

ANNALES
UNIVERSITATIS SCIENTIARUM
BUDAPESTINENSIS
DE ROLANDO EÖTVÖS NOMINATAE

SECTIO MATHEMATICA

TOMUS XXVII.
1984

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О РАЗРЕШИМОСТИ ОДНОГО КЛАССА ЭЛЛИПТИЧЕСКИХ УРАВНЕНИЙ ВЫРОЖДАЮЩИХСЯ НА ГРАНИЦЕ ОБЛАСТИ

БАГИРОВ Л. А. и МЫШКИН И. А.

Кафедра прикладной математики Московского инженерно-строительного института
Москва

(Поступило 19. 2. 1982)

1. Введение и обозначения

Пусть $G \subset \mathbf{R}^{n+1}$ ограниченная область с границей Γ , являющейся бесконечно дифференцируемым компактным n -мерным многообразием.

Рассмотрим в G уравнение

$$(1.1) \quad \mathcal{L} u(x) = \sum_{|\alpha| \leq 2m} l_\alpha(x) D^\alpha u(x) = f(x).$$

Здесь $x = (x_1, x_2, x_3, \dots, x_n, x_{n+1})$; $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ — целочисленный мультииндекс,

$$|\alpha| = \sum_{k=1}^{n+1} \alpha_k; \quad D_{x_k}^{\alpha_k} = (-i)^{\alpha_k} \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}, \quad D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_{n+1}}^{\alpha_{n+1}}.$$

Коэффициенты $l_\alpha(x)$ — бесконечно дифференцируемые комплекснозначные функции в G , непрерывные в G .

Условие 1. \mathcal{L} — эллиптический оператор в G , т.е. для $\forall x^0 \in G$ $\exists \delta_0 > 0$ такое, что

$$\left| \sum_{|\alpha| \leq 2m} l_\alpha(x^0) \xi^\alpha \right| \geq \delta_0 |\xi|^{2m}, \quad \xi \in \mathbf{R}^{n+1}.$$

Для описания характера вырождения оператора \mathcal{L} вблизи Γ введем необходимые локальные координаты.

Пусть $\{\Omega_r\}$ — такое конечное покрытие окрестности Γ , что в каждом из множеств $V^r = \Omega_r \cap G$ можно ввести локальную систему координат (л.с.к.) $(y^r, t^r) = (y_1^r, y_2^r, \dots, y_n^r, t^r)$ такую, что каждая точка $M \in V^r$, $M \notin \Gamma$ имеет координату $t^r > 0$ а каждая точка $M_1 \in V^r \cap \Gamma$ имеет координату $t^r = 0$.

Естественно, предполагается, что введенные л.с.к. превращают рассматриваемую окрестность Γ в $n+1$ — мерное бесконечно дифференцируемое многообразие. Будем обозначать локальные координаты в V^r через (y, t) в тех случаях, когда это не вызовет недоразумений.

Условие 2. Оператор \mathcal{L} в области V^r имеет вид

$$\mathcal{L} u(y, t) = \sum_{|\alpha| + k \leq 2m} l_{\alpha k}(y, t) (t D_t)^k D_y^\alpha u(y, t),$$

Причем:

1°. $I_{zk}(y, t) = t^{\gamma_{zk}} I_{zk}^0(y, t)$, $I_{zk}^0(y, t)$ — гладкие функции в V^r , $\gamma_{zk} = -p [2(m-l)]^{-1}(|z| + k - 2l)$ для $|z| + k > 2l$ и $\gamma_{zk} = 0$ для $|z| + k \leq 2l$ и $I_{zk}^0(y, t) = o(1)$ при $t \rightarrow 0$ для $2l < |z| + k < 2m$;

2°. $I_{zk}^0(y, 0)$ для $|z| + k = 2m$ и $|z| + k = 2l$ вещественные функции и $\exists \delta_1 > 0$ и $\delta_2 > 0$ такие, что для $\forall \sigma \in R^1$, $\forall z \in R^n$, $\forall y \in \Gamma$

$$(1.2) \quad \sum_{|z|+k=2m} I_{zk}^0(y, 0) \sigma^k z^k \geq \delta_1 (\sigma^2 + |\xi|^2)^m,$$

$$(1.3) \quad \sum_{|z|+k=2l} I_{zk}^0(y, 0) \sigma^k z^k \geq \delta_2 (\sigma^2 + |\xi|^2)^l.$$

ПРИМЕР 1. Пусть $K_{r_0} = \{x : x \in R^2, |x| \leq r_0\}$,

$$r = |x|, \quad q = \operatorname{arctg} \frac{x_2}{x_1}.$$

Рассмотрим оператор \mathcal{L}_1 вида

$$\begin{aligned} \mathcal{L}_1 u(q, r) = & |r - r_0|^p \sum_{|z|+k=2m} a_{zk} [(r - r_0) D_r]^k D_q^k u(q, r) + \\ & + [(r - r_0) D_r]^2 u(q, r) + D_q^2 u(q, r). \end{aligned}$$

Если $p > 0$, a_{zk} — вещественные постоянные,

$$\sum_{|z|+k=2m} a_{zk} \sigma^k z^k \geq \delta (\sigma^2 + |\xi|^2)^m \quad \text{для } \forall \sigma, \xi \in R^1, \delta > 0,$$

то оператор \mathcal{L}_1 удовлетворяет условиям 1 и 2.

ПРИМЕР 2. Рассмотрим в ситуации примера 1 оператор \mathcal{L}_2 имеющий вид

$$\mathcal{L}_2 u(q, r) = \mathcal{L}_1 u(q, r) + M_1 u(q, r),$$

где

$$M_1 u(q, r) = [a_1(r - r_0) D_r + a_2 D_q + a_3] u(q, r).$$

Здесь a_i , $i = 1, 2, 3$ комплексные постоянные. Оператор \mathcal{L}_2 также удовлетворяет условиям 1 и 2.

ПРИМЕР 3. Пусть $\mathcal{L}_{2m}(D_x)$ и $\mathcal{L}_{2l}(D_x)$ однородные дифференциальные эллиптические операторы в R^{n+1} с постоянными вещественными коэффициентами и положительными символами. После сферической замены переменной они будут иметь вид

$$\mathcal{L}_{2m}(D_x) = r^{-2m} \tilde{\mathcal{L}}_{2m}(\omega, r D_r, D_\omega),$$

$$\mathcal{L}_{2l}(D_x) = r^{-2l} \tilde{\mathcal{L}}_{2l}(\omega, r D_r, D_\omega),$$

где ω — точка единичной сферы в R^{n+1} . Тогда условиям 1 и 2 в единичном шаре $|x| < 1$ удовлетворяет оператор \mathcal{L}_3 вида

$$\mathcal{L}_3 = |r - 1|^p \tilde{\mathcal{L}}_{2m}(\mu, (r - 1) D_r, D_\omega) + \tilde{\mathcal{L}}_{2l}(\omega, (r - 1) D_r, D_\omega)$$

при $p > 0$. ■

Эллиптические уравнения вырождающиеся на границе области изучались многими исследователями. Отметим работы [2], [5], [7], [8], [9], посвящённые уравнениям высокого порядка. Класс операторов изучаемый нами отличается тем, что условия разрешимости выражаются в терминах спектра некоторого оператора, определенного на многообразии Γ . Близкие операторы рассматривались в работах [8] и [9]. Они получаются из рассматриваемых нами, если коэффициенты

$$I_{z,k}(y, t) = 0 \quad \text{для} \quad 2l < |z| + k \leq 2m.$$

Изучение поставленной задачи потребовало разработки новой техники, связанной с решением задачи Карлемана в полосе с рациональным коэффициентом, зависящим от n -мерного параметра (см. 4.). В простейшем варианте (коэффициент на бесконечности стремится к единице и не зависит от параметра) такая задача разобрана в монографии [3]. Авторы надеются, что аналогичная техника найдет применение и в других задачах для уравнений в частных производных.

Нами используются следующие обозначения и простые факты:

$$F[\psi(\sigma)](t) = \int_{-\infty}^{+\infty} \psi(\sigma) e^{it\sigma} d\sigma, \quad F^{-1}[\psi(t)](\sigma) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \psi(t) e^{-it\sigma} dt$$

— прямое и обратное преобразование Фурье:

$$M[u(t)](\lambda) = (2\pi)^{-1} \int_0^\infty u(t) t^{-1-i\lambda} dt, \quad \lambda = \sigma + i\tau,$$

$$M^{-1}[v(\lambda)](t) = \int_{-\infty+i\tau}^{\infty+i\tau} t^{\lambda} v(\lambda) d\lambda$$

— прямое и обратное преобразование Меллина; для

$$C_0^\infty(R_+^1) \ni u(t), \quad R_+^1 = \{t : t \in R^1, t > 0\},$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} |M[u(t)](\lambda)|^2 d\lambda = (2\pi)^{-1} \int_0^\infty t^{-1+2\tau} |u(t)|^2 dt$$

— равенство Парсеваля для преобразования Меллина;

если Γ — компактное n -мерное многообразие класса C^∞ , то $H_\lambda^s(\Gamma)$ для $s \geq 0$ — пространство функций на Γ с нормой

$$\|u(x, \lambda)\|_s = \|u(x, \lambda)\|_s + |\lambda|^s \|u(x, \lambda)\|_0,$$

где $\|\cdot\|_s$ — обычная норма в пространстве $L^s(\Gamma)$:

$$U = \{(t, y) : t \in R_+^1, y \in \Gamma\};$$

$H_{s, \gamma}(U)$ — пространство функций с нормой

$$\|v(y, t)\|_{s, \gamma}^2 = \sum_{0 \leq k \leq s, 0 \leq l \leq s} \int_0^\infty t^{2(s+k)} \|D_t^k D_y^l v(y, t)\|_0^2(\Gamma) dt$$

с помощью преобразования Меллина вводится эквивалентная норма (для простоты обозначаем так же)

$$\|v(y, t)\|_{s, \gamma}^2 = \int_{-\infty + i\tau_0}^{+\infty + i\tau_0} M[v(y, t)](\lambda) \frac{2}{s}(I)^{-2}(\Gamma) d\lambda, \quad \tau_0 = \gamma + \frac{1}{2};$$

$\tilde{H}_t^s(\mathbf{R}^1)$ — пространство функций $f(\xi, \sigma)$ с нормой

$$\|f(\xi, \sigma)\|_s^2 = \int_{-\infty}^{+\infty} |f(\xi, \sigma)|^2 (1 + \sigma^2 + |\xi|^2)^s d\sigma;$$

$$H_a^b = \{\lambda = \sigma + i\tau : a < \tau < b\};$$

$\tilde{H}_{\xi, a, b}^{s_1, s_2}$ — пространство функций $g(\xi, \lambda)$ голоморфных в полосе H_a^b граничные значения которых принадлежат $\tilde{H}_\xi^{s_1}(\mathbf{R}^1)$ при $\tau = a$ и $\tilde{H}_\xi^{s_2}(\mathbf{R}^1)$ при $\tau = b$, норму в этом пространстве определим так

$$\|g(\xi, \lambda)\|_{s_1, s_2}^2 (H_a^b) = \int_{-\infty}^{+\infty} [|g(\xi, \sigma + ia)|^2 (1 + |\xi|^2 + |\sigma + ia|^2)^{s_1} +$$

$$+ |g(\xi, \sigma + ib)|^2 (1 + |\xi|^2 + |\sigma + ib|^2)^{s_2}] d\sigma.$$

$H_{\gamma_1, \gamma_2}^{s_1, s_2}(U) = H_{s_1, \gamma_1}(U) \cap H_{s_2, \gamma_2}(U)$ — это пространство можно описать с помощью преобразования Меллина следующим образом: $v(y, t) \in H_{\gamma_1, \gamma_2}^{s_1, s_2}(U)$, если $M[v(y, t)](\lambda)$ голоморфная функция в полосе $\gamma_1 \leq \tau - \frac{1}{2} \leq \gamma_2$, граничные значения которой удовлетворяют оценке $\left(\tau_i = \gamma_i + \frac{1}{2} \right)$

$$\int_{-\infty + i\tau_1}^{+\infty + i\tau_1} M[v(y, t)](\lambda) \frac{2}{s_1}(I)^{-2}(\Gamma) d\lambda + \int_{-\infty + i\tau_2}^{+\infty + i\tau_2} M[v(y, t)](\lambda) \frac{2}{s_2}(I)^{-2}(\Gamma) d\lambda =$$

$$= \|v(y, t)\|_{s_1, s_2, \gamma_1, \gamma_2}^2 (U) < +\infty.$$

Последнее выражение очевидно является нормой в пространстве $H_{\gamma_1, \gamma_2}^{s_1, s_2}(U)$.

$H_{\Omega, \gamma, p}^{s_1, s_2}(G)$ — пространство функций в области $G \subset \mathbf{R}^{n+1}$, для которых конечна следующая норма

$$\|u\|_{s_1, s_2, \gamma, p} = \sum_r \langle q_r u \rangle_{s_1, s_2, \gamma, r} + \|q_r u\|_{s_2}.$$

Здесь $\|\cdot\|_{s_2}$ — норма пространства $H^{s_2}(\mathbf{R}^{n+1})$, $\{q_r, q_\tau\}$ — разбиение единицы, подчиненное покрытию $\{\Omega_r, \Omega_\tau\}$ области G (Ω_r уже введенное пок-

рытие Γ и $\Omega \cap \Gamma = \emptyset$),

$$\langle q_r u \rangle_{s_1, s_2, \gamma, \gamma+p}^2 = \int_{\mu^{n+1}} \left[\sum_{s_1 \leq k + |x_i| \leq s_2} t^{2(k+\gamma) + 2p - \frac{s_1 - (k+\gamma)}{s_1 - s_2}} |D_t^k D_y^{\gamma} q_r u(y, t)|^2 + \right. \\ \left. + \sum_{k + |x_i| < s_1} t^{2\gamma + 2k} |D_t^k D_y^{\gamma} q_r u(y, t)|^2 \right] dy dt$$

$(y_1, y_2, \dots, y_n, t)$ — локальные координаты в Ω_r ;

$$H_{\gamma, \gamma}^{s_1, s_2}(G) = H_{s_2, \gamma}(G) \quad \text{и} \quad \|\cdot\|_{s_1, s_2, \gamma, \gamma} = \|\cdot\|_{s_2, \gamma}(G).$$

Многочисленные постоянные, встречающиеся в оценках приведенных в работе, мы будем обозначать (одной и той же) буквой C всегда, когда это не может вызвать недоразумений.

Авторы благодарят В. А. Кондратьева за полезные обсуждения результатов работы, а также Г. К. Тарасову за помощь при проведении выкладок 4.

2. Основные результаты

В работе [8] доказано существование в окрестности Γ покрытия $\{U_\mu\}$, обладающего следующими свойствами:

1° $\{U_\mu\}$ — достаточно мелкое конечное покрытие окрестности Γ , вписанное в $\{\Omega_r\}$, причём каждое множество $U'_\mu = U_\mu \cap G$ имеет вид $U'_\mu = \bigcup_{x \in \Gamma \cap U_\mu} I_{x, \varrho_0}$, где I_{x, ϱ_0} — полуинтервал длины ϱ_0 , ортогональный Γ и восстановленный из точки x , а ϱ_0 — достаточно малое фиксированное число.

2° в каждом U'_μ можно ввести координаты (y'', t'') , причём координата t'' любой точки $s \in U'_\mu$ равна расстоянию по нормали от s до $\Gamma \cap U'_\mu$ и координаты y'' точек, лежащих на одной нормали, равны.

В той же работе доказано, что условие 2 (т.е. характер вырождения оператора \mathcal{L}) в координатах (y'', t'') , в которых оно формулировалось, переходит в такое же условие 2 в новых координатах (y'', t'') . Это конечно означает и инвариантность условий (1.2) и (1.3). ■

Пусть $t_{zk}(y, t)$ — коэффициенты оператора \mathcal{L} , записанные в новых локальных координатах.

Рассмотрим на Γ дифференциальный оператор

$$(2.1) \quad B_{2l}(y, D_y, \lambda) = \sum_{|x_i| = k = 2l} b_{zk}(y) D_y^z \lambda^k,$$

где $b_{zk}(y) = t_{zk}(y, 0)$, λ — комплексный параметр. Из условия (1.2) следует, что это эллиптический оператор с параметром на многообразии Γ .

В работе [1] показано, что обратный оператор $B_{2l}^{-1}(\lambda) : H^s(\Gamma) \rightarrow H^{s-2l}(\Gamma)$ является мероморфной функцией $\lambda \in \mathbb{C}$, причём в области

$$\Omega_{\delta, N} = \{\lambda : |\arg \lambda - k\pi| < \delta, k = 0, 1, |\lambda| > N\}$$

(при некоторых положительных δ и N) нет полюсов $B_{2l}^{-1}(\lambda)$. Там же доказана оценка

$$(2.2) \quad \|u(y, \lambda)\|_{s-2l}(I) \leq C \|B_{2l}(y, D_y, \lambda) u(y, \lambda)\|_s(I),$$

для $\forall u(y, \lambda) \in H^{s+2l}(I)$. Постоянная C — не зависит от $\lambda \in \Omega_{\delta, N}$.

Основным результатом настоящей работы является

Теорема 1. Пусть выполнены условия 1 и 2 и пусть на прямых $\operatorname{Im} \lambda = \gamma + \frac{1}{2}, \gamma + \frac{1}{2} + p$ нет полюсов оператора $B_{2l}^{-1}(\lambda)$. Тогда для

$$s \geq \max \{0, p^{-1}(2\gamma + 1 + p)(m - l)\},$$

$s_1 = s + 2l, s_2 = s + 2m$ и $p > 0$ (из условия 2)

(i) \exists ограниченный оператор $R : H_{s, \gamma}(G) \rightarrow H_{\gamma, \gamma+p}^{s_1, s_2}(G)$ такой, что

$$(2.3) \quad \mathcal{L} R f(x) = f(x) + T f(x), \quad \forall f(x) \in H_{s, \gamma}(G),$$

где $T : H_{s, \gamma}(G) \rightarrow H_{s, \gamma}(G)$ вполне непрерывный оператор.

(ii) для $\forall u(x) \in H_{\gamma, \gamma+p}^{s_1, s_2}(G)$ выполнена оценка

$$(2.4) \quad \|u(x)\|_{s_1, s_2, \gamma, \gamma+p}(G) \leq C [\|\mathcal{L} u\|_{s, \gamma}(G) + \|u\|_{s+2m-l}(G)],$$

где $G_1 \subset G$ и постоянная C — не зависит от $u(x)$.

Из теоремы 1 легко следует

Теорема 2. При выполнении условий теоремы 1 оператор

$$\mathcal{L} : H_{\gamma, \gamma+p}^{s_1, s_2}(G) \rightarrow H_{s, \gamma}(G)$$

нешероб.

Замечание 1. Утверждение о конечномерности ядра оператора \mathcal{L} справедливо без предположения об отсутствии полюсов оператора $B_{2l}^{-1}(\lambda)$ на соответствующей прямой.

Замечание 2. Условие отсутствия полюсов $B_{2l}^{-1}(\lambda)$ на определённой прямой являются в некотором смысле необходимыми. Легко показать, что при нарушении этого условия образ оператора \mathcal{L} незамкнут.

Замечание 3. Условия на s в теореме 1 означают, что либо результат получен для всех $s \geq 0$, но в классе убывающих к границе функций, либо в классе функций без ограничений на рост, но с достаточно высокой гладкостью.

Замечание 4. Полюса $B_{2l}^{-1}(\lambda)$ в ряде случаев выписываются. В примере 1 оператор $B_{2l}(\lambda)$ имеет вид $D_\phi^2 + \lambda^2$ и полюса $B_{2l}^{-1}(\lambda)$ это просто числа вида ik $k = 0, \pm 1, \pm 2, \dots$. В примере 2 оператор $B_{2l}(\lambda)$ имеет вид $D_\phi^2 + \lambda^2 + a_1 \lambda + a_2 D_\phi + a_3$ и определение полюсов $B_{2l}^{-1}(\lambda)$ сводится к решению легко выписываемого трансцендентного уравнения.

В примере 3 оператор $B_{2l}(\lambda) = \tilde{\mathcal{L}}_{2l}(\omega, \lambda, D_\omega)$ и, если исходный оператор $\mathcal{L}_{2l}(D_x) = (-1)^l A^{2l}$, то полюса $B_{2l}^{-1}(\lambda)$ находятся в точках $\lambda = ik$,

$k = 0, \pm 1, \pm 2, \dots$ при $2l - (n+1) \geq 0$ и в точках $\lambda = ik$ $k = 0, 1, 2, 3, \dots$ и $k = 2l - (n+1), 2l - (n+1) - 1, 2l - (n+1) - 2, \dots$ ■

Главное место в доказательстве теоремы 1 — изучение модельной задачи в цилиндре \mathcal{U} .

Рассмотрим в \mathcal{U} уравнение

$$(2.5) \quad [t^p A_{2m}(y, D_y, tD_t) + B_{2l}(y, D_y, tD_t)]v(y, t) = f(y, t).$$

Здесь оператор $B_{2l}(y, D_y, tD_t)$ определен формулой (2.1),

$$A_{2m}(y, D_y, tD_t) = \sum_{|x|+k=2m} a_{xk}(y) D_y^k(tD_t)^k, \quad \text{где } a_{xk}(y) = l_{xk}^0(y, 0).$$

На каждой прямой $\operatorname{Im} \lambda = \tau_0$, на которой $B_{2l}^{-1}(\lambda)$ не имеет полюсов, мы можем определить оператор

$$(2.6) \quad B_{2l}^{-1}(y, D_y, tD_t)g(y, t) = \int_{-\infty + i\tau_0}^{+\infty + i\tau_0} t^{\mu} B_{2l}^{-1}(\lambda) (Mg)(y, \lambda) d\lambda,$$

который в силу оценок (2.2) действует ограниченным образом из $H_{s, \gamma}(\mathcal{U})$ в $H_{s+2l, \gamma}(\mathcal{U})$, где $\gamma = \tau_0 - \frac{1}{2}$ (см. [6]). Из (2.5) следует

$$(2.7) \quad \begin{aligned} & \mathcal{L}^0 g(t, y) = \\ & = [t^p A_{2m}(y, D_y, tD_t) B_{2l}^{-1}(y, D_y, tD_t) + 1]g(y, t) = f(y, t), \end{aligned}$$

где

$$g(y, t) = B_{2l}(y, D_y, tD_t)v(y, t).$$

Очевидно, что оператор $\mathcal{L}^0 : H_{s, \gamma+p}^{s, s+2(m-l)}(\mathcal{U}) \rightarrow H_{s, \gamma}(\mathcal{U})$ ограничен, если на прямой $\operatorname{Im} \lambda = \gamma + p + \frac{1}{2}$ нет полюсов $B_{2l}^{-1}(\lambda)$.

Теорема 3. Пусть выполнены условия (1.2) и на прямой $\operatorname{Im} \lambda = \gamma + \frac{1}{2} + p$ нет полюсов оператора $B_{2l}^{-1}(\lambda)$. Тогда для

$$\forall s \geq \max\{0, p^{-1}(2\gamma + p + 1)(m - l)\} \quad u = \forall s > 0$$

- (i) \exists ограниченный оператор $R_r^0 : H_{s, \gamma}(\mathcal{U}) \rightarrow H_{s, \gamma+p}^{s, s+2(m-l)}(\mathcal{U})$ такой, что $\mathcal{L}^0 R_r^0 f(y, t) = f(y, t) + T_r^0 f(y, t)$ для $\forall f(y, t) \in H_{s, \gamma}(\mathcal{U})$, где

$$(2.8) \quad \begin{aligned} & \|T_r^0 f(y, t)\|_{s, \gamma}(\mathcal{U}) = \\ & = \epsilon \|f(y, t)\|_{s, \gamma}(\mathcal{U}) + C \|R_r^0 f(y, t)\|_{s+2(m-l)-1, \gamma+p}(\mathcal{U}), \end{aligned}$$

постоянная C — не зависит от $f(y, t)$.

- (ii) для $\forall g(y, t) \in H_{s, \gamma+p}^{s, s+2(m-l)}(\mathcal{U})$ справедлива оценка

$$(2.9) \quad \begin{aligned} & \|g(y, t)\|_{s, s+2(m-l), \gamma, \gamma+p} \leq \\ & \leq C \left\{ \|\mathcal{L}^0 g(y, t)\|_{s, \gamma} + \|g(y, t)\|_{s-1+2(m-l), \gamma+p} \right\}, \end{aligned}$$

где постоянная C — не зависит от $g(y, t)$.

После преобразования Меллина (2.7) имеет вид

$$(2.10) \quad A_{2m}(y, D_y, \lambda + ip) B_{2l}^{-1}(\lambda + ip) (Mg)(y, \lambda + ip) + (Mg)(y, \lambda) = \\ = M(\mathcal{L}^0 g(y, t))(\lambda).$$

Мы доказываем теорему 3 подробно исследуя выражение (2.10). Основой этого исследования является решение задачи Карлемана.

Задача К. Найти функцию $\Gamma(\xi, \lambda) \in \tilde{H}_{\xi, a, b}^{s_1, s_2}$, граничные значения которой удовлетворяют условию

$$(2.11) \quad p_{2m}(\xi, \sigma + ib) q_{2l}^{-1}(\xi, \sigma + ib) \Gamma(\xi, \sigma + ib) + \Gamma(\xi, \sigma + ia) = f(\sigma, \xi)$$

для $f(\xi, \sigma) \in \tilde{H}_\xi^{s_1}$.

Здесь $p_{2m}(\xi, \sigma)$ и $p_{2l}(\xi, \sigma)$ — однородные полиномы по σ, ξ с вещественными коэффициентами степени $2m$ и $2l$, соответственно, удовлетворяющие условию

$$(2.12) \quad p_{2m}(\xi, \sigma) = \delta_1 (\xi^2 + \sigma^2)^m, \quad q_{2l}(\xi, \sigma) = \delta_2 (\sigma^2 + |\xi|^2)^l$$

для $\sigma \in R^1$, $\xi \in R^n$, где δ_1 и δ_2 — положительные постоянные.

Разрешимость задачи К устанавливает

Теорема 4. Пусть выполнены условия (2.12) и $s_2 = \left(1 - \frac{2b}{b-a}\right)(m-l)$.

Тогда, для $|\xi| \geq N$, где N — достаточно большое число и для $f(\xi, \sigma) \in \tilde{H}_\xi^s$ существует единственное решение задачи К из $\tilde{H}_{\xi, a, b}^{s, s+2(m-l)}$ причём выполнена оценка

$$(2.13) \quad \|\Gamma(\xi, \lambda)\|_{s, s+2(m-l)} \sim C \|f(\xi, \sigma)\|_s,$$

где постоянная C — не зависит от ξ .

Технически сложному доказательству теоремы 4 посвящен **4**, теоремы 1 и 3 доказываются в **3**.

3. Доказательство теорем 1 и 3

Сначала докажем теорему 3.

Из результатов работы [1] следует, что оператор $B_{2l}^{-1}(\lambda)$ имеет вид

$$(3.1) \quad B_{2l}^{-1}(\lambda) = R(\lambda) - B_{2l}^{-1}(\lambda) T_l(\lambda).$$

Здесь

$$R(\lambda) = \sum_{j=1}^N q_j(y) R_j(\lambda) q_j(v),$$

$\{q_j(y)\}$ — достаточно мелкое разбиение единицы на Γ ,

$$R_j(\lambda) = F_{\bar{\xi}, y}^{-1} |b_{2l}^0(y_j, \bar{\xi}, \lambda)|^{-1} F_{y \rightarrow \cdot}, \quad y_j \in \text{supp } q_j(y), \quad \bar{\xi} = \bar{\xi} - \frac{z_j + d}{1 - z_j}.$$

$d > 0$, b_{2l}^0 — главная часть символа оператора $B_{2l}(y, D_y, tD_l)$. Для оператора $T_1(\lambda)$ справедлива оценка

$$(3.2) \quad \|T_1(\lambda)F(y, \lambda)\|_s \leq \eta_1 \|F(y, \lambda)\|_s + C \|F(y, \lambda)\|_{s-1},$$

где C — не зависит от λ , а $\eta_1 \rightarrow 0$, если $\max_j \operatorname{diam} \operatorname{supp} q_j(y) \rightarrow 0$. ■

Обозначим $v(y, \lambda) = (Mg)(y, \lambda)$ и

$$G(y, \lambda) = M(\mathcal{L}^0 g)(y, \lambda), \quad a = \gamma + \frac{1}{2}, \quad b = \gamma + p + \frac{1}{2}.$$

Тогда (2.10) перепишется в виде

$$(3.3) \quad A_{2m}(y, D_y, \sigma + ib) B_{2l}^{-1}(\sigma + ib) v(y, \sigma + ib) + \\ + v(y, \sigma + ia) = G(y, \sigma + ia).$$

Подставляя (3.1) в (3.3) получим

$$(3.4) \quad A_{2m}(y, D_y, \sigma + ib) R(\sigma + ib) v(y, \sigma + ib) + v(y, \sigma + ia) + \\ + T_2 v(y, \sigma + ib) = G(y, \sigma + ia),$$

где

$$T_2 = -A_{2m}(y, D_y, \sigma + ib) B_{2l}^{-1}(\sigma + ib) T_1(\sigma + ib).$$

Из теоремы 4 следует существование операторов

$$K_j^{-1} : \tilde{H}_{\xi, a, b}^s(R^1) \rightarrow \tilde{H}_{\xi, a, b}^{s, s+2(m-l)},$$

нормы которых ограничены при всех $\xi \in \mathbb{R}^n$, если число d в определении $\hat{\xi}$ достаточно большое число. Причем $v_j(\xi, \lambda) = K_j^{-1} W(\xi, \sigma + ia)$ аналитична в P_a^b и удовлетворяет соотношениям

$$(3.5) \quad a_{2m}(y, \xi, \sigma + ib) [b_{2l}^0(y, \xi, \sigma + ib)]^{-1} v_j(\xi, \sigma + ib) + \\ + v_j(\xi, \sigma + ia) = W(\xi, \sigma + ia).$$

Будем искать решение уравнения (3.4) в виде

$$(3.6) \quad v(y, \lambda) = SF(y, \sigma + ia) =$$

$$= \sum_{r=1}^N q_r(y) F_{y-r}^{-1} K_r^{-1} F_{y-r} q_r(y) F(y, \sigma + ia),$$

где

$$F(y, \lambda) = (Mf)(y, \lambda).$$

Из (2.13) следует оценка

$$(3.7) \quad \int_{-\infty}^{+\infty} [\|v(y, \sigma + ib)\|_{s+2(m-l)}^2(F) + \|v(y, \sigma + ia)\|_s^2(F)] d\sigma \leq \\ \leq C \int_{-\infty}^{+\infty} \|F(y, \sigma + ia)\|_s^2(F) d\sigma.$$

Стандартными приёмами (см. например [1], [6]) приведём (3.4) к виду

$$(3.8) \quad \sum_{j=1}^N q_j(v) F_{\tilde{v}-\tilde{\gamma}}^{-1} \{a_{2m}(v_p, \tilde{\xi}, \sigma+ib) [b_{2l}^0(v_p, \tilde{\xi}, \sigma+ib)]^{-1} F_{v-\tilde{\gamma}} q_j(v) v(v, \sigma+ib) + \\ + F_{v-\tilde{\gamma}} q_j(v) v(v, \sigma+ia)\} + [T_2 + T_3] v(v, \sigma+ib) = G(v, \sigma+ia).$$

По предположению теоремы на прямой $\sigma+ib$ нет полюсов оператора $B_{2l}^{-1}(\tilde{\lambda})$. Из этого следует, что для оператора T_2 справедлива оценка

$$(3.9) \quad T_2 v(v, \sigma+ib) \|_s(I) \leq \eta_2 \|v(v, \sigma+ib)\|_{s+2m-n} + \\ + C \|v(v, \sigma+ib)\|_{s+2m-n-1}(I).$$

Выписав оператор T_3 в явном виде, легко получить и для него оценку

$$(3.10) \quad T_3 v(v, \sigma+ib) \|_s(I) \leq \eta_3 \|v(v, \sigma+ib)\|_{s+2m-n} + \\ + C \|v(v, \sigma+ib)\|_{s+2m-n-1}(I).$$

Здесь η_2 и η_3 обладают тем же свойством, что и η_1 .

Подставим (3.6) в (3.8). Используя соотношения (3.5) и стандартные приемы (см. например [1], [6]) мы получим уравнение

$$(3.11) \quad F(v, \sigma+ia) + T_1 v(v, \sigma+ib) = G(v, \sigma+ia).$$

Из (3.9), (3.10) и (3.6) следует, что для T_1 выполнена оценка аналогичная (3.10). Запишем её в виде

$$(3.12) \quad T_1(SF)(v, \sigma+ib) \|_s(I) \leq \eta_4 \|SF(v, \sigma+ib)\|_{s+2m-n}(I) + \\ + C \|SF(v, \sigma+ib)\|_{s+2m-n-1}(I),$$

где $\eta_4 \rightarrow 0$, когда $\max_j \operatorname{diam} \operatorname{supp} q_j(v) \rightarrow 0$.

После преобразования Меллина в (3.11), (3.12) и используя (3.7) получим пункт (i) теоремы 3. ■

Априорная оценка (2.9) доказывается аналогичными рассуждениями.

Перейдем к доказательству теоремы 1.

Докажем сначала априорную оценку (2.4). Пусть $q_1(x)$ и $q_2(x)$ функции из $C^\infty(G)$ такие, что

$$(i) \quad q_1(x) + q_2(x) \equiv 1, \quad x \in G,$$

$$(ii) \quad q_1(x) \equiv 1, \quad \text{если } x \in G_c = \{x : x \in G, q(x, I) < \epsilon\} \text{ и } q_1(x) \equiv 0 \text{ на } G \setminus G_c.$$

Тогда $u(x) = q_1(x) u_1(x) + (1 - q_1(x)) u_2(x) = u_1(x) + u_2(x)$ и по условию 1

$$(3.13) \quad \|u_2(x)\|_{s_1, s_2, r, r+p}(G) \leq C [\|q_2 \mathcal{L} u\|_{s_1}(G) + \|u\|_{s+2m-1}(G \setminus G_c)].$$

Займемся оценкой $u_1(x)$. Переходя к локальным координатам, введенным в начале настоящего параграфа, мы можем считать (не вводя для простоты новых обозначений), что $u_1 = u_1(y, t)$ функция на цилиндре H ,

последний член в правой части (3.14) имеет норму в области $\mathcal{U}_{\varepsilon_1} = \{(y, t) : y \in \Gamma, 0 < t < \varepsilon_1\}$, где $\varepsilon_1(\varepsilon) \rightarrow 0$ при $\varepsilon \rightarrow 0$.

Выражение $\mathcal{L} u_1$ представим в виде

$$(3.14) \quad \mathcal{L} u_1 = \mathcal{L}^0 B_{2t}(y, D_y, tD_t) u_1 + [\mathcal{L} - \mathcal{L}^0 B_{2t}(y, D_y, tD_t)] u_1.$$

Обозначим $B_{2t}(y, D_y, tD_t) u_1 = g(y, t)$. Тогда из (2.9) следует оценка

$$(3.15) \quad \|g(y, t)\|_{s, s+2(m-l), \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \leq C \left[\|\mathcal{L}^0 g(y, t)\|_{s, \gamma}(\mathcal{U}_{\varepsilon_1}) + \|g(y, t)\|_{s+2(m-l)-1, \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \right].$$

Так как

$$\|g(y, t)\|_{s+2(m-l)-1, \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) < \varepsilon_2 \|g(y, t)\|_{s, s+2(m-l), \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}),$$

где $\varepsilon_2 \rightarrow 0$ при $\varepsilon \rightarrow 0$, то

$$(3.16) \quad \|g(y, t)\|_{s, s+2(m-l), \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \leq C(1 + \varepsilon_2)^{-1} \|\mathcal{L}^0 g(y, t)\|_{s, \gamma}(\mathcal{U}_{\varepsilon_1}).$$

Из предположений теоремы об отсутствии полюсов оператора $B_{2l}^{-1}(\lambda)$

$$(3.17) \quad \|u_1\|_{s+2l, \gamma}(\mathcal{U}_{\varepsilon_1}) \leq C \|B_{2l}(y, D_y, t, D_t) u_1\|_{s, \gamma}(\mathcal{U}_{\varepsilon_1}).$$

$$\|u_1\|_{s+2m, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \leq C \|B_{2l}(y, D_y, t, D_t) u_1\|_{s+2(m-l), \gamma+p}(\mathcal{U}_{\varepsilon_1}).$$

Объединяя оценки (3.16) и (3.17), получим

$$(3.18) \quad \|u_1\|_{s+2l, s+2m, \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \leq C_1 \|g(y, t)\|_{s, s+2(m-l), \gamma, \gamma+p}(\mathcal{U}_{\varepsilon_1}) \leq C_2 \|\mathcal{L}^0 B_{2l}(y, D_y, tD_t) u_1\|_{s, \gamma}(\mathcal{U}_{\varepsilon_1}).$$

Второй член в правой части (3.14) имеет в $\mathcal{U}_{\varepsilon_1}$ малую норму в соответствующих пространствах (см. [6]) и поэтому из оценок (3.18) и (3.13) и стандартных рассуждений следует неравенство (2.4). ■

Первая часть теоремы 1 доказывается построением оператора R обычным в эллиптической теории способом склейки при помощи разбиения единицы из регуляризатора в области $G \setminus G_\varepsilon$ для равномерно эллиптического оператора \mathcal{L} и оператора R_l^0 в области G_ε , существование которого доказано в теореме 3. При этом для оператора T из (2.3) получается оценка

$$(3.19) \quad \|Tf\|_{s, \gamma}(G) \leq z(\varepsilon) \|f\|_{s, \gamma}(G) + \\ + C \left[\|Rf\|_{s+2m-1, \gamma+p}(G) + \|f\|_{s-1}(G \setminus G_\varepsilon) \right],$$

где $z(\varepsilon) \rightarrow 0$ при $\varepsilon \rightarrow 0$.

Докажем вполне непрерывность оператора T . Так как

$$\|Rf\|_{s+2m-1, \gamma+p}(G) \leq z_1(\varepsilon) \|f\|_{s, \gamma}(G) + C \|f\|_{s-1}(G \setminus G_\varepsilon),$$

то из (3.19) следует

$$(3.20) \quad \|Tf\|_{s, \gamma}(G) \leq z_2(\varepsilon) \|f\|_{s, \gamma}(G) + C \|f\|_{s-1}(G \setminus G_\varepsilon).$$

Здесь $z_i(\varepsilon) \rightarrow 0$ при $\varepsilon \rightarrow 0$ $i = 1, 2$.

Пусть $\{f_k\}$ — множество функций равномерно ограниченное в $H_{s, \varepsilon}(G)$, тогда $\{f_k\}$ по теореме вложения компактно в $H^{s-1}(G \setminus G_\varepsilon)$ для $\forall \varepsilon > 0$. Из (3.20) следует компактность $\{Tf_k\}$. ■

4. Доказательство теоремы 4

Теорема 4 будет выведена из ряда лемм.

Лемма 1. Справедливы формулы:

$$F[\ln(\sigma + i\theta)](t) = (2\pi I''(1) + i\pi^2)\delta(t) - 2\pi t_-^{-1},$$

$$F[\ln(\sigma - i\theta)](t) = (2\pi I''(1) - i\pi^2)\delta(t) - 2\pi t_+^{-1}.$$

(Заметим, что мы придерживаемся обозначений монографии [4].)

Доказательство. Так как

$$\ln(\sigma + i\theta) = \ln|\sigma| + \pi i \Theta(\cdot - \sigma)$$

и

$$\ln(\sigma - i\theta) = \ln|\sigma| - \pi i \Theta(\cdot - \sigma),$$

то доказываемые формулы выводятся из формул, приведённых в [4] (стр. 124, 129, 222).

Лемма 2. Пусть $p(\sigma + i\gamma, \xi) = (\sigma + i\gamma - i\chi(\xi))(\sigma + i\gamma + i\chi(\xi))$ и пусть $\chi(\xi) = \mu(\xi) + i\nu(\xi)$, $|\mu(\xi)| < |\gamma|$, $|\nu(\xi)| \leq \mu(\xi) \leq r_1|\xi|$, $r_1, r_2 > 0$.

Тогда

$$(4.1) \quad F[\ln p(\sigma + i\gamma, \xi)](t) = 4\pi I''(1)\delta(t) - 2\pi e^{\gamma t}[e^{-\gamma t}t_-^{-1} + e^{\gamma t}t_+^{-1}].$$

Доказательство. Так как

$$F[\ln(\sigma + i\gamma + i\chi)](t) = e^{\chi t + \gamma t}F[\ln(\sigma + i\theta)]$$

и

$$F[\ln(\sigma + i\gamma - i\chi)](t) = e^{-\chi t + \gamma t}F[\ln(\sigma - i\theta)],$$

то (4.1) следует из леммы 1.

Лемма 3. Для $\forall \xi \in \mathbf{R}^n$ таких, что $\mu(\xi) < a$, функция $m(\lambda, \xi)$ такая, что

$$(4.2) \quad \begin{aligned} F[\ln m(\sigma + i\gamma, \xi)](t) &= 4\pi(b - a)^{-1}I''(1)\delta'(t) - \\ &- 4\pi I''(1)(b - a)(b - a)^{-1}\delta(t) - 4\pi I''(1)\delta(t) - \\ &- 2\pi(1 - e^{(b-a)t})^{-1}e^{\gamma t}[e^{-\gamma t}t_-^{-1} + e^{\gamma t}t_+^{-1}]e^{(b-a)t}, \end{aligned}$$

аналитична в полосе $a < \operatorname{Im} \lambda < b$, непрерывна в полосе $a \leq \operatorname{Im} \lambda \leq b$ и удовлетворяет соотношению

$$(4.3) \quad p(\sigma + ib, \xi) = m(\sigma + ia, \xi)m^{-1}(\sigma + ib, \xi),$$

где $p(\sigma + ib, \xi)$ — функция введённая в лемме 2.

Доказательство. Правая часть (4.2) есть $O(e^{-(\gamma + |b|)t})$ при $t \rightarrow +\infty$ и $O(e^{(\gamma + |b| - a)t})$ при $t \rightarrow -\infty$. Значит функция $\ln m(\lambda, \xi)$, и следовательно $m(\lambda, \xi)$ аналитична в полосе $a - (|b| - |\mu|) < \operatorname{Im} \lambda < b + (|b| + |\mu|)$.

Для доказательства формулы (4.3) положим в (4.2) $\gamma = a$ и $\mu = b$. Затем вычтем полученные выражения друг из друга. Получим в результате

$$\begin{aligned} F[\ln m(\sigma + ia, \xi)](t) - F[\ln m(\sigma + ib, \xi)](t) = \\ = 4\pi I'(1)\delta(t) - 2\pi e^{bt}[e^{-\sigma t}t^{-1} + e^{\sigma t}t^{-1}] = F[\ln p(\sigma + ib, \xi)](t). \quad \blacksquare \end{aligned}$$

Лемма 4. Пусть $W_a = \{\xi : \xi \in \mathbf{R}^n, \mu(\xi) > 1 + |b|\}$. Для $\gamma \neq \xi \in W_a$ и $\forall \sigma \in \mathbf{R}^1$ справедливы оценки:

$$(4.4) \quad K_2(1 + |\sigma| + |\xi|)^{-1 - \frac{2b}{b-a}} \leq |m(\sigma + ia, \xi)| \leq$$

$$< K_1(1 + |\sigma| + |\xi|)^{-1 - \frac{2b}{b-a}},$$

$$\begin{aligned} (4.5) \quad K_4(1 + |\sigma| + |\xi|)^{-1 - \frac{2b}{b-a}} &\leq |m(\sigma + ib, \xi)| \leq \\ &\leq K_3(1 + |\sigma| + |\xi|)^{-1 - \frac{2b}{b-a}}. \end{aligned}$$

Доказательство. Запишем $\operatorname{sh}^{-1}(y)$ в виде: $\operatorname{sh}^{-1}(y) = y^{-1}e^{-3y} + + \operatorname{sgn} y[2e^{-y} + e^{-3y}] + q(y)$, где $q(y) = O(e^{-3y})$ при $|y| \rightarrow \infty$ и $q(y) = O(y)$ при $|y| \rightarrow 0$.

Так как $(1 - e^{ia t})^{-1} = 2e^{-2} \operatorname{sh}\left(\frac{\tau_0 t}{2}\right)$, то обозначив $b - a = \tau_0$, $\gamma - a = \tau$, из (4.2) получим

$$\begin{aligned} (4.6) \quad F[\ln m(\sigma + ia + i\tau, \xi)](t) = \\ = e^{bt}[2\pi \tau_0^{-1} t_v^{-2} e^{-(\varepsilon - \tau + 2\tau_0)t} + 2\pi t_v^{-1} e^{-(\varepsilon - \tau + \tau_0)t} + \pi t_v^{-1} e^{-(\varepsilon - \tau + 2\tau_0)t} + \\ + 2\pi \tau_0^{-1} t_v^{-2} e^{(\varepsilon + \tau + \tau_0)t} - 2\pi t_v^{-1} e^{(\varepsilon + \tau)t} - \pi t_v^{-1} e^{(\varepsilon + \tau + \tau_0)t}] + \psi_1(t) + \psi_2(t, \xi). \end{aligned}$$

Здесь

$$\psi_1(t) = 4\pi \tau_0^{-1} I'(1)\delta'(t) - 4\pi I'(1)\left(\frac{\tau_0}{\tau} + 1\right)\delta(t) + C\delta(t),$$

$$\psi_2(t, \xi) = \pi q\left(\frac{\tau_0 t}{2}\right) \left[t_v^{-1} e^{-\left(\varepsilon - \tau - \frac{\tau_0}{2}\right)t} + t_v^{-1} e^{\left(\varepsilon + \tau - \frac{\tau_0}{2}\right)t} \right] e^{bt}.$$

Легко показать, что для $\xi \in W_a$

$$\int_{-\infty}^{+\infty} |\psi_2(t, \xi)| dt < C$$

причём C — не зависит от ξ .

Следовательно

$$(4.7) \quad |F^{-1}[\psi_2(t, \xi)](\lambda)| \leq C_1,$$

где C_1 – не зависит от $\xi \in W_n$, $\sigma \in R^1$, $\tau \in [0, b-a]$. Из вида $\psi_1(t)$ следует, что

$$(4.8) \quad \operatorname{Re} F^{-1}[\psi_1(t)] \leq C_2,$$

где C_2 – не зависит от $\sigma \in R^1$ и $\tau \in [0, b-a]$.

Обратное преобразование Фурье явно выписанных членов правой части (4.6) подсчитываются по формулам, приведённым в [4] (стр. 223, 224). После применения F^{-1} к обеим частям (4.6) и элементарных выкладок получим

$$(4.9) \quad \begin{aligned} & \operatorname{Re} \ln m(\sigma + ia + i\tau, \xi) = \tau_0^{-1}(I'(1) + 1)(2b + 2\tau - \tau_0) + \\ & + \frac{\sigma + r}{\tau_0} \left[\operatorname{arctg} \left(-\frac{\sigma + r}{\mu + b + \tau + \tau_0} \right) - \operatorname{arctg} \left(-\frac{\sigma + r}{\mu - b - \tau + 2\tau_0} \right) \right] + \\ & + \ln \left[\left[(\sigma + r)^2 + (\mu + b + \tau + \tau_0)^2 \right]^{\frac{\mu - b - \tau + \tau_0}{2\tau_0}} \times \right. \\ & \times \left[(\sigma + r)^2 + (\mu - b + \tau + \tau_0)^2 \right]^{\frac{\mu + b + \tau}{2\tau_0}} \left[(\sigma + r)^2 + (\mu + b - \tau)^2 \right]^{\frac{1}{2}} \times \\ & \times \left. \left[(\sigma + r)^2 + (\mu - b - \tau + \tau_0)^2 \right]^{\frac{1}{2}} \right] + \operatorname{Re} F^{-1}[\psi_1(t) + \psi_2(t, \xi)]. \end{aligned}$$

Первый и второй члены суммы в правой части (4.9) равномерно ограничены при $\sigma \in R^1$, $\xi \in W_n$, $\tau \in [0, b-a]$, а последний член этой суммы ограничен в силу (4.7), (4.8).

Следовательно справедлива оценка

$$(4.10) \quad \begin{aligned} & C[(\sigma + r)^2 + 1 + \mu^2]^{\frac{-b-\tau+\tau_0}{2\tau_0}-\frac{1}{2}} \leq |m(\sigma + ia + i\tau, \xi)| \leq \\ & \leq \tilde{C}[(\sigma + r)^2 + 1 + \mu^2]^{\frac{-b+\tau+\tau_0}{2\tau_0}+\frac{1}{2}}, \end{aligned}$$

где C и \tilde{C} – постоянные не зависящие от $\xi \in W_n$, $\sigma \in R^1$ и $\tau \in [0, \tau_0]$. Так как $\varepsilon_2|\xi| \leq \mu(\xi) \leq \varepsilon_1|\xi|$, то лемма доказана. ■

Обозначим Wb , $p = \{\xi : \xi \in R^n, p_{2m}(\lambda, \xi) \neq 0 \text{ для } |\operatorname{Im} \lambda| \leq 1 + |b|\}$.

Лемма 5. Пусть $p_{2m}(\lambda, \xi)$ полином с вещественными коэффициентами удовлетворяющий условию (2.12). Тогда \exists функция $M(\lambda, \xi)$, аналитическая в полосе $a \leq \operatorname{Im} \lambda \leq b$ такая, что

$$(4.11) \quad p_{2m}(\sigma + ia, \xi) = M(\sigma + ia, \xi) M^{-1}(\sigma + ib, \xi)$$

и справедлива оценка

$$(4.12) \quad \begin{aligned} & C_1(1 + |\sigma|^2 + |\xi|^2)^{\left[\frac{1}{2} - \frac{b+\tau}{b-a} \right]m} \leq |M(\sigma + ia + i\tau, \xi)| \leq \\ & \leq \tilde{C}_1(1 + |\sigma|^2 + |\xi|^2)^{\left[\frac{1}{2} - \frac{b+\tau}{b-a} \right]m} \end{aligned}$$

где C_1 и \tilde{C}_1 – постоянные, не зависящие от $\xi \in Wb$, p , $\sigma \in R^1$ и $\tau \in [0, \tau_0]$.

Доказательство. Из условия (2.12) следует представление

$$p_{2m}(\sigma, \xi) = p_{2m}(1, 0) \prod_{k=1}^m (\sigma - i z_k(\xi)) (\sigma + i \bar{z}_k(\xi)),$$

где $z_k(\xi) = \mu_k(\xi) + i r_k(\xi)$, $\mu_k(\xi) = e^{|\xi|}$. Значит лемма 5 получается по-следовательным применением лемм 2, 3, 4. Заметим, что

$$Wb, p = \bigcap_{k=1}^m W_{z_k}. \blacksquare$$

Разрешимость задачи K в пространстве с фиксированными s_1 и s_2 устанавливает

Теорема 5. Пусть выполнены условия (2.12) и

$$s_1 + s = - \left(1 - \frac{2b}{b-a} \right) (m-l), \quad s_2 = \left(1 + \frac{2b}{b-a} \right) (m-l).$$

Тогда для $\tau \in Wb, p \subset Wb, q$ и для $\tau f(\sigma, \xi) \in \dot{H}_{\xi}^s(R^1)$ – единственное решение задачи K из $\dot{H}_{\xi, a, b}^{s_1, s_2}$.

Причем для $\tau v(\lambda, \xi) \in \dot{H}_{\xi, a, b}^{s_1, s_2}$ и $\tau \xi \in Wb, p \subset Wb, q$ выполнена оценка

$$(4.13) \quad C \|f(\sigma, \xi)\|_s^2 \leq \|v(\lambda, \xi)\|_{s_1, s_2}^2 \cdot \hat{C} \|f(\sigma, \xi)\|_s^2,$$

где постоянные C, \hat{C} – не зависят от ξ .

Доказательство. Из леммы 5 следует существование функций $M(\lambda, \xi)$ и $N(\lambda, \xi)$, которые удовлетворяют равенствам (4.11) для полиномов p_{2m} и q_{2l} соответственно. Тогда (2.11) перепишется в виде

$$(4.14) \quad \begin{aligned} & M^{-1}(\sigma + ia, \xi) N(\sigma + ia, \xi) v(\sigma + ia, \xi) + \\ & + M^{-1}(\sigma + ib, \xi) N(\sigma + ib, \xi) v(\sigma + ib, \xi) = \\ & = M^{-1}(\sigma + ia, \xi) N(\sigma + ia, \xi) f(\sigma, \xi). \end{aligned}$$

Обозначим

$$V(\lambda, \xi) = M^{-1}(\lambda, \xi) N(\lambda, \xi) v(\lambda, \xi),$$

$$G(\sigma, \xi) = M^{-1}(\sigma + ia, \xi) N(\sigma + ia, \xi) f(\sigma, \xi).$$

Тогда (4.14) примет вид

$$(4.15) \quad V(\sigma + ia, \xi) + V(\sigma + ib, \xi) = G(\sigma, \xi).$$

Заметим, что из оценок для $M(\sigma + ib, \xi), N(\sigma + ib, \xi)$ следует, что $G(\sigma, \xi) \in L^2(R_\sigma^1)$.

После преобразования Фурье по σ в (4.15) получим

$$e^{at} F[V(\sigma, \xi)](t) + e^{bt} F[V(\sigma, \xi)](t) = F[G(\sigma, \xi)](t).$$

Следовательно

$$(4.16) \quad F[V(\sigma, \xi)](t) = (e^{at} + e^{bt})^{-1} F[G(\sigma, \xi)](t).$$

Значит $V(\lambda, \xi)$ аналитична в полосе $a < \operatorname{Im} \lambda < b$ и выполнена оценка

$$(4.17) \quad \int_{-\infty}^{+\infty} (|V(\sigma + ia, \xi)|^2 + |V(\sigma + ib, \xi)|^2) d\sigma \leq 2 \int_{-\infty}^{+\infty} |G(\sigma, \xi)|^2 d\sigma.$$

Из (4.17) легко следует (4.13), а значит теорема 5 доказана. ■

Переходим наконец к доказательству теоремы 4. Теорема 4 доказана нами при

$$s = s_0 = -\left(1 - \frac{2b}{b-a}\right)(m-l)$$

(см. Теорему 5). Покажем справедливость оценки (2.13) для всех $s = s_0$. Это и будет доказывать теорему 4. Пусть $s = s_0 + 1$. Продифференцировав (4.16) по t получим

$$[e^{at} F[V](t)]' = [e^{at} [e^{at} + e^{bt}]^{-1}]' F[G](t) + e^{at} [e^{at} + e^{bt}]^{-1} [F[G](t)]'.$$

Откуда после преобразования Фурье следует

$$\int_{-\infty}^{+\infty} |\sigma V(\sigma + ia, \xi)|^2 d\sigma \leq C \int_{-\infty}^{+\infty} (|\sigma G(\sigma, \xi)|^2 + |G(\sigma, \xi)|^2) d\sigma,$$

где C – зависит лишь от a и b . Аналогичная оценка получается для $V(\sigma + ib, \xi)$. После их объединения и применения леммы 5 получим (2.13) для $s = s_0 + 1$. Так же доказывается (2.13) для $s = s_0 + k$, $k = 2, 3, 4, \dots$

Докажем теперь (2.13) для $s = s_0 + d$, где $0 < d < 1$. По известной формуле для дробной производной имеем

$$(4.18) \quad \begin{aligned} & ad \int_{-\infty}^{+\infty} |\sigma|^{2d} |V(\sigma + ia, \xi)|^2 d\sigma = \\ & = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\tau|^{-(1+2d)} |e^{a(t+\tau)} F[V](t+\tau) - e^{at} F[V](t)|^2 dt d\tau. \end{aligned}$$

Из (4.16) следует, что правая часть (4.18) оценивается сверху выражением

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [1 + e^{(b-a)(t+\tau)}]^{-2} |F[G](t+\tau) - F[G](t)|^2 |\tau|^{-1-2d} dt d\tau + \\ & + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \{(1 + e^{(b-a)(t+\tau)})^{-1} - (1 + e^{(b-a)t})^{-1}\}^2 |\tau|^{-1-2d} dt d\tau = I_1 + I_2. \end{aligned}$$

Очевидно, что

$$I_1 \leq \int_{-\infty}^{+\infty} |G(\sigma, \xi)|^2 |\sigma|^{2d} d\sigma,$$

$$I_2 \leq C \int_{-\infty}^{+\infty} |G(\sigma, \xi)|^2 d\sigma,$$

где C – не зависит от ξ .

Из полученных оценок и (4.18), (4.17) следует (2.13) для $s = s_0 + d$. Общий результат получает комбинированием предыдущих рассуждений. ■

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ON THE "LARGE VALUE" ESTIMATE OF HECKE POLYNOMIALS

By

M. MAKNYS

Department of Probability Theory and Number Theory
of the V. Kapsukas University, Vilnius and Department of Numerical Methods
and Computer Science of the L. Eötvös University, Budapest

(Received February 24, 1982)

1. Introduction. It is well-known that the mean and "large" value estimates of Dirichlet polynomials

$$(1.1) \quad \sum_{N \leq n} a(n) \chi(n) n^{-s}$$

are very useful for obtaining zero-density results for Riemann zeta and Dirichlet L -functions in region $3/4 < \sigma < 1$. See for example [1], [5], [6], [12].

In this paper we shall consider Dirichlet polynomials in quadratic imaginary number field $Q(\sqrt{d})$ ($d < 0$ is a square-free number) which is widened to an ideal number system K [2], [8]. We shall prove "large value" estimate for so called Hecke polynomials, i.e.

$$(1.2) \quad \sum'_{N \neq N}^* C(z) \Xi(z) N z^{-s},$$

where z is an ideal number from K , Nz denotes the norm of z , $s = \sigma + it$, $N \geq 1$, $\Xi(z)$ is Hecke's character of the second kind (or Größencharakter) mod η , ($\eta \neq 0$) is an integer ideal from our field) with exponent m , $m \in \mathbf{Z}$, i.e.

$$(1.3) \quad \Xi_m(z) = \chi(z) e^{igm \arg z},$$

where $\chi(z)$ is an abelian character of the multiplicative group mod η , g is the number of units mod η , $C(z)$ in (1.2) are complex numbers depending on z . The ' $'$ in (1.2) and as follows denotes that z runs over a set of non-associated non-zero numbers from K .

By using the methods due to HUXLEY [3], JUTILA [5, 6] and HEATH-BROWN [1], we shall prove the following

THEOREM. Suppose that we have sequences $\Xi_r(z)$ and $s_{mr} = \sigma_{mr} + it_{mr}$, $r = 1, 2, \dots, R$, $\sigma_{mr} > 0$. Assume that for $r \neq q$ or $\Xi_r \neq \Xi_q$ or $|t_{rm} - t_{qm}| \geq 1$, $|t_{rm}| < T$, $T > 2$, $|m| < M$, $M > 1$. Let $V_{mt} = \left(\frac{1}{4} g^2 m^2 + t^2 \right)^{1/2}$, $1 < N \leq V_{MT}^{7/6}$, $\sigma = \min \sigma_{mr}$, $C(z)$ be complex numbers depending only on z .

Then for every fixed integer $k \geq 1$ we have

$$(1.4) \quad \left(\sum_{r \leq R} \sum_{N \leq N \leq 2N}^{*} |c(\alpha) \Xi_r(\alpha) N \alpha^{-2mr}| \right)^2 \ll \\ \ll V_{MT}^{\epsilon} \sum_{N \leq N \leq 2N}^{*} |c(\alpha)|^2 N \alpha^{-2\sigma} \times \\ \times (RN + R^{2-1/2k} V_{MT} + R^{2-1/4k} V_{MT}^{1/2k} + \sqrt{N} R^2 E(N)),$$

where $\epsilon > 0$ is an arbitrary small constant depending on k and the field, the constant implied in symbol \ll depends on ϵ, k , the ideal η and the field and

$$E(N) = \begin{cases} 1, & \text{for } N \leq V_{MT}^{7/6}, \\ V_{MT}^{-1/6k} & \text{for } N > V_{MT}. \end{cases}$$

This theorem is similar to lemma 6 of HEATH-BROWN [1].

The letter ϵ denotes an arbitrarily small positive constant not the same at every occurrence, C_1, C_2, \dots , are absolute constants that may depend on the field, k , the ideal η and ϵ . The same is true for symbols Q and \ll .

2. Lemmas. We shall use some lemmas to prove (1.4). First we need some properties of Hecke Z -functions. For $\sigma \geq 1$ the Hecke Z -functions are defined by the series

$$(2.1) \quad \sum_{\alpha=0}^{*} \Xi(\alpha) N \alpha^{-s} = Z(s, \Xi).$$

We restate some properties of $Z(s, \Xi)$ [2] in the following.

LEMMA 1. Function $Z(s, \Xi)$ is an entire function excepting the case of trivial character and has a functional equation of Dirichlet type

$$(2.2) \quad Z(s, \Xi) = \chi(\Xi) \Psi(s, \Xi) Z(1-s, \Xi),$$

where $|\chi(\Xi)| = 1$ and

$$(2.3) \quad \Psi(s, \Xi) = \left(\frac{1}{2\pi} \sqrt{DN} \eta \right)^{1-2s} \frac{\Gamma\left(\frac{1}{2} g|m| + 1 - s\right)}{\Gamma\left(\frac{1}{2} g|m| + s\right)}.$$

LEMMA 2. In the region $\sigma \geq 1$, $V_{mt} \geq 2$ the relation

$$(2.4) \quad |\Psi(s, \Xi)| \ll (\sqrt{DN} \eta V_{mt})^{1-2\sigma}$$

holds. Here the constant implied in symbol \ll is an absolute one.

PROOF. The assertion follows from Stirling's formula for $\ln \Gamma(S)$ if we rewrite $\Psi(s, \Xi)$ in the form

$$\Psi(s, \Xi) = \chi(\Xi) \left(\frac{1}{2\pi} \sqrt{DN} \eta \right)^{1-2s} \exp \left\{ \ln \Gamma\left(\frac{1}{2} g|m| + 1 - s\right) - \ln \Gamma\left(\frac{1}{2} g|m| + s\right) \right\}.$$

LEMMA 3. Suppose that $h = \ln^2 DN \eta V_{MT}$, $0 \leq \sigma \leq 1$, $T > 2$, $M \geq 1$, $|t| \geq h$, $1 < N < V_{MT}^{2/3}$,

$$(2.5) \quad H(s, \Xi) = \sum_{\alpha}^* \left\{ \exp \left(- \left(\frac{N\alpha}{2N} \right)^h \right) - \exp \left(- \left(\frac{N\alpha}{N} \right)^h \right) \right\} \Xi(\alpha) N \alpha^{-s}.$$

Then

$$(2.6) \quad |H(s, \Xi)| \ll E_1(\Xi) N + 1 + \sqrt{N} \int_{-h^2}^{h^2} \left| \sum_{N\alpha \leq P}^* \Xi(\alpha) N \alpha^{-\frac{1}{2}+it+i\tau} \right| d\tau,$$

where

$$P = \frac{DN \eta V_{MT}^2}{N} \ln c_1 DN \eta V_{MT}$$

and

$$E_1(\Xi) = \begin{cases} \operatorname{res}_{s=1} Z(s, \Xi) & \text{if } \Xi = \Xi_0 \text{ is a trivial character,} \\ 0, & \text{otherwise} \end{cases}$$

and the constants C_1 and that involved by \ll in (2.6) depend only on our field.

PROOF. We have [12], [9]

$$(2.7) \quad H(s, \Xi) := \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z(s+w, \Xi) \Gamma \left(\frac{w}{h} + 1 \right) \frac{(2N)^w - N^w}{w} dw.$$

Moving the line of integration in (2.7) to $\operatorname{Re}(s+w) = \frac{1}{2}$, we pass the pole at point $s+w = 1$ in case of trivial character with residue

$$\left[\operatorname{res}_{s=1} Z(s, \Xi_0) \right] \cdot \Gamma \left(\frac{1-s}{h} + 1 \right) \frac{(2N)^{1-s} - N^{1-s}}{1-s} \ll E_1(\Xi) N.$$

Applying the functional equation for $Z(s, \Xi)$ (2.2) and removing to the line $\operatorname{Re}(s+w) = -1/2$ we have

$$\begin{aligned} |H(s, \Xi)| &\ll E_1(\Xi) N + \left| \int_{\left(\operatorname{Re}(s+w) = -\frac{1}{2} \right)} \Gamma \left(\frac{w}{h} + 1 \right) \times \right. \\ &\quad \times \left. \frac{(2N)^w - N^w}{w} \sum_{N\alpha \leq P}^* N \alpha^{-1+s+w} \Xi(\alpha) \Psi(1-s-w, \Xi) dw \right| + \\ &\quad + \left| \int_{\left(\operatorname{Re}(s+w) = -\frac{1}{2} \right)} \Gamma \left(\frac{w}{h} + 1 \right) \times \right. \\ &\quad \times \left. \frac{(2N)^w - N^w}{w} \sum_{N\alpha > P}^* N \alpha^{-1+s+w} \Xi(\alpha) \Psi(1-s-w, \Xi) dw \right| = \\ &= E_1(\Xi) + J_1 + J_2. \end{aligned}$$

Estimating J_1 we move the line of integration to $\operatorname{Re}(s + it) = \frac{1}{2}$ and rewrite J_1 as sum of j_1 and j_2 with limits of integration $|\tau| \leq h^2$ and $|\tau| > h^2$ respectively. Then j_1 gives us the needed integral term in (2.6), from Lemma 2 and from Stirling's formula it follows that $|j_2| \ll 1$. Estimating J_2 we move the line of integration to $\operatorname{Re} w = -\frac{h}{2}$ and we have from Lemma 2 and Stirling's formula that

$$|J_2| \ll \int_{-\infty}^{\infty} \sum_{N \geq P}^* N \tau^{-1-\frac{h}{2}+\sigma} \left[DN \eta \left(\frac{1}{2} g^2 m^2 + (t+\tau)^2 \right) \right]^{1+h-\sigma - \frac{1}{2} \cdot \frac{1}{h}} N^{-\frac{h}{2}} d\tau.$$

For $|\tau| > |t|$ we get obviously that $|J_2| \ll 1$. If $|\tau| \leq |t|$ then $|J_2| \ll 1$ from the condition $P = N^{-1} DN \eta V_{MT}^2 \ln^{C_1} DN \eta V_{MT}$.

LEMMA 4. Let $M > 1$, $T > 2$, for each Ξ we associate natural number R_Ξ and suppose that $|t_{am} - t_{bm}| \asymp \ln^2 V_{MT}$ for $1 \leq a, b \leq R_\Xi$, $a \neq b$, $|t_{cm}| \asymp T$, $|m| \asymp M$.

Then for $\frac{1}{2} - \sigma \ll \frac{1}{2} + (\ln V_{MT})^{-1}$

$$(2.8) \quad \sum_{m \asymp M} \sum_{t=1}^{R_\Xi} |Z(\sigma + it, \Xi)|^4 \ll V_{MT}^2 \ln^{C_2} V_{MT}.$$

For the proof see [10].

LEMMA 5. For $V_{mt} \geq 2$ we have

$$|Z(s, \Xi)| \ll \begin{cases} V_{MT}^{2/3(1-\sigma)} \ln^{C_3} V_{mt}, & \frac{1}{2} - \sigma \geq 1, \\ V_{MT}^{1-\frac{1}{3}\sigma} \ln^{C_4} V_{mt}, & 0 \leq \sigma \leq \frac{1}{2}. \end{cases}$$

PROOF. It follows from the inequalities

$$|Z(1+it, \Xi)| \ll \ln V_{mt},$$

$$|Z(0+it, \Xi)| \ll V_{mt} \ln V_{mt},$$

$$|Z\left(\frac{1}{2}+it, \Xi\right)| \ll V_{mt}^{1/3} \ln^{C_5} V_{mt},$$

see [8], [7] and by taking into account the convexity of Dirichlet series of Hecke Z -functions.

LEMMA 6. Let $|t| < T$, $T > 2$, $|m| \leq M$, $M > 1$, $a = (\ln V_{MT})^{-1}$. Then for a natural number $N > 1$ we have

$$(2.9) \quad \sum_{N \geq N}^* \Xi(\alpha) N x^{-\frac{1}{2}+it} = \\ = E_1(\Xi) \frac{N^{1/2+it}}{\frac{1}{2}+it} + \frac{1}{2\pi i} \int_{a-iV_{MT}}^{a+iV_{MT}} Z\left(\frac{1}{2}-it+w, \Xi\right) \cdot \frac{N^w}{w} dw + \\ + O(E_2(N) \ln V_{MT}),$$

where

$$E_2(N) = \begin{cases} 1, & N \leq V_{MT}^{7/6}, \\ V_{MT}^{-1/6}, & N > V_{MT}. \end{cases}$$

PROOF. For $b = a + \frac{1}{2} > \frac{1}{2}$ and a natural number N we can apply a well-known contour-integration formula [13]. Thus

$$(2.10) \quad \sum_{N \geq N}^* \Xi(\alpha) N x^{-\frac{1}{2}+it} = \\ = \frac{1}{2\pi i} \int_{b-iV_{MT}}^{b+iV_{MT}} Z\left(\frac{1}{2}-it+w, \Xi\right) \frac{N^w}{w} dw + \\ + O\left(\frac{N^b}{V_{MT}\left(b-\frac{1}{2}\right)}\right) + O\left(\frac{\sqrt{N} \ln 2N}{V_{MT}}\right).$$

From Cauchy theorem we have

$$(2.11) \quad \frac{1}{2\pi i} \int_{b-iV_{MT}}^{b+iV_{MT}} Z\left(\frac{1}{2}-it+w, \Xi\right) \frac{N^w}{w} dw = E_1(\Xi) \frac{N^{1/2+it}}{\frac{1}{2}+it} - \frac{1}{2\pi i} \cdot \\ - \left(\int_{a-iV_{MT}}^{b-iV_{MT}} + \int_{b+iV_{MT}}^{a+iV_{MT}} + \int_{a+iV_{MT}}^{a-iV_{MT}} \right) Z\left(\frac{1}{2}-it+w, \Xi\right) \frac{N^w}{w} dw = \\ = E_1(\Xi) \frac{N^{1/2+it}}{\frac{1}{2}+it} + \frac{1}{2\pi i} \int_{a-iV_{MT}}^{a+iV_{MT}} Z\left(\frac{1}{2}-it+w, \Xi\right) \frac{N^w}{w} dw + J_{11} + J_{12}.$$

Furthermore

$$(2.12) \quad |J_{11}| \ll |J_{12}| \ll \left| \int_a^b Z\left(\frac{1}{2}-it+u+iV_{MT}, \Xi\right) \frac{N^u}{u+iV_{MT}} du \right| \ll \\ \ll \frac{N^b}{b+V_{MT}} \left(b-a \max_{a \leq u \leq b} |Z\left(\frac{1}{2}+u+iV_{MT}, \Xi\right)| \right).$$

The lemma follows from (2.10), (2.11) and (2.12) if we estimate

$$\max_{a \leq u \leq b} Z\left(\frac{1}{2} + it + iV_{MT}, \Xi\right).$$

by Lemma 5.

LEMMA 7. Let $a(z)$ be complex numbers depending on $z \in K$, t_r be real numbers, $r = 1, 2, \dots, R$ and $\max_{1 \leq N z \leq N} |a(z)| \leq A$, $N \geq 1$. Then

$$\begin{aligned} & \sum_{r, q=1}^R \left| \sum_{1 \leq N z \leq N}^* a(z) \Xi_r(z) \Xi_q(z) N z^{-\frac{1}{2} + it_r - it_q} \right|^2 \ll \\ & \ll A^2 \sum_{r, q=1}^R \left| \sum_{1 \leq N z \leq N}^* \Xi_r(z) \Xi_q(z) N z^{-\frac{1}{2} + it_r - it_q} \right|^2. \end{aligned}$$

PROOF. Lemma 7 is similar to Lemma 2 of JUTILA [6] and it follows by the same way as in [6]. It is enough to put

$$Z_r(z) = \Xi_r(z) N z^{-\frac{1}{2} + it_r}$$

and then rewrite the terms $|\sum \dots|^2$ as products of conjugates and finally after changing the order of summation refer to conditions of our Lemma.

LEMMA 8. Let $T \geq 2$, r, q run over the integers $1, 2, \dots, R$. Assume that t_r are real numbers satisfying the condition $|t_r| \leq T$ and Ξ_r are characters belonging to the same modulus η_r . Assume furthermore that for each pair of $r \neq q$ one of the relations $\Xi_r \neq \Xi_q$ or $|t_r - t_q| \leq \ln^4 V_{MT}$ holds.

Then for natural $N \leq V_{MT}^{7/6}$

$$\begin{aligned} & \sum_{r, q \leq R} \left| \sum_{N z \leq N}^* \Xi_r(z) \Xi_q(z) N z^{-\frac{1}{2} + it_r - it_q} \right|^2 \ll \\ & \ll (R^{3/2} V_{MT} + RN + E_3(N) R^2) \ln^{C_6} V_{MT}, \end{aligned}$$

where

$$E_3(N) = \begin{cases} 1, & N \leq V_{MT}^{7/6}, \\ V_{MT}^{-1/3}, & N > V_{MT}. \end{cases}$$

PROOF. From Lemma 6 and Schwarz-Cauchy inequality we have

$$\begin{aligned} & \sum_{r, q \leq R} \left| \sum_{N z \leq N}^* \Xi_r(z) \Xi_q(z) N z^{-\frac{1}{2} + it_r - it_q} \right|^2 \ll RN + E_3(N) R^2 \ln^2 V_{MT} + \\ & + \int_{a-iV_{MT}}^{a+iV_{MT}} R \left(\sum_{r, q \leq R} \left| Z\left(\frac{1}{2} - it_r + it_q + w, \Xi_r \Xi_q\right) \right|^2 \right)^{1/2} |dw| \cdot \int_{a-iV_{MT}}^{a+iV_{MT}} \frac{N^w}{w^2} dw \ll \\ & \ll (RN + E_3(N) R^2 + R^{3/2} V_{MT}) \ln^{C_7} V_{MT} \end{aligned} \tag{2.13}$$

if we estimate the sum in the first integral by Lemma 4.

3. Proof of theorem. HALÁSZ-MONTGOMERY lemma [12] for quadratic imaginary number fields [9] gives us an estimate

$$(3.1) \quad \left(\sum_{r \leq R} \left| \sum_{N \leq N_2 \leq 2N}^* C(\alpha) \Xi_r(\alpha) N \alpha^{-it_r - it_q} \right|^2 \right)^{1/2} \ll \\ \ll \sum_{r, q \leq R} |H(\sigma_r + \sigma_q + it_r - it_q, \Xi_r \Xi_q)| \cdot \sum_{N \leq N_2 \leq 2N}^* |C(\alpha)|^2 N \alpha^{-2\sigma}.$$

From Lemma 3 we have

$$(3.2) \quad \sum_{r, q \leq R} |H(\sigma_r + \sigma_q + it_r - it_q, \Xi_r \Xi_q)| \ll \sum_{r, q \leq R} (E_1(\Xi) N + 1) + \\ + \sqrt{N} \int_{R^2}^{R^2} \sum_{r, q \leq R} \left| \sum_{N \leq N_2 \leq P}^* \Xi_r(\alpha) \Xi_q(\alpha) N \alpha^{-\frac{1}{2} + it_r - it_q} N \alpha^{-it_r} \right| d\tau \ll \\ \ll RN + R^2 + \sqrt{N} J(\Sigma).$$

Applying first Lemma 7 with $a(\alpha) = N \alpha^{-it_r}$, after then Hölder's inequality and again Lemma 7 with $a(\alpha) \sim \tau_k(\alpha) \ll V_{MT}^k$ we get

$$\sum \ll \left(\sum_{r, q \leq R} 1 \right)^{1+1/2k} \left(\sum_{r, q \leq R} \left| \left(\sum_{N \leq N_2 \leq P}^* \Xi_r(\alpha) \Xi_q(\alpha) N \alpha^{-\frac{1}{2} + it_r - it_q} \right)^k \right|^2 \right)^{1/2k} \ll \\ \ll R^{2+1/k} \left(\sum_{r, q \leq R} \left| \sum_{N \leq N_2 \leq P^k}^* a(\alpha) \Xi_r(\alpha) \Xi_q(\alpha) N \alpha^{-\frac{1}{2} + it_r - it_q} \right|^2 \right)^{1/2k} \ll \\ \ll V_{MT}^{rk} R^{2+1/k} \left(\sum_{r, q \leq R} \left| \sum_{N \leq N_2 \leq P^k}^* \Xi_r(\alpha) \Xi_q(\alpha) N \alpha^{-\frac{1}{2} + it_r - it_q} \right|^2 \right)^{1/2k}.$$

For the last sum we can apply Lemma 8. We have

$$(3.3) \quad \sqrt{N} J(\Sigma) \ll V_{MT}^k R^{2+1/k} \left(R^{3/2} V_{MT} + R \cdot \frac{V_{MT}^{2k}}{N^k} + E_3(N) R^2 \right)^{1/2k} \ll \\ \ll (R^{2+1/4k} V_{MT}^{1/2k} + R^{2+1/2k} N^{-1/2} V_{MT} + R^2 (E_3(N))^{1/2k}) V_{MT}^k.$$

Theorem follows from (3.1), (3.2) and (3.3) immediately.

4. Remarks. We shall use Theorem of this paper for sharpening zero-density theorems for non-trivial zeros of Hecke functions [9, 10]. In later papers we shall estimate

$$\sum_{m \leq M} N(\sigma, T, \Xi),$$

where $N(\sigma, T, \Xi)$ is the number of non-trivial zeros $\rho = \beta + i\gamma$ of Hecke Z-functions in rectangle $\sigma \geq \beta \geq 1$, $|\gamma| \leq T$, $3/4 < \sigma < 1$, $T > 2$. Such estimates shall lead us to new results in angular distribution of prime ideal numbers of quadratic imaginary number fields in sectors [4], [8], [9], [10].

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A GEOMETRIC CHARACTERIZATION OF K -MINIMA

By

JURI S. PARVANOV

H. Department of Analysis, of L. Eötvös University, Budapest
and Department of Advanced Mathematics, Angel Kanchev College, Russe, Bulgaria

(Received June 7, 1982)

In this paper we deal with a characterization of K -minima in the range of the function considered which reduces a vector optimization problem to optimization of real-valued functions along halflines lying in the range space. Our investigation essentially generalizes the results of J. G. LIN [1] (see also [2]) to the infinite dimensional case.

In our investigations the following definitions will be needed.

DEFINITION 1. Let Z be a real Hilbert space, $K \subset Z$ a convex cone. The pair (Z, K) is said to be an ordered Hilbert space.

In connection with this ordering the following notations will be used.

For any $z_1, z_2 \in Z$ we shall write $z_1 \leq z_2$, $z_1 < z_2$ and $z_1 \prec z_2$, if $z_2 - z_1 \in K$, $z_2 - z_1 \in K \setminus \{0\}$ and $z_2 - z_1 \in \text{int } K$ respectively.

DEFINITION 2. Let $\mathcal{U} \neq 0$ be an arbitrary set, (Z, K) an ordered Hilbert space and $f: \mathcal{U} \rightarrow Z$ a given function. An element $u_0 \in \mathcal{U}$ is said to be a K -minimum point of the function f , if there exists no $u_1 \in \mathcal{U}$ with $f(u_1) \leq f(u_0)$.

Further on by the Riesz theorem the dual space Z^* of the Hilbert space Z will be identified with Z .

DEFINITION 3. Let $p \in Z$ be arbitrary. For the function f considered in Definition 2 an element $u_0 \in \mathcal{U}$ is said to be a minimum point with respect to the direction p (briefly a p -minimum point), if the point

$$D_p := \{t \in \mathbf{R} : f(u_0) + tp \in R_f\}$$

is minimum point of the function

$$\tilde{p}: D_{\tilde{p}} \rightarrow \mathbf{R}, \quad \tilde{p}(t) := \langle p, f(u_0) + tp \rangle.$$

First we prove a necessary condition for K -optimality.

THEOREM. Using the above notations, suppose the cone K is closed and

$$(1) \quad \langle z_1, z_2 \rangle \geq 0 \quad (z_1, z_2 \in K^\perp).$$

Then, if $u_0 \in \mathcal{U}$ is K -minimum point of the function f then for each vector $p \in K^\perp$ u_0 is a p -minimum point of f .

PROOF. Since for $p = 0$ \hat{p} is constant we can suppose that $p \in K^+ \setminus \{\mathbf{0}\}$. First, we show that

$$(2) \quad p \in K \setminus \{\mathbf{0}\}.$$

Suppose the contrary, i.e. $p \notin K$. Since K is convex and closed, by the separation theorem ([3] 3.4, theorem) there exists a $q \in Z$ such that

$$(3) \quad a := \sup \{\langle q, z \rangle : z \in K\} < \langle q, p \rangle.$$

Since $0 \in K$, obviously $a \geq 0$. If $a > 0$, then there exists a vector $z_0 \in K$ such that $\langle q, z_0 \rangle > 0$. But thus the set

$$\{\langle q, \lambda z_0 \rangle : \lambda \in \mathbf{R}_+\}$$

cannot be bounded above, in contradiction with the equality (3). Hence $a = 0$. Therefore,

$$\langle q, z \rangle \leq 0 \quad (z \in K).$$

Thus $q \in K^\perp$ and on the other hand $a = 0$. Hence, by (3), we have

$$\langle -q, p \rangle < 0,$$

which is a contradiction with the condition (1) and (2) is proved.

Now, suppose that for some $p \in K^+ \setminus \{\mathbf{0}\}$ $0 \in D_{\hat{p}}$ is not a minimum point of the function \hat{p} . Then there exists a $t \in \mathbf{R}$ such that

$$(4) \quad f(u_0) + tp \in R_f$$

and

$$\langle p, f(u_0) + tp \rangle < \langle p, f(u_0) \rangle.$$

Hence we obtain that $t \|p\|^2 < 0$ which implies $t < 0$. According to (4) there exists a $u \in \mathcal{U}$ such that

$$f(u) = f(u_0) + tp.$$

Since by (2)

$$-tp \in K \setminus \{\mathbf{0}\},$$

therefore

$$f(u_0) - f(u) \in K \setminus \{\mathbf{0}\},$$

which is contradiction because u_0 is K -minimum point of the function f . ■

Now, we provide a simple sufficient condition for K -minimality.

STATEMENT. Using the symbols of Definition 1–3, let $K \subset K^+$ and suppose that an element $u_0 \in \mathcal{U}$ is a p -minimum point of f for every $p \in K^+$. Then u_0 is K -minimum point of the function f .

PROOF. Suppose the contrary. Then there exists a $u_1 \in \mathcal{U}$ such that

$$f(u_1) < f(u_0).$$

Hence, for the vector

$$(5) \quad p := f(u_0) - f(u_1)$$

we have

$$p \in K^\perp \setminus \{0\},$$

thus

$$(6) \quad \langle p, f(u_0) - f(u_1) \rangle = \|p\|^2 > 0.$$

By (5) it is obvious that $-1 \notin D_{\tilde{p}}$ and according to (6)

$$\tilde{p}(-1) = \langle p, f(u_1) \rangle - \langle p, f(u_0) \rangle = \tilde{p}(0),$$

which is a contradiction, because $0 \in D_{\tilde{p}}$ is a minimum point of the function \tilde{p} . ■

Conferring the Theorem and the Statement we obtain the following necessary and sufficient condition.

COROLLARY. Suppose that for an ordered Hilbert space K satisfies $K = K^\perp$. Then for a point $u_0 \in \mathcal{U}$ to be a K -minimum point of f it is necessary and sufficient that for every $p \in K^\perp$ u_0 be a p -minimum point of f .

REMARK. If for some ordered Hilbert space (Z, K) $K = K^\perp$, then the condition (1) is obviously satisfied; therefore we can apply the Corollary. Below we show examples for such ordered Hilbert spaces.

EXAMPLE 1. Let $n \in \mathbb{N}$, $[t_0, t_1] \subset \mathbf{R}$, $Z := L_2^n[t_0, t_1]$ and

$$K := \{z \in L_2^n[t_0, t_1] : z(t) \in \mathbf{R}_{+0} \text{ e.a. } t \in [t_0, t_1]\}.$$

We show that $K = K^\perp$. The inclusion $K \subset K^\perp$ is obvious. Let $z \in K^\perp$ and suppose that $z \notin K$, i.e. there exists a number $i \in \overline{1, n}$ and a set $A \subset [t_0, t_1]$, such that

$$\operatorname{mes} A > 0, \quad z_i(t) < 0 \quad (t \in A).$$

Define the function $y \in L_2^n[t_0, t_1]$ as follows. For each $j \in \overline{1, n}$ put

$$y_j := \begin{cases} z_A, & \text{if } j = i, \\ y_j = 0, & \text{if } j \in \overline{1, n} \setminus \{i\}. \end{cases}$$

It is obvious that $y \in K$ and

$$\langle z, y \rangle = \int_A z_i < 0,$$

therefore $z \notin K^\perp$ in contradiction with the choice of z .

EXAMPLE 2. Let $n \in \mathbb{N}$, $[t_0, t_1] \subset \mathbf{R}$ and

$$Z := H^{1,n}[t_0, t_1] := \{z : [t_0, t_1] \rightarrow \mathbf{R} : z \text{ is absolutely continuous, } \dot{z} \in L_2^n[t_0, t_1]\}.$$

$H^{1,n}[t_0, t_1]$ is a Hilbert space with the following scalar product

$$\langle z_1, z_2 \rangle := \langle z_1(t_0), z_2(t_0) \rangle + \int_{t_0}^{t_1} \langle \dot{z}_1, \dot{z}_2 \rangle,$$

$$(z_1, z_2 \in H^{1,n}[t_0, t_1]).$$

Define

$$K := \{z \in Z : z(t_0) \in \mathbf{R}_{\geq 0}^n, \dot{z}(t) \geq 0 \text{ for e.a. } t \in [t_0, t_1]\}.$$

As in the previous example it can be proved that for this ordered Hilbert space (Z, K) the condition $K \vdash K^-$ is also satisfied.

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SPACE-FILLING CURVES AND UNIVERSAL NORMED SPACES

By

DAVID YOST

Mathematics Department, La Trobe University, Bundoora, Australia

(Received July, 26, 1982)

The existence of a space-filling curve ([4] p. 122) is a topological result which is, at first sight, surprising. The existence of a separable normed space which contains an isometric copy of every finite-dimensional normed space ([1] p. 185) is a much ignored result from functional analysis. We point out the relationship between these two facts.

By a space-filling curve, we mean a continuous map from the unit interval I onto I^2 . The sup-normed space of continuous functions from I to \mathbf{R} , denoted as usual by $C(I)$, actually contains a copy of every separable normed space. For simplicity, we will call a normed space universal if it contains a copy of every finite-dimensional normed space. To illustrate the connection between space-filling curves and universal normed spaces, it is appropriate to consider compact topological spaces other than the unit interval.

So let X be a compact topological space (assumed Hausdorff), and let $C(X)$ be the normed space of continuous functions from X to \mathbf{R} . There are two pertinent questions we may ask concerning X . Firstly, is there a continuous map $\psi: X \rightarrow E$, for some finite-dimensional normed space E , such that $\psi(X)$ has an interior point? The existence of such a map when $X = I$ and E is two-dimensional is equivalent to the existence of a space-filling curve. Secondly, is $C(X)$ universal?

If E is any normed space, we will denote its dual by E^* and its unit ball by $U(E)$. If E is finite-dimensional, then $U(E^*)$ is a compact metric space. If $\psi: X \rightarrow U(E^*)$ is continuous and onto, then $C(X)$ contains a copy of E . Indeed, the map $\psi^*: E \rightarrow C(X)$ defined by $\psi^*(a)(x) = \psi(x)(a)$, for a in E and x in X , is an isometric embedding. It is this which motivates us to consider the two questions just raised, and their relationship. Our first result is a refinement of this observation.

LEMMA 1. Let E be a finite-dimensional normed space. Then E embeds isometrically in $C(X)$ if and only if there is a map $\psi: X \rightarrow U(E^*)$ such that $-\psi(X) \cup \psi(X)$ contains every extreme point of $U(E^*)$.

PROOF. Sufficiency can be proved in the same manner as above. To establish necessity, we assume that E is a subspace of $C(X)$, with basis

$\{a_1, \dots, a_n\}$. Then E^* may be identified with \mathbf{R}^n : to a functional f in E^* we associate the n -tuple $(f(a_1), \dots, f(a_n))$. Let $\Psi: X \rightarrow E^*$ be the evaluation map, $\Psi(x) = (a_1(x), a_2(x) \dots a_n(x))$. Then $K = \Psi(X)$ is a compact subset of $U(E^*)$. For any function a in E we have $\|a\| = \max\{|a(x)| : x \in X\} = \max\{|f(a)| : f \in K\}$. A routine application of the separation theorem then shows that $U(E^*)$ is the convex hull of $-K \cup K$. Immediately we deduce that every extreme point of $U(E^*)$ lies in $-K \cup K$.

We will use this duality lemma to establish results of two types. First we will present a result of Donoghue's, which essentially says that the smooth subspaces of $C(I)$ must contain some pathological functions. On the other hand, every two-dimensional normed space can be represented by quite well-behaved functions in $C(I)$.

Let n be a positive integer. We will say that X fills n -space if there is an n -dimensional normed space E , and a continuous map $\Psi: X \rightarrow E$, such that $\Psi(X)$ has an interior point. Since any cube in \mathbf{R}^n is a retract, we see that X fills n -space if and only if there is a continuous map from X onto I^n . A normed space E is said to be smooth, if, for every non-zero x in E , there is a unique functional f in E^* satisfying $f(x) = \|x\|$ and $\|f\| = 1$. When E is finite-dimensional, this is the same as requiring every norm one functional in E^* to be an extreme point of $U(E^*)$.

PROPOSITION 2. For $n > 2$, the following are equivalent.

- (i) $C(X)$ contains a smooth n -dimensional subspace,
- (ii) X fills $(n-1)$ -space,
- (iii) $C(X)$ contains every n -dimensional normed space.

PROOF. (i) \Rightarrow (ii) [3]. Let $\{a_1, \dots, a_n\}$ be a basis for a smooth subspace E . Then, by lemma 1, $S(E^*) = \{f \in E^* : \|f\| = 1\}$ is contained in $-K \cup K$. Now define a projection P on \mathbf{R}^n by $P(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_{n-1}, 0)$. Then F , the range of P , is an $(n-1)$ -dimensional normed space. For any x in F , we have $\|x + (0, \dots, 0, \lambda)\| \geq 1$, for all sufficiently large λ . It follows from the intermediate value theorem that $U(F) \subseteq P(S(E^*)) \subseteq P(K) \cup P(K)$. Thus $P(K)$, considered as a subset of F , contains an interior point. This is precisely the assertion that X fills $(n-1)$ -space, via the map

$$x \mapsto (a_1(x), a_2(x) \dots a_{n-1}(x)).$$

(ii) \Rightarrow (iii) Let E be any n -dimensional normed space, and choose $x \in E \setminus \{\mathbf{0}\}$. Then $K = \{f \in S(E^*) : f(x) = 0\}$ is easily shown to be homeomorphic to I^{n-1} . But $-K \cup K$ is just $S(E^*)$, so lemma 1 tells us that E embeds in $C(X)$.

It is necessary to reduce the dimension in this proof. For example, let X be the unit circle, $\{(z, \beta) \in \mathbf{R}^2 : z^2 + \beta^2 = 1\}$, and define f and g in $C(X)$ by $f(z, \beta) = z$ and $g(z, \beta) = \beta$. Then $\|\lambda f + \mu g\|^2 = \lambda^2 + \mu^2$, for any λ and μ in \mathbf{R} , so the subspace spanned by f and g is a two-dimensional Hilbert space. However, the map $x \mapsto (f(x), g(x))$ does not fill 2-space.

The next result is obvious.

COROLLARY 3. A necessary and sufficient condition for $C(X)$ to be universal is that X fills n -space, for every n .

Taking $X = I$ in Corollary 3, we find that the following four assertions are equivalent:

- (i) $C(I)$ is universal,
- (ii) $C(I)$ contains a 3-dimensional Euclidean space,
- (iii) there is a continuous map from I onto I^2 ,
- (iv) for every n , there is a continuous map from I onto I^n .

The implications (iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) are clear. Induction shows that (iii) \Rightarrow (iv). Thus, universality of $C(I)$ is equivalent to the existence of a space-filling curve. Of course, these four assertions are not merely equivalent; they are all true! This gives the following refinement of Corollary 3.

THEOREM 4. *The following three conditions are equivalent:*

- (i) $C(X)$ is universal,
- (ii) $C(X)$ contains a smooth 2-dimensional subspace,
- (iii) there is a continuous map from X onto I .

We remark that the usual proof that $C(I)$ is universal uses the fact that every compact metric space is a continuous image of the Cantor set ([4] p. 127). Instead of this, we have deduced universality of $C(I)$ from the existence of a space-filling curve. However, the existence of a space-filling curve can also be deduced from this property of the Cantor set ([4] p. 128).

Intuitively speaking, space-filling curves are somewhat pathological. For example, they cannot be differentiable, or even Lipschitz continuous. It is an easy exercise to show that if $\psi: I \rightarrow \mathbf{R}^2$ is a Lipschitz continuous map, then $\psi(I)$ has plane measure zero. Thus Donoghue's result asserts that the subalgebra of Lipschitz continuous functions in $C(I)$ does not contain any smooth 3-dimensional subspace. BOSZNAJ [2] showed that the subalgebra of analytic functions in $C(I)$ does not contain every 2-dimensional normed space. In contrast to these results, we show that every 2-dimensional normed space embeds isometrically in the algebra of Lipschitz continuous functions.

THEOREM 5. *If E is any 2-dimensional normed space, then we can find Lipschitz continuous functions $f_1, f_2 \in C(I)$ such that E is isometric to the linear span of $\{f_1, f_2\}$.*

PROOF. Identify E and E^* with \mathbf{R}^2 , under the obvious duality. Then $K = \{(z, \beta) \in S(E^*); \beta \geq 0\}$ is easily seen to be homeomorphic to I . Let $\psi: I \rightarrow K$ be any continuous surjection. Routine calculations show that $\psi^*(E)$ is the linear span of $\{f_1, f_2\}$, where $f_i \in C(I)$ are the unique functions which satisfy the identity $\psi(x) = (f_1(x), f_2(x))$. To say that each f_i is Lipschitz continuous is the same as saying that ψ is Lipschitz continuous.

Thus, by lemma 1, we have only to show that K is a Lipschitz continuous image of I . This can be done in a number of ways. For example, we could introduce arc length along the curve K as a parameter (normalizing so that K has length one), and let f_1 and f_2 be the projections onto the co-ordinate axes. Or we could prove this for a simple case, such as the Euclidean norm, then use the fact that all norms on \mathbf{R}^2 are uniformly equivalent.

Another refinement is to choose our co-ordinate system so that $(1, 0)$ is an exposed point of $U(E^*)$. This means that $\|(1, z)\| \geq 1$ unless $z = 0$. If $f_1(x) = 2x - 1$, and $f_2(x)$ is determined by the condition $(f_1(x), f_2(x)) \in K$, then f_2 is obviously a convex function. It follows that every function in $\varphi^*(E)$ will be differentiable at all but countably many points in I . The derivative of f_2 may be unbounded in this case, but a simple change of scale will give us functions with bounded derivatives. Again we have Lipschitz continuity.

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Current address: Mathematics Department, Institute of Advanced Studies, Australian National University, Canberra, Australia.

A NEW BOUND ON MULTIVARIATE EXTREME VALUE DISTRIBUTIONS

By

JANOS GALAMBOS

Department of Mathematics, Temple University, Philadelphia, U.S.A.

(Received September 15, 1982)

Let $\mathbf{X} = (X^{(1)}, X^{(2)}, \dots, X^{(d)})$ be a d -dimensional random vector with distribution function

$$F(x_1, x_2, \dots, x_d) = P(X^{(j)} \leq x_j, 1 \leq j \leq d).$$

Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be independent copies of \mathbf{X} . We put

$$Z_n^{(j)} = \max(X_1^{(j)}, X_2^{(j)}, \dots, X_n^{(j)}),$$

and

$$\mathbf{Z}_n = (Z_n^{(1)}, Z_n^{(2)}, \dots, Z_n^{(d)}).$$

Assume that there are sequences $\mathbf{a}_n = (a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(d)})$ and $\mathbf{b}_n = (b_n^{(1)}, b_n^{(2)}, \dots, b_n^{(d)})$ of (non-random) vectors such that, for each j , $1 \leq j \leq d$, $b_n^{(j)} > 0$, and that the distribution function $F^n(a_n^{(1)} + b_n^{(1)}z_1, \dots, a_n^{(d)} + b_n^{(d)}z_d)$ of the normalized vector $(\mathbf{Z}_n - \mathbf{a}_n)/\mathbf{b}_n$, where each operation is understood to be component by component, converges weakly to a distribution function $H(z_1, z_2, \dots, z_d)$, whose univariate marginals $H_j(z_j)$, $1 \leq j \leq d$, are non-degenerate. Such a multivariate distribution function $H(z_1, z_2, \dots, z_d)$ is called an extreme value distribution function (for the maxima). We consider here the case of maxima only. For the significance of extreme value distributions in model building, see GALAMBOS (1985), which, together with Chapter 5 of GALAMBOS (1978), can also be consulted for references to the asymptotic theory of multivariate extremes.

One interesting property of multivariate extreme value distributions is the following inequality

$$(1) \quad H(z_1, z_2, \dots, z_d) \geq H_1(z_1)H_2(z_2)\dots H_d(z_d),$$

which implies that the components of a random vector with distribution function $H(z_1, z_2, \dots, z_d)$ are positively correlated.

By means of bivariate marginals, upper inequalities can also be established on H . In fact, when simple Bonferroni type inequalities are applied to F^n , then a passage to limit leads to the inequality

$$(2) \quad H(z_1, z_2, \dots, z_d) \leq \prod_{j=1}^d H_j(z_j) \prod_{1 \leq i < j \leq d} r_{ij}(z_i, z_j),$$

where $H_{ij}(z_i, z_j)$ is the bivariate marginal of $H(z_1, z_2, \dots, z_d)$ corresponding to the i -th and j -th components, and

$$(3) \quad r_{ij}(z_i, z_j) = \frac{H_{ij}(z_i, z_j)}{H_i(z_i) H_j(z_j)}.$$

Our aim in the present paper is to sharpen (2).

Let us pause for a moment for giving some references. The inequality (1) was recognized early when the foundations of the multivariate extreme value theory was laid down (see Chapter 5 of GALAMBOS (1978) for references). In the 1960s, J. TIAGO DE OLIVEIRA developed a number of inequalities, including (2), using a unique technique. For a summary of his inequalities, see TIAGO DE OLIVEIRA (1975). Both (1) and (2) are special cases of inequalities obtained by the present author (Theorem 5.3.1 in GALAMBOS (1978)), which are in terms of arbitrary marginals, not just univariate and bivariate ones. Notice that (1) and (2) imply that if the components of a vector with distribution function $H(z_1, z_2, \dots, z_d)$ are pairwisely independent then they are completely independent. Namely, for pairwisely independent components r_{ij} of (3) is one, and thus the two bounds in (1) and (2) coincide. This remarkable similarity of multivariate extreme value distributions to the family of normal distributions has not yet been exploited to a full extent.

Let us now state the result of the present paper.

THEOREM. Put k_0 for the integer part of the expression

$$(4) \quad \begin{aligned} & \frac{2 \log \prod_{1 \leq i < j \leq d} r_{ij}(z_i, z_j)}{\log \prod_{i=1}^d H_i(z_i)}, \\ & -k_0 \end{aligned}$$

and set $s = k_0 + 2$. Then

$$(5) \quad H(z_1, z_2, \dots, z_d) \leq \left\{ \prod_{j=1}^d H_j(z_j) \right\}^{2-s} \prod_{1 \leq i < j \leq d} \{r_{ij}(z_i, z_j)\}^{2s(s-1)}.$$

REMARK. Notice that when $k_0 = 0$, i.e. $s = 2$, (5) reduces to (2). In all other cases, (5) is sharper than (2), which easily follows by comparing the two bounds (use the definition of k_0 at (4), and the fact that, in view of (1), each $r_{ij}(z_i, z_j) \geq 1$). In fact, (5) is valid for all $s \geq 2$, as it will be seen in the course of the proof. Our choice of s is, in some sense, optimal.

The proof of the Theorem is based on the following inequality (see GALAMBOS (1977)).

LEMMA. Let A_1, A_2, \dots, A_n be events on a given probability space. Put

$$S_1(n) = \sum_{j=1}^n P(A_j) \quad \text{and} \quad S_2(n) = \sum_{1 \leq i < j \leq n} P(A_i A_j).$$

Then, for any integer $k \geq 2$,

$$P(\text{at least one } A_j \text{ occurs}) \geq \frac{2}{k} S_1(n) - \frac{2}{k(k-1)} S_2(n).$$

The best bound is achieved when $k=2$ is the integer part of $2S_2(n)/S_1(n)$.

PROOF OF THE THEOREM. First notice that if a single term on the right hand side of (5) is zero, then, by basic properties of distribution functions, $H(z_1, z_2, \dots, z_d) = 0$, and thus (5) holds. We therefore assume that each term on the right hand side of (5) is positive. This, by (1), implies that $H(z_1, z_2, \dots, z_d)$ is positive as well.

Now, by the definition of H , there exist constants $(a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(d)})$ and $(b_n^{(1)}, b_n^{(2)}, \dots, b_n^{(d)})$ such that

$$(6) \quad \lim_{n \rightarrow +\infty} F^n(z_{1,n}, z_{2,n}, \dots, z_{d,n}) = H(z_1, z_2, \dots, z_d),$$

where

$$z_{j,n} = a_n^{(j)} + b_n^{(j)} z_j.$$

A particular consequence of (6) is that, as $n \rightarrow +\infty$,

$$\lim F(z_{1,n}, z_{2,n}, \dots, z_{d,n}) = 1.$$

Hence, by taking logarithm in (6), and applying the asymptotic relation

$$\log u_n \sim \log \{1 - (1 - u_n)\} = -(1 + o(1))(1 - u_n), \quad u_n \rightarrow 1,$$

with $u_n := 1 - F(z_{1,n}, z_{2,n}, \dots, z_{d,n})$, we get

$$(7) \quad \lim_{n \rightarrow +\infty} n[1 - F(z_{1,n}, z_{2,n}, \dots, z_{d,n})] = -\log H(z_1, z_2, \dots, z_d).$$

Furthermore, if we denote by $F_j(x_j)$ and $F_{ij}(x_i, x_j)$, respectively, the univariate and bivariate marginals of $F(x_1, x_2, \dots, x_d)$, then, by repeating the argument above, we obtain

$$(8) \quad \lim_{n \rightarrow +\infty} n[1 - F_j(z_{j,n})] = -\log H_j(z_j)$$

and

$$(9) \quad \lim_{n \rightarrow +\infty} n[1 - F_{ij}(z_{i,n}, z_{j,n})] = -\log H_{ij}(z_i, z_j).$$

On the other hand, $1 - F(z_{1,n}, z_{2,n}, \dots, z_{d,n})$ is the probability that at least one of the events $A_j : \{X^{(j)} \geq z_{j,n}\}$, $1 \leq j \leq d$, occurs. Hence, by the Lemma, for any integer $k \geq 2$,

$$(10) \quad 1 - F(z_{1,n}, z_{2,n}, \dots, z_{d,n}) \geq \frac{2}{k} S_{1,n} - \frac{2}{k(k-1)} S_{2,n},$$

where

$$(11) \quad S_{1,n} = S_{1,n}(d) = \sum_{j=1}^d [1 - F_j(z_{j,n})]$$

and

$$(12) \quad S_{2,n} = S_{2,n}(d) = \sum_{1 \leq i < j \leq d} P(X^{(i)} \geq z_{i,n}, X^{(j)} \geq z_{j,n}).$$

By the following well-known elementary formula of probability theory for a random vector (U, V)

$$P(U \geq u, V \geq v) = 1 - P(U < u) \cdot P(V < v) + P(U < u, V < v),$$

$S_{2,n}$ of (12) can also be written as

$$(13) \quad \begin{aligned} S_{2,n} &= \sum_{1 \leq i < j \leq d} \{1 - F_i(z_{i,n}) - F_j(z_{j,n}) + F_{ij}(z_{i,n}, z_{j,n})\} = \\ &= \sum_{1 \leq i < j \leq d} \{[1 - F_i(z_{i,n})] + [1 - F_j(z_{j,n})] - [1 - F_{ij}(z_{i,n}, z_{j,n})]\}. \end{aligned}$$

We first record that, in view of (8), (9), (11) and (13), as $n \rightarrow +\infty$,

$$(14) \quad \lim n S_{1,n} = - \sum_{j=1}^d \log H_j(z_j) = - \log \prod_{j=1}^d H_j(z_j)$$

and

$$(15) \quad \begin{aligned} \lim n S_{2,n} &= \sum_{1 \leq i < j \leq d} \{-\log H_i(z_i) - \log H_j(z_j) + \log H_{ij}(z_i, z_j)\} = \\ &= - \log \prod_{1 \leq i < j \leq d} r_{ij}(z_i, z_j). \end{aligned}$$

Next observe that, although the inequality (10) is valid for all integers $k \geq 2$, the best bound is obtained for a fixed n if $k-2$ is the integer part of

$$\frac{2S_{2,n}}{S_{1,n}} = \frac{2n S_{2,n}}{n S_{1,n}}.$$

This, however, in view of (14) and (15), asymptotically equals the expression in (4). Hence, we choose $k = s = k_0 + 2$ as stated in the Theorem at (4). With this k , we turn to (10). Multiplying (10) by n and letting $n \rightarrow +\infty$, (7), (14) and (15) immediately yield (5), which completes the proof.

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ON A CHARACTERIZATION OF FROBENIUS AND $L_2(2^k)$ GROUPS

By

P. CSÖRGÖ

Department of Numerical Methods and Computer Science
of the L. Eötvös University, Budapest

(Received November 13, 1978)

This paper is a generalization of author's earlier one: "On the Theory of Frobenius groups" [3]. In that we gave a characterization of Frobenius groups of abelian kernel and abelian complement. The generalization of these characterization properties leads, in this paper, to a characterization of Frobenius and $L_2(2^k)$ groups, $k \geq 2$. The notations are standard. See GORENSTEIN [1] or HUPPERT [2].

In this paper all groups are finite.

DEFINITION. Let G be a group. We now define a binary relation R on $G \setminus \{e\}$, $x, y \in G \setminus \{e\}$ by definition if there exists a chain:

$$x = x_0, x_1, \dots, x_n = y$$

with $x_i \in C_G(x_{i+1})$ for all $0 \leq i \leq n-1$.

It is easy to see, that R is an equivalence relation. In the following equivalence class in a group G means that of above R on $G \setminus \{e\}$.

STATEMENT 1. Let \bar{M} be an equivalence class in a group G , then $C_G(u) \subseteq \bar{M} \cup \{e\}$ is true for all $u \in \bar{M}$.

The proof is trivial.

STATEMENT 2. Let M be an equivalence class in a group G . Denote $\bar{M} := M \cup \{e\}$ and $N := N_G(M)$. Suppose $N \neq G$. Let $t \in G \setminus N$ be arbitrary. Then we have

$$M \cap M^t = E.$$

PROOF. It is clear, that \bar{M}^t is an equivalence class too in G , hence the statement follows by definition.

STATEMENT 3. Let \bar{M} be an equivalence class in a group G . Denote $M := \bar{M} \cup \{e\}$. Suppose M is a subgroup of G , then $(|M|, |G : M|) = 1$ follows.

PROOF. Suppose the contrary. Then there exists an S_p -subgroup S of M , with $S < S^*$ where S^* is a S_p -subgroup of G . Thus there exists an element $g \neq e$ with $g \in Z(S^*)$, which is a contradiction.

STATEMENT 4. Let \bar{M} be an equivalence class in a group G . Suppose $M := \bar{M} \cup \{e\}$ is a subgroup of G . Then $N_G(M)$ is a Frobenius group with kernel M , and M is nilpotent.

PROOF. Denote $N := N_G(M)$. By the Statement 3, M is a Hall subgroup of G . By the Theorem of ZASSENHAUS (Th. 6.2.1. in [1]) there exists a subgroup K with $KM \subset N$ and $K \cap M = E$. By the Statement 1, all the elements of K induce by conjugation a fixed-point-free automorphism of M , hence N is a Frobenius group with kernel M .

M is nilpotent by the Theorem of THOMPSON (Th. 10.2.1. in [1]).

THEOREM 1. Let G be a group, $|G|$ is odd. H_1, H_2, \dots, H_n are all the equivalence classes in G . Suppose, that (i)–(ii) below hold:

- (i) $H_i := H_i \cup \{e\}$ is a subgroup of G for all i ,
- (ii) $n \leq 2$.

Then G is a Frobenius group.

PROOF. Let G be a counter-example of smallest order.

- (1) Suppose there exists an H_i with $H_i \triangleleft G$ for some i . By the Statement 4, G is a Frobenius group, a contradiction.
- (2) Suppose there exists an H_i with $N_G(H_i) = H_i$ for some i , then by the Statement 2, $H_j \cap H_i^k = E$ for all $k \in G \setminus H_i$ hence G is a Frobenius group, a contradiction.

Thus $H_i \triangleleft N_G(H_i) \triangleleft G$ holds for all i . Then $N_G(H_i)$ is a Frobenius group and H_i is nilpotent for all i , by the Statement 4. Hence $C_G(a)$ is nilpotent for all $a \in G$. In this case G is solvable by the Theorem 14.3.1. in [1]. Let N be a minimal normal subgroup of G . N is abelian, hence $N \triangleleft H_j$ for some j by the Statement 1. Thus $H_j \triangleleft G$ by the Statement 2., a contradiction.

THEOREM 2. Let G be a group of even order. $\bar{H}_1, H_2, \dots, H_z$ are the all equivalence classes in G . Suppose, that (i)–(ii) below hold.

- (i) $H_1 := H_1 \cup \{e\}$ containing an S_3 -subgroup of G is a abelian subgroup of G ,
- (ii) $z \leq 2$.

Then G is isomorphic to one of the following groups:

- (a) a Frobenius group,
- (b) $L_2(2^k)$ where $k \geq 2$.

PROOF. Let G be a counter-example of smallest order. Denote $K = H_1$. Suppose, that $K \triangleleft G$, then G is a Frobenius group by the Statement 4., a contradiction. Suppose $N_G(K) = K$, then by the Statement 2, $K \cap K^l = E$ for all $l \in G \setminus K$ hence G is a Frobenius group, a contradiction. Thus $K \triangleleft N_G(K) \triangleleft G$. $N_G(K)$ is a Frobenius group and K is nilpotent by the Statement 4. Denote $N := N_G(K)$, we show that N is strongly embedded in G .

1. $K \setminus \{e\}$ is an equivalence class in N , K is a Hall subgroup of N by the Statement 3., hence $C_G(b) \trianglelefteq N$ holds for any involution b of N .
2. If we consider an S_2 -subgroup V of N then $V \trianglelefteq K$. So $N_G(V) \trianglelefteq N$ by the Statement 2.
3. Let x be an involution of K , then $x^t \notin N$ holds for some $t \in G \setminus N$ by the Statement 2.

Let $u \neq e$ be an arbitrary element of K . We consider $C_G^*(u)$. If $u^k = u^{-1}$ holds for some involution k of G , then $k \in N$ follows by the Statement 2., hence $k \in K$. Thus $C_G^*(u) \trianglelefteq K$. By the Theorem of SUZUKI (Th. 9.3.2. in [1]) one the following holds:

- I. An S_2 -subgroup of G is cyclic or generalized quaternion.
- II. $\Omega_1(O_2(K)) \trianglelefteq G$.
- III. Under the permutation representation of G on the right cosets of N , G is a Zassenhaus group of degree $|K| + 1$.

In the first case, the S_2 -subgroup S with $S \trianglelefteq K$ has only one involution, hence $N = K$, which is a contradiction. In the second case $\Omega_1(O_2(K)) \trianglelefteq K$, hence $K \trianglelefteq G$ by the Statement 2., a contradiction. Thus the third case holds. N is a Frobenius group with kernel K and with complement H . Put $|H| = h$, $|K| = k$ so $|G| = h \cdot k(k+1)$ holds and $h/k-1$ is true.

We show, that G is a simple group. Suppose the contrary. We consider a minimal normal subgroup L in G . Suppose $L \neq K$, then $K \trianglelefteq G$ follows by the Statement 2., which is a contradiction.

(*) Thus $L \neq K$ holds.

We consider the subgroup KL .

Suppose $KL \neq G$. The number of equivalence classes in KL is at least 2, because $L \neq K$ by (*). It is clear, that $K \setminus \{e\}$ is an equivalence class in KL , and K is containing an S_2 -subgroup S of KL . Thus KL satisfies the conditions of our theorem, hence by minimality of G , KL is either a Frobenius group or it is a $L_2(2^k)$ for some natural number $k \geq 2$. $L_2(2^k)$ ($k \geq 2$) is a simple group, by the Theorem 15.1.2. in [1], hence KL is a Frobenius group. Let M be the kernel of KL . $M \cap L \neq E$ because in the opposite case $ML = M \times L$ which is a contradiction. M is a nilpotent normal Hall-subgroup in KL , so $M \cap L \trianglelefteq G$, hence $M \cap L = L$ follows by minimality of L . If $M \cap K \neq E$ then by nilpotency of M there exist $m \in M \cap K$ and $t \in L \setminus K$ with $t \in C_G(m)$, but $L \neq K$, which is contradicting to the Statement 1. Thus $M \cap K = E$, hence $K \trianglelefteq L = E$. $KL = KM$ hold, so $M = L$ follows because KL is a Frobenius group. Thus K is a complement in KL . It is known, that S is either cyclic or it is a generalized quaternion, hence S has only one involution, so $N = K$, which is a contradiction.

(**) Thus $KL = G$.

If $L \cap H \neq H$, then $KL \cap N = K(L \cap H) \cdot N$ follows, which is a contradiction by (**). So $L \cap H = H$.

If $L \cap K = K$, then $LK = L \trianglelefteq G$, a contradiction. Thus $L \cap K \neq K$.

We consider the subgroup $L \cap N$. Denote $L_0 = L \cap N$, $L_0 = H(L \cap K)$ ^{def} holds. It is clear, that L_0 is a Frobenius group with the kernel $L \cap K$. Let $k_1 \in K \setminus L$ be arbitrary, $L_0 < N$ hence $H^{k_1} < L_0$ follows. It is known, that there exists a $k_2 \in L_0 \cap K$ with $H^{k_1} = H^{k_2}$ (by the Theorem of ZASSENHAUS hence $k_1 k_2^{-1} \in N_N(H)$ but $k_1 k_2^{-1} \notin K$ which is a contradiction. Thus G is simple, hence by the Theorem 14.4.1. in [1], G is a $L_2(2^k)$ group ($k \geq 2$), a contradiction.

On the basis of well-known characteristics of Frobenius and $L_2(2^k)$ groups it can be easily controlled that these groups satisfy conditions of the Theorem 1. and 2.

Acknowledgement. The author would like to express her deep gratitude to Professor K. CORRÁDI for his valuable suggestions and comments.

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ON NORMAL-COMPLEMENTS IN FINITE GROUPS

By

P. CSÖRGŐ

Department of Numerical Methods and Computer Science
of the L. Eötvös University, Budapest

(Received January 26, 1979)

In 1962 G. ZAPPA introduced the notion of a designated set of representatives of some subgroup in a finite group. He studied the following problem. Let G be a finite group, and let H be a π -Hall subgroup of G , there exists a designated set of representatives of H in G . The question arises: What properties must H have, for the existence of a normal-complement to H in G . This paper is a generalization of his results. The notations are standard. See GORENSTEIN [1] or HUPPERT [2].

In this paper we consider only finite groups.

DEFINITION. Let G be a group, and let H be a subgroup of G . R is a designated set of representatives of H in G by definition, if R is a complete set of right coset representatives of H in G , and $R^t = R$ for all $t \in H$.

STATEMENT 1. Let G be a group, and let H be a subgroup of G . If R is a designated set of representatives of H in G , then R is a set of left coset representatives of H in G .

PROOF. Clear.

STATEMENT 2. Let G be a group, and let H be a subgroup of G . If R is a designated set of representatives of H in G , then $g \in C_G(H \cap H^g)$ is true for all $g \in R$.

PROOF. Let $g \in R$ be, and let $h_1, h_2 \in H$ be, such that $h_1^g = h_2$ is true. So $h_2^{g^{-1}} = h_1 = h_2 h_3$ follows for some $h_3 \in H$. Thus $g^{h_2} = h_3 g$ holds, but $g^{h_2} \in R$ by definition of R , from which $h_3 = e$ follows. Thus $h_1 = h_2$.

LEMMA 1. Let U be a group, and let W be a subgroup of U . Let u and v be elements of U with $u, v \in N_U(W)$. $w^u = w^v$ holds for all $w \in W$, and $(|u|, |v|) = 1$ is true. Then $v, u \in C_U(W)$.

PROOF. We prove, that $w^{(u^r)} = w^{(v^r)}$ holds for all natural numbers r , and for all $w \in W$, by induction on r .

In case $r = 1$, $w^u = w^v$ is true for all $w \in W$ by the supposes of our Lemma. We suppose that the statement is true in case $r = n$ so $w^{(u^n)} = w^{(v^n)}$ holds for all $w \in W$. Since $u, v \in N_U(W)$, then $w^{(u^n)} \in W$, consequently $w^{(u^{n+1})} =$

$\pi^{(w^q)} \vdash$. Let $|u| = s$, hence $\pi^{(u)} = \pi^{(w^q)} = w$ follows for all $w \in W$. $(|v|, s) = 1$ is true, from which $\pi^u = w^q = w$ holds for all $w \in W$. Thus the proof of Lemma is complete.

LEMMA 2. Let G be a group, and let H be a subgroup of G . R is a designated set of representatives of H in G . Let a be a q -element of G and let B a p -subgroup of G with following properties:

- (1) $B \triangleleft H$
- (2) $(q, |H|) = 1$, $a \notin N_G(B)$.

Then $a \notin C_G(B)$.

PROOF. Since $a \notin N_G(B)$, then $B \triangleleft H \cap H^a$ follows. Let $g \in Ha$ be with $g \notin R$. By Statement 2, $g \notin C_G(H \cap H^a)$, hence $g \notin C_G(B)$ follows. $a \sim hg$ holds with some $h \in H$. Let $b \in B$ be arbitrary. Since $b^a = b^{hg}$ is true, then $b^a \in B$, consequently $b^a = b^h$ holds for all $b \in B$. Since $a, h \in N_G(B)$ and $(|a|, |h|) = 1$ are true, then $a \in C_G(B)$ by Lemma 1.

DEFINITION. A finite one step non-nilpotent group is a non-nilpotent group, all of whose proper subgroups are nilpotent.

THEOREM 1. [3] If G is a finite one step non-nilpotent group, then there exist a p -Sylow subgroup P of G and a q -Sylow subgroup Q of G for some distinct primes p and q such that

- (i) $G = PQ$,
- (ii) $P \triangleleft G$,
- (iii) Q is cyclic,
- (iv) G is solvable,
- (v) $P = G'$,
- (vi) G/P' is one step non-abelian,
- (vii) P/P' is an elementary abelian p -group, and
- (viii) $|P/P'| = p^n$ where n denotes the least positive integer n such that $p^n \equiv 1 \pmod{q}$.

DEFINITION. Let p and q be distinct primes. A group G is called a (p, q) -group if:

- a) the order of G involves only the prime factors p and q ,
- b) G is a one step non-nilpotent group,
- c) the derived group G' is the p -Sylow subgroup of G .

DEFINITION. We shall use the notation $(p, q) \in G$ for a prime pair p and q to express the fact that G contains no (p, q) -group. Here p and q are fixed prime divisors of $|G|$ and their order is essential.

LEMMA 3. Let G be a group, and let H be a π -Hall subgroup of G . R is a designated set of representatives of H in G . Then G contains no (p, q) -group for every pair $\{p, q\}$ with $q \nmid |G : H|$ and $p \nmid |H|$.

PROOF. Suppose the contrary, there exists a (p, q) -group U , such that $U \triangleleft G$, $p \nmid |H|$ and $q \nmid |G : H|$ hold. By definition of (p, q) -group $U = U_p U_q$,

where U_p is a p -Sylow subgroup of U , and U_q is a q -Sylow subgroup of U , $U_p = U'$ and $U_q = \langle a \rangle$. The Theorems of Sylow imply that we may assume $U_p \leq H$, $U_q \cap H = E$ holds by conditions. Since $(q, |H|) = 1$ and $a \in N_G(U_p)$ then $a \in C_G(U_p)$ by Lemma 2. So U is a nilpotent group, a contradiction.

LEMMA 4. [4]. Let G be a group. G has a normal p -complement if and only if $(p, q) \nmid G$ for all $q \in \pi(G) \setminus \{p\}$.

DEFINITION. Let G be a group, and let H be a subgroup of G . G has normal-complement to H by definition if there exists a subgroup K in G with following properties:

- a) $K \triangleleft G$,
- b) $K \cap H = E$,
- c) $KH = G$.

STATEMENT 3. Let G be a group, and let H be a π -Hall subgroup of G . R is a designated set of representatives of H in G . We consider the transfer of G into H . Put $K := \ker v_{G \rightarrow H}$, then $K \cap H = H'$.

PROOF. $G = Hg_0 + Hg_1 + Hg_2 + \dots + Hg_n$ holds with $g_i \in R$ for all $0 \leq i \leq |G:H|-1$. By properties of R $g_i h = hg_j$, $g_j \in R$ hold for all $h \in H$ and $g_i \in R$. So $v:h \mapsto h^{g_i:H} H'$ is true for all $h \in H$. Since H is a π -Hall subgroup, then $h^{g_i:H} \in H'$ if and only if $h \in H'$. Thus $K \cap H = H'$.

In [1962] G. ZAPPA [5] proved that if G is a finite group, and H is a π -Hall subgroup of G , H is nilpotent, or has a Sylow tower, there exists a designated set of representatives of H in G , then G has normal-complement to H .

We now prove the following Theorem.

THEOREM 2. Let G be a group, let H be a π -Hall subgroup of G , H' is nilpotent and there exists a designated set of representatives of H in G . Then G has normal-complement to H .

PROOF. Put $K := \ker v_{G \rightarrow H}$. $K \cap H = H'$ holds by the Statement 3. It is known that $G/K \cong H/H' \cap K$, hence $|K:H \cap K| = |G:H|$ follows. So $H \cap K$ is a Hall-subgroup in K . By the conditions $H' = P_1 \times \dots \times P_k$ holds where P_i is a p_i -Sylow subgroup of K for all i , $1 \leq i \leq k$. We consider an arbitrary P_i , $1 \leq i \leq k$.

We show that K has normal p_i -complement. Suppose the contrary. By the Lemma 4, in K there exists a (p_i, q) -group U for some $q \in \pi(G) - \{p_i\}$. By the Lemma 3, $q \mid |H'|$. So $|U| \mid |H'|$, hence $U \leq H'^a$ follows for some $a \in K$ by the Theorem of Wielandt, see Theorem 11.8.1 in [6]. H'^a is nilpotent, from which U is nilpotent, a contradiction. Thus K has normal p_i -complement N_i for all i , $1 \leq i \leq k$.

Put $N := \bigcap_{i=1}^k N_i$. N has the following properties:

- a) $N \triangleleft K$,
- b) since $N_i \cap P_i = E$ then $N \cap H' = E$,

c) since $NH' = \left\langle \bigcap_{i=1}^k N_i \right\rangle (P_1 \times \dots \times P_k)$ then $NH' = K$, N is normal-complement to H is G , because

1. since $N \cap H' = E$ and $K \cap H = H'$, then $N \cap H = E$,
2. since $K \triangleleft G$ and N is a normal Hall-subgroup in K , then $N \triangleleft G$,
3. since $KH = G$, $K = NH'$, then $HN = G$.

The proof is complete.

THEOREM 3. Let G be a group, and let H be a subgroup of G . Suppose, that (i)–(iii) below hold.

- (i) R is a designated set of representatives of H in G .
- (ii) K is a subgroup of G with $K \triangleleft G$ and $KH = G$.
- (iii) Put $H_1 := H \cap K$, there exists a subgroup H_2 with $H_2 \triangleleft H_1$, $H_1 = H_2 P$, $H_2 \cap P = E$ where P is a p -Sylow subgroup of G .

Then K has normal p -complement if and only if $K \cap N_G(P)$ has normal p -complement.

PROOF. We first prove, if $K \cap N_G(P)$ has normal p -complement, then K has too. Suppose the contrary, then in K there exists a (p, q) -group U for some $q \in \pi(G) - \{p\}$. Lemma 3. implies, that $q \nmid |H|$. Denote U_p the p -Sylow subgroup of U , and denote U_q the q -Sylow subgroup of U , $U_q = \langle a \rangle$. It is easy to see that in U_p there exists a set of generators A with $A^a \neq A$.

(*) **STATEMENT.** Let L be a subset of U_p , let Q be a p -Sylow subgroup of G , such that $L \subset P \cap Q$. If x is a p -element of $N_K(P \cap Q)$, then $L^x = L^y$ for some $y \in P$, with $xy^{-1} \in C_K(L)$.

PROOF. a) If $x \in H_1 \setminus P$, then $x = bc$ for some $b \in H_2$ and $c \in P$. Since $L^x \subset P$ then $L^b \subset P$. So $b \in C_G(L)$ by (iii). Thus $L^x = L^c$ with $xc^{-1} \in C_K(L)$.

REMARK. In this case we did not use the fact x is a p -element.

b) If $x \in K \setminus H_1$, then $x = zg$ with $z \in H$, $g \in R$. $L^x \subset P \cap P^x \subset H \cap H^x = H \cap H^g$ holds. By the Statement 2. $g \in C_G(H \cap H^g)$, so $L^x = L^z$ with $xz^{-1} \in C_K(L)$.

b/1) If $z \in H_1$ then the case a) of our Statement is applicable for z by the Remark. Thus there exists an element u of P with $L^x = L^z = L^u$, and $zu^{-1} \in C_K(L)$ holds. Using $xz^{-1} \in C_K(L)$, $xu^{-1} \in C_K(L)$ is true.

b/2) Suppose $z \in H \setminus H_1$, $L^x \subset P \cap Q$, $x \in N_K(P \cap Q)$ are true. So $(P \cap Q)^x \subset H \cap H^x = H \cap H^g$, and $(P \cap Q)^x = (P \cap Q)^g$. Thus $g^x = g^z$ for all $g \in P \cap Q$. If $(|z|, p) = 1$, then $x \in C_K(g)$ follows, for all $g \in P \cap Q$ by the Lemma 1. So $L^x = L^z$ is true, and $x \in C_K(L)$. If $|z| = p^m m$ with $(m, p) = 1$, $z \geq 1$, then $g^{(z^m)} = g^{(z^m)}$, where $|z^m| = p^m$, $z^m \in H_1$ by (ii). Since $(m, p) = 1$ then there exists a natural number r with $x^{mr} = x$. So $g^{(z^m)} = g^{(z^m)}$ for all $g \in P \cap Q$. Thus $x(z^{mr})^{-1} \in C_K(L)$ and $z^{mr} \in H_1$. Case a) of our Statement is applicable by the Remark, so $L^x = L^{(z^{mr})} = L^v$ for some $v \in P$. Using $z^{mr} v^{-1} \in C_K(L)$, $xv^{-1} \in C_K(L)$ follows. The proof of Statement is complete.

Now we continue the proof of our Theorem. We apply Theorem of ALPERIN [7] in K to A and $B = A^a$. There exist elements x_1, \dots, x_n, y with $x_i \in K$ for all i , $1 \leq i \leq n$ and $y \in N_K(P)$, there exist subgroups Q_1, \dots, Q_n where Q_i is a p -Sylow subgroup of K for all $1 \leq i \leq n$, $P \cap Q_i$ is a tame intersection, x_i is a p -element of $N_K(P \cap Q_i)$ for all $1 \leq i \leq n$.

$a = x_1, \dots, x_n, y$ holds, and $A \subset P \cap Q_1$ while $A^{x_1} \cdots x_n \subset P \cap Q_{i+1}$ are true for all $1 \leq i \leq n-1$. By the (*) Statement $A^{x_1} = A^{y_1}$ holds for some $y_1 \in P$, and $x_1 y_1^{-1} \in C_K(A)$. $A^{x_1} \subset P \cap Q_2$.

x_2 is a p -element of $N_K(P \cap Q_2)$. We apply (*) Statement to A^{x_1} and to $(A^{x_1})^{x_2}$, we get $(A^{x_1})^{x_2} = A^{x_1 x_2} = A^{y_1 y_2}$ for some $y_2 \in P$, and $x_2 y_2^{-1} \in C_K(A^{x_1})$. Continuing this process we get: $A^{x_1 \cdots x_n} = A^{y_1 \cdots y_n}$ where $y_i \in P$ for all $1 \leq i \leq n$, and $h^{x_1 \cdots x_n} = h^{y_1 \cdots y_n}$ holds for all $h \in A$. Denote $\omega := y_1 y_2 \cdots y_n y$, then $h^\omega = h^\omega$ is true for all $h \in A$. By conditions $N_K(P)$ has normal p -complement M , hence $N_K(P) = P \times M$. Thus $\omega = sl$ with $s \in P$, $l \in M$. Since $M \triangleleft C_K(P)$ then $A^s = A^l = A^s$ and $h^s = h^l$ hold for all $h \in A$. A is a set of generators of U_p , so $u^\omega = u^s$ is true for all $u \in U_p$. $(|u|, |s|) = 1$ holds, hence $u \in C_K(U_p)$ follows by the Lemma 1., a contradiction.

We now prove the converse of Theorem. Let N be a normal p -complement in K , put $N_1 = N \cap N_K(P)$. Clearly, N_1 has following properties:

- a) $N_1 \triangleleft N_K(P)$,
- b) $N_1 \cap P = E$,
- c) $N_1 P = N_K(P)$.

Thus N_1 is a normal p -complement in $N_K(P)$. The proof of our Theorem is complete.

Acknowledgement. The author wishes to thank her Professor K. CORRÁDI for his advice and encouragement.

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A SUFFICIENT CONDITION FOR TWO SETS TO BE DISTANT

By

ADAM P. BOSZNAY

Department of Mathematics, Faculty of Mechanical Engineering,
Technical University of Budapest

(Received January 28, 1981)

Introduction. Let $(X, \|\cdot\|)$ be a normed linear space, $0 \neq K \subset X$ a bounded set. We call K a remotal set if for each $x \in X$ the set

$$Q_K(x) = \{y \in K, \|x - y\| = \sup_{k \in K} \|x - k\|\}$$

is non-empty.

We say that the non-empty bounded sets $K_1, K_2 \subset X$ are distant [2], if there exist $x_1 \in K_1, x_2 \in K_2$ with the property

$$d(K_1, K_2) \equiv \sup_{k_1 \in K_1, k_2 \in K_2} \|k_1 - k_2\| = \|x_1 - x_2\|.$$

In [1] AHUJA, NARANG, and SWARAN TREHAN proved that K_1 and K_2 are distant in the case when K_1 is remotal and K_2 is compact (they proved this in a general metric space).

The question naturally arises whether the remotality of K_1 and K_2 implies their distantness. In this paper we give an affirmative answer for certain subsets of c_0 . Also we give a counterexample dropping a condition of the theorem.

The result. First, let us introduce the following.

DEFINITION. We say that the sets K_1 and K_2 have property A), if

$$d(K_1, K_2) > \sup_{k_1, k'_1 \in K_1} \|k_1 - k'_1\| + \sup_{k_2, k'_2 \in K_2} \|k_2 - k'_2\|.$$

THEOREM. Let K_1, K_2 be remotal sets in c_0 with property A). Then K_1 and K_2 are distant sets.

PROOF. Indirect. Let $x_i^1 \in K_1, x_i^2 \in K_2$ ($i = 1, 2, 3, \dots$) have the property

$$\lim_{i \rightarrow \infty} \|x_i^1 - x_i^2\| = d(K_1, K_2).$$

Let us introduce the notation $f_j : c_0 \rightarrow \mathbf{R}$ by

$$f_j((y_1, y_2, \dots, y_j, y_{j+1}, \dots)) = y_j.$$

Now, there are two possible cases.

Case I. There exist subsequences $x_{i_j}^1$ and $x_{i_j}^2$ with the property

$$\lim_{j \rightarrow \infty} |f_{l_j}(x_{i_j}^1 - x_{i_j}^2)| = d(K_1, K_2)$$

where

$$\lim_{j \rightarrow \infty} l_j = \infty.$$

Case II. There exist a fixed $s \in \mathbb{N}$ and subsequences $x_{i_j}^1$ and $x_{i_j}^2$ such that

$$\lim_{j \rightarrow \infty} |f_s(x_{i_j}^1 - x_{i_j}^2)| = d(K_1, K_2).$$

In case I, let

$$0 < \varepsilon < \frac{d(K_1, K_2) - d^*(K_1) - d^*(K_2)}{4},$$

where

$$d^*(K_i) = \sup_{k_i, k'_i \in K_i} \|k_i - k'_i\| \quad (i = 1, 2).$$

There exists a j such that

$$(1) \quad |f_{l_j}(x_{i_j}^1 - x_{i_j}^2)| > d(K_1, K_2) - \varepsilon.$$

Now, let us choose a j^* such that

$$(2) \quad |f_{l_{j^*}}(x_{i_j}^1)| < \varepsilon,$$

$$(3) \quad |f_{l_{j^*}}(x_{i_j}^2)| < \varepsilon,$$

and

$$|f_{l_{j^*}}(x_{i_{j^*}}^1 - x_{i_{j^*}}^2)| > d(K_1, K_2) - \varepsilon.$$

Using (2) and (3),

$$|f_{l_{j^*}}(x_{i_j}^1 - x_{i_j}^2 - (x_{i_{j^*}}^1 - x_{i_{j^*}}^2))| > d(K_1, K_2) - 3\varepsilon.$$

This implies

$$\|x_{i_j}^1 - x_{i_j}^2 - (x_{i_{j^*}}^1 - x_{i_{j^*}}^2)\| > d(K_1, K_2) - 3\varepsilon.$$

So,

$$\|x_{i_j}^1 - x_{i_{j^*}}^1\| + \|x_{i_j}^2 - x_{i_{j^*}}^2\| > d(K_1, K_2) - 3\varepsilon.$$

Since

$$\|x_{i_j}^1 - x_{i_{j^*}}^1\| < d^*(K_1),$$

$$\|x_{i_j}^2 - x_{i_{j^*}}^2\| < d^*(K_2),$$

we have

$$d^*(K_1) + d^*(K_2) > d(K_1, K_2) - 3\varepsilon$$

and this contradicts the definition of ε .

Case II. We can choose the subsequences $x_{i_j}^1$ and $x_{i_j}^2$ so that

$$(4) \quad \lim_{j \rightarrow \infty} f_s(x_{i_j}^1) = a_1$$

and

$$(5) \quad \lim_{j \rightarrow \infty} f_s(x_{ij}^2) = a_2$$

exist.

We can assume without loss of generality $a_1 > a_2$, so

$$(6) \quad d(K_1, K_2) = a_1 - a_2.$$

Now, let

$$x^1 = (0, 0, \dots, \underbrace{0}_{s-1}, 3 \sup_{k_1 \in K_1} \|k_1\|, 0, \dots)$$

and

$$x^2 = (0, 0, \dots, \underbrace{0}_{s-1}, 3 \sup_{k_2 \in K_2} \|k_2\|, 0, \dots).$$

By the remoteness of K_1 and K_2 ,

$$\sup_{k_1 \in K_1} \|x^1 - k_1\| = \|x^1 - \tilde{x}^1\|$$

and

$$\sup_{k_2 \in K_2} \|x^2 - k_2\| = \|x^2 - \tilde{x}^2\|$$

for some $\tilde{x}^1 \in K_1$, $\tilde{x}^2 \in K_2$. By the definition of the c_0 norm,

$$\|x^1 - \tilde{x}^1\| = |f_{t_1}(x^1 - \tilde{x}^1)|,$$

$$\|x^2 - \tilde{x}^2\| = |f_{t_2}(x^2 - \tilde{x}^2)|$$

or some $t_1, t_2 \in \mathbb{N}$. Using the definition of x^1 and x^2 , for any $t \neq s$ we have

$$|f_t(x^1 - \tilde{x}^1)| \leq \sup_{k_1 \in K_1} \|k_1\|,$$

$$|f_t(x^2 - \tilde{x}^2)| \leq \sup_{k_2 \in K_2} \|k_2\|$$

and

$$|f_s(x^1 - \tilde{x}^1)| \geq 2 \sup_{k_1 \in K_1} \|k_1\|,$$

$$|f_s(x^2 - \tilde{x}^2)| \geq 2 \sup_{k_2 \in K_2} \|k_2\|.$$

These imply

$$\|x^1 - \tilde{x}^1\| = |f_s(x^1 - \tilde{x}^1)|,$$

$$\|x^2 - \tilde{x}^2\| = |f_s(x^2 - \tilde{x}^2)|.$$

By the above two equalities and the definitions of \tilde{x}^1 and \tilde{x}^2 ,

$$\sup_{k_1 \in K_1} f_s(k_1) = f_s(\tilde{x}^1),$$

$$\inf_{k_2 \in K_2} f_s(k_2) = f_s(\tilde{x}^2).$$

Applying (4) and (5),

$$a_1 < f_s(\hat{x}^1), \quad a_2 \geq f_s(\hat{x}^2).$$

So,

$$\|\hat{x}^1 - \hat{x}^2\| \geq f_s(\hat{x}^1 - \hat{x}^2) \geq a_1 - a_2.$$

By (6),

$$\|\hat{x}^1 - \hat{x}^2\| = d(K_1, K_2)$$

and this contradicts the indirect hypothesis. Qu.e.d.

REMARK 1. The theorem remains valid with the same proof in that subspace c'_0 of c_0 , which consists of all elements of c_0 with finitely many non-zero co-ordinates.

REMARK 2. The following example shows that in c'_0 the property A) is essential.

Namely, we shall construct remotal sets K_1, K_2 in c'_0 which are not distant.

Put

$$e_i = (0, 0, \dots, 0, \overset{i-1}{\overbrace{0}}, 1, 0, \dots) \in c'_0,$$

$$K_1 = \{e_{2i}; i = 1, 2, \dots\} \cup \left\{ \left(1 - \frac{1}{2i+1}\right) e_{2i+1}; i = 1, 2, \dots \right\},$$

$$K_2 = \{e_{2i}; i = 1, 2, \dots\} \cup \left\{ \left(1 - \frac{1}{2i+1}\right) e_{2i+1}; i = 1, 2, \dots \right\}.$$

Clearly,

$$d(K_1, K_2) \leq \sup_{k_1 \in K_1} \|k_1\| + \sup_{k_2 \in K_2} \|k_2\| = 2.$$

On the other hand

$$d(K_1, K_2) \geq 2 \left(1 - \frac{1}{2i+1}\right) e_{2i+1}$$

for all $i \in \mathbb{N}$, and this implies

$$d(K_1, K_2) = 2.$$

It is elementary to show that K_1 and K_2 are not distant sets.

We prove that K_1, K_2 are remotal. We prove the remotality of K_1 , the proof for K_2 is similar.

Let $x \in c'_0$ be arbitrary. Then there exists $i \in \mathbb{N}$ such that, for all $j \geq i$, $f_j(x) = 0$. Thus for $j \geq \frac{i}{2}$

$$(7) \quad g_{2j}(x) \equiv \|x - e_{2j}\| = \max \{1, \|x\|\},$$

$$(8) \quad g_{2j+1}(x) \equiv \|x - \left(1 - \frac{1}{2j+1}\right) e_{2j+1}\| = \max \left\{1 - \frac{1}{2j+1}, \|x\|\right\}.$$

By (7) and (8), K_1 is remotal since $g_{2j}(x)$ is constant for $j \geq \frac{i}{2}$ and

$$g_{2j+1}(x) \leq g_{2j}(x)$$

for all $j \geq \frac{i}{2}$.

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A REMARK ON THE FARTHEST POINT PROBLEM IN HILBERT SPACE

By

ADAM P. BOSZNAY

Department of Mathematics, Faculty of Mechanical Engineering,
Technical University of Budapest

(Received January 28, 1981)

Introduction. Let us recall the farthest point problem [3].

Let $0 \neq K \subseteq X$ be a bounded set in the normed linear space $(X, \|\cdot\|)$. Assume that for each $x \in X$ the set

$$Q_K(x) = \{y_0 \in K; \|x - y_0\| = \sup_{y \in K} \|x - y\|\}$$

consists of exactly one element. In this case we call K a uniquely remotal set. It is natural to ask whether there exists a non-singleton set K with this property.

This problem — as far as we know — is generally unsolved, even in Hilbert space. In [4] we proved that for every normed linear space there exists an equivalent renorming with the following property: in the new norm every uniquely remotal set is a singleton.

In this paper for each $\varepsilon > 0$ we give such a renorming $\|\cdot\|_\varepsilon$ of the real separable infinite-dimensional Hilbert space with the additional property

$$1 - \varepsilon \leq \frac{\|x\|_\varepsilon}{\|x\|} \leq 1$$

for all $x \neq 0$.

The result.

THEOREM. For any $\varepsilon > 0$ there exists a norm $\|\cdot\|_\varepsilon$ on the real space l_2 such that

$$1 - \varepsilon \leq \frac{\|x\|_\varepsilon}{\|x\|} \leq 1$$

for all $x \neq 0$, and that all uniquely remotal sets in $(l_2, \|\cdot\|_\varepsilon)$ are singletons.

PROOF. By a theorem of ASPLUND [1], we must only prove that there exists a countable set $f_1, f_2, \dots \in l_2^*$ such that $\|f_1\| = \|f_2\| = \dots = 1$ and

- (i) $\max_n f_n(x)$ exists for all $x \in l_2$,
- (ii) $\max_n f_n(x) \leq (1 + \varepsilon) \|x\|$ for all $x \in l_2$,
- (iii) $\max_n f_n(x) = \max_n f_n(-x)$ for all $x \in l_2$.

(Asplund's theorem reads: let $(X, |\cdot|)$ be a normed linear space where $|\cdot|$ is the maximum of a countable set of continuous linear functionals. Then all uniquely remotest sets in $(X, |\cdot|)$ are singletons.)

First, let us construct a sequence $x_1, x_2, \dots, \|x_1\| = \|x_2\| = \dots = 1$, in l_2 such that

- A) x_1, x_2, \dots is a $\sqrt{2}\varepsilon$ -net on the unit sphere,
- B) for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$(1) \quad -x_n = x_m$$

and

- C) the set $\{x_n; n \in \mathbb{N}\}$ is closed.

Defining the (non-convex) cone C by

$$C = \{\lambda x_n; \lambda \geq 0, n \in \mathbb{N}\}$$

it is clear that C is a closed set in l_2 .

Using a theorem of PAPINI [2] we obtain that for any $x \in l_2$, $\|x\| = 1$ there exists a nearest element in C . (The theorem of Papini is the following: let C be a closed (non-convex) cone in a uniformly convex Banach space $(X, |\cdot|)$. Then for any $x \in X$, C has an element which is nearest to x .)

Let $\hat{x}_n x_n$ be the nearest element to x . Then for all $k \in \mathbb{N}$ we have

$$\begin{aligned} 1 + \hat{x}_n^2 - 2\hat{x}_n \langle x, x_n \rangle &= \langle x - \hat{x}_n x_n, x - \hat{x}_n x_n \rangle = \\ &\leq \|x - \hat{x}_n x_n\|^2 \leq \|x - \hat{x}_n x_k\|^2 = 1 + \hat{x}_n^2 - 2\hat{x}_n \langle x, x_k \rangle \end{aligned}$$

and therefore

$$\langle x, x_k \rangle \leq \langle x, x_n \rangle$$

for all $k \in \mathbb{N}$.

Let us define

$$f_k(y) = \langle y, x_k \rangle$$

for all $k \in \mathbb{N}$, $y \in l_2$. Then (i) is satisfied.

Since $\{x_k; k \in \mathbb{N}\}$ is a $\sqrt{2}\varepsilon$ -net on the unit sphere,

$$(\sqrt{2}\varepsilon)^2 \leq \|x - x_n\|^2 = \langle x - x_n, x - x_n \rangle = 2 - 2 \langle x, x_n \rangle, \quad \langle x, x_n \rangle \geq 1 - \varepsilon,$$

i.e. we have (ii). The validity of (iii) follows from (1).

The theorem is proved.

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GENERALIZATION OF A THEOREM OF JOÓ

By

V. KOMORNÍK

II. Department of Analysis of the L. Eötvös University, Budapest

(Received December 28, 1981)

Let $G = [a, b]$ be a compact interval and consider the formal differential operator

$$(1) \quad Lu = u^{(n)} + p_2 u^{(n-2)} + \dots + p_{n-1} u' + p_n u,$$

where $p_2, \dots, p_n \in L^1(G)$ are arbitrary complex functions. Let λ be a complex number. The function $u_{-1} : G \rightarrow \mathbb{C}$, $u_{-1} \equiv 0$ is called an eigenfunction of order -1 (of the operator L) with the eigenvalue λ . A function $u_i : G \rightarrow \mathbb{C}$, $u_i \neq 0$ ($i = 0, 1, \dots$), as it is usual, is said to be an eigenfunction of order i (of the operator L) with the eigenvalue λ if u_i , together with its first $n-1$ derivatives is absolutely continuous on G and if for almost all $x \in G$ the equations

$$(2) \quad (L u_i)(x) = \lambda u_i(x) + u_{i-1}(x)$$

hold, where u_{i-1} is an eigenfunction of order $i-1$ with the eigenvalue λ .

Developing the ideas and results of the paper of Joó [1], we have proved in [5] the following assertion.

THEOREM A. Consider the above operator (1). There exist then such constants $K_{n, m}$ depending only on n, m that given any eigenfunction u_m of order $m \geq 0$ of the operator (1) with the eigenvalue $\lambda = \varrho^n$, we have for all $0 \leq j \leq m$, $0 \leq i \leq n$, $1 \leq r \leq \infty$ the estimates

$$\|u_{m-j}^{(i)}\|_\infty \leq K_{n, m} \cdot S^{n+j+i+r} \cdot \|u_m\|_r,$$

where

$$(3) \quad S \equiv \max \left(|\varrho|, \frac{1}{b-a}, \|p_2\|_1, \sqrt{\|p_3\|_1}, \dots, \sqrt[n-1]{\|p_n\|_1} \right).$$

Moreover, if $p_2, \dots, p_n \in L^q(G)$ for some $1 \leq q \leq \infty$, then there exist such constants $K_{n, m}$ ($b-a$), depending only on the indicated arguments that for all $0 \leq j \leq m$ and $0 \leq i \leq n$,

$$\|u_{m-j}^{(i)}\|_q \leq K_{n, m}(b-a) \cdot Q^{nj+i} \|u_m\|_q,$$

where

$$(4) \quad Q = \max(|\varrho|, 1, \|p_2\|_q, \sqrt{\|p_3\|_q}, \dots, \sqrt[n-1]{\|p_n\|_q})$$

and

$$1/q + 1/q' = 1.$$

In case $n \geq 3$ these estimates are exact from the point of view of dependence on the quantities λ, p_2, \dots, p_n . However, in case $n = 2$ our result is not exact. Recently, in [6], developping his previous result [1], L. Joó has obtained exact estimates in case $n = 2, q = 1$ for the eigenfunctions, having real negative eigenvalues.

In the present paper we show that our method, developped in [4] and [5], can also be applied in case $n = 2$. Moreover, we obtain exact estimates for arbitrary $\lambda \in \mathbb{C}$ and $1 < q < \infty$. The case $q = 2$ will also improve a previous result of V. A. Il'in [2]. We shall prove the following assertion.

THEOREM. *Let $G = [a, b]$ be a compact interval, $p \in L^1(G)$ an arbitrary complex function and consider the formal operator*

$$(5) \quad L u = u'' + p \cdot u.$$

There exist then constants $K_{2, m}$ depending only on m such that given any eigenfunction u_m of order $m \geq 0$ of the operator (5) with the eigenvalue $\lambda + \varrho^2$, we have for all $0 \leq j \leq m, 0 \leq i \leq 2$ and $1 \leq r \leq \infty$ the estimates

$$\|u_{m-j}^{(i)}\|_\infty \leq K_{2, m} S_1^{j+i} S_2^{j+1-r} \|u_m\|_r$$

where

$$(6) \quad S_1 = \max\left(|\varrho|, \frac{1}{b-a}, \|p\|_1\right),$$

$$S_2 = \max\left(|\operatorname{Re} \varrho|, \frac{1}{b-a}, \|p\|_1\right).$$

Moreover, if $p \in L^q(G)$ for some $1 < q < \infty$, then there exist constants $K_{2, m}(b-a)$ depending only on the indicated arguments such that for all $0 \leq j \leq m$ and $0 \leq i \leq 1$,

$$\|u_{m-j}^{(i)}\|_{q'} \leq K_{2, m}(b-a) Q_1^{j+i} Q_2^{j+1} \|u_m\|_q,$$

where

$$(7) \quad Q_1 = \max(|\varrho|, 1, \|p\|_q), \quad Q_2 = \max(|\operatorname{Re} \varrho|, 1, \|p\|_q),$$

$$1/q + 1/q' = 1.$$

Our proof will be based on the following Proposition, following from Theorem 1 and its proof in [5]:

PROPOSITION. Given any integer $m \geq 0$ there exist such meromorphic on \mathbb{C} functions f_{jik} and entire functions h_j ($0 \leq j \leq m, 0 \leq i \leq 1, 1 \leq k \leq N \equiv 2m+2$) that given any eigenfunction u_m of order $\leq m$ of the operator (5) with the eigenvalue $\lambda + \varrho_2 \in \mathbb{C}$, introducing for $j \leq m$ the continuous functions u_j by

(2), we have for all $j \geq 0$, $0 \leq i \leq 1$, $x \in G$ and $xNt \in G$, whenever f_{jik} (ϱt) is defined,

$$(8) \quad t^{2j+i} u_{m-i}^{(i)}(x) = \sum_{k=1}^N f_{jik}(\varrho t) u_m(x+kt) + \\ + \sum_{k=1}^N f_{jik}(\varrho t) \sum_{l=0}^m \int_x^{x+kt} (x+kt-\tau)^{2r+1} h_r(\varrho(x+kt-\tau)) p(\tau) u_{m-l}(\tau) d\tau.$$

Furthermore, the functions f_{jik} , h_r have the following properties ($0 \leq j$, $r \leq m$, $0 \leq i \leq 1$, $1 \leq k \leq N$):

$$(9) \quad f_{jik}(z) \equiv (\sinh z)^{-2(m+1)^2} \sum_{s=j}^{2j+1} C_{js} z^{2j+i-s} P_{ks}(\varrho z, e^{-z})$$

with some constants C_{js} and polynomials P_{ks} :

(10) f_{jik} has no poles in the region $|z| < \pi$;

(11) for $j = i = 0$, the functions f_{00k} are entire;

$$(12) \quad h_r(z) \equiv z^{-2r-1} (P_r(z) e^z + Q_r(z) e^{-z})$$

with some polynomials P_r, Q_r of degree r .

One can see easily (see also [1]) that for any $z \in \mathbb{C}$, $|\operatorname{Re} z| \approx 1$, $|z| \approx 2$, $|\sinh(z/z)| \approx 1/3$ for some $1/2 \leq z \leq 1$. Using this property, we obtain from (9)–(12) the following estimates with some constant $C > 0$ ($0 \leq j$, $r \leq m$, $0 \leq i \leq 1$, $1 \leq k \leq N$):

for any $z \in \mathbb{C}$, $|\operatorname{Re} z| \approx 1$, there exists some $1/2 \leq z \leq 1$ with

$$(13) \quad |f_{jik}(z)| \leq C \max(1, |z|)^{j+i};$$

for any $t \in \mathbb{R}$, $\varrho \in \mathbb{C}$, $|\operatorname{Re} \varrho t| = N$,

$$(14) \quad |t^{2r+1} h_r(\varrho t)| \leq C t^r \min\left(t, \frac{1}{|\varrho|}\right)^{r+1};$$

$$(15) \quad \text{for any } z \in \mathbb{C}, |\operatorname{Re} z| \approx 1, |f_{00k}(z)| \leq C.$$

Let us now turn to the proof of the theorem. Assume $p \in L^q(G)$, $1 \leq q \leq \infty$ and define

$$(16) \quad K \equiv \begin{cases} C^2 N^{N+1} \max(2, b-a) & \text{if } q > 1, \\ 2 C^2 N^{N+1} & \text{if } q = 1, \end{cases}$$

$$(17) \quad R \equiv \min\left(\frac{1}{|\operatorname{Re} \varrho|}, \frac{1}{K \cdot \|p\|_q}, \frac{b-a}{2N}\right).$$

We note that

$$(18) \quad K \geq 2 C^2 N^{N+1} \left(\frac{b-a}{2}\right)^{1/q'} \quad (1/q + 1/q' = 1).$$

Fixing an arbitrary $a \leq x \leq \frac{a+b}{2}$, we have by (8), (13), (14), (17) and the Hölder inequality for some

$$(19) \quad R/2 \leq t \leq R:$$

$$\begin{aligned} t^{2j+i} |u_{m-j}^{(i)}(x)| &\leq C \max(1, |\varrho t|)^{j+i} \sum_{k=1}^N |u_m(x+kt)| + \\ &+ C \max(1, |\varrho t|)^{j+i} \sum_{k=1}^N \sum_{r=0}^m C(kt)^r \min\left(kt, \frac{1}{|\varrho|}\right)^{r+1} \times \\ &\times \|p\|_q \|u_{m-r}\|_q, \leq C \max(1, |\varrho t|)^{j+i} \sum_{k=1}^N |u_m(x+kt)| + \\ &+ C^2 N^N \max(1, |\varrho t|)^{j+i} \|p\|_q \sum_{r=0}^m t^r \min\left(t, \frac{1}{|\varrho|}\right)^{r+1} \|u_{m-r}\|_q, \\ &(0 \leq j \leq m, 0 \leq i \leq 1). \end{aligned}$$

Taking the $L^{q'}\left(a, \frac{a+b}{2}\right)$ norm of both sides and applying the triangle inequality, we obtain

$$\begin{aligned} t^{2j+i} \|u_{m-j}^{(i)}\|_{L^{q'}\left(a, \frac{a+b}{2}\right)} &\leq C N \max(1, |\varrho t|)^{j+i} \|u_m\|_{q'} + \\ &+ C^2 N^N \left(\frac{b-a}{2}\right)^{1/q'} \max(1, |\varrho t|)^{j+i} \|p\|_q \times \\ &\times \sum_{r=0}^m t^r \min\left(t, \frac{1}{|\varrho|}\right)^{r+1} \|u_{m-r}\|_{q'}. \end{aligned}$$

The same estimate can be obtained by a similar way for the quantity

$$t^{2j+i} \|u_{m-j}^{(i)}\|_{L^{q'}\left(\frac{a+b}{2}, b\right)}$$

too. Summing these two estimates, using on the left side the triangle inequality and finally dividing both sides by $\max(1, |\varrho t|)^{j+i}$, we obtain for all $0 \leq j \leq m$ and $0 \leq i \leq 1$,

$$\begin{aligned} t^{2j+i} \max(1, |\varrho t|)^{-j-i} \|u_{m-j}^{(i)}\|_{q'} &\leq 2C N \|u_m\|_{q'} + \\ &+ 2C^2 N^N \left(\frac{b-a}{2}\right)^{1/q'} \|p\|_q \sum_{r=1}^m t^r \min\left(t, \frac{1}{|\varrho|}\right)^{r+1} \|u_{m-r}\|_{q'}. \end{aligned}$$

Introducing the notation

$$(20) \quad M_{q'} = \max \left\{ t^i \min\left(t, \frac{1}{|\varrho|}\right)^{j+i} \|u_{m-j}^{(i)}\|_{q'} : 0 \leq j \leq m, 0 \leq i \leq 1 \right\}$$

and taking into account (17)–(19), we can write

$$\begin{aligned} M_{q'} &\leq 2C N \|u_m\|_{q'} + 2C^2 N^N \left(\frac{b-a}{2} \right)^{1/q'} \|p\|_q (N/2) t M_{q'} \\ &\leq 2C N \|u_m\|_{q'} + 1/2 M_{q'}, \\ M_{q'} &\leq 4C N \|u_m\|_{q'}, \end{aligned}$$

and in view of (19)–(20) for all $0 < j < m$ and $-m \leq i \leq 1$,

$$(21) \quad \|u_{m-jl}^{(i)}\|_{q'} \leq 4C N (2/R)^j \max(2/R, |\varrho|)^{j+i} \|u_m\|_{q'}.$$

(21), (17) and (16) yield (7) and the part $r = \infty$ of (16).

To finish the proof of the theorem, it suffices to show (6) for $j - i = 0$, $1 < r < \infty$. We have by (8), (14), (17), (15) and (21) for all $a \leq x \leq \frac{a+b}{2}$, $0 \leq t \leq R/2$,

$$\begin{aligned} |u_m(x)| &\leq C \sum_{k=1}^N |u_m(x+kt)| + \sum_{k=1}^N C \sum_{r=0}^m C(kt)^r \times \\ &\quad \times \min \left(kt, \frac{1}{|\varrho|} \right)^{r+1} \|p\|_1 4C N \left(\frac{2}{R} \right)^r \max \left(\frac{2}{R}, |\varrho| \right)^r \|u_m\|_\infty \leq \\ &\leq C \sum_{k=1}^N |u_m(x+kt)| + 2C^3 N^{N+2} \|p\|_1 t \|u_m\|_\infty. \end{aligned}$$

Applying to both sides the operation $R_1^{-1} \int_0^{R_1} \cdot dt$ where

$$(22) \quad R_1 \equiv \min(R/2, (2C^3 N^{N+2} \|p\|_1)^{-1}),$$

we obtain

$$|u_m(x)| \leq C R_1^{-1} \sum_{k=1}^N \int_0^{R_1} |u_m(x+kt)| dt + 1/2 \|u_m\|_\infty.$$

But, using the Hölder inequality,

$$\begin{aligned} R_1^{-1} \int_0^{R_1} |u_m(x+kt)| dt &= (k R_1)^{-1} \int_0^{kR_1} 1 \cdot |u_m(x+\tau)| d\tau \leq \\ &\leq (k R_1)^{-1+1/r} \|u_m\|_r, \end{aligned}$$

whence

$$|u_m(x)| \leq C N R_1^{-1/r} \|u_m\|_r + 1/2 \|u_m\|_\infty.$$

This is true for all $a \leq x \leq \frac{a+b}{2}$, but a similar consideration gives this result for all $\frac{a+b}{2} \leq x \leq b$, too. Therefore

$$\|u_m\|_{\infty} \leq C N R_1^{-1/r} \|u_m\|_r + 1 \cdot 2 \|u\|_r,$$

$$2\|u_m\|_{\infty} \leq 2C N R_1^{-1/r} \|u_m\|_r;$$

in view of (16), (17) and (22), hence the required part of (6) follows.

The Theorem is proved.

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MOMENT TYPE THEOREMS FOR *-SEMIGROUPS OF OPERATORS ON HILBERT SPACE II.

By

ZOLTÁN SEBESTYÉN

I. Department of Math. Analysis of the L. Eötvös University, Budapest

(Received January 14, 1982)

Introduction. Given a (complex) C^* -algebra A with unity e , a multiplicative semigroup G of it which is closed under involution, contains e and spans A , briefly a *-semigroup, an operator valued function F of G into $B(H)$ – the C^* -algebra of all bounded linear operators on the complex Hilbert space H – has an A -dilation S on a Hilbert space K (in a general sense) if there is a continuous linear map $V : K \rightarrow H$ with

$$(1) \quad F(g) = VS_g V^* \quad \forall g \in G$$

where $S : A \rightarrow B(K)$ is a *-representation of A on K . In case when $F(e) := I_H$, the identity operator on H or equivalently $VV^* = I_H$, S is called a strong A -dilation of F on K .

Let further be given an H -valued function x on G , it is natural to ask:

1. Under what condition does there exist an A -dilatable H -valued function F on G such that

$$(2) \quad x_g = F(g)x_e \quad \forall g \in G$$

holds. Our main aim is to answer this question. The problem of existence of a strong A -dilation remains open. Only two special cases will be treated:

11. When does exist a *-representation T of A on H for which

$$(3) \quad x_g = T_g x_e \quad \forall g \in G$$

is satisfied. As a corollary we treat the unitary and normal operator case with restricted spectra. To be more special we treat the following problem:

III. Given a sequence $\{x_n^{n'}\}_{n', n}$ of H such that $\{x_n^0\}_n$ spans H , under what condition does there exist a subnormal operator T on H satisfying

$$(4) \quad x_n^{n'} = T^{*n'} T^n x_n^0 \quad \text{for } n', n = 0, 1, 2, \dots$$

and such that the spectrum of its dilation is included in some prescribed compact set Ω of the complex plane \mathbb{C} .

The present note serves a continuation to our previous papers [4], [5].

For normal extension (dilation) of subnormal operators we refer to BRAM [1], HALMOS [2] and SZ. NAGY [3].

Results. THEOREM 1. Problem 1 has a solution if and only if

$$(5) \quad M^{-1} \left\| \sum_g c_g x_g \right\|^2 \leq \sum_{g, h} c_g c_h (x_{h^* g}, x_e) \leq M \left\| \sum_g c_g g \right\|^2$$

holds with some constant $M > 0$ and for all finite sequence $\{c_g\}$ of complex numbers indexed by elements of G .

PROOF. Assuming an A -dilatable solution $F : G \rightarrow B(H)$ with (1) and (2) we have for any finite sequence $\{c_g\}$ in \mathbb{C}

$$\begin{aligned} \left\| \sum_g c_g x_g \right\|^2 &\leq \|V\|^2 \left\| \sum_g c_g S_g V^* x_e \right\|^2 = \|V\|^2 \sum_{g, h} c_g c_h (V S_{h^* g} V^* x_e, x_e) = \\ &= \|V\|^2 \sum_g c_g c_h (x_{h^* g}, x_e), \\ \sum_{g, h} c_g c_h (x_{h^* g}, x_e) &= \sum_g c_g \bar{c}_h (V S_{h^* g} V^* x_e, x_e) = \\ &= \left\| \sum_g c_g S_g V^* x_e \right\|^2 \leq \left\| \sum_g c_g S_g \right\|^2 \|V^* x_e\|^2 \leq \\ &\leq \|V^* x_e\|^2 \left\| \sum_g c_g g \right\|^2. \end{aligned}$$

The last inequality is a consequence of a known property of a *-representation of a C^* -algebra. These together show (5) as desired.

We shall continue analogously but in a more delicate manner with respect to the representation as we want to show representation for A rather than for G . This is the reason why boundedness not only for $\{x_g\}$ holds as in [5], (4), Theorem A. Assume that we are familiar with the proof of the cited theorem: the linear space Y of $\sum_g c_g \delta_g$ (finite sum) functions on G of finite support, where δ_g is 1 in g and 0 otherwise, with semi-inner product

$$\langle \sum_h c_h \delta_h, \sum_k d_k \delta_k \rangle := \sum_{h, k} c_h d_k (x_{k^* h}, x_e)$$

and a map of Y into H defined by

$$V(\sum_h c_h \delta_h) := \sum_h c_h x_h,$$

gives a Hilbert space K and a continuous linear map V of K into H satisfying

$$V^* x_e = \delta_e.$$

We have also a shift operator S_g for any g in G given on Y by

$$S_g (\sum_h c_h \delta_h) := \sum_h c_h \delta_{gh} \quad \forall g \in G$$

which defines a continuous linear operator on K . Our aim is two show

$$(6) \quad \left\| \sum_g \lambda_g S_g \right\| \leq \left\| \sum_g \lambda_g g \right\|$$

for any finite sum $\sum_g \lambda_g g$ in the linear span of G in A , which is norm dense by

assumption. Moreover, since S is a \ast -representation of G as shown in [5]; it generates a \ast -representation of A as well, since it is given on the linear span of G by

$$S_a := \sum_g \lambda_g S_g \quad \text{for } a = \sum_g \lambda_g g.$$

For

$$a = \sum_g \lambda_g g, \quad y = \sum_h c_h \delta_h \in V$$

we have

$$\begin{aligned} \|S_a y\|^2 &= \left\| \sum_{g,h} \lambda_g c_h \delta_{gh} \right\|^2 = \sum_{g',g,h,k} \lambda_g \lambda_{g'} c_h c_k \langle x_{k^* g'^* g h}, x_e \rangle = \\ &= \langle S_{a^* a} y, y \rangle \leq \|S_{a^* a} y\| \|y\|, \end{aligned}$$

and by induction also

$$\begin{aligned} \|S_a y\|^{2^n-1} &\leq \|S_{(a^* a)^{2^{n-1}}} y\|^2 \|y\|^{2^{n-1}-2} = \\ &= \|y\|^{2^{n-1}-2} \langle S_{(a^* a)^{2^{n-1}}} y, S_{(a^* a)^{2^{n-1}}} y \rangle = \\ &= \|y\|^{2^{n-1}-2} \langle \sum_{h,s} c_h d_s \delta_{g(s)h}, \sum_{h,s} c_h d_s \delta_{g(s)h} \rangle \end{aligned}$$

if we write $(a^* a)^{2^{n-1}} = \sum_s d_s g(s)$. But (5) implies

$$\begin{aligned} \|S_a y\|^{2^n-1} &\leq M \left\| \sum_{h,s} c_h d_s g(s) h \right\|^2 \|y\|^{2^{n-1}-2} \leq \\ &\leq \left\| \sum_s d_s g(s) \right\|^2 \|y\|^{2^{n-1}-2} M \left\| \sum_h c_h h \right\|^2 = \\ &= \|(a^* a)^{2^{n-1}}\|^2 \|y\|^{2^{n-1}-2} M \left\| \sum_h c_h h \right\|^2 = \\ &= \|a\|^{2^n-1} \|y\|^{2^n-1} M \|y\|^{-2} \left\| \sum_h c_h h \right\|^2 \end{aligned}$$

for any $n = 0, 1, 2, \dots$ and thus (6) too.

Finally $F(g) = VS_g V^*(g \in G)$ defines the desired A -dilatable $F: G \rightarrow B(H)$ function since for any g in G

$$F(g)x_e = VS_g V^*x_e = VS_g \delta_e = V \delta_g = x_g$$

holds indeed. The proof is complete.

As a corollary we have a solution to Problem II.

THEOREM II. *There exists a \ast -representation T of A on H with (3) if and only if*

$$(7) \quad (x_h, x_k) = (x_{k^* h}, x_e) \quad \forall h, k \in G$$

and

$$(8) \quad \left\| \sum_g c_g x_g \right\| \leq \left\| \sum_g c_g g \right\| \|x_e\|$$

holds for any finite sequence $\{c_g\}$ of complex numbers indexed by elements of G .

PROOF. The necessity of (7), (8) are obvious:

$$(x_h, x_k) = (T_h x_e, T_k x_e) = (T_{k^*h} x_e, x_e) \Rightarrow (x_{h^*k}, x_e),$$

$$\left\| \sum_g c_g x_g \right\|^2 = \sum_{g,h} c_g c_h (x_g, x_h) = \sum_{g,h} c_g \tilde{c}_h (x_{h^*g}, x_e),$$

$$\left\| \sum_g c_g x_g \right\|^2 = \left\| \sum_g c_g T_g x_e \right\|^2 \cong \left\| \sum_g c_g T_g \right\|^2 \|x_e\|^2 = \left\| \sum_g c_g g \right\|^2 \|x_e\|^2.$$

For sufficiency we refer to the proof of Theorem 1. Since (7) implies

$$\left\| \sum_g c_g x_g \right\|^2 = \sum_{g,h} c_g \tilde{c}_h (x_{h^*g}, x_e)$$

for any finite sequence $\{c_g\}$, V is isometry from K into H (in the proof of Theorem 1) satisfying also

$$(9) \quad V^* x_g = \delta_g \quad \forall g \in G.$$

Indeed we have

$$\begin{aligned} \left\langle \sum_h c_h \delta_h, V^* x_g \right\rangle &= \left(V \left(\sum_h c_h \delta_h \right), x_g \right) = \sum_h c_h (x_h, x_g) \\ &= \sum_h c_h (x_{g^*h}, x_e) = \left\langle \sum_h c_h \delta_h, \delta_g \right\rangle \end{aligned}$$

for any $\sum_h c_h \delta_h$ from Y . But (9) implies that

$$T_a := VS_a V^* \quad \forall a \in A$$

defines the desired *-representation of A because of $V^*V = I_K$, since $V^*V \delta_g = V x_g = \delta_g$ for any g in G , as

$$T_{gh} = VS_{gh} V^* = VS_g V^* VS_h V^* = T_g T_h$$

holds for any g, h in G indeed.

As a corollary we have a strengthened version of Corollary B in [5]:

COROLLARY 1. Given a sequence $\{x_n\}_{n=-\infty}^{\infty}$ in the Hilbert space H and a compact subset Ω in the unit circle of the complex plane, there is a unitary operator U on H with spectrum contained in Ω and such that

$$x_n = U^n x_0 \text{ for } n = \dots, -2, -1, 0, 1, 2, \dots$$

if and only if

$$(x_m, x_n) = (x_{m-n}, x_0) \quad (m, n = \dots, -1, 0, 1, \dots)$$

and

$$\left\| \sum_n c_n x_n \right\| \cong \|x_0\| \max_{z \in \Omega} \left| \sum_n c_n z^n \right|$$

hold for any finite sequence $\{c_n\}_{n=-\infty}^{\infty}$ of complex numbers.

PROOF. The *-semigroup of functions $\{z^n\}_{n=-\infty}^{\infty}$ ($z \in \Omega$) (isomorphic to \mathbb{Z}) span the C^* -algebra of all complex continuous functions on Ω by the Stone-Weierstrass theorem. Thus the *-representation on H ensured by Theorem 2

gives the desired unitary operator $U = T_{\varphi}$, the image of the identity function on Ω under the representation T , the spectrum of which is contained in Ω of course. Hence the proof ends.

As a further corollary to Theorem 2 we have an improved form of Proposition of [4].

COROLLARY 2. *Let $\{x_n^{n'}\}_{n,n'=0}^\infty$ be a sequence in the Hilbert space H and Ω be a compact subset of the complex plane. There is a normal operator T on H with spectrum contained in Ω such that*

$$x_n^{n'} = T^{*n'} T^n x_0^0 \text{ for } n = 0, 1, 2, \dots$$

if and only if

$$(x_m^{m'}, x_n^{n'}) = (x_{m+n}^{m+n}, x_0^0) \quad (m', m, n', n = 0, 1, 2, \dots)$$

and

$$\left\| \sum_{n' \neq n} c_{n', n} x_n^{n'} \right\| \leq \|x_0^0\| \max \left| \sum_{n' \neq n} c_{n', n} z^{n'} z^n \right|$$

hold for any finite sequence $\{c_{n', n}\}$ of complex numbers indexed by pairs of natural numbers.

PROOF. The *-semigroup G of functions $\{z^{n'} z^n\}_{n', n=0}^\infty$ ($z \in \Omega$) span $C(\Omega)$, the C^* -algebra of all complex continuous functions on Ω again by the Stone-Weierstrass theorem. As a consequence of Theorem 2 we have a *-representation T of $C(\Omega)$ on H with

$$T z^{n'} z^n x_0^0 = x_n^{n'} \quad (n', n = 0, 1, 2, \dots)$$

Hence $T = T_\varphi$ is a normal operator the spectrum of which belongs to Ω and for which $T^{*n'} T^n = T z^{n'} z^n$ holds. The proof is complete.

Our final result improves Theorem C in [5] as we have restriction on the spectrum of the operator in question.

THEOREM III. *Problem III has a solution if and only if*

$$(III_1) \quad (x_n^{n'}, x_m^0) = (x_n^{m+n'}, x_0^0) \quad (m, n, n' = 0, 1, 2, \dots)$$

and

$$(III_2) \quad \begin{aligned} \left\| \sum_{n' \neq n} c_{n', n} x_n^{n'} \right\|^2 &\leq \sum_{m' \neq m, n' \neq n} c_{m', m} c_{n', n} (x_{m+n}^{m+n}, x_0^0) \leq \\ &\leq \|x_0^0\|^2 \max_{z \in \Omega} \left| \sum_{n' \neq n} c_{n', n} z^{n'} z^n \right|^2 \end{aligned}$$

hold for any finite double sequence $\{c_{n', n}\}_{n', n=0}^\infty$ of complex numbers.

PROOF. If we have a subnormal operator T satisfying (4), since T has a (minimal) normal dilation N with spectrum in Ω , on some Hilbert space K :

$$T^{*n'} T^n x = P N^{*n'} N^n x \quad (x \in H; n', n = 0, 1, 2, \dots)$$

holds with orthogonal projection P of K onto H ,

$x_n^{n'} := T^{*n'} T^n x_0$ clearly implies (III₁), (III₂):

$$\begin{aligned} (x_n^{n'}, x_m^0) &= (T^{*n'} T^n x_0^0, T^m x_0^0) = (T^{*(m+n')} T^n x_0^0, x_0^0) = (x_n^{m+n'}, x_0^0), \\ \left\| \sum_{n', n} c_{n', n} x_n^{n'} \right\|^2 &\leq \left\| \sum_{n', n} c_{n', n} N^{*n'} N^n x_0^0 \right\|^2 \leq \|x_0^0\|^2 \sum_{n', n} |c_{n', n}|^2 N^{*n'} N^n \leq \\ &\leq \|x_0^0\| \max_{z \in Sp(N)} \left| \sum_{n', n} c_{n', n} z^{n'} z^n \right| = \|x_0^0\| \max_{z \in \Omega} \left| \sum_{n', n} c_{n', n} z^{n'} z^n \right| \end{aligned}$$

by the spectral theorem of normal operators:

$$\begin{aligned} \left\| \sum_{n', n} c_{n', n} x_n^{n'} \right\|^2 &\leq \left\| \sum_{n', n} c_{n', n} N^{*n'} N^n x_0^0 \right\|^2 = \\ &= \sum_{\substack{m', m \\ n', n}} c_{m', m} c_{n', n} (N^{*(m'+n)} N^{m+n'} x_0^0, x_0^0) = \\ &= \sum_{\substack{m', m \\ n', n}} c_{m', m} c_{n', n} (T^{*(m'+n)} T^{m+n'} x_0^0, x_0^0) = \sum_{\substack{m', m \\ n', n}} c_{m', m} c_{n', n} (x_{m+n}^{m'+n}, x_0^0). \end{aligned}$$

To prove the converse assume (III₁), (III₂) and let G be $\mathbf{N} \times \mathbf{N}$ as a *-semigroup *-isomorphic to functions $\{z^{n'} z^n\}_{n', n \geq 0}$ ($z \in \Omega$) with usual operations, as before. Theorem 1 says there exists an $C(\Omega)$ -dilatable H -valued function

$$F(z^{n'} z^n) = V S_{z^{n'} z^n} V^* \text{ where } S : C(\Omega) \rightarrow B(K)$$

is a *-representation on a Hilbert space K constructed for the linear space of complex functions of finite support on $\mathbf{N} \times \mathbf{N}$ by semi-inner product

$$\left\langle \sum_{m', m} c_{m', m} \delta_m^{m'}, \sum_{n', n} d_{n', n} \delta_n^{n'} \right\rangle = \sum_{\substack{m', m \\ n', n}} c_{m', m} d_{n', n} (x_{m+n}^{m'+n}, x_0^0)$$

where $\delta_m^{m'}$ is the function 1 in (m', m) and 0 otherwise. V is a continuous linear operator from K into H above, the origin of which is defined on Y by the formula

$$V \left(\sum_{m', m} c_{m', m} \delta_m^{m'} \right) = \sum_{m', m} c_{m', m} x_m^{m'}.$$

Our aim is to prove $VV^* = I_H$ (the identity operator on H) for which it is enough to see

$V^* x_n^0 = \delta_n^0$ for $n = 0, 1, 2, \dots$, since then $VV^* x_n^0 = V \delta_n^0 = x_n^0$ and $\{x_n^0\}_{n=0}^\infty$ spans the space H . But we have

$$\begin{aligned} \left\langle \sum_{m', m} c_{m', m} \delta_m^{m'}, V^* x_n^0 \right\rangle &= \left\langle V \left(\sum_{m', m} c_{m', m} \delta_m^{m'} \right), x_n^0 \right\rangle = \\ &= \sum_{m', m} c_{m', m} (x_m^{m'}, x_n^0) = \sum_{m', m} c_{m', m} (x_{m+n}^{m+n}, x_0^0) = \left\langle \sum_{m', m} c_{m', m} \delta_m^{m'}, \delta_n^0 \right\rangle \end{aligned}$$

which yields the statement. The operator T on H , $T = VN^*V^*$, where $N = S_z$ is a normal operator on K with spectrum included in Ω , is a subnormal operator because of $VV^* = I_H$ and since

$$T^*T = VN^*NV^*.$$

To prove this we have to check for any x_m^0 ($m = 0, 1, 2, \dots$)

$$T^* T x_m^0 = V N^* N V^* x_m^0$$

where

$$V N^* N V^* x_m^0 = V S_{zz} V^* x_m^0 = V S_{zz} \delta_m^0 = V \delta_{m+1}^1 = x_{m+1}^1.$$

But we have indeed:

$$T x_m^0 = V N V^* x_m^0 = V S_z \delta_m^0 = V \delta_{m+1}^0 = x_{m+1}^0,$$

$$T^* T x_m^0 = V N^* V^* T x_m^0 = V N^* V^* x_{m+1}^0 = V N^* \delta_{m+1}^0 = V \delta_{m+1}^1 = x_{m+1}^1.$$

Identifying H with $V^*(H)$ in K, N is a normal extension of T , in particular a normal dilation such that

$$T^{*n'} T^n = V N^{*n'} N^n V^* \quad (n' = 0, 1, 2, \dots)$$

holds. In consequence

$$T^{*n'} x_0^0 = V N^{*n'} N^n V^* x_0^0 = V S_{z^{n'}} z^n \delta_0^0 = V \delta_n^{n'} = x_n^{n'}$$

holds also for any n, n natural numbers. The proof is thus complete.

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MOMENT TYPE THEOREMS FOR *-SEMIGROUPS OF OPERATORS ON HILBERT SPACE III

By

ZOLTÁN SEBESTYÉN

II. Department of Math. Analysis of the L. Eötvös University, Budapest

(Received January 14, 1982)

Introduction. Our present note is a direct continuation of the author's previous works [1] and [2]. Our main aim is to solve problems mentioned in [1], [2] in case of strong dilatability (A -dilatability) of an operator valued function (on a Hilbert space H) F of a *-semigroup G with identity (of a C^* -algebra spanned by G respectively) such that

$$(1) \quad F(g)x = P S_g x \quad (g \in G, x \in H)$$

with a *-representation $S : G \rightarrow B(K)$ ($S : A \rightarrow B(K)$ respectively).

$$(2) \quad x_g = F(g)x_e \quad \forall g \in G$$

holds, where K is a Hilbert space containing H as a subspace which is the range of the orthogonal projection P in K , and $\{x_g\}_{g \in G}$ is a given family of vectors spanning H . Such questions are treated in [1], [2] only in special cases.

Results. **THEOREM 1.** *There exists a $B(H)$ -valued function F on a *-semigroup G with identity e satisfying (1) and (2) if and only if there exists a family $\{x_h^h\}_{h', h \in G}$ in H such that*

$$(3) \quad x_h^e = x_e^h = x_h \quad \forall h \in G$$

$$(4) \quad \|x_h^{h'}\| \leq \|x_e\| p(h') p(h) \quad \forall h', h \in G$$

with submultiplicative function $p : G \rightarrow \mathbb{R}^+$ satisfying a C^* -property

$$p(g^*g) = p(g)^2 \quad \forall g \in G$$

implying

$$p(g^*) \leq p(g) \quad \forall g \in G$$

and such that

$$(5) \quad \left\| \sum_{h', h} c_{h', h} x_h^{h'} \right\|^2 \leq \sum_{h', h} |c_{h', h}|^2 c_{k', k} (x_{k'h}^{h'}, x_k)$$

holds for any finite sequence $\{c_{h', h}\}$ of complex numbers indexed by pairs of elements in G .

PROOF. To prove the necessity of conditions (3–5) let

$$x_h^{h'} = F(h)x_{h'} \quad \forall h', h \in G,$$

then we have (3) immediately by (2), (4) by (1) and (2), since

$$\|x_h^{h'}\| \leq \|F(h)\| \|x_{h'}\| \leq \|F(h)\| \|F(h')\| \|x_e\| \leq \|x_e\| \|S_h\| \|S_{h'}\|$$

and finally (5) as

$$\begin{aligned} \left\| \sum_{h', h} c_{h', h} x_h^{h'} \right\|^2 &\leq \left\| \sum_{h', h} c_{h', h} S_h x_{h'} \right\|^2 = \sum_{\substack{h', h \\ k', k}} c_{h', h} \bar{c}_{k', k} (S_{k'*h} x_{h'}, x_k) = \\ &= \sum_{\substack{h', h \\ k', k}} c_{h', h} c_{k', k} (F(k*h) x_{h'}, x_k) = \sum_{\substack{h', h \\ k', k}} c_{h', h} c_{k', k} (x_{k*h}^{h'}, x_k). \end{aligned}$$

For sufficiency assume (3–5) and let Y be the linear space of complex valued functions of finite support on $G \times G$ each of which is of the form (finite sum) $\sum_{h', h} c_{h', h} \delta_h^{h'}$, where $\delta_h^{h'}$ denotes the function 1 in (h', h) and 0 otherwise. Let further $\langle \cdot, \cdot \rangle$ be a semi-inner product on Y given by

$$\sum_{h', h} c_{h', h} \delta_h^{h'} \cdot \sum_{k', k} d_{k', k} \delta_k^{k'} := \sum_{\substack{h', h \\ k', k}} c_{h', h} d_{k', k} (x_{k*h}^{h'}, x_k)$$

for any two functions $\sum_{h', h} c_{h', h} \delta_h^{h'}$ and $\sum_{k', k} d_{k', k} \delta_k^{k'}$ in Y . We have also two maps on Y given for $\sum_{h', h} c_{h', h} \delta_h^{h'}$ by

$$V(\sum_{h', h} c_{h', h} \delta_h^{h'}) := \sum_{h', h} c_{h', h} x_h^{h'}$$

and with values in H such that by (5) V is a contraction and

$$S_g(\sum_{h', h} c_{h', h} \delta_h^{h'}) := \sum_{h', h} c_{h', h} \delta_{gh}^{h'} \quad \forall g \in G$$

mapping Y in Y continuously with respect to $\langle \cdot, \cdot \rangle$ as we shall see later on.

First of all we have thus a Hilbert space K by factoring Y with respect to the null space of $\langle \cdot, \cdot \rangle$ and completing with respect to the norm inherited from the inner product on this factor space. For simplicity the same symbols denote the elements of Y in K , the maps induced by V and S_g ($g \in G$) and the inner product on K respectively, as was done in [1], [2]. We are now going to prove

$$(6) \quad V^* x_{k'} = \delta_e^{k'} \quad \forall k' \in G$$

$$(7) \quad VV^* = T_H \text{ (the identity operator on } H),$$

$$(8) \quad \|S_g y\| \leq \|\rho(g)\| \|y\| \quad (g \in G, y \in Y \subset K)$$

and that $S : G \rightarrow B(K)$ is a $*$ -representation with the desired function

$$(9) \quad F(g) = VS_g V^* \quad \forall g \in G$$

which is the same as (1) if we identify H with a subspace of K by V^* since then V serves as an orthogonal projection P of K onto this subspace indeed. A routine calculation implies (6) as for any $v = \sum_{h', h} c_{h', h} \delta_h^{h'}$ in Y we have

$$\langle y, V^* x_k \rangle = (V y, x_k) = \sum_{h', h} c_{h', h} (x_h^{h'}, x_k) = \left\langle \sum_{h', h} c_{h', h} \delta_h^{h'}, \delta_e^k \right\rangle = \langle y, \delta_e^k \rangle.$$

We have then (7) since by definition of V and by (3)

$$VV^* x_k = V \delta_e^k = x_e^k = x_{k'} \quad \forall k' \in G$$

and $\{x_k\}_{k' \in G}$ spans H by assumption. To show (8) let $y = \sum_{h', h} c_{h', h} \delta_h^{h'}$ be an element of Y in K for which

$$\begin{aligned} \|S_g y\|^2 &= \left\| \sum_{h', h} c_{h', h} \delta_h^{h'} \right\|^2 = \sum_{\substack{h', h \\ k', k}} c_{h', h} \bar{c}_{k', k} (x_{k'*h}^{h'}, x_k) = \\ &= \langle S_{g*g} y, y \rangle \leq \|S_{g*g} y\| \|y\| \end{aligned}$$

and by induction for any $n = 0, 1, 2, \dots$

$$\begin{aligned} (10) \quad \|S_g y\|^{2^n-1} &\leq \|S_{(g*g)^{2^n-1}} y\|^2 \|y\|^{2^n-1-2} = \\ &= \|y\|^{2^n-1-2} \sum_{\substack{h', h \\ k', k}} c_{h', h} \bar{c}_{k', k} (x_{k'*g^{2^n} h}^{h'}, x_k) \leq \\ &\leq \|y\|^{2^n-2-2} M^2 p((g*g)^{2^n}) \left(\sum_{h', h} |c_{h', h}| p(h') p(h) \right)^2 \leq \\ &\leq p(g)^{2^n-1} \|y\|^{2^n-1} \|y\|^{2^n-1-2} M^2 \left(\sum_{h', h} |c_{h', h}| p(h') p(h) \right)^2 \end{aligned}$$

showing (8) by letting n to infinity. To prove that the function S on G is a *-representation we need only routine machinery so we omit this. The function F on G defined by (9) satisfies (2) as well since

$$F(g)x_e = VS_e V^* x_e = VS_g \delta_e^g = V \delta_g^e = x_g \quad \forall g \in G$$

holds by (6) and (3). The proof is ended.

THEOREM 2. *There exists a $B(H)$ -valued function F on a unital C^* -algebra spanned by a *-semigroup G with identity e satisfying (1) and (2) if and only if there exists a family $\{x_h^{h'}\}_{h', h \in G}$ in H with (3) and such that*

$$\left\| \sum_{h', h} c_{h', h} x_h^{h'} \right\|^2 \leq \sum_{\substack{h', h \\ k', k}} c_{h', h} \bar{c}_{k', k} (x_{k'*h}^{h'}, x_k) \leq \|x_e\|^2 \sum_{h'} \|h'\| \left\| \sum_h c_{h', h} h \right\|^2$$

holds for any finite sequence $\{c_{h', h}\}_{h', h \in G}$ of complex numbers.

PROOF. Since (10) clearly implies (4) with $p(h) = \|h\|$ for any h in G we shall go throughout the proof of Theorem 1 to prove that S is a *-representation of A on K as well. For this we have to show

$$(11) \quad \left\| \sum_g \lambda_g S_g \right\| \leq \left\| \sum_g |\lambda_g| g \right\|$$

for any finite sequence $\{\lambda_g\}_{g \in G}$ of complex numbers, as was done in proof of Theorem I in [2]. The further properties of S are seen analogously as there. For if $a = \sum_g \lambda_g g \in A$ and $y = \sum_{h', h} c_{h', h} \delta_h^{h'} \in Y$ are given, we have

$$\|S_a y\|^2 = \langle S_{a^* a} y, y \rangle \leq \|S_{a^* a} y\| \|y\|$$

by denoting $\sum_g \lambda_g S_g$ with S_a . By induction we have

$$\begin{aligned} \|S_a y\|^{2^{n+1}} &\leq \|S_{(a^* a)^{2^n}} y\|^2 \|y\|^{2^{n+1}-2} \leq \|y\|^{2^{n+1}-2} \|\sum_{h', h, s} c_{h', h} d_s \delta_s^{h'}(s) y\|^2 \leq \\ &\leq \|y\|^{2^{n+1}-2} \|x_e\|^2 \left(\sum_{h'} \|h'\| \left\| \sum_{h', h, s} c_{h', h} d_s g(s) h \right\|^2 \right) \leq \\ &\leq \|y\|^{2^{n+1}-2} \|x_e\|^2 \left(\sum_{h'} \|h'\| \left\| \sum_h c_{h', h} h \right\|^2 \left\| \sum_s d_s g(s) \right\|^2 \right) = \\ &= \|(a^* a)^{2^{n+1}}\|^2 \|y\|^{2^{n+1}-2} \|x_e\|^2 \left(\sum_{h'} \|h'\| \left\| \sum_h c_{h', h} h \right\|^2 \right)^2 = \\ &= \|a\|^{2^{n+1}} \|y\|^{2^{n+1}-2} \|x_e\|^2 \left(\sum_{h'} \|h'\| \left\| \sum_h c_{h', h} h \right\|^2 \right)^2, \end{aligned}$$

where $(a^* a)^{2^{n+1}} = \sum_s d_s g(s)$ was used for a complex polynomial with variables g and g^* . This shows (11) by letting n to infinity, indeed.

Sufficiency of the second inequality in (10) needs yet some observation as the following

$$\begin{aligned} \left\| \sum_{h', h} c_{h', h} x_h^{h'} \right\|^2 &\leq \sum_{h', h} c_{h', h} c_{h', h} (x_{h^* h}, x_{h'}) \leq \left\| \sum_{h', h} c_{h', h} S_h x_h \right\|^2 \leq \\ &\leq \left(\sum_{h'} \|x_{h'}\| \left\| \sum_h c_{h', h} S_h \right\| \right)^2 \leq \left(\sum_{h'} \|S_h x_h\| \left\| \sum_h c_{h', h} h \right\| \right)^2 \leq \\ &\leq \|x_e\|^2 \left(\sum_{h'} \|h'\| \left\| \sum_h c_{h', h} h \right\|^2 \right). \end{aligned}$$

The proof is thus complete.

The following two theorems generalize our previous results in [2], [3] with respect to subnormal operators to a *-semigroup of type described below.

Given a commutative semigroup with identity G the set $G \times G$ becomes a commutative *-semigroup with operations

$$(g, g') (h, h') = (gh, g' h'),$$

$$(g, h)^* = (h, g) \quad : g, g', h', h \in G.$$

If now $T : G \rightarrow B(K)$ is a multiplicative map of G into the normal operators on a Hilbert space K then

$$S : G \times G \rightarrow B(K), S_{(g, h)} := T_h^* T_g \quad (g, h \in G)$$

is easily seen to be a *-representation of $G \times G$ on K .

The verification needs the theorem of FUGLEDE on commutation property of normal operators on Hilbert space. If further H is a (closed) subspace

in K invariant for all operators $\{T_g\}_{g \in G}$ we have a $B(H)$ -valued operator function on $G \times G$ given by

$$(12) \quad F(g, h) = P T_h^* T_g \quad (g, h \in G)$$

such that $F(g, e) = T_g|_H$ (the restriction of T_g to H) holds for any g in G .

Given a family $\{x_g^e\}_{g, h \in G}$ in a Hilbert space H , the subfamily $\{x_g^e\}_{g \in G}$ spanning H , it is natural to ask: when does exist a $B(H)$ -valued function F on $G \times G$ of type (12) such that

$$(13) \quad F(g, h)x_g^e = x_g^h \quad (g, h \in G)$$

holds. The answer is the following

THEOREM 3. *There exists a $B(H)$ -valued function F on $G \times G$ with (12) if and only if*

$$(14) \quad (x_g^h, x_k^e) = (x_g^{hk}, x_e^e) \quad (g, h, k \in G),$$

$$(15) \quad \|x_g^h\| \leq M p(g, h)$$

for some constant $M \geq 0$ and submultiplicative map $p : G \times G \rightarrow \mathbb{R}^+$ with C^* -property

$$(16) \quad p((g, h)^*(g, h)) = p(g, h)^2 \quad (g, h \in G)$$

implying

$$p((g, h)^*) = p(g, h)$$

and such that

$$\left\| \sum_{g, h} c_{g, h} x_g^h \right\|^2 \leq \sum_{\substack{g, h \\ g', h'}} c_{g, h} \bar{c}_{g', h'} (x_{g'h'}^{g'h}, x_e^e)$$

holds for any finite sequence $\{c_{g, h}\}_{g, h \in G}$ of complex numbers.

PROOF. The necessity of (14–16) is simple

$$\begin{aligned} (x_g^h, x_k^e) &= (P T_h^* T_g x_e^e, x_k^e) = (T_h^* T_g x_e^e, T_k x_k^e) = (T_{hk}^* T_g x_e^e) = \\ &= (P T_{hk}^* T_g x_e^e, x_e^e) := (x_g^{hk}, x_e^e), \end{aligned}$$

$$\|x_g^h\| \leq \|T_h^* T_g x_e^e\| \leq \|x_e^e\| \|S_{(g, h)}\|,$$

$$\begin{aligned} \left\| \sum_{g, h} c_{g, h} x_g^h \right\|^2 &\leq \left\| \sum_{g, h} c_{g, h} T_h^* T_g x_e^e \right\|^2 = \sum_{\substack{g, h \\ g', h'}} c_{g, h} \bar{c}_{g', h'} (T_{g'h'}^* x_e^e, x_e^e) = \\ &= \sum_{\substack{g, h \\ g', h'}} c_{g, h} \bar{c}_{g', h'} (P T_{g'h'}^* T_{g'h'} x_e^e, x_e^e) = \sum_{\substack{g, h \\ g', h'}} c_{g, h} \bar{c}_{g', h'} (x_{g'h'}^{g'h}, x_e^e). \end{aligned}$$

For sufficiency assume (14–16) and let Y be the linear space of complex valued functions of finite support on $G \times G$ describing in the form $\sum_{h, k} c_{h, k} \delta_h^k$ where δ_h^k denotes the function 1 in (h, k) and 0 otherwise. We have by (16) a semi-inner product on Y given by

$$(17) \quad \left\langle \sum_{h, k} c_{h, k} \delta_h^k, \sum_{h', k'} d_{h', k} \delta_{h'}^{k'} \right\rangle := \sum_{\substack{h, k \\ h', k'}} c_{h, k} \bar{d}_{h', k'} (x_{h'k'}^{h'k}, x_e^e)$$

for $\sum_{h,k} c_{h,k} \delta_h^k$ and $\sum_{h',k'} c_{h',k'} \delta_{h'}^{k'}$ in Y and two familiar maps as

$$(18) \quad V : Y \rightarrow H, \quad V\left(\sum_{h,k} c_{h,k} \delta_h^k\right) := \sum_{h,k} c_{h,k} x_h^k,$$

$$(19) \quad S_{(g,g')} : Y \rightarrow Y, \quad S_{(g,g')}\left(\sum_{h,k} c_{h,k} \delta_h^k\right) := \sum_{h,k} c_{h,k} \delta_{g'h}^{g'k}$$

for any g, g' in G .

So we have a Hilbert space K arising from Y as before, by (16) a contraction V from K into H such that

$$(20) \quad V^* x_{h'}^e = \delta_{h'}^e \quad (h' \in G)$$

since by (14), (17)

$$\begin{aligned} \sum_{h,k} c_{h,k} \delta_h^k, V^* x_{h'}^e \rangle &= \left(V\left(\sum_{h,k} c_{h,k} \delta_h^k\right), x_{h'}^e\right) = \sum_{h,k} c_{h,k} (x_h^k, x_{h'}^e) = \\ &= \sum_{h,k} c_{h,k} (x_h^{h'k}, x_e^e) = \sum_{h,k} c_{h,k} \delta_h^k, \delta_{h'}^e \rangle. \end{aligned}$$

Hence $VV^* x_{h'}^e = V \delta_{h'}^e = x_{h'}^e$ holds for any h' in G ,

$$(21) \quad VV^* = I_H$$

follows by the assumption that $\{x_e^e\}_{e \in G}$ spans H and thus H may be viewed as a subspace of K such that V represents then the orthogonal projection of K onto H . Moreover

$$(22) \quad \|S_{(g,g')} y\| \leq p(g, g') \|y\| \quad (g, g' \in G; y \in Y)$$

holds also since for $\sum_{h,k} c_{h,k} \delta_h^k$ we have

$$\begin{aligned} \|S_{(g,g')} y\|^2 &= \sum_{h,k} c_{h,k} \delta_{g'h}^{g'k} \|^2 = \sum_{h,k,h',k'} c_{h,k} c_{h',k'} (x_{h'g'gk}^{h'g'gk}, x_e^e) = \\ &= \|S_{(g,g')*(g,g')} y, y\| \leq \|S_{(g,g')*(g,g')} y\| \|y\| \end{aligned}$$

and by induction (from (15))

$$\begin{aligned} \|S_{(g,g')} y\|^{2^{n+1}} &\leq \|S_{((g,g')*(g,g))^{2^n-1}} y\|^2 \|y\|^{2^{n+1}-2} = \\ &= \|y\|^{2^{n+1}-2} \sum_{h,k,h',k'} c_{h,k} c_{h',k'} (x_{h(g'g)^{2^n-1}k}^{h(g'g)^{2^n-1}k}, x_e^e) \leq \\ &\leq \|y\|^{2^{n+1}-2} M \|x_e^e\| \sum_{h,k,h',k'} c_{h,k} c_{h',k'} p(h(g'g)^{2^n-1} k', h'(g'g)^{2^n-1} k) \leq \\ &\leq \|y\|^{2^{n+1}-2} M \|x_e^e\| p(g, g')^{2^{n+1}} \sum_{h,k} c_{h,k} p(h, k) \leq \end{aligned}$$

showing (22) as n tends to ∞ . It is easy to see that S is a *-representation of $G \times G$ on K . Finally

$$F(g, g') = VS_{(g, g')} V^* \quad (g, g' \in G)$$

is the desired function since (by (20), (19), (18))

$$F(g, g') x_e^e = VS_{(g, g')} \delta_e^e = V \delta_g^{g'} = x_g^{g'}.$$

The proof is complete.

To generalize the previous situation assume further that $G \times G$ is contained in a C^* -algebra A such that spans A and S let the *-representation of A on some Hilbert space K as well. The existence problem with respect to a $B(H)$ -valued function F satisfying (12) and (13) is solved in the following

THEOREM 4. *There exists a $B(H)$ -valued function F on $G \times G$ with (12–13), where S is a *-representation of the C^* -algebra A if and only if (14) and*

$$(23) \quad \left\| \sum_{h, k} c_{h, k} x_h^k \right\|^2 \leq \sum_{\substack{h, k \\ h', k'}} c_{h, k} \bar{c}_{h', k'} (x_{h'k}^{h'k}, x_e^e) \leq \|x_e^e\|^2 \left\| \sum_{h, k} c_{h, k} (h, k) \right\|^2$$

hold for any finite sequence $\{c_{h, k}\}_{h, k \in G}$ of complex numbers.

PROOF. Since (23) implies (15) with $p(g, h) = \|(g, h)\|$ ($g, h \in G$) we can use Theorem 3. We have only to show that S is a *-representation of A as well. To see this we shall prove

$$(24) \quad \left\| \sum_{g, g'} \lambda_{g, g'} S_{(g', g)} \right\| \leq \left\| \sum_{g, g'} \lambda_{g, g'} (g, g') \right\|$$

for any finite sequence $\{\lambda_{g, g'}\}_{g, g' \in G}$ of complex numbers.

Let $a = \sum_{h, k} \lambda_{g, g'} (g, g')$ be given and $y = \sum_{g, g'} c_{h, k} \delta_h^k \in V$ be given. Then we have for $S_a = \sum_{g, g'} \lambda_{g, g'} S_{(g, g')}$

$$\|S_a y\|^2 = \langle S_{a*} y, y \rangle \leq \|S_{a*} y\| \|y\|,$$

and by induction

$$\begin{aligned} \|S_a y\|^{2^n+1} &\leq \|S_{(a*a)^{2^n+1}} y\|^2 \|y\|^{2^n+1-2} = \|y\|^{2^n+1-2} \left\| \sum_{h, k, s} c_{h, k} d_s \delta_{g(s)h}^{g'(s)k} \right\|^2 \leq \\ &\leq \|y\|^{2^n+1-2} \|x_e^e\|^2 \left\| \sum_{h, k, s} c_{h, k} d_s (g(s)h, g'(s)k) \right\|^2 \leq \\ &\leq \|y\|^{2^n+1-2} \|x_e^e\|^2 \left\| \sum_{h, k} c_{h, k} (h, k) \right\|^2 \left\| \sum_s d_s (g(s), g'(s)) \right\|^2 = \\ &= \|(a^* a)^{2^n+1} \|y\|^{2^n+1-2} \|x_e^e\|^2 \left\| \sum_{h, k} c_{h, k} (h, k) \right\|^2 = \\ &= \|a\|^2 \|y\|^{2^n+1-2} \|x_e^e\|^2 \left\| \sum_{h, k} c_{h, k} (h, k) \right\|^2 \end{aligned}$$

where $(a^* a)^{2^n+1} = \sum_s d_s (g(s), g'(s))$ denotes a suitable complex polynomial with variable (g, g') and (g', g) . By n tending to ∞ this shows then (24).

Finally the sufficiency of the right hand side in (23) needs some suitable observation as

$$\begin{aligned} \left\| \sum_{h,k} c_{h,k} x_h^k \right\|^2 &\leq \sum_{h,k} |c_{h,k}| |c_{h',k'}(x_{hk'}^{h'k}, x_e^e)| \leq \left\| \sum_{h,k} c_{h,k} S_{(h,k)} x_e^e \right\|^2 \leq \\ &\leq \|x_e^e\|^2 \left\| \sum_{h,k} c_{h,k} S_{(h,k)} \right\|^2 \leq \|x_e^e\|^2 \left\| \sum_{h,k} c_{h,k} (h,k) \right\|^2 \end{aligned}$$

where the last step uses a known property of the *-representation of a C^* -algebra. The proof is complete.

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SOFT SUBGROUPS OF p -GROUPS

By

L. HÉTHÉLYI

Department of Philosophy of the L. Eötvös University, Budapest

(Received February 8, 1982)

It is a well known fact (see e.g. [1], III, 14.23) that a p -group P is of maximal class iff it contains a self-centralizing subgroup A of order p^2 . Then obviously $|N_P(A) : A| = p$ holds provided $P \triangleright A$. Our concept of soft subgroup generalizes this situation.

DEFINITION. A subgroup A of a (nonabelian) p -group P is called a soft subgroup if $C_P(A) = A$ and $|N_P(A) : A| = p$.

The aim of the present paper is to formulate some properties of p -groups possessing soft subgroups. It turns out that a soft subgroup A is contained in a unique maximal subgroup of P , namely in $M = P \bar{A}$. We establish some interesting properties of the either central series of M . If $|P : A| = p^n$, then the nilpotence class of M is exactly n , while that of P is at least $n+1$.

As a rule we use the notations of [1]. Moreover for $X = Y$, $\text{core}(X; Y)$ denotes the largest normal subgroup of Y contained in X and $\text{n.cl.}(X; Y)$ stands for the smallest normal subgroup of Y containing X .

Our induction arguments will be based on the following.

LEMMA 1. Let A be a soft subgroup in P , $N = N_P(A)$, $K = \text{core}(A; P)$, $\bar{P} = P/K$. Then N is soft in P and $N = \bar{A} \times Z(\bar{P})$. Moreover $|N'| = p$ if $|p : A| \geq p^2$.

PROOF. Obviously, $\bar{N} = N_P(A) \cong A\bar{Z}(\bar{P})$. Now $\bar{A} \cap \bar{Z}(\bar{P})$ is normal in \bar{P} , hence trivial. Then $|\bar{N} : \bar{A}| = p$ implies the second assertion. We have $C_{\bar{P}}(\bar{N}) = C_P(\bar{A}) \leq N_P(\bar{A}) = \bar{N}$, i.e. \bar{N} is self-centralizing. Let us denote $N_2 = N_P(N)$, so $N_P(N) \subset N_2$. For $x \in N_2 \setminus N$, $A^x \neq A$ is a maximal subgroup in N , hence $A^x \cap A = Z(N)$, $|N : Z(N)| = p^2$, $|N'| = p$. Counting the conjugate subgroups A^x , $x \in N_2 \setminus N$ we obtain $|N_2 : N| = p$, thus N is soft in \bar{P} .

Now we fix the notation we use throughout the paper. Let A be a soft subgroup in P , $|P : A| = p^n$. Define $N_0 = A$, $N_i = N_P(N_{i-1})$ for $i = 1, \dots, n$. For sake of brevity we shall denote N_{n+1} by M .

LEMMA 2. The subgroups of P containing A form a chain $A = N_0 \subset N_1 \subset \dots \subset N_{n-1} = M \subset N_n = P$, $|N_i : N_{i-1}| = p$ ($i = 1, \dots, n$).

PROOF. By induction Lemma 1 yields $|N_i : N_{i-1}| = p$. Now if $A \leq X \leq P$ then let N_j be the largest subgroup contained in X . We have

$$N_j \leq N_X(N_j) = X \cap N_P(N_j) = X \cap N_{j+1} < N_{j+1},$$

hence $N_X(N_j) = N_j$, thus $X = N_j$.

COROLLARY 1. For any $v \in P \setminus M$ we have $\langle A, A^v \rangle = M$.

PROOF. By Lemma 2, $\langle A, A^v \rangle = N_j$ for some $j \leq n-1$. Since $A \leq N_j$ and $A \leq N_j^{v-1}$, it follows that $v \in N_P(N_j) = N_{j+1}$ so we are done.

COROLLARY 2. There exists an $a \in A$ such that $C_P(a) = A$.

PROOF. Since $A = C_P(A) = \bigcap_{a \in A} C_P(a)$, the assertion obviously follows from Lemma 2.

Our aim now is to investigate some chains of subgroups in P .

PROPOSITION 1. For the upper central series of M we have:

- (i) $Z_i(M) = \text{core}(N_{i-1}; P) = N_{i-1} \cap N_{i-1}^v$ ($v \in P \setminus M$), $i = 1, \dots, n$;
- (ii) $Z_{i+1}(M)/Z_i(M)$ is elementary abelian of order $\leq p^2$ for $i = 1, \dots, n-1$;
- (iii) $Z_i(P) \leq Z_i(M)$, $i = 1, \dots, n$.

PROOF. (i). For $i = 1$ we have $N_{i-1} = A$. Let $v \in P \setminus M$ be fixed. By Corollary 1, $M = \langle A, A^v \rangle$. Now

$$Z(M) = C_M(A) \cap C_M(A^v) = A \cap A^v.$$

Finally, Lemma 1 proves (i) by induction.

$$(ii) \quad Z_i(M) = N_{i-1} \cap N_{i-1}^v = N_{i-1} \cap N_i^v = N_i \cap N_i^v = Z_{i-1}(M).$$

The index in both steps is 1 or p , so $|Z_{i+1}(M) : Z_i(M)| \leq p^2$. If the index is p^2 , inserting also $N_i \cap N_{i-1}^v$ in the middle, we see that the factor is elementary abelian.

(iii) For $i = 1$ we have $Z_1(P) \leq \text{core}(A; P) = Z_1(M)$. Since $Z_i(P) \leq Z_{i-1}(P)$, Lemma 1 is again applicable.

Now as a corollary to (i) and (iii) we obtain

THEOREM 1. Let A be a soft subgroup in P with $|P : A| = p^n$. Then there is a unique maximal subgroup M of P which contains A . The nilpotence class of M is n , the nilpotence class of P is at least $n+1$.

Since A is also a soft subgroup in N_i ($i = 1, \dots, n-1$), every statement is also true for N_i mutatis mutandis. We mention only the following.

COROLLARY 3. The nilpotence class of N_i is $i+1$ for $i = 0, \dots, n-1$.

COROLLARY 4. $|A : \text{core}(A; P)| \leq p^{n-1}$.

PROOF. Making use of (i) and (ii) we obtain $|A : \text{core}(A; P)| = p^{-n+1} |M : \text{core}(A; P)| = p^{-n+1} |Z_n(M) : Z_1(M)| \leq p^{-n+1} p^{2(n-1)} = p^{n-1}$.

PROPOSITION 2. For the lower central series of M we have:

- (i) $K_i(M) \simeq n$, $\text{cl}_i(N'_{n+1-i}; P) = \langle N'_{n+1-i}, N'^v_{n+1-i} \rangle$ ($v \in P \setminus M$),
 $i = 2, \dots, n$;
- (ii) $K_i(M)/K_{i+1}(M)$ is elementary abelian of order $\leq p^2$ for $i = 2, \dots,$
 \dots, n ;
- (iii) $|K_2(M) : K_3(M)| = p$ if $n \geq 2$.

PROOF. (i). Let $i = n$. Since $K_n(M) \triangleleft P$, $K_n(M) \leq Z_1(M) \leq A$ and $\text{cl}_n(M/K_n(M)) = n-1$, Lemma 1 implies that $N_1/K_n(M)$ is soft in $P/K_n(M)$, hence $N'_1 \leq K_n(M)$. Conversely for $T = \langle N'_1, N'^v_1 \rangle$ we have $Z_2(M)/T = (N_1 \cap N'^v_1)/T \leq Z(\langle N_1, N'^v_1 \rangle/T) = Z(M/T)$, hence $\text{cl}_n(M/T) \leq \text{cl}_n(M)$, so $T \leq K_n(M)$ also holds. Now for $i < n$ (i) follows by backwards induction.

- (ii) For $i = n$, $|K_n(M)| = |\langle N'_1, N'^v_1 \rangle| \leq |N'_1|^2 \leq p^2$,

by Lemma 1. The result follows by induction.

(iii). Consider $\bar{P} := P/K_3(M)$. Now \bar{N}_{n-2} is a soft subgroup in \bar{P} by (i) and Lemma 1. So we may suppose without loss of generality that $n = 2$. Then $K_2(M) = M' = N_P(A)'$ has order p by Lemma 1.

PROPOSITION 3. For the series of commutator subgroups N'_i we have:

- (i) $N'_i A = N_{i-1}$, $i = 1, \dots, n$, hence $N'_1 \triangleleft N'_2 \triangleleft \dots \triangleleft N'_n = P'$;
- (ii) $[N'_i, A] = N'_{i-1}$, $i = 1, \dots, n$ (of Prop. 5.);
- (iii) $N'_i \triangleleft N_j$ for $i-1 \leq j \leq n-i$;
- (iv) $|N'_{i+1} : N'_i| = p$ iff $N'_i \triangleleft N'_{i+2}$ ($i = 1, \dots, n-2$). Especially we have

$$|N'_{i+1} : N'_i| = p \text{ for } i = 1, \dots, \left[\frac{n}{2} \right] - 1;$$

$$(v) N'_i \text{ char } N_{n-i-1}, i = 1, \dots, \left[\frac{n-1}{2} \right].$$

PROOF. (i). Clearly $N'_i A \leq N_{i-1}$ and $N'_i A \triangleleft N_i$. Now Lemma 2 yields $N'_i A = N_{i-1}$.

(ii). Denote $H := [N'_i, A] = [N'_i A, A] = [N_{i-1}, A]$. Then $H \triangleleft N_{i-1}$, $AH/H \leq Z(N_{i-1}/H)$, $AH \triangleleft N_{i-1}$, hence $|N_{i-1} : AH| \leq p$. The centrality of AH/H yields $AH = N_{i-1}$, therefore $N'_{i-1} \leq H$. The reverse inclusion is obvious.

(iii) By Proposition 2 (i) and Theorem 1, $N'_1 \leq K_n(M) \leq Z(M)$, hence $N'_1 \triangleleft M = N_{n-1}$. Now N_1/N'_1 is a soft subgroup of M/N'_1 and the assertion $N'_i \triangleleft N_{n-1} \left(i = 1, \dots, \left[\frac{n}{2} \right] \right)$ follows by induction. Finally $N'_i \leq N_{i-1}$ by (i) hence $N'_i \triangleleft N_j$ holds for any $i-1 \leq j \leq n-i$.

(iv). We may suppose w.l.o.g. that $i = n-2$. Then we have $N_{i+2} = P$, $N_{i+1} = M$, therefore $N'_i = N'_{n-2} \leq \langle N'_{n-2}, N'^v_{n-2} \rangle = K_3(M) \triangleleft M'$ for any $v \in P \setminus M$. Since $|M' : K_3(M)| = p$ we obtain that $|M' : N'_{n-2}| = p$ iff $N'^v_{n-2} = N'_{n-2}$, i.e. $N_P(N'_{n-2}) = P$.

(v). By Proposition 2 (i) we have

$K_{n-1}(N_{n-2}) = n \cdot \text{cl.}(N'_1; N_{n-1}) = N'_1$, since $N'_1 < N_{n-1} = M$ by (iii). Hence N'_1 is characteristic in N_{n-2} . The assertion then follows by induction.

Finally we formulate some additional remarks.

PROPOSITION 4. M contains a subgroup T generated by two elements, such that the nilpotence class of T is also n . Moreover we have $Z_i(T) = T \cap Z_i(M)$ ($i = 1, \dots, n$) and $K_i(T) = K_i(M)$ ($i = 2, 3, \dots, n+1$). $TA = M$.

PROOF. Let $a \in A$ with $C_P(a) = A$ (see Corollary 2) and $v \in P \setminus M$. We claim that the nilpotence class of $T = \langle a, a^v \rangle$ is n . Clearly $Z(T) = C_T(a) \cap C_T(a^v) = T \cap A \cap A^v = T \cap Z(M)$. Considering the homomorphism $P \rightarrow \bar{P} = P/Z(M)$ we see that $\bar{g} \in C_{\bar{P}}(\bar{a})$ implies $g^{-1}ag \in A$ whence $A \leq C_P(g^{-1}ag) = g^{-1}C_P(a)g = g^{-1}Ag$ i.e. $g \in N_P(A)$. This proves $C_P(a) = N_1$ and now it follows by induction, that $Z_i(T) = T \cap Z_i(M)$ and $C_{P/Z_i(M)}(aZ_i(M)) = N_i/Z_i(M)$. This shows also that $a \in M \setminus Z_{n-1}(M)$, hence the nilpotence class of T is n .

A useful consequence of this is that $\langle a^v, A \rangle = M$. Denote $\langle a^v, A \rangle = K$, $v \in P \setminus M$. $M'K \triangleleft M$. $M'K = N_i$ for some i . As $M'A \leq M'K$, $i \geq n+2$. However $a^v \notin N_{n-2}$ so $M'K = M$ and thus $K = M$.

We now prove that $T' = M'$. Denote $U = TM'$ and $\bar{M} = M/U$. Then \bar{T} is abelian, so $C_M(\bar{a}) = \langle T, A \rangle = M$, and similarly $C_{\bar{M}}(\bar{a}^v) = M$, hence $\bar{T} \leq Z(M)$. Now it follows that $M = \langle \bar{T}, A \rangle$ is abelian, therefore $M' \leq U$, hence $U = TM' = T\Phi(U)$, so $U = T$, and $M' \triangleleft U' = T' \triangleleft M'$, i.e. $T' = M'$, as we claimed.

By the same method we get the following result.

PROPOSITION 5. There are subgroups $T_i \triangleleft M$ such that $T_0 \triangleleft T_1 \triangleleft \dots \triangleleft T_{n+1} = T$ and

1. $|T_i : \Phi(T_i)| \leq p^2$;
2. $\text{cl.}(N_i) = \text{cl.}(T_i)$ and $N'_i = T'_i$;
3. $T_i \triangleleft T_{i+1}$ iff $N'_i \triangleleft N_{i+2}$, $i = 1, \dots, n+2$.

In that case $|T_{i+1}/T_i| = p$.

PROOF. All but the last statement are trivial. We shall in fact show that there is exactly one maximal subgroup of T_j which contains T_k for any $k < j$. It is trivial that $C_{T_j}(a)$ is soft in T_j where a is such that $C_P(a) = A$.

As $|T_j : \Phi(T_j)| \leq p^2$ there is exactly one maximal subgroup of T_j which contains a so if $T_k \triangleleft T_j$ then $|T_j : T_k| = p$ for $k < j$. However $T'_j \langle a \rangle \triangleleft T_j$ and $T_k \leq T'_j \langle a \rangle$ so $T'_j \langle a \rangle = T_k = T_{j-1}$. Thus $|T'_j : T'_{j-1}| = p$ and $N'_{j-1} \triangleleft N_{j+1}$. If $N'_i \triangleleft N_{i+2}$ then it is trivial that $T_i \triangleleft T_{i+1}$.

This method yields the following sharpened version of the result in Prop. 3 (ii).

COROLLARY 5. $[N'_i, a] = N'_{i+1}$ ($i = 1, \dots, n$), where $a \in A$ such that $C_P(a) = A$.

PROOF. Let $H = [N_{i-1}, \langle a \rangle] \triangleleft N_{i-1}$, then $aH \in Z(N_{i-1}/H)$. Since $N_{i-1} = \langle a, A^w \rangle$, where $w \in N_i \setminus N_{i-1}$, it follows that N_{i-1}/H is abelian, so $N'_{i-1} \leq [N_{i-1}, \langle a \rangle] \triangleleft [N_{i-1}, a] = [N'_i, A, a] = [N'_i, a] \trianglelefteq [N'_i, A] = N'_{i-1}$, by Prop. 3. (i), (ii).

PROPOSITION 6. If $|N_1 : \Phi(N_1)| = p^2$ then $|P : \Phi(P)| = p^2$.

PROOF. Suppose $|N_1 : \Phi(N_1)| = p^2$. Then we have $\Phi(N_1) \triangleleft N_2$ and $|N_2 : \Phi(N_1)| = p^3$. But $N_2/\Phi(N_1)$ is not abelian since $N'_2 A = N_1$. This implies $|N_2 : \Phi(N_2)| = p^2$. The result then follows by induction.

PROPOSITION 7. P contains at most one maximal subgroup L such that $L' \cong K_3(M)$.

PROOF. Let L_1, L_2 be maximal subgroups of P . Denote $H = \langle L'_1, L'_2 \rangle$. Then $H \triangleleft P$ and P/H contains two distinct maximal abelian subgroups, hence the nilpotence class of P/H is at most 2.

If P/H is nonabelian then M/H contains a soft subgroup of P/H , hence M/H is in any case abelian (see Theorem 1), so $H \trianglelefteq M'$. By symmetry we may suppose that $AL'_1 \trianglelefteq AL'_2 = N_t$ ($t = n-1$). Then $N'_t \trianglelefteq L'_2$ and $N_t \trianglelefteq \langle L'_1, L'_2 \rangle = H \trianglelefteq M'$, therefore $N_t \cong M' A = N_{n-2}$ hence $L'_2 \cong n$, cl. $(N'_t : P) \cong n$, cl. $(N'_{n-2} : P) = K_3(M)$ (see Prop. 2. (i)).

PROPOSITION 8. If $\text{cl}(P) > n+1$ then all abelian normal subgroups are contained in M .

PROOF. Let $N \triangleleft P$, $N' = 1$, $N \not\cong M$. Then $NA = P$ so $P' \cong N$. We prove by induction that $K_i(P) \cong K_{i-1}(M)$ for $i \geq 2$. For $i = 2$ we have $P' \cong M$, further $K_{i+1}(P) = [K_i(P), P] = [K_i(P), MN] = [K_i(P), M] \cong [K_{i-1}(M), M] = K_i(M)$. Hence $\text{cl}(P) \cong \text{cl}(M) + 1 = n+1$.

COROLLARY 6. There is at most one maximal subgroup L of P such that $L \not\cong M$ and $L' \cong K_3(P)$.

PROOF. By Prop. 7, all but one maximal subgroups L satisfy $L' \cong K_3(M)$. Then $\text{cl}(M/L') \leq 2$, L/L' is abelian, therefore Prop. 8. forces $\text{cl}(P/L') \leq 3$, i. e. $L' \cong K_1(P)$.

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INVESTIGATIONS WITH RESPECT TO THE VILENKNIN SYSTEM

By

P. SIMON

Department for Numerical Analysis of the L. Eötvös University, Budapest

(Received February 11, 1982)

Introduction. In this work we are concerned with some questions with respect to the orthonormal system of Vilenkin type [16]. Thus are investigated a Bernstein type inequality, the $(C, 1)$ -summation in the Hardy space and the relation between the Hardy space and the conjugation, introduced earlier to the Vilenkin system [12], [13]. It is known (see. e.g. [9]) that in the investigations with respect to the Vilenkin system a boundedness condition plays an important part. Thus e.g. in the "bounded" case an analogous statement [3], as the inequality of Bernstein for the trigonometric system [19], is true and a simple example shows that this is not true in the "unbounded" case. The situation is similar with respect to the (H^1, L^1) -type of the maximal operator σ^* of the $(C, 1)$ -summation. The operator σ^* is of type (H^1, L^1) in the "usual" H^1 -space, when the above mentioned boundedness criterion is fulfilled [4]. In this case the H^1 -space is atomic [1] and this fact has an important role in the proofs. In the "unbounded" case the above H^1 -space is not atomic, but — modifying the concept of the "atoms" — we can define a new atomic space. It is proved that the operator $\sigma^* : H^1 \rightarrow L^1$ is not bounded (without the above boundedness condition) — either in the previous H^1 -space, or in the atomic space.

In the theory of the trigonometric series it is well-known [2] that the classical H^1 -space contains exactly those L^1 -functions, whose (trigonometric) conjugate function is integrable. We investigate the analogous question for the Vilenkin system with respect to the conjugation defined in [12]. We prove that this conjugation is of type (H^1, L^1) in the above mentioned atomic H^1 -space. It is an open question: if the conjugate function belongs to L^1 , then the function is in H^1 ? We remark that this problem is connected with an open problem in the theory of the trigonometric series: whether the integrability of Paley's quadratic variation [19] characterizes the classical H^1 -space. This characterization of the martingale H^1 -space is well-known [5].

1. In this section we introduce some notations and definitions. Let

$$m = (m_0, m_1, \dots, m_k, \dots) \quad (2 \leq m_k, m_k \in \mathbb{N}, k \in \mathbb{N} := \{0, 1, \dots\})$$

be a sequence of natural numbers and denote by Z_{m_k} the m_k th discrete cyclic group, i.e.

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\} \quad (k \in \mathbb{N}).$$

If we define the group G_m as the direct product of the groups Z_{m_k} , then G_m is a compact Abelian group. Thus the elements of G_m are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $0 \leq x_k \leq m_k$ ($k \in \mathbb{N}$) and for x, y in G_m their sum $x + y$ is obtained by adding the n th coordinates of x and y modulo m_n ($n \in \mathbb{N}$). (Let $\bar{\cdot}$ be the inverse of the operation \cdot) The topology of G_m is completely determined by the following subgroups of G_m :

$$I_n(x) := \{y \in G_m : y = (x_0, \dots, x_{n-1}, y_n, \dots)\} \quad (x \in G_m, n \in \mathbb{N}).$$

For a fixed $x \in G_m$ and for $n \in \mathbb{N}$ let

$$I_n(x, k) := \{y \in G_m : y = (x_0, \dots, x_{n-1}, k, y_{n+1}, \dots)\} \quad (k \in Z_{m_n}).$$

It is evident that $I_n(x) = \bigcup_{k \in Z_{m_n}} I_n(x, k)$ and this decomposition contains disjoint sets, furthermore

$$[I_n(x, k)]^c = m_n^{-1} [I_n(x)] \quad (k \in Z_{m_n}).$$

($|A|$ denotes the measure of a Haar-measurable set $A \subset G_m$.)

Let

$$K_n^j(x) := \begin{cases} \bigcup_{k=0}^{\left[\frac{m_n}{2}\right]-1} I_n(x, k) & \left(x_n < \left[\frac{m_n}{2}\right]\right) \\ \bigcup_{k=\left[\frac{m_n}{2}\right]}^{m_n-1} I_n(x, k) & \left(x_n \geq \left[\frac{m_n}{2}\right]\right). \end{cases}$$

and if

$$K_n^j(x) = \bigcup_{k=k_j}^{h_j} I_n(x, k) \quad (j \in \mathbb{N}; k_j, h_j \in \mathbb{N}; k_j \leq h_j),$$

then let

$$K_n^{j+1}(x) := \begin{cases} \bigcup_{k=k_j}^{\left[\frac{k_j+h_j}{2}\right]-1} I_n(x, k) & \left(x_n < \left[\frac{k_j+h_j}{2}\right]\right) \\ \bigcup_{k=\left[\frac{k_j+h_j}{2}\right]}^{h_j} I_n(x, k) & \left(x_n \geq \left[\frac{k_j+h_j}{2}\right]\right). \end{cases}$$

($[c]$ denotes the entire part of the real number c .)

For all $x \in G_m$ and $n \in \mathbb{N}$ there exists a unique $j_n \in \mathbb{N}$ such that $K_n^{j_n}(x) = I_{n+1}(x)$. Let $K_n(x)$ ($n \in \mathbb{N}$) be the sequence of the $K_n^j(x)$ ($n \in \mathbb{N}$, $j = 1, \dots, j_n$) lexicographically ordered by pairs (n, j) , then

- (i) $1 \leq \frac{|K_n(x)|}{|K_{n+1}(x)|} \leq 3 \quad (n \in \mathbb{N}),$
- (ii) $\lim_n K_n(x) = \{x\}.$

The sets $K_n(x)$ ($n \in \mathbb{N}$, $x \in G_m$) are called "intervals".

Next, let $\{\psi_n : n \in \mathbb{N}\}$ denote the character group of G_m . We enumerate its elements as follows. For $k \in \mathbb{N}$ let r_k be the function defined by

$$r_k(x) := \exp \left(-\frac{2\pi i x_k}{m_k} \right) \quad (x \in G_m, i = \sqrt{-1}).$$

If we define the sequence $(M_k, k \in \mathbb{N})$ by $M_0 := 1$ and $M_{k+1} := m_k M_k$ ($k \in \mathbb{N}$), then each $n \in \mathbb{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (n_k \in Z_{m_k}).$$

For such $n \in \mathbb{N}$ we define the function ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}.$$

We remark that $\{\psi_n : n \in \mathbb{N}\}$ — the so-called Vilenkin system [16] — is a complete orthonormal system with respect to the normalized Haar measure dx on G_m . (In the special case $m_n = 2$ ($n \in \mathbb{N}$) $\{\psi_n : n \in \mathbb{N}\}$ is the Walsh-Paley system [8].)

For $f \in L^1(G_m)$ let

$$\hat{f}(k) := \int_{G_m} f \cdot \psi_k \quad (k \in \mathbb{N}),$$

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) \cdot \psi_k \quad (n \in \mathbb{N}),$$

$$\sigma_n(f) := \frac{1}{n} \cdot \sum_{k=1}^n |S_k(f)| \quad (n = 1, 2, \dots),$$

$$\sigma^*(f) := \sup_n |\sigma_n(f)|,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}),$$

$$K_n := \frac{1}{n} \cdot \sum_{k=1}^n D_k \quad (n = 1, 2, \dots).$$

Then $S_n(f) = f * D_n$, $\sigma_n(f) = f * K_n$ ($n \in \mathbb{N}$), where for the functions g , $h \in L^1(G_m)$ the convolution $g * h$ is defined by

$$g * h(x) := \int_{G_m} g(t) h(x - t) dt \quad (x \in G_m).$$

The concept of derivative over G_m is defined as follows [6]. The function $f \in L^1(G_m)$ has a derivative $f^{(1)} \in L^1(G_m)$, if

$$\lim_n \|f^{(1)} - d_n f\|_1 = 0,$$

where

$$d_n f(x) := \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} \frac{k}{m_j} \sum_{s=0}^{m_j-1} r_j(se_j)^{-k} \cdot f(x + se_j)$$

($x \in G_m$, $n \in \mathbb{N}$, $e_j := (0, \dots, 0, \overset{j}{1}, 0, \dots) \in G_m$, $se_j := e_j + \dots + e_j$). The following relations are easily verified:

$$\psi_n^{(1)} = n \cdot \psi_n \quad (n \in \mathbb{N}),$$

$$K_n = D_n - \frac{1}{n} D_n^{(1)} \quad (n = 1, 2, \dots).$$

Let

$$A_k := \begin{cases} 1 & (m_k = 2) \\ \left[\frac{m_k - 1}{2} \right] & (m_k > 2) \end{cases} \quad (k \in \mathbb{N})$$

and

$$L_k := \left(- \sum_{j=1}^{A_k} r_k^j + \sum_{j=A_k+1}^{m_k-1} r_k^j \right) D_{M_k}.$$

We define the V -conjugate function of $f \in L^1(G_m)$ as follows:

$$\tilde{f} := \sum_{k=0}^{\infty} f * L_k.$$

The series $\sum_{k=0}^{\infty} f * L_k$ converges in measure and the V -conjugation is L^p -bounded, if $1 < p < \infty$ [12], [13].

The definition of Hardy space is possible in several ways. The question is only, which of these possibilities is useful in respect of the Vilenkin-Fourier analysis. So e.g. the concept of the Hardy space in the martingale theory is defined in the following way [5]: a function $f \in L^1(G_m)$ belongs to the H^1 -space if and only if the quadratic variation of f

$$q(f) := \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} |S_{M_{n+1}}(f) - S_{M_n}(f)|^2 \right)^{1/2}$$

is in $L^1(G_m)$. It is known [5] that $\|q(f)\|_p$ is equivalent to $\|f\|_p$ ($1 < p < \infty$) and q is of weak-type (1,1). From these it follows in the special case $m_n = 2$ ($n \in \mathbb{N}$) — i.e. for the Walsh-Paley system — that $\{\psi_n : n \in \mathbb{N}\}$ is a basis in $L^p(G_m)$ ($1 < p < \infty$) [8]. This deduction is not possible in the general case, we must modify the definition of the quadratic variation, taking into consideration the finer structural properties of $\{\psi_n : n \in \mathbb{N}\}$. Hence let

$$Q(f) := \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} |S_{(j+1)M_n}(f) - S_{jM_n}(f)|^2 \right)^{1/2}$$

($f \in L^1(G_m)$). If the sequence m is bounded — the “bounded” case —, then also for Q are valid the above (for q) mentioned statements [18]. Then from this it follows that $\{\psi_n : n \in \mathbb{N}\}$ is a basis in $L^p(G_m)$ ($1 < p < \infty$) in the “bounded” case [18]. (It is immediate that

$$(1) \quad q(f) \leq C \cdot Q(f) \quad (f \in L^1(G_m)),$$

if $m_n \leq C$ ($n \in \mathbb{N}$), but q and Q are not equivalent.)

Furthermore, it is known [18] that in the case $\sup_n m_n = \infty$ there exists a function $f \in L^p(G_m)$ ($0 < p < 2$), resp. $g \notin L^p(G_m)$ ($p > 2$), for which $Q(f) = \infty$ a.e., resp. $Q(g)$ is bounded.

We define the Hardy space $H^1(G_m)$ by means of the atoms. An atom is either the function $a = 1$ or a function $a \in L^\infty(G_m)$, satisfying [1]:

- (i) the support of a is contained in an interval $K_n(x) =: J_a$,
- (ii) $\|a\|_\infty \leq |J_a|^{-1}$,
- (iii) $\int_{G_m} a = 0$.

Let us denote the set of the atoms by $A(G_m)$ and let

$$H^1(G_m) := \left\{ \sum_{i=0}^{\infty} \lambda_i a_i : a_i \in A(G_m), \sum_{i=0}^{\infty} |\lambda_i| < \infty \right\},$$

and for $f \in H^1(G_m)$

$$\|f\|_{H^1} := \inf \sum_{i=0}^{\infty} |\lambda_i|.$$

(The infimum is taken over all decompositions $\sum \lambda_i a_i$.) It is known [1] that $H^1(G_m)$ with the above norm is a Banach space.

The martingale H^1 -space (determined by q) has an atomic structure only for bounded m . This is a simple consequence of a Calderon-Zygmund decomposition [1] and of the fact that $\|q(f)\|_1$ and $\|\sup_n |S_{M_n}(f)|\|_1$ ($f \in L^1(G_m)$) are equivalent [5]. Let us introduce a modified version of the maximal function:

$$f^*(x) := \sup_n |K_n(x)|^{-1} \int_{K_n(x)} f \quad (f \in L^1(G_m)).$$

(We remark that f^* is a martingale maximal function and is of weak type (1,1) [5], [14].) Since

$$S_{M_n}(f)(x) = |I_n(x)|^{-1} \int_{I_n(x)} f \quad (x \in G_m, n \in \mathbb{N}, f \in L^1(G_m)),$$

it is evident that $\sup_n |S_{M_n}(f)| \leq f^*$.

For a bounded sequence m a function f belongs to $H^1(G_m)$ if and only if $q(f) \in L^1(G_m)$. Indeed, let $m_n = C$ ($n \in \mathbb{N}$) and $q(f) \in L^1(G_m)$ ($f \in L^1(G_m)$). Then we can decompose the function f in the following way [1]:

$$f = \sum_{i=1}^{\infty} \mu_i \cdot b_i, \quad \sum_{i=0}^{\infty} |\mu_i| < \infty$$

and for $b_i \in L^\infty(G_m)$ ($i \in \mathbb{N}$)

- (i) $\text{supp } b_i \subset I_{n_i}(x_i) \quad (x_i \in G_m, n_i \in \mathbb{N})$,
- (2) (ii) $\|b_i\|_{L^\infty} \leq M_{n_i}$,
- (iii) $\int_{G_m} b_i = 0$.

Since at the same time $b_i \in A(G_m)$ ($i \in \mathbb{N}$) is true, thus $f \in H^1(G_m)$. (Furthermore, $\|f\|_H = O(1) \cdot \|q(f)\|_1$.)

Inversely, if $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H^1(G_m)$ ($a_i \in A(G_m)$, $\sum_{i=0}^{\infty} |\lambda_i| < \infty$), then evidently there exists a sequence of coefficients $\{z_i, i \in \mathbb{N}\}$, for which $z_i = \frac{1}{C}$ ($i \in \mathbb{N}$) is true and the functions $z_i \cdot a_i$ ($i \in \mathbb{N}$) have the properties (2). Let $b_i := z_i \cdot a_i$ ($i \in \mathbb{N}$), then $f = \sum_{i=0}^{\infty} \frac{\lambda_i}{z_i} \cdot b_i$ and $\sum_{i=0}^{\infty} \frac{|\lambda_i|}{z_i} = C \cdot \sum_{i=0}^{\infty} |\lambda_i| < \infty$, i.e. it follows that $q(f) \in L^1(G_m)$ [1]. (Furthermore, $\|q(f)\|_1 = O(1) \|f\|_H$.)

Hence for a bounded sequence m $\|q(f)\|_1$ and $\|f\|_H$ ($f \in L^1(G_m)$) are equivalent.

2. §. In the first theorem we give a characterization of the space $H^1(G_m)$.

THEOREM 1. *A function $f \in L^1(G_m)$ belongs to $H^1(G_m)$ if and only if $f^* \in L^1(G_m)$. Furthermore, $\|f\|_H$ is equivalent to $\|f^*\|_1$.*

(Hence our space $H^1(G_m)$ is a martingale H -space.)

We have seen that in the "bounded" case $H^1(G_m) := \{f \in L^1(G_m) : q(f) \in L^1(G_m)\}$ and $\|q(f)\|_1$ is equivalent to $\|\sup_n |S_{M_n} f|\|_1$. Therefore $H^1(G_m)$ and the by q defined H^1 -space are the same, if the sequence m is bounded.

If m is not bounded, then the space $H^1(G_m)$ cannot be characterized by the L^1 -boundedness of Q . This follows from the next theorem.

THEOREM 2. *Let $\sup_n m_n = \infty$. Then*

$$\sup_{a \in A(G_m)} \|Q(a)\|_1 < \infty$$

For a bounded sequence $m \|q(f)\|_1$ and $\|Q(f)\|_1$ ($f \in L^1(G_m)$) are but equivalent. This follows – taking into consideration (1) – from

THEOREM 3. *For all bounded sequences m there exists a constant $C > 0$ such that*

$$\|Q(f)\|_1 \leq C \|q(f)\|_1 \quad (f \in L^1(G_m)).$$

In the next theorem we are concerned with the V -conjugation. It is proved that the V -conjugation is of type (H^1, L^1) .

THEOREM 4. *There exists an absolute constant $C > 0$, for which*

$$\|\tilde{f}\|_1 \leq C \|f\|_{H^1} \quad (f \in H^1(G_m))$$

holds.

It is an open question, whether $\|\tilde{f}\|_1$ and $\|f^*\|_1$ ($f \in L^1(G_m)$) are equivalent. But from Theorem 2 and 4 it follows evidently that $\|\tilde{f}\|_1$ and $\|Q(f)\|_1$ ($f \in L^1(G_m)$) are not equivalent. Namely, if the sequence m is not bounded, then a constant $C > 0$ cannot be given, for which $\|Q(f)\|_1 \leq C \cdot \|\tilde{f}\|_1$ ($f \in L^1(G_m)$) holds.

For the maximal operator σ^* we have proved in [7] the weak-type (1,1) properties in the “bounded” case. In his work [4] N. FUJII proved that for bounded m the operator $\sigma^* : H^1(G_m) \rightarrow L^1(G_m)$ is bounded. In the following theorem we shall prove that the boundedness of m plays an important role in the above statement.

THEOREM 5. *The operator $\sigma^* : H^1(G_m) \rightarrow L^1(G_m)$ is bounded if and only if $\sup_n m_n < \infty$.*

The original proof of N. FUJII of the sufficiency part of Theorem 5 is rather complicated and a simplified version is presented in this paper.

The necessity of the previous theorem is connected with an inequality of Bernstein type, which was proved for the Walsh-Paley system by H. J. WAGNER [17]. If the sequence m is bounded, then the proof of WAGNER can be generalized for $\{\psi_n : n \in \mathbb{N}\}$ and S. FRIDL proved the following statement [3]:

There exists a constant $C > 0$ (depending only on n) that for all “polynomials” $P = \sum_{k=0}^n c_k \psi_k$ ($n \in \mathbb{N}$) the inequality

$$\|P^{(1)}\|_1 \leq C n \|P\|_1$$

holds.

In the case $\sup_n m_n = \infty$ the above inequality cannot be true. This follows taking into consideration $\|D_{M_n}\|_1 = 1$ ($n \in \mathbb{N}$) – from the next theorem.

THEOREM 6. *For all sequences m and for all natural numbers $n \in \mathbb{N}$*

$$\frac{1}{2\pi} \sum_{j=0}^{n-1} M_{j+1} \log m_j + \frac{M_n - 1}{2} \leq \|D_{M_n}\|_1 \leq \frac{1}{2} \sum_{j=0}^{n-1} M_{j+1} \log m_j + \frac{M_n - 1}{2}.$$

Since $K_n = D_n - \frac{1}{n} D_n^{(1)}$ ($n = 1, 2, \dots$), thus from Theorem 6

$$\|K_{M_n}\|_1 \geq \frac{1}{M_n} \cdot \|D_{M_n}^{(1)}\|_1 - \|D_{M_n}\|_1 \geq \frac{1}{2\pi} \log m_{n+1} - 1 \quad (n = 1, 2, \dots)$$

follows, i.e. if $\sup_n m_n = \infty$, then $\sup_n \|K_n\|_1 = \infty$ [9]. We remark that in the case $\sup_n m_n < \infty$ we proved the boundedness of $\|K_n\|_1$'s ($n \in \mathbb{N}$) [7]. (An analogous estimation, as the above for $\|D_{M_n}\|_1$, can be found in [15] for the L^1 -norm of the imaginary part of K_{M_n} .)

3. §. PROOF OF THEOREM 1.

1. We prove that for all $a \in A(G_m)$

$$\|a^*\|_1 = 1$$

holds. (From this it follows evidently that $\|f^*\|_1 = \|f\|_{H^1}$ ($f \in H^1(G_m)$).

If $a = 1$, then $\|a^*\|_1 = 1$ is trivial hence we can assume that $a \neq 1$. Let $x \in G_m \setminus J_a$, then for all $n \in \mathbb{N}$ $K_n(x) \cap J_a = 0$ or $K_n(x) \cap J_a = J_a$, therefore $a^*(x) = 0$. On the other hand for $x \in J_a$

$$|K_n(x)|^{-1} \int_{K_n(x)} a \approx |J_a|^{-1} \quad (n \in \mathbb{N})$$

holds evidently, thus $a^*(x) = |J_a|^{-1}$ and $\|a^*\|_1 = 1$.

2. Let us assume that for the function $f \in L^1(G_m)$ the maximal function f^* belongs to $L^1(G_m)$. We shall prove that $f \in H^1(G_m)$. Since the original idea of the proof can be found in [1], we give only the sketch of the proof.

For s an integer we define the set $U_s := \{x \in G_m : f^*(x) > 2^s\}$. Then U_s can be decomposed in the form $U_s = \bigcup_i K_{n_i}(x_i(s)) =: \bigcup_i K_{n_i}^s$, where for

a fixed s the sets $K_{n_i}^s$ are disjoint and $|K_{n_i}^s|^{-1} \int_{K_{n_i}^s} f \approx 2^s$, but $|K_{n_{i+1}}^s|^{-1} \int_{K_{n_{i+1}}^s} f \approx 2^s$. (Without loss of generality we can assume $\int_{G_m} f = 0$.) Furthermore,

the function f can be represented as $f = f_s + w_s$, where the following assumptions are valid: $\|f_s\|_\infty \leq 2^s$, $\|f_s\|_1 \leq \|f\|_1$, $w_s = \sum_i f_i^s$, $\text{supp } f_i^s \subset K_{n_{i+1}}^s$,

$\int_{G_m} f_i^s = 0$, $|U_s| \leq 2^{-s} \cdot \|f\|_1$. It is easy to see that $f = \sum_{s=-\infty}^{\infty} (f_{s+1} - f_s)$ and

$$a_i^s := \frac{f_{s+1} - f_s}{3 \cdot 2^s \cdot |K_{n_i}^s|} \cdot \chi_{K_{n_i}^s} \in A(G_m).$$

(χ_B denotes the characteristic function of the set $B \subset G_m$.) Therefore

$$f = \sum_{s=-\infty}^{\infty} 3 \cdot 2^s \sum_i |K_{n_i}^s| a_i^s$$

is an atomic decomposition of f , i.e. $f \in H^1(G_m)$. On the other hand

$$\begin{aligned}\|f^*\|_1 &= \int_0^\infty |\{f^* > y\}| dy = \sum_{s=-\infty}^\infty \int_{2^s}^{2^{s+1}} |\{f^* > y\}| dy \geq \sum_{s=-\infty}^\infty 2^s |U_{s+1}| = \\ &= \sum_{s=-\infty}^\infty 2^{s-1} |U_s| \geq \sum_{s=-\infty}^\infty 2^{s-1} \cdot 3^{-1} \sum_l |K_{n_l-1}^s| > 18^{-1} \|f\|_{H^1}.\end{aligned}$$

PROOF of THEOREM 2. We define a sequence of atoms in the following way: for $n \in \mathbb{N}$ let $J_n := I_{n+1}(0) \cup I_n(0,1)$ and

$$a_n(x) := \begin{cases} \frac{M_{n+1}}{3} & (x \in I_{n+1}(0)), \\ -\frac{M_{n+1}}{3} & (x \in I_n(0,1)), \\ 0 & (\text{otherwise}). \end{cases}$$

It is evident that $a_n \in A(G_m)$ ($n \in \mathbb{N}$). Since

$$D_{(j+1)M_n} - D_j M_n = r^j D_{M_n} \quad (n \in \mathbb{N}, j = 1, \dots, m_n - 1), \text{ thus}$$

$$\begin{aligned}\|Q(a_n)\|_1 &\geq \int_{G_m} \left(\sum_{j=1}^{m_n-1} |S_{(j+1)M_n}(a_n) - S_j M_n(a_n)|^2 \right)^{1/2} = \\ &= \int_{G_m} \left(\sum_{j=1}^{m_n-1} r_n^j(x) \int_{G_m} a_n(t) \bar{r}_n^j(t) D_{M_n}(x-t) dt \right)^{1/2} dx = \\ &\geq M_n \int_{G_m} \left(\sum_{j=1}^{m_n-1} \left| \int_{I_n(x) \cap J_n} a_n(t) \bar{r}_n^j(t) dt \right|^2 \right)^{1/2} dx \geq M_n \int_{I_n(0)} \left(\sum_{j=1}^{m_n-1} \left| \int_{J_n} a_n \bar{r}_n^j \right|^2 \right)^{1/2} = \\ &= 3^{-1} \left(\sum_{j=1}^{m_n-1} \left| 1 - \exp \frac{2\pi i j}{m_n} \right|^2 \right)^{1/2} = \frac{2}{3} \cdot \left(\sum_{j=1}^{m_n-1} \sin^2 \frac{\pi j}{m_n} \right)^{1/2} \geq C \sqrt{m_n},\end{aligned}$$

where $C > 0$ is an absolute constant.

PROOF of THEOREM 3. It is enough to prove that for a bounded sequence m the operator $Q : H^1(G_m) \rightarrow L^1(G_m)$ is bounded. (Since in the "bounded" case $\|q(f)\|_1$ is equivalent to $\|f\|_{H^1}$, $f \in H^1(G_m)$.)

Before we start the proof of the above statement we make the following general remark [10], [11].

Let the operator T be sublinear and we assume that

$$(3) \quad \sup_{a \in A(G_m)} \|T(a)\|_1 < \infty.$$

Then $T : H^1(G_m) \rightarrow L^1(G_m)$ is evidently bounded. If there exists a constant $C > 0$ such that $\|T(f)\|_2 \leq C\|f\|_2$ ($f \in L^2(G_m)$) and

$$(4) \quad \sup_{a \in A(G_m)} \int_{G_m \setminus J_a} |T(a)| < \infty,$$

then (3) is true. Indeed,

$$\begin{aligned} \|T(a)\|_1 &= \int_{G_m \setminus J_a} |T(a)| + \int_{J_a} |T(a)| \leq \int_{G_m \setminus J_a} |T(a)| + \sqrt{|J_a|} \sqrt{\int_{G_m} |T(a)|^2} \leq \\ &\leq \int_{G_m \setminus J_a} |T(a)| + C \quad (a \in A(G_m)). \end{aligned}$$

If for all $a \in A(G_m)$ there exists a set $\tilde{J}_a \subset G_m$ such that $J_a \subset \tilde{J}_a$ and

$$\sup_{a \in A(G_m)} \frac{|\tilde{J}_a|}{|J_a|} < \infty,$$

then from

$$(5) \quad \sup_{a \in A(G_m)} \int_{G_m \setminus \tilde{J}_a} |T(a)| < \infty$$

(4) follows immediately.

We prove that for the operator Q the assumption (5) is true. Let $a \in A(G_m)$ (we can assume that $a \neq 1$), $J_a = \bigcup_k I_N(y, k)$ and we define

$$\tilde{J}_a := I_N(y) \ (N \in \mathbb{N}, y \in G_m). \text{ Then } \sup_{a \in A(G_m)} |\tilde{J}_a| |J_a|^{-1} \leq \sup_n m_n < \infty$$

and for $x \in G_m \setminus \tilde{J}_a$

$$\begin{aligned} S_{(j+1)M_n}(a)(x) - S_{jM_n}(a)(x) &= r_n^j(x) \int_{G_m} a(t) r_n^j(t) D_{M_n}(x-t) dt \\ &= r_n^j(x) M_n \int_{J_a \cap I_n(x)} a r_n^j \ (n \in \mathbb{N}, j = 1, \dots, m_n - 1). \end{aligned}$$

For $n \geq N$ it is easy to see that $J_a \cap I_n(x) = \emptyset$. On the other hand in the case $n < N$ $J_a \cap I_n(x) = \emptyset$ or $J_a \cap I_n(x) = J_a$ and r_n^j is constant over J_a , therefore for all $n \in \mathbb{N}$

$$\int_{J_a \cap I_n(x)} a r_n^j = 0.$$

Thus $Q(a)(x) = 0$ ($x \in G_m \setminus \tilde{J}_a$) and (5) is trivially true.

Since the operator Q is sublinear and of type (2,2), the proof is complete.

PROOF OF THEOREM 4. We shall prove that the V -conjugation satisfies (5). For an $a \in A(G_m)$ let us define \tilde{J}_a in the following manner: if $n \in \mathbb{N}$ is the largest natural number, for which an $I_n(\xi)$ ($\xi \in G_m$) contains J_a as a proper subset, then — consider $I_n(\xi)$ as a circle — let \tilde{J}_a denote the “interval” inside $I_n(\xi)$ which contains J_a at its center and $|\tilde{J}_a| = 3|J_a|$. (If this is not possible, then let \tilde{J}_a be equal to $I_n(\xi)$.)

Let $a \in A(G_m)$ be an arbitrary atom (we can assume that $a \neq 1$), $x \in G_m \setminus \tilde{J}_a$ and $k \in \mathbb{N}$. Then

$$\begin{aligned} a * L_k(x) &= \int_{G_m} a(t) \left(\sum_{j=-J_k+1}^{m_k-1} r_k^j(x+t) - \sum_{j=1}^{J_k} r_k^j(x-t) \right) D_{M_k}(x \pm t) dt = \\ &= M_k \cdot \int_{J_a \cap I_k(x)} a(t) \delta_k(x \pm t) dt, \end{aligned}$$

where

$$\delta_k := \sum_{j=-J_k+1}^{m_k-1} r_k^j - \sum_{j=1}^{J_k} r_k^j.$$

If $J_a = \bigcup_k I_n(\xi, k)$, then for $x \notin I_n(\xi)$ and for $k > n$

$$a * L_k(x) = 0.$$

(See the proof of Theorem 3.) From this it follows that we can assume $x \in I_n(\xi) \setminus \tilde{J}_a$. Then

$$\begin{aligned} \hat{a}(x) &= M_n \int_{J_a \cap I_n(x)} a(t) \delta_n(x \pm t) dt = M_n \int_{J_a} a(t) \delta_n(x \pm t) dt = \\ &= M_n \int_{J_a} a(t) [\delta_n(x \pm t) - \delta_n(x \pm \tilde{t})] dt, \end{aligned}$$

where \tilde{t} denotes a point of J_a .

Hence

$$\begin{aligned} \int_{G_m \setminus \tilde{J}_a} |\hat{a}| &\leq M_n \int_{I_n(\xi) \setminus \tilde{J}_a} \int_{J_a} |a(t)| [\delta_n(x \pm t) - \delta_n(x \pm \tilde{t})] dt dx \leq \\ &= \int_{J_a} |a(t)| \left(M_n \int_{I_n(\xi) \setminus \tilde{J}_a} |\delta_n(x \pm t) - \delta_n(x \pm \tilde{t})| dx \right) dt. \end{aligned}$$

We know [12] that there is an absolute constant $C > 0$, for which

$$M_n \int_{I_n(\xi) \setminus \tilde{J}_a} |\delta_n(x \pm t) - \delta_n(x \pm \tilde{t})| dx \leq C \quad (t \in J_a),$$

thus

$$\int_{G_m \setminus \tilde{J}_a} |\hat{a}| \leq C \int_{J_a} |a| \leq C.$$

Taking into consideration that the V -conjugation is linear and of type (2,2), the proof is complete.

PROOF OF THEOREM 5.

1. Let us assume that for a constant $C > 0$

$$m_n < C \quad (n \in \mathbb{N}).$$

We prove that for σ^* (5) is true. (Since σ^* is sublinear and of type (2,2) (see [7]), thus we complete the proof of Theorem 5.)

For $a \in A(G_m)$ let \tilde{J}_a be defined as in the proof of Theorem 3. If $\tilde{J}_a = I_N(y)$ ($N \in \mathbb{N}$, $y \in G_m$), then for all $n = 0, 1, \dots, M_N - 1$ the function φ_n is constant over J_a , thus $\hat{a}(n) = 0$, i.e. $S_n(a) = 0$. Therefore $\sigma^*(a) = \sup_{n \in M_N} |\sigma_n(a)|$.

Let $n \in \mathbb{N}$ and $M_{s-1} \leq n \leq M_s$ ($N \in s > N$), then (see [7])

$$|K_n(x)| \leq C_1 M_s^{-1} \sum_{j=0}^{s-1} M_j \sum_{i=j}^{s-1} \sum_{l=0}^{m_j-1} D_{M_l}(x + l e_j)$$

($x \in G_m$, the constant $C_1 > 0$ depends only on C).

Thus for $x \in G_m \setminus \tilde{J}_a$ we have

$$|\sigma_n(a)(x)| \leq C_1 M_s^{-1} \sum_{j=0}^{s-1} M_j \sum_{i=j}^{s-1} \sum_{l=0}^{m_j-1} |a(t)| D_{M_l}(x + l e_j - t) dt.$$

On the other hand for $i \geq N, j \geq N, l = 0, \dots, m_j - 1$

$$\int_{G_m} |a(t)| D_{M_l}(x + l e_j - t) dt = 0,$$

therefore

$$|\sigma_n(a)(x)| \leq C_1 M_N^{-1} \sum_{j=0}^{N-1} M_j \sum_{i=j}^{N-1} \sum_{l=0}^{m_j-1} \int_{G_m} |a(t)| D_{M_l}(x + l e_j - t) dt,$$

i.e.

$$\begin{aligned} \int_{G_m \setminus \tilde{J}_a} \sigma^*(a) &\leq C_1 M_N^{-1} \sum_{j=0}^{N-1} M_j \sum_{i=j}^{N-1} \sum_{l=0}^{m_j-1} \|a\|_1 \|D_{M_l}\|_1 \leq \\ &\leq C_1 C \sum_{j=0}^{N-1} \frac{(N-j) M_j}{M_N} \leq C_1 C \sum_{j=0}^{\infty} j 2^{-j} < \infty. \end{aligned}$$

2. For the proof of the converse direction we assume that $\sup_n m_n = \infty$ and prove

$$\sup \left\{ \frac{1}{\|f\|_{H^1}} \|\sigma^*(f)\|_1 : f \in H^1(G_m), \|f\|_{H^1} \neq 0 \right\} = \infty.$$

It is easy to see that we need to prove only

$$\sup \{\|\sigma_n(a)\|_1 : n \in \mathbb{N}, a \in A(G_m)\} = \infty.$$

Let $n \in \mathbb{N}$ and $f_n := r_n D_{M_n}$, then $f_n \in A(G_m)$. Since $K_n = D_n - \frac{1}{n} D_n^{(1)}$ ($n = 1, 2, \dots$), thus

$$\|\sigma_{2M_n}(f_n)\|_1 \geq \frac{1}{2M_n} \|S_{2M_n}^{(1)}(f_n)\|_1 = \|S_{2M_n}(f_n)\|_1 \quad (n \in \mathbb{N}).$$

Furthermore, $\|S_{2M_n}(f_n)\|_1 = \|f_n\|_1 = \|D_{M_n}\|_1 = 1$ and

$$\begin{aligned} S_{2M_n}^{(1)}(f_n) &= f_n^{(1)} = \sum_{l=0}^{M_n-1} \psi_{M_n+l}^{(1)} = \sum_{l=0}^{M_n-1} (M_n+l) \psi_{M_n} \cdot \psi_l = \\ &= r_n M_n D_{M_n} + r_n D_{M_n}^{(1)}. \end{aligned}$$

We have also

$$\|\sigma_{2M_n}(f_n)\|_1 \geq \frac{1}{2M_n} \|D_{M_n}^{(1)}\|_1 - 3/2 \quad (n \in \mathbb{N})$$

and by Theorem 6

$$\|D_{M_n}^{(1)}\|_1 \geq \frac{M_n}{2\pi} \log m_{n-1} \quad (n = 1, 2, \dots),$$

thus

$$\|\sigma_{2M_n}(f_n)\|_1 \geq \frac{1}{4\pi} \log m_{n-1} - 3/2 \quad (n = 1, 2, \dots).$$

Since $\sup_n m_n = \infty$, this completes the proof.

PROOF OF THEOREM 6. We can compute the derivative of D_{M_n} ($n \in \mathbb{N}$) in a simple way, since $D_{M_j}(x) = D_{M_j}(x + le_j)$ ($x \in G_m$; $j, i \in \mathbb{N}$, $j \geq i$; $l = 0, \dots, m_j - 1$) and

$$\sum_{l=0}^{m_j-1} r_j(l e_j)^k = 0 \quad (k = 1, 2, \dots).$$

From these and from the definition of the derivative we have for $n \in \mathbb{N}$ and for $x \in G_m$

$$D_{M_n}^{(1)}(x) = d_n D_{M_n}(x) = \sum_{j=0}^{n-1} M_j \sum_{k=1}^{m_j-1} \frac{k}{m_j} \sum_{l=0}^{m_j-1} r_j(l e_j)^k D_{M_n}(x + l e_j).$$

Let $j = 0, \dots, n-1$ and $l = 1, \dots, m_j - 1$, then

$$D_{M_n}(x + l e_j) = \begin{cases} M_n & (x \in I_n((m_j - l) e_j)), \\ 0 & (\text{otherwise}). \end{cases}$$

On the other hand for $x \in I_n(0)$

$$D_{M_n}^{(1)}(0) = \sum_{j=0}^{n-1} M_j \sum_{k=1}^{m_j-1} \frac{k}{m_j} M_n = M_n \sum_{j=0}^{n-1} \frac{m_j-1}{2} M_j = \frac{M_n(M_n-1)}{2}.$$

Summarizing the above facts we have also

$$\begin{aligned} \|D_{M_n}^{(1)}\|_1 &= \int_{I_n(0)} |D_{M_n}^{(1)}| + \sum_{j=0}^{n-1} \sum_{l=1}^{m_j-1} \int_{I_n(l\epsilon_j)} |D_{M_n}^{(1)}| = \frac{M_n-1}{2} + \\ &+ \sum_{j=0}^{n-1} \sum_{l=1}^{m_j-1} M_j \left| \sum_{k=1}^{m_j-1} \frac{k}{m_j} r_j(l\epsilon_j)^k \right| = \frac{M_n-1}{2} + \sum_{j=0}^{n-1} M_j \sum_{l=1}^{m_j-1} \frac{1}{|r_j(l\epsilon_j)-1|} = \\ &= \frac{M_n-1}{2} + \sum_{j=0}^{n-1} M_j \sum_{l=1}^{m_j-1} \frac{1}{2 \sin \frac{\pi l}{m_j}}. \end{aligned}$$

From this and from the relations

$$\frac{2}{\pi} \alpha \leq \sin \alpha \leq \alpha \left(0 < \alpha < \frac{\pi}{2} \right), \log k < \sum_{j=1}^k j^{-1} < 1 + \log k \quad (k = 1, 2, \dots)$$

our theorem follows.

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A CHARACTERIZATION OF COMPLETE METRIC SPACES AND OTHER REMARKS TO A THEOREM OF EKELAND

By

T. SZILÁGYI

II. Department of Math. Analysis of the L. Eötvös University, Budapest

(Received February 8, 1982)

1. Let us recall J. Ekeland's theorem (Theorem 1.1. in [5], or Theorem 1 in [6]):

THEOREM 1. *If (X, d) is a complete metric space, $q : X \rightarrow [-\infty, +\infty]$ is a lower semicontinuous and lower bounded function which has at least one finite value, ε and λ are positive numbers and $\inf q \leq q(u) \leq \inf q + \varepsilon$, then there is a $v \in X$ such that $q(v) \leq q(u)$, $d(u, v) \leq \lambda$ and for all $w \in X \setminus \{v\}$*

$$q(w) - q(v) \geq -\frac{\varepsilon}{\lambda} \cdot d(v, w).$$

Interesting applications of this theorem can be found for example in [3], in [5], or in [6].

If (X, d) is a metric space, let us denote by Φ the set of all lower semi-continuous lower bounded functions $q : X \rightarrow \mathbb{R}$.

M. HEGEDÜS (private communication) made the following remark and asked the question below. From Ekeland's theorem it is clear that the completeness of (X, d) implies the following

PROPERTY E. For each $q \in \Phi$,

$$X_q := \{x \in X : \exists y \in X \setminus \{x\} \quad d(x, y) = q(x) - q(y)\} \neq \emptyset.$$

Is it true that property E implies the completeness of the space (X, d) ?

We give an affirmative answer to this question.

THEOREM 2. *A metric space is complete if and only if it has property E.*

PROOF. Suppose (X, d) is not complete and let (v_n) be a Cauchy sequence which is not convergent. Then we can choose a sequence of positive integers

$$n_1 < n_2 < \dots < n_k < \dots$$

such that $d(v_n, v_m) < 2^{-k}$ whenever $k \in \mathbb{N}$, $n \geq n_k$ and $m \geq n_k$. Introducing the notation $u_k := v_{n_k}$ it is easy to see that (u_k) is a Cauchy sequence, (u_k) is not convergent and

$$A := \sum_{k=1}^{\infty} d(u_{k+1}, u_k) < 1.$$

Furthermore we may assume $u_k \neq u_{k-1}$ for all $k \in \mathbb{N}$. We must prove that (X, d) does not have property E. To this aim we show that the function $\varphi : X \rightarrow \mathbf{R}_+$ defined by

$$\varphi(x) := \begin{cases} \sum_{i=k}^{\infty} d(u_{i-1}, u_i) & \text{if } x = u_k, \\ d(x, u_1) + A & \text{if } x \notin \bigcup_{k=1}^{\infty} \{u_k\} =: H \end{cases}$$

is lower semicontinuous and, for each $x \in X$, $x \in X_\varphi$. Since the function $x \mapsto d(x, u_1)$ is continuous on X and H is closed (no subsequence of (u_k) is convergent) φ is continuous on $X \setminus H$. If $x \in H$ then φ has a local minimum in x therefore φ is lower semicontinuous in x . Since for each $k \in \mathbb{N}$ $d(u_k, u_{k+1}) = q(u_k) - q(u_{k+1})$ and $u_{k+1} \neq u_k$, $u_k \in X_\varphi$. Finally, $d(x, u_1) = \varphi(x) - q(u_1)$ whenever $x \in X \setminus H$ that is $X \setminus H \subset X_\varphi$.

The "only if part" easily follows from Theorem 1: If $q \in \Phi$, $u \in X$ arbitrary, $\varepsilon := \lambda := q(u) - \inf q$, the element $v \in X$ obtained from Theorem 1 does not belong to X_φ .

2. The following theorem is rather trivial.

THEOREM 3. *Let (X, d) be complete and $H \subset X$. $H = X$ if and only if there exists a $q \in \Phi$ such that $H \subset X_\varphi$.*

PROOF. If there exists a $y \in X \setminus H$ then, for the $q \in \Phi$ defined by

$$q(x) := d(x, y),$$

$H \subset X_\varphi$ (if $x \in H$ then $y \neq x$ and $d(x, y) = q(x) - q(y)$), while the other part of the theorem follows from Theorem 2.

REMARK 1. Theorem 3 enables us to state various existence theorems in an "if and only if" form. We give two examples, the first of them shows that the fixed point theorem of Caristi-Wong (Proposition in [10]) is the most general one in complete metric spaces.

THEOREM 4. *If (X, d) is a complete metric space, $f : X \rightarrow X$ and the set of all fixed points of f is denoted by $\text{Fix}(f)$ then $\text{Fix}(f) \neq \emptyset$ if and only if there exists a lower semicontinuous and lower bounded $q : X \rightarrow \mathbf{R}$ such that for each $x \in X \setminus \text{Fix}(f)$ there is a $y \in X \setminus \{x\}$ with $d(x, y) \leq q(x) - q(y)$ (that is if and only if $X \setminus \text{Fix}(f)$ is contained in some X_φ).*

The second example is an existence theorem on stable points of set-valued dynamic systems. In [9] the following result is proved (Remark

3.21): If (X, d) is a compact metric space, $F : X \rightarrow (2^X \setminus \{\emptyset\})$ and there exists a continuous function $\Psi : X \rightarrow \mathbf{R}$ such that $\Psi(x) - \Psi(y) \geq d(x, y)$ whenever $x \in X$ and $y \in F(x)$, then there exists an $x \in X$ with $F(x) = \{x\}$. We have a somewhat stronger result which follows immediately from Theorem 3:

THEOREM 5. Suppose (X, d) is a complete metric space, $F : X \rightarrow (2^X \setminus \{\emptyset\})$ and $H := \{x \in X \mid \exists x \neq y \in F(x)\}$. There exists an $x \in X$ with $F(x) = \{x\}$ (that is $H \neq X$) if and only if there exists a $q \in \Phi$ and for every $x \in H$ an $y \in X \setminus \{x\}$ such that $q(x) - q(y) \geq d(x, y)$. In particular, if there is a $q \in \Phi$ for which $q(x) - q(y) \geq d(x, y)$ whenever $x \in X$ and $y \in F(x)$, then there exists an $x \in X$ with $F(x) = \{x\}$.

REMARK 2. From the proof of Theorem 3 it is obvious that instead of the lower semicontinuity of q in Theorem 3, in Theorem 4 and in Theorem 5 we may assume that q is continuous or q is a Lipschitz function with Lipschitz constant 1.

Finally we recall Caristi's fixed point theorem (see [1]) and a generalization of it and give a new, very simple proof of the latter theorem.

Caristi's theorem states that if (X, d) is a complete metric space, $f : X \rightarrow X$ and there is a $q \in \Phi$ such that for every $x \in X$

$$d(x, f(x)) \leq q(x) - q(f(x))$$

then f has a fixed point. For applications of this theorem see [1], [2], [7], or [8]. This theorem is an easy consequence of the fact that (X, d) has property E.

The generalization mentioned above (Theorem 2.1 in [4]) is the following:

THEOREM 6. Let $(X_1, d_1), (X_2, d_2)$ be complete metric spaces and $g : X_1 \rightarrow X_1$. If there exist a closed map $h : X_1 \rightarrow X_2$, a lower semicontinuous function $\Psi : h(X_1) \rightarrow \mathbf{R}^+$ and a positive number c such that for each $x \in X_1$

$$d_1(x, g(x)) \leq \Psi(h(x)) - \Psi(h(g(x)))$$

and

$$cd_2(h(x), h(g(x))) \leq \Psi(h(x)) - \Psi(h(g(x)))$$

then g has a fixed point.

PROOF. Let us denote the graph of h by X and the restriction to $X \times X$ of the metric

$$((x_1, x_2), (y_1, y_2)) \mapsto d_1(x_1, y_1) + cd_2(x_2, y_2)$$

defined on $(X_1 \times X_2)^2$ by d . (X, d) is complete since h is a closed map. We may apply Caristi's theorem to the map f defined by

$$(x, h(x)) \mapsto (g(x), h(g(x)))$$

and to the function q defined by

$$(x, h(x)) \mapsto 2\Psi(h(x))$$

which is obviously lower semicontinuous and lower bounded. If $(x, h(x))$ is a fixed point of f then x is a fixed point of g .

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**EIN ELEMENTARER BEWEIS
FÜR DIE ISOPERIMETRISCHE UNGLEICHUNG IN DER
EUKLIDISCHEN UND HYPERBOLISCHEN EBENE**

Von

K. BEZDEK

Lehrstuhl für Geometrie der L. Eötvös Universität, Budapest
(Eingegangen am 11. Februar 1982)

Für die isoperimetrische Ungleichung wurden im Laufe der Zeit zahlreiche verschiedene Beweise gegeben, die sich auf verschiedene Bereichsklassen beziehen und sich oft auf verschiedene Umfangsbegriffe stützen. Es seien hier nur die folgenden Werke erwähnt: B. Sz. NAGY [5], G. BOL [1], H. HADWIGER [4/a], L. FEJES TÓTH [3], A. DINGHAS [2], E. SCHMIDT [6]. Unsere Betrachtungen beziehen sich — nach E. SCHMIDT — auf nichtleere, beschränkte und abgeschlossene Punktmengen der Ebene (die nennen wir Bereiche) und auf ihre gewöhnlichen Minkowskischen Umfangsmaße (die nennen wir ganz einfach den Umfang von den Bereichen) und auf ihre Lebesgueschen Maße (die nennen wir ganz einfach den Flächeninhalt von den Bereichen) (siehe dazu HADWIGER [4/b]).

SATZ I. Unter den Bereichen (der euklidischen bzw. hyperbolischen Ebene), die gleichen positiven Flächeninhalt besitzen, hat der Kreis den kleinstmöglichen Umfang.

BEMERKUNG. Wegen der Allgemeinheit der Bereichsklasse hat nicht nur der Kreis den kleinstmöglichen Umfang.

Beschränken wir uns aber auf zusammenhängende Bereiche, so erhalten wir den

SATZ II. Unter den zusammenhängenden Bereichen (der euklidischen bzw. hyperbolischen Ebene), die gleichen positiven Flächeninhalt besitzen, hat der Kreis und nur der Kreis den kleinstmöglichen Umfang.

BEWEIS von SATZ I. Wegen der Definitionen des Umfangs und des Flächeninhalts eines Bereiches ist für uns genug das folgende Lemma zu beweisen.

Lemma I. Es sei L der Umfang eines konvexen Polygons mit dem Flächeninhalt $T (> 0)$. Dann ist $L > \bar{L}$ wobei \bar{L} den Umfang eines Kreises vom Flächeninhalt T bedeutet.

Dieses Lemma ist aber gleichbedeutend mit dem

LEMMA II. Es sei T der Flächeninhalt eines konvexen Polygons mit dem Umfang $L (> 0)$. Dann ist $T < \bar{T}$ wobei \bar{T} den Flächeninhalt eines Kreises vom Umfang L bedeutet.

Wir beweisen nun das folgende Lemma, aus dem das Lemma II. ganz einfach folgt.

LEMMA III. Es sei $L_0 > 0$ bzw. $n_0 \geq 3$ eine beliebige positive reelle Zahl bzw. eine beliebige positive ganze Zahl. Unter den konvexen Polygonen mit dem Umfang $L \leq L_0$ und mit der Seitenzahl $n \leq n_0$, hat das reguläre (konvexe) n_0 -Eck vom Umfang L_0 und nur das den größtmöglichen Flächeninhalt.

Und nun zum Beweis vom Lemma III. Nachdem die Existenz eines besten konvexen n -Ecks T_n auf Grund des Weierstrassschen Satzes feststeht, läßt sich mit Rücksicht auf die untenstehende triviale Behauptung leicht zeigen, daß der Umfang von T_n L_0 ist und die Seitenzahl von T_n n_0 ist und die Seitenlängen von T_n gleich sind.

BEHAUPTUNG. Sind für die Dreiecke $ABC \triangle$ bzw. $ABC' \triangle |AC - BC| < |AC' - BC'|$ und $AC + BC = AC' + BC'$, so gilt für die Flächeninhalte $ABC \triangle > ABC' \triangle$.

Schließlich beweisen wir, daß die Winkel von T_n gleich sind. Zu diesem Zwecke bezeichnen wir mit α_i den Winkel bei dem Eckenpunkt E_i ($1 \leq i \leq n_0$) des konvexen Polygons $T_n = E_1 E_2 \dots E_{n_0}$. So gilt

1. $\alpha_i = \alpha_j$ wenn $i = j \bmod (2)$.

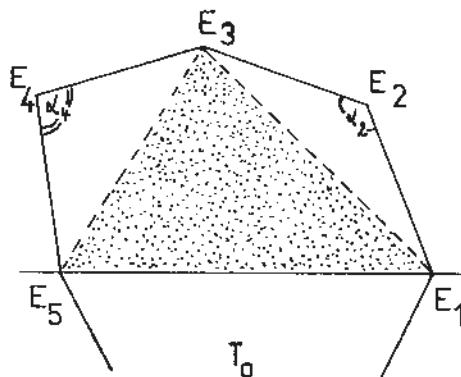


Abb. 1.

Andernfalls ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß $\alpha_2 > \alpha_1$. So gilt im Fünfeck $E_1 E_2 E_3 E_4 E_5$ $E_1 E_3 > E_3 E_5$ wegen $L_0/n_0 = E_1 E_2 = E_2 E_3 = E_3 E_4 = E_4 E_5$. (Abb. 1.) Dann nehmen wir das Fünfeck $E_1^* E_2^* E_3^* E_4^* E_5^*$, wobei $E_1^* E_2^* = E_1 E_3$; $E_1^* E_3^* = E_3 E_5$; $E_1 E_3 + E_3 E_5 = E_1^* E_3^* + E_3^* E_5^*$; $L_0/n_0 = E_1^* E_3^* = E_2^* E_3^* = E_3^* E_4^* = E_4^* E_5^*$ (Abb. 2.).

Mit Rücksicht auf die Behauptung folgt

(a) $E_1^* E_3^* E_5^* \triangle > E_1 E_3 E_5 \triangle$.

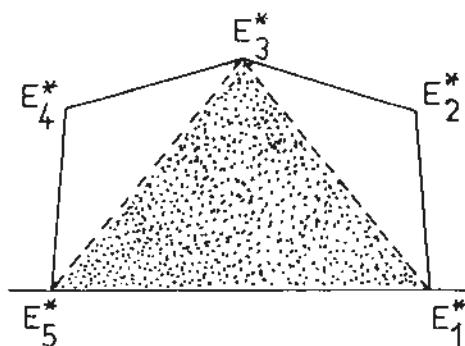


Abb. 2.

Sogar wird die folgende Ungleichung gezeigt

$$(b) \quad E_1^* E_2^* E_3^* \Delta + E_3^* E_4^* E_5^* \Delta > E_1 E_2 E_3 \Delta + E_3 E_4 E_5 \Delta.$$

In der Tat: wir werden die gleichschenkligen Dreiecke $E_1 E_2 E_3 \Delta$; $E_3 E_4 E_5 \Delta$ entlang eines gemeinsamen Schenkels aneinanderfügen (Abb. 3.). Also $OP = OQ = OR = L_0/n_0$ und $PQ = E_3 E_5 < E_1 E_3 = QR$.

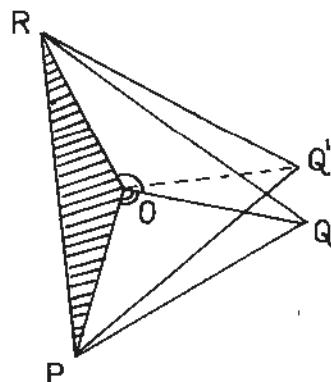


Abb. 3.

Und nun betrachten wir das Dreieck $PQ'R$, worauf $PQ' = Q'R$ und $PQ' + Q'R = PQ + QR$ besteht. Wegen der Behauptung gilt $OPQ' \Delta + OQ'R \Delta > E_1 E_2 E_3 \Delta + E_3 E_4 E_5 \Delta$. Gleichzeitig ist aber $OQ' < L_0/n_0$ und $PQ' = Q'R = E_1^* E_3^* = E_3^* E_5^*$, woraus $E_1^* E_2^* E_3^* \Delta + E_3^* E_4^* E_5^* \Delta > OPQ' \Delta + OQ'R \Delta$ folgt. So wurde die Ungleichung (b) erbracht. Aus (a) und (b) resultiert für die Flächeninhalte der Fünfecke: $E_1^* E_2^* E_3^* E_4^* E_5^* > E_1 E_2 E_3 E_4 E_5$. Ersetzen wir dann das Fünfeck $E_1 E_2 E_3 E_4 E_5$ durch das Fünfeck $E_1^* E_2^* E_3^* E_4^* E_5^*$, so nimmt der Flächeninhalt von T_0 zu, was aber unmöglich ist. Zum Schluß beweisen wir, daß

2. $z_i = z_j$ eigentlich gilt.

In der Tat: ist $n_0 = 2k+1$, so folgt aus 1. durch den Umtausch der Indizes $z_i = z_j$ unmittelbar. Ist aber $n_0 = 2k$, so bilden die Punkte $E_1 E_3 \dots E_{2k-1}$ und $E_2 E_4 \dots E_{2k}$ ein reguläres (konvexes) k -Eck. Also die Winkelhalbierenden von T_n schneiden sich in einem Punkt S . Jetzt nehmen wir an, daß $SE_i \neq SE_{i-1}$ ($i = 1, \dots, (2k-1)$). Dann wird ein Punkt E_i^* auf der Halbgeraden SE_i ($i = 1, \dots, 2k$) so gewählt, daß die Punkte $E_1^* E_2^* \dots E_{2k}^*$ ein reguläres konvexes Polygon mit dem Umfang L_0 und mit dem Zentrum S bilden. Selbstverständlich gilt dann $E_i E_{i-1} = E_i^* E_{i-1}^* = L_0/n_0$ ($i = 1, 2, \dots, 2k$), $E_{2k+1}^* = E_1^*$, $E_{2k+1} = E_1$ und $SE_i^* E_{i-1}^* \triangleleft SE_i E_{i-1} \triangleleft$. Folglich ist $E_1^* E_2^* \dots E_{2k}^* > E_1 E_2 \dots E_{2k}$, was aber unmöglich ist. Also muß $SE_i = SE_{i-1}$ ($i = 1, \dots, (2k-1)$) sein, daraus 2. unmittelbar folgt. Damit ist des Lemma III. beweisen.

BEWEIS von SATZ II. Der Einfachheit halber eine beschränkte abgeschlossene und konvexe ebene Punktmenge mit inneren Punkten nennen wir einen konvexen Bereich. Wegen der Definitionen des Umfangs und des Flächeninhalts eines zusammenhängenden Bereiches ist für uns genug das folgende Lemma zu beweisen.

LEMMA IV. Unter den konvexen Bereichen mit gleichem positiven Flächeninhalt hat der Kreis und nur der Kreis den kleinstmöglichen Umfang.

Dieses Lemma ist aber gleichbedeutend mit dem

LEMMA V. Unter den konvexen Bereichen mit gleichem positiven Umfang hat der Kreis und nur der Kreis den größtmöglichen Flächeninhalt.

Nun zum Beweis vom Lemma V. Es sei $L > 0$ eine beliebige positive reelle Zahl und es sei T ein konvexer Bereich mit dem Umfang L . Mit Hilfe vom Lemma II. ergibt sich $T \cong \tilde{T}$, wobei \tilde{T} den Flächeninhalt eines Kreises vom Umfang L bedeutet. So müssen wir noch die Eindeutigkeit beweisen. Aus diesem Grunde es sei \hat{T} ein konvexer Bereich mit dem Umfang L , wo aber $\hat{T} \neq \tilde{T}$ gilt. Es wird gezeigt, daß \hat{T} ein Kreis ist. Wir bezeichnen mit Γ die Randpunkte des konvexen Bereiches \hat{T} . So ist Γ eine geschlossene konvexe Kurve. Nun betrachten wir den Punkt H_1 von Γ eine zugehörige Stützgerade e_1 . Dann können wir von H_1 ausgehen und Γ in der positiven Richtung durchlaufen, während wir die Paare $(H_2, e_2), (H_3, e_3), \dots, (H_n, e_n)$ mit den folgenden Eigenschaften konstruieren können:

1. H_i ist ein Punkt von Γ und e_i ist eine zugehörige Stützgerade ($1 \leq i \leq n$),
2. $e_i \perp e_{i-1}$ ($2 \leq i \leq n$),
3. $e_n \cap e_1 = M$ ist ein Punkt.

Ist $H_n = H_1 = M$, so ist unser Verfahren beendet. Ist aber $H_n \neq H_1$ und $M \notin \Gamma$, so sei $H_{n+1} = M$. Und jetzt müssen wir noch zwei Fälle unterscheiden:

- (a) $H_n \neq H_1$, $M \notin \Gamma$ und der Winkelbereich $\angle H_n MH_1$ (der den konvexen Bereich \hat{T} enthält) ein Spitzwinkel ist. Dann wählen wir einen Punkt

H_{n+1} auf Γ zwischen den Punkten H_n, H_1 so, daß die zugehörige Stützgerade e_{n+1} mit den Geraden e_n und e_1 ein spitzwinkliges Dreieck bilden kann.

- (b) $H_n \neq H_1$, $M \notin \Gamma$ und der Winkelbereich $\angle H_n M H_1$ (der den konvexen Bereich \hat{T} enthält) rechtwinklig oder stumpfwinklig ist. In diesem Falle müssen wir keinen neuen Punkt konstruieren.

Letztthin entstehen k verschiedene H_i -Punkte ($1 \leq i \leq k$), die die konvexe geschlossene Kurve Γ in k verschiedene konvexe Bogen $C_i = \widehat{H_i H_{i-1}}$ ($i = 1, 2, \dots, k$, $H_{k+1} = H_1$) zerlegen. Wichtigst ist aber die folgende Festlegung: Durchläuft ein Punkt P den Bogen C_i in der positiven (bzw. negativen) Richtung, so nimmt PH_i (bzw. PH_{i-1}) streng monoton zu. Sogar besitzt jeder konvexe Teilbogen von C_i diese Eigenschaft.

Und jetzt sei h eine beliebige reelle Zahl, worauf aber

$$0 < h \leq \frac{1}{6} \cdot \min \{H_i H_{i+1} \mid i = 1, 2, \dots, k; H_{k+1} = H_1\}$$

gilt, ferner sei $S_1 \in C_1$ ein Punkt mit $S_1 H_1 = S_1 H_2$. So wir gehen von S_1 aus und durchlaufen Γ in der positiven Richtung, während wir mit S_2 den ersten Punkt auf Γ bezeichnen, worauf $S_2 S_1 = h$ gilt und im allgemeinen, wenn der Punkt S_i gegeben ist, bedeutet S_{i+1} den ersten Punkt von Γ (in der positiven Richtung), worauf $S_{i+1} S_i = h$ steht. Endlich sei S_m der erste Punkt auf Γ , wovon (ausgehend in der positiven Richtung) wir zuerst den Punkt S_1 dann den Punkt S_{m+1} erreichen können. Jetzt betrachten wir das konvexe m -Eck $S_1 S_2 \dots S_m$ in dem $S_1 S_2 = S_2 S_3 = \dots = S_{m-1} S_m = h$ und $0 < S_m S_1 \leq h$ gilt. Mit Hilfe von der Methode des Beweises vom Lemma III. ergibt sich

- (i) Im Falle $m = 2l$ sind die Winkel bei den Ecken $S_2, S_4, \dots, S_{2l-2}$ (bzw. bei den Ecken $S_3, S_5, \dots, S_{2l-1}$) gleich, also ist $S_1 S_3 \dots S_{2l-1}$ ein einem Kreis einbeschriebenes Polygon, wobei $2h \geq S_1 S_3 = S_3 S_5 = \dots = S_{2l-3} S_{2l-1} \geq S_{2l-1} S_1$
- (ii) Im Falle $m = 2l+1$ sind die Winkel bei den Ecken S_2, S_4, \dots, S_{2l} (bzw. bei den Ecken $S_3, S_5, \dots, S_{2l-1}$) gleich und so ist $S_1 S_3 \dots S_{2l+1}$ ein einem Kreis einbeschriebenes Polygon, wobei $2h \geq S_1 S_3 = S_3 S_5 = \dots = S_{2l-1} S_{2l+1} > S_{2l+1} S_1$.

In beiden Fällen gilt: ist P ein Punkt von Γ , so enthält der Kreis mit dem Radius $2h$ und mit dem Zentrum P in seinem Inneren mindestens eine der Ecken des konstruierten einem Kreis einbeschriebenen Polygons. Zum Schluß sei h_n eine streng monoton abnehmende Folge, wobei $0 < h_n < h$ und $\lim_{n \rightarrow +\infty} h_n = 0$. Wie wir es in den Fällen (i), (ii) gesehen haben, gehört zu jeder Zahl h_n ein einem Kreis k_{h_n} einbeschriebenes Polygon S_{h_n} . Ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß $\lim k_{h_n} = k$, wo k einen Kreis bezeichnet. Daraus ganz einfach, wegen der obigen Eigenschaft des konstruierten Polygons, $k = \Gamma$ folgt, womit unser Beweis beendet ist.

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AUSFÜLLUNGEN IN DER HYPERBOLISCHEN EBENE DURCH ENDLICHE ANZAHL KONGRUENTER KREISE

Von

K. BEZDEK

Lehrstuhl für Geometrie der L. Eötvös Universität, Budapest

(Eingegangen am 11. Februar 1952)

Das Ergebnis von THUE [1] ist in der diskreten Geometrie gut bekannt; ist d die Dichte eines beliebigen Systems von kongruenter, nicht übereinandergrifffenden Kreisen¹ der euklidischen Ebene, so ist $d \leq \pi/\sqrt{12} = 0,906899 \dots$. Im Jahre 1953 hat L. FEJES TÓTH [2] analoge Schranke auf der Kugel und in der hyperbolischen Ebene für die Dichte der Ausfüllung durch kongruente Kreise gegeben. Jetzt formulieren wir exakt dieses Ergebnis von L. FEJES TÓTH:

Wird eine Kugel von der Krümmung χ^2 durch wenigstens drei Kreise vom Radius r unterdeckt, so ist die Unterdeckungsdichte \leq

$$\leq d^*(a) = \frac{3 \operatorname{cosec} \frac{\pi}{a}}{a - 6}, \quad d^*(6) = \lim_{a \rightarrow 6} d^*(a) = \frac{\pi}{\sqrt{12}}$$

wo a durch $\operatorname{cosec} \frac{\pi}{a} = 2 \cos \sqrt{\chi} r$ gegeben ist.

Gleichheit gilt in der obigen Ungleichung für alle ganzzahligen Werte von $a \geq 2$. Für die nicht ganzen Zahlen ergibt der folgende Satz von J. MOLNÁR [4] eine bessere obere Schranke:

Wird eine Kugel von der Krümmung χ durch wenigstens drei Kreise vom Radius r unterdeckt, so ist

¹ Unter Kreis verstehen wir immer eine geschlossene Kreisscheibe.

² Wir legen unseren Betrachtungen die „allgemeine Kugel“ der Krümmung χ zugrunde, d.h. den Fällen $\chi \leq 0$ entsprechend die gewöhnliche Kugelfläche, die euklidische Ebene oder die hyperbolische Ebene.

(1) die Unterdeckungsdichte $d(a)$

$$3 \operatorname{cosec} \frac{\pi}{a} - 6$$

$$\cdot [a] + 3 = \frac{6}{\pi} \cdot \operatorname{arctg} \left\{ \sqrt{3} \operatorname{tg} \frac{\pi}{a} \cdot \operatorname{ctg} \left(1 - \frac{|a|}{a} \right) \pi \right\}$$

wobei $d(6) = \lim_{a \rightarrow 6} d(a) = \frac{\pi}{\sqrt{12}}$ und $\operatorname{cosec} \frac{\pi}{a} = 2 \cos \frac{\pi}{a}$ ist.

Wir müssen jetzt hier bemerken, daß wir die gewöhnliche Definition der Dichte bezüglich der ganzen euklidischen Ebene im Falle der hyperbolischen Ebene nicht anwenden können (K. BÖRÖCZKY [6]). Wegen dieser Schwierigkeit hat L. FEJES TÓTH vorgeschlagen, die Ausfüllungen in beschränkten Bereichen zu untersuchen. Wir werden hier zwei Sätze (von L. FEJES TÓTH [3] bzw. von J. MOLNÁR [5]) erwähnen.

Sind in einem konvexen Gebiet der euklidischen Ebene wenigstens zwei kongruente nicht übereinandergreifende Kreise eingelagert, so ist die Lagerungsdichte $\approx \pi/\sqrt{12}$.

Sind in einem konvexen sphärischen Gebiet wenigstens drei kongruente nicht übereinandergreifende Kugelkappen eingelagert, so ist die Lagerungsdichte $\approx \pi/\sqrt{12}$.

Die hier aufgeworfene Frage ist in der hyperbolischen Ebene noch interessanter. Nehmen wir zwei Kreise vom Radius r in der hyperbolischen Ebene, die einander berühren, so gilt für die Dichte $d(r)$ bezüglich der konvexen Hülle von zwei Kreisen: $\lim d(r) = 1$. Trotzdem hat L. FEJES TÓTH die folgende Vermutung ausgesprochen, die der Verfasser bewiesen hat [7]:

Sind in einem Kreis der hyperbolischen Ebene wenigstens zwei kongruente nicht übereinandergreifende Kreise eingelagert, so ist die Lagerungsdichte $\approx \pi/\sqrt{12}$.

Wir erwähnen hier, daß für die Funktion von (1) $d(a) > \pi/\sqrt{12}$ gilt, wenn a genug groß ist. Eine sehr wichtige Bemerkung ist das, daß wir die obere Schranke des erwähnten Satzes nicht verbessern können, also sie die „beste“ obere Schranke ist. In der vorliegenden Arbeit beweisen wir den folgenden allgemeineren Satz.

SATZ. Sind $n \geq 2$ einander nicht überdeckende Kreise K_1, \dots, K_n vom Radius $r > 0$ in der hyperbolischen Ebene gegeben, so gilt

$$\frac{\sum_{i=1}^n K_i}{T_r} < \frac{\pi}{\sqrt{12}}^3$$

³ Wir werden den Flächeninhalt eines Bereiches ebenfalls mit demselben Symbol bezeichnen wie den Bereich.

wo T die konvexe Hülle der Mittelpunkte von K_1, \dots, K_n und T_r die äußere Parallelmenge⁴ von T im Abstand r bedeutet.

BEWEIS VOM SATZ.

LEMMA. Bedeutet G die abgeschlossene Hülle der Menge $T_r - T$, so gilt⁵ $G = (e^r - 1)T$.

Nun zum Beweis vom Lemma. Hat T keinen inneren Punkt, so ist $T = 0$ und die Ungleichung ist trivial. Also wir können voraussetzen, daß $T \neq 0$ steht. Es sei m die Anzahl der Seiten von T und z_1, z_2, \dots, z_m die Winkel von T und L der Umfang von T . Aus den Gleichungen

$$T = (m-2)\pi \cdot \sum_{i=1}^m z_i \text{ und } G = L \operatorname{sh} r + \frac{1}{2\pi} \cdot 4\pi \operatorname{sh}^2 \frac{r}{2}$$

folgt $G = L \operatorname{sh} r + 2(T + 2\pi) \cdot \operatorname{sh}^2 \frac{r}{2}$, das heißt

$$\frac{G}{T} = \frac{L}{T} \operatorname{sh} r + \frac{4\pi}{T} \operatorname{sh}^2 \frac{r}{2} + 2 \operatorname{sh}^2 \frac{r}{2}.$$

Nun sei x der Radius eines Kreises, dessen Umfang L ist. Also gilt $L = \approx 2\pi \operatorname{sh} x$. Wegen der isoperimetrischen Eigenschaft des Kreises [8] ergibt sich $T \approx 4\pi \operatorname{sh}^2 \frac{x}{2}$. Folglich haben wir

$$\begin{aligned} \frac{G}{T} &= \frac{2\pi \operatorname{sh} x}{4\pi \operatorname{sh}^2 \frac{x}{2}} \operatorname{sh} r = \frac{4\pi}{4\pi \operatorname{sh}^2 \frac{x}{2}} \operatorname{sh}^2 \frac{r}{2} + 2 \operatorname{sh}^2 \frac{r}{2} = \\ &= 2 \left(\operatorname{sh} \frac{r}{2} \cdot \operatorname{cth} \frac{x}{2} \right) \operatorname{ch} \frac{r}{2} + \operatorname{sh}^2 \frac{r}{2} \cdot \operatorname{cth}^2 \frac{x}{2} + \operatorname{ch}^2 \frac{r}{2} - 1 = \\ &= \left[\operatorname{sh} \frac{r}{2} \cdot \operatorname{cth} \frac{x}{2} + \operatorname{ch} \frac{r}{2} \right]^2 - 1 + \left[\operatorname{sh} \frac{r}{2} + \operatorname{ch} \frac{r}{2} \right]^2 - 1 = \\ &= e^r - 1 \quad \text{w. b. z. w.} \end{aligned}$$

jetzt folgen die Abschätzungen. Zu diesem Zweck führen wir die Symbole O_1, \dots, O_n ein, die die Mittelpunkte der Kreise K_1, \dots, K_n bedeuten.

ABSCHÄTZUNG O. Es sei $\dim T = 1$. Ohne Beschränkung der Allgemeinheit können wir dann voraussetzen, daß $T = \overline{O_1 O_2}$ gilt. Jetzt werden wir mit t die Gerade der Punkte O_1, O_2 bezeichnen. So ist $t \cap K_i$ ($i = 1, \dots, n$) ein Durchmesser von K_i . Die zugehörigen Tangente begrenzen einen Strei-

⁴ T_r ist die Vereinigungsmenge der Kreise vom Radius r , deren Mittelpunkte zu T gehören.

⁵ Von nun an wir nehmen an, daß die Raumkonstante $= 1$ ist.

fen, dessen Durchschnitt mit G durch das Symbol D_i bezeichnet ist. Offenbar steht $\min_i \{D_i\} = D_1 = D_2$ und so

$$\frac{\sum_{i=1}^n K_i}{G} \leq \frac{K_1 + K_2}{D_1 + D_2} = \frac{\frac{4\pi \operatorname{sh}^2 \frac{r}{2}}{2}}{\frac{2\pi \operatorname{sh}^2 \frac{r}{2} + 2r \operatorname{sh} r}{2}} = \frac{2\pi}{\pi + 4 \cdot \frac{\operatorname{th} \frac{r}{2}}{\frac{r}{2}}} = \frac{2\pi}{\pi + 4 - \frac{\pi}{\sqrt{12}}}.$$

ABSCHÄTZUNG 1. Es sei $\dim T = 2$ und $r \geq 0,93$. Dann ist $K_{i,j}$ eine zweidimensionale zusammenhängende Komponente von $K_i \cap G$ ($i = 1, \dots, n$) wenn solche existiert, die selbstverständlich ein Kreissegment oder ein Kreissektor ist. Der Streckenzug $K_{i,j} \cap G \cap T$ mit den Endpunkten $V_{i,j}^1, V_{i,j}^2$ besteht aus einer Strecke oder zwei Strecken von Länge r . Ferner sei $t_{i,j}^1$ (bzw. $t_{i,j}^2$) die den Punkt $V_{i,j}^1$ (bzw. $V_{i,j}^2$) enthaltene Gerade, die den Streckenzug $K_{i,j} \cap G \cap T$ senkrecht schneidet. Schließlich enthält die durch Gerade $t_{i,j}^1$ (bzw. $t_{i,j}^2$) begrenzte geschlossene Halbebene $S_{i,j}^1$ (bzw. $S_{i,j}^2$) den Mittelpunkt O_i . Dann bilden wir die Menge $\tilde{D}_{i,j} := G \cap S_{i,j}^1 \cap S_{i,j}^2$, deren eine zusammenhängende Komponente $D_{i,j}$ die Menge $K_{i,j}$ enthält. Einfache Überlegungen zeigen, daß

$$\frac{K_{i,j}}{\tilde{D}_{i,j}} \leq \frac{\frac{4\pi \operatorname{sh}^2 \frac{r}{2}}{2}}{\frac{2\pi \operatorname{sh}^2 \frac{r}{2} + 2r \cdot \operatorname{sh} r}{2}} = \frac{2\pi}{\pi + 2r \operatorname{eth} \frac{r}{2}}$$

und so

$$\frac{\sum_{i=1}^n K_i}{T+G} \leq \frac{2\pi}{\pi + 2r \operatorname{eth} \frac{r}{2}}.$$

gilt. Daraus mit Hilfe vom Lemma folgt

$$\frac{T+G}{T+G} = \frac{\frac{2\pi}{\pi + 2r \operatorname{eth} \frac{r}{2}}}{1 + (e^r - 1)} = \frac{\frac{2\pi}{\pi + 2r \operatorname{eth} \frac{r}{2}}}{e^r} = \mathcal{J}(r)$$

Offenbar nimmt die Funktion $\tilde{f}(r)$ in $(0, +\infty)$ monoton ab, daher gilt

$$\tilde{f}(r) \leq \tilde{f}(0,93) = 0,906838\dots < 0,906899\dots = \frac{\pi}{\sqrt{12}}, \text{ wenn } r \geq 0,93 \text{ ist.}$$

ABSCHÄTZUNG 2. Es sei $\dim T = 2$ und $0 < r < 0,93$. Wir ordnen jedem Kreis K_i ($i = 1, \dots, n$) die Menge D_i aller Punkte von T zu, die nicht weiter bei O_i liegen als bei jedem anderen Mittelpunkt O_j ($j \neq i$). Wir behaupten, daß die konvexen Polygone D_i ein Mosaik bilden, daß sie also das konvexe Polygon T , ohne übereinanderzugreifen, vollständig überdecken. Liegt K_i in T , so gilt $K_i \subset D_i$. Es sei $I_1 = \{i \mid i \in \{1, \dots, n\} \text{ und } K_i \subset T\}$ und $H_1 = \bigcup_{i \in I_1} D_i$.

Mit Rücksicht auf den bekannten Satz von J. MOLNÁR (siehe dazu [4] S. 234) haben wir jetzt:

$$(2) \quad \frac{\sum_{i \in I_1} K_i}{H_1} \leq \max_{i \in I_1} \left\{ \frac{K_i}{D_i} \right\} \equiv d(a)$$

(siehe dazu (1)). Ferner sei $I_2 = \{j \mid j \in \{1, \dots, n\} \text{ und } K_j \not\subset T\}$ und $H_2 = \bigcup_{j \in I_2} D_j \subset G$. Im nachfolgenden werden wir für den Wert von

$$\frac{\sum_{j \in I_2} K_j}{H_2}$$

eine obere Schranke geben.

Es sei $i \in I_2$. Dann ist K_i^j eine zweidimensionale zusammenhängende Komponente von $K_i \cap G$, die selbstverständlich ein Kreissegment oder ein Kreissektor ist. Wir können also das System von $\{K_i^j\}$ betrachten, wo $i \in I_2$ und $1 \leq j \leq r(i)$ und $r(i)$ die Anzahl der verschiedenen zweidimensionalen Komponenten von $K_i \cap G$ bedeutet. Nun sei K_i^j eine beliebige zweidimensionale zusammenhängende Komponente. Dann existieren solche eindeutig bestimmte Mengen $K_i^{j*}, K_i^{j**}, K_i^{j***}$, die im beliebigen Umlaufssinn (am Rande von T) die Nachbarn von K_i^j sind (Abb. 1).

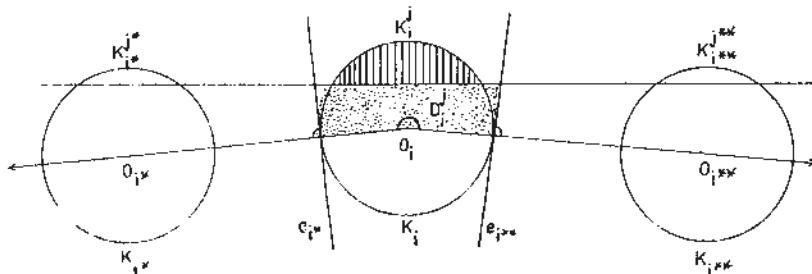


Abb. 1.

Jetzt definieren wir den Winkelbereich $\angle O_i* O_i O_i**$ so, daß $\angle O_i* O_i O_i** \supset K_i^j$. Ferner nehmen wir die Tangente e_i* (bzw. e_i**) des Kreises K_i , deren Berührungsgeraden zur Halbgeraden $O_i \vec{O}_i*$ (bzw. $\vec{O}_i O_i**$) gehört. Bezeich-

nen wir die durch c_{i*} (bzw. c_{i**}) begrenzte und den Punkt O_i enthaltene geschlossene Halbebene mit S_{i*} (bzw. S_{i**}), so kann man das folgende Viereck konstruieren

$$D_i^j = \langle O_{i*} O_i O_{i**} \cap S_{i*} \cap S_{i**} \rangle \cap T,$$

Es läßt sich leicht beweisen, daß $D_i^j \subseteq D_i$. Sogar ist es leicht zu zeigen, daß die Mengen D_i^j vom System $\{D_i^j\}_{i \in I_2}$ ein fixierter Index aus I_2 und $1 \leq j \leq r(i)$ einander nicht überdecken können. Und nun betrachten wir die Zelle

$$\hat{D}_i = (\bigcup_{1 \leq j \leq r(i)} D_i^j) \cup (K_i \cap T),$$

woraus $\hat{D}_i \subseteq D_i$ folgt. Also unbedingt gilt

$$(3) \quad H_2 = \sum_{i \in I_2} K_i \stackrel{?}{=} \sum_{i \in I_2} K_i + \left(\sum_{i \in I_2} \hat{D}_i \right) + G.$$

Eine weitere Abschätzung zu erreichen, konstruieren wir neuere Zellen. Es seien K_i^k und K_j^l am Rande des konvexen Polygons benachbart ($i, j \in I_2$) (Abb. 2.).

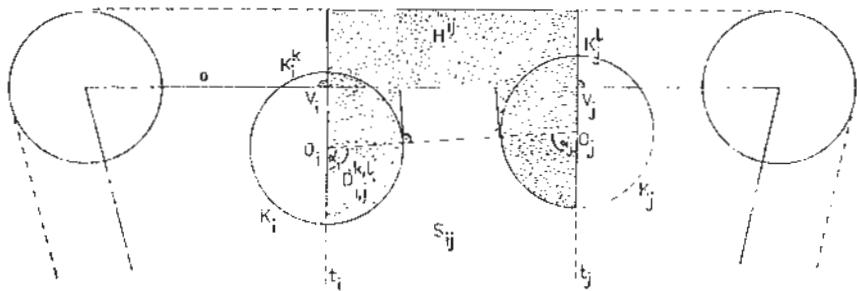


Abb. 2.

Wir bezeichnen die Seite von T mit a , die mindestens eine Teilstrecke sowohl von $K_i^k \cap G \cap T$ als auch von $K_j^l \cap G \cap T$ enthält. Ferner sei t_i (bzw. t_j) die den Punkt O_i (bzw. O_j) enthaltene Gerade, die die Seite a im Punkt V_i (bzw. V_j) senkrecht schneidet. Dann können wir den durch t_i , t_j begrenzten geschlossenen Streifen S_{ij} nehmen. So wird der zur Strecke $V_i V_j$ gehörige Hyperzyklussektor vom Abstand r mit H^{ij} bezeichnet. Schließlich enthält die durch Gerade $O_i O_j$ begrenzte geschlossene Halbebene S^{ij} den Hyperzyklussektor H^{ij} . Also kann die folgenden Mengen definiert werden:

$$D_{i,j}^{kl} = H^{ij} \cup (D_i^k \cap S_{ij} \cap S^{ij}) \cup (D_j^l \cup S_{ij} \cap S^{ij}) \cup (K_i \cap S_{ij}) \cup (K_j \cap S_{ij}),$$

$$K_{i,j}^{kl} = (K_i \cap S_{ij}) \cup (K_j \cap S_{ij}).$$

Es ist leicht einzusehen:

$$(\sum K_{i,j}^{k,l}) - (\sum_{i \in I_2} K_i) = (\sum D_{i,j}^{k,l}) - (G + \sum_{i \in I_2} \hat{D}_i) \geq 0$$

deshalb folgt

$$(4) \quad \frac{\sum_{i \in I_2} K_i}{(\sum_{i \in I_2} \hat{D}_i) + G} \leq \frac{\sum K_{i,j}^{k,l}}{\sum D_{i,j}^{k,l}} \leq \max \left\{ \frac{K_{i,j}^{k,l}}{D_{i,j}^{k,l}} \right\}.$$

Im nachfolgenden beschäftigen wir uns mit der Abschätzung von $K_{i,j}^{k,l}/D_{i,j}^{k,l}$. Gesucht wird diejenige Menge $D_{i,j}^{k,l}$, für die der Flächeninhalt $D_{i,j}^{k,l}$ den kleinstmöglichen Wert erreicht. Wir werden zeigen, daß es im Falle $O_i = V_i$, $O_j = V_j$, $O_i O_j = 2r$ eintreten wird. Und jetzt der Beweis: selbstverständlich muß $O_i O_j = 2r$ sein. Und nun sei S^{ij} die komplementäre geschlossene Halbebene von S^{ij} . Dann besteht $K_{i,j}^{k,l} \cap S^{ij}$ aus den Kreissektoren z_i, z_j , deren Inhaltssumme ihr Minimum auch in Falle $O_i = V_i$, $O_j = V_j$, $O_i O_j = 2r$ erreicht. Darum ist es für uns genug, das Minimum des Inhalts der Menge $D_{i,j}^{k,l} \cap S^{ij}$ zu bestimmen ($O_i O_j = 2r$). Durch einfache Überlegungen ergibt sich, daß entweder $O_i = V_i$ oder $O_j = V_j$ gelten muß. Ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß $O_i = V_i$ und $O_i O_j = 2r$ gilt. Jetzt werden die Endpunkte des Hyperzyklusbogens von H^{ij} mit W_i, W_j bezeichnet ($W_i \in t_i, W_j \in t_j$). Ferner sei $V_j O_j = x$ (Abb. 3.)

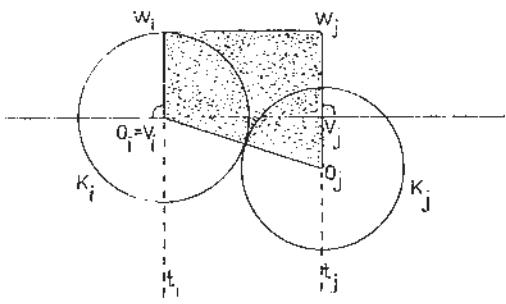


Abb. 3.

Durch einfache Rechnung ergibt sich für den Flächeninhalt

$$D_{i,j}^{k,l} \cap S^{ij} = H^{ij} \cup O_i O_j V_j \Delta = \operatorname{sh} r \cdot \operatorname{arch} \left(\frac{\operatorname{ch} 2r}{\operatorname{ch} x} \right) +$$

$$+ 2 \operatorname{arctg} \left(\operatorname{th} \frac{x}{2} \sqrt{\frac{\operatorname{ch} 2r - \operatorname{ch} x}{\operatorname{ch} 2r + \operatorname{ch} x}} \right) + f(x)$$

wo $0 \leq x \leq r$. So erhalten wir

$$f'(x) = -\frac{\operatorname{sh} r \cdot \operatorname{ch} 2r \cdot \operatorname{th} x}{\sqrt{\operatorname{ch}^2 2r - \operatorname{ch}^2 x}} + \frac{2 \sqrt{\frac{\operatorname{ch} 2r - \operatorname{ch} x}{\operatorname{ch} 2r + \operatorname{ch} x}}}{1 + \operatorname{th}^2 \left(\frac{x}{2} \right) \frac{\operatorname{ch} 2r + \operatorname{ch} x}{2 \operatorname{ch} 2r + \operatorname{ch} x}}.$$

$$\cdot \left(\frac{1}{2 \operatorname{ch}^2 \left(\frac{x}{2} \right)} - \frac{2 \operatorname{sh}^2 \left(\frac{x}{2} \right) \cdot \operatorname{ch} 2r}{\operatorname{ch}^2 2r - \operatorname{ch}^2 x} \right).$$

Nach einiger Rechnung folgt für $0 < x < r$:

$$\frac{1}{2 \operatorname{ch}^2 \left(\frac{x}{2} \right)} - \frac{2 \operatorname{sh}^2 \left(\frac{x}{2} \right) \cdot \operatorname{ch} 2r}{\operatorname{ch}^2 2r - \operatorname{ch}^2 x} > 0,$$

$$\frac{2 \sqrt{\frac{\operatorname{ch} 2r - \operatorname{ch} x}{\operatorname{ch} 2r + \operatorname{ch} x}}}{1 + \operatorname{th}^2 \left(\frac{x}{2} \right) \frac{\operatorname{ch} 2r + \operatorname{ch} x}{2 \operatorname{ch} 2r + \operatorname{ch} x}} > 0,$$

$$\left(\frac{1}{2 \operatorname{ch}^2 \left(\frac{x}{2} \right)} - \frac{2 \operatorname{sh}^2 \left(\frac{x}{2} \right) \cdot \operatorname{ch} 2r}{\operatorname{ch}^2 2r - \operatorname{ch}^2 x} \right)' < 0,$$

$$\left(\frac{2 \sqrt{\frac{\operatorname{ch} 2r - \operatorname{ch} x}{\operatorname{ch} 2r + \operatorname{ch} x}}}{1 + \operatorname{th}^2 \left(\frac{x}{2} \right) \frac{\operatorname{ch} 2r + \operatorname{ch} x}{2 \operatorname{ch} 2r + \operatorname{ch} x}} \right)' < 0,$$

$$\left(-\frac{\operatorname{sh} r \cdot \operatorname{ch} 2r \cdot \operatorname{th} x}{\sqrt{\operatorname{ch}^2 2r - \operatorname{ch}^2 x}} \right)' < 0.$$

Deshalb gilt $f''(x) < 0$ für $0 < x < r$. Folglich haben wir die folgende Gleichung

$$\inf_{x \in [0, r]} f(x) = \min \{f(0), f(r)\}.$$

HILFSSATZ. $f(0) < f(r)$.

Nun zum Beweis vom Hilfssatz. In üblicher Weise bezeichnen wir mit $\frac{2\pi}{a}$ die Winkel eines gleichseitigen Dreiecks mit der Seitenlänge $2r$.

Fall I. Es sei $\frac{2\pi}{a} \geq \frac{\pi}{4}$ d.h. $0 < r \leq r_1 = 0,764\dots$. Der Einfachheit halber seien $O_i^0 O_j^0 W_j^0 W_i^0 = f(0)$ und $O_i^r O_j^r W_j^r W_i^r = f(r)$, sogar wir werden annehmen, daß $O_i^0 = O_i^r, W_j^0 = W_j^r$ und der Hyperzyklusbogen $\widehat{W_i^r W_j^r}$ ein Teil des Hyperzyklusbogens $\widehat{W_j^0 W_i^0}$ ist (Abb. 4.). Dann ist $V_j^r = O_j^r W_j^r \cap O_i^r O_j^0$ und eine Halbdrehung um den Punkt V_j^r führt den Punkt O_i^r in einen Punkt N_1 über. Hier ist $\angle O_i^r O_j^r V_j^r = \frac{2\pi}{a}$, so gilt $\angle O_i^r O_j^r N_1 = \frac{4\pi}{a} \geq \frac{\pi}{2}$. Ferner sei e die innere Winkelhalbierende von $\angle O_j^0 O_i^0 O_j^r$ ($<\pi$) und n^\perp eine Gerade, die den Punkt O_j^r enthält und die Gerade $O_i^r O_j^r$ senkrecht schneidet. Offenbar ist $e \cap n^\perp \neq 0$, daher sei $M_1 = e \cap n^\perp$. Selbstverständlich gehört der Punkt M_1 zum Dreieck $N_1 V_j^r O_j^r$ und gehört zur Geraden $O_j^0 W_j^0$. Schließlich sei $M_2 = O_j^r W_j^0 \cap N_1 W_i^r$. Also gilt

$$\frac{r}{2} = \frac{O_j^r V_j^r}{2} < \frac{O_i^r O_j^0}{2} = \angle O_j^0 M_1 \leq O_j^0 M_2$$

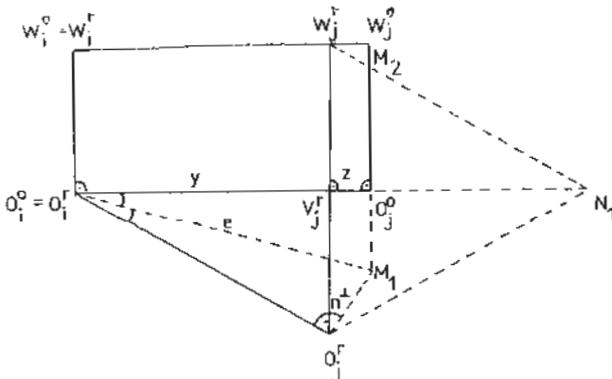


Abb. 4.

woraus $O_j^r M_2 > M_2 W_i^0$ folgt. So können wir aber den durch den Hyperzyklusbogen $\widehat{W_j^0 W_i^0}$ und die Strecken $W_i^0 M_2, W_i^0 M_1$ begrenzten Bereich durch das rechtwinklige Dreieck $M_2 O_j^0 N_1$ überdecken. Daraus eigentlich $f(0) < f(r)$ folgt.

Fall II. Es sei $r_1 < r < 0,93$.

Mit Hilfe von der Abbildung 4. sei $O_i^r V_j^r = y$ und $V_j^r O_j^0 = z$. Dann ist

$$e(r) = f(r) - f(0) = \frac{\pi}{2} - 3 \operatorname{arc} \sin \left(\frac{1}{2 \operatorname{ch} r} \right) - (2r - y) \operatorname{sh} r =$$

$$= \frac{\pi}{2} + y \operatorname{sh} r - 3 \operatorname{arc} \sin \left(\frac{1}{2 \operatorname{ch} r} \right) - 2r \operatorname{sh} r.$$

Nachdem wir erhalten

$$\varepsilon'(r) = \operatorname{sh} r \left[y' + \frac{3}{\operatorname{ch} r \cdot \sqrt{4 \operatorname{ch}^2 r - 1}} - 2 \right] - z \operatorname{ch} r.$$

Wegen $\operatorname{ch} y = \frac{\operatorname{ch} 2r}{\operatorname{ch} r}$ ergibt sich $y' = \frac{\operatorname{th} r(2 \operatorname{ch}^2 r + 1)}{4 \operatorname{ch}^4 r - 5 \operatorname{ch}^2 r + 1}$.

Also gilt

$$\varepsilon'(r) = g(r) \operatorname{sh} r - z \operatorname{ch} r$$

wobei

$$g(r) = \frac{\operatorname{th} r \cdot (2 \operatorname{ch}^2 r - 1)}{\sqrt{4 \operatorname{ch}^4 r - 5 \operatorname{ch}^2 r + 1}} + \frac{3}{\operatorname{ch} r \cdot \sqrt{4 \operatorname{ch}^2 r - 1}} - 2.$$

Aus $r_1 < r$ folgt $g'(r) > 0$ und so erhalten wir, daß $g(r) > 0,351$ steht. Gleichzeitig ist

$$z' + 2 - y' = 2 - \frac{\operatorname{th} r \cdot (2 \operatorname{ch}^2 r + 1)}{4 \operatorname{ch}^4 r - 5 \operatorname{ch}^2 r + 1} > 0$$

folglich haben wir wegen $r_1 < r$ die Ungleichung $z > 0,304$. Womit wir erhalten haben

$$\varepsilon'(r) > 0,351 \cdot \operatorname{sh} r - 0,304 \cdot \operatorname{ch} r > 0$$

(denn es steht $1,36 \cdot \operatorname{ch} r$ wegen $r < 0,93$). Gleichzeitig ist aber $\varepsilon(0,93) = 0,086$, also muß $\varepsilon(r) > 0$ für $r_1 < r < 0,93$ sein so ist der Beweis vom Hilfssatz beendet.

Also mit Rücksicht auf (3) und (4) erhalten wir

$$(5) \quad \frac{\sum_{i=1}^n K_i}{H_2} = \frac{\frac{4\pi \operatorname{sh}^2 \frac{r}{2}}{2}}{\frac{2\pi \operatorname{sh}^2 \frac{r}{2} + 2r \cdot \operatorname{sh} r}{2}} = \frac{2\pi}{\frac{\pi + 4 + \frac{2}{\operatorname{th} \frac{r}{2}}}{2}}.$$

Wenn $I_1 = 0$, dann gilt

$$\frac{\sum_{i=1}^n K_i}{T + G} = \frac{\sum_{i=1}^n K_i}{H_2} \approx \frac{2\pi}{\frac{\pi + 4 + \frac{2}{\operatorname{th} \frac{r}{2}}}{2}} \approx \frac{\pi}{\sqrt{12}}.$$

Ist aber $I_1 \neq \emptyset$, dann bekommen wir mit Rücksicht auf (2), (5)

$$(6) \quad \frac{\sum_{i=1}^n K_i}{T+G} = \frac{\sum_{i \in I_1} K_i + \sum_{i \in I_2} K_i}{H_1 + H_2} \leq \dots \leq \frac{d(a) H_1 + \frac{2\pi}{\pi+4} \cdot \frac{r/2}{\operatorname{th} \frac{r}{2}} \cdot H_2}{H_1 + H_2},$$

wobei $\frac{\pi}{a} := \arcsin\left(\frac{1}{2 \operatorname{ch} r}\right)$.

Zuerst sei $0 < r \approx 0,49$ d.h. $6 < a \approx 6,804815, \dots$ Hier wissen wir aber, daß die Funktion $d(a)$ in (6, 7) zuerst streng monoton abnimmt dann zunimmt, sogar ist $d(6) \approx \frac{\pi}{\sqrt{12}}$ und $d(6,804815, \dots) \approx 0,906035 \approx 0,906899, \dots = \frac{\pi}{\sqrt{12}}$. Folglich muß $d(a) \approx \frac{\pi}{\sqrt{12}}$ sein, wenn $6 < a \approx 6,804815, \dots$ Daraus mit (6) ergibt sich

$$\frac{\sum_{i=1}^n K_i}{T+G} \leq \frac{\pi}{\sqrt{12}}.$$

Zum Schluß sei $0,49 < r < 0,93$ d.h. $6,804, \dots < a < 9,016, \dots$ Dann wegen $d(a) \approx d(10) \approx 0,927051$ und $e^r - 1 \approx \frac{G}{T} = \frac{H_2}{H_1}$ erhalten wir aus (6)

$$\frac{0,927051 + (e^r - 1) \cdot \frac{2\pi}{\pi+4} \cdot \frac{r/2}{\operatorname{th} \frac{r}{2}}}{T+G} \leq \frac{\sum_{i=1}^n K_i}{e^r} \leq \frac{\pi}{\sqrt{12}} =: \bar{\mathcal{F}}^*(r).$$

Offenbar nimmt die Funktion $\bar{\mathcal{F}}^*(r)$ in $(0, \dots)$ monoton ab, daher gilt $\bar{\mathcal{F}}^*(r) \approx \bar{\mathcal{F}}^*(0,49) \approx 0,905 \approx \frac{\pi}{\sqrt{12}}$, womit der Beweis vom Satz beendet ist.

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UPPER ESTIMATES FOR THE EIGENFUNCTIONS OF A LINEAR DIFFERENTIAL OPERATOR

By

V. KOMORNÍK

II. Department of Math. Analysis of the I. Eötvös University, Budapest

(Received February 11, 1982)

In the convergence theory of Fourier series an important role is played by those orthonormal systems which are uniformly bounded on a given compact interval. As it is well-known, the trigonometric system and the system of the Jacobi polynomials have this property. These results can be easily deduced from the following, more general theorem of V. A. Il'IN and L. Joó ([2], [4]):

Let $[a, b] \subset \mathbb{R}$ be an arbitrary compact interval and $q : [a, b] \rightarrow \mathbb{C}$ an arbitrary Lebesgue integrable function. There exists then a positive constant C_1 , depending only on $(b-a)$ and $\|q\|_1$ such that given any eigenfunction $u : [a, b] \rightarrow \mathbb{C}$ of the Schrödinger operator

$$Lu = u'' + q u$$

with the eigenvalue $\lambda \in \mathbb{C}$, for any number $1 \leq p < \infty$,

$$\|u\|_\omega \leq C_2(1 + |\operatorname{Re} \bar{\lambda}|)^{1/p} \|u\|_p.$$

This result is exact from the point of view of dependence on λ : the author of the present paper has proved in [5] that under the same conditions the estimate

$$\|u\|_\omega \geq C_2(1 + |\operatorname{Re} \bar{\lambda}|)^{1/p} \|u\|_p$$

holds, too, for a suitable positive constant C_2 , depending only on $(b-a)$ and $\|q\|_1$.

We mention that both results are proved for eigenfunctions of higher order, too ([4], [5]).

The aim of the present paper is to extend the theorem of V. A. Il'IN and L. Joó for the eigenfunctions of an arbitrary linear differential operator. We shall prove the following result.

THEOREM. *Let $G = [a, b] \subset \mathbb{R}$ be an arbitrary compact interval and $p_1, p_2, \dots, p_n : G \rightarrow \mathbb{C}$ arbitrary Lebesgue integrable functions. There exists then a*

positive constant C_n , depending only on n such that given any eigenfunction $u : G \rightarrow \mathbb{C}$ of the operator

$$Lu := u^{(n)} + p_2 \cdot u^{(n-2)} + \dots + p_n \cdot u$$

with the eigenvalue $\lambda \in \mathbb{C}$, for any integer $0 \leq i \leq n$ and for any number $1 < \rho < \infty$, we have

$$\|u^{(i)}\|_{\rho} \leq C_n R^{i+1} \rho^i \|u\|_{\rho},$$

where

$$R = \max \left(\sqrt[n]{|\lambda|} + \frac{1}{b-a} \cdot |p_{n-1}|, \sqrt[n-1]{|p_3|}, \dots, \sqrt[n-i+1]{|p_n|} \right).$$

We recall that a function $u : G \rightarrow \mathbb{C}$ is called an eigenfunction of L with the eigenvalue λ if u , together with its first $n-1$ derivatives is absolutely continuous on G and if for almost every $x \in G$, the equation

$$u^{(n)}(x) + p_2(x) u^{(n-2)}(x) + \dots + p_n(x) u(x) = \lambda u(x)$$

holds.

The theorem of V. A. LEVIN and L. JÓÓ is the case $n = 2$, $i = 0$ of this theorem (it turns out during the proof that in this case, $\sqrt[2]{\lambda}$ can be replaced by $\operatorname{Re} \sqrt[2]{\lambda}$).

The proof of the estimates for the Schrödinger operator was based on the mean value formulas of TRICUMARSH and MOISEEV. These formulas were extended for an arbitrary linear differential operator of order n (see [3]), but the result is rather complicated and its application to prove our theorem seems to be hard. We choose another way: using instead of the mean value formula the method of variation of constants, we trace the problem back to a question in determinant theory. As we shall show in another publication, this approach also works in the case of eigenfunctions of higher order.

For the brevity, let us introduce the notations

$$M(u, t) := (Lu)(t) - u^{(n)}(t) \quad (t \in G),$$

$$r(x) = \begin{cases} u(x) + \int_a^x \sum_{q=1}^n \frac{\varrho_q}{n \cdot \lambda} e^{\varrho_q(x-\tau)} M(u, \tau) d\tau & \text{if } \lambda \neq 0, \\ u(x) + \int_a^x \frac{(x-\tau)^{n-1}}{(n-1)!} M(u, \tau) d\tau & \text{if } \lambda = 0, \end{cases}$$

where u is an eigenfunction of L with the eigenvalue λ , and $\varrho_1, \varrho_2, \dots, \varrho_n$ are the different n -th roots of λ if $\lambda \neq 0$.

In the sequel, let u denote an arbitrary eigenfunction of L with the eigenvalue λ .

LEMMA 1. There exist complex numbers a_1, \dots, a_n such that

$$v^{(i)}(x) = \begin{cases} \sum_{p=1}^n a_p q_p^i e^{apx} & \text{if } \lambda \neq 0, \\ \sum_{p=i+1}^n a_p \frac{(p-1)!}{(p-i-1)!} x^{p-i-1} & \text{if } \lambda = 0, \end{cases}$$

$0 \leq i \leq n$, $x \in G$.

PROOF. It's enough to verify the formula for $i = 0$, the others follow from it by repeated derivation.

In case $\lambda \neq 0$ we have by the definition of $v(x)$ and $M(n, t)$,

$$v(x) = u(x) + \int_a^x \sum_{q=1}^n \frac{q_q}{n\lambda} e^{aq(x-\tau)} (\lambda u(\tau) + u^{(n)}(\tau)) d\tau.$$

One can easily see, integrating by parts n times that

$$\begin{aligned} \int_a^x \sum_{q=1}^n \frac{q_q}{n\lambda} \cdot e^{aq(x-\tau)} u^{(n)}(\tau) d\tau &= u(x) - \\ &- \sum_{k=1}^n \sum_{q=1}^n \frac{q_q^k}{n\lambda} e^{aq(x-a)} u^{(n-k)}(a) + \int_a^x \sum_{q=1}^n \frac{q_q^{1-n}}{n\lambda} e^{aq(x-\tau)} u(\tau) d\tau; \end{aligned}$$

hence our assertion follows at once ($q_q^n = \lambda$) with the numbers

$$a_p = \sum_{k=1}^n \frac{q_q^k}{n\lambda} \cdot e^{-ap} u^{(n-k)}(a), \quad 1 \leq p \leq n.$$

In case $\lambda = 0$ similarly

$$v(x) = u(x) - \int_a^x \frac{(x-\tau)^{n-1}}{(n-1)!} u^{(n)}(\tau) d\tau,$$

$$\int_a^x \frac{(x-\tau)^{n-1}}{(n-1)!} u^{(n)}(\tau) d\tau = u(x) - \sum_{k=1}^n \frac{(x-a)^{n-k}}{(n-k)!} = u^{(n-k)}(a),$$

whence the assertion follows with

$$a_p = \sum_{k=1}^{n+1-p} \binom{n-k}{p-1} \frac{1}{(n-k)!} (-a)^{n+1-p-k} u^{(n-k)}(a), \quad 1 \leq p \leq n. \blacksquare$$

LEMMA 2. For any $0 < i < n$ and $x \in G$,

$$v^{(i)}(x) = \begin{cases} u^{(i)}(x) + \int_a^x \sum_{q=1}^n \frac{g_q^{1+i}}{n\lambda} e^{g_q(x-\tau)} \cdot M(u, \tau) d\tau & \text{if } \lambda \neq 0, \\ u^{(i)}(x) + \int_a^x \frac{(x-\tau)^{n-1-i}}{(n-1-i)!} M(u, \tau) & \text{if } \lambda = 0. \end{cases}$$

PROOF. For $i = 0$ it is just the definition of $v(x)$. Suppose the formula is true for some $0 \leq i - 1 < n - 1$, then taking the derivative of both sides, we obtain

$$v^{(i)}(x) = \begin{cases} u^{(i)}(x) + \int_a^x \sum_{q=1}^n \frac{g_q^{1+i}}{n\lambda} e^{g_q(x-\tau)} \cdot M(u, \tau) d\tau + \\ + \sum_{q=1}^n \frac{g_q^i}{n\lambda} M(u, x) & \text{if } \lambda \neq 0, \\ u^{(i)}(x) + \int_a^x \frac{(x-\tau)^{n-i}}{(n-1-i)!} M(u, \tau) d\tau + \\ + \frac{(x-x)^{n-i}}{(n-i)!} M(u, x) & \text{if } \lambda = 0. \end{cases}$$

Here the last terms vanish due to $\sum_{q=1}^n g_q^i = 0$ and $i < n$, and we obtain the required formula for i . ■

Let $D(t)$ denote the determinant

$$\begin{array}{cc} e^{gt} & \dots & e^{gn} \\ e^{2gt} & \dots & e^{2gn} \\ \vdots & \vdots & \vdots \\ e^{ngt} & \dots & e^{ngn} \end{array} \quad \text{resp.} \quad \begin{array}{c} 1 & t & \dots & t^{n-1} \\ 1 & 2t & \dots & (2t)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & nt & \dots & (nt)^{n-1} \end{array}.$$

if $\lambda \neq 0$ resp. $\lambda = 0$ ($t \in \mathbb{R}$). Obviously $D(t) \neq 0$ for $t \neq 0$. Let $D_{kp}(t)$ denote the minor of $D(t)$, corresponding to the p -th element of the k -th row.

LEMMA 3. For any $t \neq 0$, $0 < i < n$ and $x, x+nt \in G$,

$$v^{(i)}(x) = \begin{cases} \sum_{k=1}^n \left(\sum_p^n g_q^i \frac{D_{kp}(t)}{D(t)} \right) v(x+kt) & \text{if } \lambda \neq 0, \\ \sum_{k=1}^n \frac{D_{k, i+1}(t)}{D(t)} v(x+kt) & \text{if } \lambda = 0. \end{cases}$$

PROOF. Consider first the case $\lambda \neq 0$. We have by Lemma 1

$$v(x+kt) = \sum_{p=1}^n (a_p e^{ip} p^{ht}), \quad 1 \leq k \leq n,$$

$$v^{(i)}(x) = \sum_{p=1}^n \varrho_q^i (a_p e^{ip} p^x).$$

From the first n equations we obtain

$$a_p e^{ip} p^x = \sum_{k=1}^n \frac{D_{kp}(t)}{D(t)} V(x+kt), \quad 1 \leq p \leq n.$$

Substituting these expressions into the last equation we arrive at the required formula.

In case $\lambda = 0$ analogously

$$v(x+kt) = \sum_{p=1}^n a_p (x+kt)^{p-1} = \sum_{q=1}^n (kt)^{q-1} \left(\sum_{p=q}^n a_p \binom{p-1}{p-1} x^{p-q} \right), \quad 1 \leq k \leq n,$$

$$v^{(i)}(x) = \sum_{p=i+1}^n a_p \frac{(p-1)!}{(p-i-1)!} x^{p-i-1} + i! \sum_{p=i+1}^n a_p \binom{p-1}{i} x^{p-i-1},$$

whence the required formula follows at once. ■

PROPOSITION 1. For any $t \neq 0$, $0 \leq i \leq n$ and $x, x+nt \in G$,

$$u^{(i)}(x) = \begin{cases} \sum_{k=1}^n \left(\sum_{p=1}^n \frac{\varrho_p^i D_{kp}(t)}{D(t)} \right) u(x+kt) + \sum_{k=1}^n \left(\sum_{p=1}^n \varrho_p^i \frac{D_{kp}(t)}{D(t)} \right) \\ \cdot \int_x^{x+kt} \sum_{q=1}^n \frac{\varrho_q}{n\lambda} e^{\varrho_q(x+kt-\tau)} M(u, \tau) d\tau, \quad \lambda \neq 0, \\ \sum_{k=1}^n \frac{D_{k, i-1}(t)}{D(t)} u(x+kt) + \sum_{k=1}^n \frac{D_{k, i-1}(t)}{D(t)}, \\ \cdot \int_x^{x+kt} \frac{(x+kt-\tau)^{n-1}}{(n-1)!} M(u, \tau) d\tau, \quad \lambda = 0. \end{cases}$$

PROOF. It suffices to show by Lemmas 2 and 3 that

$$\int_u^x \sum_{q=1}^n \frac{\varrho_q^{1+i}}{n\lambda} e^{\varrho_q(x-\tau)} M(u, \tau) dt =$$

$$= \sum_{k=1}^n \left(\sum_{p=1}^n \varrho_p^i \frac{D_{kp}(t)}{D(t)} \right) \int_u^x \sum_{q=1}^n \frac{\varrho_q}{n\lambda} e^{\varrho_q(x+kt-\tau)} M(u, \tau) d\tau \text{ if } \lambda \neq 0,$$

$$\begin{aligned} & \int_u^x \frac{(x-\tau)^{n-1-i}}{(n-1-i)!} M(u, \tau) d\tau = \\ & = \sum_{k=1}^n i! \frac{D_{k, i+1}(t)}{(n-1)!} \int_u^x \frac{(x+kt-\tau)^{n-1}}{(n-1)!} M(u, \tau) d\tau \text{ if } \lambda = 0. \end{aligned}$$

These equations follow from the relations

$$\begin{aligned} \varrho_q^{1-i} &= \sum_{k=1}^n \left(\sum_{p=1}^n \varrho_p^i \frac{D_{kp}(t)}{D(t)} \right) \varrho_q e^{qkt} \quad (1 \leq q \leq n) \text{ if } \lambda \neq 0, \\ \frac{(x-\tau)^{n-1-i}}{(n-1-i)!} &= \sum_{k=1}^n i! \frac{D_{k, i+1}(t)}{D(t)} \cdot \frac{(x+kt-\tau)^{n-1}}{(n-1)!} \quad \text{if } \lambda = 0, \end{aligned}$$

or from the relations

$$\begin{aligned} \sum_{k=1}^n e^{qkt} \frac{D_{kp}(t)}{D(t)} &= \delta_{pq} \quad (1 \leq p, q \leq n) \text{ if } \lambda \neq 0, \\ \sum_{k=1}^n (kt)^j \frac{D_{k, i+1}(t)}{D(t)} &= \delta_{ij} \quad (0 \leq j \leq n) \text{ if } \lambda = 0. \end{aligned}$$

But these equations are direct consequences of the definition of $D(t)$ (ordinary and skew expansion of the determinant). ■

We shall need in the sequel some special properties of $D(t)$. For this, introduce the notations

$$D(x_1, \dots, x_n) = V(x_1, \dots, x_n) = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix},$$

$$V^p(x_1, \dots, x_n) = H\{x_j - x_i : 1 \leq i < j \leq n, i \neq p \neq j\},$$

and denote by $S_m(x_1, \dots, x_n)$ (resp. $S_m^p(x_1, \dots, x_n)$) the m -th elementary symmetric polynomial of the variables x_1, \dots, x_n (resp. $x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n$), with the coefficient 1.

LEMMA 4. For the minor $D_{kp} = D_{kp}(x_1, \dots, x_n)$ of $D(x_1, \dots, x_n)$ corresponding to the p -th element of the k -th row, we have

$$D_{kp} = (-1)^{k+p} S_{n-1}^p(x_1, \dots, x_n) V^p(x_1, \dots, x_n) S_{n-k}^p(x_1, \dots, x_n).$$

PROOF. Introducing the new variables $y_1 = x_1, \dots, y_{p-1} = x_{p-1}, y_p = x_{p+1}, \dots, y_{n-1} = x_n$, D_{kp} is a homogeneous polynomial of the variables y_1, \dots, y_{n-1} . For any $1 \leq i \leq n$, $y_i \neq 0$ and for any $1 \leq i < j \leq n$, $y_i = y_j$ the

determinant D_{kp} vanishes, there exists therefore a homogeneous polynomial P of the variables y_1, \dots, y_{n-1} such that

$$D_{kp} = S_{n-1}(y_1, \dots, y_{n-1}) V(y_1, \dots, y_{n-1}) P(y_1, \dots, y_{n-1}).$$

For any $1 < i < j < n$, interchanging y_i and y_j , D_{kp} and $V(y_1, \dots, y_{n-1})$ change their sign but not their absolute value, while $S_{n-1}(y_1, \dots, y_{n-1})$ does not change. Consequently the polynomial P is symmetric. The main term of P with respect to the antilexicographic order is

$$\frac{(-1)^{k+p} y_1 y_2^2 \cdots y_{k-1}^{k-1} y_k^{k-1} \cdots y_{n-1}^n}{y_1 y_2^2 \cdots y_{k-1}^{k-1} y_k^k \cdots y_{n-1}^{n-1}} = (-1)^{k+p} y_k y_{k+1} \cdots y_{n-1}.$$

It follows from here by the basic theorem of symmetric polynomials that

$$P(y_1, \dots, y_{n-1}) = (-1)^{k+p} S_{n-k}(y_1, \dots, y_{n-1}),$$

and hence

$$D_{kp} = (-1)^{k+p} S_{n-1}(y_1, \dots, y_{n-1}) V(y_1, \dots, y_{n-1}) S_{n-k}(y_1, \dots, y_{n-1});$$

but this is equivalent to the assertion of the Lemma. ■

LEMMA 5. For any $0 < i < n$ and $1 \leq k \leq n$,

$$\sum_{p=1}^n \varrho_p^i \frac{D_{kp}(t)}{D(t)} = O(t^{-i}) \quad (t \sqrt{|\lambda|} \rightarrow 0) \text{ if } \lambda \neq 0,$$

$$\text{if } \frac{D_{k,i+1}(t)}{D(t)} = Ct^{-i} \text{ for some constant } C \text{ if } \lambda = 0.$$

PROOF. The case $\lambda = 0$ is obvious: $D(t)$ is a constant multiple of

$$t^{1+2+\cdots+(n-1)} = t^{\frac{(n-1)n}{2}},$$

and $D_{k,i+1}(t)$ is a constant multiple of

$$t^{1+2+\cdots+(i-3)+\cdots+(n-1)} = t^{\frac{(n-1)n}{2}-i},$$

In case $\lambda \neq 0$ we show that the multiplicity of the root 0 in

$$\sum_{p=1}^n (\varrho_p t)^i D_{kp}(t)$$

is greater or equal to its multiplicity in $D(t)$ (both functions are holomorphic in t). Introducing the notation $x_p = \varrho_p t$, it suffices to show by Lemma 4 that the function

$$\begin{aligned} f(x_1, \dots, x_n) &\equiv \sum_{p=1}^n x_p^i (-1)^{k+p} S_{n-1}^p \left(\sum_{l=0}^{\infty} \frac{x_1^l}{l!}, \dots, \sum_{l=0}^{\infty} \frac{x_n^l}{l!} \right) \times \\ &\times V^p \left(\sum_{l=0}^{\infty} \frac{x_1^l}{l!}, \dots, \sum_{l=0}^{\infty} \frac{x_n^l}{l!} \right) S_{n-k}^p \left(\sum_{l=0}^{\infty} \frac{x_1^l}{l!}, \dots, \sum_{l=0}^{\infty} \frac{x_n^l}{l!} \right) \end{aligned}$$

vanishes, whenever $x_i = x_j$ for some $1 \leq i < j \leq n$. And this is in fact true: in the above sum the addends, corresponding to $p \neq i, j$, vanish (because $V^p(\cdot) = 0$), while the addends, corresponding to $p = i, j$ are eliminated by each other because $S_{n-1}^j(\cdot) = S_{n-1}^i(\cdot)$, $S_{n-k}^j(\cdot) = S_{n-k}^i(\cdot)$, $V^j(\cdot) = (-1)^{j-i-1} V^i(\cdot)$, $(-1)^{k+j} = (-1)^{j-i} (-1)^{k+i}$. ■

LEMMA 6. We have

$$\sum_{q=1}^n \frac{\varrho_q}{n\lambda} e^{\varrho_q t} = O(t^{n-1}) \quad (t \sqrt{|\lambda|} \rightarrow 0) \quad \text{if } \lambda \neq 0.$$

PROOF.

$$\begin{aligned} \sum_{q=1}^n \frac{\varrho_q}{n\lambda} e^{\varrho_q t} &= \sum_{q=1}^n \frac{\varrho_q}{n\lambda} \sum_{i=0}^{\infty} \frac{(\varrho_q t)^i}{i!} = \sum_{i=0}^{\infty} \frac{t^i}{n\lambda i!} \sum_{q=1}^n \varrho_q^{1+i} - \\ &= \sum_{j=1}^{\infty} \frac{t^{nj-1}}{n\lambda(nj-1)!} + n\lambda j t^{nj-1} \sum_{j=1}^{\infty} \frac{(tn\lambda)^{j-1}}{(nj-1)!}, \end{aligned}$$

and hence

$$\sum_{q=1}^n \frac{\varrho_q}{n\lambda} e^{\varrho_q t} \leq |t|^{n-1} \sum_{j=1}^{\infty} \frac{|t| \sqrt{|\lambda|}^{-nj+1}}{(nj-1)!} = |t|^{n-1} e^{|t| \sqrt{n|\lambda|}}. \quad \blacksquare$$

PROPOSITION 2. There exist positive numbers C_1, C_2, ε , depending only on n , such that for all $0 < i < n$, $0 < |t| < \frac{\varepsilon}{n\sqrt{|\lambda|}}$ ($\frac{\varepsilon}{n} \equiv +\infty$) and $x, x+nt \in G$,

$$|u^{(k)}(x)| \leq C_1 |t|^{-i} \sum_{k=1}^n |u(x+kt)| + C_2 |t|^{n-1-i} \sum_{k=1}^{n-2} \|p_{n-k}\|_1 \|tu^{(k)}\|_\infty.$$

PROOF. It follows directly from Lemmas 4, 5 and Proposition 1, taking into account that in the formulas of Proposition 1,

$$|x+kt-\tau| \leq n |t| \quad \text{for all } x-k|t| \leq \tau \leq x+k|t|. \quad \blacksquare$$

Let us turn now to the proof of the theorem. Proposition 2 can be applied for all

$$a \leq x \leq \frac{a+b}{2}, \quad 0 < t < \min \left(\frac{\varepsilon}{n\sqrt{|\lambda|}}, \frac{b-a}{2n} \right)$$

and for all

$$\frac{a+b}{2} \leq x \leq b, \quad 0 < -t < \min \left(\frac{\varepsilon}{n\sqrt{|\lambda|}}, \frac{b-a}{2n} \right).$$

We have therefore for all $0 \leq i < n$, $x \in G$ and $0 < |t| < \min \left(\frac{\varepsilon}{n|\lambda|}, \frac{b-a}{2n} \right)$,

$$(*) \quad |t|^i |u^{(i)}(x)| \leq C_1 \sum_{k=1}^n |u(x+kt)| + C_2 |t|^{n-1} \sum_{k=0}^{n-2} \|p_{n-k}\|_1 \|u^{(k)}\|_\infty.$$

Consider first the case $p = +\infty$, then we obtain from $(*)$

$$|t|^i \|u^{(i)}\|_\infty \leq n C_1 \|u\|_\infty + C_2 |t|^{n-1} \sum_{k=0}^{n-2} \|p_{n-k}\|_1 \|u^{(k)}\|_\infty,$$

for all $0 \leq i < n$, $0 < |t| < \min \left(\frac{\varepsilon}{n|\lambda|}, \frac{b-a}{2n} \right)$.

Introducing the notation $M \equiv \max \{|t|^i \|u^{(i)}\|_\infty : 0 \leq i < n\}$, we obtain from this

$$M \leq n C_1 \|u\|_\infty + C_2 \sum_{k=0}^{n-2} |t|^{n-k-1} \|p_{n-k}\|_1 M.$$

Choosing $|t|$ so as to satisfy over and above

$$0 < |t| < \min \left(\frac{\varepsilon}{n|\lambda|}, \frac{b-a}{2n} \right)$$

also the conditions

$$|t|^{k-1} \|p_k\|_1 \leq \frac{1}{2n C_2}, \quad k = 2, 3, \dots, n,$$

we obtain

$$M \leq n C_1 \|u\|_\infty + \frac{1}{2} M,$$

whence

$$M \leq 2n C_1 \|u\|_\infty,$$

$$\|u^{(i)}\|_\infty \leq 2n C_1 |t|^{-i} \|u\|_\infty, \quad i = 0, 1, \dots, n-1.$$

The part $p = +\infty$ of the theorem hence follows at once.

Consider now the case $1 \leq p < +\infty$. Applying to both sides of $(*)$ the operation

$$\frac{1}{t} \int_0^t \cdot dt,$$

we obtain

$$(i+1)^{-1} |t|^i \|u^{(i)}(x)\| \leq C_1 t^{-1} \sum_{k=1}^n \int_0^t |u(x+kt)| dt + \\ + C_2 n^{-1} |t|^{n-1} \sum_{k=0}^{n-2} \|p_{n-k}\|_1 \|u^{(k)}\|_\infty$$

for all $0 \leq i < n$, $x \in G$ and $0 < |t| < \min \left(\frac{\varepsilon}{\sqrt{|\lambda|}}, \frac{b-a}{2n} \right)$.

Using the Hölder inequality,

$$t^{-1} \int_0^t |u(x+kt)| dt = t^{-1} \int_0^{kt} |u(x+\tau)| d\tau k^{-1} \leq |kt|^{-1} \|u\|_p \|kt\|^{1-1/p} \leq \\ \leq |t|^{-1/p} \|u\|_p,$$

and therefore

$$(i+1)^{-1} |t|^i \|u^{(i)}(x)\| \leq nC_1 |t|^{-1/p} \|u\|_p + \\ + C_2 n^{-1} \sum_{k=0}^{n-2} |t|^{n-k-1} \|p_{n-k}\|_1 \|t\|^k \|u^{(k)}\|_\infty,$$

whence – introducing again the notation $M = \max \{ |t|^i \|u^{(i)}\|_\infty : 0 \leq i < n \} =$

$$M + n^2 C_1 |t|^{-1/p} \|u\|_p + C_2 \sum_{k=2}^n |t|^{k-1} \|p_k\|_1 M,$$

Choosing $|t|$ so as to satisfy not only the condition

$$0 < |t| < \min \left(\frac{\varepsilon}{\sqrt{|\lambda|}}, \frac{b-a}{2n} \right),$$

but also the conditions

$$|t|^{k-1} \|p_k\|_1 \leq \frac{1}{2nC_2}, \quad k = 2, 3, \dots, n,$$

we obtain

$$M + n^2 C_1 |t|^{-1/p} \|u\|_p + \frac{1}{2} M,$$

$$M \leq 2n^2 C_1 |t|^{-1/p} \|u\|_p,$$

and finally

$$\|u^{(i)}\|_\infty \leq 2n^2 C_1 |t|^{-i-1/p} \|u\|_p, \quad i = 0, 1, \dots, n-1.$$

The part $p < +\infty$ of the theorem hence follows at once, and the whole theorem is proved.

REMARK. It can be shown easily that in case $n = 2, i = 0$ we can write in Lemmas 5 and 6 $|\operatorname{Re} \sqrt[4]{\lambda}|$ instead of $|\sqrt[4]{\lambda}|$. Thus the theorem of Il'IN and Joó, mentioned at the beginning of this paper, is a special case of our theorem indeed.

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A CHARACTERIZATION OF INFINITELY DIVISIBLE MARKOV CHAINS WITH FINITE STATE SPACE

By

F. GÖNDÖCS, G. MICHALETZKY, T. F. MÓRI and G. J. SZÉKELY

Department of Probability Theory, L. Eötvös University, Budapest

(Received April 7, 1982)

1. Introduction. The Lévy–Hinčin formula characterizes the infinitely divisible probability distributions if the operation is the convolution. We give a similar characterization for the infinitely divisible Markov chains with finite state space.

DEFINITION 1. A Markov chain and its transition matrix P is infinitely divisible if for any natural number n there exists a transition matrix $P\left(\begin{array}{c|c} 1 & \\ \hline & n \end{array}\right)$ such that P is the n -th power of $P\left(\begin{array}{c|c} 1 & \\ \hline & n \end{array}\right)$.

DEFINITION 2. A transition matrix P is pseudo-Poissonian if

$$P = e^{-\lambda} \left[H_0 + \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \cdot H^i \right],$$

where $\lambda > 0$, H_0 and H are transition matrices, H_0 is idempotent and

$$H_0 H = H H_0 = H.$$

It is obvious that a pseudo-Poissonian P is infinitely divisible. In this paper we prove the converse statement for the case of finite state space.

REMARKS 1. The problem of embedding of infinitely divisible Markov chains is as old as the papers [2] and [3] of ELEVING. Further results are contained e.g. in [5]–[16]. These papers show that our theorem is well-known if we suppose the nonsingularity of P (in this case, obviously, H_0 is the identity matrix). On the other hand it was mentioned even by CHUNG ([4] II, 1) that if P can be embedded in a time-continuous (measurable) Markov chain, then a reduction of the state space always leads to a Markov chain the transition matrix of which is nonsingular. Thus the main point of our paper can be summarized as follows: we have proved that, even in the singular case, every infinitely divisible Markov chain is embeddable into a time-continuous one.

2. In the terminology of [1] we obtain the following result: If $P(t)_{t \geq 0}$ is a measurable semigroup then $P(t) = P(0)Q(t) = Q(t)P(0)$, where $P(0) = -\lim_{t \rightarrow 1^0} P(t)$ is idempotent and $Q(t)$ is a standard semigroup.

3. FELLER shows ([4] p. 342) that every continuous semigroup of contractions is the limit of pseudo-Poissonian semigroups. In our case the main problem is just the embedding of the infinitely divisible P into a continuous semigroup. In general a direct embedding is impossible but we can reduce the state space and modify the Markov chain so that this new Markov chain can be embedded into a continuous semigroup of transition matrices.

2. Some preliminary results. Denote by ID the set of infinitely divisible transition matrices with fixed finite state space. First observe that $P^k \in ID$ (for some natural number $k \geq 1$) does not imply that $P \in ID$ (e.g. if P is a permutation matrix with negative determinant, then P does not have a square root but there exists a k such that $P^k = I$ (identity) which is infinitely divisible.) On the other hand we prove

LEMMA 1. If $P \in ID$ then for any natural number k there exists a $P\left(\frac{1}{k}\right) \in ID$ such that $P\left(\frac{1}{k}\right)^k = P$, i.e. for any k $P \in ID$ always has an infinitely divisible k -th root.

PROOF. Let k be an arbitrary fixed natural number and denote

$$Q_s = \{Q^m : Q \text{ is a transition matrix, } Q^{km} = P \text{ and } s|m\}.$$

The filter base $Q = \{Q_1, Q_2, \dots\}$ has an accumulation point A (by the compactness) which is a transition matrix such that $A^k = P$, i.e. A is a k -th root of P . We show that this root is infinitely divisible. Take the common refinement of Q and the neighbourhood base of A . In this refinement the s -th roots of the elements of Q_s also constitute a net with an accumulation point B_s such that $B_s^s = A$ (by continuity of matrix multiplication); this holds for every natural number s , thus $A \in ID$.

In the following the roots will always be supposed to be infinitely divisible.

If the state space has n elements then the transition matrix P transforms the n -dimensional space to some $k (\leq n)$ -dimensional space.

LEMMA 2. If $P \in ID$ then P is regular on the k -dimensional space, i.e. P is regular restricted to its image space.

PROOF. Denote by $P\left(\frac{1}{n}\right) \in ID$ an n -th root of P . The image spaces of $P\left(\frac{1}{n}\right)^k$ for $k = 0, 1, 2, \dots, n$ form a decreasing sequence of sets. In this sequence the first space is n -dimensional and the last one (the $(n+1)$ -th one) is at least 1-dimensional thus at least two of these spaces are equal i.e. $P\left(\frac{1}{n}\right)$ is regular

on this space, consequently it is regular on the image space of P , thus P is also regular on the same space-what was to be proved.

The proof shows that the image of $P \in ID$ is the same as the image of $P\left(\frac{1}{n}\right) \in ID$ for any natural number n . Denote this image space by IM and denote by ' the restrictions of the transition matrices to this image space.

Let $P(1) = P$. An infinitely divisible $(k+1)$ -th root of $P\left(\frac{1}{k!}\right)$ will be denoted by $P\left(\frac{1}{(k+1)!}\right)$ $k = 1, 2, \dots$ and the powers $P\left(\frac{1}{k!}\right)^m$ by $P\left(\frac{m}{k!}\right)$. Thus the transition matrix $P(r)$ is defined for any positive rational r with the property:

$$(*) \quad P(r+s) = P(r)P(s)$$

for any positive rationals r and s .

LEMMA 3. The above defined restrictions $P(r)'$ of $P(r)$ have the following continuity property:

$$\lim_{r \rightarrow +0} P(r)' = I' \text{ (=: identity).}$$

PROOF. The matrix $P(r)'$ is regular thus $\det P(r)' \neq 0$. The infinite divisibility implies that $\det P(r)' > 0$. By the semigroup property $(*)$ $\det P(r)' = (\det P(1)')^r \rightarrow 1$ if $r \rightarrow +0$. This relation will be used below. The transition matrices $P(r)$ transform the simplex

$$I = \left\{ (x_1, x_2, \dots, x_n) : x_i \geq 0 \text{ and } \sum_{i=1}^n x_i = 1 \right\}$$

into itself. Denote $I' = \bigcup_{i=1}^n IA_i$. This is a convex compact set with at most 2^n vertices. (The number of different subsimplices of I is 2^n and it is obvious that for different vertices of I' the minimal containing subsimplices are also different.) This property will be also used below.

By compactness argument $P(r)'$ has an accumulation point if $r \rightarrow +0$ thus it is enough to show that only the identity can be an accumulation point. Denote by A an accumulation point of $P(r)'$ and let $r_m \rightarrow +0$ be a sequence such that $P(r_m)'' \rightarrow A$. By the continuity of the determinant $\det A = 1$ thus $A' = I' \cap A$ by definition and the Lebesgue measure of A' equals that of I' i.e. A is a permutation of the vertices of I' . Now let $N = 2^n!$ and denote by A^* an accumulation point of $P(r_m/N)$. The same argument as above shows that A^* is also a permutation of the vertices of I' . But the order of the permutation A^* is a divisor of N thus $I = (A^*)^N = A$ what was to be proved.

LEMMA 4. $\lim_{r \rightarrow +0} P(r) = H_0$ exists and $P(r)H_0 = H_0 P(r) = P(r)$; this latter equation implies that $H_0^2 = H_0$ i.e. H_0 is idempotent.

PROOF. If $0 < r < s$ are small rational numbers then $P(s-r)$ is close to I thus $P(r)$ and $P(s) = P(s-r)P(r)$ are close to each other thus by the Cauchy criterion $\lim_{r \rightarrow 0} P(r) = H_0$ exists. The matrix H_0 is obviously a transition matrix and $P(r)H_0 = H_0P(r) = P(r)$ is also obvious.

3. The main result.

THEOREM. *If P is an infinitely divisible transition matrix with finitely many states then P is pseudo-Poissonian, i.e.*

$$P = e^{-\lambda} \left[H_0 + \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} H^i \right]$$

where $\lambda > 0$, H_0 and H are transition matrices, H_0 is idempotent and $H_0H = HH_0 = H$.

PROOF. Consider a vertex V of Γ and the smallest subsimplex of Γ which contains V . Since H_0 is idempotent the H_0 -image of this subsimplex is V . The minimal subsimplices containing different vertices of Γ are disjoint thus the „vertex vectors” of Γ are independent. These „vertex vectors” form a basis B of the H_0 -image space. The elements of this basis can be considered as „mixed states”. These „mixed states” constitute a new state space $\{1, 2, \dots, k\}$.

Since $P(r) \subset I$ we get $P(r)'H_0 \subset H_0I$. In the above mentioned basis B of the H_0 -image space, $P(r)'$ is a transition matrix since

$$H_0I = \left\{ (y_1, y_2, \dots, y_k) : y_i \geq 0 \text{ and } \sum_{i=1}^k y_i = 1 \right\}.$$

By a well-known theorem of Markov processes (see e.g. [17] p. 42)

$$\lim_{r \rightarrow 0} \frac{P(r)' - I}{r} \approx A$$

exists and $P' = e^A$. In the basis B the elements a_{ij} of $A = (a_{ij})$ are nonnegative if $i \neq j$ and $\sum_{j=1}^k a_{ij} > 0$, $i = 1, 2, \dots, k$, thus if λ is large enough then

$$H' = \frac{A}{\lambda} + I$$

is a transition matrix and $P' = e^{-\lambda} e^{\lambda H'}$. Now let $H := H' H_0$. Since H' is a transition matrix in the basis B therefore H is a transition matrix in the basis of R^n . Finally

$$P = P' H_0 e^{-\lambda} \left[H_0 + \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} H^i \right].$$

4. An open problem. Characterize the infinitely divisible Markov chains if the state space is not necessarily finite. If the state space is not finite then simple examples show that not every divisible transition matrix is pseudo-Poissonian.

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ÜBER EINIGE OPTIMALE KONFIGURATIONEN VON KREISEN

Von

KÁROLY BEZDEK

Lehrstuhl für Geometrie der I. Eötvös Universität, Budapest

(Eingegangen am 16. April 1982)

Unser Thema ist mit der folgenden Frage der diskreten Geometrie verbunden. Wir nehmen endlich viele Kreise in der euklidischen Ebene, dann betrachten wir diejenige Kreise, die durch die vorgegebenen Kreise überdeckt werden können, nachdem wir fragen können: Wie groß ist der Radius von dem größten, überdeckten Kreis? Auf welche Weise kann der erwähnte Kreis überdeckt werden? (Unter Kreis verstehen wir immer eine geschlossene Kreisscheibe, und wir werden den Rand des Kreises Kreislinie heißen.)

Im Falle, daß drei (bzw. vier) Kreise gegeben sind, hat J. MOLNÁR [2] das Folgende gezeigt: Konstruieren wir ein Dreieck (bzw. eingeschriebenes Viereck) aus den Durchmessern der gegebenen Kreise! Wenn das Dreieck (bzw. Viereck) das Zentrum seines angeschriebenen Kreises in seinem Inneren enthält, liefert der erwähnte, eingeschriebene Kreis den größten, überdeckten Kreis. (Fig. 1.). Sonst, der gesuchte Kreis ist nicht größer als die gegebenen Kreise.

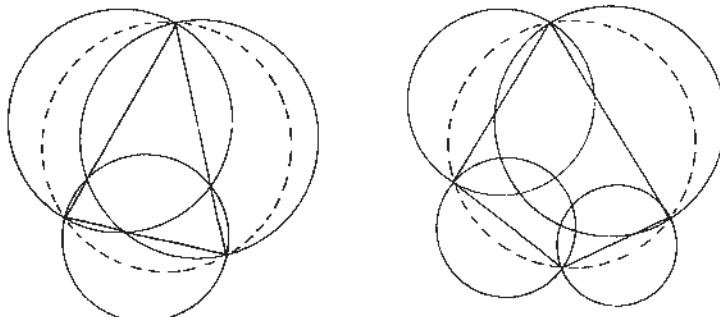


Fig. 1.

Von nun an beschäftigen wir uns mit dem Fall von kongruenten Kreisen. Zuerst erwähnen wir, daß der Fall von 7 kongruenten Kreisen trivial ist. Die beste Konfiguration wird durch die Figur 2. veranschaulicht.

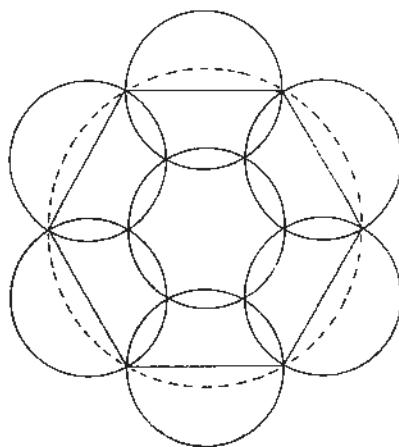


Fig. 2.

Unsere neuen Ergebnisse lauten folgendermaßen:

SATZ. Betrachten wir in der euklidischen Ebene solche Kreise, die durch fünf Einheitskreise überdeckt werden können, so beträgt der Radius des größtmöglichen, überdeckten Kreises $1,640\dots$. Die Überdeckung wird durch die Figur 3. veranschaulicht. (Höchstens drei Einheitskreise haben einen gemeinsamen Punkt; M ist der gemeinsame Randpunkt von den Kreisen e_1, e_2, e_3 ; M liegt im Äußeren der Kreise $k_1 k_2$; $H_1 H_5 = H_1 H_2 < 2$; $H_3 H_4 < 2$; $H_4 H_5 = H_2 H_3 = 2$; natürlichweise die Punkte $H_1 H_2 H_3 H_4 H_5$ liegen am Rande des überdeckten Kreises und eine Spiegelung an der Geraden MH_1 führt das System der Kreise in sich über.)

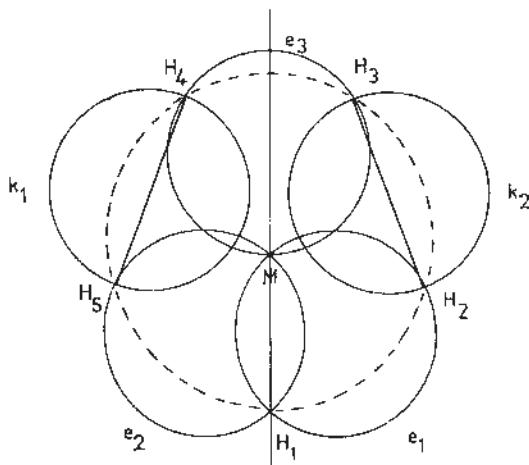


Fig. 3.

Wir haben jetzt zu bemerken: nach einigen Hinweisen der mathematischen Fachliteratur hatte E. H. NEVILLE das erwähnte Problem von fünf Einheitskreisen im Aufsatz [3] gelöst. In Wirklichkeit beschäftigt sich E. H. NEVILLE im erwähnten Aufsatz hauptsächlich mit einer neuen numerischen Methode, nachdem er sich ganz kurz auch mit dem Problem von fünf kongruenten Kreisen beschäftigt, die aber jeder mathematischen Grundlage entbehrt.

BEWEIS DES SATZES. (Wir skizzieren nur die wichtigsten Schritte vom Beweis.) Wir heißen die beste Lagerung der fünf Einheitskreise (die nach dem Satz von Weierstrass existiert) extreme Lagerung, und wir nennen den ihr gehörigen überdeckten Kreis extremen Kreis. Es kann vorausgesetzt werden, daß der Radius des extremen Kreises $> \sqrt{2}$ ist, sogar daß, der Teil der Kreislinie vom beliebigen Einheitskreis der extremen Lagerung, der durch den extremen Kreis nicht überdeckt ist, nicht länger als π ist. Nach diesen Voraussetzungen führen wir die geometrischen Eigenschaften der extremen Lagerung auf. (Wenn ein Hilfssatz mit Hilfe von den synthetischen Methoden leicht bewiesen werden kann, so verzichten wir hier auf die Durchführung des Beweises.)

Wir wählen einen Einheitskreis und betrachten jenen Teil seiner Kreislinie, dessen Punkte nicht zu den übrigen (vier) Einheitskreisen gehören.

HILFSSATZ 1. Dieser Teil ist entweder leer, oder ein einziger, offener Kreisbogen, der 2 Endpunkte hat.

DEFINITION. Man nennt den erwähnten, offenen Kreisbogen unbedeckten Kreisbogen des betreffenden Einheitskreises.

HILFSSATZ 2. Nehmen wir die Ergänzung eines unbedeckten Kreisbogens bezüglich der entsprechenden Kreislinie, so gehören ihre innenpunkte zum Inneren der extremen Lagerung.

KOROLLAR. Die Grenzlinie der extremen Lagerung besteht aus den unbedeckten Kreisbögen und aus ihren Endpunkten.

HILFSSATZ 3. In Jedem Endpunkt von den unbedeckten Kreisbögen sind genau zwei unbedeckte Kreisbögen zu finden.

HILFSSATZ 4. Die unbedeckten Kreisbögen sind nicht länger als π , und sie bilden – zusammen mit ihren Endpunkten – ein zusammenhängendes System. (Selbstverständlich die Anzahl der unbedeckten Kreisbögen ist gleich die Anzahl ihrer Endpunkte.)

DEFINITION. Zwei Einheitskreise der extremen Lagerung sind benachbart, wenn ihre unbedeckten Kreisbögen einen gemeinsamen Endpunkt haben. (Die benachbarten Kreise überschneiden sich.)

HILFSSATZ 5. Beliebiger unbedeckter Kreisbogen und der extreme Kreis haben keinen gemeinsamen Punkt, und die Kreislinie des extremen Kreises enthält wenigstens drei Endpunkte von den unbedeckten Kreisbögen.

HILFSSATZ 6. Beliebiger Einheitskreis der extremen Lagerung hat einen unbedeckten Kreisbogen.

KOROLLAR. Die Grenzlinie der extremen Lagerung besteht aus 5 unbedeckten Kreisbögen und aus ihren 5 Endpunkten.

HILFSSATZ 7. Unter allen unbedeckten Kreisbögen der extremen Lagerung gibt es wenigstens einen, der kürzer als π ist.

HILFSSATZ 8. Wir betrachten einen beliebigen Einheitskreis der extremen Lagerung. Nimmt man seine zwei Nachbarn, so haben sie entweder keinen gemeinsamen Punkt, oder berühren einander, wann der Berührungs punkt zum betrachteten Einheitskreis gehört, oder überschneiden sich, wann wenigstens einer unter den zwei Schnittpunkten zum Inneren des betrachteten Einheitskreises gehört.

DEFINITION. Einen der Punkte der euklidischen Ebene heißt man n -gradigen, wenn er durch genau n Einheitskreise der extremen Lagerung überdeckt wird ($0 \leq n \leq 5$).

HILFSSATZ 9. Es gibt keinen 5-gradigen Punkt.

Der Beweis des letzteren Hilfssatzes lautet folgendermaßen. Wenn es (wenigstens) einen 5-gradigen Punkt gibt, wählen wir (auf Grund des Hilfssatzes 7.) einen solchen Einheitskreis, dessen unbedeckter Kreisbogen kürzer als π ist. Dieser Einheitskreis kann wegen unserer Voraussetzung so verschoben werden, daß das neue System der 5 Einheitskreise die folgenden Eigen schaften hat: Die Grenzlinie des neuen Systems hat gleiche topologische Eigenschaften wie im Ausgangsfall; das neue System von 5 Einheitskreisen überdeckt den extremen Kreis; die Endpunkte des unbedeckten Kreisbo gens vom verschobenen Kreis liegen im Äußeren des extremen Kreises. So könnten wir aber den extremen Kreis vergrößern, was unmöglich ist.

HILFSSATZ 10. Es gibt keinen 4-gradigen Punkt.

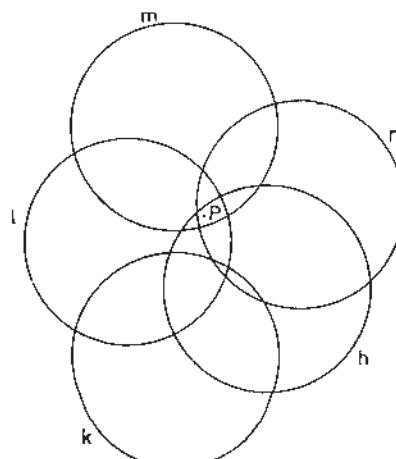


Fig. 4.

Jetzt werden wir den Beweis dieser Aussage in kurzen Umrissen darstellen: Wir nehmen an, daß der Punkt P zu den Einheitskreisen h, l, m, n gehört, und im Äußeren des Einheitskreises k liegt (Fig. 4.).

Mit Hilfe von Verschiebungen kann man nachweisen, daß die Kreise m , n , k unbedeckte Kreisbögen von π Länge haben, und P der gemeinsame Randpunkt der Kreise h , l , m , n ist (Fig. 5.).

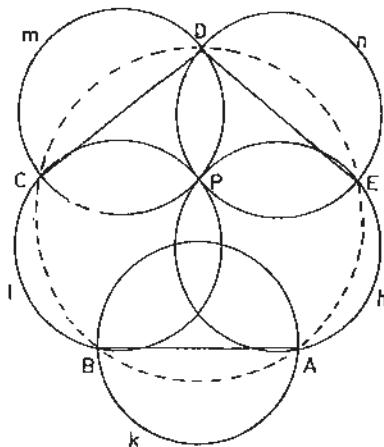


Fig. 5.

So liegen die Endpunkte ($A; B; C; D; E$) der unbedeckten Kreisbögen am Rande des extremen Kreises. Das Zentrum des Kreises k liegt auf der Geraden DP .

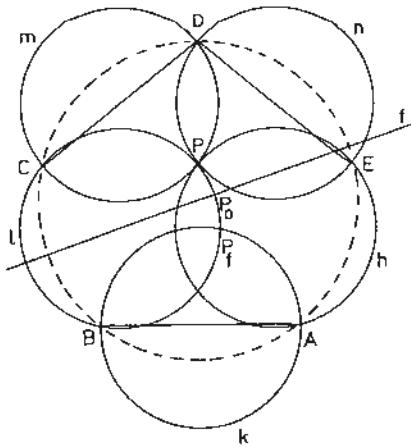


Fig. 6.

Jetzt betrachten wir die Figur 6.

Die Spiegelung an der Geraden f führt B in C ; P in P_f über. Der offene Kreisbogen $\widehat{PP_f}$ von Kreislinie des Kreises l enthält die Punkte $B; C$ nicht.

Wenn $\widehat{PP_f} \cap$ Gerade $DA = 0$ so $P_0 := f^{-1}\widehat{PP_f}$.

wenn $\widehat{PP_f} \cap$ Gerade $DA \neq 0$ so $P_0 \in \widehat{PP_f} \cap$ Gerade DA und $\widehat{PP_0} \leq \widehat{P_0 P_f}$. Man kann den Kreis n (bzw. h) um den Punkt D (bzw. A) so drehen, daß der neue Kreis n' (bzw. h') die folgende Eigenschaft hat: P gehört zum Inneren von n' (bzw. h') liegt im Inneren von h' und P_0 liegt am Rande des Einheitskreises n' (bzw. h'). Sogar, wir können die Richtung und die Größe dieser Drehungen so wählen, daß man über das System der Kreise k, l, m, n', h' sagen kann: die Grenzlinie des neuen Systems hat gleiche topologische Eigenschaften wie im Ausgangsfall; das neue System von 5 Einheitskreisen überdeckt den extremen Kreis; der Schnittpunkt E' der Kreise n', h' (hier $E' \neq P_0$) liegt im Äußeren des extremen Kreises. Hier haben wir das folgende Lemma benutzt.

LEMMA. $XZY; XYZ'$ sind Dreiecke und $\widehat{XZ'} \geq \widehat{XZ}$; $\widehat{XZ'} + \widehat{Z'Y} > \widehat{XZ} + \widehat{ZY}$; $2 \geq \widehat{XZ} \geq \widehat{ZY}$; $2 \geq \widehat{XZ'} \geq \widehat{Z'Y}$. Wenn $\widehat{XZ}; \widehat{ZY}; \widehat{XZ'}; \widehat{Z'Y}$ die geeigneten, geschlossenen Kreisbögen von Kreislinie des Einheitskreises sind, die nicht länger als π sind (Fig. 7.), so gilt:

$$\widehat{XZ'} + \widehat{Z'Y} > \widehat{XZ} + \widehat{ZY}.$$

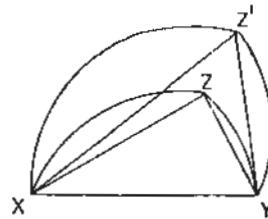


Fig. 7.

Also können wir den extremen Kreis durch eine kleine Drehung vom Kreis h' (um den Punkt P_0) vergrößern, was unmöglich ist.

HILFSSATZ 11. Es gibt drei Einheitskreise (e_1, e_2, e_3) der extremen Lagerung, die gemeinsamen Punkt haben, und unter denen genau zwei (e_1, e_2) benachbart sind. Diese drei Einheitskreise (e_1, e_2, e_3) sollen einen gemeinsamen Randpunkt M haben.

HILFSSATZ 12. Die Einheitskreise k_1, k_2 der extremen Lagerung, die den Punkt M nicht enthalten, haben unbedeckte Kreisbögen von π Länge. Mit Hilfe vom Lemma kann man nachweisen:

HILFSSATZ 13. Die gemeinsame Sekante der Kreise e_1, e_2 ist die Spiegelachse der extremen Lagerung.

KOROLLAR. Selbstverständlich liegen die Endpunkte der unbedeckten Kreisbögen am Rande des extremen Kreises.

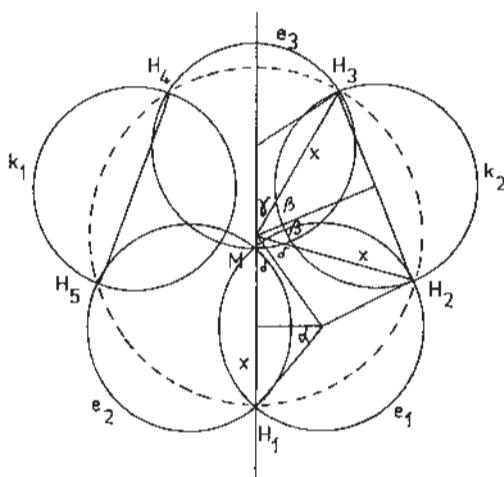


Fig. 8.

Bezeichnen wir den Radius des extremen Kreises mit x (Fig. 8.), so gilt

$$\cos \gamma = \frac{x^2 + (2 \sin \alpha + 1 - x)^2 - 1}{2x(2 \sin \alpha + 1 - x)},$$

$$\cos \delta = -\frac{x - \sin \alpha}{\sqrt{(x - \sin \alpha)^2 + \cos^2 \alpha}},$$

$$\sin \delta = \frac{\cos \alpha}{\sqrt{(x - \sin \alpha)^2 + \cos^2 \alpha}},$$

$$\cos \beta = \frac{\sqrt{x^2 - 1}}{x},$$

$$\sin \beta = \frac{1}{x}$$

und

$$\cos \gamma = -\cos(2\beta + 2\delta) = -\cos 2\beta \cdot \cos 2\delta + \sin 2\beta \cdot \sin 2\delta.$$

Woraus

$$(1) \quad \begin{aligned} \frac{x^2 + (2 \sin \alpha + 1 - x)^2 - 1}{2x(2 \sin \alpha + 1 - x)} &= -\frac{x^2 - 2}{x^2} \cdot \frac{(x - \sin \alpha)^2 - \cos^2 \alpha}{(x - \sin \alpha)^2 + \cos^2 \alpha} + \\ &+ \frac{2\sqrt{x^2 - 1}}{x^2} \cdot \frac{2(x - \sin \alpha) \cos \alpha}{(x - \sin \alpha)^2 + \cos^2 \alpha} \end{aligned}$$

und

$$(2) \quad 2 \sin \alpha + 1 > x > \sqrt{2}$$

folgt. (1) ist äquivalent mit der folgenden Gleichung:

$$0 = (80 \sin^2 z + 64 \sin z) x^6 + (-416 \sin^3 z - 384 \sin^2 z - 64 \sin z) x^5 + \\ + (848 \sin^4 z + 928 \sin^3 z + 352 \sin^2 z + 32 \sin z) x^4 + \\ + (-768 \sin^5 z - 992 \sin^4 z - 736 \sin^3 z - 288 \sin^2 z - 96 \sin z) x^3 + \\ + (256 \sin^6 z + 384 \sin^5 z + 592 \sin^4 z + 480 \sin^3 z + 336 \sin^2 z + \\ + 96 \sin z + 16) x^2 + \\ + (-128 \sin^5 z - 192 \sin^4 z - 256 \sin^3 z - 160 \sin^2 z - 96 \sin z - 32) x \\ - (64 \sin^2 z + 64 \sin z + 16).$$

Folglich gilt

$$x = \max_{0 < z < \frac{\pi}{2}} \left\{ z : z \text{ ist die größte Wurzel von (3), die auch die Ungleichung (2) befriedigt} \right\}$$

Mit Hilfe der Methode von Sturm haben wir gefunden, daß $x = 1,640 \dots$ ist.

BEMERKUNG 1. Es läßt sich den folgenden Satz beweisen (siehe dazu [1]): Betrachten wir in der euklidischen Ebene solche Kreise, die durch sechs Einheitskreise überdeckt werden können, so beträgt der Radius des größtmöglichen, überdeckten Kreises $1,7988\dots$. Die Überdeckung wird durch die Figur 9. veranschaulicht. (Höchstens drei Einheitskreise haben ei-

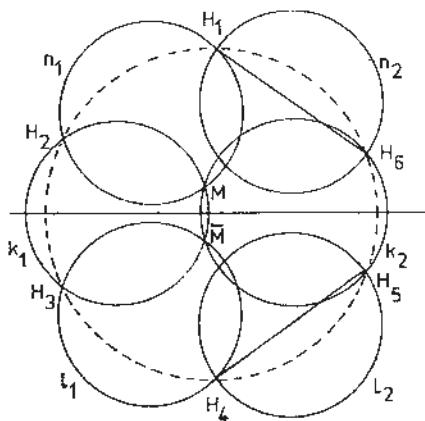


Fig. 9.

nen gemeinsamen Punkt; M (bzw. M) ist der gemeinsame Randpunkt von den Kreisen k_1, k_2, n_1 (bzw. k_1, k_2, l_1) $H_1, H_6 = H_4, H_5 = 2$; die Punkte $H_1, H_2, H_3, H_4, H_5, H_6$ liegen am Rande des überdeckten Kreises und eine Spiegelung an der Zentralachse der Kreise k_1, k_2 führt das System der Kreise in sich über.) Der Beweis dieses Satzes ist kompliziert und weitläufig; deshalb verzichten wir hier auf die Durchführung des Beweises. Jetzt möchten wir nur das Folgende betonen: Laut dieses Satzes ist die Vermutung von

GRÜNBAUM [4] falsch. (Diese Vermutung wird durch die Figur 10. veranschaulicht, wobei der Radius des überdeckten Kreises $1,7952\dots$ beträgt.)

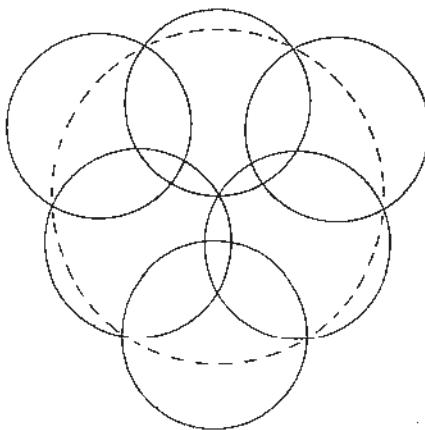


Fig. 10.

BEMERKUNG 2. Zum Schluß möchten wir noch einen Satz erwähnen: Es seien fünf Kreise mit dem Radius r in der hyperbolischen Ebene gegeben. Wir betrachten in der hyperbolischen Ebene solche Kreise, die durch diese (fünf) Kreise überdeckt werden können. Ist $r = \text{arch}(\text{ctg } 36^\circ)$, so liefert der angeschriebene Kreis des eingeschriebenen Fünfecks mit der Seitenlänge $2r$ den größtmöglichen, überdeckten Kreis, sonst [$r < \text{arch}(\text{ctg } 36^\circ)$] hat die Überdeckung des größtmöglichen, überdeckten Kreises gleiche geometrische Eigenschaften, wie im euklidischen Fall (Fig. 3.) Dieser Satz kann auf ähnliche Weise bewiesen werden, wie wir es im euklidischen Fall gemacht haben (siehe dazu [1]).

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WEIGHTED (0,2)-INTERPOLATION ON THE ROOTS OF HERMITE POLYNOMIALS

By

L. SZILJ

II. Department of Analysis of the L. Eötvös University, Budapest

(Received July 23, 1982)

I. Introduction

Firstly E. EGERVÁRY and P. TURÁN have started the study of (0,2)-interpolation in order to get approximate solution of the differential equation

$$(I.1) \quad y'' + f \cdot y = 0.$$

Their results ([2], [3], [4]) had been developed by some other mathematicians. These — so called — lacunary interpolation polynomials usually cannot be determined uniquely. An other difficulty is that they have no simple explicit form and therefore the convergence theorems related to these polynomials are rather complicated.

K. K. MATHUR and A. SHARMA showed ([5]) the uniqueness of (0,2)-interpolation polynomials if the fundamental points are the roots of Hermite polynomials and n is even. They gave the explicit form of these polynomials too, but they have not proved any convergence theorem because the constants in their formulas were very complicated.

In order to avoid these difficulties P. TURÁN suggested to study the following modified problem.

Weighted (0,2)-interpolation. Let (a, b) be a finite or infinite interval,

$$(I.2) \quad -\infty < a < \xi_{n,n} < \dots < \xi_{1,n} < b < +\infty \quad (n \in \mathbb{N})$$

distinct fundamental points and $\varrho \in C^2(a, b)$ a weight function. How can a polynomial R_n of lowest possible degree satisfying the conditions

$$(I.3) \quad R_n(\xi_{i,n}) = z_{i,n}, \quad (\varrho R_n)''(\xi_{i,n}) = \beta_{i,n} \quad (i = 1, 2, \dots, n),$$

be determined where $z_{i,n}$ and $\beta_{i,n}$ are arbitrarily given real numbers?

J. BALÁZS [1] was the first who investigated this problem. He proved, that if the fundamental points (I.2) are the roots of the ultraspherical polynomial $P_n^{(\alpha)}(x) = (-1)^n P_n(x)$, and the weight function is $\varrho(x) = (1-x^2)^{\frac{\alpha+1}{2}}$ ($x \in [-1, 1]$) then generally there does not exist any polynomial of degree $\leq 2n-1$ satisfying the requirements (I.3). But he could show, that under the condition (I.4) below there exists a unique polynomial of degree $\leq 2n$. (If n is odd then

the uniqueness is not true.) He gave the explicit form of this polynomial and proved the following convergence theorem.

THEOREM A. *Let the function $f: [-1,1] \rightarrow \mathbb{R}$ be differentiable, $f' \in \text{Lip } \mu$ ($1/2 < \mu \leq 1$) and*

$$z_{i,n} = f(\xi_{i,n}), \quad \beta_{i,n} = o(\sqrt{n})(1 - \xi_{i,n}^2)^{-\frac{\mu-3}{2}} \quad (i = 1, 2, \dots, n),$$

where $z_i > 0$. In this case the sequence of the weighted (0,2)-interpolation polynomials R_n ($n = 4, 6, \dots$) satisfying (1.3) and

$$(1.4) \quad R_n(0) = \sum_{i=1}^n z_{i,n} l_{i,n}^2(0)$$

converges to f on $(-1,1)$ and the convergence is uniform on $[-1+\delta, 1-\delta]$ for each $\delta > 0$.

In this note we want to study some analogous problems in that case when the fundamental points are the roots of Hermite polynomials and the weight function is

$$(1.5) \quad w(x) = e^{-x^2} \quad (x \in \mathbb{R}).$$

2. Results

Let H_n be the n -th Hermite polynomial with the usual normalization

$$(2.1) \quad \int_{-\infty}^{+\infty} H_n(t) H_m(t) e^{-t^2} dt = \pi^{1/2} 2^n n! \delta_{n,m} \quad (n, m \in \mathbb{N}).$$

These polynomials are orthogonal with respect to the weight function $w^2(x) = e^{-x^2}$ ($x \in \mathbb{R}$). It is well-known that the roots of H_n (which we denote by $x_{i,n}$ in this note everywhere) satisfy the following relations:

$$(2.2) \quad \begin{aligned} -\infty &< x_{n,n} < \dots < x_{\frac{n}{2}+1,n} < 0 < x_{\frac{n}{2},n} < \dots < x_{1,n} < +\infty \quad (n = 2m), \\ -\infty &< x_{n,n} < \dots < x_{n+1,\frac{n}{2}},n = 0 < \dots < x_{1,n} < +\infty \quad (n = 2m+1), \\ x_{i,n} &= -x_{n-i+1,n} \quad \left(i = 1, 2, \dots, \left[\frac{n}{2} \right] \right). \end{aligned}$$

Let us denote by $l_{i,n}$ the Lagrange-fundamental polynomials corresponding to the nodal point $x_{i,n}$, i.e.

$$(2.3) \quad l_{i,n}(x) = -\frac{H_n(x)}{H'_n(x_{i,n})(x - x_{i,n})} \quad (i = 1, 2, \dots, n).$$

The following theorems are true.

THEOREM 1. If the nodal points are the roots of the Hermite polynomial H_n and the weight function is $w(x) = e^{-x^2}$ ($x \in \mathbb{R}$), then in this case there does not exist — in general — a polynomial R_n of degree $\leq 2n-1$ satisfying the conditions (1.3).

THEOREM 2. Let n be even and

$$(2.4) \quad A_{i,n}(x) = l_{i,n}^2(x) + \frac{H_n(x)}{H'_n(x_{i,n})} \int_0^x \frac{l_{i,n}(t)(a_{i,n}t + b_{i,n}) - l'_{i,n}(t)}{t - x_{i,n}} dt,$$

where

$$(2.5) \quad a_{i,n}x_{i,n} + b_{i,n} = l'_{i,n}(x_{i,n}) = x_{i,n}, \quad a_{i,n} + \frac{w''(x_{i,n})}{w(x_{i,n})} = 1 - x_{i,n}^2,$$

and

$$(2.6) \quad B_{i,n}(x) = \frac{H_n(x)}{2w(x_{i,n})H'_n(x_{i,n})} \int_0^x l_{i,n}(t) dt \\ (i = 1, 2, \dots, n).$$

Then

$$(2.7) \quad R_n(x) = \sum_{i=1}^n y_{i,n} A_{i,n}(x) + \sum_{i=1}^n y'_{i,n} B_{i,n}(x)$$

are the uniquely determined polynomials of degree $\leq 2n$ satisfying the following requirements:

$$(2.8) \quad R_n(x_{i,n}) = y_{i,n}, \quad (wR_n)''(x_{i,n}) = y''_{i,n} \quad (i = 1, 2, \dots, n), \\ R_n(0) = \sum_{i=1}^n y_{i,n} l_{i,n}^2(0),$$

where $y_{i,n}$ and $y'_{i,n}$ are arbitrary real numbers.

THEOREM 3. If n is odd then there are infinitely many polynomials of degree $\leq 2n$ satisfying (2.8).

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ we shall use a special form of modulus of continuity $\omega(f; \delta)$ introduced by G. FREUD ([6]):

$$(2.9) \quad \omega(f; \delta) := \sup_{0 \leq t \leq \delta} \|w(x+t)f(x+t) - w(x)f(x)\| + \|\tau(\delta x)w(x)f(x)\|,$$

where

$$\tau(x) := \begin{cases} |x|, & \text{if } |x| < 1 \\ 1, & \text{if } |x| \geq 1 \end{cases}$$

and $\|\cdot\|$ denotes the sup-norm in $C(\mathbb{R})$. If $f \in C(\mathbb{R})$ and

$$(2.10) \quad \lim_{|x| \rightarrow +\infty} w(x)f(x) = 0,$$

then $\lim_{\delta \rightarrow 0} \omega(f, \delta) = 0$.

The main result of this note is the following

THEOREM 4. *If the interpolated function $f: \mathbf{R} \rightarrow \mathbf{R}$ is continuously differentiable,*

$$(2.11) \quad \lim_{|x| \rightarrow +\infty} x^{2r} w(x) f(x) = 0 \quad (r = 0, 1, \dots) \text{ and } \lim_{|x| \rightarrow +\infty} w(x) f'(x) = 0,$$

furthermore

$$(2.12) \quad y_{i,n} = f(x_{i,n}), \quad y'_{i,n} = O\left(e^{\beta x_{i,n}} \sqrt{n} \omega\left(f'; \frac{1}{\sqrt{n}}\right)\right)$$

$$\left(i = 1, 2, \dots, n; \quad 0 \leq \beta < \frac{1}{2} \right),$$

then the weighted (0,2)-interpolation polynomials R_n ($n = 2, 4, 6, \dots$) given by (2.7) satisfy the following estimate:

$$(2.13) \quad e^{-\gamma n^2} |f(x) - R_n(x)| = O\left(\log n \omega\left(f'; \frac{1}{\sqrt{n}}\right)\right),$$

which holds on the whole real line, where $\gamma > 1$ and O does not depend on n and x .

We want to prove only our main Theorem 4, because the proofs of the other theorems are quite similar to that of the theorems in [1].

3. Basic estimates with respect to the fundamental polynomials $B_{i,n}$ and $A_{i,n}$

We shall use the following notations:

$$x_i := x_{i,n}, \quad l_i := l_{i,n}, \quad A_i := A_{i,n} \quad \text{and} \quad B_i := B_{i,n}.$$

LEMMA 1. Let n be even, $0 \leq \beta < 1/2$ and $\gamma > 1$, then we can estimate the Lebesgue-function of the "second-kind" polynomials (2.6) as follows

$$(3.1) \quad \sum_{i=1}^n e^{\beta x_i^2} |B_i(x)| = O\left(e^{\gamma x^2} \frac{\log n}{\sqrt{n}}\right) \quad (x \in \mathbf{R}),$$

where O does not depend on n and x .

PROOF. Firstly we mention some basic relations with respect to the Hermite polynomials which will be used later.

For the roots of the Hermite polynomial H_n we have

$$(3.2) \quad x_i^2 \sim \frac{i^2}{n} \quad (i = 1, 2, \dots, n)$$

* $a_n \sim b_n$ means that $|a_n| = O(b_n)$ and $|b_n| = O(a_n)$.

([7], (6.31.16) and (6.31.17)). The Hermite polynomial satisfies the relation

$$(3.3) \quad H_n(x) = O\left(n^{-\frac{1}{4}} \sqrt{2^n n!} (1 + \sqrt{|x|}) e^{\frac{x^2}{2}}\right) \quad (x \in \mathbb{R})$$

([7], Theorems 8.91.3 and 8.22.9).

We shall use the following estimate

$$(3.4) \quad |H'_n(x_i)| \geq c_1 2^{n+1} \left(\frac{n}{2}\right)! e^{\frac{\delta x_i^2}{2}} \quad (i = 1, 2, \dots, n),$$

where $0 < \delta < 1$ is an arbitrary real number.

For the proof of (3.4) firstly we consider the case when $|x_i| \leq \sqrt{n}$. Since the Hermite polynomials can be expressed by the Laguerre polynomials ([7], (5.6.1)):

$$H_n(x) = (-1)^n 2^n \left(\frac{n}{2}\right)! L_{n/2}^{(-1/2)}(x^2),$$

thus from the relation (see [7], (8.9.11) and (8.22.9))

$$|(L_{n/2}^{(-1/2)})'(x_i^2)| \sim \frac{\sqrt{n}}{i} e^{\frac{x_i^2}{2}} \quad (0 < x_i \leq \sqrt{n})$$

we obtain

$$|H'_n(x_i)| = 2^n \left(\frac{n}{2}\right)! 2|x_i| |(L_{n/2}^{(-1/2)})'(x_i^2)| \geq c_2 2^{n+1} \left(\frac{n}{2}\right)! e^{\frac{x_i^2}{2}}.$$

Let $|x_i| > \sqrt{n}$. Then from

$$(3.5) \quad \frac{2^n \left[\left(\frac{n}{2}\right)!\right]^2}{(n+1)!} \sim n^{-1/2} \quad (n = 1, 2, \dots)$$

and from the inequality (see [12])

$$\sum_{i=1}^n \frac{e^{\delta^* x_i^2}}{H'_n(x_i)^2} = O\left(\frac{1}{2^{n+1} n!}\right) \quad (0 < \delta^* < 1)$$

we get (the number δ^* can be chosen so, that $0 < \delta < \delta^* < 1$)

$$\begin{aligned} |H'_n(x_i)| &\geq c_3 \sqrt{2^{n+1} n!} e^{\frac{\delta^* x_i^2}{2}} \geq c_3 \sqrt{2^{n+1} n!} e^{\frac{\delta^* - \delta}{2} x_i^2} \cdot e^{\frac{\delta}{2} x_i^2} \geq \\ &\geq c_4 \sqrt{2^{n+1} n!} n^{1/4} e^{\frac{\delta}{2} x_i^2} \geq c_1 2^{n+1} \left(\frac{n}{2}\right)! e^{\frac{\delta}{2} x_i^2}. \end{aligned}$$

Thus the inequality (3.4) is proved.

In the first step of the proof of (3.1) we estimate the integral of the Lagrange interpolation. Since ([7], (5.5.9))

$$(3.6) \quad \sum_{i=0}^{n-1} \frac{1}{2^i i!} H_i(y) H_i(x) = \frac{1}{2^n (n-1)!} \frac{H_n(y) H_{n-1}(x) - H_{n-1}(y) H_n(x)}{y-x}$$

and ([7], (5.5.10))

$$(3.7) \quad H'_n(x) = 2^n n H_{n-1}(x),$$

therefore

$$(3.8) \quad I_i(x) = \frac{H_n(x)}{H'_n(x_i)(x-x_i)} = \frac{2^{n+1} n!}{H'_n(x_i)^2} \sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k(x_i) H_k(x),$$

so using the relations (3.8), (3.7) and (3.3) we obtain

$$(3.9) \quad \begin{aligned} \int_0^x I_i(t) dt &= \frac{2^{n+1} n!}{H'_n(x_i)^2} \sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k(x_i) \int_0^x H_k(t) dt = \\ &= \frac{2^{n+1} n!}{H'_n(x_i)^2} \sum_{k=0}^{n-1} \frac{1}{2^{k+1} (k+1)!} H_k(x_i) (H_{k+1}(x) - H_{k+1}(0)) = \\ &= O(1) \frac{2^{n+1} n!}{H'_n(x_i)^2} \sum_{k=0}^{n-1} \frac{1}{2^{k+1} (k+1)!} |H_k(x_i) H_{k+1}(x)| = O(1) \frac{2^{n+1} n!}{H'_n(x_i)^2} \cdot A. \end{aligned}$$

Since from (3.3) it follows that

$$\frac{1}{2^{k+1} (k+1)!} |H_k(x_i) H_{k+1}(x)| = O\left(e^{\gamma_1} \frac{x^2 + x_i^2}{2} \frac{1}{k}\right),$$

where $\gamma_1 > 1$ is again an arbitrary real number, thus

$$(3.10) \quad A = O\left(e^{\gamma_1} \frac{x^2 + x_i^2}{2} \log n\right).$$

The Lebesgue-function of the fundamental polynomials of second kind (3.1) can be estimated using (3.9) and (3.10):

$$(3.11) \quad \sum_{i=1}^n e^{\beta x_i^2} |B_i(x)| = \frac{|H_n(x)|}{2} \sum_{i=1}^n \frac{e^{(\beta+1/2)x_i^2}}{|H'_n(x_i)|} \int_0^x I_i(t) dt = \\ = O(1) \left\{ |H_n(x)| e^{\gamma_1 \frac{x^2}{2}} \cdot 2^{n+1} \cdot n! (\log n) \sum_{i=1}^n \frac{e^{(\beta+\frac{\gamma_1}{2}+\frac{1}{2})x_i^2}}{|H'_n(x_i)|^3} \right\}.$$

Since the number $\gamma_1 > 1$ can be chosen so, that $\varepsilon := \frac{-3}{2} \delta - \left(\beta + \frac{\gamma_1}{2} + \frac{1}{2}\right) > 0$ and $\gamma_1 < \gamma$, then applying the inequalities (3.3), (3.4) and

$$\sum_{i=1}^n \frac{1}{e^{\varepsilon x_i^2}} = O(\sqrt{n}),$$

we have

$$\begin{aligned} \sum_{i=1}^n e^{2xt^2} |B_i(x)| &= O(1) |H_n(x)| e^{-\frac{x^2}{2}} \frac{2^{n+1} n!}{\left[2^{n+1} \left(\frac{n}{2}\right)!\right]^3} \log n \sum_{i=1}^n \frac{1}{e^{ex_i^2}} \\ &= O\left(e^{cx^2} \frac{\log n}{\sqrt{n}}\right), \end{aligned}$$

which completes the proof of Lemma 1.

To the estimation of the Lebesgue-function of the fundamental polynomials (2.4) we need some other representations of these polynomials.

LEMMA 2. The “first-kind” fundamental polynomials can be written in the following form:

$$\begin{aligned} A_i(x) &= \frac{l_i^2(x)}{2} + a_i \frac{H_n(x)}{H'_n(x_i)} \int_0^x l_i(t) dt + \frac{n H_n(x)}{H'_n(x_i)} \int_0^x l_i(t) dt + \frac{H'_n(x)}{2H'_n(x_i)} l_i(x) - \\ (3.12) \quad &- x \frac{H_n(x)}{H'_n(x_i)} l_i(x) - \frac{H_n(x)}{2H'_n(x_i)} l'_i(0) \quad (i = 1, 2, \dots, n). \end{aligned}$$

PROOF. The Hermite polynomial H_n satisfies the following differential equation

$$H''_n(x) - 2x H'_n(x) + 2n H_n(x) = 0 \quad (x \in \mathbf{R}),$$

thus we have

$$\begin{aligned} x l_i(x) - l'_i(x) &= \frac{x-x_i}{2} [l''_i(x) - 2x l'_i(x) + 2n l_i(x)] \quad (x \in \mathbf{R}; i = 1, \dots, n). \\ (3.13) \quad & \end{aligned}$$

Then from (2.4), (2.5) and (3.13) we get

$$\begin{aligned} A_i(x) &= l_i^2(x) + \frac{H_n(x)}{H'_n(x_i)} \int_0^x \frac{l_i(t)(a_i t + b_i) - l'_i(t)}{t - x_i} dt = \\ &= l_i^2(x) + \frac{H_n(x)}{H'_n(x_i)} a_i \int_0^x l_i(t) dt + \frac{H_n(x)}{H'_n(x_i)} \int_0^x \frac{l_i(t)(a_i x_i + b_i) - l'_i(t)}{t - x_i} dt = \\ &= l_i^2(x) + \frac{H_n(x)}{H'_n(x_i)} a_i \int_0^x l_i(t) dt - \frac{H_n(x)}{H'_n(x_i)} \int_0^x l_i(t) dt + \\ &\quad + \frac{H_n(x)}{2H'_n(x_i)} \int_0^x (l''_i(t) - 2tl'_i(t) + 2nl_i(t)) dt, \end{aligned}$$

and so by integration by parts we get (3.12).

LEMMA 3. Let n be even and $\gamma > 1$. The Lebesgue-function of the fundamental polynomials (2.4) allows the following estimate:

$$(3.14) \quad \sum_{i=1}^n \frac{|A_i(x)|}{w(x_i)} = O(e^{\gamma x^2} \sqrt{n} \log n)$$

where O does not depend on n and x .

PROOF. It is well-known (see e.g. [13], (2.36)) that

$$(3.15) \quad \sum_{i=1}^n e^{ix_i^2} l_i^2(x) = O(e^{x^2}).$$

Since $|a_i| = |1 - x_i^2| \leq e^{\beta x_i^2}$ ($0 < \beta < 1/2$) — see (2.5) —, thus from (3.1)

$$(3.16) \quad \sum_{i=1}^n \frac{|a_i| |H_n(x)|}{w(x_i) |H'_n(x_i)|} \left| \int_0^x l_i(t) dt \right| = O(1) \sum_{i=1}^n e^{\beta x_i^2} |B_i(x)| = O\left(e^{\beta x^2} \frac{\log n}{\sqrt{n}}\right).$$

It follows similarly that

$$(3.17) \quad n \sum_{i=1}^n \frac{|H_n(x)|}{w(x_i) |H'_n(x_i)|} \left| \int_0^x l_i(t) dt \right| = O(e^{\gamma x^2} \sqrt{n} \log n).$$

From (3.8) and (3.3) we get that

$$|l_i(x)| = O(1) e^{\frac{x^2 + x_i^2}{2}} \frac{2^{n+1} n!}{|H'_n(x_i)|^2} \sqrt{n},$$

where $\gamma_1 < 1$ is an arbitrary real number.

Thus by (3.4), (3.5) and (3.7) we have

$$(3.18) \quad \begin{aligned} \frac{1}{2} \sum_{i=1}^n \frac{|H'_n(x)|}{w(x_i) |H'_n(x_i)|} |l_i(x)| &= \\ &= \frac{1}{2} n |H_{n-1}(x)| e^{\frac{\gamma_1 x^2}{2}} \frac{2^{n+1} n!}{\sqrt{n}} \sum_{i=1}^n \frac{e^{\frac{x_i^2}{2}}}{|H'_n(x_i)|^3} = O(e^{\gamma x^2} \sqrt{n}). \end{aligned}$$

Similarly

$$(3.19) \quad |x| |H_n(x)| \sum_{i=1}^n \frac{1}{w(x_i) |H'_n(x_i)|} |l_i(x)| = O(e^{\gamma x^2}).$$

From the obvious equation

$$\frac{H_n(x)}{2H'_n(x_i)} l'_i(0) = \frac{-H_n(x)}{2H'_n(0)} l_i^2(0)$$

and from (3.3), (3.5) and (3.14) we obtain that

$$(3.20) \quad \begin{aligned} \frac{1}{2} \sum_{i=1}^n \frac{|H_n(x_i)|}{w(x_i) |H'_n(x_i)|} |P_i'(0)| &= \frac{|H_n(x)|}{2|H_n(0)|} \sum_{i=1}^n P_i^2(0) = \\ &= O(1) n^{-\frac{1}{4}} \sqrt{2^{n+1}} n! \cdot \frac{\left(\frac{n}{2}\right)!}{n!} e^{nx^2} = O(e^{nx^2}) \end{aligned}$$

From the relations (3.15)–(3.20) we get our statement (3.14).

4. Weighted polynomial approximation on the real line

G. FREUD proved ([9], Theorem 4 and [6], Theorem 1) the following statement.

THEOREM B. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be continuously differentiable,*

$$(4.1) \quad \lim_{|x| \rightarrow +\infty} x^{2r} f(x) w(x) = 0 \quad (r = 0, 1, \dots) \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} f'(x) w(x) = 0,$$

then there exist polynomials p_n of degree $\leq n$ such that

$$(4.2) \quad \begin{aligned} w(x) |f(x) - p_n(x)| &= O(1) \frac{1}{\sqrt{n}} \omega\left(f'; \frac{1}{\sqrt{n}}\right) \quad (x \in \mathbf{R}), \\ w(x) |f'(x) - p_n'(x)| &= O(1) \omega\left(f'; \frac{1}{\sqrt{n}}\right) \quad (x \in \mathbf{R}), \end{aligned}$$

where ω is the modulus of continuity (2.9).

For the proof of Theorem 4 we need

LEMMA 4. Denote by p_n the polynomial of degree $\leq n$ satisfying the conditions (4.2). In this case the following estimations hold:

$$(4.3) \quad w(x) |p_n(x)| = O(1) \quad (x \in \mathbf{R}),$$

$$(4.4) \quad w(x) |p_n'(x)| = O(1) \quad (x \in \mathbf{R}),$$

$$(4.5) \quad w(x) |p_n''(x)| = O(1) \sqrt{n} \omega\left(f'; \frac{1}{\sqrt{n}}\right) \quad (|x| \leq \sqrt{2n+1}).$$

PROOF. We obtain (4.3) and (4.4) from the inequalities (4.1) and (4.2):

$$(4.6) \quad w(x) |p_n(x)| \leq w(x) |f(x) - p_n(x)| + w(x) |f(x)| = O(1),$$

$$w(x) |p_n'(x)| \leq w(x) |f'(x) - p_n'(x)| + w(x) |f'(x)| = O(1).$$

For the proof of the relation (4.5) we use some other results of G. FREUD ([6], [10]).

Let $m = 18n$. Then the polynomial

$$q_{m-1}(x) = \sum_{k=1}^m \frac{x^k}{k!}$$

satisfies

$$(4.7) \quad c_1 e^x \leq q_{m+1}(x) \leq c_2 e^x$$

$$(4.8) \quad c_1 e^x \leq q'_{m+1}(x) = q_m(x) = c_2 e^x \quad \left(-m/4 \leq x \leq 0 \right)$$

(see [6], Lemma 3). By (4.4) and (4.7) we have

$$(4.9) \quad q_m\left(-\frac{x^2}{2}\right) |p'_n(x)| = O(1) \quad \left(|x| \leq \sqrt{\frac{m}{2}}\right).$$

If Q_n is a polynomial of degree at most n , then the following relation is valid

$$(4.10) \quad |Q'_n(x)| = \sqrt{\frac{m}{2-x^2}} \omega\left(Q_n, \frac{\sqrt{m}}{n}\right) \quad \left(|x| < \sqrt{\frac{m}{2}}\right),$$

where ω means the modulus of continuity over the interval $[-\sqrt{\frac{m}{2}}, \sqrt{\frac{m}{2}}]$ (see [11]).

If we use this inequality on the interval $[-\frac{1}{2}\sqrt{\frac{m}{2}}, \frac{1}{2}\sqrt{\frac{m}{2}}]$ for the polynomial (4.9), then we obtain that

$$\begin{aligned} \frac{d}{dx} \left[q_m\left(-\frac{x^2}{2}\right) \right] p'_n(x) &= -x q'_m\left(-\frac{x^2}{2}\right) p'_n(x) + q_m\left(-\frac{x^2}{2}\right) p''_n(x) = \\ &= O(1)\sqrt{n} \omega\left(q_m p'_n, \frac{1}{\sqrt{n}}\right) \quad (|x| < \sqrt{2n+1}). \end{aligned}$$

By (4.7) and (4.8) we get

$$(4.11) \quad \begin{aligned} |w(x) p''_n(x)| &\leq q_m\left(-\frac{x^2}{2}\right) |p''_n(x)| \leq c_1 \sqrt{n} \omega\left(q_m p'_n, \frac{1}{\sqrt{n}}\right) + \\ &+ |x w(x) p'_n(x)| \quad (|x| < \sqrt{2n+1}) \end{aligned}$$

and by (4.2), (2.9) and (4.1)

$$\omega\left(q_m p'_n, \frac{1}{\sqrt{n}}\right) = \sup_{\substack{0 \leq t \leq 1 \\ |x|+t < \sqrt{\frac{m}{2}}}} q_m\left[-\left(\frac{x+t}{2}\right)^2\right] p'_n(x+t) \cdot q_m\left(-\frac{x^2}{2}\right) p'_n(x) \leq$$

$$\begin{aligned}
& \leq 2 \sup_{|x| < \sqrt{\frac{m}{2}}} q\left(-\frac{x^2}{2}\right) |p_n'(x) - f'(x)| + \sup_{\substack{0 \leq t \leq 1, |t| \\ |x| + t < \sqrt{\frac{m}{2}}}} q_m\left[-\left(\frac{x+t}{2}\right)^2\right] f'(x+t) - \\
& - q_m\left(-\frac{x^2}{2}\right) f'(x) \leq c_2 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + \sup_{\substack{0 \leq t \leq 1, |t| \\ |x| + t < \sqrt{\frac{m}{2}}}} |w(x+t) f'(x+t) - w(x) f'(x)| + \\
& + \sup_{\substack{0 \leq t \leq 1, |t| \\ |x| + t < \sqrt{\frac{m}{2}}}} \sum_{k=m+1}^{\infty} \left[-\frac{\left(\frac{x+t}{2}\right)^2}{k!} \right]^k f'(x+t) - \sum_{k=m+1}^{\infty} \left(-\frac{x^2}{2} \right)^k f'(x) \leq \\
& \leq c_3 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + 2 \sup_{|x| < \sqrt{\frac{m}{2}}} \sum_{k=m+1}^{\infty} \frac{x^{2k}}{2^k k!} f'(x) \leq c_3 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + \\
(4.12) \quad & + 2 \sup_{|x| < \sqrt{\frac{m}{2}}} |f'(x)| \left[\sum_{k=m+1}^{\infty} \frac{m^k}{2^{2k} k!} \right] \leq c_3 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + \\
& + c_4 \sup_{|x| < \sqrt{\frac{m}{2}}} |f'(x)| \sum_{k=m+1}^{\infty} \frac{1}{\sqrt{k}} \left(\frac{em}{2^{2k} k} \right)^k \leq c_3 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + \\
& + c_4 \sup_{|x| < \sqrt{\frac{m}{2}}} |f'(x)| \frac{1}{\sqrt{m}} \left(\frac{e}{4} \right)^m \leq c_3 \omega\left(f'; \frac{1}{\sqrt{n}}\right) + \\
& + c_5 \frac{1}{m} \left(\frac{e^{5/4}}{4} \right)^m \leq c_6 \omega\left(f'; \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

On the other hand we have on the interval $\left[-\frac{1}{2}, \sqrt{\frac{m}{2}}; \frac{1}{2}, \sqrt{\frac{m}{2}}\right]$

$$\begin{aligned}
& |xw(x) p_n'(x)| \leq |xw(x) (p_n'(x) - f'(x))| + |xw(x) f'(x)| \leq \\
(4.13) \quad & \leq c_7 \sqrt{n} \omega\left(f'; \frac{1}{\sqrt{n}}\right) + c_8 \sqrt{n} \left| \frac{1}{\sqrt{n}} xw(x) f'(x) \right| \leq \\
& \leq c_7 \sqrt{n} \omega\left(f'; \frac{1}{\sqrt{n}}\right) + c_8 \sqrt{n} \left| \tau\left(\frac{x}{\sqrt{n}}\right) w(x) f'(x) \right| = O(1) \sqrt{n} \omega\left(f'; \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

The relations (4.11), (4.12) and (4.13) show the validity of (4.5), thus the statement is proved.

5. Proof of Theorem 4

Let n be even. From Theorem 2 it follows, that every polynomial Q_n of degree $\leq 2n$ satisfies the equality

$$(5.1) \quad Q_n(x) = \sum_{i=1}^n Q_n(x_{i,n}) A_{i,n}(x) + \sum_{i=1}^n (w Q_n)''(x_{i,n}) B_{i,n}(x) + c_n H_n(x),$$

where

$$(5.2) \quad C_n = \frac{1}{H_n(0)} \left\{ Q_n(0) - \sum_{i=1}^n Q_n(x_{i,n}) B_{i,n}(0) \right\}.$$

Let p_n be a polynomial of degree $\leq 2n$ satisfying the inequality (4.2), then we have

$$\begin{aligned} e^{-\gamma x^2} |f(x) - R_n(x)| &= O(1) \left\{ w(x) |f(x) - p_n(x)| + e^{-\gamma x^2} \sum_{i=1}^n (f(x_{i,n}) - \right. \\ &\quad \left. - p_n(x_{i,n})) w(x_{i,n}) \cdot \frac{A_{i,n}(y)}{w(x_{i,n})} + e^{-\gamma x^2} \sum_{i=1}^n [(wp_n)''(x_{i,n}) - y_{i,n}''] B_{i,n}(x) + \right. \\ &\quad \left. + e^{-\gamma x^2} |C_n H_n(x)| \right\}. \end{aligned}$$

From (4.2), Lemmas 2 and 3 we get

$$\begin{aligned} e^{-\gamma x^2} |f(x) - R_n(x)| &= O(1) \left\{ \omega \left(f'; \frac{1}{\sqrt{n}} \right) + \omega \left(f'; \frac{1}{\sqrt{n}} \right) \log n + \right. \\ &\quad + e^{-\gamma x^2} \sum_{i=1}^n |y_{i,n}'' B_{i,n}(x)| + e^{-\gamma x^2} \sum_{i=1}^n w(x_{i,n}) |p_n''(x_{i,n}) B_{i,n}(x)| + \\ &\quad + e^{-\gamma x^2} \sum_{i=1}^n |w'(x_{i,n}) p_n'(x_{i,n}) B_{i,n}(x)| + e^{-\gamma x^2} \sum_{i=1}^n |w''(x_{i,n}) p_n(x_{i,n}) B_{i,n}(x)| + \\ &\quad \left. + e^{-\gamma x^2} |C_n H_n(x)| \right\}. \end{aligned}$$

By Lemmas 1 and 4 we have

$$e^{-\gamma x^2} \sum_{i=1}^n |w''(x_{i,n})| |p_n(x_{i,n}) B_{i,n}(x)| = O(1) \frac{\log n}{\sqrt{n}},$$

$$e^{-\gamma x^2} \sum_{i=1}^n |w'(x_{i,n})| |p_n'(x_{i,n}) B_{i,n}(x)| = O(1) \frac{\log n}{\sqrt{n}},$$

$$e^{-\gamma x^2} \sum_{i=1}^n w(x_{i,n}) |p_n''(x_{i,n}) B_{i,n}(x)| = O(1) \omega \left(f'; \frac{1}{\sqrt{n}} \right) \log n$$

and therefore

$$(5.3) \quad e^{-\gamma x^2} |f(x) - R_n(x)| = O(1) \omega\left(f'; \frac{1}{\sqrt{n}}\right) \log n + e^{-\gamma x^2} |C_n H_n(x)|.$$

We need only an estimate for the second term on the right hand side of (5.3). By (5.2) we get

$$(5.4) \quad \begin{aligned} e^{-\gamma x^2} |C_n H_n(x)| &= e^{-\gamma x^2} \left| \frac{H_n(x)}{H_n(0)} \right|^{\frac{1}{n}} \left| p_n(0) - \sum_{i=1}^n p_n(x_{i,n}) l_{i,n}^2(0) \right|^{\frac{1}{n}} = \\ &= e^{-\gamma x^2} \left| \frac{H_n(x)}{H_n(0)} \right|^{\frac{1}{n}} G. \end{aligned}$$

Since n is even, thus $H_n(0) = \frac{n!}{\left(\frac{n}{2}\right)!}$ therefore from (3.5) and (3.3) we have

$$e^{-\gamma x^2} \left| \frac{H_n(x)}{H_n(0)} \right| = O(1) \cdot \frac{\sqrt{2^n \cdot n!} \cdot \left(\frac{n}{2}\right)!}{n!} n^{-1/4} = O(1).$$

The expression G can be written in another form

$$(5.5) \quad G = \left| \sum_{i=1}^n (p_n(0) - p_n(x_{i,n})) l_{i,n}^2(0) + p_n(0) \left(1 - \sum_{i=1}^n l_{i,n}^2(0) \right) \right|.$$

From the equality

$$\sum_{i=1}^n \left(1 - \frac{H_n''(x_{i,n})}{H_n'(x_{i,n})} (x - x_{i,n}) \right) l_{i,n}^2(x) = 1$$

we can see that

$$(5.6) \quad \begin{aligned} \left| 1 - \sum_{i=1}^n l_{i,n}^2(0) \right| &= O(1) \sum_{i=1}^n x_{i,n}^2 l_{i,n}^2(0) = O(1) \sum_{i=1}^n \frac{H_n^2(0)}{H_n'(x_i)^2} = \\ &\approx O(1) \left[\frac{n!}{\left(\frac{n}{2}\right)!} \right]^2 \sum_{i=1}^n \frac{1}{H_n'(x_i)^2} = O(1) \left[\frac{n!}{\left(\frac{n}{2}\right)!} \right]^2 \cdot \frac{1}{2^n n!} = O(n^{-1/2}). \end{aligned}$$

Similarly we obtain from Lemma 4 that the first term in (5.5) is

$$(5.7) \quad \begin{aligned} \sum_{i=1}^n |p_n(0) - p_n(x_{i,n})| l_{i,n}^2(0) &= O(1) \sum_{i=1}^n |x_{i,n}| |p_n'(\xi_{i,n})| l_{i,n}^2(0) = \\ &= O(1) \left[\sum_{i=1}^n |x_{i,n}| e^{\frac{x_{i,n}^2 n}{2}} l_{i,n}^2(0) \right] = O(1) H_n^2(0) \sum_{i=1}^n \frac{e^{\frac{x_{i,n}^2 n}{2}}}{H_n'(x_{i,n})^2 |x_{i,n}|} = : B. \end{aligned}$$

From (3.4) and (3.5) it follows that

$$B = O(1) \left[\left(\frac{n!}{2^n} \right)^2 - \left(\frac{\sqrt{n}}{2^n} \left(\frac{n}{2} \right) ! \right)^2 \right] \sum_{i=1}^n \frac{1}{e^{ex_i^2}} + \frac{1}{i} + O(1) \frac{\log n}{\sqrt{n}} \\ \left(0 < \varepsilon < \frac{1}{2} \right),$$

and since $1/\sqrt{n} = o\left(f', \frac{1}{\sqrt{n}}\right)$, thus from (5.3)–(5.7) we get

$$e^{-x^2} |C_n H_n(x)| = O(1)(\log n) o\left(f', \frac{1}{\sqrt{n}}\right),$$

and so our Theorem 4 is proved.

I wish to express my thanks to DR. J. SZABADOS for his helpful remarks.

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ON THE SUMMABILITY OF EIGENFUNCTION EXPANSIONS II.

By

I. JOÓ

II. Department of Math. Analysis of the L. Eötvös University, Budapest

(Received May 17, 1982)

DEDICATED TO PROFESSOR PÁL ERDŐS ON THE OCCASION OF HIS 70th BIRTHDAY

The aim of the present paper is the investigation of Riesz summability of eigenfunction expansions associated with the Schrödinger operator, in any bounded N -dimensional domain ($N \geq 3$). The results obtained generalize that of the paper [6] for more general potential and for more general class of functions (for the Nikolskii class instead the Liouville one).

Let Ω be an arbitrary bounded domain in \mathbf{R}^N ($N \geq 1$) with C^∞ — smooth boundary, q be an arbitrary non — negative function from the class $L_2(\Omega)$. Consider the Schrödinger operator

$$L = L(x, D) = -A + q(x), D_L = C_0^\infty(\Omega).$$

Denote by \hat{L} an arbitrary selfadjoint extension of L with discrete spectrum. According to a well known theorem of K. O. FRIEDRICHS ([22] p. 356.) there exists such an extension. Indeed, it follows from the trivial estimate

$$(Lu, u) = (\nabla u, \nabla u) + (qu, u) \geq c \|u\|_{W_1^2(\Omega)}^2 \quad (u \in C_0^\infty(\Omega))$$

that the domain of the Friedrichs's extension of L is a subset of $W_2^1(\Omega)$. Rellich's theorem [20] states the compactness of the imbedding $W_2^1 \rightarrow L_2$.

Consequently the Friedrichs's extension $\hat{L} + I$ of $L + I$ has compact inverse and we arrive at a selfadjoint extension of L with discrete spectrum.

Denote by $0 < \lambda_1 \leq \lambda_2 \leq \dots$ the sequence of eigenvalues and by $\{u_n\}_1^\infty$ the complete (in $L_2(\Omega)$) orthonormal system of the corresponding eigenfunctions of the operator \hat{L} . For any $s > 0$ and $f \in L_2(\Omega)$ consider the s -th Riesz means of the spectral expansion of f

$$E_i^s f(x) = \sum_{n \leq i} \left(1 - \frac{\lambda_n}{\lambda} \right)^s (f, u_n) u_n(x).$$

¹ We have used the Friedrichs-Poincaré inequality:

$$\|u\|_{L_2} = c(\Omega) \|\nabla u\|_{L_2} \quad (u \in C_0^\infty(\Omega))$$

([23] p. 62), and the notations

$$(\nabla u, \nabla u) = \|\nabla u\|_{L_2}^2 \stackrel{\text{def}}{=} \sum_{i=1}^N (D_i u, D_i u).$$

It is assumed in this work that the potential q has the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|} + q_1(x) = q_0(x) + q_1(x) \quad (x_0 \in \Omega)$$

where $a \in C^\infty(0, \infty)$ is such a non-negative function for which

$$(1) \quad t^k |a^{(k)}(t)| \leq c_\tau t^{\tau-1} \quad (t > 0; k = 0, 1, \dots, [N])$$

hold with some $\tau > 0$ (if $N > 3$ then assume $\tau > 1/2$); and $q_1(x) \in C^N(\Omega)$ is such a non-negative function for which the estimate

$$(2) \quad |q_1(x)| \leq c|x-x_0|^l \quad (l > (N-4)/2)$$

holds in a neighbourhood of x_0 .

We shall prove the following theorems.

THEOREM 1. Let $p \geq 1$, $s > 0$, $\alpha > 0$, $\alpha + s \geq (N-1)/2$, $p\alpha \leq N$. Then for any $f \in \dot{H}_p^s(\Omega)$

$$(3) \quad \lim_{\lambda \rightarrow \infty} E_\lambda^s f(x) = f(x), \quad x \in \Omega.$$

THEOREM 2. Let $s > 0$, $\alpha > 0$, $\alpha + s \geq (N-1)/2$. Then for any $f \in \dot{H}_2^s(\Omega)$

$$(4) \quad \lim_{\nu \rightarrow \infty} E_\nu^s f(x) = f(x), \quad x \in \Omega \setminus \text{supp } f.$$

REMARKS. These theorems were proved for $q \equiv 0$ (for any selfadjoint extension) in [13]. Earlier, they were proved by V. A. IL'IN [10], [12] for $q \equiv 0$, $s = 0$, $\alpha \in \mathbb{N}$ (integer); These theorems were proved by S. A. ALIMOV [2] for $q_0 \equiv 0$, $N = 3$ and for $q_1 = 0$ they were proved in [6] (by S. A. ALIMOV and the author).

Our theorems are not refinable, namely they are not refinable if $q \equiv 0$ (Cf. [13]).

The proofs of the theorems are the developments of the ideas of the papers [5], [6], [14]. First we prove a mean value formula for the eigenfunctions u_n with centrum x_0 ; using this formula we prove the estimate

$$\sum_{|\lambda_n - \mu| \geq 1} |u_n(x_0)|^2 \approx c \mu^{N-1} \quad (\mu \geq 1)$$

by the method of V. A. IL'IN [12]. (Another method for the proof of such estimate is due to B. M. LEVITAN [15]; this method was further developed by L. HÖRMANDER [8, 9]). Later on our theorems follow by applying Hörmander's Tauber type theorem [8]. For the application of this theorem we need estimation for the Fourier coefficients of functions from the Nikolskii's class \dot{H}_p^s further we must estimate the resolvent of the operator \hat{L} outside of an angular domain which contains the spectrum $\{\lambda_n\}$. This last estimation is given in [6] for $q_1 \equiv 0$, but this method works also in our case, so we omit the estimation of the Green's function.

Next we use the following condition, which is weaker than (I):

$$(5) \quad t^{k+1} \cdot |a^{(k)}(t)| \leq c \omega(t) \left(t < 0, 0 \leq k \leq \left[\frac{N}{2} \right] \right)$$

where the function $\omega(t)$ increases for $t > 0$, further

$$(6) \quad \int_0^{\infty} \frac{\omega(t)}{t} dt < \infty, \quad 0 < \omega(t) < 1.$$

I. The mean value formula and its applications

For $0 < t \leq r$ define

$$W(t, r) = \frac{1}{4} \pi^{-p} \Gamma\left(\frac{N}{2}\right) [J_p(t) Y_p(r) - Y_p(t) J_p(r)], \quad p \stackrel{\text{def}}{=} \frac{N}{2} - 1,$$

where J_p and Y_p denotes the p -th Bessel and Neumann function, respectively.

LEMMA 1.1. We have for $0 < t \leq r$

$$(1.1) \quad |w(t, r)| \leq \frac{c_1}{\sqrt{tr}} \quad (t \geq 1),$$

$$(1.2) \quad |w(t, r)| \leq \frac{c_1}{t^p \sqrt{r}}, \quad (0 < t \leq 1 \leq r),$$

$$(1.3) \quad |w(t, r)| \leq c_1 \left(\frac{t}{r} \right)^p \quad (0 < t \leq r \leq 1).$$

PROOF. These estimates follow from [7], 7.2.1(2), (4); 7.13.1(3), (4). Define

$$h(t) = \min \left\{ 1, t^{-\frac{p-1}{2}} \right\} \quad (t > 0),$$

$$b(r) = \int_0^r a(t) dt.$$

LEMMA 1.2. The estimate

$$(1.4) \quad r^{-p} \int_0^r h(t \mu) |w(t \mu, r \mu)| t^p a(t) dt \leq c_2 b(r) h(r \mu) \quad (r > 0)$$

holds with some constant c_2 , which does not depend on r and on μ .

PROOF. Denote by I the left hand side of (1.4). If $r\mu \approx 1$, then using (1.3) we get

$$I \approx c_1 \int_0^r \left(\frac{t}{r} \right)^p \cdot \left(\frac{r}{t} \right)^p a(t) dt \approx c_1 b(r) = c_1 b(r) h(r\mu).$$

If $r\mu \ll 1$, then set

$$I = \int_0^{1/4} + \int_{1/4}^r = I_1 + I_2$$

and apply (1.2) and (1.1) respectively. It follows

$$I_1 \approx c_1 r^{-p} \int_0^{1/4} (t\mu)^{-p} \sqrt{\frac{1}{r\mu}} t^p a(t) dt \approx c_1 (r\mu)^{-p - \frac{1}{2}} \int_0^{1/p} a(t) dt = c_1 b(r) h(r\mu)$$

and

$$\begin{aligned} I_2 &\leq c_1 r^{-p} \int_{1/\mu}^r (t\mu)^{-p} \left(\frac{1}{\sqrt{t\mu}} \right)^2 \cdot \frac{1}{\sqrt{t\mu}} t^p a(t) dt \\ &= c_1 (r\mu)^{-p - \frac{1}{2}} \int_{1/\mu}^r \frac{a(t)}{t\mu} dt \leq c_1 h(r\mu) \int_0^r a(t) dt = c_1 b(r) h(r\mu). \end{aligned}$$

Lemma 1.2 is proved.

Define

$$v_0(r, \mu) = 2^p F\left(\frac{N}{2}\right) (r\mu)^{-p} f_p(r\mu),$$

$$v_k(r, \mu) = r^{-p} \int_0^r v_{k-1}(t, \mu) W(t\mu, r\mu) t^p a(t) dt.$$

LEMMA 1.3.

$$(1.5) \quad |v_k(r, \mu)| \leq c_2 h(r\mu) [c_1 b(r)]^k \quad (k = 0, 1, \dots), \quad (r > 0, \mu > 0).$$

PROOF. Use induction in k . The case $k = 0$ is trivial, further, using the induction hypothesis we get

$$\begin{aligned} |v_k(r, \mu)| &\leq \int_0^r |v_{k-1}(t, \mu)| \left(\frac{t}{r} \right)^p |W(t\mu, r\mu)| a(t) dt \leq \\ &\leq c_2 [c_1 b(r)]^{k-1} \cdot r^{-p} \int_0^r h(t\mu) |W(t\mu, r\mu)| t^p a(t) dt \leq \\ &\leq c_2 [c_1 b(r)]^{k-1} \cdot c_1 b(r) h(r\mu) = c_2 h(r\mu) [c_1 b(r)]^k. \end{aligned}$$

We used (1.4) in the last step. Lemma 1.3 is proved.

Define

$$\begin{aligned} w_0(r, \mu) &= \int_0^r \left(\int_{x_0+t\theta} u(x_0+t\theta) q_1(x_0+t\theta) d\theta \right) W(t\mu, r\mu) \left(\frac{t}{r} \right)^p t dt, \\ w_k(r, \mu) &= \int_0^r w_{k-1}(t, \mu) W(t\mu, r\mu) \left(\frac{t}{r} \right)^p a(t) dt, \end{aligned}$$

where $u(x) = u(x, \mu) \in W_2^1(\Omega)$ is an arbitrary normed ($\|u\|_{L_2} = 1$) eigenfunction of the operator $-I + q$, with eigenvalue μ^2 ; for $0 < t < \text{dist}(x_0, \Omega)$ set

$$\begin{aligned} \Phi(x, t) &= \begin{cases} q_1(x) & \text{if } |x - x_0| < t \\ 0 & \text{if } |x - x_0| > t \end{cases}, \\ \hat{\Phi}(t, \mu) &= \int_{x_0+t\beta} q_1(y) u(y, \mu) dy, \quad B = \{x \in \mathbf{R}^N, \|x\| = 1\}. \end{aligned}$$

LEMMA 1.4. We have for $\mu > 1$

$$(1.6) \quad |w_k(r, \mu)| \leq c_3 \left(\int_0^r |\hat{\Phi}(t, \mu)| t^{-2p-1} dt \right) [c_1 b(r)]^k,$$

$(k = 0, 1, 2, \dots; 0 < 2R < \text{dist}(x_0, \partial \Omega); R < r < 2R)$

The constant c_3 depends on R but it does not depend on μ .

PROOF. Use induction in k . First consider the case $k = 0$. Integrating by parts we get

$$\begin{aligned} w_0(r, \mu) &= \int_0^r \frac{1}{\omega_N t^{2p+1}} \cdot \left[\frac{d}{dt} \int_{x_0+t\beta} q_1(y) u(y, \mu) dy \right] W(t\mu, r\mu) \left(\frac{t}{r} \right)^p \cdot t dt = \\ (1.7) \quad &= -\frac{1}{\omega_N} \int_0^r \left(\frac{d}{dt} [(tr)^{-p} W(t\mu, r\mu)] \right) \left(\int_{x_0+t\beta} q_1(y) u(y, \mu) dy \right) dt, \end{aligned}$$

where $\omega_N = \text{mes}(\partial B)$. Now we prove the estimate

$$(1.8) \quad \left| \frac{d}{dt} [(tr)^{-p} W(t\mu, r\mu)] \right| \leq c_4 \cdot t^{-2p-1},$$

$(0 < t < r, \quad R < r < 2R).$

where the constant c_4 depends only on R . To this we need the asymptotic formulas

$$(1.9) \quad J_p(x) = \sqrt{\frac{2}{\pi x}} \left\{ \cos \left(x - p \frac{\pi}{2} - \frac{\pi}{4} \right) + O \left(\frac{1}{x} \right) \right\},$$

$$(1.10) \quad Y_p(x) = \sqrt{\frac{2}{\pi x}} \left\{ \sin \left(x - p \frac{\pi}{2} - \frac{\pi}{4} \right) + O \left(\frac{1}{x} \right) \right\},$$

for $x \geq \delta > 0$ (Cf. [7], 7.13.1 (3), (4)) further the equalities

$$(1.11) \quad J_p(x) = x^p \cdot j_p(x),$$

$$(1.12) \quad Y_p(x) = x^{-p} \cdot y_p(x),$$

for $0 < x \leq 1$, where $j_p(x)$ and $y_p(x)$ are analytical functions (Cf. [7], 7.2.1. (2), (4)).

The desired estimate (1.8) follows from the identity

$$\begin{aligned} \frac{d}{dt} [(tr)^{-p} W(t\mu, r\mu)] &= [\mu J_p(r\mu) t^{-p} Y_{p+1}(t\mu) + \mu Y_p(r\mu) t^{-p} J_{p+1}(t\mu)] \cdot \\ &\quad \cdot r^{-p} \frac{1}{4} \pi^{-p} \Gamma \left(\frac{N}{2} \right) \end{aligned}$$

taking into account (1.9) – (1.12). From (1.7) and (1.8) we obtain (1.6) for $k = 0$. Now using (1.1) – (1.3) and the induction hypothesis we obtain

$$\begin{aligned} |w_k(r, \mu)| &\leq r^{-p} \int_0^r |w_{k-1}(t, \mu)| |W(t\mu, r\mu)| t^p a(t) dt \leq \\ &\leq c_3 \left(\int_0^r |\hat{\Phi}(t, \mu)| t^{-2p-1} dt \right) [c_1 b(r)]^{k-1} r^{-p} \int_0^r |W(t\mu, r\mu)| t^p a(t) dt \leq \\ &\leq c_3 \left(\int_0^r |\hat{\Phi}(t, \mu)| t^{-2p-1} dt \right) [c_1 b(r)]^k. \end{aligned}$$

Lemma 1.4 is proved.

COROLLARY. From (1.5) and (1.6) we obtain for the functions

$$\alpha(r, \mu) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} v_k(r, \mu),$$

$$\beta(r, \mu) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} w_k(r, \mu),$$

the following estimates:

$$(1.13) \quad |\alpha(r, \mu)| < c_4 b(r) h(r, \mu),$$

$$(1.14) \quad |\beta(r, \mu)| \leq c_5 \left(\int_0^r |\hat{\Phi}(t, \mu)| t^{-2p-1} dt \right)$$

for $R \leq r \leq 2R$, $0 < R < r_0$ (where $r_0 > 0$ is such that $2 \cdot c_1 b(r_0) < 1$), $\mu \geq 1$.

LEMMA 1.5. We have

$$(1.15) \quad \int_{\Theta} u(x_0 + r\theta, \mu) d\theta = u(x_0, \mu) \left[c_N \frac{J_p(r\mu)}{(r\mu)^p} + \alpha(r, \mu) \right] + \beta(r, \mu) \quad (0 < r < r_0, \mu \geq 1).$$

PROOF. It is easy to see that the function

$$v(r, \mu) = u(x_0, \mu) [v_0(r, \mu) + \alpha(r, \mu)] + \beta(r, \mu)$$

is the only solution in $L_2(0, r_0)$ of the integral equation

$$f(r) = u(x_0, \mu) \cdot v_0(r, \mu) + \int_0^r f(t) W(t\mu, r\mu) \left(\frac{t}{r} \right)^p a(t) dt + w_0(r, \mu),$$

and hence, taking into consideration the Titchmarsh formula (Cf. [19] p. 232)

$$\begin{aligned} \int_{\Theta} u(x_0 + r\theta, \mu) d\theta &= c_N \frac{J_p(r\mu)}{(r\mu)^p} u(x_0, \mu) + \\ &+ \int_0^r \left(\int_{\Theta} q(x_0 + t\theta) u(x_0 + t\theta, \mu) d\theta \right) W(t\mu, r\mu) \left(\frac{t}{r} \right)^p t dt, \\ &\quad \left(c_N = 2^p \Gamma \left(\frac{N}{2} \right) \right) \end{aligned}$$

the equation

$$v(r, \mu) = \int_{\Theta} u(x_0 + r\theta, \mu) d\theta$$

follows. (The uniqueness of the solution of the integral equation given above, follows from its contractivity, i.e. from

$$\int_0^r |W(t\mu, r\mu) \left(\frac{t}{r} \right)^p a(t)| dt < 1 \quad (0 < r < r_0),$$

Cf. [22]). Lemma 1.5 is proved.

LEMMA 1.6. We have

$$(1.16) \quad \sum_{n \in \mathbb{Z}^N - 1} |u_n(x_0)|^2 = c_6 \mu^{N-1} \quad (\mu \geq 1; u_n \stackrel{\text{def}}{=} \sqrt{\lambda_n}).$$

The constant c_6 does not depend on μ .

PROOF. Consider the Fourier coefficients of the function

$$d(r, \mu) = \begin{cases} \mu^{N/2} \frac{J_p(r\mu)}{r^p} & \text{if } R < r < 2R \\ 0 & \text{if } r \notin (R, 2R) \end{cases}$$

with respect to the system $\{u_n\}$. Using (1.15) we obtain

$$\begin{aligned} d_n &= \int_{\Omega} d(|y - x_0|; \mu) u_n(y) dy = \int_R^{2R} d(r, \mu) \left(\int_{\omega} u_n(x_0 + r\Theta) d\Theta \right) r^{N-1} dr = \\ &= \mu^{N/2} u_n(x_0) \left[c_N \int_R^{2R} J_p(r\mu) J_p(r\mu_n) (r\mu_n)^{-p} r^{N/2} dr + \right. \\ &\quad \left. + \int_R^{2R} J_p(r\mu) \alpha(r, \mu_n) r^{N/2} dr \right] + \mu^{N/2} \int_R^{2R} J_p(r\mu) \beta(r, \mu_n) r^{N/2} dr \\ &= A_n^{(1)} + A_n^{(2)} + A_n^{(3)}. \end{aligned}$$

Applying the formula (1.9), using (1.13) and (1.14) further the Bessel inequality, we obtain for $\mu \geq \mu_0$, $|\mu_n - \mu| \leq 1$ the following estimates

$$\begin{aligned} \sqrt{\mu\mu_n} \int_R^{2R} J_p(r\mu) J_p(r\mu_n) rdr &= \frac{2}{\pi} \int_R^{2R} \cos(r\mu + q) \cos(r\mu_n + q) dr + \\ &+ O\left(\frac{1}{\mu}\right) = \frac{1}{\pi} \int_R^{2R} \cos r(\mu - \mu_n) dr + O\left(\frac{1}{\mu}\right) \geq \frac{R}{\pi} \cos 2R + O\left(\frac{1}{\mu}\right), \\ \int_R^{2R} J_p(r\mu) \alpha(r, \mu_n) r^{N/2} dr &\leq \text{const} \int_R^{2R} \frac{1}{\sqrt{\mu r}} (r\mu_n)^{-p-\frac{1}{2}} b(r) r^{N/2} dr \leq \\ &\leq \text{const} \mu^{-N/2} \int_R^{2R} b(r) dr < \text{const} \mu^{-N/2} b(2R) \cdot R, \end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} |A_n^{(3)}|^2 &\leq \sum_{n=1}^{\infty} \mu^N \left[\int_R^{2R} |J_p(r\mu)| \cdot |\beta(r, \mu_n)| r^N dr \right]^2 \leq \\
&\leq \text{const } \mu^{N-1} \sum_{n=1}^{\infty} \left[\int_R^{2R} \left(\int_0^r \hat{\Phi}(t, \mu_n)^\frac{1}{l} t^{1-N} dt \right) dr \right]^2 \leq \\
&= c(\varepsilon) \mu^{N-1} \int_R^{2R} \int_0^r \left(\sum_{n=1}^{\infty} |\hat{\Phi}(t, \mu_n)|^2 \right) t^{2-2N+1-\varepsilon} dt dr = \\
&= c(\varepsilon) \mu^{N-1} \int_R^{2R} \int_0^r \left(\int_{x_0+t}^r |q_1(y)|^2 dy \right) t^{3-2N-\varepsilon} dt dr = \\
&= O(\mu^{N-1}) \int_R^{2R} \int_0^r t^{N+2l+3-2N-\varepsilon} dt dr = O(\mu^{N-1}).
\end{aligned}$$

(We used here the assumption (2).) Consequently

$$\sum_{|a_n - \mu| \leq 1} |u_n(x_0)|^2 \leq \text{const} \sum_{n=1}^{\infty} (|\alpha_n|^2 + |A_n^{(3)}|^2) \leq \text{const } \mu^{N-1} \text{ if } R \leq R_0,$$

$\mu \geq \mu_0(R_0),$

Lemma 1.6 is proved.

2. Estimates for the Fourier coefficients of functions from the Nikolskii's classes $\dot{H}_p^u(\Omega)$

Assume in this paragraph for the sake of simplicity: $x_0 = 0$.

LEMMA 2.1. For any natural number $m \in [0, N/2]$ we have

$$(2.1) \quad (\mathcal{A} - q)^m = \sum_{|\alpha| \leq 2m} c_{m, \alpha}(x) D^\alpha$$

and

$$(2.2) \quad |D^\beta c_{m, \alpha}(x)| \leq \text{const} \omega(|x|) |x|^{|\alpha|-2m-|\beta|} (0 \leq |\beta| \leq N).$$

PROOF. Use induction in m . The case $m = 0$ is trivial, further, using the induction hypothesis, we get

$$\begin{aligned}
(\mathcal{A} - q)^{m+1} &= (\mathcal{A} - q) \sum_{|\alpha| \geq 2m} c_{m, \alpha}(x) D^\alpha = \\
&= \sum_{|\alpha| \geq 2m} [c_{m, \alpha}(x) \mathcal{A} D^\alpha + 2(\vee c_{m, \alpha}) \triangleright D^\alpha + (\triangle c_{m, \alpha}) D^\alpha - q c_{m, \alpha} D^\alpha].
\end{aligned}$$

Using the trivial estimates

$$\omega^2(f) \leq \omega(t), |D^\alpha q(x)| \leq \text{const } \omega(|x|) |x|^{|a|-2-|\beta|},$$

we obtain:

$$1. \quad c_{m,a} \cdot ID^a = \sum_{i=1}^N \tilde{c}_{m+1,a(i)} D^{a(i)},$$

$$\begin{aligned} (\alpha(i)) &\stackrel{\text{def}}{=} (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 2, \alpha_{i+1}, \dots, \alpha_N) \\ |D^\beta \tilde{c}_{m+1,a(i)}(x)| &= |D^\beta c_{m,a}(x)| \leq \text{const } \omega(|x|) |x|^{|a|-2m-|\beta|} = \\ &= \text{const } \omega(|x|) \cdot |x|^{(|a|+2)-2(m+1)-|\beta|}; \end{aligned}$$

$$2. \quad (\nabla c_{m,a}) \nabla D^a = \sum_{i=1}^N \tilde{c}_{m+1,a(i)} D^{a(i)},$$

$$(\alpha(i)) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_N),$$

$$\begin{aligned} |D^\beta \tilde{c}_{m+1,a(i)}(x)| &= |D^\beta \sum_{j=1}^N \frac{\partial}{\partial x_j} c_{m,a}(x)| \leq \text{const } \omega(|x|) \cdot |x|^{|a|-2m-(|\beta|+1)} = \\ &= \text{const } \omega(|x|) \cdot |x|^{(|a|+1)-2(m+1)-|\beta|}; \end{aligned}$$

$$3. \quad (\Delta c_{m,a}) D^a = \left(\sum_{i=1}^N \tilde{c}_{m+1,a(i)} \right) D^a,$$

$$\left(\tilde{c}_{m+1,a(i)} \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x_i^2} c_{m,a} \right),$$

$$\begin{aligned} |D^\beta \tilde{c}_{m+1,a(i)}(x)| &= |D^\beta \left(\frac{\partial^2}{\partial x_i^2} c_{m,a}(x) \right)| \leq \text{const } \omega(|x|) |x|^{|a|-2m-(C|\beta|+2)} = \\ &= \text{const } \omega(|x|) |x|^{(|a|+1)-2(m+1)-|\beta|}; \end{aligned}$$

$$4. \quad qc_{m,a} D^a = \tilde{c}^{m+1,a} D^a,$$

$$|D^\beta \tilde{c}_{m+1,a}(x)| = |D^\beta (q(x) c_{m,a}(x))| \leq \text{const } \omega(|x|) \cdot |x|^{(|a|+2(m+1)-|\beta|)}.$$

Lemma 2.1 is proved.

LEMMA 2.2. For every $1 < p \leq 2$ and every natural number $m \in [0, N/p]$ the estimate

$$(2.3) \quad \|(\Delta - q)^m f\|_{L_p} \leq C_7 \|f\|_{W_p^{2m}} (f \in \dot{W}_p^{2m}(\Omega))$$

holds. The constant C_7 does not depend on f .

PROOF. Taking into consideration (2.1) and (2.2), it is enough to prove that

$$(2.4) \quad \omega(|x|) |x|^{|a|-2m} D^a f \in L_p \text{ if } |x| < 2m.$$

To this use the imbedding $W_p^{2m} \rightarrow W_{p_1}^r$, p_1 is defined by the equality $(N/p_1) - |z| = (N/p) - 2m$ and q_1 is defined by $(1/p_1) + (1/q_1) = 1/p$. Then, using the generalized Hölder inequality

$$\begin{aligned} & \|\omega(|x|)|x|^{-n-2m} D^\alpha f\|_{L_p} \leq \|D^\alpha f\|_{L_{p_1}} \cdot \\ & \cdot \left(\int_{\Omega} [\omega(|x|)|x|^{-n-2m}]^{q_1} dx \right)^{1/q_1}. \end{aligned}$$

On the other hand, $q_1(|z| - 2m) = -N$ and therefore

$$\begin{aligned} \int_{\Omega} [\omega(|x|)|x|^{-n-2m}]^{q_1} dx &= \int_{\Omega} [\omega(|x|)]^{q_1} \cdot |x|^{-N} dx \leq \\ &\leq \text{const} \int_0^{\infty} [\omega(r)]^{q_1} \frac{dr}{r} < \infty. \end{aligned}$$

Lemma 2.2 is proved.

LEMMA 2.3. For any natural number $k \in \{0, N/2\}$ the estimate

$$(2.5) \quad \|\hat{L}^k f\|_{L_2} \leq c_8 \|f\|_{W_k^k} \quad (f \in \dot{W}_2^k)$$

holds. The constant C_8 does not depend on f .

PROOF. If $k = 2m$, then (2.5) is equivalent to the case $p = 2$ of (2.3). If $k = 2m+1$, then, using the spectral theorem we get

$$\begin{aligned} \sum_{n=1}^{\infty} |\lambda_n^k (f, u_n)|^2 &= \int_{\Omega} [(1-q)(1-q)^m f] [(1-q)^m f] = \\ &= - \int_{\Omega} [-(1-q)^m f] \cdot [(1-q)^m f] + \int_{\Omega} [\sqrt{q}(1-q)^m f]^2 = \\ &= \|\nabla(1-q)^m f\|_{L_2}^2 + \|\sqrt{q}(1-q)^m f\|_{L_2}^2 \end{aligned}$$

for every $f \in C_0^{\infty}(\Omega)$. (Here we used also the Green's formula). It is enough to prove (2.5) for functions from $C_0^{\infty}(\Omega)$, because $\dot{W}_2^k(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in the space $W_2^k(\Omega)$.

First consider

$$\nabla(1-q)^m f = \nabla(1)^m f + \sum_{|\alpha| \leq 2m-2} \{ (\nabla c_{m,\alpha}) D^\alpha f + c_{m,\alpha} \nabla D^\alpha f \}$$

take into account (2.2) and remark

$$(2.6) \quad \|\omega(|x|)|x|^{-n-k} D^\alpha f\|_{L_2} \leq \text{const} \|f\|_{W_2^k} \quad \left(|z| \leq k \leq \left[\frac{N}{2} \right] \right).$$

(This estimate follows from (2.4)). On the other hand

$$\sqrt{q}(1-q)^m f = \sqrt{q}(1)^m f + \sum_{|\alpha| \leq 2m-2} \sqrt{q} c_{m,\alpha} D^\alpha f$$

hence, taking into consideration the trivial estimates

$$\begin{aligned} |(\sqrt{q} A^m f)(x)| &\leq \text{const } \sqrt{\frac{a(|x|)}{|x|}} \cdot |A^m f(x)| \leq \\ &\leq \text{const } \frac{\sqrt{\omega(|x|)}}{|x|} \cdot |A^m f(x)| \leq \text{const } \sum_{\alpha=2m} \sqrt{\omega(|x|)} |x|^{\alpha-k} |D^\alpha f(x)| \end{aligned}$$

and (by the case $\beta = 0$ of (2.2), we get)

$$\begin{aligned} |(\sqrt{q} c_{m,\alpha} D^\alpha f)(x)| &\leq \text{const } \sqrt{\frac{a(|x|)}{|x|}} \cdot \omega(|x|) \cdot |x|^{\alpha-2m} |D^\alpha f(x)| \leq \\ &\leq \text{const } \frac{\sqrt{\omega(|x|)}}{|x|} \cdot \omega(|x|) |x|^{\alpha-2m} |D^\alpha f(x)| \leq \\ &\leq \text{const } [\omega(|x|)]^{3/2} \cdot |x|^{\alpha-k} \cdot |D^\alpha f(x)| \end{aligned}$$

and also (2.6), the desired estimate (2.5) follows also in the case of $k = 2m+1$. Lemma 2.3 is proved.

LEMMA 2.4. Let A be a positive selfadjoint operator in the Hilbert space \mathcal{H} let $\|f\|_{\mathcal{D}(A)}^2 := \|f\|_{\mathcal{H}}^2 + \|A^\alpha f\|_{\mathcal{H}}^2$ and W be a dense subset in \mathcal{H} which is also a Hilbert space with some norm. Suppose

$$(2.7) \quad W \subset \mathcal{D}(A) \subset \mathcal{H}$$

and

$$(2.8) \quad \|f\|_{\mathcal{D}(A)} \leq \text{const} \|f\|_W \quad (f \in W)$$

are fulfilled. Then for every $\theta \in (0, 1)$, the interpolating space $(\mathcal{H}, W)_{\theta, 2}$ is imbedded in $\mathcal{D}(A^\alpha)$ and the following inequality holds:

$$(2.9) \quad \|A^\alpha f\|_{\mathcal{H}} \leq c_\theta \|f\|_{(\mathcal{H}, W)_{\theta, 2}} \quad (f \in (\mathcal{H}, W)_{\theta, 2}).$$

The constant c_θ does not depend on f .

PROOF. We use below in this work the notations of the book of H. TRIEBEL [20]. Denote by $K(t, f, \mathcal{H}, W)$ the Peetre functional, i.e.

$$K(t, f, \mathcal{H}, W) \stackrel{\text{def}}{=} \inf_{\substack{f_1 + f_2 = f \\ f_1 \in \mathcal{H}, f_2 \in W}} (\|f_1\|_{\mathcal{H}} + t \|f_2\|_W).$$

Using (2.8) we get

$$K(t, f, \mathcal{H}, \mathcal{D}(A)) \leq \text{const } K(t, f, \mathcal{H}, W),$$

hence

$$(2.10) \quad \|f\|_{(\mathcal{H}, \mathcal{D}(A))_{\theta, 2}} \leq \text{const } \|f\|_{(\mathcal{H}, W)_{\theta, 2}}.$$

After this it is enough to remark that $(\mathcal{K}, \mathcal{D}(A))_{\theta, 2} = \mathcal{D}(A^\theta)$ (Cf. TRIEBEL [20], 1.18.10) and

$$(2.11) \quad \|A^\theta f\|_{\mathcal{K}} \leq \text{const} \|f\|_{(\mathcal{K}, \mathcal{D}(A))_{\theta, 2}}.$$

From (2.10) and (2.11) the desired estimate (2.9) follows. Lemma 2.4 is proved.

LEMMA 2.5. For any real number $s \in [0, N/2]$

$$(2.12) \quad \|\hat{L}^{\frac{s}{2}} f\|_{L_2} \leq c_{10} \|f\|_{L_2^s} \quad (f \in \dot{L}_2^s(\Omega)).$$

PROOF. Apply Lemma 2.4 for $\mathcal{B} = L_2(\Omega)$, $W = \dot{W}_2^{[\frac{N}{2}]}(\Omega)$, $A = \hat{L}^{2[\frac{N}{2}]}$ and use TRIEBEL [20], 4.3.2/2. It follows $(L_2, \dot{W}_2^{[\frac{N}{2}]})_{\theta, 2} = L_2^{[\frac{N}{2}]}$. Now let $\Theta = \frac{s}{[N/2]}$ ($0 \leq \Theta \leq 1$), then we get $(L_2, \dot{W}_2^{[\frac{N}{2}]}) = \dot{L}_2^s$ and $A^\theta = \hat{L}^{\frac{\theta}{2}[\frac{N}{2}]} = \hat{L}^{\frac{s}{2}}$.

Now assume (1) is fulfilled and denote $L_0 = -A$, $L = -A + q(x)$. We know that $|D^\alpha q(x)| \leq C_\alpha |x|^{\tau-2-\alpha}$. Let $m = [N/4]$. First we prove the

STATEMENT A. For any $f \in C_0^\infty(\Omega)$

$$\|I(L^m - L_0^m) f(x)\|_{L_p} \leq c \|f\|_{W_2^{\frac{N}{2}}} \quad \text{if } \frac{N}{p} > 2m + 2 - \tau.$$

PROOF. Let $N/p > 2m + 2 - \tau$ i.e. $N/p = 2m + 2 - \tau + \epsilon$ ($\epsilon > 0$). Obviously

$$\|I(L^m - L_0^m) f(x)\| \leq c \sum_{\alpha \leq 0} |x|^{\alpha + \tau - 2m - 2} |D^\alpha f(x)|$$

and

$$\||x|^{\alpha + \tau - 2m - 2} |D^\alpha f(x)|\|_{L_p} \leq c \||x|^{\alpha + \tau - 2m - 2}\|_{L_{q_1}} \cdot \|D^\alpha f\|_{L_{p_1}}$$

$$\text{if } \frac{N}{p} = \frac{N}{p_1} + \frac{N}{q_1}. \quad \text{Let } \frac{N}{p_1} = |\alpha| + \frac{\epsilon}{2}, \quad \frac{N}{q_1} = 2m - |\alpha| - \tau + 2 + \frac{\epsilon}{2}.$$

Then the imbedding $W_2^{\frac{N}{2}} \rightarrow W_{p_1}^{\frac{N}{2}}$ holds, further $q_1(2m + 2 - |\alpha| - \tau) < N$. Hence

$$\||x|^{\alpha + \tau - 2m - 2} |D^\alpha f(x)|\|_{L_p} \leq c \|f\|_{W_2^{\frac{N}{2}}}$$

and

$$\frac{N}{p} = \frac{N}{p_1} + \frac{N}{q_1} = \left(|\alpha| + \frac{\epsilon}{2}\right) + \left(2m - |\alpha| - \tau + 2 + \frac{\epsilon}{2}\right) = 2m - \tau + 2 + \epsilon.$$

Statement A is proved.

COROLLARY 1. For any $f \in C_0^\infty(\Omega)$

$$\|(L^m - L_0^m) f\|_{W_p^{2\delta}} \leq c \|f\|_{W_2^{\frac{N}{2}}} \text{ if } \frac{N}{p} > 2m + 2 - \tau.$$

COROLLARY 2. For any $f \in C_0^\infty(\Omega)$ and $0 < \delta < \frac{N}{4} - m + \frac{\tau}{2}$, $\delta \ll 1$

$$\|(L^m - L_0^m) f\|_{W_2^{2\delta}} \leq c \|f\|_{W_2^{\frac{N}{2}}}.$$

(For the proof take into account the imbedding $W_p^2 \hookrightarrow W_2^{2\delta}$).

STATEMENT B. Let $0 \ll \delta < \frac{N}{4} + \frac{\tau}{2}$. Then for any $f \in C_0^\infty(\Omega)$

$$\|L^\sigma f\|_{L_2} \leq c \|f\|_{W_2^{2\sigma}}.$$

PROOF. Let $\sigma = m + \delta$, $m = [N/4]$, $0 < \delta \ll 1$. Then we have

$$\begin{aligned} \|L^\sigma f\|_{L_2} &= \|L^{m+\delta} f\|_{L_2} \leq \|L^m f\|_{W_2^{2\delta}} \leq \|(L^m - L_0^m) f\|_{W_2^{2\delta}} + \\ &\quad + \|L_0^m f\|_{W_2^{2\delta}} \leq c \|f\|_{W_2^{\frac{N}{2}}} + c \|f\|_{W_2^{2\delta+2m}} \leq c \|f\|_{W_2^{2\delta}}. \end{aligned}$$

Statement B is proved.

COROLLARY. If $\sigma > N/2$ then for any $f \in W_2^{\sigma}$ the Fourier series $\Sigma(f, U_n) \cdot U_n(x)$ converges absolutely and uniformly on every compact subset of Ω . Lemma 2.5 is proved.

REMARK. Taking into account the spectral theorem, (2.12) means that for any $s \in [0, N/2]$ and $f \in L_2^s(\Omega)$ we have the estimate

$$(2.13) \quad \sum_{n=1}^{\infty} |(f, u_n)|^2 \lambda_2^n \leq c_{10} \|f\|_{L_2^s}^2.$$

For the estimation of the Fourier coefficients of functions from the Nikolskii's class $\dot{H}_p^s(\Omega)$ we need the

LEMMA 2.6. Let A be a positive selfadjoint operator in the Hilbert space \mathcal{H} , $W \subset \mathcal{H}$ be a dense subset, which is also a Hilbert space with some norm. Assume $W \subset D(A) \subset \mathcal{H}$ and $\|A f\|_{\mathcal{H}} \leq \text{const} \|f\|_W$ ($f \in W$). Define $q(t) = t/(1+t)$ for $t > 0$. Then we have

$$(2.14) \quad \|q(hA) f\|_{\mathcal{H}} \leq c_{11} h^\theta \|f\|_{W_\Theta} \quad (0 < \theta < 1, h > 0, f \in W_\Theta)$$

where

$$(2.15) \quad W_\Theta = (\mathcal{H}, W)_\Theta, \quad$$

PROOF. According to the spectral theorem we have

$$\|q(hA)f\|_{\mathcal{H}}^2 = \int_0^\infty \left(\frac{h\lambda}{1+h\lambda} \right)^2 d(E_\lambda f, f) \leq \|f\|_{\mathcal{H}}^2$$

and

$$\|q(hA)f\|_{\mathcal{H}}^2 = h^2 \int_0^\infty \frac{\lambda^2 d(E_\lambda f, f)}{(1+h\lambda)^2} \leq h^2 \int_0^\infty \lambda^2 d(E_\lambda f, f) = h^2 \|Af\|_{\mathcal{H}}^2.$$

Now consider the Peetre functional $K(t, f, \mathcal{H}, W) = K(t, f)$. Obviously

$$\|q(hA)f\|_{\mathcal{H}}^p \leq \text{const } K(h, f)$$

and

$$\|f\|_{W_\Theta} = \sup_{0 < t < \infty} t^{-\Theta} K(t, f)$$

(Cf. TRIEBEL [21], p. 23). Consequently

$$K(h, f) \asymp h^\theta \|f\|_{W_\Theta}$$

and

$$\|q(hA)f\|_{\mathcal{H}} \leq \text{const } h^\theta \|f\|_{W_\Theta}.$$

Lemma 2.6 is proved.

COROLLARY.

$$(2.16) \quad \left(\int_{-\mu}^{2\mu} d(E_\lambda f, f) \right)^{1/2} \leq c_{12} \mu^{-\Theta} \|f\|_{W_\Theta} (\mu \geq 1, f \in W_\Theta).$$

Now we give an important special case of (2.16). Let $A = \hat{L}^s \left(0 < s < \frac{N}{4} \right)$, $W = L_2(\Omega)$, $W = W_2^{2s}(\Omega)$. According to the Theorem 2, p. 2.4.2 (16) (page 223) of Triebel's book [20] we have in this special case

$$W_\Theta = (L_2, W_2^{2s})_{2, -\Theta} = H_2^{2s\Theta}$$

and the estimate (2.16) gives

$$(2.17) \quad \left(\int_{-\mu}^{2\mu} d(E_\lambda f, f) \right)^{1/2} \leq c_{12} \mu^{-\Theta s} \|f\|_{H_2^{2s\Theta}} \left(0 < \Theta < 1, 0 < s < \frac{N}{4} \right)$$

if $\{E_\lambda\}$ denotes the spectral family of the operator \hat{L} , i.e. $\hat{L} = \int \lambda dE_\lambda$. (We have used here also the estimate (2.5)). We can write (2.17) also in the form

$$(2.18) \quad \left(\int_{-\mu}^{2\mu} d(E_\lambda f, f) \right)^{1/2} \leq c_{12} \mu^{-\alpha/2} \|f\|_{H_2^\alpha} \left(0 < \alpha < \frac{N}{2}, \mu \geq 1 \right).$$

Hence we obtain the following estimate

$$(2.19) \quad \|f - E_\lambda f\|_{L_2} \leq c_{13} \lambda^{-\alpha/2} \|f\|_{H_2^\alpha} \left(0 < \alpha < \frac{N}{2}, \lambda \geq 1\right).$$

Indeed, using (2.18) we get

$$\begin{aligned} \|f - E_\lambda f\|_{L_2}^2 &= \int_0^\infty d(E_\lambda f, f) = \sum_{k=0}^\infty \int_{2^k \lambda}^{2^{k+1}} d(E_\lambda f, f) = \\ &\leq \text{const} \sum_{k=0}^\infty (2^k \lambda)^{-\alpha} \|f\|_{H_2^\alpha}^2 = \\ &= \text{const} \lambda^{-\alpha} \sum_{k=0}^\infty 2^{-ka} \|f\|_{H_2^\alpha}^2 = \text{const} \lambda^{-\alpha} \|f\|_{H_\mu^\alpha}^2. \end{aligned}$$

and (2.19) is proved.

LEMMA 2.7. Suppose $f \in H_2^\alpha$, $0 < \alpha < N/2$, $0 < a < 1$, $t \geq 1$. Then we have

$$(2.20) \quad |q[(t+a)^2] - q(t^2)| \leq c_{14} \|f\|_{H_2^\alpha} (t+a)^{\frac{N-1}{2}-\alpha}$$

where

$$q(t) \stackrel{\text{def}}{=} (E_t f)(x_0).$$

PROOF. Using (1.16), (2.19) and the Schwartz inequality we obtain

$$\begin{aligned} &|q[(t+a)^2] - q(t^2)| \leq \\ &\leq \left(\sum_{n \in \mathbb{Z}_N \setminus (t+a)^2} |f_n|^2 \right)^{1/2} \left(\sum_{n \in \mathbb{Z}_N \setminus (t+a)^2} |u_n(x_0)|^2 \right)^{1/2} \leq \|f - E_{t^2} f\|_{L_2} \cdot \\ &\cdot \left(\left[\sum_{k=[t]}^{[t+a]} \left[\sum_{n \in \mathbb{Z}_N \setminus t^2} |u_n(x_0)|^2 \right] \right]^{1/2} \leq \text{const} \cdot t^{-\alpha} \|f\|_{H_2^\alpha} \left(\sum_{k=[t]}^{[t+a]} k^{N-\alpha} \right)^{1/2} \leq \\ &\leq \text{const} (t+a)^{\frac{N-1}{2}-\alpha} \cdot \|f\|_{H_2^\alpha}. \end{aligned}$$

Lemma 2.7 is proved.

Denote by $D'(\Omega)$ the space of distributions with support in Ω . We need

LEMMA 2.8 Suppose $u \in D'(\Omega)$ and $|u| \in L_p^{\text{loc}}(\Omega)$, $p > N/2$. Then we have $u \in W_p^{2,\text{loc}}(\Omega)$.

PROOF. Let $\Omega_1 \subset \mathbb{R}^N$ be such a domain for which $\bar{\Omega}_1 \subset \Omega$, and $\text{supp } u \subset \Omega_1$. Then we have $u \in D'(\Omega_1)$ and $|u| \in L_p(\Omega_1)$. Denote $|u| = f$ and define

$$v(x) = c_N \int_{\Omega_1} \frac{f(y)}{|x-y|^{N-2}} dy.$$

We have $v \in W_p^2(\Omega_1)$ and $\mathcal{A}v = f$. Hence, $v \in D'(\Omega_1)$ and $\mathcal{A}(u-v) = 0$ follows, consequently, $u-v$ is a harmonic function in Ω_1 . (We used the result (6.4.1) of Triebel's book [5]). Lemma 2.8 is proved.

LEMMA 2.9. Suppose $q \in L_p(\Omega)$, $p > N/2$. Denote \hat{L} an arbitrary selfadjoint extension of the operator $L(x, D) = -\mathcal{A} + q(x)$. (with domain $C_0^\infty(\Omega)$). Then for any $u \in \mathcal{D}(\hat{L})$: if $\hat{L}u = f$ belongs to the class $L_p(\Omega)$, then $u \in W_p^{2,\text{loc}}(\Omega)$.

PROOF. Let $g \in C_0^\infty(\Omega) = D(\Omega)$, then taking into consideration $C_0^\infty(\Omega) \subset \mathcal{D}(\hat{L})$, we have

$$\begin{aligned} (f, g) &= (\hat{L}u, g) = (u, \hat{L}g) = (u, L(x, D)g) = \\ &= (u, -\mathcal{A}g + q \cdot g) = (u, -\mathcal{A}g) + (qu, g), \end{aligned}$$

i.e.

$$(u, -\mathcal{A}g) = (qu, g) - (f, g) = (qu - f, g)$$

for every $g \in D(\Omega)$. Thus we have the equality

$$\mathcal{A}u = qu - f$$

in the sense of distribution (in the space $D'(\Omega)$).

Now we prove that if $u \in L_{p_1}^{\text{loc}}(\Omega_1)$ and $p_1 < 2N/(2p-N)$, then

$$(2.21) \quad u \in L_{p_2}^{\text{loc}}(\Omega), \text{ where } (1/p_2) = (1/p_1) - [(2/N) - (1/p)].$$

Indeed, if $u \in L_{p_1}^{\text{loc}}(\Omega)$ and $p_1 < pN/(2p-N)$, then we have $q \cdot u \in L_{p_0}^{\text{loc}}(\Omega)$, where $(1/p_0) = (1/p_1) + (1/p)$. So applying Lemma 2.8 and taking into consideration $q \cdot u - f \in L_{p_0}^{\text{loc}}(\Omega)$, we obtain $u \in W_{p_0}^{2,\text{loc}}(\Omega)$, and hence, by an imbedding theorem $u \in L_{p_2}^{\text{loc}}(\Omega)$ follows, where $(N/p_2) = (N/p_0) - 2$, i.e.

$$\frac{1}{p_2} = \frac{1}{p_0} - \frac{2}{N} = \frac{1}{p_1} - \left(\frac{2}{N} - \frac{1}{p} \right).$$

Now apply the statement (2.21) for our case, when $u \in L_2(\Omega)$, because $\mathcal{D}(\hat{L}) \subset L_2(\Omega)$. A finite iterative application of (2.21) gives the statement of Lemma 2.9.

Lemma 2.9 is proved.

After this preparation the theorems follows by the method of the work [6].

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ON SOME SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR

By

I. JOÓ

II. Department of Math. Analysis of the L. Eötvös University, Budapest

(Received May 28, 1982)

DEDICATED TO PROFESSOR ANDRÁS HAJNAL ON THE OCCASION OF HIS 50-th BIRTHDAY

Let $G = (a, b)$ be an arbitrary (finite or infinite) open interval, $q \in L_1^{\text{loc}}(G)$ be an arbitrary (complex) function (i.e. q is integrable on every compact subset of G). Consider the Schrödinger operator

$$l : u \mapsto -\frac{d^2}{dx^2} + q(x).$$

A function $u = u_0$ is said to be an eigenfunction of the operator l (of order 0) with (complex) eigenvalue λ , if u is absolutely continuous with its derivative on every compact subinterval of G and if it satisfies for almost all $x \in G$ the equation

$$lu(x) = \lambda u(x).$$

A function u_i is said to be an eigenfunction of the operator l of order i (≥ 1) with (complex) eigenvalue λ , if u_i is absolutely continuous with its derivative on every compact subinterval of G and if there exist eigenfunctions u_j ($0 \leq j < i$) of order j with the same eigenvalue λ of the operator l such that for almost all $x \in G$ the equations

$$lu_j(x) = \lambda u_j(x) + u_{j-1}(x) \quad (0 \leq j \leq i)$$

hold. Here $u_{-1} = 0$.

In the sequel we shall use the notation

$$\|\cdot\|_{L_p(G)} = \|\cdot\|_p \quad (1 \leq p \leq \infty).$$

In [1] the following is proved

THEOREM A. Suppose $q \in L_1(G)$ and G finite. Then an arbitrary eigenfunction u_i of order i (≥ 0) of the operator l with (complex) eigenvalue λ is necessarily absolutely continuous on G .

(These facts are proved in [3] also for more general differential operators). Furthermore the following estimates hold:

$$(1) \quad \|u_{i-1}\|_\infty \leq C_i(1 + |\sqrt{\lambda}|)(1 + |\operatorname{Im}\sqrt{\lambda}|) \|u_i\|_\infty,$$

$$(2) \quad \|u_i\|_\infty \leq C_i(1 + |\operatorname{Im}\sqrt{\lambda}|)^{1/p} \|u_i\|_p \quad (1 \leq p \leq \infty),$$

$$(3) \quad \|u'_i\|_\infty \leq C_i(1 + |\sqrt{\lambda}|) \|u_i\|_\infty, \quad (i = 0, 1, \dots).$$

The constants $C_i = C_i(b-a, \|q\|_1)$ do not depend on λ .

For $i = 0$, $\lambda \geq 0$ we have (Cf. [1])

$$(4) \quad \|u_0\|_\infty \leq 12 \sqrt{\frac{1}{b-a} + \|q\|_1 \|u_0\|_2}.$$

For fixed i , a , b and q the order of the estimates (1), (2), (3) in λ cannot be improved (this was proved by V. KOMORNÍK in [2]). The estimate (4) shows the dependence of $\|u_0\|_\infty/\|u_0\|_2$ on q . The author showed, that the method of the work [1] gives the following more exact estimates

$$(5) \quad \|u_{i+1}\|_\infty \leq C_i^* \max \left(|\sqrt{\lambda}|, \frac{1}{b-a}, \|q\|_1 \right) \cdot \\ \cdot \max \left(|\operatorname{Im} \sqrt{\lambda}|, \frac{1}{b-a}, \|q\|_1 \right) \|u_i\|_\infty,$$

$$(6) \quad \|u_i\|_\infty \leq C_i^* \left[\max \left(|\operatorname{Im} \sqrt{\lambda}|, \frac{1}{b-a}, \|q\|_1 \right) \right]^{1/p} \|u_i\|_p \quad (1 \leq p \leq \infty)$$

$$(7) \quad \|u'_i\|_\infty \leq C_i^* \max \left(|\sqrt{\lambda}|, \frac{1}{b-a}, \|q\|_1 \right) \|u_i\|_\infty, \quad (\lambda \in \mathbb{R}_+).$$

These estimates were obtained also by V. KOMORNÍK [4] for $\lambda \in \mathbb{C}$ by a slightly different method. Namely V. KOMORNÍK generalized the results of [1] for differential operators of n -th order; to this he got some new ideas which give also the estimates (5), (6), (7).

The aim of the present paper is to investigate the exactness of (5), (6), (7) from point of view of the dependence on q , further, developing the ideas of the papers [1], [2], [6], [10] to prove an equiconvergence theorem for the operator t .

1. First we prove the

STATEMENT 1. If $q \in L_1(G)$ and G is finite then u_i is absolutely continuous on G .

PROOF. This follows from a more general result of [3] (p. 149), but the ideas of the work [1] give a new and simple proof. (Our proof works also in the case of differential operators of n -th order).

We need the mean value formula of E. C. TITCHMARSH [7] (see also [1])

$$(8) \quad u_i(x-t) + u_i(x+t) = 2u_i(x) \cos \sqrt{\lambda}t + \\ + \int_{x-t}^{x+t} [q(\xi)u_i(\xi) - u_{i-1}(\xi)] \frac{\sin \sqrt{\lambda}(t - |x-\xi|)}{\sqrt{\lambda}} d\xi, \quad \text{if } \lambda \neq 0;$$

$$(9) \quad u_i(x-t) + u_i(x+t) = 2u_i(x) + \\ + \int_{x-t}^{x+t} [q(\xi)u_i(\xi) - u_{i-1}(\xi)] (t - |x-\xi|) d\xi, \quad \text{if } \lambda = 0;$$

assuming $x-t, x+t \in G$.

We prove the Statement 1 only for $\lambda \neq 0$ (the case $\lambda = 0$ is similar). Using (8) at $x+t$ in place of x and the trivial estimates

$$|\cos z| \leq 2, |\sin z| \leq 2, |\sin z| \leq 2|z| \quad (\text{if } |\sin z| \leq 1)$$

we obtain for any u_i ,

$$0 < \delta < R \stackrel{\text{def}}{=} \min \left\{ \frac{b-a}{6}, \frac{1}{|\operatorname{Im} \sqrt{\lambda}|}, \frac{1}{4\|q\|_1} \right\}$$

and $a+\delta \leq x \leq (a+b)/2$

$$\begin{aligned} |u_i(x)| &\leq |u_i(x+2R)| + 4|u_i(x+R)| + 2R\|q\|_1 \|u_i\|_{L^\infty(a+\delta, b-\delta)} + \\ &+ 4R^2\|u_{i-1}\|_\infty \leq 5\|u_i\|_{L^\infty(a+R, b-R)} + \frac{1}{2}\|u_i\|_{L^\infty(a+\delta, b-\delta)} + 4R^2\|u_{i-1}\|_\infty. \end{aligned}$$

This inequality is valid by an analogous argument for all $(a+b)/2 \leq x \leq b-\delta$ too, therefore

$$\|u_i\|_{L^\infty(a+\delta, b-\delta)} \leq 5\|u_i\|_{L^\infty(a+R, b-R)} + \frac{1}{2}\|u_i\|_{L^\infty(a+\delta, b-\delta)} + 4R^2\|u_{i-1}\|_\infty.$$

Taking the limit $\delta \rightarrow +0$, we obtain

$$\|u_i\|_\infty \leq 10\|u_i\|_{L^\infty(a+R, b-R)} + 8R^2\|u_{i-1}\|_\infty.$$

Now it follows from $u_{i-1} \in L^\infty(G)$ that $u_i \in L^\infty(G)$ and $u_i'' = (q - \lambda) u_i - u_{i-1}' \in L^1(G)$ if G is a finite interval, i.e. u_i and u_i' are absolutely continuous on \bar{G} . The Statement 1 follows by induction on i .

REMARKS. (1) if G is an infinite interval and u_i is an arbitrary eigenfunction of order i of the operator t with eigenvalue $\lambda > 0$,

$$\sqrt{\lambda} > \int_a^b |q(x)| dx$$

then u_i belongs to the class $L^\infty(G)$. If

$$\sqrt{\lambda} = \int_a^b |q(x)| dx,$$

then this is not true.

The proof is analogous to that of the Statement 1, i.e. we use induction in i , only in this case we need the non-symmetrical Titchmarsh formula, i. e.

$$\begin{aligned} u_i(x+t) &= u_i(x) \cos \mu t + u_i'(x) \frac{\sin \mu t}{\mu} + \\ &+ \int_x^{x+t} [q(\xi)u_i(\xi) - u_{i-1}(\xi)] \frac{\sin \mu(x+t-\xi)}{\mu} d\xi \quad \text{if } \mu := \sqrt{\lambda} \neq 0, \\ u_i(x+t) &= u_i(x) + u_i'(x) \cdot t + \int_x^{x+t} [q(\xi)u_i(\xi) - u_{i-1}(\xi)](x+t)d\xi \\ &\quad \text{if } \mu := \sqrt{\lambda} = 0, x, x+t \in G. \end{aligned}$$

The example $G = (-\infty, +\infty)$, $q \equiv 0$, $\lambda = 0$ shows the validity of the second part of the remark.

(2) The exactness of (1) in λ follows from the example:

$$G = (0, 1), q \equiv 0, u_0(x) = 2i\sqrt{\lambda}e^{i\sqrt{\lambda}x}, u_1(x) = (1-x)e^{i\sqrt{\lambda}x}.$$

Here we have

$$\|u_0\|_\infty/\|u_1\|_\infty = 2|\sqrt{\lambda}| \cdot |\operatorname{Im}\sqrt{\lambda}|.$$

2. Now investigate the estimate (4). It is trivial, that (4) is exact in $(b-a)$. Now we show that it is exact also in q . For this consider the special case of the Legendre functions [8]. They are the eigenfunctions of the Schrödinger operator

$$l = -\frac{d^2}{d\theta^2} + \frac{-1}{4\sin^2 \theta}, \quad G = (0, \pi),$$

namely

$$u_n(\theta) = \sqrt{\sin \theta} p_n(\cos \theta), \quad \lambda_n = \left(n + \frac{1}{2}\right)^2 \quad (n = 0, 1, 2, \dots)$$

where $p_n(x)$ denotes the n -th normalized Legendre polynomial, i.e.

$$\int_{-1}^1 p_n(x) p_m(x) dx = \delta_{n,m}.$$

Laplace proved the following non-refinable estimate [8]

$$(10) \quad |p_n(x)| \leq \frac{\pi}{\sqrt{1-x^2}} \quad (-1 < x < 1; n = 0, 1, \dots)$$

or

$$|\sqrt{\sin \theta} p_n(\cos \theta)| \leq \frac{\pi}{\sqrt{\sin \theta}} \quad (0 < \theta < \pi; n = 0, 1, 2, \dots).$$

On the other hand we obtain from (4) the estimate

$$\|\sqrt{\sin \theta} p_n(\cos \theta)\|_{L^\infty(\epsilon, \pi-\epsilon)} \leq 24 \left| \int_{\epsilon}^{\pi-\epsilon} \frac{d\xi}{(1-\xi^2)^{3/2}} \right|^{1/2} \\ (0 < \epsilon < \pi/4; n = 0, 1, \dots)$$

and hence

$$|\sqrt{\sin \epsilon} p_n(\cos \epsilon)| \leq 24 \left| \int_{\epsilon}^{\pi-\epsilon} \frac{d\xi}{(1-\xi^2)^{3/2}} \right|^{1/2}$$

i.e.

$$(11) \quad |p_n(x)| \leq 24 \left| \frac{1}{\sqrt{1-x^2}} \int_{-x}^x \frac{d\xi}{(1-\xi^2)^{3/2}} \right|^{1/2} \\ (-1 < x < 1; n = 0, 1, 2, \dots).$$

It is easy to see that (10) and (11) are equivalent in order with respect to x , namely, using the notations

$$f(x) = (1-x^2)^{-1/2},$$

$$g(x) = (1-x^2)^{-1/2} \left\{ \int_x^\infty \frac{dz}{(1-z^2)^{3/2}} \right\}^{1/2}$$

and the L'Hospital rule we obtain

$$\lim_{|x| \rightarrow 1^-} \frac{f(x)}{g(x)} = \text{const} \neq 0.$$

According to the equivalence of (10) and (11) we can consider the estimate (4) as a wide generalization of (10) for general Schrödinger operator I . (4) gives (among others) the uniform boundedness of the classical orthonormal systems (Jacobi, Laguerre, Hermite etc.) on the compact subsets of the fundamental interval. Earlier this fact was proved e.g. for the Jacobi functions using (10) and applying the Korovkin theorem. Our proof is much more simple and more general.

In connection with the estimates above occurs PROBLEM 1. Let $G = (0, 1)$, $q \in C(G)$ and suppose

$$q(x) = O\left(\frac{1}{x^\alpha}\right) \quad (x \rightarrow +0, -\infty < \alpha < \infty).$$

What kind of estimates can we state for an arbitrary eigenfunction u_i of the operator I without any apriori assumption on u_i (or with the assumption of type $u_i \in L_p$)?

Probably, the key of the solution of this problem is the method of our proof for the Statement 1. This method allows one to prove the existence of such a function $f(t)$ ($0 < t < \varepsilon$) for which $\delta(t) \leq f(t)$ implies

$$m(t) \stackrel{\text{def}}{=} \|u_i\|_{L^\infty(t, 1/2)} \leq 6\|u_i\|_{L^\infty(t+\delta(t), 1/2)}$$

and hence we obtain information on the behaviour of $m(t)$ for $t \rightarrow +0$. E.g. in the case of $q(x) = O(1/x)$ ($x \rightarrow +0$) we obtain in this way for an arbitrary eigenfunction u_i of the operator I

$$(12) \quad u_i \in L^\infty(+0).$$

On the other hand we obtain from (6)

$$(13) \quad \|u_0\|_\infty \leq \text{const} \|q\|_1 \cdot \|u_0\|_1$$

c. in the case of $u_0 \in L^1(G)$ we can state for $q(x) = O(1/x^2)$ ($x \rightarrow +0$) the estimate

$$(14) \quad u_0(x) = O\left(\frac{1}{x}\right) \quad (x \rightarrow +0).$$

3. At last we prove an equiconvergence theorem, which generalizes the Theorem 1 of [9].

Let G be a finite or infinite open interval and consider the Schrödinger operators

$$L_k u = -u'' + q_k \cdot u \quad (q_k \in L_1^{\text{loc}}(G); k = 1, 2).$$

Let $\{u_n^{(k)}\}_{n=1}^{\infty}$ be a complete in $L_2(G)$ orthonormal system of eigenfunctions (of order 0) of the operator L_k with (complex) eigenvalues $\{\lambda_n^{(k)}\}_{n=1}^{\infty}$ ($k = 1, 2$). For any $f \in L_2(G)$ consider the partial sums

$$\sigma_{\mu}^{(k)}(f, x) := \sum_{\substack{n \\ |\operatorname{Re} \sqrt{\lambda_n^{(k)}}| < \mu}} (f, u_n^{(k)}) u_n^{(k)}(x) \quad (k = 1, 2).$$

Define

$$Z_1 = Z_1(c) = \{z \in \mathbb{C} : |1-z| < c\},$$

$$Z_2 = Z_2(c) = \left\{ z \in \mathbb{C} : \frac{|z|}{|\operatorname{Im} z|} \leq c \right\},$$

$$Z = Z(c) = Z_1 \cup Z_2.$$

We shall prove the following

THEOREM. *If $\{\sqrt{\lambda_n^{(k)}}\} \subset Z(c)$ for $k = 1, 2$ with some $c > 0$, then*

$$(15) \quad \sigma_{\mu}^{(1)}(f, x) - \sigma_{\mu}^{(2)}(f, x) \rightarrow 0 \quad (\mu \rightarrow \infty)$$

uniformly in x on every compact subset of G .

PROOF. We need some lemmas. First introduce the notations

$$\lambda_n^{(k)} = (u_n^{(k)})^2, \quad u_n^{(k)} = x_n^{(k)} + i\beta_n^{(k)}.$$

In the following for the sake of simplicity we omit the index k and $\{\lambda_n\}$ means e.g. $\{\lambda_n^{(1)}\}$ or $\{\lambda_n^{(2)}\}$.

LEMMA I.

$$(16) \quad \sum_{\substack{n \\ |\mu - \lambda_n| \leq 1 \\ |\beta_n| \leq C}} \|u_n\|_{L^{\infty}(K)}^2 \leq C_1 \quad (x \in K, \mu \geq 1)$$

holds, for any compact subset K of G . The constant $C_1 = C_1(K, c)$ depends on K and on c but it does not depend on μ .

PROOF. (16) was stated and used in [6], but the proof given there contains a mistake. The main idea of [6] works and the proof given here is an improvement of that in [6].

Use the Titchmarsh's formula (8) at $i = 0$ and $x+t$ in place of x for u_n . Fix an arbitrary compact subinterval $K = [a, b]$ of G , and define

$$R := \min \left\{ \frac{b-a}{4}, \frac{1}{4C_0C_2(1 + \|q\|_{L^1(K)}\sqrt{b-a})} \right\}$$

(see (19) for c_2) where C_0 is the constant given in (2) for $i = 0$ (also in the following part of this work). We obtain for $x \in K$ and $0 \leq t \leq R$

$$(17) \quad u_n(x) = 2u_n(x+t) \cos \mu_n t - u_n(x+2t) + \\ + \int_x^{x+2t} q(\xi) u_n(\xi) \frac{\sin \mu_n(t - |x+t-\xi|)}{\mu_n} d\xi,$$

and hence, integrating by t from 0 to R

$$Ru_n(x) = \int_0^R [2u_n(x+t) \cos \mu_n t - u_n(x+t)] dt + \\ + \int_0^R 2u_n(x+t) [\cos \mu_n t - \cos \mu_n t] dt + \\ + \int_0^R \int_x^{x+2t} q(\xi) u_n(\xi) \frac{\sin \mu_n(t - |x+t-\xi|)}{\mu_n} d\xi dt = A_n + B_n + C_n,$$

consequently

$$(18) \quad R^2 \sum_{\substack{|\mu - |\alpha_n|| \leq 1 \\ |\beta_n| \leq C}} \|u_n\|_{L^2(K)}^2 \leq 2 \int_a^b \left(\sum_{n=1}^{\infty} |A_n|^2 + \sum_{\substack{|\mu - |\alpha_n|| \leq 1 \\ |\beta_n| \leq C}} |B_n + C_n|^2 \right) dx.$$

By the Bessel inequality we get

$$\sum_{n=1}^{\infty} |A_n|^2 \leq 5R,$$

further, using the trivial estimates

$$(19) \quad |\sin z|, |\cos z| \leq c_2; |\sin z| \leq c_2 |z|; |\cos u - \cos v| \leq c_2 |u - v| \\ (\text{if } z, u, v \in Z_1(c))$$

and also (2) we obtain

$$|B_n| \leq C_0 C_2 R^2 \|u_n\|_{L^2(K)}, \\ |C_n| \leq C_0 C_2 R^2 \|q\|_{L^1(K)} \|u_n\|_{L^2(K)}.$$

Summarizing our estimates and taking into account the definition of R we obtain

$$\int_a^b \left(\sum'_{\substack{|\mu - |\alpha_n|| \leq 1 \\ |\beta_n| \leq C}} |B_n + C_n|^2 \right) dx \leq \frac{1}{4} R^2 \sum'_{\substack{|\mu - |\alpha_n|| \leq 1 \\ |\beta_n| \leq C}} \|u_n\|_{L^2(K)}^2,$$

¹ Σ' denotes that we consider only finite sum. Hence (20) follows, where infinitely many terms are also admitted.

consequently

$$(20) \quad \sum_{\substack{|\mu_n| < |\beta_n| \leq 1 \\ |\beta_n| \neq c}} \|u_n\|_{L^2(K)} < \frac{20(b-a)}{R}.$$

Now applying (2) again, the desired estimate (16) follows. Lemma 1 is proved.

LEMMA 2. For every $x \in G$ there exists a compact neighbourhood $K = K_x$ of x and a number $R = R_x > 0$ such that the following estimate holds

$$(21) \quad \|u_n^*\|_{L^\infty(K_R)} \leq 10 \|u_n^*\|_{L^\infty(K_R \setminus K)}, \quad (n = 1, 2, \dots; |\beta_n| \geq c),$$

where

$$(22) \quad u_n^*(x) := u_n(x) \operatorname{ch}[\beta_n \varrho(x)],$$

$$\varrho(x) := \operatorname{dist}(x, \partial K_R),$$

$$(23) \quad K_R := \{x \in G : \operatorname{dist}(x, K) \leq R\}.$$

PROOF. Let $K = [a, b]$ be a sufficiently small compact connected neighborhood of the fixed $x \in G$. Then we have for $R := (b-a)/4 : K_R \subset G$ and

$$(24) \quad \operatorname{sh}(cR) \geq 3, \quad \operatorname{cth}(cR) \leq 2,$$

$$\|q\| := \|q\|_{L^1(K_R)} \leq \min \left\{ \frac{c}{20}, \frac{1}{2000R} \right\}.$$

We shall often use the trivial estimates

$$\operatorname{sh}(\operatorname{Im} z) \leq |\sin z| \leq \operatorname{ch}(\operatorname{Im} z)$$

$$\operatorname{sh}(\operatorname{Im} z) \leq |\cos z| \leq \operatorname{ch}(\operatorname{Im} z).$$

By the second inequality we get

$$|u_n^*(x)| \leq |\operatorname{cth}[\beta_n \varrho(x)]| \cdot |u_n(x) \cos[\mu_n \varrho(x)]|,$$

and hence, applying the Titchmarsh formula (8) we obtain (using also (24))

$$\begin{aligned} |u_n^*(x)| &\leq |\operatorname{cth}[\beta_n \varrho(x)]| \left\{ |u_n(x - \varrho(x))| + |u_n(x + \varrho(x))| + \frac{\|q\|}{|\mu_n|} \|u_n^*\|_{L^\infty(K_R)} \right\} \leq \\ &\leq 2 \left\{ 2 \|u_n^*\|_{L^\infty(K_R \setminus K)} + \frac{1}{\operatorname{cth} \beta_n R} \|u_n^*\|_{L^\infty(K)} + \frac{\|q\|}{c} \|u_n^*\|_{L^\infty(K_R)} \right\} \leq \\ &\leq 4 \|u_n^*\|_{L^\infty(K_R \setminus K)} + \frac{1}{\operatorname{sh} \beta_n R} \|u_n^*\|_{L^\infty(K)} + \frac{\|q\|}{c} [\|u_n^*\|_{L^\infty(K_R \setminus K)} + \|u_n^*\|_{L^\infty(K)}] \leq \\ &\leq 5 \|u_n^*\|_{L^\infty(K_R \setminus K)} + \frac{1}{2} \|u_n^*\|_{L^\infty(K)}, \end{aligned}$$

i.e.

$$\|u_n^*\|_{L^\infty(K)} \leq 5 \|u_n^*\|_{L^\infty(K_R \setminus K)} + \frac{1}{2} \|u_n^*\|_{L^\infty(K)}.$$

Lemma 2 is proved.

LEMMA 3. Under the assumptions of the Lemma 2 we have

$$(25) \quad \|u_n^*\|_{L^\infty(K_R \setminus K)} \leq \frac{6}{\sqrt{R}} \|u_n^*\|_{L^2(K)} \quad (n \in \mathbb{N}, \beta_n \geq c).$$

PROOF. Suppose $x \in [a-R, a]$, $R \leq t \leq 2R$. Multiply both sides of (17) by $\operatorname{ch} [\beta_n \varrho(x)]$ and take into account

$$(\operatorname{ch} x)(\operatorname{ch} y) = \operatorname{ch}(x+y) \quad (x, y \geq 0)$$

and

$$\operatorname{ch} [\beta_n(t - |x+t-z| + \varrho(x))] = \operatorname{ch} [\beta_n \varrho(z)] \quad (x \leq z \leq x+2t).$$

We obtain

$$|u_n^*(x)| \leq |u_n^*(x+t)| + |u_n^*(x+2t)| + \frac{\|q_n^*\|}{c} \|u_n^*\|_{L^\infty(K_R)},$$

Integrating both sides by t from R to $2R$ and applying the Schwarz inequality, we obtain

$$|u_n^*(x)| \leq \frac{3}{\sqrt{R}} \|u_n^*\|_{L^2(K)} + \|q_n^*\|_{L^\infty(K_R)}.$$

This inequality is valid by an analogous argument for all $b \leq x \leq b+R$ too, therefore

$$\|u_n^*\|_{L^\infty(K_R \setminus K)} \leq \frac{3}{\sqrt{R}} \|u_n^*\|_{L^2(K)} + \frac{\|q_n^*\|}{c} \|u_n^*\|_{L^\infty(K_R)},$$

and hence, taking into account (21) and (24)

$$\|u_n^*\|_{L^\infty(K_R \setminus K)} \leq \frac{3}{\sqrt{R}} \|u_n^*\|_{L^2(K)} + \frac{1}{2} \|u_n^*\|_{L^\infty(K_R \setminus K)}.$$

Lemma 3 is proved.

LEMMA 4. Under the assumptions of the Lemma 1 we have

$$(26) \quad \|u_n^*\|_{L^2(K)} \leq 2 \|u_n^{**}\|_{L^2(K)},$$

where

$$u_n^{**}(x) := u_n(x) \operatorname{sh} [\beta_n \varrho(x)].$$

PROOF. It is enough to remark that $x \notin K$ implies $\varrho(x) \geq R$ and to take into account (24).

COROLLARY.

$$(27) \quad \|u_n^*\|_{L^\infty(K_R)} \leq \frac{120}{\sqrt{R}} \|u_n^{**}\|_{L^2(K)}.$$

LEMMA 5. Under the conditions of the Lemma 1 we have

$$(28) \quad \sum_{\substack{\beta_n > C \\ |\alpha_n| \leq C \beta_n}} [\|u_n\|_{L^\infty(K)} \operatorname{ch} (\beta_n R)]^2 \leq C_3(K).$$

PROOF. Calculate the following integral, using the Titchmarsh's formula (8) at $i = 0$:

$$D_n := \int_0^{\varrho(x)} u_n(x+t) \frac{u_n(x-t)}{2} dt - \int_0^{\varrho(x)} u_n(x) \cos \mu_n t dt + \\ + \int_0^{\varrho(x)} \int_{x-t}^{x+t} q(z) u_n(z) \frac{\sin \mu_n(t-x-z)}{\mu_n} dz dt \quad (x \in K).$$

Hence, using the Bessel inequality, (24) and (27) further taking into account that $\varrho(x) < 3R$, we obtain

$$\int_0^{\varrho(x)} u_n(x) \cos \mu_n t dt \leq |u_n(x)| \cdot \frac{\operatorname{ch} \beta_n \varrho(x)}{\mu_n} \leq |D_n| \leq \\ \leq 3\|q\|R \frac{1}{|\mu_n|} \|u_n^{**}\|_{L^\infty(K_R)} \leq |D_n| \leq \frac{360}{2000 |\mu_n| \sqrt{R}} \|u_n^{**}\|_{L^2(K)},$$

i.e.,

$$\sum_{|\beta_n| \leq C} \left[\|u_n^{**}\|_{L^2(K)} \frac{1}{|\mu_n|} \right]^2 \leq 8R \sum_{n=1}^{\infty} |D_n|^2 \leq \frac{1}{2} \sum_{|\beta_n| \leq C} \left[\|u_n^{**}\|_{L^2(K)} \frac{1}{|\mu_n|} \right]^2,$$

or

$$\sum_{|\beta_n| \leq C} \left[\|u_n^{**}\|_{L^2(K)} \frac{1}{|\mu_n|} \right]^2 \leq 100 R^2.$$

Hence, using (27) we obtain

$$\sum_{|\beta_n| \leq C} \left[\|u_n^{**}\|_{L^\infty(K_R)} \frac{1}{|\mu_n|} \right]^2 \leq 1000(120)^2 R.$$

Taking into consideration the inequalities

$$|\mu_n| = c |\beta_n|, \\ |u_n(x) \cdot \operatorname{ch} \beta_n R| \leq |u_n^*(x)| \quad (x \in K); \\ \operatorname{ch}(x) \leq \frac{\operatorname{ch}(2x)}{2x} \quad (x > 0),$$

the desired estimate (28) follows.

Lemma 5 is proved.

Consider the function

$$v = v_R(|x-y|, \mu) = \begin{cases} \frac{1}{\pi} \frac{\sin \mu|x-y|}{|x-y|} & \text{if } |x-y| \leq R \\ 0 & \text{if } |x-y| > R, \end{cases}$$

$$\mu > 0, 0 < 2R < \operatorname{dist}(K, \partial G), x \in K$$

for any fixed compact interval $K \subset G$, and calculate its Fourier coefficients with respect to the system $\{u_n\}$. We obtain by (8)

$$\begin{aligned} v_n := \int_a^b v_R(|x-y|, \mu) u_n(y) dy &= \frac{2}{\pi} u_n(x) \int_0^R \frac{\sin \mu t \cos \mu_n t}{t} dt + \\ &+ \int_{x-R}^{x+R} q(\xi) u_n(\xi) \gamma''_{\mu_n}(R, |x-\xi|) d\xi, \\ \gamma''_{\mu_n}(R, |x-\xi|) &:= \int_{x-\xi}^R \frac{\sin \mu t \sin \mu_n(t-|x-\xi|)}{\pi t \mu_n} dt. \end{aligned}$$

Using the notation

$$S_n(f, x) := \int_G v_R(|x-y|, \mu) f(y) dy = \frac{1}{\pi} \int_{x-R}^{x+R} f(y) \frac{\sin \mu(x-y)}{(x-y)} dy$$

we have for an arbitrary $f \in L_2(G)$ the equation (cf. [9])

$$\begin{aligned} (29) \quad S_n(f, x) - \sigma_n(f, x) &= \frac{1}{2} \sum_{|\alpha_n| = \mu} (f, u_n) \cdot u_n(x) + \\ &+ \sum_{n=1}^{\infty} \left\{ \frac{2}{\pi} \int_0^R \frac{\sin \mu t \cos \mu_n t}{t} dt - \delta_{\mu_n}^{\mu} \right\} (f, u_n) u_n(x) + \\ &+ \sum_{n=1}^{\infty} \left[\int_{x-R}^{x+R} q(\xi) u_n(\xi) \gamma''_{\mu_n}(R, |x-\xi|) d\xi \right] (f, u_n), \\ \delta_{\mu}^{\mu} &:= \int_0^{\pi} \frac{\sin \mu t \cos \mu t}{t} dt \left(= \begin{cases} 1, & \mu > 0 \\ \frac{1}{2}, & \mu = 0 \\ 0, & \mu < 0 \end{cases} \right), \quad (\mu, v > 0), \end{aligned}$$

where $x \in G$ is an arbitrary fixed point, K denotes a compact connected neighborhood of x and $R > 0$. Suppose they are chosen so that our lemmas 1-5 are fulfilled. We shall prove that the right hand side of (29) has the order $O(1)$ ($\mu > 1$) uniformly in $K = K_x$ and hence the statement of the Theorem follows by Borel's covering theorem.

To this we need the following estimates:

$$(30) \quad E_n := \frac{2}{\pi} \int_0^R \frac{\sin \mu t \cos \mu_n t}{t} dt - \delta_{\mu_n}^{\mu} \approx \frac{c_4(R)}{1 + |\mu - |\alpha_n||} \operatorname{ch}(\beta_n R),$$

$$(31) \quad |\gamma''_{\mu_n}(R, \theta)| \approx \frac{c_3(R)}{1 + |\alpha_n|^{1-\epsilon}} \operatorname{ch}(\beta_n R) \quad (0 < \theta < R, \epsilon > 0).$$

First we prove (30). An easy calculation gives

$$\begin{aligned} E_n &= \frac{2}{\pi} \int_0^R \frac{\sin \mu t}{t} [\cos \alpha_n t \operatorname{ch} \beta_n t - i \sin \alpha_n t \operatorname{sh} \beta_n t] dt - \delta''_{\alpha_n} = \\ &\leq \frac{2}{\pi} \left\{ \int_0^R \frac{\sin \mu t \cos \alpha_n t}{t} dt + \int_0^R \frac{\sin \mu t \cos \alpha_n t \operatorname{ch} (\beta_n t) - 1}{t} dt + \right. \\ &\quad \left. + \int_0^R \frac{\sin \mu t \sin \alpha_n t \operatorname{sh} \beta_n t}{t} dt \right\} = \{I_1 + I_2 + I_3\} \frac{2}{\pi}, \\ I_1 &= \frac{2}{1 + |\mu - |\alpha_n|| \cdot R} \end{aligned}$$

([9]). Integrating by parts we get

$$\begin{aligned} I_1 &= \frac{2}{|\mu - |\alpha_n||} \left[\frac{\operatorname{ch} \beta_n R - 1}{R} + \int_0^R \left(\frac{\operatorname{ch} \beta_n t - 1}{t} \right)' dt \right] = \frac{4}{R|\mu - |\alpha_n||} \operatorname{ch} \beta_n R, \\ I_2 &= \frac{2}{|\mu - |\alpha_n||} \left[\frac{\operatorname{sh} \beta_n R}{R} + \int_0^R \left(\frac{\operatorname{sh} \beta_n t}{t} \right)' dt \right] = \frac{4}{R|\mu - |\alpha_n||} \operatorname{sh} \beta_n R. \end{aligned}$$

Summarizing the estimates (30) follows. For the proof of (31) first remark that obviously

$$|\gamma''_{\alpha_n}(R, \theta)| \leq \int_0^R \frac{\sin \mu_n t}{\pi \mu_n t} dt,$$

hence, in the case of $|\alpha_n| \leq 1/R$ we obtain

$$|\gamma''_{\alpha_n}(R, \theta)| \leq \int_0^R \operatorname{ch} \beta_n t dt \leq R \operatorname{ch} (\beta_n R),$$

and in the case of $|\alpha_n| \geq 1/R$

$$\begin{aligned} |\gamma''_{\alpha_n}(R, \theta)| &\leq \int_0^{1/|\alpha_n|} \operatorname{ch} (\beta_n t) dt + \int_{1/|\alpha_n|}^R \frac{\operatorname{ch} (\beta_n t)}{|\alpha_n| t} dt = \\ &= \frac{\operatorname{ch} (\beta_n R)}{|\alpha_n|} [1 + \ln R + \ln |\alpha_n|]. \end{aligned}$$

The estimates (30) and (31) are proved. Using these estimates, (16), (28) and the Schwarz inequality we obtain

$$\begin{aligned} |S_\mu(f, x) - \sigma_\mu(f, x)|^2 &\leq \sum_{n=1}^{\infty} |(f, u_n)|^2 \left\{ \frac{1}{2} \sum_{|x_n|=\mu} |u_n(x)|^2 + \sum_{n=1}^{\infty} |E_n u_n(x)|^2 + \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left| \int_{|x-R|}^{x+R} q(\xi) u_n(\xi) \gamma''_{\mu n}(R, |x-\xi|) d\xi \right|^2 \right\} \leq \\ &\leq C_6(K, R, \varepsilon) \|f\|_{L^2(G)} \sum_{n=1}^{\infty} \left\{ \frac{1}{1 + |\alpha_n|^{2+2\varepsilon}} + \frac{1}{1 + |\mu - |\alpha_n||^2} \right\} \text{ch}(\beta_n R) \|u_n\|_{L^2(K)} \leq \\ &\leq C(K) \|f\| \sum_{k=0}^{\infty} \left\{ \frac{1}{1 + k^{2+\varepsilon}} + \frac{1}{1 + |\mu - k|^2} \right\} \left[\sum_{k \neq |\alpha_n| \leq k+1} (\|u_n\|_{L^2(K_R)} \text{ch}(\beta_n R))^2 \right] \leq \\ &\leq C(K) \cdot \|f\|. \end{aligned}$$

We obtained: every $x \in G$ has a compact connected neighborhood $K = K_x$ such that

$$|S_\mu(f, x) - \sigma_\mu(f, x)| \leq C(K) \|f\|_{L^2(G)} \quad (x \in K, \mu \approx 1).$$

Hence by Borel's covering theorem the last estimate follows for every compact set $K \subset G$. For $f = u_n$ we have

$$S_\mu(f, x) - \sigma_\mu(f, x) \rightarrow 0 \quad (\mu \rightarrow \infty)$$

and using the Banach-Steinhaus theorem we obtain

$$(S_\mu - \sigma_\mu)(f, x) \rightarrow 0 \quad (\mu \rightarrow \infty, f \in L^2(G))$$

uniformly in x on every compact subset of G . Hence the Theorem follows by the triangle inequality. The Theorem is proved.

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MULTI-SYMMETRIC PACKING OF EQUAL CIRCLES ON A SPHERE

By

T. TARNAI

Hungarian Institute for Building Science, Budapest

(Received July 12, 1982)

1. Introduction

The arrangement of the openings on the surface of certain spherical pollen grains has inspired the following mathematical problem (Tammes problem) [12]: To distribute n points on the sphere so that the least distance between the points should be as great as possible. This problem is equivalent to the determination of the largest diameter d of n equal circles which can be packed on the surface of a sphere without overlapping.

The literature of the Tammes problem is surveyed, e.g., in FEJES TÓTH's books [4], [5], and more recent results up to $n = 60$ are summarized in SZÉKELY's paper [11]. Some of the spherical circle-packings listed in [11] have been improved by DANZER [3] ($n = 17, 32$) and, more recently, by TARNAI and GÁSPÁR [13] ($n = 18, 27, 34, 35, 40$); and a new packing for $n = 80$ has also been constructed [14].

The aim of this paper is to present constructions for the packing of 54, 72 and 132 equal circles on a sphere and, in this way, to give lower bounds for the extremal density of packing.

2. Structure of virus coats and circle-packings of Robinson

Electron microscope observations show that many viruses have a shape of a regular icosahedron and their structure units, in general, form a regular pattern on them. Studying the structure of virus coats CASPAR and KLUG [1] discovered that a regular triangular plane net can be folded into a polyhedral net having icosahedral symmetry in rotation. This fact was also discovered earlier by GOLDBERG [6].

COXETER [2] has shown that this operation can be done in the following way. "Consider the regular tessellation {3, 6}, which consists of equilateral triangles, six at each vertex, filling and covering the Euclidean plane. Proceed from a vertex A along one edge, continuing in the same direction until b edges have been traversed, then change direction by 60° and proceed straight along c edges to a new vertex B " (Fig. 1).

The starting point A and the end point B of this "knight's move" determine a straight line-segment which can be considered as an edge of an equilateral

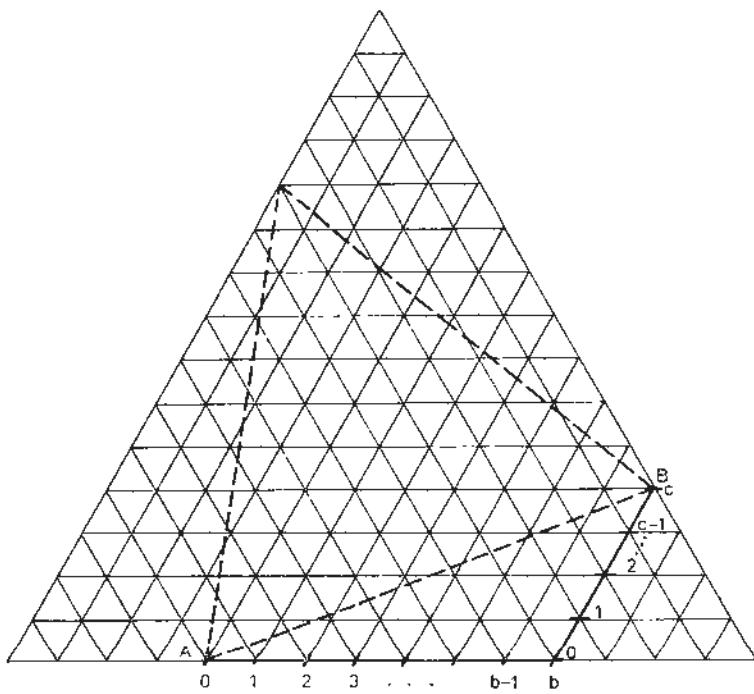


Fig. 1.

triangle (dashed lines in Fig. 1). This "large" equilateral triangle defines a regular plane tessellation $\{3, 6\}$ every vertex of which is a vertex of the original "small" tessellation $\{3, 6\}$. The edge net of an icosahedron can be given by the edge net of the "large" tessellation $\{3, 6\}$. Because of the rotational symmetries of the "large" tessellation supplemented by the "small" one, a "small" triangular tessellation is obtained on the surface of the icosahedron. This was denoted by

$$\{3, 5+\}_{b,c}$$

by COXETER. In this symbol, number 3 means that the tessellation consists of triangles, and notation $5+$ refers to the fact that five and more than five (i.e., six) triangles meet in the vertices of the tessellation.

Repeating the previous argument and using an analogy with the Coxeter symbol $\{3, 5+\}_{b,c}$ we can define tessellations on the regular octahedron:

$$\{3, 4+\}_{b,c}$$

and also on the regular tetrahedron:

$$\{3, 3+\}_{b,c}$$

When the polyhedral tessellation $\{p, q+\}_{b,c}$ ($p = 3; q = 3, 4, 5$; b and c are integers, positive or zero, not both zero) is "blown up" onto a sphere then

the edge-lengths in the obtained spherical tessellation are, in general, different. But if the way of "blowing up" is not fixed, then some of the edges can be chosen to be of the same length. And if the edges of equal length are chosen properly, then the system of these edges can be considered as the graph of a packing of equal circles on the sphere. (The vertices of the graph are the centres of the spherical circles and the edges of the graph are great-circle arcs joining the centres of the touching spherical circles.)

In his paper ROBINSON [9] used the tessellation $\{3, q+\}_{2, 1}$, $q = 3, 4, 5$, for the packing of 12, 24, 60 circles on a sphere (Fig. 2 (a)) and the tessellation $\{3, q+\}_{3, 1}$, $q = 3, 4, 5$, for the packing of 24, 48, 120 circles on a sphere (Fig. 2 (b)). Fig. 2 shows a part of the graphs where the triangle composed of dashed lines is a face of the regular tetrahedron $\{3, 3\}$, octahedron $\{3, 4\}$ and icosahedron $\{3, 5\}$, respectively.

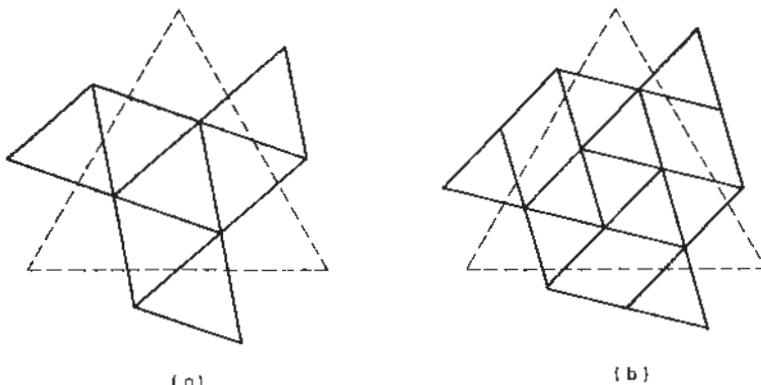


Fig. 2.

3. New results

Consider the graphs of the circle-packings of ROBINSON (Fig. 2). Modify them in such a way that, while the character of the base tessellations is preserved, new q -valent vertices are added to the graphs at the vertices of the polyhedra $\{3, q\}$, $q = 3, 4, 5$, and then the regular q -gons around these vertices are removed. In this way the edge-systems in Fig. 3 are obtained, where the triangle composed of dashed lines is again a face of the regular polyhedra $\{3, q\}$, $q = 3, 4, 5$.

On the spherical tetrahedron, the edge-system in Fig. 3 cannot determine good spherical circle-packings. For this reason, packing of 16 and 28 circles in tetrahedral symmetry will not be considered further.

On the spherical octahedron, the arrangements according to Fig. 3(a) and Fig. 3(b) result in packings of 30 circles of diameter $d = 37^\circ 28' 43.0''$ and 54 circles of diameter $d = 28^\circ 16' 32.7''$, respectively. The result for $n = 30$ is not of interest since the obtained diameter is less than that in packings due to GOLDBERG [7] and STROHMAIER [10]. The diameter obtain-

ed for $n = 54$, apart from a difference of $0.1''$, is the same as that of SZÉKELY [11]. This slight difference is presumably due to inaccuracies in his numerical computations. Since the graph of the obtained packing in a stereographic projection has the properties of a spiral of four branches (Fig. 4) as described in [11], it is very probable that we have found the missing graph of SZÉKELY's packing of 54 circles. For, the figure of the graph for $n = 54$ was unfortunately left out of the paper of SZÉKELY [11] and this graph has been questioned by MELNYK, KNOP and SMITH [8].

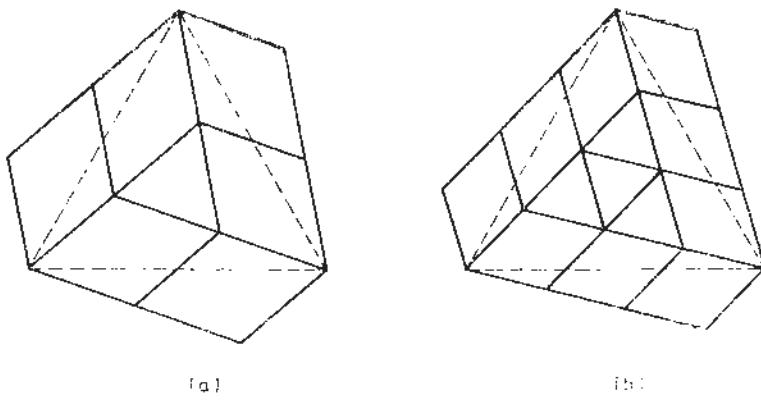


Fig. 3.

Considering the nets in Fig. 3(a) and Fig. 3(b) as parts of the graphs of circle-packings on the spherical icosahedron we obtain packings of 72 circles of diameter $d = 24^\circ 50' 23.1''$ and 132 circles of diameter $d = 18^\circ 21' 59.5''$, respectively.

Let e denote the number of edges of the graph. It is easy to see that $2n - 3 + e$ holds for the graphs presented in the cases of $n = 54, 72, 132$, and so it seems very probable that the graphs have no degree of freedom in the Danzerian sense [3], and so they are rigid. Consequently the packings cannot be improved by moving the graphs [13].

Table I.

n	d	D	v	e	$2n - 3 + e$	tessellation	graph
54	$28^\circ 16' 32.7''$	0.8178	120	13	{3,4+}\$_{3,1}\$		Fig. 3(b), Fig.4
72	$24^\circ 50' 23.1''$	0.8425	150	9	{3,5+}\$_{2,1}\$		Fig.3(a)
132	$18^\circ 21' 59.5''$	0.8459	300	39	{3,5+}\$_{3,1}\$		Fig.3(b)

The numerical data of the newly discovered circle-packings are collected in Table I. Here the quantity D , which represents the density of packing

(defined as the ratio of the total area of the surface of the spherical caps and the surface area of the sphere), has been computed by the expression

$$D = \frac{n}{2} \left(1 - \cos \frac{d}{2} \right).$$

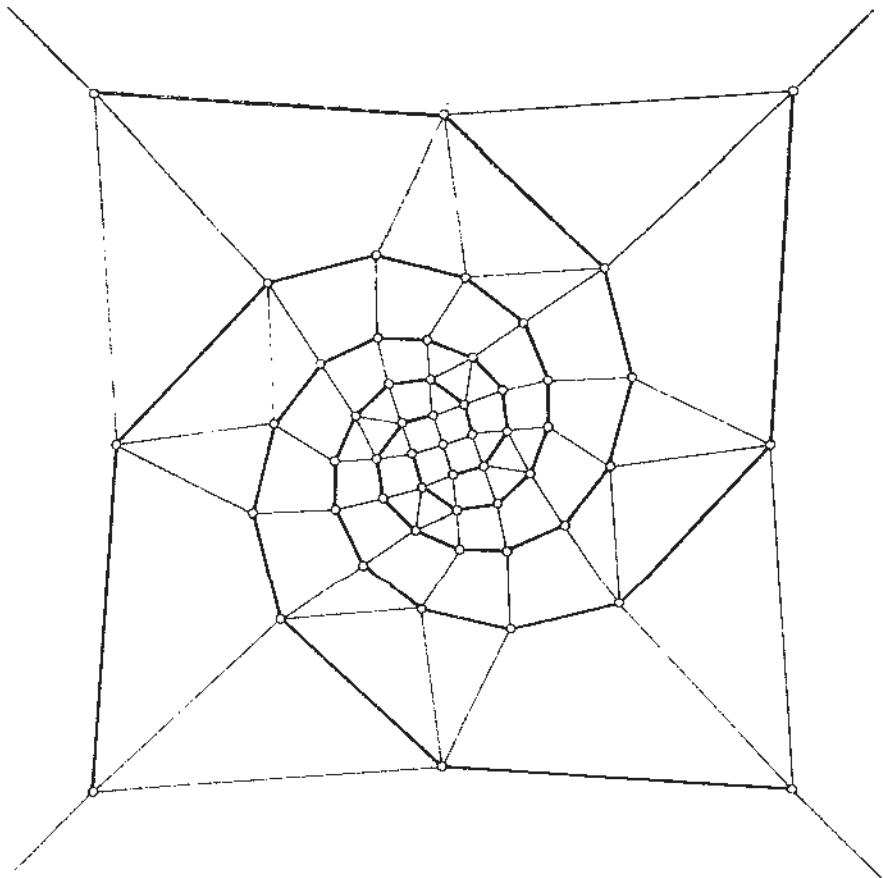


Fig. 1.

Acknowledgements. I thank DR. C. R. CALLEDINE, who aroused my interest in morphology of viruses and made many suggestions for improvement of the text. I also thank Mr. J. GYURKÓ for the assistance in the numerical computations.

Note added in proof. After the manuscript went to press, it came to our knowledge that the result presented here for 72 circles was also obtained earlier by MACKAY et al. [15].

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ON THE \mathcal{K}_ϕ -SPACES WITH GENERAL YOUNG-FUNCTION Φ

By

N. L. BASSILY and J. MOGYORÓDI

Department of Probability Theory of L. Eötvös University, Budapest

(Received August 4, 1983)

1. We continue our work [1] and we use the notions and notations of it. The main purpose of the present note is to give a maximal inequality for the random variables belonging to the space \mathcal{K}_ϕ . This is a direct generalization of the spaces \mathcal{K}_p defined in [1].

Let $X \in L_1(\Omega, \mathcal{A}, P)$ and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ be an increasing sequence of σ -fields of events. Here (Ω, \mathcal{A}, P) is a probability space. Put

$$X_n = E(X|\mathcal{F}_n)$$

and suppose that $X_0 = 0$ a.e., further that $\mathcal{F}_\infty = \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n) = \mathcal{A}$.

It is known that if X belongs to L^ϕ then a necessary and sufficient condition for the validity of the maximal inequality

$$E(\Phi(X_n^*)) \leq E(\Phi(a|X_n|)),$$

where

$$X_n^* = \max_{0 \leq k \leq n} |X_k|$$

and a is a positive constant, is that the conjugate Young-function Ψ be of moderated growth [2].

The relevant knowledge about the theory of Young-functions and Orlicz spaces can be found e.g. in [4].

We cannot prove in general that X belongs to L^ϕ if it belongs to \mathcal{K}_ϕ . For this reason it is of interest to establish a maximal inequality for the \mathcal{K}_ϕ martingales.

For the random variables belonging to \mathcal{K}_ϕ we shall show among others the maximal inequality

$$\|X_n^*\|_\phi \leq C_\phi \|X\|_{\mathcal{K}_\phi},$$

which is true if Φ and Ψ , the conjugate of Φ , have finite power. Here $C_\phi > 0$ is a constant depending only on Φ .

2. Let Φ be a Young-function and consider the family Γ_X^ϕ of the variables γ defined by the formula

$$\Gamma_X^\phi = \{\gamma : \gamma \in L^\phi, E(|X - X_{n-1}| \mid \mathcal{F}_n) \leq E(\gamma \mid \mathcal{F}_n) \text{ a.e. } \forall_n\}.$$

If Γ_X^ϕ is not empty then we set

$$\|X\|_{\mathcal{K}_\phi} = \inf_{\gamma \in \Gamma_X^\phi} \|X\|_\phi.$$

It is easy to see that $\|X\|_{\mathcal{K}_\phi}$ is a norm on the set \mathcal{K}_ϕ .

DEFINITION 1. We say that $X \in \mathcal{K}_\phi$ with norm $\|X\|_{\mathcal{K}_\phi}$, if Γ_X^ϕ is not empty.

THEOREM 1. Let $X \in \mathcal{K}_\phi$ and suppose that $X > 0$ a.e. Let Φ be a Young function having finite power p , i.e. the quantity

$$p := \sup_{x>0} \frac{xq(x)}{\Phi(x)}$$

is finite. For arbitrary $k \geq 1$ set

$$X^{*(k)} := \sup_{n \geq k} |X_n - X_{k-1}|.$$

Then for any constants $q > 1$ and $c > 1$ we have

$$E\left(\Psi\left(q\left(\frac{X^{*(k)}}{\frac{c}{c-1}A(c)\|X\|_{\mathcal{K}_\phi}}\right)\right)\right) \leq \frac{1}{q-1},$$

where $A = A(c)$ is the constant such that

$$q(cx) \leq Aq(x).$$

PROOF. Since $\Psi(q(x)) = xq(x) - \Phi(x)$ increases in x , by the monotone convergence theorem it suffices to prove the inequality of the assertion with

$$\max_{k \leq t \leq n} |X_t - X_{k-1}|$$

instead of $X^{*(k)}$. Also, by using the idea of Theorem 1 and Corollary 1 of [1] it is enough to show the validity of our inequality with

$$\max_{1 \leq t \leq n} |X_t| = X_n^*.$$

By the result of Theorem 1 of [1] this random variable for arbitrary $\beta > \alpha > 0$ satisfies the inequality

$$(\beta - \alpha)P(X_n^* \geq \beta) \leq E(\gamma \zeta(X_n^* \geq \alpha)),$$

where $\gamma \in \Gamma_X^\phi$ is arbitrary and $\zeta(B)$ denotes the indicator of the event B .

For arbitrary $a > 0$ define

$$X_n^{**} = \min(X_n^*, a).$$

Then $X_n^{**} \in L_\infty$ and for arbitrary $\lambda > 0$ we have

$$\zeta(X_n^{**} > \lambda) = \begin{cases} 0 & \text{if } \lambda > a \\ \zeta(X_n^* > \lambda), & \text{if } \lambda \leq a. \end{cases}$$

Consequently it follows that

$$(\beta - z) P(X_n^{**} > \beta) \leq E(\gamma q(X_n^{**} > z)).$$

Choose $\beta = c z$, where $c > 1$ is a constant and integrate the obtained inequality with respect to the measure generated by $q(z)$. Using the Fubini theorem we get

$$(c-1)E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right] \leq E(\gamma q(X_n^{**})).$$

For arbitrary $c > 1$ there exists $A = A(c)$ satisfying the inequality

$$q(cx) \leq Aq(x), \quad x \geq 0,$$

since Φ has finite power p . Thus from the preceding inequality it follows that

$$(c-1)E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right] \leq A(c)E\left[\gamma q\left(\frac{X_n^{**}}{c}\right)\right].$$

Now apply to the right hand side the Young-inequality to get

$$(c-1)E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right] \leq A(c)b\left[E\left(\Phi\left(\frac{\gamma}{b}\right)\right) + E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right]\right],$$

where $b > 0$ is a constant to be determined later. Since $X_n^{**} \in L_\infty$ we can rearrange this inequality to obtain

$$\{c-1 - A(c)b\}E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right] \leq A(c)bE\left(\Phi\left(\frac{\gamma}{b}\right)\right).$$

Let us choose $b > 0$ in such a way that

$$A(c)b = \frac{c-1}{\varrho}$$

be valid. Then the preceding inequality gives

$$(\varrho-1)E\left[\Psi\left(q\left(\frac{X_n^{**}}{c}\right)\right)\right] \leq E\left(\Phi\left(\frac{\gamma}{b}\right)\right),$$

where

$$b = \frac{c-1}{A(c)\varrho}.$$

Letting $a \uparrow +\infty$ by the monotone convergence theorem we obtain

$$(\varrho-1)E\left[\Psi\left(q\left(\frac{X_n^*}{c}\right)\right)\right] \leq E\left(\Phi\left(\frac{\gamma}{b}\right)\right),$$

since in this case $X_n^{**} \uparrow X_n^*$ and since $\Psi(q(x))$ increases. Apply this inequality with the new martingale

$$\left(\frac{X_k}{\|\gamma\|_\Psi} b, (\mathcal{F}_k)\right), \quad k \geq 1.$$

This implies

$$(q-1)E\left(\Psi\left(q\left(\frac{X_n^*}{\frac{c}{q-1}A(c)\|\gamma\|_{\mathcal{K}_\phi}}\right)\right)\right)\leq E\left(\Phi\left(\frac{\gamma}{\|\gamma\|_{\mathcal{K}_\phi}}\right)\right)\leq 1.$$

Here $\gamma \in \Gamma_X^\phi$ is arbitrary. Since $\|X\|_{\mathcal{K}_\phi} > 0$ by turning to the infimum for $\gamma \in \Gamma_X^\phi$ we finally get

$$(q-1)E\left(\Psi\left(q\left(\frac{X_n^*}{\frac{c}{q-1}A(c)\|X\|_{\mathcal{K}_\phi}}\right)\right)\right)\leq 1.$$

This was to be proved.

COROLLARY 1. Suppose that both Φ and its conjugate Ψ have finite power p and q , resp. Then for $X \in \mathcal{K}_\phi$ we have

$$\|X_n^*\|_\phi \leq qA(c)\frac{c}{c-1}\|X\|_{\mathcal{K}_\phi},$$

where $c > 1$ is an arbitrary constant and $A = A(c)$ is defined above.

In fact, if q , the power of Ψ , is finite, then

$$(q-1)\Psi(q(x)) \leq \Phi(x).$$

Consequently, from the inequality of Theorem 1 we get

$$E\left(\Phi\left(\frac{X_n^*}{q\frac{c}{c-1}A(c)\|X\|_{\mathcal{K}_\phi}}\right)\right) \leq 1,$$

where we have chosen $q = q$. This means that

$$\|X_n^*\|_\phi \leq q\frac{c}{c-1}A(c)\|X\|_{\mathcal{K}_\phi}.$$

Especially, if $\Phi(x) = x^{p/q}$ then $q(x) = x^{p-1}$ and $\Psi(x) = x^{q/p}$, where $p^{-1} + q^{-1} = 1$. Thus, if $X \in \mathcal{K}_\phi = \mathcal{K}_p$ (cf. [1]) we obtain

$$\|X_n^*\|_p \leq q\frac{c^p}{c-1}\|X\|_{\mathcal{K}_p}.$$

This is the inequality of GARSIA ([5], Theorem III, 5.2.). The constant of Garsia is, however, other than ours.

The constant $c > 1$ can be used to optimize the coefficient on the right hand side in this inequality. The minimal value of $c^p/(c-1)$ is obtained when we take $c = p/(p-1)$. Thus we get

$$\|X_n^*\|_p \leq pq^p\|X\|_{\mathcal{K}_p} \leq pqe\|X\|_{\mathcal{K}_p},$$

an inequality which gives a constant sharper than Garsia's one.

3. Is there some connection between the space \mathcal{K}_ϕ and the Hardy space \mathcal{H}_ϕ ? This last is defined as the set of those random variables $X \in L_1$ for which the quadratic variation

$$S = \left(\sum_{i=1}^{\infty} di^2 \right)^{1/2}$$

belongs to L^ϕ . Here $d_0 = 0$, d_1, d_2, \dots is the difference sequence of the martingale

$$X_n = E(X | \mathcal{F}_n), \quad n \geq 0, \quad X_0 = 0 \text{ a.e.}$$

In this case we define

$$\|X\|_{\mathcal{H}_\phi} = \|S\|_\phi.$$

When the power of ϕ is finite then $\mathcal{H}_\phi \subset \mathcal{K}_\phi$. In fact, the Burkholder, Davis, Gundy inequality (cf. [6], Theorem 15.1.) in this case guarantees that $X \in \mathcal{H}_\phi$ implies

$$X^* = \sup_{n \geq 1} |X_n| \in L^\phi.$$

From this for arbitrary $n \geq 1$ we have

$$E(|X - X_{n-1}| | \mathcal{F}_n) \leq E(2X^* | \mathcal{F}_n) \quad \text{a.e.}$$

Consequently,

$$X \in \mathcal{K}_\phi$$

and

$$\|X\|_{\ell_q} \leq 2\|X^*\|_\phi.$$

The following assertion gives a sufficient condition which ensures that the spaces \mathcal{H}_ϕ and \mathcal{K}_ϕ coincide.

THEOREM 2. Suppose that ϕ and its conjugate ψ have finite power p and q respectively. Then the spaces \mathcal{K}_ϕ and \mathcal{H}_ϕ coincide. More precisely, there exist positive constants c_ϕ and C_ϕ such that

$$c_\phi \|X\|_{\mathcal{K}_\phi} \leq \|X\|_{\mathcal{H}_\phi} \leq C_\phi \|X\|_{\ell_q}.$$

PROOF. Suppose that $X \in \mathcal{K}_\phi$. Then by Corollary 1 we have for all $n \geq 1$ that

$$\|X_n^*\|_\phi \leq q \frac{c}{c-1} A(c) \|X\|_{\mathcal{K}_\phi}.$$

Consequently, for the random variable

$$X^* = \sup_{n \geq 1} |X_n|$$

by the monotone convergence theorem we have

$$\|X^*\|_\phi \leq q \frac{c}{c-1} A(c) \|X\|_{\mathcal{K}_\phi},$$

where $c > 1$ is an arbitrary constant and $A = A(c)$ is defined above. So, we get that $X^* \in L^\phi$. By the Burkholder, Davis and Gundy inequality it then follows that $X \in \mathcal{K}_\phi$ since Φ has finite power and with some constant $c'_\phi > 0$ we have

$$c'_\phi \|X\|_{\mathcal{K}_\phi} \leq \|X^*\|_\phi \leq q \cdot \frac{c}{c-1} A(c) \|X\|_{\mathcal{K}_\phi}.$$

This proves the right hand side of our inequality.

Conversely, suppose that $X \in \mathcal{K}_\phi$. Then by the remarks which precede the present theorem and by the Burkholder, Davis and Gundy inequality we get with some constant $c''_\phi > 0$ that

$$\|X\|_{\mathcal{K}_\phi} \leq 2\|X^*\|_\phi \leq 2c''_\phi \|X\|_{\mathcal{K}_\phi}.$$

This proves the left hand side of our inequality.

As an application consider the following martingale: let Y_1, Y_2, \dots be independent random variables with zero mean. Suppose that the series

$$Y = \sum_{i=1}^{\infty} Y_i$$

converges a.e. and $E(|Y|)$ is finite. Let $\bar{\mathcal{F}}_n$ be the σ -field generated by the random variables Y_1, Y_2, \dots, Y_n , $n \geq 1$ and let $\bar{\mathcal{F}}_0 = (\emptyset, \Omega)$. Then the martingale

$$X_n = E(Y | \bar{\mathcal{F}}_n)$$

in such that

$$X_n = Y_1 + \dots + Y_n, \quad n \geq 1$$

and

$$X_0 = 0 \quad \text{a.e.}$$

We have the following

LEMMA 1. Suppose that the conjugate Ψ of the Young-function Φ has finite power q . Then $Y \in \mathcal{K}_\phi$ if and only if

$$\sup_{k \geq 1} |Y_k| \in L^\phi.$$

In this case

$$\gamma = \sup_{k \geq 1} |Y_k| + E(|Y|) \in I_Y^\phi$$

and

$$\|X\|_{\mathcal{K}_\phi} \leq \left\| \sup_{k \geq 1} |Y_k| \right\|_\phi + \frac{E(|Y|)}{x_0},$$

where $x_0 > 0$ satisfies $\Phi(x_0) = 1$.

PROOF. Suppose that

$$\sup_{k \geq 1} |Y_k| \in L^\phi.$$

Then for arbitrary $k \geq 1$ we have a.e.

$$E(|Y - X_{k-1}| | \bar{\mathcal{F}}_k) \leq |Y_k| + E(|Y - X_k|) \leq E(\sup_{k \geq 1} |Y_k| | \bar{\mathcal{F}}_k) + E(|Y|).$$

Thus with

$$\gamma = \sup_{i \geq 1} |Y_i| + E(|Y|)$$

we have

$$E(|Y - X_{k-1}| \mid \mathcal{F}_k) \leq E(\gamma \mid \mathcal{F}_k) \quad \text{a.e.}$$

and consequently, $\gamma \in L^\phi$. From this it follows that

$$\|Y\|_{\mathcal{L}_\phi} \leq \|\gamma\|_\phi \leq \left\| \sup_{i \geq 1} |Y_i| \right\|_\phi + \frac{E(|Y|)}{x_0},$$

since $x_0 > 0$ is the number such that $\phi(x_0) = 1$.

Conversely, suppose that for some $\gamma \in L^\phi$ we have for all $k \geq 1$

$$E(|Y - X_{k-1}| \mid \mathcal{F}_k) \leq E(\gamma \mid \mathcal{F}_k) \quad \text{a.e.}$$

From this

$$|Y_k| \leq E(\gamma \mid \mathcal{F}_k) \quad \text{a.e.}$$

and consequently

$$\sup_{k \geq 1} |Y_k| \leq \sup_{k \geq 1} E(\gamma \mid \mathcal{F}_k).$$

Since the power q of the conjugate Young-function of ϕ is finite by the maximal inequality (cf. [7]) we get

$$\left\| \sup_{k \geq 1} |Y_k| \right\|_\phi \leq \left\| \sup_{k \geq 1} E(\gamma \mid \mathcal{F}_k) \right\|_\phi \leq q \sup_{k \geq 1} \|E(\gamma \mid \mathcal{F}_k)\|_\phi \leq q \|\gamma\|_\phi.$$

This proves the assertion.

On the basis of this assertion we can deduce the following interesting

THEOREM 3. *Let*

$$Y = \sum_{i=1}^{\infty} Y_i$$

be an a.e. convergent and finitely integrable series of independent random variables with zero mean. Consider the martingale

$$(X_n = Y_1 + \dots + Y_n \mid \mathcal{F}_n), \quad n \geq 0,$$

where $\mathcal{F}_0 = (\phi, \Omega)$ and $X_0 = 0$ a.e. If ϕ and its conjugate ψ have finite power p and q , resp. and if

$$\sup_{i \geq 1} |Y_i| \in L^\phi$$

then $X^ \in L^\phi$. More precisely, we have*

$$\frac{1}{2} \left\| \sup_{i \geq 1} |Y_i| \right\|_\phi \leq \|X^*\|_\phi \leq q \frac{c}{c-1} \cdot A(c) \left[\left\| \sup_{i \geq 1} |Y_i| \right\|_\phi + \frac{E(|Y|)}{x_0} \right].$$

Here $c > 1$ is an arbitrary constant.

PROOF. The left hand side of this inequality is trivial since for arbitrary $i \geq 1$ we have

$$|Y_i| = |X_i - X_{i-1}| \leq 2X^*.$$

To prove the right hand side use the result of Corollary 1 and Lemma 1. According to these we have

$$\|X^*\|_\phi \leq q - \frac{c}{c+1} A(c) \|Y\|_{\psi_\phi} \leq q - \frac{c}{c+1} A(c) \left[\left\| \sup_{i \geq 1} |Y_i| \right\|_\phi + \frac{E(|Y|)}{x_0} \right].$$

This proves the assertion.

The result of this theorem is to be compared with that of the following

THEOREM 4. *Let Y_1, Y_2, \dots be independent random variables with zero mean. Then for arbitrary Young function Φ we have*

$$\|X_n^*\|_\phi \leq 4\|X_n\|_\phi$$

where $S_k = Y_1 + \dots + Y_k$, $k = 1, 2, \dots$, $X_k = |S_k|$ and $X_n^* := \max_{1 \leq k \leq n} |S_k|$.

PROOF. We can suppose that $\|X_n\|_\phi$ is positive. In fact, the norm of the submartingale $\{X_k\}$ increases. Consequently, in the case $\|X_n\|_\phi = 0$ for all $k \leq n$ we have $\|X_k\|_\phi = 0$. This means that X_n^* vanishes. In this case we have nothing to prove. In case $\|X_n\|_\phi > 0$ let $g(x)$ be defined by the formula

$$g(x) = \begin{cases} \Phi(x), & \text{if } x \geq 0 \\ \Phi(-x), & \text{if } x < 0. \end{cases}$$

$g(x)$ is even nonnegative and convex. BICKEL [9] has proved that if the random variables Y_1, Y_2, \dots are independent and symmetrically distributed then for arbitrary $\lambda > 0$ we have

$$P(\max_{1 \leq k \leq n} g(S_k) \geq \lambda) \leq 2P(g(S_n) \geq \lambda).$$

From this by integrating with respect to λ we get

$$E(\max_{1 \leq k \leq n} g(S_k)) \leq 2E(g(S_n)).$$

This means that

$$E(\max_{1 \leq k \leq n} \Phi(X_k)) \leq 2E(\Phi(X_n)).$$

Now

$$\max_{1 \leq k \leq n} \Phi(X_k) = \Phi(X_n^*),$$

since Φ is increasing and continuous. From these using the inequality

$$\Phi\left(\frac{x}{2}\right) \leq \frac{1}{2}\Phi(x)$$

we get

$$E\left(\Phi\left(\frac{X_n^*}{2}\right)\right) \leq E(\Phi(X_n)).$$

Apply this inequality to the variables $Y_i^* = Y_i/\|X_n\|_\phi$ to get

$$E\left(\Phi\left(\frac{X_n^*}{2\|X_n\|_\phi}\right)\right) \leq E\left(\Phi\left(\frac{X_n}{\|X_n\|_\phi}\right)\right) \leq 1.$$

We conclude that

$$\|X_n^*\|_\phi \leq 2\|X_n\|_\phi.$$

In the non-symmetrical case let us consider the random variables Y'_1, Y'_2, \dots, Y'_n such that Y_i has the same distribution as Y'_i , $i = 1, 2, \dots, n$ and the sequence

$$Y_1, Y'_1, Y_2, Y'_2, \dots, Y'_n, Y'_n$$

is an independent sequence of random variables. Put

$$Z_i = Y_i - Y'_i$$

and

$$S'_i = \sum_{k=1}^i Y'_k.$$

Then the variables Z_i , $i = 1, 2, \dots$ are symmetrically distributed and independent. Consequently, by what we have obtained above

$$(*) \quad E\left(\Phi\left(\frac{\max_{1 \leq k \leq n} |Z_1 + \dots + Z_k|}{2}\right)\right) \leq E\left(\Phi\left(\frac{\max_{1 \leq k \leq n} |S_k - S'_k|}{2}\right)\right) \leq \\ \leq E(\Phi(|Z_1 + \dots + Z_n|)) = E(\Phi(|S_n - S'_n|)).$$

We show that

$$E\left(\Phi\left(\frac{\max_{1 \leq k \leq n} |S_k - S'_k|}{2}\right)\right) \leq E\left(\Phi\left(\frac{X_n^*}{2}\right)\right).$$

For this purpose let $\bar{\mathcal{F}}_n$ be the σ -field generated by the random variables Y_1, Y_2, \dots, Y_n . Using the Jensen inequality for the convex functions Φ and $|x|$ resp., we get

$$E\left(\Phi\left(\frac{|S_k - S'_k|}{2}\right) \mid \bar{\mathcal{F}}_n\right) \leq \Phi\left(E\left(\frac{S_k - S'_k}{2} \mid \bar{\mathcal{F}}_n\right)\right) = \\ = \Phi\left(\frac{|S_k|}{2}\right) = \Phi\left(\frac{X_k}{2}\right) \quad \text{a.e.}$$

since S_k is $\bar{\mathcal{F}}_n$ -measurable while S'_k is independent of $\bar{\mathcal{F}}_n$ and has zero mean. From this

$$\max_{1 \leq k \leq n} E\left(\Phi\left(\frac{|S_k - S'_k|}{2}\right) \mid \bar{\mathcal{F}}_n\right) \leq \max_{1 \leq k \leq n} \Phi\left(\frac{X_k}{2}\right) \quad \text{a.e.}$$

or

$$E\left(\max_{1 \leq k \leq n} \Phi\left(\frac{|S_k - S'_k|}{2}\right) \mid \bar{\mathcal{F}}_n\right) \leq \max_{1 \leq k \leq n} \Phi\left(\frac{X_k}{2}\right) \quad \text{a.e.}$$

By the monotonicity of Φ we get

$$E\left(\Phi\left(\max_{1 \leq k \leq n} \frac{|S_k - S'_k|}{2}\right) \mid \bar{\mathcal{F}}_n\right) \leq \Phi\left(\frac{X_n^*}{2}\right).$$

Turning to the expectation on both sides we obtain

$$E\left(\Phi\left(\max_{1 \leq k \leq n} \frac{|Z_1 + \dots + Z_k|}{2}\right)\right) \geq E\left(\Phi\left(\frac{X_n^*}{2}\right)\right).$$

Comparing the so obtained result with (*) we obtain

$$E(\Phi(|S_n - S'_n|)) \geq E\left(\Phi\left(\frac{X_n^*}{2}\right)\right),$$

or

$$E\left(\Phi\left(\frac{X_n^*}{2}\right)\right) \leq E(\Phi(|S_n| + |S'_n|)).$$

Not that S_n and S'_n have the same distribution. So their common L^ϕ -norm is $\|X_n\|_\phi$. This means that

$$E\left(\Phi\left(\frac{X_n^*}{4\|X_n\|_\phi}\right)\right) \leq E\left(\Phi\left(\frac{|S_n| + |S'_n|}{2\|X_n\|_\phi}\right)\right) \leq 1$$

and we finally obtain that

$$\|X_n^*\|_\phi \leq 4\|X_n\|_\phi.$$

This proves the assertion.

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ON THE BMO_p -SPACES WITH GENERAL YOUNG FUNCTION

By

N. L. BASSILY and J. MOGYORÓDI

Department of Probability Theory of L. Eötvös University, Budapest

(Received August 4, 1982)

1. Introduction. The \mathcal{K}_p -spaces are treated e.g. in the book by A. M. GARSIA [1]. Let $X \in L_1$ be a random variable defined on the probability space (Ω, \mathcal{A}, P) and consider the regular martingale

$$X_n = E(X|\mathcal{F}_n), \quad n \geq 0;$$

where (\mathcal{F}_n) , $n \geq 0$, is an increasing sequence of σ -fields of events such that

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right) = \mathcal{A}.$$

We suppose that

$$X_0 = 0 \quad \text{a.e.}$$

For $1 \leq p \leq +\infty$ set

$$\mathcal{I}_X^{(p)} = \{\gamma : \gamma \in L_p, E(|X - X_{n-1}| |\mathcal{F}_n) \leq E(\gamma |\mathcal{F}_n) \text{ a.e. } \forall n \in \mathbb{N}\}.$$

We say that $X \in \mathcal{K}_p$ if the set $\mathcal{I}_X^{(p)}$ is not empty. In this case let

$$\|X\|_{\mathcal{K}_p} = \inf_{\gamma \in \mathcal{I}_X^{(p)}} \|\gamma\|_p.$$

It is easily seen that $\|\cdot\|_{\mathcal{K}_p}$ is a semi-norm. The space \mathcal{K}_∞ is the well-known BMO_1 -space.

The aim of the present note is to give a general inequality for the maximum of the martingale corresponding to $X \in \mathcal{K}_p$, further, to give better constants in the maximal inequality for BMO_1 -random variables and to generalize the notion of the BMO_p -spaces, $1 < p < +\infty$, cf. [1].

2. A general maximal inequality. In this section we consider random variables $X \in L_1$ such that for every $n \geq 1$ we have

$$E(|X - X_{n-1}| |\mathcal{F}_n) \leq E(\gamma |\mathcal{F}_n) \quad \text{a.e.}$$

where $\gamma \in L_1$ is a random variable.

On the basis of this we easily deduce that for arbitrary $n \geq k \geq 1$ the inequality

$$E(|X_n - X_{k-1}| |\mathcal{F}_k) \leq E(\gamma |\mathcal{F}_k)$$

holds. For this purpose let us remark that the conditional expectations

$$E(|X_n - X_{k-1}| \mid \mathcal{F}_k), \quad n = k, k+1, \dots,$$

increase as n increases and k is fixed. This follows from the fact that the sequence

$$(|X_n - X_{k-1}|), \quad n = k, k+1, \dots$$

is a submartingale. This submartingale converges in L_1 and a.e. to

$$|X - X_{k-1}|.$$

Consequently, the same is true for the above conditional expectations. Therefore, for arbitrary $n \geq k$ we have

$$E(|X_n - X_{k-1}| \mid \mathcal{F}_k) \leq E(|X - X_{k-1}| \mid \mathcal{F}_k) = E(\gamma \mid \mathcal{F}_k) \text{ a.e.}$$

Conversely, if we suppose that for some $\gamma \in L_1$ and for arbitrary n which is not smaller than k , where k is arbitrary but fixed, we have

$$E(|X_n - X_{k-1}| \mid \mathcal{F}_k) = E(\gamma \mid \mathcal{F}_k) \text{ a.e.}$$

then

$$E(|X - X_{k-1}| \mid \mathcal{F}_k) = E(\gamma \mid \mathcal{F}_k) \text{ a.e.}$$

This assertion also follows from the above remarks.

These conclusions enable us to prove the following

THEOREM 1. *If $\gamma \in L_1$ is random variable such that for all $n \geq k \geq 1$ we have*

$$E(|X_n - X_{k-1}| \mid \mathcal{F}_k) = E(\gamma \mid \mathcal{F}_k) \text{ a.e., } k = 1, 2, \dots, n$$

then for arbitrary $\beta \geq z > 0$ we have

$$(\beta - z)E(\chi(X_n^* - \beta) \mid \mathcal{F}_1) \leq (\gamma \chi(X_n^* - z) \mid \mathcal{F}_1) \quad \text{a.e.,}$$

where

$$X_n^* = \max_{1 \leq k \leq n} |X_k|.$$

PROOF. Define the stopping time r_β for arbitrary fixed $\beta > 0$ by the formula

$$r_\beta = \begin{cases} \inf(k : 1 \leq k \leq n, X_k^* > \beta), & \text{if } X_n^* > \beta, \\ n+1, & \text{if } X_n^* \leq \beta. \end{cases}$$

Then, as it is easily seen, $r_\beta \leq r_n$ if $\beta \geq z$. We have

$$E(\chi(X_n^* - \beta) \mid \mathcal{F}_1) = \sum_{k=1}^n E(\chi(r_\beta = k) \mid \mathcal{F}_1) =$$

$$= \sum_{k=1}^n \sum_{l=1}^k E(\chi(r_\beta = k, r_n = l) \mid \mathcal{F}_1).$$

On the event $(v_\beta = k, v_a = l)$ we have $X_k^* > \beta$ and $X_{l-1}^* < \alpha$, i.e. $X_k^* - X_{l-1}^* > \beta - \alpha$. On the same event $X_k^* = |X_k|$, since $X_{l-1}^* < \beta$. Consequently,

$$\begin{aligned} E(\chi(X_n^* > \beta) | \mathcal{F}_1) &\equiv \sum_{k=1}^n \sum_{l=1}^k E\left(\frac{|X_k| - X_{l-1}^*}{\beta - \alpha} \chi(v_\beta = k, v_a = l) | \mathcal{F}_1\right) \leq \\ &\leq \sum_{k=1}^n \sum_{l=1}^k E\left(\frac{|X_k| \cdot |X_{l-1}|}{\beta - \alpha} \chi(v_\beta = k, v_a = l) | \mathcal{F}_1\right). \end{aligned}$$

Since $|X_k| \in E(|X_n| | \mathcal{F}_k)$ and $|X_{l-1}|$ is \mathcal{F}_k -measurable from the preceding inequality we get

$$\begin{aligned} (\beta - \alpha) E(\chi(X_n^* > \beta) | \mathcal{F}_1) &\leq \\ &\leq \sum_{k=1}^n \sum_{l=1}^k E(E(|X_n - X_{l-1}| | \mathcal{F}_k) \chi(v_\beta = k, v_a = l) | \mathcal{F}_1) \leq \\ &\leq \sum_{k=1}^n \sum_{l=1}^k E(E(|X_n - X_{l-1}| | \mathcal{F}_k) \chi(v_\beta = k, v_a = l) | \mathcal{F}_1) = \\ &= \sum_{k=1}^n \sum_{l=1}^k E(|X_n - X_{l-1}| \chi(v_\beta = k, v_a = l) | \mathcal{F}_1) \end{aligned}$$

given that $\chi(v_\beta = k, v_a = l)$ is trivially \mathcal{F}_k -measurable. A reordering then implies the inequality

$$\begin{aligned} (\beta - \alpha) E(\chi(X_n^* > \beta) | \mathcal{F}_1) &\geq \sum_{l=1}^n E\left(|X_n - X_{l-1}| \sum_{k=1}^n \chi(v_\beta = k, v_a = l) | \mathcal{F}_1\right) \geq \\ &\geq \sum_{l=1}^n E(|X_n - X_{l-1}| \chi(v_a = l) | \mathcal{F}_1) = \sum_{l=1}^n E(E(|X_n - X_{l-1}| | \mathcal{F}_l) \chi(v_a = l) | \mathcal{F}_1) \end{aligned}$$

since $\chi(v_a = l)$ is \mathcal{F}_l -measurable. Using our assumption we finally get

$$\begin{aligned} (\beta - \alpha) E(\chi(X_n^* > \beta) | \mathcal{F}_1) &\geq \sum_{l=1}^n E(E(\gamma | \mathcal{F}_l) \chi(v_a = l) | \mathcal{F}_1) = \\ &= E(\gamma \chi(X_n^* > \beta) | \mathcal{F}_1). \end{aligned}$$

This was to be proved.

REMARK. GARSIA in [1] (Theorem III, 2, 1, and Lemma III, 5, 1.) has proved similar a inequality. Our inequality is a unified version of these and leads at the same time to better constants in the estimations below.

REMARK. For arbitrary $l \geq 1$ consider the random variable $X - X_{l-1} \in L_1$. Then the sequence

$$X'_p = X_{l-1+p} - X_{l-1}, \quad p \geq 0,$$

of random variables is a martingale with respect to the sequence

$$G_p = \mathcal{F}_{l-1+p}, \quad p \geq 0$$

of σ -fields. Further, with arbitrary $\gamma \in I_1$ satisfying the condition of Theorem 1 for arbitrary $p \geq q \geq 1$ we have

$$\begin{aligned} E(|X'_p - X'_{q-1}| | G_q) &= E(|X_{t-1+p} - X_{t-1+q-1}| | \mathcal{F}_{t-1+q}) \leq \\ &\leq E(\gamma | \mathcal{F}_{t-1+q}) = E(\gamma | G_q) \quad \text{a.e.} \end{aligned}$$

Consequently, (X'_p, G_p) also satisfies the conditions of Theorem 1 and so for arbitrary $\beta > z > 0$ have

$$\begin{aligned} (\beta - z) E(\chi(\max_{0 \leq p \leq n-t+1} |X'_p| > \beta) | G_1) &\leq \\ &\leq E(\gamma \chi(\max_{0 \leq p \leq n-t+1} |X'_p| > z) | G_1) \quad \text{a.e.} \end{aligned}$$

After relabeling this leads to the following formulation of Theorem 1.

COROLLARY 1. Suppose that the conditions of Theorem 1 are satisfied. Then

$$(\beta - z) E(\chi(\max_{t \leq k \leq n} |X_k - X_{t-1}| > \beta) | \mathcal{F}_t) \leq E(\gamma \chi(\max_{t \leq k \leq n} |X_k - X_{t-1}| > z) | \mathcal{F}_t) \quad \text{a.e.}$$

3. A sharper constant in the John-Nirenberg theorem.

THEOREM 2. Suppose that $X \in \text{BMO}_1$ with

$$\|X\|_{\text{BMO}_1} = B.$$

Then for arbitrary $t \geq 1$ and $n \geq t$ we have

$$E(\max_{t \leq k \leq n} |X_k - X_{t-1}| | \mathcal{F}_t) \leq 4B \quad \text{a.e.}$$

PROOF. By means of Corollary 1 it suffices to prove the assertion for $t = 1$. Set $\beta = z + 2B$ in the inequality of Theorem 1 and estimate γ by B . Then we have

$$2E(\chi(X_n^* - 2B > z) | \mathcal{F}_1) \leq E(\chi(X_n^* - z) | \mathcal{F}_1) \quad \text{a.e.}$$

Integrating this with respect to z from 0 to $+\infty$ we get

$$2E((X_n^* - 2B)^+ | \mathcal{F}_1) \leq E(X_n^* | \mathcal{F}_1) \quad \text{a.e.}$$

From this

$$E(X_n^* | \mathcal{F}_1) \leq E((X_n^* - 2B)^+ | \mathcal{F}_1) + 2B \leq \frac{1}{2} E(X_n^* | \mathcal{T}) + 2B \quad \text{a.e.}$$

and consequently

$$E(X_n^* | \mathcal{F}_1) \leq 4B \quad \text{a.e.}$$

This proves the assertion.

REMARK. GARSIA in [1], Theorem III, 2, 1., gives the constant 8 instead of our constant 4. The sharp constant seems to be 2.

4. The conditional L^p -norm. This notion will be used for purposes to extend the notion of the BMO p -spaces, where $1 < p < +\infty$.

In what follows we suppose the knowledge of the theory of the Young-functions as well as the theory of the Orlicz-space generated by a Young-function. Let (Ω, \mathcal{A}, P) be a probability space.

Let Φ be a Young-function, X a random variable and $\mathcal{F} \subset \mathcal{A}$ a σ -field. Consider the set $F_X^{\Phi, \mathcal{F}}$ of random variables γ defined by the formula

$$F_X^{\Phi, \mathcal{F}} = \left\{ \gamma : \gamma > 0 \text{ a.e., } \mathcal{F}\text{-measurable, } E\left(\Phi\left(\frac{|X|}{\gamma}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.} \right\}$$

We say that $X \in L_\phi^p$, if $F_X^{\Phi, \mathcal{F}}$ is not empty. In this case we define

$$\|X\|_\phi^p = \text{ess inf } F_X^{\Phi, \mathcal{F}}.$$

Amongst the elements of $F_X^{\Phi, \mathcal{F}}$ the ordering is the following: $\gamma_1 \geq \gamma_2$ if this inequality holds a.e. It is known (cf. e.g. [2], Proposition VI-1-1) that the random variable $\|X\|_\phi^p$ exists and under the above ordering it can be obtained as the a.e. limit of a decreasing sequence from $F_X^{\Phi, \mathcal{F}}$. Consequently, $\|X\|_\phi^p$ is an \mathcal{F} -measurable random variable and for every element $\gamma \in F_X^{\Phi, \mathcal{F}}$ we have

$$\gamma \geq \|X\|_\phi^p \quad \text{a.e.}$$

If X belongs to the Orlicz space generated by the Young-function Φ , i.e. if $X \in L^p(\Omega, \mathcal{A}, P)$, then $X \in L_\phi^p$ where \mathcal{F} is an arbitrary sub σ -field of \mathcal{A} . (Here we use the Luxemburg norm when defining L^p). To show this we can suppose that $\|X\|_\phi = 0$. For, $\|X\|_\phi = 0$ implies that $X = 0$ a.e. and consequently, with arbitrary \mathcal{F} -measurable and a.e. positive γ we have

$$E\left(\Phi\left(\frac{|X|}{\gamma}\right) | \mathcal{F}\right) = 0 \quad \text{a.e.}$$

This means that $F_X^{\Phi, \mathcal{F}}$ contains all the \mathcal{F} -measurable and a.e. positive random variables. Therefore,

$$\|X\|_\phi^p = \text{ess inf } F_X^{\Phi, \mathcal{F}} = 0 \quad \text{a.e.}$$

So we suppose that

$$0 < \sigma \Rightarrow \|X\|_\phi < +\infty.$$

Let us take

$$\gamma = \sigma \max\left(1, E\left(\Phi\left(\frac{|X|}{\sigma}\right) | \mathcal{F}\right)\right).$$

This is a positive and \mathcal{F} -measurable random variable. We show that $\gamma \in F_X^{\Phi, \mathcal{F}}$. For this purpose remark that if $c \geq 1$ is an arbitrary number then

$$\Phi\left(\frac{\gamma}{c}\right) \leq \frac{1}{c} \Phi(\gamma)$$

since $\phi(0) = 0$ and $\phi(y)$ is convex. Consequently, using this and the fact that " $E(UV|\bar{\mathcal{F}}) = U E(V|\bar{\mathcal{F}})$ if U is $\bar{\mathcal{F}}$ -measurable", we get

$$E\left(\phi\left(\frac{|X|}{\gamma}\right)|\bar{\mathcal{F}}\right) = \frac{1}{\max\left(1, E\left(\phi\left(\frac{|X|}{\sigma}\right)|\bar{\mathcal{F}}\right)\right)} E\left(\phi\left(\frac{|X|}{\sigma}\right)|\bar{\mathcal{F}}\right) \leq 1 \quad \text{a.e.}$$

So we deduce that $F_X^{\phi, \bar{\mathcal{F}}}$ is not empty.

The properties of the random variable $\|X\|_{\phi}^{\bar{\mathcal{F}}}$ are presented in the following assertion:

THEOREM 4. Suppose that $X \in L_{\phi}^{\bar{\mathcal{F}}}$. Then

- a) $\|X\|_{\phi}^{\bar{\mathcal{F}}} = 0$ a.e. if and only if $X = 0$ a.e.
- b) for any $\bar{\mathcal{F}}$ -measurable random variable Y we have $YX \in L_{\phi}^{\bar{\mathcal{F}}}$ and $\|YX\|_{\phi}^{\bar{\mathcal{F}}} = |Y| \|X\|_{\phi}^{\bar{\mathcal{F}}}$ a.e.
- c) if $X \in L_{\phi}^{\bar{\mathcal{F}}}$ and $Y \in L_{\phi}^{\bar{\mathcal{F}}}$ then $X + Y \in L_{\phi}^{\bar{\mathcal{F}}}$ and we have

$$\|X + Y\|_{\phi}^{\bar{\mathcal{F}}} \leq \|X\|_{\phi}^{\bar{\mathcal{F}}} + \|Y\|_{\phi}^{\bar{\mathcal{F}}} \quad \text{a.e.}$$

PROOF. a) If $X = 0$ a.e. then with arbitrary $\bar{\mathcal{F}}$ -measurable and a.e. positive γ we have

$$E\left(\phi\left(\frac{|X|}{\gamma}\right)|\bar{\mathcal{F}}\right) = 0 \quad \text{a.e.}$$

Therefore, $\|X\|_{\phi}^{\bar{\mathcal{F}}} = 0$ a.e. Suppose now conversely that $\|X\|_{\phi}^{\bar{\mathcal{F}}} = 0$ a.e. Then let γ_n be a decreasing sequence from $F_X^{\phi, \bar{\mathcal{F}}}$ such that

$$\lim_{n \rightarrow +\infty} \gamma_n = \|X\|_{\phi}^{\bar{\mathcal{F}}} = 0 \quad \text{a.e.}$$

Note that for $x > 0$ the function $\frac{\phi(x)}{x}$ is increasing in x and tends to $+\infty$ as $x \rightarrow +\infty$. Thus on the set $\{|X| \geq \gamma_k\}$ we can write for arbitrary $n \geq k$ that

$$\frac{|X|}{\gamma_k} \geq \frac{\gamma_k}{\gamma_n} \frac{\phi\left(\frac{|X|}{\gamma_n}\right)}{\phi\left(\frac{\gamma_k}{\gamma_n}\right)},$$

From this

$$\begin{aligned} P(|X| \geq \gamma_k | \bar{\mathcal{F}}) &\equiv E\left(\frac{|X|}{\gamma_k} \cdot \mathbb{1}(|X| \geq \gamma_k) | \bar{\mathcal{F}}\right) \leq \\ &\leq \frac{\gamma_k}{\phi\left(\frac{\gamma_k}{\gamma_n}\right)} E\left(\phi\left(\frac{|X|}{\gamma_n}\right) | \bar{\mathcal{F}}\right) \equiv \frac{\gamma_k}{\phi\left(\frac{\gamma_k}{\gamma_n}\right)} \quad \text{a.e.} \end{aligned}$$

since

$$E\left(\Phi\left(\frac{|X|}{\gamma_n}\right)\right) \leq 1 \quad \text{a.e.}$$

Letting $n \rightarrow +\infty$ from these we get

$$P(|X| \geq \gamma_k | \mathcal{F}) = 0 \quad \text{a.e.}$$

since $\gamma_n \downarrow 0$. It follows that for arbitrary fixed k we have

$$P(|X| \geq \gamma_k) = 0.$$

Since $\gamma_k \downarrow 0$ as $k \rightarrow +\infty$ we get that $X = 0$ a. e.

b) For arbitrary $\gamma \in F_X^{\phi, \mathcal{F}}$ and $n \geq 1$ we have

$$E\left(\Phi\left(\frac{|XY|}{\gamma\left(|Y| + \frac{1}{n}\right)}\right) | \mathcal{F}\right) \leq E\left(\Phi\left(\frac{|X|}{\gamma}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

Consequently, $\gamma\left(|Y| + \frac{1}{n}\right) \in F_{XY}^{\phi, \mathcal{F}}$ and letting $n \rightarrow +\infty$ from this we deduce that $XY \in L_\phi^{\mathcal{F}}$ and that

$$\|XY\|_\phi^{\mathcal{F}} \leq |Y| \|X\|_\phi^{\mathcal{F}} \quad \text{a.e.}$$

To prove the converse inequality we first show that if $X \in L_\phi^{\mathcal{F}}$ then $\|X\|_\phi^{\mathcal{F}} + \varepsilon \in F_X^{\phi, \mathcal{F}}$ whatever be $\varepsilon > 0$. In fact, let $\gamma_n \in F_X^{\phi, \mathcal{F}}$ be a decreasing sequence of random variables tending to $\|X\|_\phi^{\mathcal{F}}$. Then

$$E\left(\Phi\left(\frac{|X|}{\gamma_n + \varepsilon}\right) | \mathcal{F}\right) \leq E\left(\Phi\left(\frac{|X|}{\gamma_n}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

Therefore, by the monotone convergence theorem for conditional expectations we have

$$E\left(\Phi\left(\frac{|X|}{\|X\|_\phi^{\mathcal{F}} + \varepsilon}\right) | \mathcal{F}\right) \leq 1 \quad \text{a.e.}$$

Now let us turn to the proof of the converse inequality. The random variable

$$\gamma = \frac{\max(\|XY\|_\phi^{\mathcal{F}} + \varepsilon, \varepsilon(\|X\|_\phi^{\mathcal{F}} + \varepsilon))}{\max(|Y|, \varepsilon)}$$

defined for arbitrary $\varepsilon > 0$, is positive and \mathcal{F} -measurable. We show that $\gamma \in F_X^{\phi, \mathcal{F}}$. In fact,

$$\begin{aligned} E\left(\Phi\left(\frac{|X|}{\gamma}\right) | \mathcal{F}\right) &= E\left(\Phi\left(\frac{|X|}{\gamma} \chi(|Y| < \varepsilon)\right) | \mathcal{F}\right) + \\ &+ E\left(\Phi\left(\frac{|X|}{\gamma} \chi(|Y| \geq \varepsilon)\right) | \mathcal{F}\right) \leq E\left(\Phi\left(\frac{|X| \varepsilon}{\varepsilon(\|X\|_\phi^{\mathcal{F}} + \varepsilon)}\right) | \mathcal{F}\right) \chi(|Y| < \varepsilon) + \\ &+ E\left(\Phi\left(\frac{|XY|}{\|XY\|_\phi^{\mathcal{F}} + \varepsilon}\right) | \mathcal{F}\right) \chi(Y \geq \varepsilon) \leq 1 \quad \text{a.e.} \end{aligned}$$

Thus the inequality

$$\|X\|_{\phi}^{\tilde{\mathcal{F}}} \leq \gamma \quad \text{a.e.}$$

is satisfied for arbitrary $\epsilon > 0$. From this letting $\epsilon \rightarrow 0$ we get

$$\|X\|_{\phi}^{\tilde{\mathcal{F}}} |Y| \leq \|XY\|_{\phi}^{\tilde{\mathcal{F}}} \quad \text{a.e.}$$

This and the preceding inequality together prove the property b)

c) Suppose that $X, Y \in L_{\phi}^{\tilde{\mathcal{F}}}$ and let $\gamma \in F_X^{\Phi, \tilde{\mathcal{F}}}, \gamma' \in F_Y^{\Phi, \tilde{\mathcal{F}}}$ be arbitrary. Then by the monotonicity and the convexity of Φ we have

$$\Phi\left(\frac{|X+Y|}{\gamma+\gamma'}\right) \leq \Phi\left(\frac{|X|+|Y|}{\gamma+\gamma'}\right) \leq \frac{\gamma}{\gamma+\gamma'} \Phi\left(\frac{|X|}{\gamma}\right) + \frac{\gamma'}{\gamma+\gamma'} \Phi\left(\frac{|Y|}{\gamma'}\right)$$

and taking conditional expectations on both sides with respect to $\tilde{\mathcal{F}}$ we get

$$E\left(\Phi\left(\frac{|X+Y|}{\gamma+\gamma'}\right) | \tilde{\mathcal{F}}\right) \leq 1 \quad \text{a.e.}$$

since γ and γ' are $\tilde{\mathcal{F}}$ -measurable. From this we deduce

$$\|X+Y\|_{\phi}^{\tilde{\mathcal{F}}} \leq \|X\|_{\phi}^{\tilde{\mathcal{F}}} + \|Y\|_{\phi}^{\tilde{\mathcal{F}}} \quad \text{a.e.}$$

and thus c) is also proved.

As an example let $\Phi(x) = x^p$ with p such that $1 < p < +\infty$. Suppose that $X \in L_p$. Then by the definition of $F_X^{\Phi, \tilde{\mathcal{F}}}$ we have

$$F_X^{\Phi, \tilde{\mathcal{F}}} = \{\gamma: \gamma > 0 \text{ a.e., } \gamma \text{-} \tilde{\mathcal{F}}\text{-measurable, } (E(|X|^p | \tilde{\mathcal{F}}))^{1/p} \leq \gamma \text{ a.e.}\}$$

We show that

$$\text{ess inf } F_X^{\Phi, \tilde{\mathcal{F}}} = (E(|X|^p | \tilde{\mathcal{F}}))^{1/p} \quad \text{a.e.}$$

In fact, for $n \geq 1$ let

$$\gamma_n = \begin{cases} (E(|X|^p | \tilde{\mathcal{F}}))^{1/p}, & \text{if } E(|X|^p | \tilde{\mathcal{F}}) > 0, \\ \frac{1}{n}, & \text{if } E(|X|^p | \tilde{\mathcal{F}}) = 0. \end{cases}$$

Then trivially $\gamma_n \in F_X^{\Phi, \tilde{\mathcal{F}}}$ and letting $n \rightarrow +\infty$ we get that

$$\lim_{n \rightarrow +\infty} \gamma_n = (E(|X|^p | \tilde{\mathcal{F}}))^{1/p}.$$

On the other hand the random variable

$$(E(|X|^p | \tilde{\mathcal{F}}))^{1/p}$$

is trivially a lower bound of $F_X^{\Phi, \tilde{\mathcal{F}}}$.

5. The BMO_{ϕ} -space. Let $X \in L_1$ and consider the corresponding martingale $(X_n, \tilde{\mathcal{F}}_n)$ defined in section 1. Let Φ be a Young-function. We introduce the following

DEFINITION. We say that X belongs to BMO_ϕ if the quantity

$$\|\sup_{n \geq 1} \|X - X_{n-1}\|_{\phi^n}^{\tilde{\mathcal{F}}_n}\|_\infty$$

is finite.

REMARK. Let $\Phi(x) = x^p$, where $1 < p < +\infty$. Then by the preceding section it follows that

$$\|X - X_{n-1}\|_{\phi^n}^{\tilde{\mathcal{F}}_n} = (E(|X - X_{n-1}|^p)|\tilde{\mathcal{F}}_n)^{1/p}.$$

So the above definition of BMO_ϕ reduces to the well-known BMO_p -space.

Our definition of BMO_ϕ permits to treat some results in a very natural way.

THEOREM 5. Suppose that $X \in \text{BMO}_\phi$ where Φ is a Young-function. Then the quantity

$$\|X\|_{\text{BMO}_\phi} = \|\sup_{n \geq 1} \|X - X_{n-1}\|_{\phi^n}^{\tilde{\mathcal{F}}_n}\|_\infty$$

is a semi-norm.

PROOF. If C is any real number then by what we proved in the preceding section and by the properties of the L^∞ -norm we easily see that

$$|\epsilon X\|_{\text{BMO}_\phi} = |\epsilon| \|X\|_{\text{BMO}_\phi}.$$

The triangle inequality is also a consequence of these properties. Finally, when $X = 0$ a.e. then trivially $\|X\|_{\text{BMO}_\phi} = 0$. Now suppose that

$$\|X\|_{\text{BMO}_\phi} = 0.$$

Then

$$\sup_{n \geq 1} \|X - X_{n-1}\|_{\phi^n}^{\tilde{\mathcal{F}}_n} = 0 \quad \text{a.e.}$$

Since $X_0 = 0$ a.e. from this taking $n = 1$ we get

$$\|X\|_{\phi^1}^{\tilde{\mathcal{F}}_1} = 0 \quad \text{a.e.}$$

In the preceding section we have proved that this is equivalent to $X = 0$ a.e. This proves the assertion.

Now we prove the following

THEOREM 6. Suppose that $X \in \text{BMO}_\phi$, where Φ is a Young-function. If ϕ^* is another Young-function such that the integral

$$I = \int_0^{+\infty} \frac{\varphi^*(\lambda)}{\Phi(\lambda)} d\lambda$$

converges then $X \in \text{BMO}_{\phi^*}$. Moreover, we have

$$\|X\|_{\text{BMO}_{\phi^*}} \leq \max(1, I) \|X\|_{\text{BMO}_\phi}.$$

Here φ^* denotes the right hand side derivative of Φ^* .

PROOF. We can suppose that $\|X\|_{\text{BMO}_\phi} > 0$. (In the contrary case we see from the definition of the BMO_ϕ -space that $X = 0$ a.e. and then trivially $X \in \text{BMO}_{\phi^*}$) We first show that for all $t \geq 1$ the inequality

$$E\left(\Phi\left(\frac{|X - X_{t-1}|}{\|X\|_{\text{BMO}_\phi}}\right) | \mathcal{F}_t\right) \leq 1$$

holds a.e. Let for this purpose

$$\gamma_n \in F_{X-X_{t-1}}^{\phi, \mathcal{F}_t}$$

be a decreasing sequence of random variables tending to $\|X - X_{t-1}\|_\phi^{\mathcal{F}_t}$. Then for arbitrary $\varepsilon > 0$ we have

$$E\left(\Phi\left(\frac{|X - X_{t-1}|}{\gamma_n + \varepsilon}\right) | \mathcal{F}_t\right) \leq 1 \quad \text{a.e.}$$

Here $n \geq 1$ is arbitrary. Letting $n \rightarrow +\infty$ from this by the monotone convergence theorem for conditional expectations we get

$$E\left(\Phi\left(\frac{|X - X_{t-1}|}{\|X - X_{t-1}\|_\phi^{\mathcal{F}_t} + \varepsilon}\right) | \mathcal{F}_t\right) \leq 1 \quad \text{a.e.}$$

This implies

$$E\left(\Phi\left(\frac{|X - X_{t-1}|}{\|X\|_{\text{BMO}_\phi} + \varepsilon}\right) | \mathcal{F}_t\right) \leq 1 \quad \text{a.e.}$$

since with probability 1 we have

$$\|X - X_{t-1}\|_\phi^{\mathcal{F}_t} \leq \|X\|_{\text{BMO}_\phi} + \varepsilon \quad \text{a.e.}$$

Since $\varepsilon > 0$ is arbitrary and $\|X\|_{\text{BMO}_\phi}$ is positive from the preceding inequality we obtain

$$E\left(\Phi\left(\frac{|X - X_{t-1}|}{\|X\|_{\text{BMO}_\phi}}\right) | \mathcal{F}_t\right) \leq 1 \quad \text{a.e.}$$

Now for any $\lambda > 0$ with probability 1 we have

$$P\left(\frac{|X - X_{t-1}|}{\|X\|_{\text{BMO}_\phi}} \geq \lambda | \mathcal{F}_t\right) \leq \frac{1}{\Phi(\lambda)} E\left(\Phi\left(\frac{|X - X_{t-1}|}{\|X\|_{\text{BMO}_\phi}}\right) | \mathcal{F}_t\right) \leq \Phi^{-1}(\lambda).$$

Using the regular version of the conditional probability on the left side with arbitrary $c \geq 1$ we have

$$\begin{aligned} E\left(\Phi^*\left(\frac{|X - X_{t-1}|}{c \|X\|_{\text{BMO}_\phi}}\right) | \mathcal{F}_t\right) &= \int_0^{+\infty} P\left(\frac{|X - X_{t-1}|}{c \|X\|_{\text{BMO}_\phi}} \geq \lambda | \mathcal{F}_t\right) d\Phi^*(\lambda) \leq \\ &\leq \int_0^{+\infty} \Phi^{-1}(c\lambda) q^*(\lambda) d\lambda = \frac{1}{c} \int_0^{+\infty} \frac{q^*(\lambda)}{q^*(c\lambda)} \frac{q^*(c\lambda)}{\Phi(c\lambda)} d(c\lambda) \leq \\ &\leq \frac{1}{c} \int_0^{+\infty} \frac{q^*(c\lambda)}{\Phi(c\lambda)} d(c\lambda) = 1/c. \end{aligned}$$

Here we have used the fact that $c \geq 1$. Now the right side is ≤ 1 if $c \geq 1$. Taking $c = \max(1, I)$ this inequality means that for arbitrary $t \geq 1$

$$\|X - X_{t-1}\|_{\Phi^*} \leq \max(1, I) \|X\|_{\text{BMO}_\phi} \quad \text{a.e.}$$

holds with probability 1. Consequently,

$$\|X\|_{\text{BMO}_{\Phi^*}} \leq \max(1, I) \|X\|_{\text{BMO}_\phi}$$

and this proves the theorem.

As it is known $X \in \text{BMO}_1$ implies that for arbitrary $t \geq 1$ the random variable

$$E(\exp(t|X - X_{t-1}|) | \mathcal{F}_t)$$

belongs to L_∞ for suitably chosen values of $t > 0$. In the following assertion this fact will be imbedded in a natural way in the present theory of the BMO_ϕ spaces. The result will allow us to show that for a rich class of Young-functions, larger than the convex power functions, the BMO_1 -norm is equivalent to the BMO_ϕ -norm.

THEOREM 7. Suppose that $X \in \text{BMO}_1$. Then $X \in \text{BMO}_\phi$ where Φ is the Young-function

$$\Phi(x) := e^x - x - 1.$$

Moreover, we have

$$\|X\|_{\text{BMO}_\phi} \leq S \|X\|_{\text{BMO}_1},$$

Proof. We can suppose that $B = \|X\|_{\text{BMO}_1} > 0$. Otherwise the assertion is trivial. The result of Theorem 2 combined with the method of GARSIA (see [1], Theorem III, 2, 1.) for every $t \geq 1$ gives with probability 1

$$E(\exp(t \max_{1 \leq k \leq n} |X_k - X_{k-1}|) | \mathcal{F}_t) \leq (1 + 4tB)^{-1} \quad \text{a.e.}$$

provided that $t \in (0, (4B)^{-1})$. Letting $n \rightarrow +\infty$ from this we get

$$E(\exp(t \sup_{k \geq t} |X_k - X_{k-1}|) | \mathcal{F}_t) \leq (1 + 4tB)^{-1} \quad \text{a.e.}$$

Since

$$|X - X_{t-1}| \equiv \sup_{k \geq t} |X_k - X_{k-1}|$$

we deduce that

$$E(\exp(t|X - X_{t-1}|) | \mathcal{F}_t) \leq (1 + 4tB)^{-1} \quad \text{a.e.}$$

From this with $t = (8B)^{-1}$ we obtain

$$E\left(\exp\left(\frac{|X - X_{t-1}|}{8B}\right)^4 | \mathcal{F}_t\right) \leq 2 \quad \text{a.e.}$$

Finally

$$\begin{aligned} E\left(\Phi\left(\frac{|X - X_{t-1}|}{8B}\right) \mid \mathcal{F}_t\right) &= E\left(\exp\left(\frac{|X - X_{t-1}|}{8B}\right) \mid \mathcal{F}_t\right) - \\ &- E\left(\frac{|X - X_{t-1}|}{8B} \mid \mathcal{F}_t\right) + 2 - E\left(\frac{|X - X_{t-1}|}{8B} \mid \mathcal{F}_t\right) = 1 \approx 1 \quad \text{a.e.} \end{aligned}$$

since $X \in \text{BMO}_1$ with $B = \|X\|_{\text{BMO}_1}$. This proves the assertion.

A consequence of the preceding two theorems is the following

THEOREM 8. Suppose that $X \in \text{BMO}_1$. Then $X \in \text{BMO}_\phi$, where ϕ is any Young-function for which the integral

$$I := \int_0^\infty e^{\lambda} \frac{q(\lambda)}{\lambda - 1} d\lambda$$

converges. In this case we have

$$\|X\|_{\text{BMO}_\phi} \leq 8 \max(1, I) \|X\|_{\text{BMO}_1}.$$

PROOF. Combine the results of the preceding two theorems.

Now we prove that if ϕ has finite power then $X \in \text{BMO}_1$ implies $X \in \text{BMO}_\phi$.

THEOREM 9. Let $X \in \text{BMO}_1$. If ϕ^* is a Young-function having finite power p then with some constants $c_{\phi^*} > 0$ and $C_{\phi^*} > 0$ we have

$$c_{\phi^*} \|X\|_{\text{BMO}_1} \leq \|X\|_{\text{BMO}_{\phi^*}} \leq C_{\phi^*} \|X\|_{\text{BMO}_1}.$$

PROOF. The left side of this inequality can be easily proved. To prove the right-hand side we first establish that with the power

$$p = \sup_{x>0} \frac{x \phi^*(x)}{\phi^*(x)}$$

of the Young-function ϕ^* , the inequality

$$\|X\|_{\text{BMO}_{\phi^*}} \leq \max\left(1, \phi^*(1) \left[1 + \int_0^\infty \frac{pu^{p-1}}{\phi(u)} du\right]\right) \|X\|_{\text{BMO}_1},$$

is satisfied, where

$$\phi(x) = e^x - x - 1.$$

To this end we use the assertion and the ideas of proof of Theorem 6. Using the regular version of the conditional probability for all $t \geq 1$ and for

any $c \geq 1$ we have

$$\begin{aligned} E\left(\Phi^*\left(\frac{|X - X_{t-1}|}{c \|X\|_{\text{BMO}_\phi}}\right) \mid \mathcal{F}_t\right) &= \int_0^{+\infty} P\left(\frac{|X - X_{t-1}|}{c \|X\|_{\text{BMO}_\phi}} \geq \lambda \mid \mathcal{F}_t\right) d\Phi^*(\lambda) \leq \\ &\leq \Phi^*\left(\frac{1}{c}\right) + \int_{1/c}^{+\infty} \frac{\varphi^*(\lambda)}{\Phi(c\lambda)} d\lambda \leq \Phi^*\left(\frac{1}{c}\right) + \frac{1}{c} \int_{1/c}^{+\infty} \frac{\varphi^*(c\lambda)}{\Phi(c\lambda)} d(c\lambda) \leq \\ &\leq \Phi^*\left(\frac{1}{c}\right) + \frac{1}{c} \int_{1/c}^{+\infty} \frac{p\Phi^*(c\lambda)}{c\lambda\Phi(c\lambda)} d(c\lambda) = \Phi^*\left(\frac{1}{c}\right) + \frac{1}{c} \int_1^{+\infty} \frac{p\Phi^*(u)}{u\Phi(u)} du \leq \\ &\leq \Phi^*\left(\frac{1}{c}\right) + \frac{1}{c} \int_1^{+\infty} \frac{p\Phi^*(1)u^p}{u\Phi(u)} du = \Phi^*\left(\frac{1}{c}\right) + p \frac{\Phi^*(1)}{c} \int_1^{+\infty} \frac{u^{p-1}}{\Phi(u)} du \quad \text{a.e.} \end{aligned}$$

since

$$\frac{x\varphi^*(x)}{\Phi^*(x)} \leq p$$

and with $x \geq 1$ we have

$$\Phi^*(x) \leq x^p \Phi^*(1).$$

Choose $c \geq 1$ in such a way that

$$\Phi^*\left(\frac{1}{c}\right) + p \frac{\Phi^*(1)}{c} \int_1^{+\infty} \frac{u^{p-1}}{\Phi(u)} du \leq 1$$

hold. Then

$$\|X - X_{t-1}\|_{\Phi^*} \leq c \|X\|_{\text{BMO}_\phi} \quad \text{a. e.}$$

from which

$$\|X\|_{\text{BMO}_{\Phi^*}} \leq c \|X\|_{\text{BMO}_\phi}.$$

Finally, use the assertion of Theorem 7.

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A REMARK ON ADDITIVE FUNCTIONS SATISFYING A RELATION

By

IMRE KÁTAI

Department of Numerical Methods and Computer Science of the L. Eötvös University,
Budapest

(Received September 7, 1982)

We shall prove the following

THEOREM. If f, g, h are real-valued completely additive functions such that

$$(1) \quad f(n)+g(n+1)+h(n+2) \equiv 0 \pmod{1}$$

for every $n \geq 1$, then f, g, h assume integer values for every n .

COROLLARY. If f, g, h are real-valued completely additive functions such that

$$(2) \quad f(n)+g(n+1)+h(n+2) = 0$$

for every n , then $f(n) = g(n) = h(n) = 0$ identically.

The Corollary is a straightforward consequence of our Theorem. Indeed, if (2) is satisfied for f, g, h , then it is satisfied for $\tau f, \tau g, \tau h$ with an arbitrary constant τ . So (1) holds for $\tau f, \tau g, \tau h$, and Theorem involves that they assume only integer values. This is possible only if f, g, h are identically zero functions.

The proof of our Theorem is very simple. First we prove the following

LEMMA. If the conditions of Theorem are satisfied and $f(n), g(n), h(n)$ assume integer values for $n = 2$ and 3 , then f, g, h assume integer values for every $n \geq 1$.

PROOF. Assume the contrary. Let P be the smallest number for which one of the functions f, g, h takes on a non-integer value. Then P has to be a prime, $P \geq 5$. From (1) we get that $h(P) \equiv 0 \pmod{1}$.

Since $P+1$ is a composite number, its largest prime divisor is smaller than P , consequently $g(P+1) \equiv 0 \pmod{1}$, $h(P+1) \equiv 0 \pmod{1}$, and so $g(P) \equiv 0 \pmod{1}$. From our assumption we have that $f(P) \not\equiv 0 \pmod{1}$, and from (1) that $h(P+2) \not\equiv 0 \pmod{1}$. This is possible only if $P+2$ is a prime. Let us consider now the relations

$$(3) \quad h(2P+2)+g(2P+1)+f(2P) \equiv 0 \pmod{1},$$

$$(4) \quad h(2P+4)+g(2P+3)+f(2P+2) \equiv 0 \pmod{1}.$$

Since $h(2P+2) \equiv 0 \pmod{1}$, $f(2P+2) \equiv 0 \pmod{1}$, hence we deduce that $g(2P+1) \equiv -f(P) \not\equiv 0 \pmod{1}$, $g(2P+3) \equiv -h(P+2) \not\equiv 0 \pmod{1}$, so the numbers $2P+1$, $2P+3$ have to have prime divisors greater than P , consequently they are primes. So $2P+1$ is a composite number, and from

$$h(2P+1) + g(2P) + f(2P-1) \equiv 0 \pmod{1}$$

we deduce that $h(2P+1) \equiv 0 \pmod{1}$. Consequently, from

$$h(4P+2) + g(4P+1) + f(4P) \equiv 0 \pmod{1}$$

we deduce that $g(4P+1) \not\equiv 0 \pmod{1}$. Since P , $P+2$, $2P+1$ are primes, therefore $P \equiv -1 \pmod{3}$ and so $4P+1 = 3R$, $(R, 2) = 1$, and $0 \not\equiv g(R) \equiv -f(P) \pmod{1}$. Then $R+1$, $R-1$ are even numbers, so the largest prime factors of them are smaller than P , consequently $h(R+1) \equiv 0 \pmod{1}$, $f(R-1) \equiv 0 \pmod{1}$, whence by

$$h(R+1) + g(R) + f(R-1) \equiv 0 \pmod{1}$$

it follows that $g(R) = 0 \pmod{1}$ which is a contradiction. ■

It remains to prove that the functions f , g , h take on integer values for $n = 2$ and 3.

Let L_n denote the expression

$$(L_n =) f(n) + g(n+1) + h(n+2).$$

From $L_1 \equiv 0 \pmod{1}$ we deduce that

$$(5) \quad g(2) + h(3) \equiv 0 \pmod{1}.$$

Hence, and from $L_7 \equiv 0 \pmod{1}$ we get that

$$(6) \quad g(2) + f(7) \equiv 0 \pmod{1}.$$

Since $L_{63} - L_{11} \equiv 0 \pmod{1}$, therefore by (6) and (5) we get

$$\begin{aligned} 0 &\equiv h(5) + h(13) + 6g(2) + 2f(3) + f(7) - h(13) - 2g(2) - g(3) + f(11) \equiv \\ &\equiv (h(5) - L_3) + 6g(2) + 2f(3) - g(2) - 2g(2) - g(3) + L_2 - f(11) \equiv \\ &\equiv g(2) + 2h(2) + f(2) - f(3) \equiv -h(3) + 2h(2) + f(2) - f(3) \pmod{1}. \end{aligned}$$

Hence we get that

$$(7) \quad f(11) \equiv 2h(2) - h(3) + f(2) + f(3) \pmod{1}.$$

Starting from $L_{11} \equiv 0 \pmod{1}$ we get that

$$\begin{aligned} f(4) + f(11) &\equiv -g(45) - h(46) \equiv -2g(3) + 2L_2 - g(5) + L_1 - h(2) - h(23) \equiv \\ &\equiv -2g(3) + 2L_2 - g(5) + L_1 - h(2) - h(23) \equiv \\ &\equiv 4h(2) + 2f(2) + h(2) + h(3) + 2f(2) - h(2) + f(21) + g(22) \equiv \\ &\equiv 4h(2) + 4f(2) + h(3) + f(3) + (f(7) + g(2)) + g(11) \pmod{1}. \end{aligned}$$

Hence we get that

$$f(11) \equiv 4h(2) + 2f(2) + h(3) + f(3) + g(11) - L_{10} \pmod{1},$$

and so that

$$\begin{aligned} f(11) &\equiv 4h(2) + 2f(2) + h(3) + f(3) - f(2) - f(5) - 2h(2) - h(3) \equiv \\ &\equiv -f(5) + 2h(2) + f(2) + f(3) \pmod{1}, \end{aligned}$$

i.e.

$$(8) \quad f(11) + f(5) \equiv 2h(2) + f(2) + f(3) \pmod{1}.$$

Let us consider the relations $L_{11} \equiv L_{24} \equiv 0 \pmod{1}$. So we have

$$\begin{aligned} 0 &= L_{24} - L_{11} \equiv h(2 \cdot 13) + g(5^2) + f(2^3 \cdot 3) - h(13) - g(2^3 \cdot 3) - f(11) \equiv \\ &\equiv h(2) + 2g(5) - 2L_4 + 3f(2) + f(3) - 2g(2) - g(3) + L_2 - f(11) \equiv \\ &= h(2) - 2h(2) - 2h(3) - 4f(2) + 3f(2) + f(3) - 2g(2) - g(3) + 2h(2) + f(2) - \\ &\quad - f(11) \pmod{1}. \end{aligned}$$

whence, by observing that $-g(3) \equiv f(2) + 2h(2)$, we get that

$$(9) \quad f(11) = h(2) + f(3) \pmod{1}.$$

From (7) and (9) we get that

$$(10) \quad h(3) \equiv h(2) + f(2) \pmod{1}.$$

From $L_{35} \equiv 0 \pmod{1}$ we deduce that

$$\begin{aligned} f(55) &\equiv -g(36) - h(57) = -3g(2) - g(7) + L_6 - h(3) - h(19) \equiv \\ &\equiv -3g(2) + 3h(2) + 2f(2) - h(3) + h(2) - h(38) + L_{36} \equiv \\ &\equiv 4h(2) + 2f(2) - h(3) + f(36) + g(37) \equiv \\ &\equiv 4h(2) + 4f(2) + 2f(3) - h(3) - g(3) + L_2 + g(111) - L_{110} \equiv \\ &\equiv 6h(2) + 5f(2) + 2f(3) - h(3) - f(110) - h(112). \end{aligned}$$

Hence, by (9) we get that

$$\begin{aligned} 2f(55) &\equiv 6h(2) + 5f(2) + 2f(3) - h(3) - f(2) - h(7 \cdot 16) \equiv \\ &\equiv 2h(2) + 4f(2) + 2f(3) - h(3) - h(7) + L_5 \equiv \\ &\equiv h(2) + 3f(2) + 2f(3) + f(5) + g(2) + g(3) - L_2 \equiv \\ &= -h(2) + 2f(2) + 2f(3) + f(5) - h(3) \equiv -2h(2) + f(2) + 2f(3) + f(5) \pmod{1}. \end{aligned}$$

After substituting (8) into the left hand side, we get that

$$2(2h(2) + f(2) + f(3)) \equiv -2h(2) + f(2) + 2f(3) + f(5) \pmod{1},$$

whence we deduce that

$$(11) \quad f(5) \equiv 6h(2) + f(2) \pmod{1}.$$

From (6) and $L_{14} \equiv 0 \pmod{1}$ we deduce that

$$\begin{aligned} g(2) - f(2) &\equiv -f(14) \equiv g(15) - h(16) \equiv -g(3) - L_2 + g(5) - L_4 + 4h(2) \equiv \\ &\equiv h(4) - f(2) - h(6) - f(4) + 4h(2) \equiv h(2) - h(3) - 3f(2) \pmod{1}, \end{aligned}$$

whence by (5) we get that

$$(12) \quad h(2) \equiv 2f(2) \pmod{1}.$$

Hence, and from (11) and (10) it follows that

$$(13) \quad f(5) \equiv 13f(2) \pmod{1}$$

$$(14) \quad h(3) \equiv 3f(2) \pmod{1}.$$

Furthermore,

$$\begin{aligned} f(3) + f(11) &\equiv -g(34) - h(35) \equiv -g(2) + g(17) + L_{16} - h(5) + L_3 - h(7) + L_5 \equiv \\ &\equiv -g(2) + f(16) : h(18) - f(3) + g(4) - f(5) + g(6) \equiv \\ &\quad + 2g(2) + g(3) + 4f(2) + h(2) + 2h(3) + f(3) - f(5) \equiv \\ &\equiv g(3) + 4f(2) + h(2) + f(3) + f(5) \equiv f(2) + f(3) + f(5) \equiv \\ &\equiv 14f(2) + f(3) \pmod{1}. \end{aligned}$$

and so

$$f(11) \equiv 14f(2) \pmod{1}.$$

Comparing this with (9), we have

$$(15) \quad f(3) \equiv 12f(2) \pmod{1}.$$

Let us consider now the relation

$$\begin{aligned} 0 \equiv L_{50} &\equiv 2f(5) + f(2) + g(3) + g(17) + L_{16} + 2h(2) - h(13) - L_{11} \equiv \\ &\equiv 2f(5) + f(2) + g(3) + h(18) - f(16) + 2h(2) + f(11) - g(12) \equiv \\ &\equiv 2f(5) + 3f(2) + g(3) + h(2) - 2h(3) - f(11) - 2g(2) - g(3) \equiv \\ &\equiv 2f(5) - f(11) + h(2) - 3f(2) \equiv 2f(5) - f(11) - f(2) \pmod{1}. \end{aligned}$$

Hence, after replacing (13),

$$(16) \quad f(11) \equiv 25f(2) \pmod{1}.$$

On the other hand, from (9) and (15) we get that

$$(17) \quad f(11) \equiv 14f(2) \pmod{1}.$$

Consequently

$$(18) \quad 14f(2) \equiv 0 \pmod{1},$$

and so

$$(19) \quad f(3) \equiv f(2) \pmod{1}.$$

Furthermore

$$\begin{aligned} 0 &\equiv L_{25} = 2f(5) + g(2) + g(13) - L_{12} + 3h(3) \equiv \\ &\equiv 2f(5) + g(2) - h(2) - h(7) - 2f(2) - f(3) + 3h(3) \equiv \\ &\equiv 2f(5) + 2h(3) - h(2) - 2f(2) - f(3) + g(6) + f(5) \pmod{1}, \end{aligned}$$

and so by (19), (14), (12), (15) we get that

$$(20) \quad 3f(5) \equiv 7f(2) \pmod{1}.$$

Hence, by (19) we get that

$$(21) \quad 32f(2) \equiv 0 \pmod{1},$$

and so by (18) that

$$(22) \quad f(2) \equiv 0 \pmod{1}.$$

Now we get immediately that

$$\begin{aligned} g(2) &\equiv 0 \pmod{1}, \quad h(2) \equiv 0 \pmod{1}, \quad f(3) \equiv 0 \pmod{1}, \\ g(3) &\equiv 0 \pmod{1}, \quad h(3) \equiv 0 \pmod{1}. \end{aligned}$$

Consequently the conditions of our lemma are satisfied. By this the proof of our theorem is finished. ■

ON A CLASS OF p -GROUPS

By

K. CORRÁDI and L. HÉTHELYI

Department of Numerical Methods and Computer Science and
Department of Philosophy of the L. Eötvös University, Budapest

(Received September 27, 1982)

1. Introduction. A nonabelian 2-group of maximal class has the following properties.

1. It contains a maximal subgroup which is abelian.
2. The Frattini subgroup is cyclic.
3. It has a subgroup of order 4 which is selfcentralizing.

The last property putting p^2 in place of 4 characterizes all p -groups of maximal class. In this paper we characterize those finite p -groups which have weakened forms of 2. and 3. In an interesting way they all have 1.

In order to state our main result we need some notations and terminology. A nonabelian group is called Rédeian if all of its proper subgroups are abelian. For $m \geq 3$, $0 \leq s \leq p-1$, p a prime, s a natural number let $P_{m,s}$ denote the following group:

$$P_{m,s} = \langle x, y; x^{p^m} = y^p = z^p \cdots e, z = [x, y], x^z = x^{1+p^{m-1}}, [y, z] = [x, z^s] \rangle.$$

The main result of this paper is the following.

THEOREM I. Let P be a finite p -group, $A \in SC(P)$, $N = N_p(A)$. Suppose $N \triangleleft P$ and

1. N is a Rédei group.
2. $\Phi(P)$ is metacyclic.
3. $\overline{\Omega}_1(P)$ is cyclic.

Then we have the following possibilities:

- a) $p = 2$, $P \cong D_m, Q_n, S_m$, $n \geq 4$, *
- b) $p \geq 2$, $|P| = p^3$, $\text{cl}(P) = 3$,
- c) $p \geq 2$ and $P \cong P_{m,s}$.

If instead of 3. we suppose that P' is cyclic then there are even metacyclic p -groups which satisfy the conditions of Theorem I.

* D_n , Q_n and S_n stands for the dihedral, the generalized quaternion and semidihedral group of order 2^n respectively.

EXAMPLE:

$$P = \langle x, y; x^{p^3} = y^{p^2} = e, x^y = x^{1+p}, n \geq 2 \rangle.$$

Then $P' = \langle x^p \rangle$

$$Z(P) = \langle x^{p^2}, y^{p^2} \rangle = \langle x^{p^2} \rangle \times \langle y^{p^2} \rangle.$$

$P' \ncong Z(P)$ thus $\text{cl}(P) \geq 3$.

It is clear that $A = \langle x^{p^2} \rangle \times y / \in SC(P)$

$N = \langle x^p, y \rangle$ is a Rédei group and $N_p(A) = N$.

Lemmas needed for the proof of Theorem 1.

2. LEMMA 1. Let P be a p -group, $p \geq 2$. If $|P| = p^4$ and $|P'| = p^2$ then P' is an abelian group of type (p, p) .

PROOF. Let $U \trianglelefteq P$, $|U| = p^2$, U not cyclic. Then $P' \trianglelefteq U$, because $|P/U| = p^2$.

LEMMA 2. Let P be a p -group, $p \geq 2$. If $|P| = p^5$ and P' is an abelian group of type (p^2, p) then $|\Omega_1(P)| \neq p$.

PROOF. Suppose $|\Omega_1(P)| = p$. Set $R = \Omega_1(P)$.

1. $R \ncong Z(P)$. If $R \leq Z(P)$ then choose a $V \trianglelefteq R$ with $|V| = p$ so that P/V is cyclic. Then if $P = P/V$, so $|P| = p^4$ and $|P'| = p^2$ and P' is cyclic. This contradicts Lemma 1.

2. Conclusion of the proof.

Set $C = C_p(R)$. Then $C \in \mathfrak{M}(P)$, $R \trianglelefteq Z(C)$ and $|C : R| = p^2$. Thus $\text{cl}(C) \geq 2$ and C is a regular group. Hence $|\Omega_1(C)| \leq |C : \Omega_1(C)|$ and $\exp(\Omega_1(C)) \geq p$.

$P' \trianglelefteq \Phi(P) \trianglelefteq C$ thus $\exp(C) \geq p^2$. So because of $|\Omega_1(P)| = p$, $\Omega_1(C) \trianglelefteq \Omega_1(P)$. Hence $|\Omega_1(C)| = p^3$. By $\Omega_1(C) \trianglelefteq P$ and by $|P : \Omega_1(C)| = p^2$, $P' \trianglelefteq \Omega_1(C)$ follows. This contradicts to $\exp(\Omega_1(C)) = p$.

LEMMA 3. Let P be a p -group, $M \in \mathfrak{M}(P)$, $M' = E$. Then $|M| = |M \cap Z(P)| \cdot |P'|$.

PROOF. See HUPPERT [4].

LEMMA 4. Let P be a p -group, $A \in SC(P)$, $N = N_p(A)$. Set $N_1 = N_p(A)$, $N_i = N_p(N_{i-1})$, $i \geq 2$. If $|N : A| = p$ then

1. $N_{i-1} = N_i A$, $i \geq 2$,
2. $|N_i : N_{i-1}| = p$, $i \geq 2$,
3. $|N_i : N'_i| \leq |N_1 : N'_1|$, $i \geq 2$,
4. $|N_i : \Phi(N_i)| \leq |N_1 : \Phi(N_1)|$.

PROOF. See HÉTHELYI [3].

LEMMA 5. Let P be a p -group. Let $Z(P)$ be cyclic. If $\chi(1) = p$ for all nonlinear $\chi \in \text{Irr}(P)$, then there is an $M \in \mathfrak{M}(P)$ with $M' = E$.

PROOF. Let $\chi \in \text{Irr}(P)$, $\deg(\chi) = p$. P is an M -group* thus there is an $M \in \mathfrak{M}(P)$ and a $\sigma \in \text{Irr}(M)$, with $\chi = \sigma P$. By $M' \trianglelefteq \text{Ker } \sigma$ and by $M \triangleleft P$, $M' \trianglelefteq \text{Ker } \chi$ follows. $\Omega_1(Z(P)) \leq M'$ and $M' \triangleleft P$ leads to

$$\Omega_1(Z(P)) \leq \bigcap_{\chi \in \text{Irr}(P)} \text{Ker } \chi = E.$$

This is a contradiction.

LEMMA 6. Let P be a p -group, $\chi \notin \text{Irr}(P)$. Then

$$|P : Z(P)| \equiv 0 \pmod{(\chi(1))^2}.$$

PROOF. See GORENSTEIN [2].

LEMMA 7. Let P be a p -group, $A \in SC(P)$ and $N = N_P(A)$. If $A \triangleleft P$ and

1. $\Phi(P)$ is cyclic,
2. N is Rédei group,

then $p = 2$ and $P \cong D_n, Q_n, S_n$, $n \geq 4$.

REMARK. If P is one of the groups mentioned above then $\Phi(P)$ is cyclic and it contains an $A \in SC(P)$ with $|A| = 4$ and $N_P(A)$ is a Rédei group.

PROOF. Since N is a Rédei group $|N : \Phi(N)| = p^2$. Thus $|P : \Phi(P)| = p^2$ by Lemma 4. Since $A \in SC(P)$, $Z(P) \trianglelefteq A$. Since $A \triangleleft P$, $P' \trianglelefteq A$. Thus $\text{cl}(P) > 3$, and so there is at least one $M \in \mathfrak{M}(P)$ such that $M' \neq E$. Let $H = \Phi(P)$, $M \in \mathfrak{M}(P)$, $M' \neq E$, $H \in \mathfrak{M}(M)$, H is cyclic and $C_M(H) = H$. Thus $M \cong D_n, Q_n, S_n, M_n(p)$ (see [2]) or $|P| = p^4$. If $|P| = p^4$, so $|P'| = p^2$ and P' is cyclic. Hence $p = 2$ follows by Lemma 1. If $p \neq 2$ then $M \cong M_n(p)$ for all $M \in \mathfrak{M}(P)$, $M' \neq E$. Then $Z(P) \cong \bar{\Omega}_1(H)$ and thus $|P/Z(P)| = p^2$. Then Lemma 6. shows that $\chi(1) = p$ for all $\chi \in \text{Irr}(P)$, $\chi(1) \neq 1$. Since $Z(P)$ is cyclic so by Lemma 5. there is an $L \in \mathfrak{M}(P)$ such that $L' = E$. Then by Lemma 3. $|L| = |Z(P)| \cdot |P'|$. Thus $|P'| = p^2$. On the other hand $|P| \geq p^5$ and thus $|Z(P)| \geq p^2$.

H is cyclic and P' , $Z(P) \trianglelefteq H$. Thus $P' \trianglelefteq Z(P)$ and so $\text{cl}(P) \leq 2$ which is a contradiction. Thus $p = 2$ and in consequence $\bar{\Omega}_1(P) = \Phi(P)$. As P has a cyclic maximal subgroup and as $\text{cl}(P) \geq 3$, $P \cong Q_n, D_n, S_n$.

LEMMA 8. Let p be a prime, $p \neq 2$. Let m be a given natural number. Let s be a natural number with $0 \leq s \leq p-1$. Set

$$P_s = \langle x, y; x^{p^m} = y^p = z^p = e, z = [x, y], x^z = e^{1+p^{m-1}}, [y, z] = [x, z]^s \rangle.$$

Then if t is a natural number with $0 \leq t \leq p-1$, then $P_s = P_t$ iff $\left(\frac{s}{p}\right) = \left(\frac{t}{p}\right)$.

Here $\left(\frac{s}{p}\right)$ is the Legendre symbol defined by

$$\left(\frac{s}{p}\right) = \begin{cases} 0 & \text{if } s = 0, \\ 1 & \text{if } s \text{ is a quadratic residue,} \\ -1 & \text{if } s \text{ is not a quadratic residue.} \end{cases}$$

* By an M -group we mean a monomial group. This notation is standard.

PROOF. Let $P_s = P_t$. Then there are $x_1, y_1, z_1 \in P_s$ such that

$$P_s = \langle x_1, y_1; x_1^{p^m} = y_1^p = z_1^p = e, z_1 = [x_1, y_1], x_1^{z_1} = x_1^{1+p^{m-1}} \rangle,$$

$$[y_1, z_1] = [x_1, z_1]^p.$$

Let $x_1 = x^a y^r z^w, y_1 = x^s y^t z^v, z_1 = x^l z^r$.

Then using the commutator identities and the fact that $\text{cl}(P) = 3$ we get the following relations

$$(u + sv)r \equiv u(p),$$

$$(*)(u + sv)t \equiv \beta rs(p),$$

$$r \equiv u\beta(p).$$

(*) gives that in case of $P_s = P_t$

$$\begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} t \\ p \end{pmatrix}.$$

If on the other hand $\begin{pmatrix} s \\ p \end{pmatrix} \neq \begin{pmatrix} t \\ p \end{pmatrix}$ then the congruences of (*) are satisfied putting $u = 1, \beta^2 = st^{-1}, r = \beta, v = t^{-1}\beta(1-\beta)$ and so $P_s \neq P_t$ follows.

3. The PROOF of THEOREM 1. As $\text{cl}(P) \geq 3$ in the case $|P| = p^4$ there is nothing to prove. Let $|P| > p^5$. If $p = 2$ then $\Phi(P) = \bar{\Omega}_1(P)$ and thus by Lemma 7, $P \cong Q_n, D_m, S_n$.

Suppose that $p > 2$. Set $R = \Omega_1(\Phi(P))$. Since $p > 2$ and $\Phi(P)$ is metacyclic, so $\Phi(P)$ is regular. $\Phi(P)$ is not cyclic because of Lemma 7. By the regularity of $\Phi(P)$, R is an abelian group of type (p, p) . Since $\bar{\Omega}_1(\Phi(P))$ is cyclic and $|R| = |\Phi(P) : \bar{\Omega}_1(\Phi(P))|$, $\Phi(P)$ has a cyclic maximal subgroup. Thus $\Phi(P)$ is either abelian or $\Phi(P) \cong M_n(p)$. On the other hand $C_P(R) \triangleleft P$ and $|P : C_P(R)| < p$. If $\Phi(P) = M_n(p)$ then $R \cong Z(\Phi(P))$. This forces that $\Phi(P)$ is an abelian group. Set $\bar{P} = P/R$. $\Phi(\bar{P})$ is cyclic, $|\bar{P}/\Phi(\bar{P})| = p^2$ because of Lemma 4. As $\Phi(\bar{P}) \neq \bar{P}' \cdot \bar{\Omega}_1(\bar{P})$ and $\Phi(\bar{P})$ is cyclic thus either

1. $\Phi(\bar{P}) \neq \bar{\Omega}_1(\bar{P})$, or 2. $\Phi(\bar{P}) = \bar{\Omega}_1(\bar{P})$.

1. $\Phi(\bar{P}) \neq \bar{\Omega}_1(\bar{P})$. Set $\bar{C} = C_{\bar{P}}(\bar{P}')$. Then $\bar{C} \neq \bar{P}'$. Namely \bar{P}' is cyclic and $p > 2$ so by $\bar{C} = \bar{P}', \bar{P}/\bar{P}'$ would be cyclic. Hence \bar{P} would be cyclic which is not the case. Thus $\bar{P}' \triangleleft \bar{C} \triangleleft \bar{P}$ and as $|\bar{P}/\bar{P}'| = p^2$ there is an $M \in \mathfrak{M}(\bar{P})$ with $M \leq \bar{C}$. As \bar{M}/\bar{P}' is cyclic, $M' = E$. By $Z(\bar{P}) \cong M$, $|Z(\bar{P})| = p$ follows using Lemma 3. If $y \in \bar{P} \setminus \bar{M}$, $B \stackrel{\text{def}}{=} C_{\bar{P}}(y)$ then $|B| = p^2$, $B \in SC(\bar{P})$ and $N_{\bar{P}}(B)$ is a Rédei group. If $\bar{B} \triangleleft \bar{P}$ then $p = 2$ follows by Lemma 7., which is not the case. If $\bar{B} \triangleleft \bar{P}$ then $|\bar{P}| = p^3$ and $|P| = p^5$. As $\Phi(P) \neq \bar{\Omega}_1(\bar{P})$, $\bar{\Omega}_1(\bar{P}) = \bar{E}$. Hence $|\bar{\Omega}_1(P)| = p$. So $\Phi(P) = P'$ and P' is an abelian group of type (p^2, p) . This contradicts to Lemma 2.

2. $\bar{\Omega}_1(\bar{P}) = \Phi(\bar{P})$ and $\bar{P}' \neq \Phi(\bar{P})$. There is an $\bar{M} \in \mathfrak{M}(\bar{P})$ which is cyclic. As $p > 2$, \bar{P} is either abelian or $\bar{P} \cong M_n(p)$. \bar{P} is not cyclic. Thus the number

of the \bar{M}_i -s in $\mathfrak{M}(\bar{P})$, which are cyclic, is exactly p . Let M denote the inverse image of \bar{M} in P . As \bar{M} is cyclic $|M'| < p$ and thus $M' \leq Z(P)$. Hence $M' \leq Z(M)$ and $\text{cl}(M) \leq 2$. Thus M is regular. As $\text{cl}(P) \geq 3$ there is at most one $M \in \mathfrak{M}(P)$ with $M' = E$. So $R \not\cong Z(P)$ is a consequence. If $M \in \mathfrak{M}(P)$ and M is cyclic but $M' \neq E$ then $M' = R \cap Z(P)$. Set $D = R \cap Z(P)$. Then $P/D = \tilde{P}$ is generated by two elements. On the other hand \tilde{P} has at least $p-1 \geq 2$ maximal subgroups which are abelian. Thus \tilde{P} is a Rédei group. $|\Omega_1(\tilde{P})| \leq p^3$ and since \tilde{P} is regular $\exp(\Omega_1(\tilde{P})) = p$. If $|\Omega_1(\tilde{P})| = p^2$ then \tilde{P} has a cyclic maximal subgroup and then $\tilde{P} \cong M_n(p)$. Thus \tilde{P} has p maximal subgroups which are cyclic. The inverse images of these are abelian because of $D \leq Z(P)$. This contradicts to $\text{cl}(P) \geq 3$. Thus $|\Omega_1(\tilde{P})| = p^3$. Let T denote the inverse image of $\Omega_1(\tilde{P})$ in P then $|T| = p^4$. $\exp(T) \leq p^2$. Assume $\exp(T) = p$. By $|\Phi(P) \cap T| \leq p^2$ and by $|P : \Phi(P)| = p^2$, $P = \Phi(P) \cdot T$ follows. So $P = T$ which cannot be the case because of $|P| > p^5$. Thus $\exp(T) = p^2$, $T \not\cong P$ and T is regular. Since $\bar{\Omega}_1(P)$ is cyclic $|\bar{\Omega}_1(T)| = p$. Thus $\Omega_1(P) = \Omega_1(T)$, $|\Omega_1(T)| = p^3$ and $\exp(\Omega_1(P)) = p$ because T is regular. There is exactly one $L \in \mathfrak{M}(P)$ with $\Omega_1(P) \leq L$. If $M \in \mathfrak{M}(P)$, $M \neq L$ then $\Omega_1(M) = R$. M has a cyclic maximal subgroup because M is regular and $\bar{\Omega}_1(M)$ is cyclic. Thus M is abelian or $M \cong M_n(p)$. Since there are $p-1 \geq 2$ maximal subgroups of P with $M' = E$ there are at least two maximal subgroups of P which are isomorphic to $M_n(p)$. If M is such a maximal subgroup then $\bar{\Omega}_1(M) = \bar{\Omega}_1(P)$, $\bar{\Omega}_1(M) = Z(M)$ and $|M/\bar{\Omega}_1(M)| = p^2$. Since there are at least two such subgroups $\bar{\Omega}_1(M) \leq Z(P)$ and thus $|P/Z(P)| \leq p^3$ and $Z(P)$ is cyclic. Thus the non-linear elements of $\text{Irr}(P)$ have degree p and thus the conditions of Lemma 5. are satisfied. Thus there is an $M \in \mathfrak{M}(P)$ with $M' = E$. P has a cyclic subgroup Q with $|P : Q| = p^2$. Since $|\Omega_1(P)| = p^3$ and $\exp(\Omega_1(P)) = p$, $P = \Omega_1(P) \cdot Q$. Thus $P' \leq \Omega_1(P)$. Thus $P' \leq \Phi(P) \cap \Omega_1(P) \leq R$. Thus $R = P'$ because $\text{cl}(P) \geq 3$. In the following we have to distinguish two cases

I. $L' = E$. Since $R = P'$ it is easy to prove that the elements of $\mathfrak{M}(P)$ are L and $M_j = \langle c^j a, b \rangle$, $0 \leq j \leq p-1$, where

$$P = \langle a, b, c \rangle = \langle a, c \rangle, M_0 = \langle a, b; a^{p^m} = b^p = e, a^b = a^{1+p^{m-1}} \rangle$$

and $L = \langle a^p \rangle \times \langle b \rangle \times \langle c \rangle$ further $M_j \cong M_{m+1}(p)$, $0 \leq j \leq p-1$. Now it is easy to see that P is of the form of Lemma 8.

2. $L' \neq E$. Just as in the previous case by using $R = P'$ we first have to prove that the elements of $\mathfrak{M}(P)$ are L and $M_j = \langle c^j a, b \rangle$, $0 \leq j \leq p-1$. There is exactly one j such that $M'_j = E$. Now it is easy to prove that P is of the form of Lemma 8.

We mention a consequence of the Theorem I.

THEOREM II. *Let P be a p -group of maximal class, $p > 2$. If $|P| = p^5$ then the following two conditions cannot be satisfied at the same time.*

1. $\Phi(P)$ is metacyclic

2. $\bar{\Omega}_1(P)$ is cyclic.

A well-known result of BLACKBURN asserts that 3-groups of maximal class have metacyclic Frattini subgroups [1]. On the other hand examples of p -groups P of maximal class are known with cyclic $\bar{O}_1(P)$.

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ON ELLIPTIC DIFFERENTIAL EQUATIONS IN \mathbf{R}^n

BY

L. SIMON

II. Department of Analysis of the L. Eötvös University, Budapest

(Received December 13, 1982)

Introduction

Let $P = P(D)$ be an elliptic differential operator of order $2m$ with constant coefficients $\left(D = \left[-i \frac{\partial}{\partial x_1}, \dots, -i \frac{\partial}{\partial x_n} \right] \right)$ and $Q = Q(x, D)$ a differential operator of order $2m$ with smooth coefficients which vanish for $|x| > a$. Consider the elliptic equation

$$(P+Q)u = f \quad \text{in } \mathbf{R}^n.$$

In [1] this equation has been considered when $P(\xi) \neq 0$ for all $\xi \in \mathbf{R}^n$. It has been shown that if for any $f \in L_a^2(\mathbf{R}^n)$ (i.e. $f \in L^2(\mathbf{R}^n)$, $f(x) = 0$ if $|x| > a$) there exists a solution u of the above equation, tending to zero at infinity then the solution is unique. Moreover, in [1] there have been formulated conditions on the differential operators $B_j(x, D)$ which guarantee that for sufficiently large $g > 0$ the boundary value problem in $B_g = \{x \in \mathbf{R}^n : |x| < g\}$

$$(P+Q)u = f \quad \text{in } B_g$$

$$B_j(x, D)u = 0 \quad \text{on } S_g, \quad j = 1, \dots, m$$

($S_g = \{x \in \mathbf{R}^n : |x| = g\}$) has a unique solution u_g in the Sobolev space $H^{2m}(B_g)$ and

$$\|u - u_g\|_{H^{2m}(B_g)} \leq c_1 \|f\|_{L_a^2(\mathbf{R}^n)} \cdot e^{-c_2 g}$$

(c_1, c_2 are positive constants which do not depend on f and g).

The aim of the present paper is to prove results of this type if $P(\xi) \neq 0$ for $\xi \in \mathbf{R}^n \setminus \{0\}$ but $P(0) = 0$. This case is important for applications since our results are valid e.g. for $P(D) = \Delta^k$. In 1. estimations on the tempered fundamental solution of $P(D)$ will be proved. In 2. the case $Q = 0$ will be considered and finally in 3. theorems for the general case will be proved.

1. Fundamental solutions

LEMMA 1. Let $P(D) = \sum_{j=t}^m P_j(D)$ be an elliptic differential operator of order $2m$ with constant coefficients ($P_j(D)$ denotes the homogeneous part of

$P(D)$ of order j), satisfying the following conditions: $P(\xi) \neq 0$ if $\xi \in \mathbf{R}^n \setminus \{0\}$ and $P_l(\xi) \neq 0$ if $\xi \in \mathbf{R}^m \setminus \{0\}$, $l < n$.

Then $P(D)$ has a (unique) fundamental solution E such that at infinity for any fixed z the estimation

$$(1.1) \quad D^\alpha E(x) = O\left(\frac{1}{|x|^{n-l+\alpha}}\right)$$

holds.

PROOF. $\frac{1}{P}$ is a locally integrable function in \mathbf{R}^n as it is continuous in $\mathbf{R}^n \setminus \{0\}$ and at zero the estimation

$$(1.2) \quad \frac{1}{P(\xi)} = O\left(\frac{1}{|\xi|^l}\right), \quad l < n$$

holds. By ellipticity of P , $P_{2m}(\xi) \neq 0$ if $\xi \in \mathbf{R}^n \setminus \{0\}$ and so $\frac{1}{P}$ is a tempered distribution. The inverse Fourier transform of

$$\frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{P}, \quad \text{i. e. } E = \frac{1}{(2\pi)^{n/2}} \cdot \mathcal{F}^{-1}\left(\frac{1}{P}\right)$$

is a fundamental solution of $P(D)$.

For the solutions u of the equation $P(D)u = 0$

$$\text{supp } (\mathcal{F}u) = \{0\}$$

($P(\xi) \neq 0$ if $\xi \in \mathbf{R}^n \setminus \{0\}$), thus $\mathcal{F}u$ has the form

$$\mathcal{F}u = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \delta$$

and so u is a polynomial. This implies the uniqueness of solutions of $P(D)u = f$ in the class of functions, vanishing at infinity.

Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be such that $\psi = 1$ in a neighbourhood of zero. Then

$$(1.3) \quad \begin{cases} E = E_1 + E_2 & \text{where} \\ E_1 = \frac{1}{(2\pi)^{n/2}} \cdot \mathcal{F}^{-1}\left(\frac{\psi}{P}\right), \quad E_2 = \frac{1}{(2\pi)^{n/2}} \cdot \mathcal{F}^{-1}\left(\frac{1-\psi}{P}\right). \end{cases}$$

It is known (see: [2]) that

$$|Q^\beta D^\alpha E_2| = \frac{1}{(2\pi)^{n/2}} \left| \mathcal{F}^{-1}\left(D^\beta Q^\alpha \frac{1-\psi}{P}\right) \right|$$

where $(Q^\beta)x = x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$. The estimation at infinity

$$\left[D^\beta Q^\alpha \frac{1-\psi}{P} \right](\xi) = O\left(\frac{1}{|\xi|^{2m+|\beta|-|\alpha|}}\right)$$

implies that

$$D^\beta Q^\alpha \frac{1-\psi}{P} \in L^1(\mathbf{R}^n) \text{ for } |\beta| > n - 2m + |\alpha|,$$

hence $Q^\beta D^\alpha E_2$ is bounded in \mathbf{R}^n . Thus for any α, β at infinity

$$(1.4) \quad (Q^\beta D^\alpha E_2)(x) = O(1).$$

Consider the term E_1 in (1.3). We have the equality

$$(1.5) \quad |Q^\beta D^\alpha E_1| = \frac{1}{(2\pi)^{n/2}} |\mathcal{F}^{-1} \left[D^\beta \frac{Q^\alpha \psi}{P} \right]|.$$

$D^\beta \frac{Q^\alpha \psi}{P}$ is smooth in $\mathbf{R}^n \setminus \{0\}$ and it has compact support, moreover at zero

$$\left[D^\beta \frac{Q^\alpha \psi}{P} \right](\xi) = O\left(\frac{1}{|\xi|^{l-a+\beta}}\right).$$

This implies that

$$D^\beta \frac{Q^\alpha \psi}{P} \in L^1(\mathbf{R}^n) \text{ if } |\beta| < n - l + |\alpha|,$$

hence by (1.5)

$$(1.6) \quad Q^\beta D^\alpha E_1 = O(1)$$

at infinity.

In order to show (1.6) for $|\beta| = n - l + |\alpha|$ it is sufficient to prove that

$$\mathcal{F}^{-1} \left[D^\beta \frac{Q^\alpha \psi}{P_l} \right] = O(1)$$

holds at infinity since

$$D^\beta \frac{Q^\alpha \psi}{P} = D^\beta \frac{Q^\alpha \psi}{P_l} + D^\beta \frac{\psi Q^\alpha (P - P_l)}{PP_l}$$

and

$$D^\beta \frac{\psi Q^\alpha (P - P_l)}{PP_l} \in L^1(\mathbf{R}^n)$$

because at zero

$$\left[D^\beta \frac{\psi Q^\alpha (P - P_l)}{PP_l} \right](\xi) = O\left(\frac{1}{|\xi|^{l-1-a+\beta}}\right) = O\left(\frac{1}{|\xi|^{n-1}}\right).$$

By use of Gauss-Ostrogradski theorem we get that for any test function φ from the Schwartz space \mathcal{S} (see: [2])

$$(1.7) \quad \left[D^\beta \frac{Q^\alpha \psi}{P_l} \right](\varphi) = \lim_{\epsilon \rightarrow 0} \int_{R^n \setminus B_\epsilon} \left[D^\beta \frac{Q^\alpha \psi}{P_l} \right](\varphi) - c\varphi(0)$$

where c is a complex number. For the functions

$$(1.8) \quad g_k(\xi) = \begin{cases} \left[D^\beta \frac{Q^\alpha \psi}{P_l} \right](\xi) & \text{if } |\xi| \geq \frac{1}{k} \\ 0 & \text{if } |\xi| < \frac{1}{k} \end{cases}$$

it follows from equality (1.7) that the sequence (g_k) tends to the distribution $D^\beta \left[\frac{Q^\alpha \psi}{P_t} \right] c\delta$ in the weak sense of the space \mathcal{S}' of tempered distributions (see: [2]). Thus the sequence $\mathcal{F}^{-1}(g_k)$ tends to

$$\mathcal{F}^{-1}\left(D^\beta \left[\frac{Q^\alpha \psi}{P_t} \right]\right) c \mathcal{F}^{-1}(\delta)$$

in the weak sense of \mathcal{S}' . Since $\mathcal{F}^{-1}(\delta)$ is a bounded (constant) function, for the boundedness of $\mathcal{F}^{-1}\left(D^\beta \left[\frac{Q^\alpha \psi}{P_t} \right]\right)$ it is sufficient to prove that there exists a constant λ such that for any x

$$|\mathcal{F}^{-1} g_k(x)| \leq \lambda \text{ if } k \geq k_0(x).$$

(The sequence $(\mathcal{F}^{-1} g_k)$ converges a.e.).

By the definition (1.8)

$$(\mathcal{F}^{-1} g_k)(x) = \frac{1}{(2\pi)^{1/2}} \int_{|\xi| \geq \frac{1}{k}} e^{i\langle x, \xi \rangle} \left[D^\beta \left[\frac{Q^\alpha \psi}{P_t} \right](\xi) d\xi,$$

Thus it is sufficient to show that there exists a number $\lambda > 0$ such that for the functions

$$h_k(x) := \int_{|\xi| \geq \frac{1}{k}} e^{i\langle x, \xi \rangle} \psi(\xi) D^\beta \left[\frac{Q^\alpha}{P_t} \right](\xi) d\xi,$$

$$(1.9) \quad |h_k(x)| \leq \lambda \text{ if } k \geq k_0(x),$$

because from the first part of the proof it follows that the functions $\mathcal{F}^{-1} g_k = \frac{1}{(2\pi)^{n/2}} h_k$ are uniformly bounded.

Let function ψ have the special form: $\psi(\xi) = \psi_0(|\xi|)$ and suppose that $\psi_0 \geq 0$, $\psi_0(r) = 0$ if $r > a$. Then

$$(1.10) \quad h_k(x) = \int_1^a \left[\int_{S_1} \int e^{i\langle x, r\theta \rangle} \psi_0(r) D^\beta \left[\frac{Q^\alpha}{P_t} \right](r\theta) r^{n-1} d\theta \right] dr = \\ = \int_{S_1} g(\theta) \left[\int_1^a \frac{e^{i\langle x, r\theta \rangle}}{r} \psi_0(r) dr \right] d\theta,$$

where

$$g(\theta) := D^\beta \left[\frac{Q^\alpha}{P_t} \right](r\theta) r^n,$$

and $D^\beta \left(\frac{Q^\alpha}{P_l} \right) (r\theta) r^n$ does not depend on r since $D^\beta \left(\frac{Q^\alpha}{P_l} \right)$ is a homogeneous function of order $|\alpha| - |\beta| - l = -n$.

Equation (1.10) implies:

$$\begin{aligned} h_k(x) &= \int_{S_1} g(\theta) \left[\int_1^a \frac{e^{i\langle x, r\theta \rangle} - 1}{r} \psi_0(r) dr \right] d\theta + \\ &\quad + \left[\int_{S_1} g(\theta) d\theta \right] \left[\int_1^a \frac{1}{r} \psi_0(r) dr \right]. \end{aligned}$$

The first term on the right and h_k are convergent in \mathcal{S}' as $k \rightarrow \infty$ so the second term is also convergent in \mathcal{S}' which implies that

$$(1.11) \quad \int_{S_1} g(\theta) d\theta \rightarrow 0$$

holds.

Thus

$$\begin{aligned} h_k(x) &= \int_{S_1} g(\theta) \left[\int_1^a \frac{e^{i\langle x, r\theta \rangle} - 1}{r} \psi_0(r) dr \right] d\theta = \\ &= \int_{S_1} g(\theta) \left[\int_{\langle x, \theta \rangle}^{\pi - \langle x, \theta \rangle} \frac{e^{i\varrho} - 1}{\varrho} (\psi_0 - 1) \left(\frac{\varrho}{\langle x, \theta \rangle} \right) d\varrho \right] d\theta, \\ (1.12) \quad h_k(x) &= \int_{S_1} g(\theta) \left[\int_{\langle x, \theta \rangle}^{\pi - \langle x, \theta \rangle} \frac{e^{i\varrho} - 1}{\varrho} (\psi_0 - 1) \left(\frac{\varrho}{\langle x, \theta \rangle} \right) d\varrho \right] d\theta + \\ &\quad + \int_{S_1} g(\theta) \left[\int_{\langle x, \theta \rangle}^{\pi - \langle x, \theta \rangle} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta. \end{aligned}$$

The first term on the right in (1.12) is uniformly bounded, because there exists a number c_1 such that

$$(\psi_0 - 1) \left(\frac{\varrho}{\langle x, \theta \rangle} \right) = 0 \text{ if } \frac{\varrho}{\langle x, \theta \rangle} < c_1$$

and so by use of inequalities

$$\frac{1}{|\varrho|} \leq \frac{1}{c_1 |\langle x, \theta \rangle|}, \quad |(e^{i\varrho} - 1)(\varphi_0 - 1)| \leq c_2$$

we get:

$$(1.13) \quad \left| \int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} (\varphi_0 - 1) \left(\frac{\varrho}{\langle x, \theta \rangle} \right) d\varrho \right| \leq$$

$$\leq c_2 \left(a + \frac{1}{k} \right) |\langle x, \theta \rangle| \frac{1}{c_1 |\langle x, \theta \rangle|} \leq \frac{c_2}{c_1} a.$$

The second term on the right in (1.12) can be written in the form

$$(1.14) \quad \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta = \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{\text{sg} \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta +$$

$$+ \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta.$$

Since $g(\theta)$ is bounded and

$$\left| \int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{\text{sg} \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right| \leq \sup_{|\varrho| \leq 1} \left| \frac{e^{i\varrho} - 1}{\varrho} \right| \quad \text{if } k \geq |\langle x, \theta \rangle|,$$

we have an estimation of the form

$$(1.15) \quad \left| \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{\text{sg} \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta \right| \leq \lambda \quad \text{if } k \geq |x|.$$

Finally we have:

$$(1.16) \quad \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho \right] d\theta = \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{e^{i\varrho}}{\varrho} d\varrho \right] d\theta -$$

$$- \int_{S_1} g(\theta) \left[\int_{\frac{\text{sg} \langle x, \theta \rangle}{k}}^{\frac{a \langle x, \theta \rangle}{k}} \frac{1}{\varrho} d\varrho \right] d\theta.$$

Equality (1.11) implies that the second term on the right in (1.16) is bounded:

$$(1.17) \quad \begin{aligned} & \left| \int_{S_1} g(\theta) \left[\int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{1}{\varrho} d\varrho \right] d\theta \right| = \left| \int_{S_1} g(\theta) \ln \left| a|x| \left\langle \frac{x}{|x|}, \theta \right\rangle \right| d\theta \right| = \\ & = \left| \int_{S_1} g(\theta) \ln \left\langle \frac{x}{|x|}, \theta \right\rangle d\theta \right| \leq \sup |g| \cdot \int_{S_1} \left| -\ln \left\langle \frac{x}{|x|}, \theta \right\rangle \right| d\theta, \end{aligned}$$

where $\int_{S_1} \left| -\ln \left\langle \frac{x}{|x|}, \theta \right\rangle \right| d\theta$ is bounded.

Furthermore

$$(1.18) \quad \begin{aligned} \int_{S_1} g(\theta) \left[\int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{\varrho} d\varrho \right] d\theta &= \int_{\left\langle x, \theta \right\rangle \geq \frac{1}{a}} g(\theta) \left[\int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{\varrho} d\varrho \right] d\theta + \\ &+ \int_{\left\langle x, \theta \right\rangle < \frac{1}{a}} g(\theta) \left[\int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{\varrho} d\varrho \right] d\theta. \end{aligned}$$

The terms on the right in (1.18) can be estimated as follows: if $|\langle x, \theta \rangle| \geq \frac{1}{a}$, i.e. $|a(x, \theta)| \geq 1$ then

$$(1.19) \quad \int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{\varrho} d\varrho = \left[\frac{e^{ia(x, \theta)}}{i(x, \theta)} - \frac{e^{i\operatorname{sg}(x, \theta)}}{\operatorname{sg}(x, \theta)} \right] + \int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{i\varrho^2} d\varrho$$

which is uniformly bounded since $\int_1^\infty \frac{1}{\varrho^2} d\varrho < \infty$. Moreover

$$(1.20) \quad \begin{aligned} \int_{\left\langle x, \theta \right\rangle < \frac{1}{a}} g(\theta) \left[\int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho}}{\varrho} d\varrho \right] d\theta &= \int_{\left\langle x, \theta \right\rangle < \frac{1}{a}} g(\theta) \int_{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}}^{\frac{a(x, \theta)}{\operatorname{sg}(x, \theta)}} \frac{e^{i\varrho} - 1}{\varrho} d\varrho d\theta + \\ &+ \int_{\left\langle x, \theta \right\rangle < \frac{1}{a}} g(\theta) \left[\ln a + \ln \left| \left\langle \frac{x}{|x|}, \theta \right\rangle \right| + \ln |x| \right] d\theta, \end{aligned}$$

where the first term on the right and $g(\theta) \ln u$ are bounded,

$$\int_{\left\langle \frac{x}{|x|}, \theta \right\rangle} g(\theta) \ln \left\langle \frac{x}{|x|}, \theta \right\rangle d\theta$$

$$\left\langle \frac{x}{|x|}, \theta \right\rangle = \frac{1}{|x|}$$

is uniformly bounded, because

$$\int_{S_1} \left| \cdot \ln \left\langle \frac{x}{|x|}, \theta \right\rangle \right| d\theta$$

is finite and does not depend on x , finally

$$\ln |x| \cdot \int_{\left\langle \frac{x}{|x|}, \theta \right\rangle} g(\theta) d\theta = \ln(|x|) \cdot O\left(\frac{1}{|x|}\right)$$

is uniformly bounded, too.

Therefore (1.12) - (1.18) imply the inequality (1.9) which completes the proof of lemma 1.

REMARK. The example $P(D) = \mathbb{I}^n$ shows that estimation (1.1) can not be generally improved and generally it is not valid for $n < l$.

The following lemma can be proved similarly.

LEMMA 2. Let $P(D)$ be an elliptic differential operator of order $2m$ with constant coefficients. Then for any tempered fundamental solution E of $P(D)$ the generalized derivate $D^\beta E$ of E are locally integrable functions in \mathbf{R}^n if $|\beta| \leq 2m-1$ and at zero the estimation

$$(1.21) \quad D^\beta E(x) = O\left(\frac{1}{|x|^{n+2m-|\beta|}}\right)$$

holds.

REMARK. It is well known (see e.g. [2]) that on $\mathbf{R}^n \setminus \{0\}$ E is a smooth function in classical sense.

2. Equations with constant coefficients

LEMMA 3. Suppose that the operator $P(D)$ satisfies the conditions of lemma 1, and $f \in L_a^2(\mathbf{R}^n)$, i.e. $f \in L^2(\mathbf{R}^n)$ and $f(x) = 0$ a.e. if $|x| > a$. Then the equation

$$(2.1) \quad P(D)u = f \quad \text{in } \mathbf{R}^n$$

has a unique solution u , tending to zero at infinity. (Moreover, at infinity $u(x) = O\left(\frac{1}{|x|^{n-l}}\right)$). This solution u can be given by $u = f * E$, where E is the

fundamental solution in lemma 1, and for any compact $K \subset \mathbf{R}^n$ the inequality

$$(2.2) \quad \|u\|_{H^{2m}(K)} = \left\| \sum_{|\alpha| \leq 2m} \int_K |D^\alpha u|^2 \right\|^{1/2} \leq c_1(K) \|f\|_{L^2_a(\mathbf{R}^n)}$$

holds.

PROOF. We have to prove only estimation (2.2) since the other statements of lemma 3, follow from lemma 1.

The formula

$$u(x) = \int_{B_a} f(y) E(x-y) dy$$

implies:

$$\|u(x)\|^2 \leq \int_{B_a} |f(y)|^2 |E(x-y)| dy \cdot \int_{B_a} |E(x-y)| dy,$$

In virtue of lemma 2, E is locally integrable in \mathbf{R}^n thus

$$\int_K |u(x)|^2 dx \leq c_2 \int_{B_a} \left[|f(y)|^2 \int_K |E(x-y)| dx \right] dy \leq c_3 \int_{B_a} |f(y)|^2 dy,$$

i.e.

$$(2.3) \quad \|u\|_{L^2(K)} \leq c_3 \|f\|_{L^2_a(\mathbf{R}^n)}$$

(c_2, c_3 are constants, c_3 depends on K).

Furthermore, from lemma 2, we get the formula

$$D^\beta u(x) = \int_{B_a} f(y) D^\beta E(x-y) dy \text{ for } |\beta| \leq 2m,$$

which implies that for $|\beta| \leq 2m$ and any $b > 0$

$$(2.4) \quad \|D^\beta u\|_{L^2(B_b)} \leq c(b) \|f\|_{L^2_a(\mathbf{R}^n)}.$$

Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be equal to 1 in a neighbourhood of K , $\text{supp } \varphi \subset B_b$. Then by the well known estimates for elliptic operators (see: [3]) inequality (2.3) implies:

$$\begin{aligned} \|u\|_{H^{2m}(K)} &\geq \|\varphi u\|_{H^{2m}(B_b)} \geq c_1 [\|P(D)(\varphi u)\|_{L^2(B_b)} + \|\varphi u\|_{L^2(B_b)}] \geq \\ &\geq c_2 [\|f\|_{L^2_a(\mathbf{R}^n)} + \|u\|_{H^{2m-1}(B_b)}]. \end{aligned}$$

From this estimation by use of (2.4) we get inequality (2.2).

For $n > 2l$ a better estimation can be proved for the solution of equation (2.1).

LEMMA 3'. Suppose that the operator $P(D)$ satisfies the conditions of lemma 1., $n > 2l$ and $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then equation (2.1) has a unique solution $u \in H^{2m}(\mathbf{R}^n)$ (the solution is unique in $L^2(\mathbf{R}^n)$, too) and the following estimation is valid:

$$(2.5) \quad \|u\|_{H^{2m}(\mathbf{R}^n)} \leq c [\|f\|_{L^1(\mathbf{R}^n)} + \|f\|_{L^2(\mathbf{R}^n)}].$$

PROOF. By lemma 3, we have only to prove that for the solution

$$u = f * E = \mathcal{F}^{-1} \left(\frac{1}{P} (\mathcal{F}f) \right)$$

of (2.1), $u \in H^{2m}(\mathbf{R}^n)$ and estimation (2.5) holds. Therefore we have to estimate the $L^2(\mathbf{R}^n)$ norm of the function

$$|\xi| \mapsto \frac{(1 + |\xi|^{4m})^{1/2}}{P(\xi)} \cdot (\mathcal{F}f)(\xi).$$

Assumptions on $P(D)$ imply that

$$(2.6) \quad \frac{1 + |\xi|^{4m}}{|P(\xi)|^2} \cdot |(\mathcal{F}f)(\xi)|^2 \leq c_1 |(\mathcal{F}f)(\xi)|^2 \text{ if } |\xi| \geq 1$$

and

$$(2.7) \quad \frac{1 + |\xi|^{4m}}{|P(\xi)|^2} \cdot |(\mathcal{F}f)(\xi)|^2 \leq \frac{c_2}{|\xi|^{2l}} \cdot |(\mathcal{F}f)(\xi)|^2 \leq \frac{c_2}{|\xi|^{2l}} \cdot \|\sup |\mathcal{F}f|^2 \leq \\ \leq \frac{c_3}{|\xi|^{2l}} \|f\|_{L^1(\mathbf{R}^n)}^2 \text{ if } |\xi| \leq 1.$$

Since $2l < n$, estimation (2.5) follows from inequalities (2.6), (2.7).

Let $P(D)$ be a homogeneous elliptic operator and consider the following Dirichlet problem in the sphere B_ϱ :

$$(2.8) \quad P(D)u_\varrho = f \text{ in } B_\varrho$$

$$(2.9) \quad \partial_r^j u_\varrho|_{S_\varrho} = 0 \text{ if } j = 0, 1, \dots, m-1,$$

where $f \in L^2_a(\mathbf{R}^n)$ and r denotes the normal to S_ϱ .

THEOREM 1. Suppose that the homogeneous operator $P(D)$ satisfies the conditions of lemma 1, and the problem (2.8), (2.9) has a unique solution $u_1 \in H^{2m}(B_1)$ for $\varrho = 1$ and arbitrary $f \in L^2_1(\mathbf{R}^n)$. Then problem (2.8), (2.9) has a unique solution $u_\varrho \in H^{2m}(B_\varrho)$ for all $\varrho > a$ and $f \in L^2_a(\mathbf{R}^n)$, moreover the difference of u_ϱ and the solution u of (2.1) (vanishing at infinity) can be estimated as follows:

$$(2.10) \quad \sup_{B_\varrho} |D^\beta u_\varrho - D^\beta u| \leq k_1 \|f\|_{L^2_a(\mathbf{R}^n)} \cdot \frac{1}{\varrho^{n+2m+\beta}},$$

$$(2.11) \quad \|D^\beta u_\varrho - D^\beta u\|_{L^2(B_\varrho)} \leq k_2 \|f\|_{L^2_a(\mathbf{R}^n)} \cdot \frac{1}{\varrho^{n/2+2m+\beta}}.$$

PROOF. The function $u_\varrho \in H^{2m}(B_\varrho)$ is a solution of problem (2.8), (2.9) if and only if function \tilde{u}_ϱ , defined by $\tilde{u}_\varrho(y) = u_\varrho(y\varrho)$ satisfies the problem

$$(2.12) \quad P(D)\tilde{u}_\varrho = \varrho^{2m} \tilde{f}_\varrho \text{ in } B_1$$

$$(2.13) \quad \partial_r^j \tilde{u}_\varrho|_{S_1} = 0 \text{ if } j = 0, 1, \dots, m-1$$

where $\tilde{v}_\varrho(y) = f(y\varrho)$. Thus problem (2.8), (2.9) has a unique solution $u_\varrho \in H^{2m}(B_\varrho)$ for all $f \in L_a^2(\mathbf{R}^n)$, $\varrho > a$.

The difference $v_\varrho = u - u_\varrho \in H^{2m}(B_\varrho)$ satisfies the following problem:

$$(2.14) \quad P(D)v_\varrho = 0 \quad \text{in } B_\varrho$$

$$(2.15) \quad \partial_r^j v_\varrho|_{S_\varrho} = \partial_r^j u|_{S_\varrho} = q_{\varrho,j} \quad \text{if } j = 0, 1, \dots, m-1,$$

furthermore for $\tilde{v}_\varrho(y) = v_\varrho(y\varrho)$, $\tilde{v}_\varrho \in H^{2m}(B_1)$ and

$$(2.16) \quad P(D)\tilde{v}_\varrho = 0 \quad \text{in } B_1,$$

$$(2.17) \quad \partial_r^j \tilde{v}_\varrho|_{S_1} = \tilde{q}_{\varrho,j}, \quad j = 0, 1, \dots, m-1$$

where

$$(2.18) \quad \tilde{q}_{\varrho,j}(y) = \varrho^j q_{\varrho,j}(\varrho y).$$

Since $f(x) = 0$ a.e. if $|x| > a$, by use of lemma 1, we get for any fixed β the estimations:

$$(2.19) \quad |D^\beta q_{\varrho,j}(x)| \leq c_1 \|f\|_{L^1_a(\mathbf{R}^n)} \cdot \frac{1}{\varrho^{n-2m+j+\beta}},$$

$$|D^\beta \tilde{q}_{\varrho,j}(y)| = \varrho^{\beta+j} |D^\beta q_{\varrho,j}(\varrho y)| \leq c_1 \|f\|_{L^1_a(\mathbf{R}^n)} \frac{1}{\varrho^{n-2m}}.$$

The problem (2.8), (2.9) has a unique solution for all $f \in L_a^2(\mathbf{R}^n)$, so (2.16), (2.17), (2.19) imply that

$$(2.20) \quad \|\tilde{v}_\varrho\|_{H^{2m}(B_1)} \leq c_2 \sum_{j=0}^{m-1} \|\tilde{q}_{\varrho,j}\|_{H^{2m-j-1/2}(S_1)} \leq c_3 \|f\|_{L^1_a(\mathbf{R}^n)} \frac{1}{\varrho^{n-2m}}.$$

As

$$\|D^\beta v_\varrho\|_{L^2(B_\varrho)} = \|D^\beta \tilde{v}_\varrho\|_{L^2(B_1)} \cdot \varrho^{n/2-\beta},$$

thus from (2.20) we get the estimation (2.11).

The inequality (2.10) follows from (2.19) and the Schauder estimates for the problem (2.16), (2.17) (see e.g. [3]):

$$\|\tilde{v}_\varrho\|_{C^{2m+\alpha}(B_1)} \leq c'_2 \sum_{j=1}^{m-1} \|\tilde{q}_{\varrho,j}\|_{C^{2m-j+\alpha}(S_1)}$$

($\alpha \in \mathbf{R}$, $0 < \alpha < 1$).

REMARK. As a consequence of theorem 1, we have

$$(2.21) \quad \lim_{\varrho \rightarrow +\infty} \|u - u_\varrho\|_{H^{2m}(B_\varrho)} = 0 \quad \text{for } n > 4m$$

and for any β

$$(2.22) \quad \lim_{\varrho \rightarrow +\infty} \sup_{B_\varrho} |D^\beta u_\varrho - D^\beta u| = 0 \quad \text{for } n > 2m.$$

The example

$$P(D) = A, \quad f(x) = \begin{cases} 1, & \text{if } x \in B_1 \\ 0, & \text{if } x \in \mathbf{R}^n \setminus B_1 \end{cases}$$

shows that in the case $n = 2m$ the equality (2.22) is not true generally because then for all fixed x

$$\lim_{q \rightarrow \pm \infty} |u_q(x) - u(x)| = \pm \infty.$$

Furthermore, if $n = 3, 4$ then this example shows that in the case $n = 4m$ the equality (2.21) is not valid generally, because for this example

$$\lim_{q \rightarrow \pm \infty} \|H - H_q\|_{H^m(B_q)} = +\infty.$$

If the operator $P(D)$ is not homogeneous then the following theorem can be proved instead of theorem 1.

THEOREM 1'. Suppose that the operator $P(D)$ satisfies the conditions of lemma 1, and the polynomial P has the form

$$P(\xi) = \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta} \xi^{\alpha+\beta}$$

where $a_{\alpha\beta} \in \mathbf{R}$, and for any complex vector (z_0, \dots, z_m, \dots) $\neq 0$ the inequalities

$$(2.23) \quad \sum_{|\alpha|, |\beta|=m} a_{\alpha\beta} z_\alpha \bar{z}_\beta > 0, \quad \sum_{|\alpha|, |\beta| < m} a_{\alpha\beta} z_\alpha \bar{z}_\beta \leq 0$$

hold.

Then for arbitrary $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, $\varrho > 0$ problem (2.8), (2.9) has a unique solution $u_\varrho \in H^{2m}(B_\varrho)$ and for every fixed compact $K \subset \mathbf{R}^n$ and β

$$(2.24) \quad \sup_K |D^\beta u_\varrho - D^\beta \tilde{u}| \leq \frac{c(K)}{\varrho^{n/2-2m-1+\beta}} [\|f\|_{L^1(\mathbf{R}^n)} + \|f\|_{L^2(\mathbf{R}^n)}].$$

PROOF. A function $u_\varrho \in H^{2m}(B_\varrho)$ is the solution of (2.8), (2.9) if and only if for $\tilde{u}_\varrho(y) = u_\varrho(y\varrho)$

$$(2.25) \quad \sum_{|\alpha|, |\beta| \leq m} \frac{a_{\alpha\beta}}{\varrho^{|\alpha|+|\beta|}} D^{\alpha+\beta} \tilde{u}_\varrho = \tilde{f}_\varrho \text{ in } B_1$$

$$(2.26) \quad \partial_r^j \tilde{u}_\varrho|_{S_1} = 0, \quad j = 0, 1, \dots, m-1$$

holds, where $\tilde{f}_\varrho(y) = f(y\varrho)$. Conditions (2.23) imply the uniqueness of the solution of (2.25), (2.26), because multiplying equation (2.25) by \tilde{u}_ϱ we get the estimation

$$\begin{aligned} \|\tilde{f}_\varrho\|_{L^2(B_1)} \|u_\varrho\|_{L^2(B_1)} &\leq \int_{B_1} \tilde{f}_\varrho \tilde{u}_\varrho = \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta} \int_{B_1} D^\alpha \tilde{u}_\varrho \left(\frac{D^\beta \tilde{u}_\varrho}{\varrho^{|\beta|}} \right) \\ &\leq \frac{1}{\varrho^{2m}} \int_{B_1} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta} (D^\alpha \tilde{u}_\varrho) (\bar{D}^\beta \tilde{u}_\varrho) \leq \frac{C_1}{\varrho^{2m}} \int_{B_1} \sum_{|\alpha| \leq m} \|D^\alpha \tilde{u}_\varrho\|^2 \leq \\ &\leq \frac{C_2}{\varrho^{2m}} \|\tilde{u}_\varrho\|_{H_0^{2m}(B_1)}^2 \end{aligned}$$

(from (2.13) it follows that $\tilde{u}_\varrho \in H_0^{2m}(B_1)$).

Therefore

$$(2.27) \quad \|\tilde{u}_\varrho\|_{L^2(B_1)} \leq c_3 \varrho^{2m} \|\tilde{f}_\varrho\|_{L^2(B_1)}.$$

Thus for any $\tilde{f}_\varrho \in L^2(B_1)$ there exists a (unique) $\hat{u}_\varrho \in H_0^m(B_1)$ such that

$$\sum_{\alpha + \beta = m} \frac{a_{\alpha\beta}}{\varrho^{|\alpha| + |\beta|}} \int_{B_1} (D^\alpha \hat{u}_\varrho)(D^\beta v) = \int_{B_1} \tilde{f}_\varrho v$$

if $v \in H_0^m(B_1)$ and so we get the existence of the solution $u_\varrho \in H^m(B_1)$ of (2.25), (2.26), too.

Equation (2.25) can be written in the form

$$(2.28) \quad \sum_{\alpha + \beta = m} a_{\alpha\beta} \varrho^{2m - |\alpha| - |\beta|} D^{\alpha+\beta} \hat{u}_\varrho + \varrho^{2m} \hat{u}_\varrho = \varrho^{2m} \tilde{f}_\varrho + \varrho^{2m} \tilde{u}_\varrho.$$

For the characteristic polynomial of the differential operator in (2.28)

$$P(\xi) + \varrho^{2m} > 0 \text{ if } \xi \in \mathbf{R}^n, \varrho \geq 0, (\xi, \varrho) \neq 0$$

hence in virtue of results of [4] on elliptic differential operators with parameter we get the estimation

$$\left\{ \sum_{k=0}^{2m} \varrho^{2k} \|\tilde{u}_\varrho\|_{H^{2m-k}(B_1)}^2 \right\}^{1/2} \leq c_4 \varrho^{2m} [\|\tilde{f}_\varrho\|_{L^2(B_1)} + \|\hat{u}_\varrho\|_{L^2(B_1)}].$$

Thus inequality (2.27) implies that

$$\|\tilde{u}_\varrho\|_{H^{2m-k}(B_1)} \leq c_5 \varrho^{4m-k} \|\tilde{f}_\varrho\|_{L^2(B_1)}$$

and so

$$(2.29) \quad \begin{aligned} \|D^\gamma \tilde{u}_\varrho\|_{L^2(S_\varrho)} &\leq c_6 \varrho^{2m+|\gamma|+1} \|\tilde{f}_\varrho\|_{L^2(B_1)} \text{ if } |\gamma| \leq 2m-1, \\ \|D^\gamma u_\varrho\|_{L^2(S_\varrho)} &\leq c_6 \varrho^{2m+1+|\gamma|} \|f\|_{L^2(B_1)}. \end{aligned}$$

Applying Green's formula, we get for $x_0 \in B_\varrho$ an estimation of the form

$$(2.30) \quad \begin{aligned} |v_\varrho(x_0)| &= \sum_{1 \leq |\gamma| + |\delta| \leq 2m} \int_{S_\varrho} g_{\gamma\delta} \left(-\frac{x}{|x|} \right) (D^\gamma v_\varrho)(x) (D^\delta E)(x_0 - x) d\sigma_x \leq \\ &\leq \sum_{1 \leq |\gamma| + |\delta| \leq 2m} \int_{S_\varrho} g_{\gamma\delta} \left(-\frac{x}{|x|} \right) (D^\gamma u_\varrho)(x) (D^\delta E)(x_0 - x) d\sigma_x + \\ &+ \sum_{1 \leq |\gamma| + |\delta| \leq 2m} \int_{S_\varrho} g_{\gamma\delta} \left(-\frac{x}{|x|} \right) (D^\gamma u_\varrho)(x) (D^\delta E)(x_0 - x) d\sigma_x \leq \\ &\leq c_7 \sum_{1 \leq |\gamma| + |\delta| \leq 2m} [\|D^\gamma u_\varrho\|_{L^2(S_\varrho)} + \|D^\gamma u_\varrho\|_{L^2(S_\varrho)}] \| (D^\delta E)(x_0 - x) \|_{L^2(S_\varrho)} \end{aligned}$$

(c_7 does not depend on x_0), since the coefficients $g_{\gamma\delta}$ are bounded functions.

If $x_0 \in K$ then by lemma 1,

$$(D^\delta E)(x_0 - x) = O\left(-\frac{1}{|x|^{n+1+\frac{1}{\alpha}+\delta}}\right),$$

thus

$$(2.31) \quad \|(D^\delta E)(x_0 - x)\|_{L^2(S_\varrho)} = O\left(\frac{1}{\varrho^{\frac{n+1}{2} + \frac{1}{\alpha} + \delta}}\right).$$

Moreover, for $|\gamma| \leq 2m-1$, $\hat{u}(y) = u(y)$

$$\begin{aligned} \|D^\gamma u\|_{L^2(S_\varrho)} &= \varrho^{-\frac{n-1-\gamma}{2}} \|D^\gamma \tilde{u}\|_{L^2(S_1)} \leq c_s \varrho^{\frac{n-1-\gamma}{2}} \|\tilde{u}\|_{H^{\gamma+1}(B_1)} = \\ &= c_s \varrho^{1/2} \|u\|_{H^{\gamma+1}(B_1)} \leq c_s \varrho^{1/2} \|u\|_{H^{2m}(\mathbb{R}^n)} \end{aligned}$$

and so lemma 3' implies that

$$(2.32) \quad \|D^\gamma u\|_{L^2(S_\varrho)} \leq c_0 \varrho^{1/2} [\|f\|_{L^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}].$$

From estimations (2.29) -- (2.32) we get the inequality (2.24) in the case $|\beta| = 0$. Similarly (2.24) can be proved for arbitrary multiindex β .

3. Equation with variable coefficients

In this paragraph an equation of the form

$$(3.1) \quad P(D)u + \lambda Q(x, D)u = f$$

will be considered, where λ is a complex number, $Q = Q(x, D)$ is a linear differential operator of order $2m$ with infinitely differentiable coefficients, vanishing for $|x| \geq a$ and $f \in L_a^2(\mathbb{R}^n)$.

Denote by \mathcal{A} the set of all $\lambda \in \mathbb{C}$ such that the operator $P(D) + \lambda Q(x, D)$ is elliptic. Obviously \mathcal{A} is an open set. Denote by \mathcal{A}_0 the set of all connected components of \mathcal{A} which contain a λ such that for all $f \in L_a^2(\mathbb{R}^n)$ the equation (3.1) has a solution u , vanishing at infinity. (\mathcal{A}_0 contains that component of \mathcal{A} which contains 0. If the order of $Q(x, D)$ is less than $2m$ then $\mathcal{A}_0 = \mathbb{C}$.)

Denote by $P^{-1}w$ the unique solution of the equation $P(D)u = w$, which vanishes at infinity (see lemma 3.), i.e. $u = E * w$, where E is the fundamental solution in lemma 1.

Using the method of [5] and [1] we get the following lemma.

LEMMA 4. Suppose that $P = P(D)$ satisfies the conditions of lemma 1, and $Q = Q(x, D)$ satisfies the above conditions. Then for any λ a function u is the solution of (3.1), vanishing at infinity, if and only if $w = P_u$ is the solution of the equation

$$(3.2) \quad w + \lambda Q P^{-1}w = f$$

in $L_a^2(\mathbb{R}^n)$.

For $\lambda \in \Lambda_0$ the index of equation (3.2) equals 0. If $\lambda \in \Lambda_0 \setminus \Lambda_1$ then for arbitrary $f \in L_a^2(\mathbf{R}^n)$ there exists a unique solution u of (3.1), vanishing at infinity, where Λ_1 is a countable set and the following estimates are valid: for any compact $K \subset \mathbf{R}^n$

$$(3.3) \quad \|u\|_{H^{2m}(K)} \leq c_2(K) \|f\|_{L_a^2(\mathbf{R}^n)}$$

and in the case $n > 2l$

$$(3.4) \quad \|u\|_{H^{2m}(\mathbf{R}^n)} \leq c_2 \|f\|_{L_a^2(\mathbf{R}^n)}.$$

Finally consider the following Dirichlet problem in B_ϱ :

$$(3.5) \quad P(D) u_\varrho + \lambda Q(x, D) u_\varrho = f \text{ in } B_\varrho,$$

$$(3.6) \quad \partial_r^j u_\varrho|_{S_\varrho} = 0 \text{ if } j = 0, 1, \dots, m-1.$$

THEOREM 2. Suppose that conditions of theorem 1, and lemma 4, are fulfilled and $\lambda \in \Lambda_0 \setminus \Lambda_1$. Then there exists $\varrho_0 > 0$ such that for all $\varrho \geq \varrho_0$ and $f \in L_a^2(\mathbf{R}^n)$ problem (3.5), (3.6) has a unique solution $u_\varrho \in H^{2m}(B_\varrho)$ and for any compact $K \subset \mathbf{R}^n$

$$(3.7) \quad \|u_\varrho - u\|_{H^{2m}(K)} \leq \frac{c_1(K)}{\varrho^{n-2m}} \|f\|_{L_a^2(\mathbf{R}^n)},$$

moreover, if $n > 4m$ then

$$(3.8) \quad \|u_\varrho - u\|_{H^{2m}(B_\varrho)} \leq \frac{c_2}{\varrho^{n/2 - 2m}} \|f\|_{L_a^2(\mathbf{R}^n)}.$$

PROOF. In virtue of theorem 1, for any $\varrho > a$ and $f \in L_a^2(\mathbf{R}^n)$ the problem

$$(3.9) \quad P(D) v_\varrho = f \text{ in } B_\varrho$$

$$(3.10) \quad \partial_r^j v_\varrho|_{S_\varrho} = 0, \quad j = 0, 1, \dots, m-1$$

has a unique solution $v_\varrho \in H^{2m}(B_\varrho)$. Denote the solution of (3.9), (3.10) by $P_\varrho^{-1} f$. Then $P_\varrho^{-1} : L_a^2(\mathbf{R}^n) \rightarrow H^{2m}(B_\varrho)$ is a bounded linear operator and from estimation (2.10) it follows that for any compact $K \subset \mathbf{R}^n$

$$(3.11) \quad \|(P_\varrho^{-1} - P^{-1})f\|_{H^{2m}(K)} \leq \frac{c_3(K)}{\varrho^{n-2m}} \|f\|_{L_a^2(\mathbf{R}^n)}.$$

Consequently, the difference of the operators

$$F = I + \lambda Q P^{-1}, \quad F_\varrho = I + \lambda Q P_\varrho^{-1},$$

mapping $L_a^2(\mathbf{R}^n)$ into itself (I denotes the identity operator in $L_a^2(\mathbf{R}^n)$), can be estimated as follows:

$$(3.12) \quad \|F_\varrho - F\| \leq \|\lambda Q(P_\varrho^{-1} - P^{-1})\| \leq \frac{c_4}{\varrho^{n-2m}}$$

(λ is a fixed number).

From lemma 4, it follows that the inverse of F exists and $F^{-1} : L_a^2(\mathbf{R}^n) \rightarrow L_a^2(\mathbf{R}^n)$ is a bounded linear operator. Therefore estimation (3.12) implies

that for sufficiently large ϱ the inverse F_ϱ^{-1} exists, too and $F_\varrho^{-1} : L_a^2(\mathbf{R}^n) \rightarrow L_a^2(\mathbf{R}^n)$ is a bounded linear operator. Since the equality $u_\varrho = P_\varrho^{-1} w_\varrho$ defines a one-to-one mapping between the solutions $u_\varrho \in H^m(B_\varrho)$ of (3.5), (3.6) and the solutions $w_\varrho \in L_a^2(\mathbf{R}^n)$ of equation $F_\varrho w_\varrho = f$, so for sufficiently large ϱ and arbitrary $f \in L_a^2(\mathbf{R}^n)$ problem (3.5), (3.6) has a unique solution.

The difference of the solutions $u_\varrho = P_\varrho^{-1} F_\varrho^{-1} f$ and $u = P^{-1} F^{-1} f$ can be estimated as follows. By (3.12) the number $\varrho_0 > 0$ can be chosen such that for $\varrho > \varrho_0$

$$\|F_\varrho + F\| \leq \frac{1}{2\|F^{-1}\|}, \quad \text{and so } \|F^{-1}(F_\varrho - F)\| \leq \frac{1}{2}.$$

Hence

$$\|F_\varrho^{-1}\| = \|(F(I + F^{-1}(F_\varrho - F))^{-1}\| = \|(I + F^{-1}(F_\varrho - F))^{-1} F^{-1}\| \leq 2\|F^{-1}\|$$

and

$$\|F_\varrho^{-1} - F^{-1}\| = \|F_\varrho^{-1}(F - F_\varrho)\| \leq 2\|F^{-1}\|^2 \cdot \|F_\varrho - F\|.$$

This estimation and (3.12) imply that

$$\|F_\varrho^{-1} - F^{-1}\| \leq \frac{c_5}{\varrho^{n-2m}}.$$

Therefore by (3.11) we get (3.7):

$$\begin{aligned} \|u_\varrho - u\|_{H^m(K)} &= \|(P_\varrho^{-1} + P^{-1}) F_\varrho^{-1} + P^{-1}(F_\varrho^{-1} - F^{-1})\| f \|_{H^m(K)} \leq \\ &\leq \frac{c_3(K)}{\varrho^{n-2m}} \|F_\varrho^{-1} f\|_{L_a^2(\mathbf{R}^n)} + c_6 \|F_\varrho^{-1} - F^{-1}\| f \|_{L_a^2(\mathbf{R}^n)} \leq \frac{c_7}{\varrho^{n-2m}} \|f\|_{L_a^2(\mathbf{R}^n)}. \end{aligned}$$

Inequality (3.8) can be proved similarly.

Finally by use of theorem 1', we get the following

THEOREM 2'. Suppose that conditions of theorem 1', and lemma 4, are fulfilled, $\lambda \in \lambda_0 \setminus \lambda_1$. Then there exists $\varrho_0 > 0$ such that for all $\varrho \geq \varrho_0$ and $f \in L_a^2(\mathbf{R}^n)$ problem (3.5), (3.6) has a unique solution $u_\varrho \in H^m(B_\varrho)$ and for any compact $K \subset \mathbf{R}^n$

$$\|u_\varrho - u\|_{H^m(K)} \leq \frac{c_3'(K)}{\varrho^{n-2-2m}} \|f\|_{L_a^2(\mathbf{R}^n)}.$$

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ON A GENERALIZATION OF q -ADDITION FUNCTIONS

By

J. FEHÉR

Teacher's Training College, Pécs

(Received December 22, 1982)

1. Let $q \geq 2$ be a fixed integer. Then every nonnegative integer n can be represented uniquely as

$$(1.1) \quad n = \sum_{i=0}^s a_i q^i, \quad a_i \in \{0, 1, \dots, q-1\}, \quad a_s \neq 0.$$

We shall say that a complex valued function f defined on the set of nonnegative integers is q -additive if $f(0) = 0$ and

$$(1.2) \quad f(n) := \sum_{i=0}^s f(a_i q^i).$$

This notion has been introduced by A. O. GELFOND [1].

As it is well known an arithmetic function g (defined on the set of natural numbers) is additive if $g(mn) = g(m) + g(n)$ for every coprime pair of integers (m, n) . Let $L := \{\log n, n = 1, 2, \dots\}$, $\lambda_p = \log p$, where p denotes a general prime. Let f be defined by $f(\log n) = g(n)$. From the unique prime factorization theorem we get that each element $\log n$ of L can be given uniquely as

$$(1.3) \quad \log n = \sum_p z(p) \cdot \lambda_p, \quad z(p) \in \{0, 1, 2, \dots\},$$

where $z(p) = 0$ with the exception of finite many primes p .

Furthermore, for additive $g(n)$ we have

$$(1.4) \quad f(\log n) = \sum_p f(z(p) \cdot \lambda_p).$$

Let us begiven an arbitrary commutative monoid R ; and let R_1, R_2, \dots be a sequence of subsets of R , all of them containing the zero element. We shall say that $\{R_1, R_2, \dots\}$ is a finite direct decomposition of R , if each element r can be represented uniquely in the form

$$(1.5) \quad r = r_{I_1} + r_{I_2} + \dots + r_{I_s}; \quad r_{I_k} \in R_{I_k} \quad (k = 1, \dots, s).$$

We shall say that a complex-valued function f is additive – with respect to \mathcal{R} and its finite direct decomposition $\{\mathcal{R}_1, \mathcal{R}_2, \dots\}$ –, if $f(0) = 0$ and

$$(1.6) \quad f(r) = \sum_{k=1}^s f(\varrho_{i_k}).$$

It is obvious that this is a straightforward generalization of the additivity and q -additivity. Introducing some topology in \mathcal{R} we may raise some questions about the characterization of additive functions. Let us assume that \mathcal{R} is a subset of the real field. It is obvious that $f(r) = cr$ (c being a real constant) is an additive function for each finite direct decomposition. Under what conditions stated for f can we assert that cr are the unique additive functions?

Now we restrict ourselves to the case when $\mathcal{R} = \mathfrak{N}_0$ is the set of non-negative integers.

Let \mathfrak{N} be the set of natural numbers. For any subset $s \subset \mathfrak{N}$ let $s^\perp = s \cup \{0\}$, and $|s|$ be the cardinality of s .

DEFINITION 1. Let $\mathcal{R}_i \subset \mathfrak{N}$ ($i = 0, 1, \dots$). We shall say that $\{\mathcal{R}_0, \mathcal{R}_1, \dots\}$ is a finite direct decomposition of \mathfrak{N}_0 if every $n \in \mathfrak{N}_0$ can be written uniquely as

$$(1.7) \quad n = r_{i_1} + r_{i_2} + \dots + r_{i_s}, \quad r_{i_k} \in \mathcal{R}_{i_k} \quad (k = 1, \dots, s).$$

DEFINITION 2. We shall say that a finite direct decomposition $\{\mathcal{R}_0, \mathcal{R}_1, \dots\}$ of \mathfrak{N}_0 is an \mathcal{R} -system if:

- a) \mathcal{R}_i is finite and nonempty: $|\mathcal{R}_i| = k_i - 1$ ($i = 0, 1, \dots$),
- b) the smallest element of \mathcal{R}_i is smaller than the smallest element of \mathcal{R}_j for every $i < j$.

DEFINITION 3. We shall say that an \mathcal{R} -system is monotonic if the smallest element of \mathcal{R}_i is larger than the greatest element of \mathcal{R}_j for every $i < j$.

It is easy to see that if the \mathcal{R} -system is monotonic and $|\mathcal{R}_i| = q-1$ ($i = 0, 1, \dots$), then $\mathcal{R}_i = \{1, q^i, \dots, (q-1)q^i\}$, and we get the q -ary decomposition.

DEFINITION 4. We shall say that a complex-valued function $f(n)$ defined on \mathfrak{N}_0 is additive with respect to the finite direct decomposition $\{\mathcal{R}_0, \mathcal{R}_1, \dots\}$ of \mathfrak{N}_0 if $f(0) = 0$ and

$$f(n) = f(r_{i_1}) + \dots + f(r_{i_s}).$$

We say that an m written in the form

$$m = t_{j_1} + t_{j_2} + \dots + t_{j_r}, \quad t_{j_k} \in \mathcal{R}_{j_k} \quad (k = 1, \dots, r)$$

and n given by (1.7) are \mathcal{R} -disjoint if the set of indices $\{i_1, i_2, \dots, i_s\}, \{j_1, \dots, j_r\}$ are disjoint.

It is obvious that if f is \mathcal{R} -additive and m and n are \mathcal{R} -disjoint then $f(m+n) = f(m)+f(n)$.

For the sake of brevity we shall use the following notations.

1. $\mathcal{R} = \bigcup_{i=0}^{\infty} \mathcal{R}_i$.
2. $\mathcal{A}_i = \{\lambda_0 + \dots + \lambda_{i-1} \mid \lambda_j \in \mathcal{R}_j\} \setminus \{0\}$
3. $\mathcal{O}_i = \{\lambda_i + \dots + \lambda_h \mid h = i+1, i+2, \dots; \lambda_j \in \mathcal{R}_j\} \setminus \{0\}$.

That is \mathcal{A}_i is the set of positive integers that have no components from \mathcal{R}_j for $j \geq i$, and \mathcal{O}_i is the set of positive integers that have no components from \mathcal{R}_j for $j \leq i-1$.

It is obvious that for $m \in \mathcal{A}_i, n \in \mathcal{O}_i$ the elements m and n are \mathcal{R} -disjoint.

2. THEOREM 1. *Given an arbitrary \mathcal{R} -system, let d be an arbitrary positive integer, f and g be additive functions concerning this \mathcal{R} -system, and*

$$(2.1) \quad \lim_{n \rightarrow \infty} (g(n+d) - f(n)) = A.$$

Then $g(n+d) = f(n) + A$ for every $n \in \mathfrak{N}_0$.

PROOF. Let $c = A/d$ and F and G defined by $F(n) = f(n) - cn$, $G(n) = g(n) - cn$.

From (2.1) we get that

$$(2.2) \quad G(n+d) - F(n) \rightarrow 0 \quad (n \rightarrow \infty).$$

It is obvious that F and G are \mathcal{R} -additive functions. Let $n \in \mathfrak{N}_0 (= \mathcal{O}_0)$ be fixed. Let i_0 be such a large index that for each $N \in \mathcal{O}_{i_0}$ the inequality $N > n+d$ holds. This involves that $G(N+d) = G(N) + G(d)$, $G(N+n+d) = -G(N) + G(n+d)$, $F(N+n) = F(N) + F(n)$ for $N \in \mathcal{O}_{i_0}$. So from (2.2) we get

$$\begin{aligned} \lim_{N \in \mathcal{O}_{i_0}} (G(N) - F(N)) &= -G(d), \\ \lim_{N \in \mathcal{O}_{i_0}} (G(N) - F(N)) &= -G(n+d) + F(n), \end{aligned}$$

which gives

$$(2.3) \quad G(n+d) - F(n) = -G(d).$$

(2.3) is true for every $n \in \mathfrak{N}_0$, so for $n = 0$ as well, consequently $G(d) = 0$, and so

$$(2.4) \quad G(n+d) - F(n) = 0 \quad (n \in \mathcal{O}_0)$$

This implies the assertion $g(n+d) = f(n) + A$ immediately. ■

THEOREM 2. *Given an arbitrary \mathcal{R} -system, let f be an arbitrary complex-valued \mathcal{R} -additive function such that*

$$(2.5) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n=x}^{x+1} |f(n+1) - f(n)| = 0.$$

Then $f(n) = 0$ identically.

PROOF. (a) Let $i_0 > 0$ be fixed, and the elements of $\epsilon \mathcal{I}_{i_0}$ in increasing order be $0 = z_1 < z_2 < \dots < z_A$. Let

$$\mathfrak{B}_i(x) := \{n|n = z_i - \beta \leq x, \beta \in \mathcal{C}_{i_0}\}, \quad |\mathfrak{B}_i(x)| = B_i(x) \quad (i = 1, \dots, A).$$

It is obvious that

$$\epsilon \mathcal{I}_{i_0} \cap \mathfrak{B}_i(x) = \emptyset \quad (i = 1, \dots, A); \quad \mathfrak{B}_i(x) \cap \mathfrak{B}_j(x) = \emptyset$$

($i \neq j$), and

$$(2.6) \quad \epsilon \mathcal{I}_{i_0} \cup \mathfrak{B}_1(x) \cup \dots \cup \mathfrak{B}_A(x) = \{n|n \leq x, n \in \mathcal{N}_0\}$$

for every large x .

So we have

$$(2.7) \quad A + B_1(x) + \dots + B_A(x) = [x] + 1.$$

Since

$$(2.8) \quad B_1(x) \geq B_2(x) \geq \dots \geq B_A(x),$$

therefore

$$(2.9) \quad A + A \cdot B_1(x) \geq [x] + 1.$$

Since $B_1(x) \leq B_A(x) + z_A$, therefore for each large x we have

$$(2.10) \quad \frac{B_1(x)}{A + A \cdot B_1(x)} \geq \frac{B_A(x)}{A + A(B_A(x) + z_A)} = \frac{1}{2A + 1}.$$

(b) Let now $m \in \mathcal{Q}$ be an arbitrary fixed integer. Let i_0 be so chosen that for every $m' < m$, $m' \in \mathcal{Q}$, we get $m' \in \epsilon \mathcal{I}_{i_0}$. Then

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| &\geq \frac{1}{x} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_0 \\ n \notin \mathcal{C}_{i_0}}} |f(\beta + m) - f(\beta + m - 1)| = \\ &= \frac{1}{x} \sum_{\substack{\beta \leq x - m \\ \beta \in \mathcal{C}_{i_0}}} |f(m) - f(m - 1)|, \end{aligned}$$

and so by (2.10) we get that the right hand side is not less than $\frac{1}{2A + 1} |f(m) - f(m - 1)|$. From (2.5) we get that $f(m) = f(m - 1)$.

(c) So we have proved that $f(m) = f(m - 1)$ for every $m \in \mathcal{Q}$. Since $f(0) = 0$ and $1 \in \mathcal{Q}$, therefore $f(1) = 0$. Now we proceed by using induction. Let $m \in \mathcal{Q}$, and assume that $f(m') = 0$ for every $m' \in \mathcal{Q}$, $m' < m$. Since $m - 1$ either belongs to \mathcal{Q} or can be written as the sum $m - 1 = r_{i_1} + \dots + r_{i_s}$ where $r_{i_k} < m$, therefore $f(m - 1) = 0$ and so $f(m) = 0$.

Hence we get the theorem immediately. ■

As an immediate consequence we remark the following

COROLLARY: Given an arbitrary \mathcal{Q} -system, let g be an arbitrary complex-valued \mathcal{Q} -additive function such that

$$\lim_{n \rightarrow \infty} (g(n+1) - g(n)) = C.$$

Then $g(n) \in Cn$.

3. THEOREM 3. Let f and g be arbitrary complex-valued q -additive functions,

$$(3.1) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |g(n+1) - f(n)| = 0$$

Then $g(n+1) = f(n)$ for every $n \in \mathfrak{N}_0$.

PROOF. Let $z \geq 1$ be an arbitrary integer, $h(n) = g(nq^z) - f(nq^z)$. It is obvious that $h(n)$ is a q -additive function and that $g(nq^z + 1) - f(nq^z) = h(n) + g(1)$. So we get that

$$\frac{1}{x} \sum_{n \leq x} |g(n+1) - f(n)| = \frac{1}{x} \sum_{nq^z \leq x} |h(n) + g(1)|$$

and so from (3.1) that

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |h(n) + g(1)| = 0.$$

Since $|h(n+1)| \leq |h(n)| + |h(n+1) + g(1)| + |h(n) + g(1)|$, from (3.2) we get

$$(3.3) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |h(n+1) - h(n)| = 0.$$

From Theorem 2, it follows that $h(n) = 0$, i.e. that

$$(3.4) \quad g(nq^z) = f(nq^z) \quad (z = 1, 2, \dots; n = 0, 1, \dots).$$

Let now k be an arbitrary integer and z be so large that $k+1 < q^z$. Since $g(nq^z + k + 1) - f(nq^z + k) = h(n) + g(k+1) - f(k)$, and the density of the integers $nq^z + k$ ($n = 1, 2, \dots$) is positive, from (3.1) it follows immediately that $g(k+1) = g(k)$.

By this the proof has been completed. ■

THEOREM 4. Take an \mathcal{Q} -system for which $\mathcal{Q}_n = \{1, \dots, k-1\}$ with a suitable $k \geq 2$. Assume that $d \in \mathcal{Q}_n$, F, G are complex-valued \mathcal{Q} -additive functions such that

$$(3.5) \quad G(n+d) = F(n) \quad n \in \mathfrak{N}_0.$$

Then

$$(3.6) \quad F(Nk) = NF(k-d) \quad (N = 1, 2, \dots)$$

and

$$(3.7) \quad G(n) = \begin{cases} F(k-d-n) - F(k-d) & \text{for } 0 \leq n < d, \\ F(n-d) & \text{for } d \leq n < k, \\ F(n) & \text{if } n \geq k. \end{cases}$$

In our case \mathcal{O}_1 is the set of all positive multiples of k . Let $F(1), F(2), \dots, F(k-1)$ be arbitrary complex numbers, $G(1), G(2), \dots, G(k-1)$ be defined by (3.7), and $G(n)$ and $F(n)$ for $n \in \mathcal{O}_2$ be defined by (3.6), (3.7).

For $n = km + s$, $0 \leq s < k$ let $F(n) = F(km) + F(s)$, $G(n) = G(km) + G(s)$. Then F, G are \mathcal{R} -additive functions satisfying (3.5).

PROOF. It is obvious that \mathcal{O}_1 is the set of all positive multiples of k . The last assertion is almost obvious. Let $n+d = kN+r$ ($0 \leq r < k$). Then $G(n+d) = G(kN)+G(r)$. Furthermore, $n = (N-1)k+k-d+r$ and $n = kN+r-d$. Then for $0 \leq r < d$ we get $F(n) = F((N-1)k+k-d+r)$ while for $d \leq r < k$ we have $F(n) = F(kN)+F(r-d)$. Taking into account (3.7), (3.6) we get (3.5).

We prove the first assertion. Since $F(0) = 0$, therefore $G(d) = 0$. Since $G(Nk+d) = G(Nk)+G(d) = G(Nk)$, therefore $G(Nk) = F(Nk)$ for $N \in \mathfrak{N}_0$. By substituting $n = k-d$ into (3.5), we get $G(k) = F(k-d)$. Now we prove that

$$(3.8) \quad G(Nk) = NF(k-d) \quad N \in \mathfrak{N}.$$

This has been proved for $N=1$. Let us assume that (3.8) is true for an $N (= 1)$. Then, from (3.5) we deduce that

$$G((N+1)k) = G(Nk+k-d+d) = F(Nk+k-d) + F(Nk)+F(k-d).$$

By using that $F(Nk) = G(Nk)$ and the induction hypothesis, we get that (3.8) is true for $N+1$ instead of N . Let $0 \leq n < d$. From the \mathcal{R} -additivity of G we get

$$G(k+n) = G(k) + G(n),$$

whence by (3.5) we deduce that

$$G(k+n) = G(k+n-d+d) = F(k-d+n).$$

Collecting the previous relations we get

$$G(n) = F(k-d+n) - F(k-d) \quad (0 \leq n < d).$$

Finally let $d \leq n < k$. Then from (3.5) we get directly that $G(n) = F(n-d)$. \blacksquare

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CHARACTERIZATION OF GENERALIZED q -ADDITIVE FUNCTIONS

By

J. FEHÉR

Teacher's Training College, Pécs

(Received December 22, 1982)

1. Let \mathfrak{N} be the set of natural numbers, \mathfrak{N}_0 be the set of nonnegative integers. For an arbitrary subset $\mathcal{S} \subset \mathfrak{N}$ let $\bar{\mathcal{S}} = \mathcal{S} \cup \{0\}$, and $|\mathcal{S}|$ be the cardinality of \mathcal{S} .

Let $\mathcal{R}_i \subset \mathfrak{N}$ ($i = 0, 1, 2, \dots$). We shall say that $\{\bar{\mathcal{R}}_0, \bar{\mathcal{R}}_1, \dots\}$ is a finite direct decomposition (FDD) of \mathfrak{N}_0 if every $n \in \mathfrak{N}$ can be written uniquely as

$$(1.1) \quad n = r_{i_1} + r_{i_2} + \dots + r_{i_s}, \quad r_{i_k} \in \mathcal{R}_{i_k} \quad (k = 1, \dots, s).$$

We shall say that this FDD is an \mathcal{R} -system if

- \mathcal{R}_i is finite and nonempty: $|\mathcal{R}_i| = k_i - 1 \geq 1$ ($i = 0, 1, \dots$)
- The smallest element of \mathcal{R}_i is smaller than the smallest element of \mathcal{R}_j , for every $i < j$.

We shall say that an \mathcal{R} -system is monotonic if the smallest element of \mathcal{R}_j is larger than the greatest element of \mathcal{R}_i for every $i < j$.

Let $f(n)$ be an arbitrary complex-valued function defined on \mathfrak{N}_0 . We call f to be \mathcal{R} -additive concerning the FDD $\{\bar{\mathcal{R}}_0, \bar{\mathcal{R}}_1, \dots\}$ if $f(0) = 0$ and for every n written in the form (1.1)

$$(1.2) \quad f(n) := f(r_{i_1}) + f(r_{i_2}) + \dots + f(r_{i_s}).$$

Given a FDD $\{\bar{\mathcal{R}}_0, \bar{\mathcal{R}}_1, \dots\}$, assume that n is represented as (1.1), and m in the form

$$(1.3) \quad m = t_{j_1} + t_{j_2} + \dots + t_{j_r}, \quad t_{j_k} \in \mathcal{R}_{j_k} \quad (k = 1, \dots, r).$$

We shall say that m and n are \mathcal{R} -disjoint if the set of indices $\{i_1, \dots, i_s\}$, $\{j_1, \dots, j_r\}$ are disjoint.

It is obvious that if f is \mathcal{R} -additive and m and n are \mathcal{R} -disjoint then $f(m+n) = f(m) + f(n)$.

For the sake of brevity we introduce the following notations:

$$(1) \quad \mathcal{R} = \bigcup_{i=0}^n \mathcal{R}_i,$$

$$(2) \quad \mathcal{A}_i = \{\lambda_0 + \dots + \lambda_{i-1} | \lambda_j \in \mathcal{R}_j\} \setminus \{0\},$$

$$(3) \quad \bar{\mathcal{D}}_i = \{\lambda_i + \dots + \lambda_n | h = i+1, i+2, \dots; \lambda_j \in \bar{\mathcal{R}}_j\} \setminus \{0\}.$$

That is, \mathcal{I}_i is the set of positive integers that have no components from \mathcal{R}_j for $j \geq i$, and \mathcal{O}_i is the set of positive integers that have no components from \mathcal{R}_j for $j \leq i-1$.

The structure of the monotonic \mathcal{R} -system is very simple:

$$\mathcal{R}_0 = \{1, \dots, k_0 - 1\},$$

$$\mathcal{R}_i = \{aK_i, a + 1, \dots, k_i - 1\}, \quad (i > 1),$$

$$K_i = k_0 \dots k_{i-1}.$$

We omit the proof of this obvious assertion.

Consequently in this case

$$\mathcal{I}_i = \{n | n \in [1, K_i - 1]\}, \quad \mathcal{O}_i = \{n | K_i | n \in \mathbb{N}\}.$$

For $k_i = q (\geq 2)$ we get the q -ary decomposition. The additive functions concerning the q -ary decomposition are called q -additive functions. This notion has been introduced by A. O. GELFOND [1].

2. In a joint paper written by L. KÁTAI [2] we proved the following assertion:

THEOREM A. *Let $g(n)$ and $f(n)$ be integer-valued q -additive functions. Assume that there exists at least one prime p for which*

- (a) $p | g(aq^j)$ holds for infinitely many pairs of a and j , where $a \in \{1, \dots, q-1\}$ and $j \in \{0, 1, 2, \dots\}$;
- (b) for every $n \in \mathbb{N}$, $p^x g(n)$ involves that $p^x | f(n)$. Then $f(n) = cg(n)$, c is a rational number.

With the same method we can prove the following

THEOREM B. *Given an \mathcal{R} -system, let $f(n)$, $g(n)$ be integer valued \mathcal{R} -additive functions. Assume that there exists at least one prime p for which*

- (a) $p | g(m)$ for infinitely many $m \in \mathcal{R}$,
- (b) there exists an index $i_0 \geq 0$ such that $n \in \mathcal{O}_{i_0}$ and $p^x | g(n)$ implies that $p^x | f(n)$.

Then $f(n) = cg(n)$ for every $n \in \mathcal{O}_{i_0}$, where c is a suitable rational constant.

By choosing $g(n) = n$ we get the following

THEOREM C. *Let $f(n)$ be an integer-valued \mathcal{R} -additive function. Assume that there exists at least one prime p for which*

- (a) $p | m$ for infinitely many $m \in \mathcal{R}$,
- (b) there exists an index $i_0 \geq 0$ such that $n \in \mathcal{O}_{i_0}$ implies that $n | f(n)$.

Then $f(n) = cn$ for every $n \in \mathcal{O}_{i_0}$, where c is a integer constant.

REMARK. It is easy to construct such \mathcal{R} -systems for which the condition (a) in Theorem C is not satisfied. This is the reason for which the next assertion seems to be interesting.

THEOREM 1. Given a monotonic \mathcal{Q} -system, let f be an integer-valued \mathcal{Q} -additive function. If for a suitable index $i_0 \geq 0$ the relation $n|f(n)$ holds for every $n \in \mathcal{O}_{i_0}$, then $f(n) = cn$ for every $n \in \mathcal{O}_{i_0}$, where c is an integer constant.

3. The proof of Theorem 1 is based on the following lemmas.

LEMMA 1. Let p and q be odd primes, $p < q$. We shall define r to be the least positive residue of $q \pmod p$ if $p \equiv 1 \pmod 4$, while for $p \equiv -1 \pmod 4$ r is the unique integer for which $q \equiv -r \pmod {4p}$, $r \equiv 1 \pmod 4$, $0 < r < 4p$.

Then for the Legendre-symbol we get $\left(\frac{p}{q}\right) = \left(\frac{r}{p}\right)$.

PROOF. See [3], Theorem 3.5.

For an arbitrary subset \mathfrak{E} of \mathfrak{N} let $\mathcal{D}(\mathfrak{E})$ denote the set of all prime-divisors of the elements of \mathfrak{E} . Let $\mathcal{D}(\mathfrak{E}) = \mathcal{P}$ be the set of all the primes.

LEMMA 2. Let k_1, k_2, \dots be an infinite sequence of arbitrary integers $k_i \geq 2$ such that $\mathcal{D}(\{k_n\})$ contains all but at most finitely many primes. Then, for the sequence $C_n = k_1 k_2 \dots k_n - 1$, $\mathcal{D}(\{C_n\})$ contains infinitely many elements.

PROOF. Assume in contrary that $\mathcal{D}(\{C_n\})$ is finite. Then $\mathcal{D}(\{C_n\})$ can be covered by a finite set of primes

$$\{2, p_1, \dots, p_u, q_1, \dots, q_s\}$$

where $p_i \equiv 1 \pmod 4$, $q_j \equiv -1 \pmod 4$. Consequently every C_n can be written in the form

$$(3.1) \quad C_n = 2^s p_1^{n_1} \dots p_u^{n_u} q_1^{m_1} \dots q_s^{m_s},$$

We shall say that q is an exceptional prime if it belongs to $\mathcal{D}(\mathcal{D}(\{k_n\}))$. Since the set of exceptional primes H is finite, therefore every large prime q is non-exceptional. If q is non-exceptional, then $q | k_{n_0}$ for at least $n_0 = n_0(q)$ consequently $C_n - 1 \equiv 0 \pmod q$ for $n \geq n_0(q)$. We shall show that there exist infinitely many q for which this condition does not satisfied.

Let x_0 denote the least positive solution of

$$(3.2) \quad 8q_1 \dots q_s x \equiv 2 \pmod{p_1 \dots p_u}$$

and let

$$(3.3) \quad Q_k = k \cdot 8p_1 \dots p_u q_1 \dots q_s + 8x_0 q_1 \dots q_s - 1$$

$k = 1, 2, \dots$. From Dirichlet theorem we get that Q_k is prime for infinitely many k . Let q be such a prime which is non-exceptional.

Observing that $q \equiv -1 \pmod 8$, we get that $\left(\frac{2}{q}\right) = 1$, furthermore from the definition of x_0 we get that $q \equiv 1 \pmod{p_i}$, $q \equiv -1 \pmod{q_j}$ and so from Lemma 1 we have that $\left(\frac{p_i}{q}\right) = 1$, $\left(\frac{q_j}{q}\right) = 1$, consequently by (3.1) we deduce

that $\left(\frac{C_n}{q}\right) = 1$. But $C_n \equiv -1 \pmod{q}$, and $\left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}} = -1$. This is a contradiction. ■

4. PROOF OF THEOREM I.

1. If there exists a prime p such that $p \nmid m$ for infinitely many $m \in \mathcal{R}$, then condition (a) in Theorem C is satisfied, consequently the assertion is true.

2. Assume now that the relation $p \mid m$ holds for all but at most finitely many $m \in \mathcal{R}$ for all primes p .

Since the \mathcal{R} -system is monotonic it can hold only if for each prime p there exists at least one k_p such that $p \mid k_p$.

3. Let n_0 be the smallest element of \mathcal{O}_{i_0} , $n_0 = K_{i_0}$. Since the conditions of Theorem I stated for $f(n)$ hold for the function $h(n) = n_0 f(n) - f(n_0)n$ as well, even we have $h(n_0) = 0$, therefore it is enough to prove our theorem under the additional condition $f(n_0) = 0$.

4. We shall prove the assertion by induction. It is known that $\mathcal{O}_{i_0} = \{Nn_0 \mid N = 1, 2, \dots\}$. Let $b = Nn_0$, $N > 1$ be fixed and assume that $f(A n_0) = 0$ for $A = 1, 2, \dots, N-1$. Let $m_r^{(1)}$ denote the smallest element of \mathcal{R}_r . Let r be so chosen that $b < m_r^{(1)}$. Let the index $s > r$ and the prime p so chosen $p \nmid m_r^{(1)}$, $p > \max\{b, |f(b)|\}$, and $p \mid k_r k_{r+1} \dots k_s - 1$. The existence of such a prime p is guaranteed by Lemma 2.

Let X denote the smallest positive solution of the congruence

$$(4.1) \quad m_r^{(1)} X \equiv -n_0 \pmod{p}.$$

Since $i_0 < r$, therefore $n_0 \nmid m_r^{(1)}$ and so $p \nmid n_0$. So $0 < X < k_r \dots k_s - 1$, and so

$$(4.2) \quad m_r^{(1)} X \in \mathcal{O}_r \setminus \{km_r^{(1)} \mid k \in [1, k_r \dots k_s - 1]\}.$$

Let j be the least integer for which $p \mid k_j$. It is clear that $j > s$, furthermore that $p \nmid m_j^{(1)} = K_{j+1}$. Consequently the set

$$\mathcal{O}_j = \{km_j^{(1)} \mid k \in [1, k_{j+1} - 1]\}$$

contains a complete residue system mod p . So there exists a positive integer Y such that

$$(4.3) \quad p \mid Ym_j^{(1)} + b - n_0 \quad \text{and} \quad Ym_j^{(1)} \in \mathcal{O}_j.$$

From the condition of our theorem, $p \mid f(Ym_j^{(1)} + b - n_0)$. Since the integers $Ym_j^{(1)}, b - n_0 = (N-1)n_0$ are \mathcal{R} -disjoint, and $f((N-1)n_0) = 0$, we get

$$(4.4) \quad p \mid f(Ym_j^{(1)}).$$

We can deduce similarly that

$$(4.5) \quad p \mid f(Xm_r^{(1)}).$$

From (4.1), (4.2) we get that $p|Ym_j^{(1)} + Xm_r^{(1)} + b$. Consequently $p|f(Ym_j^{(1)} + Xm_r^{(1)} + b) = f(Ym_j^{(1)}) + f(Xr^{(1)}) + f(b)$. Collecting (4.3), (4.4), (4.5), we deduce that $p|f(b)$, that is possible only if $f(b) = 0$. ■

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ISSN 0524—9 007

Technikai szerkesztő:

DR. SCHARNITZKY VIKTOR

A kiadásért felelős: az Eötvös Loránd Tudományegyetem rektora
A kézirat nyomdába érkezett: 1983. május. Megjelent: 1985. október

Terjedelem: 22,6 A 5 ív. Példányszám: 1000

Készült monó- és kéziszedéssel, ives magasnyomással,
az MSZ 5601—59 és MSZ 5602—55 szabványok szerint

Sz.471, Átlomi Nyomda, Budapest

Felelős vezető: Mihalek Sándor igazgató