} 0 & \text{if} \quad p = 2 \\ \log p & \text{if} \quad p \neq 2 . \end{cases} \tag{3.4}$$

Then

$$g(n) = \begin{cases} \log(n+k-1) & \text{if} \quad 2 \nmid n+k-1 \\ \log(n+k-2) & \text{if} \quad 2 \mid n+k-1 . \end{cases}$$

This example is completely additive and suitable to an arbitrary k.

Another example for $k \geq 3$:

$$f(p) = \begin{cases} \log 2 & \text{if} \quad p = 3 \\ \log p & \text{if} \quad p \neq 3 . \end{cases} \tag{3.5}$$

Then

$$g(n) = \begin{cases} \log(n+k-1) & \text{if} \quad 3 \nmid n+k-1 \\ \log(n+k-2) & \text{if} \quad 3 \mid n+k-1 . \end{cases}$$

Let us examine now the functions g which satisfy these conditions, i.e. which are not strictly monotone and are unbounded.

Because g isn't strictly monotone, g has identical values too. Then g has identical values infinitely often too. In the opposite case, g would be strictly monotone increasing from a number on, and then f must be a same kind of function i.e. f would be strictly monotone on a set having upper density one, and so f would be $c \cdot \log n$, because of the result of BIRCH and KÁTAI.

If we change the function $f = c \cdot \log n$ on less than k powers of different primes, which are greater than k, then we can obviously construct such a function f, that g have $k-1$ adjacent identical values.

We can raise the following problem: What can we say about the number of the adjacent identical values? For the time being, we can show the following weak result:

If $f(q) > 0$ and $f(q+1) > 0$, then the length of the "period" of identical values beginning from q isn't larger then q^2+1. (This is true, because $f(q^2+q) = f(q)+f(q+1) > f(q)$.)

Finally, we prove that similarly to the case $g = c$, f cannot be negative on too many prime-powers:

THEOREM 3. *If g is a monotone increasing function, then f cannot be negative on the powers of more than $k-1$ primes.*

PROOF. We assume, that f is negative on more than $k-1$ powers of different primes say $p_i^{\alpha_i}$ $(i = 1, \ldots, k)$.

The system of congruences

$$x+i \equiv p_i^{\alpha_i} \bmod p_i^{\alpha_i+1} \quad (i = 1, \ldots, k)$$

has infinitely many solutions. These solutions can be written in the following form:

$$x+i = p_i^{\alpha_i}(1+p_i t) \quad (i = 1, \ldots, k),$$

and so

$$f(x+i) = f(p_i^{\alpha_i})+f(1+p_i t) < f(1+p_i t),$$

consequently

$$g(x+1) < \max_i g\left(\frac{x+i}{p_i^{\alpha_i}}\right).$$

Thus g cannot be monotone increasing. ∎

REMARK: If the least of the prime-powers is greater than k, then f can be negative on $k-1$ powers of different primes (for example f is 0 on the other powers), but if the least p_i is smaller then k, than f can be negative only on less than $k-1$ primes. Analogously to part (b) of the proof of Theorem 2, we can assert even more: If q_i are powers of different primes, with $f(q_i) < 0$ and $q_1 < \ldots < q_r < k < q_{r+1} < \ldots < q_s$, then $s < k-r$.

REMARK: All our results remain valid, if we regard the following more general function g:

$$g(n) := \max\{f(n), f(n+r_1), \ldots, f(n+r_{k-1})\}.$$

The proofs are nearly identical.

I am indebted to ROBERT FREUD for his valuable remarks.

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ON SQUARES IN ARITHMETIC PROGRESSIONS

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1. Throughout this paper, we use the following notations:

We denote the number of distinct prime factors of a positive integer n by $\nu(n)$. The number of elements of a finite set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. We put

$$\mathcal{D} = \{1^2, 2^2, \ldots, n^2, \ldots\},$$

$$\mathcal{D}_N = \{1^2, 2^2, \ldots, ([\sqrt{N}])^2\} \quad (\text{for} \quad N = 1, 2, \ldots),$$

$$\mathcal{M}(b, q, k) = \{b+q, b+2q, \ldots, b+kq\}$$

$$(\text{for} \quad b = 0, \pm 1, \pm 2, \ldots, \quad q = 1, 2, \ldots \quad \text{and} \quad k = 1, 2, \ldots)$$

and

$$|\mathcal{M}(b, q, k) \cap \mathcal{D}| = D(b, q, k)$$

$$(\text{for} \quad b = 0, \pm 1, \pm 2, \ldots, \quad q = 1, 2, \ldots \quad \text{and} \quad k = 1, 2, \ldots).$$

P. Erdős raised the following conjecture (oral communication):

If ε is an arbitrary positive number, $N, T, q_1, q_2, \ldots, q_T, k_1, k_2, \ldots, k_T$ are positive integers and $b_1, b_2, \ldots, b_T$ are integers such that $N > N_0(\varepsilon)$,

(1) $$\mathcal{M}(b_i, q_i, k_i) \subset \{1, 2, \ldots, N\} \quad (\text{for} \quad i = 1, 2, \ldots, T)$$

and

(2) $$\mathcal{D}_N \subset \bigcup_{i=1}^{T} \mathcal{M}(b_i, q_i, k_i)$$

hold, then we have

$$T \sum_{i=1}^{T} k_i > N^{1-\varepsilon}.$$

(Roughly speaking: the sequence of squares cannot be well-covered by arithmetic progressions in the sense that if few "long" arithmetic progressions cover the sequence of squares not exceeding N, then these arithmetic progressions must cover much more integers than $\sqrt{N}$.)

The aim of this paper is to prove the following sharper form of this conjecture:

THEOREM. *There exists a number N_0 such that if N is a positive integer satisfying $N>N_0$, and $T, q_1, q_2, \ldots, q_T, k_1, k_2, \ldots, k_T$ are positive integers, $b_1, b_2, \ldots, b_T$ are integers such that* (1) *and* (2) *hold then we have*

$$T\sum_{i=1}^{T} k_i > \frac{1}{700} \cdot \frac{N}{\log^2 N}. \tag{3}$$

In Section **2**, we prove a lemma (Lemma 3) from which our theorem follows easily, as we show it in Section **3**.

2. The proof of the theorem is based on the large sieve. We use the following form of the large sieve (see e.g. [1], p. 23):

LEMMA 1. *Let U be a positive integer, M an integer, X a non-negative real number and let*

$$\mathcal{A}\subset\{M+1, M+2, \ldots, M+U\}.$$

Write

$$Z(q, h) = \sum_{\substack{a\in\mathcal{A} \\ a\equiv h \pmod q}} 1$$

(for $q=1, 2, \ldots$ and $h=0, \pm 1, \pm 2, \ldots$) and

$$Z = Z(1, 0) = |\mathcal{A}| = \sum_{a\in\mathcal{A}} 1.$$

Then we have

$$\sum_{p\le X} p \sum_{h=1}^{p} \left(Z(p, h)-\frac{Z}{p}\right)^2 \le \left(U+\frac{2}{\sqrt{3}}X^2+3\right)Z.$$

We need also the following well-known lemma:

LEMMA 2. *Let ε be an arbitrary positive number. Then for $n>n_0$, we have*

$$\nu(n) < (1+\varepsilon)\frac{\log n}{\log\log n}.$$

In fact, this can be derived easily from the prime number theorem. Also, with respect to

$$2^{\nu(n)} = \prod_{p^\alpha \| n} 2 \le \prod_{p^\alpha \| n} (\alpha+1) = d(n)$$

(where $d(n)$ denotes the number of positive divisors of n), this is a consequence of the following theorem of WIGERT (see [5]): for $n>n_1(\varepsilon)$ we have

$$d(n) < 2^{(1+\varepsilon)\frac{\log n}{\log\log n}}.$$

By using Lemmas 1 and 2, we prove the following lemma (from which our theorem can be derived easily):

LEMMA 3. *There exists a number N_0 such that if N is a positive integer satisfying $N > N_0$, and q, k are positive integers, b is a non-negative integer such that*

$$\mathcal{M}(b, q, k) \subset \{1, 2, \ldots, N\} \tag{4}$$

and

$$k \geqq 100 \log^2 N \tag{5}$$

then we have

$$D(b, q, k) \leqq 18\, k^{1/2} \log k\,. \tag{6}$$

PROOF. Let us define the sequence $\mathcal{A}$ in the following way: $a \in \mathcal{A}$ should hold if and only if $1 \leqq a \leqq k$ and $b + aq \in \mathcal{D}$. Then obviously, we have

$$Z \overset{\text{def}}{=} |\mathcal{A}| = |\mathcal{M}(a, q, k) \cap \mathcal{D}| = D(b, q, k)\,.$$

By using Lemma 1 with this sequence $\mathcal{A}$ and with $M = 0$, $U = k$ and $X = k^{1/2}$, we obtain for large N that

$$\sum_{p \leqq k^{1/2}} p \sum_{h=1}^{p} \left(Z(p, h) - \frac{D(b, q, k)}{p} \right)^2 \leqq \tag{7}$$
$$\leqq \left(k + \frac{2}{\sqrt{3}} k + 3 \right) D(b, q, k) \leqq 3k\, D(b, q, k)\,.$$

(Note that for large N, k is also large by (5).) Here we have

$$Z(p, h) = \sum_{\substack{1 \leqq a \leqq k \\ b + aq \in \mathcal{D} \\ a \equiv h \ (\mathrm{mod}\ p)}} 1\,.$$

Let p be a prime number such that $p > 2$ and $p \nmid q$, and let $\mathcal{R}_p$ denote the set of the integers h for which

$$1 \leqq h \leqq p \quad \text{and} \quad \left(\frac{b + hq}{p} \right) = -1$$

hold $\left(\text{where } \left(\frac{n}{p} \right) \text{ denotes the Legendre-symbol} \right)$. Then obviously, $h \in \mathcal{R}_p$ and $a \equiv h \pmod q$ imply that $b + aq \notin \mathcal{D}$ and thus

$$Z(p, h) = 0 \quad \text{for} \quad h \in \mathcal{R}_p\,,$$

furthermore, by $p \nmid q$, we have

$$|\mathcal{R}_p| = \frac{p-1}{2}\,.$$

Thus we have

$$(8)\qquad \sum_{p\le k^{1/2}} p\sum_{h=1}^{p}\left(Z(p,h)-\frac{D(b,q,k)}{p}\right)^2 \ge$$

$$\ge \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}} p\sum_{h=1}^{p}\left(Z(p,h)-\frac{D(b,q,k)}{p}\right)^2 \ge$$

$$\ge \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}} p\sum_{\substack{1\le h\le p\\ h\in\mathcal{R}_p}}\left(Z(p,h)-\frac{D(b,q,k)}{p}\right)^2 = \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}} p\sum_{\substack{1\le h\le p\\ h\in\mathcal{R}_p}}\frac{D^2(b,q,k)}{p^2} =$$

$$= \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}} p\,\frac{p-1}{2}\,\frac{D^2(b,q,k)}{p^2} = \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}}\frac{1-\frac{1}{p}}{2}D^2(b,q,k)\ge$$

$$\ge \sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}}\frac{D^2(b,q,k)}{3} = \frac{D^2(b,q,k)}{3}\sum_{\substack{2<p\le k^{1/2}\\ p\nmid q}}1\ge$$

$$\ge \frac{D^2(b,q,k)}{3}\left(\sum_{2<p\le k^{1/2}}1-\sum_{p\mid q}1\right)=\frac{D^2(b,q,k)}{3}\left(\pi(k^{1/2})-1-\nu(q)\right).$$

By Lemma 2, for large N we have

$$\nu(n)<2\frac{\log N}{\log\log N}\quad\text{for all}\quad 1\le n\le N.$$

Thus by the prime number theorem and with respect to (4) and (5), we obtain from (8) that for large N,

$$(9)\qquad \sum_{p\le k^{1/2}} p\sum_{h=1}^{p}\left(Z(p,h)-\frac{D(b,q,k)}{p}\right)^2 \ge$$

$$\ge \frac{D^2(b,q,k)}{3}\left(\pi(k^{1/2})-1-\nu(q)\right)\ge\frac{D^2(b,q,k)}{3}\left(\frac{1}{2}\,\frac{k^{1/2}}{\log k^{1/2}}-2\,\frac{\log N}{\log\log N}\right)\ge$$

$$\ge \frac{D^2(b,q,k)}{3}\left(\frac{k^{1/2}}{\log k}-2\,\frac{(k/100)^{1/2}}{\log (k/100)^{1/2}}\right)=$$

$$=\frac{D^2(b,q,k)}{3}\left(\frac{k^{1/2}}{\log k}-\frac{2k^{1/2}}{5\log k/100}\right)>\frac{D^2(b,q,k)}{6}\,\frac{k^{1/2}}{\log k}.$$

(Note that (4) and (5) imply that $q\le N$, furthermore, the function $\dfrac{x}{\log x}$ is increasing for $x>x_0$.)

(7) and (9) yield that

$$\frac{D^2(b,q,k)}{6}\frac{k^{1/2}}{\log k}\leqq 3k\,D(b,q,k)$$

which implies (6) and the proof of Lemma 3 is completed.

3. In this section, we complete the proof of the theorem, by deriving (3) from Lemma 3.

Assume that (1) and (2) hold for some arithmetic progressions

$$\mathcal{M}(b_1,q_1,k_1),\ \ldots,\ \ \mathcal{M}(b_T,q_T,k_T).$$

Then (1) and (2) imply that

$$\sum_{i=1}^{T} D(b_i,q_i,k_i)=\sum_{i=1}^{T}|\mathcal{M}(b_i,q_i,k_i)\cap\mathcal{D}_N|\geqq$$

$$\geqq\left|\left(\bigcup_{i=1}^{T}\mathcal{M}(b_i,q_i,k_i)\right)\cap\mathcal{D}_N\right|\geqq|\mathcal{D}_N|=[\sqrt{N}\,].$$

Thus by Lemma 3, and by using Cauchy's inequality twice we obtain for arge N that

$$T\sum_{i=1}^{T}k_i\geqq T\sum_{k_i<100\log^2 N}k_i+T\sum_{k_i\geqq 100\log^2 N}k_i\geqq$$

$$\geqq\left(\sum_{k_i<100\log^2 N}1\right)\left(\sum_{k_i<100\log^2 N}k_i\right)+\left(\sum_{k_i\geqq 100\log^2 N}1\right)\left(\sum_{k_i\geqq 100\log^2 N}k_i\right)\geqq$$

$$\geqq\left(\sum_{k_i<100\log^2 N}\frac{k_i}{100\log^2 N}\right)\left(\sum_{k_i<100\log^2 N}k_i\right)+$$

$$+\left(\sum_{k_i\geqq 100\log^2 N}1\right)\left(\sum_{k_i\geqq 100\log^2 N}\left(\frac{D(b_i,q_i,k_i)}{18\log k_i}\right)^2\right)\geqq\frac{1}{100\log^2 N}\left(\sum_{k_i<100\log^2 N}k_i\right)^2+$$

$$+\left(\sum_{k_i\geqq 100\log^2 N}1\right)\left(\sum_{k_i\geqq 100\log^2 N}\frac{D^2(b_i,q_i,k_i)}{324\log^2 N}\right)\geqq$$

$$\geqq\frac{1}{100\log^2 N}\left(\sum_{k_i<100\log^2 N}D(b_i,q_i,k_i)\right)^2+$$

$$+\frac{1}{324\log^2 N}\left(\sum_{k_i\geqq 100\log^2 N}1\right)\left(\sum_{k_i\geqq 100\log^2 N}D^2(b_i,q_i,k_i)\right)\geqq$$

$$\geqq\frac{1}{324\log^2 N}\left\{\left(\sum_{k_i<100\log^2 N}D(b_i,q_i,k_i)\right)^2+\left(\sum_{k_i\geqq 100\log^2 N}D(b_i,q_i,k_i)\right)^2\right\}\geqq$$

$$\geqq\frac{1}{2\cdot 324\log^2 N}\left(\sum_{i=1}^{T}D(b_i,q_i,k_i)\right)^2\geqq\frac{1}{648\log^2 N}([\sqrt{N}\,])^2>\frac{1}{700}\frac{N}{\log^2 N}$$

(note that (1) implies that $k_i \leqq N$) which proves (3) and this completes the proof of the theorem.

4. Finally, we note that the conjecture of Erdős could be derived also from a result of S. Uchiyama (see [4]) but by using Uchiyama's result, we would obtain a slightly weaker lower bound for the left hand side of (3) than the one given in our theorem. In fact, in [4], Uchiyama gave an upper bound for $D(b, q, k)$ in terms of b, q and k. If we want to estimate $D(b, q, k)$ in terms of k solely then Uchiyama's result yields

$$D(b, q, k) = o(k) \tag{10}$$

only for

$$k > \exp\left(c_1 \frac{\log N}{\log\log N}\right)$$

(provided that (4) holds), while our estimate given in Lemma 3 yields (10) already for $k > c_2 \log^2 N$. Accordingly, Uchiyama's result would imply only the lower bound $N \exp\left(-c_3 \frac{\log N}{\log\log N}\right)$ for the left hand side of (3) (compare with the lower bound $N(\log N)^{-2}$ given in our theorem).

Furthermore, Uchiyama asserted in [4] that he gave a second proof of an other conjecture of Erdős (see [1], Problem 16), saying that for $k > k_0$ we have

$$D(b, q, k) < \varepsilon k\,. \tag{11}$$

(The first proof was given by E. Szemerédi in [3].) This is a misunderstanding; in fact, Erdős conjectured that if $k > k_0(\varepsilon)$ then (11) holds uniformly for all b and q; in particular, also for q's much greater than k (and Szemerédi proved this) while Uchiyama proved (11) only for fixed b, q and large k, i.e., for $k > k_0(\varepsilon, b, q)$. Thus it is still an open problem to give a direct proof for (11) (without using Szemerédi's celebrated theorem on the existence of long arithmetic progressions in "dense" sequences of integers), and to sharpen (11) by showing that $D(b, q, k) < k^{1/2+\varepsilon}$ or at least $D(b, q, k) < k^{1-c}$.

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ON A PROBLEM OF R. F. GUNDY

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1. The following result of Doob is classic: let $(X_n, \mathcal{F}_n)$, $n \geqq 1$, be a non-negative submartingale, such that

$$\sup_n E(X_n \log^+ X_n) < +\infty .$$

Then

$$E\left(\sup_n X_n\right) < +\infty .$$

More precisely, we have

$$E\left(\sup_n X_n\right) \leqq \frac{e}{e-1} \left(1 + \sup_n E(X_n \log^+ X_n)\right). \tag{1}$$

R. F. Gundy has shown a sufficiently large class of non-negative martingales for which the preceding inequality cannot be sharpened. Let $(X_n, \mathcal{F}_n)$ be a non-negative martingale satisfying the so called condition of Gundy: there is a constant $C > 0$ such that

$$X_{n+1} \leqq C X_n, \quad n \geqq 1, \tag{2}$$

holds a.e. Then the condition

$$E\left(\sup_n X_n\right) < +\infty$$

implies

$$\sup_n E(X_n \log^+ X_n) < +\infty ,$$

provided that $E(X_1 \log^+ X_1) < +\infty$.

Let $\Phi(x)$, $x \geqq 0$, be an arbitrary Young-function and denote by $\Psi(x)$ its conjugate Young-function. For the definition and the properties of these we refer to [2]. E.g. the function $\Phi(x) = x \log^+ x$, $x \geqq 0$, in inequality (1), or its other form $\Phi(x) = (x+1) \log (x+1) - x$, $x \geqq 0$, are Young-functions and the conjugate of the last one is

$$\Psi(x) = e^x - x - 1 .$$

The aim of the present note is to generalize inequality (1) for general Young-functions. Also we shall be able to isolate a class of Young-functions Φ for which the so obtained inequality cannot be sharpened in the same sense as above.

2. Let $Z_1, Z_2, \ldots$ be non-negative random variables defined on the probability space $(\Omega, \mathcal{A}, P)$ such that $\sum_{i=1}^{\infty} Z_i \in L_\infty$ with $\left\| \sum_{i=1}^{\infty} Z_i \right\|_\infty = K < +\infty$. Let further $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an arbitrary increasing sequence of σ-fields of events. Then for arbitrary $t \in (0, K^{-1})$ we have

$$E(\exp(tA_\infty)) \leqq (1-tK)^{-1}, \tag{3}$$

where

$$A_\infty = \lim_{n \to +\infty} A_n; \quad A_n = \sum_{i=1}^{n} E(Z_i | \mathcal{F}_i), \quad n \geqq 1 .$$

This result is known (cf. e.g. P. A. MEYER [3], Theorem 46.). We shall use it to prove the following

LEMMA 1. Let us suppose that

$$\sum_{i=1}^{\infty} Z_i \leqq K \quad \text{a.e.}$$

and let $\Psi(x)$ be an arbitrary Young-function. If for some $t \in (0, K^{-1})$ the Laplace-transform

$$I = \int_0^{+\infty} e^{-t\lambda} d\Psi(\lambda)$$

converges then

$$E(\Psi(A_\infty)) \leqq (1-tK)^{-1} I .$$

PROOF. We have

$$E(\Psi(A_\infty)) = \int_0^{+\infty} P(A_\infty \geqq \lambda) d\Psi(\lambda) .$$

By the Markov inequality and by (3)

$$P(A_\infty \geqq \lambda) = P(e^{tA_\infty} \geqq e^{t\lambda}) \leqq e^{-t\lambda} E(e^{tA_\infty}) \leqq e^{-t\lambda}(1-tK)^{-1} .$$

Consequently,

$$E(\Psi(A_\infty)) \leqq (1-tK)^{-1} I .$$

This proves the assertion.

We use this lemma to prove the following

THEOREM 1. *Let $(X_n, \mathcal{F}_n)$, $n \geqq 1$, be a non-negative submartingale and let Φ and Ψ be conjugate Young-functions. If for some $t \in (0, 1)$ the Laplace-transform*

$$I = \int_0^{+\infty} e^{-t\lambda} d\Psi(\lambda)$$

converges then

$$E\left(\max_{1 \leqq k \leqq n} X_k\right) \leqq E(\Phi(X_n)) + (1-t)^{-1} I ,$$

provided that $E(\Phi(X_n)) < +\infty$.

PROOF. Consider the events

$$A_1 = \{X_1 = X_n^*\}, \quad A_k = \{X_{k-1}^* < X_n^*, \quad X_k = X_n^*\}, \quad k = 2, \ldots, n$$

where $X_k^* = \max_{1 \leqq l \leqq k} X_l$. Then the random variables

$$Z_k = \chi_{A_k}, \quad k = 1, \ldots, n ,$$

have the property

$$\sum_{k=1}^{n} Z_k = 1 .$$

Here χ_{A_k} denotes the indicator of the event A_k, $k = 1, \ldots, n$. So the preceding lemma can be applied to the Young-function Ψ.

Now by the submartingale property

$$E(X_n^*) = \sum_{k=1}^{n} E(X_k \chi_{A_k}) \leqq \sum_{k=1}^{n} E(\chi_{A_k} E(X_n|\mathcal{F}_k)) .$$

The conditional expectation being self-adjoint we get

$$E(X_n^*) \leqq \sum_{k=1}^{n} E(X_n E(\chi_{A_k}|\mathcal{F}_k)) = E\left(X_n \sum_{k=1}^{n} E(\chi_{A_k}|\mathcal{F}_k)\right).$$

Apply now to the ringt-hand side the Young-inequality

$$xy \leqq \Phi(x) + \Psi(y), \quad x \geqq 0, \quad y \geqq 0 ,$$

to get

$$E(X_n^*) \leqq E(\Phi(X_n)) + E\left(\Psi\left(\sum_{k=1}^{n} E(\chi_{A_k}|\mathcal{F}_k)\right)\right).$$

By the result of the preceding lemma applied to the second term on the right-hand side we obtain

$$E(X_n^*) \leqq E(\Phi(X_n)) + (1-t)^{-1} I .$$

This was to be proved.

3. For the non-negative martingales satisfying the condition of Gundy and for a large class of Young-functions the inequality of the preceding assertion cannot be sharpened. To isolate this class of Young-functions we prove the following assertion. For this purpose we use the notation

$$\xi(x) = x\varphi(x) - \Phi(x) .$$

THEOREM 2. *Suppose that*

$$\xi(x) = O(x), \quad x \to +\infty .$$

If for the non-negative martingale $(X_n, \mathcal{F}_n)$ *condition* (2) *of Gundy holds then*

$$E(\Phi(X_n)) \leqq K\left[E(X_1\varphi(X_1)) + E(X_n^*) + 1\right],$$

provided that

$$E(X_1\varphi(X_1)) < +\infty .$$

Here $K>0$ *is a constant depending only on* Φ *and* φ *denotes the "density" function of* Φ.

PROOF. GUNDY has shown that for the martingales satisfying (2) we have

$$E(X_n\chi(X_n^* \geqq \lambda)) \leqq E(X_1\chi(X_1 \geqq \lambda)) + C\lambda E(\chi(X_n^* \geqq \lambda)) .$$

Cf. e.g. J. NEVEU [2], Proposition IV–2–11. Here $\lambda > 0$ is an arbitrary constant and $\chi(A)$ denotes the indicator of the event A. Integrate this inequality on $(0, +\infty)$ with respect to the measure generated by the non-decreasing and right-continuous function $\varphi(\lambda)$. Using the theorem of Fubini we get

$$E(X_n\varphi(X_n^*)) \leqq E(X_1\varphi(X_1)) + C E(\xi(X_n^*)) ,$$

since

$$\int_0^x \lambda \, d\varphi(\lambda) = \xi(x) .$$

Note that $\varphi(x)$ increases and so

$$E(X_n\varphi(X_n^*)) \geqq E(X_n\varphi(X_n)) \geqq E(\Phi(X_n)) .$$

From this

$$E(\Phi(X_n)) \leqq E(X_1\varphi(X_1)) + C E(\xi(X_n^*)) .$$

Now if $\xi(x) = O(x)$, $x \to +\infty$, we can choose $x_0 \geqq 0$ such that

$$\xi(x) \leqq K' x$$

hold for $x \geqq x_0$. Here K' is a finite positive constant which by the supposition exists. Consequently,

$$E\left(\xi(X_n^*)\right) \leqq \xi(x_0) + K' E(X_n^*)$$

and so

$$E(\Phi(X_n)) \leqq E\left(X_1 \varphi(X_1)\right) + C\,\xi(x_0) + C K' E(X_n^*).$$

Let

$$K = \max\left(1, C\,\xi(x_0), C K'\right).$$

Then with this K we finally obtain

$$E(\Phi(X_n)) \leqq K\left[E\left(X_1 \varphi(X_1)\right) + E(X_n^*) + 1\right].$$

The assertion is thus proved.

REMARKS. Note that when $\Phi(x) = x \log^+ x$ then we trivially have

$$\xi(x) \leqq x, \quad x \geqq 0.$$

Remark that if $\mathcal{F}_1 = (\emptyset, \Omega)$, then $E\left(X_1 \varphi(X_1)\right)$ is a constant and in this case we have

$$E(\Phi(X_n)) \leqq K\left(E(X_n^*) + 1\right),$$

with some constant $K > 0$.

The condition $\xi(x) = O(x)$, $x \to +\infty$, is also necessary in the sense below. If for all non-negative submartingale $(X_n, \mathcal{F}_n)$, $n \geqq 1$, where $\mathcal{F}_0 = (\emptyset, \Omega)$, satisfying condition (2) of Gundy the inequality

$$E(\Phi(X_n)) \leqq K\left[E(X_n^*) + 1\right],$$

holds, where $K > 0$ is a constant depending only on Φ, then necessarily we have

$$\xi(x) = O(x), \quad x \to +\infty.$$

In fact, using the inequality of Doob

$$\lambda P(X_n^* \geqq \lambda) \leqq E\left(X_n \chi(X_n^* \geqq \lambda)\right), \quad \lambda > 0,$$

and integrating this on $(0, +\infty)$ with respect to the measure generated by the function $\varphi(\lambda)$ we get

$$E\left(\xi(X_n^*)\right) \leqq b\, E\left(\frac{X_n}{b} \varphi(X_n^*)\right),$$

where $0 < b < 1$ is a constant. Applying the Young-inequality to the right-hand side and remarking that $\Psi(\varphi(x)) = \xi(x)$ from this we deduce that

$$E\left(\xi(X_n^*)\right) \leqq b\left[E\left(\Phi\left(\frac{X_n}{b}\right)\right) + E\left(\xi(X_n^*)\right)\right],$$

or, in other words

$$(1-b)\, E\left(\xi(X_n^*)\right) \leqq b\, E\left(\Phi\left(\frac{X_n}{b}\right)\right).$$

The martingale $\left(\frac{X_n}{b}, \mathcal{F}_n\right)$, $n \geqq 1$, also satisfies condition (2) of Gundy. Consequently, by our supposition

$$(1-b)\,E\left(\xi(X_n^*)\right) \leqq b\,K\left[E\left(\frac{X_n^*}{b}\right)+1\right].$$

This proves the assertion by taking $X_n \equiv x$.

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ESSENTIAL COVER AND CLOSURE

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The aim of this note is to provide an answer to a question raised by ANDERSON and WIEGANDT [1] by constructing a regular class of semiprime rings whose essential cover is not essentially closed.

An ideal I of a ring R is said to be *essential* if $I \cap A \neq 0$ for each $0 \neq A \lhd R$. Given a class $\mathcal{M}$ of rings the class

$$\mathcal{M}_k = \{R \mid \exists I \lhd R \text{ with } I \text{ essential in } R \text{ and } I \in \mathcal{M}\}$$

is called the *essential cover* of $\mathcal{M}$. If $\mathcal{M} = \mathcal{M}_k$ then $\mathcal{M}$ is said to be *essentially closed*. These ideas were studied in [2] where it is proved that if $\mathcal{M}$ is a hereditary class of semiprime rings, then so also is $\mathcal{M}_k$ and $\mathcal{M}_k$ is essentially closed. In [1] it is shown that if $\mathcal{M}$ is a class of semiprime rings and regular (that is, if $R \in \mathcal{M}$ and $0 \neq A \lhd R$, then A has a non-zero image in $\mathcal{M}$) rather than hereditary, then so is $\mathcal{M}_k$, but the authors comment that they do not know of a regular class $\mathcal{M}$ of semiprime rings for which $\mathcal{M}_k$ is not essentially closed. We construct such a class here.

EXAMPLE. Let $R = \mathbf{Z}[x]$, $A = x^2R$ and $M = x^2\mathbf{Z} + x^4R$. Thus $M \lhd A \lhd R$, but $M \ntriangleleft R$. Since $x^4R \subseteq M$ it is clear that each of these extensions is essential. Let $\mathcal{D}$ be the class of semiprime rings with nonzero characteristic. Put $\mathcal{M} = \mathcal{D} \cup \{M\}$. Thus $\mathcal{M}$ is a class of semiprime rings.

To see that $\mathcal{M}$ is regular suppose $0 \neq N \lhd M$. If $N \cap pR = N$ for all primes p then $N \subseteq \cap pR = 0$. Thus there is a prime p for which $N \cap pR \subset N$. Hence $0 \neq N/(N \cap pR)$ is of characteristic p. Further, if $f, g \in N$ and $fg \in N \cap pR$, then, by Gauss' Lemma either $f \in pR$ or $g \in pR$. Thus $N/(N \cap pR) \in \mathcal{D} \subseteq \mathcal{M}$ and $\mathcal{M}$ is regular.

Now $A \in \mathcal{M}_k$ and $R \in (\mathcal{M}_k)_k$. To complete the argument we show that $R \notin \mathcal{M}_k$. If $R \in \mathcal{M}_k$, then R is an essential extension of an ideal $I \in \mathcal{M}$. Since R^+ is torsion-free this means that $I \cong M$. Suppose $\varphi : M \to I$ is a ring isomorphism. Then $x^2\varphi = f$ and $x^5\varphi = g$ for some $f, g \in I$. Hence $f^5 = g^2$ and from unique factorization in R, $f = h^2$ and $g = h^5$ for some $h \in R$. Now $fh =$

$=h^3 \in I$ so $h^3 q^{-1} = m \in M$. Therefore $m^2 = h^6 q^{-1} = f^3 q^{-1} = x^6$ and so $x^3 = \pm m \in M$, which is a contradiction. Thus $R \notin \mathcal{M}_k$ which is not essentially closed.

REMARKS. We note that a partial answer to the question under discussion has been given in [3, Theorem 3] where it is shown that for a subdirectly closed, regular class $\mathcal{M}$ of semiprime rings, $\mathcal{M} = \mathcal{M}_k$ if and only if $\mathcal{M}$ is the semisimple class, of an hereditary radical.

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