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MISZELLEN ZUM THEMA OVALE IN STEINER-TRIPEL-SYSTEMEN

Von

H. ZEITLER

Bayreuth

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. Zusammenhang mit Vorträgen bei der Tagung „Combinatorics 1984“ und an der McMaster University in Hamilton wurden viele Fragen t. Einige davon versuchen wir hier zu beantworten.

1. Einleitung

ei V mit $|V| = v > 3$ eine endliche Menge und B mit $|B| = b$ eine e dreielementiger Teilmengen aus V. Die Elemente aus V nennen wir *te*, die aus B *Geraden*. Zu zwei Punkten gebe es genau eine Gerade, der gehören. Die Inzidenzstruktur $(V, B, \in)$ heißt dann ein *Steiner-Tripel-n der Ordnung* v. Wir schreiben kurz STS (v). Durch jeden Punkt gehen $r = \frac{1}{2}(v-1)$ Geraden. Die Bedingung $v = 7+6n$ oder $v = 9+6n$ mit ist notwendig und hinreichend für die Existenz solcher Systeme. (Die lfälle $v = 1$ und $v = 3$ werden hier ausgeschlossen.) Die Menge all „Steinerzahlen“ v bezeichnen wir mit STS.

Eine nicht leere Teilmenge $O \subset V$ in einem STS (v) heißt *Oval* genau wenn es in jedem ihrer Punkte genau eine Tangente (sie schneidet enau einem Punkt) gibt und jede andere Gerade höchstens zwei Punkte ır gemeinsam hat. Neben Tangenten existieren also bezüglich O nur ten (zwei Schnittpunkte) und Passanten (kein Schnittpunkt). Die te eines Ovals nennen wir *on-Punkte*, die auf den Tangenten liegenden on den jeweiligen Berührpunkten verschiedenen *ex-Punkte*, alle übrigen *nkte*.

Vir sprechen speziell von einem *Knotenoval*, wenn sich alle Tangenten em Punkt, dem Knoten, schneiden. Die Ovalpunkte zusammen mit Knoten bilden ein *Hyperoval*. Darunter versteht man eine nicht leere enge $H \subset V$, die von jeder Geraden entweder in genau zwei Punkten gar nicht geschnitten wird. Ein Oval heißt *regulär*, wenn es durch jeden

ex-Punkt genau zwei Tangenten an das Oval gibt. Alle anderen noch denkbaren Ovale nennen wir *gemischte Ovale*. Selbstverständlich treffen die durch das Wort Oval suggerierten Vorstellungen aus der euklidischen Geometrie hier nicht zu. Die Worte Knotenoval, reguläres Oval, gemischtes Oval und Hyperoval sind nichts als Namen für gewisse Punktmengen in STS (v).

Bezüglich der so definierten Ovale gelten nun die folgenden Sätze [4, 9, 10, 11].

SATZ 1.1. *Die Anzahl der Tangenten in einem ex-Punkt ist gerade oder ungerade, je nachdem r gerade oder ungerade ist.*

SATZ 1.2. *Wenn es in einem* STS (v) *Knotenovale (bzw. Hyperovale) gibt, dann existieren keine regulären Ovale.*

Aus der Existenz regulärer Ovale folgt dann sofort die Nichtexistenz von Knotenovalen.

SATZ 1.3. *Genau für alle* $v \in$ HSTS *gibt es* STS (v) *mit Knotenovalen, und genau für alle* $v \in$ RSTS *gibt es solche mit regulären Ovalen. Dabei gilt*

$$\left.\begin{array}{l}\text{HSTS: } v = 7+12n,\ v = 15+12n\\ \text{RSTS: } v = 9+12n,\ v = 13+12n\end{array}\right\}\text{mit } n \in \mathbf{N}_0.$$

Wegen HSTS $\cup$ RSTS = STS bedeutet dies, daß für alle $v \in$ STS Systeme STS (v) mit Ovalen existieren. Selbstverständlich schließt dies nicht aus, daß es STS (v) ohne Ovale gibt.

Über die Existenz gemischter Ovale wird in 1.2 und 1.3 nichts ausgesagt.

2. Ovale in PG (d, 2) und AG (d, 3)

Frage: Gibt es in PG (d, 2) und in AG (d, 3) Knotenovale, gibt es reguläre und gemischte Ovale?

Bei den in Abschnitt 1 als Ovale bezeichneten Punktmengen in PG (d, 2) und in AG (d, 3) handelt es sich um spezielle „Kalotten" (caps). Deshalb können einige der in Abschnitt 2 bewiesenen Sätze (etwa 2.1.3 und 2.2.3) direkt aus der sehr umfangreichen „Kalottentheorie" gefolgert werden. Gerade dies aber sollte in der vorliegenden Arbeit vermieden werden. Wir wollten nicht mit „Kanonen auf Spatzen" schießen. Bezüglich der genannten Theorie verweisen wir auf B. SEGRE [8] und A. BARLOTTI [1].

2.1. Projektive Räume

Jeder endliche projektive Raum der Ordnung 2 und der Dimension $d \geqq 2$ enthält genau $2^{d+1}-1$ Punkte [2]. Es handelt sich um ein STS $(2^{d+1}-1)$.

LEMMA 2.1.1. *Bei geradem d gilt* $v = 7+12n$, *bei ungeradem d dagegen* $v = 15+12n$ *für* $n \in \mathbf{N}_0$.

Insgesamt erhalten wir also $v \in$ HSTS. Zusammen mit 1.2 und 1.3 bedeutet dies, daß in PG (d, 2) keine regulären Ovale existieren.

BEWEIS. (a) d gerade $\Rightarrow 12 \mid (2^{d+1}-8)$.

Umformung liefert $\frac{2(2^{d-2}-1)}{3} \in \mathbf{N}_0$. Für $d = 2$ ist dies trivial. Sei nun $d \geqq 4$. Weil d und deshalb auch $d-2$ gerade ist, haben wir

$$\frac{2(2^{d-2}-1)}{2^2-1} = 2(1+2^2+\ldots+2^{d-4}) \in \mathbf{N}.$$

(b) d ungerade $\Rightarrow 12 \mid (2^{d+1}-16)$.

Umformung liefert $\frac{2^2(2^{d-3}-1)}{3} \in \mathbf{N}_0$. Für $d = 3$ ist dies trivial. Sei nun $d \geqq 5$. Weil d ungerade und deshalb $d-3$ gerade ist, haben wir

$$\frac{2^2(2^{d-3}-1)}{2^2-1} = 2^2(1+2^2+\ldots+2^{d-5}) \in \mathbf{N}.$$

SATZ 2.1.2. *In* PG $(d, 2)$, $d \geqq 2$ *gibt es genau* $2^d(2^{d+1}-1)$ *Knotenovale.*

BEWEIS. In einem STS (v) bildet die Komplementärmenge eines Hyperovals zusammen mit den zugehörigen Passanten ein Untersystem STS (r). In unserem Fall handelt es sich dabei um eine Hyperebene PG $(d-1, 2)$. Umgekehrt ist die Komplementärmenge einer jeden Hyperebene ein Hyperoval. Bekanntlich [2] gibt es in PG $(d, 2)$ genau $2^{d+1}-1$ Hyperebenen, jede mit 2^d-1 Punkten. Also existieren auch genau $2^{d+1}-1$ Hyperovale. Jeder der 2^d Punkte eines solchen Hyperovals kann als Knoten gewählt werden. Die übrigbleibenden Punkte des Hyperovals bilden dann ein Knotenoval.

SATZ 2.1.3. *In* PG $(d, 2)$, $d \geqq 2$ *gibt es nur Knotenovale.*

BEWEIS.

PG (2,2)

In einem PG (2,2) gebe es ein Oval O_2 mit $|O_2| = 3$. Durch einen ex-Punkt gehen nach 1.1 entweder genau 3 oder genau 1 Tangente. Im ersten Fall haben wir ein Knotenoval. Im zweiten geht durch jeden ex-Punkt genau eine Tangente. Dies liefert zu den 3 Ovalpunkten noch weitere 6, also insgesamt 9 Punkte. Widerspruch! Wenn es in PG (2,2) ein Oval gibt, dann ist es stets ein Knotenoval.

PG (3,2)

In einem PG (3,2) gebe es ein Oval O_3 mit $|O_3| = 7$. Seien A_1, A_2 zwei Punkte von O_3. Durch die Sekante $A_1 A_2$ gehen genau drei Ebenen α_2, β_2, γ_2. In keiner dieser Ebenen liegen mehr als vier Ovalpunkte. Ansonsten wären 3 Ovalpunkte kollinear. Dies bedeutet, daß in 2 Ebenen – etwa in α_2 und in β_2 – je 4 und in einer – also in γ_2 – genau 3 Ovalpunkte liegen. Diese Punkte bilden in α_2 und β_2 jeweils ein Hyperoval und in γ_2 ein Knotenoval $O_2 = \{A_1, A_2, A_3\}$ mit Knoten K (Figur 1). Die Punkte auf den 3 Sekanten von O_2 seien mit X_1, X_2, X_3 bezeichnet. Die Tangenten in A_1, A_2, A_3 schneiden sich in K. Dies gilt bezüglich O_2 und auch O_3. Wir behaupten nun, daß alle anderen Tangenten von O_3 dies auch tun. Sei B ein Ovalpunkt nicht in γ_2. Durch die Sekante $A_1 A_3$ gehen 3 Ebenen. Eine davon, und zwar γ_2, enthält

ein Knotenoval, die anderen je ein Hyperoval. Daraus folgt, daß die Ger BX_2 Sekante bezüglich O_3 ist. Dies gilt auch für die Geraden BX_1 und B Mit BA_1, BA_2, BA_3 sind damit alle 6 Sekanten von O_3 durch B gefund Also muß BK Tangente an O_3 sein.

Wenn es in PG (3,2) ein Oval gibt, dann ist es also stets ein Knote oval. Weiter existiert dann eine Ebene PG (2,2), die das Knotenoval in ein Hyperoval H_2 schneidet.

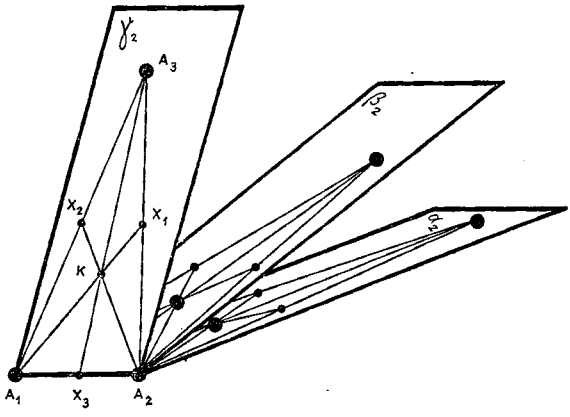

Fig. 1.

PG (d, 2), $d \geqq 3$

Der Beweis wird jetzt mit vollständiger Induktion geführt.

Induktionsannahme: Wenn es in PG $(n, 2)$ ein Oval O_n gibt, dann ist ein Knotenoval. Weiter existiert dann eine Hyperebene PG $(n-1,2)$, di O_n in einem Hyperoval H_{n-1} schneidet. Dabei gilt $n \in \{3,4,\ldots, d-1\}$.

Induktionsschluß: In einem PG $(d, 2)$ gebe es ein Oval O_d mit $|O_d| = 2^d - 1$ Dann existieren d Ovalpunkte, die eine Hyperebene aufspannen (Denn die Anzahl $2^d - 1$ der Ovalpunkte ist größer als die Anzahl $2^{d-1} - 1$ der Punkte in einem PG $(d-2,2)$.) Die Menge aller Ovalpunkte in dieser Hyper ebene bildet ein Oval O_{d-1}. Nach Induktionsannahme ist O_{d-1} ein Knotenoval, und es gibt in unserer Hyperebene einen Unterraum $S_{d-2} = \mathrm{PG}\,(d-2,2)$, der O_{d-1} in einem Hyperoval H_{d-2} mit $|H_{d-2}| = 2^{d-2}$ schneidet. Durch S_{d-2} gehen dann genau 3 Hyperebenen, nämlich α_{d-1}, β_{d-1}, γ_{d-1}. In keinem dieser Räume liegen mehr als 2^{d-1} Punkte des Ovals O_d. Dies bedeutet, daß in zwei Räumen – etwa in α_{d-1} und in β_{d-1} – je 2^{d-1} und in einem – also in γ_{d-1} – genau $2^{d-1} - 1$ Ovalpunkte liegen. Diese Punkte bilden in α_{d-1} und in β_{d-1} je ein Hyperoval und in γ_{d-1} ein Knotenoval O_{d-1} mit dem Knoten K. Zum Nachweis, daß alle Tangenten von O_d sich in H schneiden, betracht wir einen Ovalpunkt $B \notin \gamma_{d-1}$. Weiter sei in γ_{d-1} ein PG $(d-2,2)$ mit Hyperov gegeben. Die Existenz ist durch Induktionsannahme gesichert. Durch d Raum gehen genau 3 Hyperebenen. Eine davon – nämlich γ_{d-1} selber enthält ein Knotenoval, die anderen beiden – einer von ihnen gehört B an

ie ein Hyperoval. Jetzt schließen wir genau wie im Falle $d = 3$ weiter. Wenn es in PG $(d, 2)$, $d \geqq 3$ ein Oval gibt, dann ist es also stets ein Knotenoval.

2.2 Affine Räume

Jeder endliche affine Raum der Ordnung 3 und der Dimension $d \geqq 2$ enthält genau 3^d Punkte [2]. Es handelt sich um ein STS (3^d).

LEMMA 2.2.1. *Bei geradem d gilt* $v = 9 + 12n$, *bei ungeradem d dagegen* $v = 15 + 12n$ *für* $n \in \mathbf{N}_0$.

Wir erhalten also im ersten Fall $v \in$ RSTS und im zweiten $v \in$ HSTS.

BEWEIS. (a) d gerade $\Rightarrow 12 \mid (3^d - 9)$.

Umformung liefert $\dfrac{3(3^{d-2}-1)}{4} \in \mathbf{N}_0$. Für $d = 2$ ist dies trivial. Sei nun $d \geqq 4$. Weil d und deshalb auch $d-2$ gerade ist, haben wir

$$\frac{3^{d-2}-1}{3^2-1} = 1 + 3^2 + \ldots + 3^{d-4} \in \mathbf{N}.$$

Also ist $3^{d-2} - 1$ durch 8 und erst recht durch 4 teilbar.

(b) d ungerade $\Rightarrow 12 \mid (3^d - 15)$.

Umformung liefert $\dfrac{3^{d-1}-1}{4} - 1 \in \mathbf{N}_0$. Weil d ungerade und deshalb $d-1$ gerade ist, haben wir

$$\frac{3^{d-1}-1}{3^2-1} = 1 + 3^2 + \ldots + 3^{d-3} \in \mathbf{N}.$$

Also ist $3^{d-1} - 1$ durch 8 und erst recht durch 4 teilbar.

SATZ 2.2.2. *In* AG (2, 3) *gibt es weder Knotenovale noch gemischte Ovale. Es existieren genau* 54 *reguläre Ovale.*

(Siehe dazu H. LÜNEBURG [5].)

BEWEIS. Die Anzahl der Tangenten durch einen ex-Punkt beträgt nach 1.1 entweder 4 oder 2. Der erste Fall führt auf ein Knotenoval. Ein solches kann wegen $9 \in$ RSTS nach 1.2 und 1.3 nicht existieren. Im zweiten Fall erhalten wir ein reguläres Oval. Wenn also überhaupt Ovale existieren, dann nur reguläre.

Jede 4-Menge (Menge mit genau 4 Punkten) aus AG (2, 3), die keine Gerade enthält, ist ein Oval. Denn durch jeden der 4 Punkte gehen insgesamt 4 Geraden. Drei davon sind Sekanten und genau eine ist Tangente. Nach dem Vorigen muß es sich um ein reguläres Oval handeln. Insgesamt gibt es in AG (2, 3) genau $\binom{9}{4} = 126$ 4-Mengen. Weil sich die Punkte jeder Geraden auf 6 Arten zu 4-Mengen ergänzen lassen und es 12 Geraden gibt, enthalten genau $6 \cdot 12 = 72$ der angegebenen 4-Mengen eine Gerade. Also existieren genau $126 - 72 = 54$ reguläre Ovale.

SATZ 2.2.3. *In* AG $(d, 3)$ *mit* $d \geqq 3$ *gibt es keine Ovale.*

BEWEIS. Wir beweisen den Satz zunächst für $d = 3$. In AG (3, 3) gebe es ein Oval O_3. Dann gilt $|O_3| = \frac{1}{2}(v-1) = 13$. Der Raum AG(3, 3) enthält drei verschiedene, paarweise parallele Ebenen. Dabei handelt es sich jeweils um ein AG (2, 3) mit je 9 Punkten. Nach unserer Annahme muß eine dieser drei Ebenen mindestens 5 Punkte des Ovals O_3 enthalten. Diese Punktmenge bezeichnen wir mit H_2. Weil O_3 Oval ist, liegen keine 3 Punkte von H_2 auf einer Geraden. Durch jeden Punkt von H_2 gibt es in der zugehörigen Ebene genau 4 Geraden. Bezüglich H_2 sind dies Sekanten. Tangenten zu H_2 existieren also nicht. Damit ist gezeigt, daß H_2 in der zugehörigen Ebene, also in einem AG (2, 3), Hyperoval ist. Aus der Existenz eines Hyperovals folgt die von Knotenovalen und damit ein Widerspruch zu 2.2.2. Den Rest des Beweises führen wir mit vollständiger Induktion. Der Satz gelte bis zur Dimension d, also bis hin zu AG $(d, 3)$. In AG $(d+1, 3)$ gebe es ein Oval O_{d+1}. Dann gilt

$$|O_{d+1}| = \frac{1}{2}(v-1) = \frac{1}{2}(3^{d+1}-1) = 1+3+3^2+\ldots+3^d.$$

Division von $|O_{d+1}|$ durch 3 ergibt also den Rest 1. Der Raum AG $(d+1, 3)$ enthält drei verschiedene, paarweise parallele Hyperebenen. Dabei handelt es sich jeweils um AG $(d, 3)$ mit je 3^d Punkten. Nach unserer Annahme muß eine dieser Hyperebenen mindestens

$$\frac{1}{3}(|O_{d+1}|-1)+1 = \frac{1}{2}(3^d-1)+1 = \frac{1}{2}(3^d+1)$$

Punkte des Ovals O_{d+1} enthalten. Diese Punktmenge bezeichnen wir mit H_d. Weil O_{d+1} Oval ist, liegen keine 3 Punkte von H_d auf einer Geraden. Durch jeden Punkt von H_d gibt es in der zugehörigen Hyperebene genau $\frac{1}{2}(3^d-1)$ Geraden. Bezüglich H_d sind dies Sekanten. Tangenten zu H_d existieren also nicht. Damit ist gezeigt, daß H_d in der zugehörigen Hyperebene, also in einem AG $(d, 3)$, Hyperoval ist. Aus der Existenz eines Hyperovals folgt die von Knotenovalen und damit ein Widerspruch zu unserer Annahme.

3. Automorphismengruppen

Frage: Ist es möglich, unter Verwendung des Ovalbegriffes Aussagen über die Automorphismengruppe eines gegebenen STS (v) zu gewinnen?

Die Frage läßt sich in dieser Allgemeinheit vermutlich nur sehr schwer beantworten. Deshalb beschränken wir die Untersuchung auf ein sehr spezielles System, nämlich STS (13). Für diesen Fall sind die Automorphismengruppen längst bekannt und wurden 1983 zusammen mit anderen Ergebnissen nochmals publiziert [6]. Es geht also hier nur um die Gewinnung alter Resultate, jedoch unter Verwendung des Ovalbegriffes.

3.1. Ovale in STS (13)

In der Arbeit [9] werden für STS (13) Ovale untersucht und dazu einige Sätze bewiesen. Wir teilen Ergebnisse mit.

3.1.1. Ovale

Nach unserer Definition enthält jedes Oval genau 6 Punkte. Es gibt genau 6 Tangenten, genau 5 Passanten und genau 15 Sekanten. Wegen $13 \in$ RSTS existieren keine Knotenovale.

3.1.2. Reguläre Ovale

Bezüglich eines regulären Ovals gibt es genau 6 ex-Punkte und genau einen in-Punkt S. In der folgenden Tabelle ist wiedergegeben, wieviele Geraden verschiedener Klasse durch Punkte verschiedener Klasse gehen.

	Sekanten	Tangenten	Passanten
ex-Punkt	2	2	2
in-Punkt	3	0	3
on-Punkt	5	1	0

Für die fünf Passanten ergibt sich die in Figur 2 dargestellte Konfiguration. Wir sprechen von der A-Konfiguration und nennen den in-Punkt S ihren Trägerpunkt.

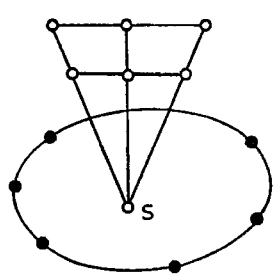

Fig. 2.

3.1.3. Gemischte Ovale

Es zeigt sich, daß nur eine einzige Art gemischter Ovale existieren kann. Bei ihnen gehen durch einen ex-Punkt Z genau 4 und durch jeden anderen ex-Punkt genau 2 Tangenten. Es gibt genau 5 ex-Punkte und genau 2 in-Punkte S_1, S_2.

In der folgenden Tabelle ist wiedergegeben, wieviele Geraden verschiedener Klasse durch Punkte verschiedener Klasse gehen.

	Sekanten	Tangenten	Passanten
ex-Punkt			
$= Z$	1	4	1
$\neq Z$	2	2	2
in-Punkt	3	0	3
on-Punkt	5	1	0

Für die 5 Passanten ergibt sich die in Figur 3 dargestellte Konfiguration. Ohne die Gerade $\{S_1, S_2, Z\}$ bleibt eine Konfiguration über, in der kein Punkt als Trägerpunkt ausgezeichnet ist. Wir sprechen von der P-Konfiguration.

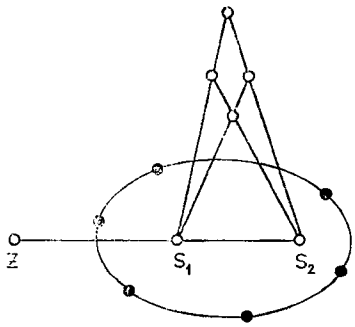

Fig. 3.

3.1.4. STS_1 und STS_2

Unter Verwendung solcher Ovale – über deren Existenz bisher noch nichts gesagt ist – wird schließlich in der zitierten Arbeit durch Konstruktion gezeigt, daß es genau zwei mit STS_1 und STS_2 bezeichnete nicht isomorphe Steiner Tripel Systeme der Ordnung 13 gibt. Auch dieses Resultat ist schon sehr lange bekannt.

Im folgenden sollen nun die Ovale in den beiden Systemen genauer untersucht werden, und zwar mit dem Ziel, Aussagen über die Automorphismengruppen zu gewinnen.

3.2. Das System STS_1

3.2.1. Die Elemente, bezüglich eines regulären Ovals

Die Punkte:

on-Punkte: 7, 8, 9, 10, 11, 12; ex-Punkte: 1, 2, 3, 4, 5, 6; in-Punkt: 0

Die Geraden:

Passanten:

{0, 1, 2}, {0, 3, 4}, {0, 5, 6}, {1, 3, 5}, {2, 4, 6}.

Tangenten:

{1, 6, 7}, {2, 5, 8}, {3, 6, 9}, {4, 5, 10}, {1, 4, 11}, {2, 3, 12}.

Sekanten:

{0, 8, 10}, {0, 9, 12}, {0, 7, 11}, {1, 8, 12}, {1, 9, 10}, {4, 7, 12},
{4, 8, 9}, {5, 7, 9}, {5, 11, 12}, {2, 9, 11}, {3, 10, 11}, {3, 7, 8},
{6, 8, 11}, {6, 10, 12}, {2, 7, 10}.

SATZ 3.2.2. *Es existieren genau* 24 *gemischte Ovale und genau* 10 *reguläre Ovale* [3]. *Die Automorphismengruppe* G_6 *hat die Ordnung* 6. *Sie wird von zwei Elementen erzeugt und ist nicht kommutativ.*

BEWEIS. Bei diesem und den folgenden Beweisen bedienen wir uns immer wieder der Ergebnisse aus [9].

P-Konfigurationen: In STS_1 gibt es genau 8 *P*-Konfigurationen. Zu jeder von ihnen lassen sich genau 3 gemischte Ovale konstruieren. Also existieren in STS_1 insgesamt 24 gemischte Ovale.

A-Konfigurationen: In der folgenden Tabelle ist die Anzahl von *A*-Konfigurationen in Trägerpunkten verschiedener Klasse angegeben.

	Trägerpunkte	Anzahl der A—Konfigurationen
ex-Punkte	$\boxed{1}$, $\boxed{3}$, $\boxed{5}$	2
ex-Punkte	②, ④, ⑥	1
in-Punkt	⓪	1
on-Punkte	7, 8, 9, 10, 11, 12,	0

Zu jeder *A*-Konfiguration läßt sich genau ein reguläres Oval konstruieren. Also existieren in STS_1 insgesamt 10 reguläre Ovale. In einer, über die zitierte Arbeit hinausgehenden Tabelle (Figur 4) geben wir den jeweiligen Trägerpunkt, die dazugehörigen Passanten und das reguläre Oval explizit an. Die beiden nicht durch den Trägerpunkt gehenden Passanten („Parallelpassanten") sind jeweils in einer eigenen Spalte angeschrieben. Alle Punkte mit genau 2 bzw. mit genau einer *A*-Konfiguration wurden – wie schon in der letzten Tabelle – besonders hervorgehoben.
Der neuen Tabelle entnehmen wir, daß das erste Oval eine Ausnahme darstellt. Nur bei ihm enthält eine der beiden „Parallelpassanten" 3 Punkte mit genau 2, die andere 3 Punkte mit genau einer *A*-Konfiguration.

Soll nun ein Automorphismus in STS_1 konstruiert werden, so müssen das erste Oval sowie jede seiner „Parallelpassanten" jeweils als Ganzes in sich abgebildet werden. Der Punkt *0* muß Fixpunkt sein.

in-Punkt	Passanten					Ovale
⓪	⓪, [1], ②	⓪, [3], ④	⓪, [5], ⑥	②, ④, ⑥	[1], [3], [5]	7, 8, 9, 10, 11, 12
[1]	⓪, [1], ②	[1], ⑥, 7	[1], ④, 11	⓪, 7, 11	②, ④, ⑥	[3], [5], 8, 9, 10, 12
[1]	[1], 8, 12	[1], ⑥, 7	[1], ④, 11	④, 7, 12	⑥, 8, 11	⓪, ②, [3], [5], 9, 10
[3]	⓪, [3], ④	[3], ⑥, 9	②, [3], 12	⓪, 9, 12	②, ④, ⑥	[1], [5], 7, 8, 10, 11
[3]	[3], 10, 11	[3], ⑥, 9	②, [3], 12	⑥, 10, 12	②, 9, 11	⓪, [1], ④, [5], 7, 8
[5]	⓪, [5], ⑥	②, [5], 8	④, [5], 10	⓪, 8, 10	②, ④, ⑥,	[1], [3], 7, 9, 11, 12
[5]	[5], 7, 9	②, [5], 8	④, [5], 10	②, 7, 10	④, 8, 9	⓪, [1], [3], ⑥, 11, 12
②	⓪, [1], ②	②, 9, 11	②, 7, 10	⓪, 7, 11	[1], 9, 10	[3], ④, [5], ⑥, 8, 12
④	⓪, [3], ④	④, 7, 12	④, 8, 9	⓪, 9, 12	[3], 7, 8	[1], ②, [5], ⑥, 10, 11
⑥	⓪, [5], ⑥,	⑥ 8, 11	⑥, 10, 12	⓪, 8, 10	[5], 11, 12	[1], ②, [3], ④, 7, 9

Fig. 4.

Von den 6 Permutationen der Punkte 1, 3, 5 ausgehend ist es möglich, unter Verwendung der eben angegebenen Forderungen die zugehörigen Automorphismen eindeutig zu ergänzen. Wir geben die Resultate in Zykelschreibweise an.

a : (1, 5, 3) (2, 6, 4) (7, 10, 12) (8, 9, 11) (0)
b : (1, 5) (2, 6) (7, 8) (9, 12) (10, 11) (0) (3) (4)
a^2 : (1, 3, 5) (2, 4, 6) (7, 12, 10) (8, 11, 9) (0)
ba : (1, 3) (2, 4) (7, 9) (8, 10) (11, 12) (0) (5) (6)
ab : (3, 5) (4, 6) (7, 11) (8, 12) (9, 10) (0) (1) (2)

Nimmt man die identische Abbildung dazu, so haben wir also insgesamt 6 Automorphismen.

Die verwendete Schreibweise macht deutlich, daß diese Automorphismen eine von a und b erzeugte, nicht kommutative Gruppe der Ordnung 6 bilden. In Figur 5 geben wir die Gruppentafel an. Diese Gruppe ist isomorph zur Gruppe der Decktransformationen eines gleichseitigen Dreiecks.

	id	a	a^2	b	ab	ba
id	id	a	a^2	b	ab	ba
a	a	a^2	id	ab	ba	b
a^2	a^2	id	a	ba	b	ab
b	b	ba	ab	id	a^2	a
ab	ab	b	ba	a	id	a^2
ba	ba	ab	b	a^2	a	id

Fig. 5.

3.3. Das System STS_2

3.3.1. Die Elemente, bezüglich eines gemischten Ovals

Die Punkte:

on-Punkte: 3, 6, 7, 9, 11, 12; ex-Punkte: 0, 1, 5, 8; in-Punkte: 2, 4; Zentrum: 10.

Die Geraden:

Passanten:

{0, 2, 8}, {1, 2, 5}, {2, 4, 10}, {4, 5, 8}, {0, 1, 4}.

Tangenten:

{0, 5, 7}, {1, 6, 8}, {1, 10, 11}, {0, 9, 10}, {5, 10, 12}, {3, 8, 10}.

Sekanten:

{6, 7, 10}, {3, 4, 7}, {4, 6, 12}, {4, 9, 11}, {2, 3, 6}, {2, 11, 12},
{2, 7, 9}, {3, 5, 11}, {5, 6, 9}, {7, 8, 11}, {8, 9, 12}, {1, 3, 9},
{1, 7, 12}, {0, 6, 11}, {0, 3, 12}.

Eine elegantere Darstellungsweise für die Geraden: $\{x, x+1, x+4\}$, $\{x, x+2, x+8\}$ mit $x \in \{0, 1, \ldots, 12\}$. (Dabei ist modulo 13 zu rechnen.) Diese Darstellung ermöglicht es, Aussagen ohne mühsames, zeitraubendes Probieren zu beweisen.

LEMMA 3.3.2. *Durch jeden Punkt gibt es genau* 6 *P-Konfigurationen.*

BEWEIS. Sei x der betreffende Punkt. Die 6 Geraden durch x:

$$\{x, x+1, x+4\},\ \{x+12, x, x+3\},\ \{x+9, x+10, x\},$$
$$\{x, x+2, x+8),\ \{x+11, x, x+6\},\ \{x+5, x+7\ , x\}.$$

Die durch x bestimmten 6 P-Konfigurationen werden nun gefunden durch Überprüfen aller 15 Paare dieser 6 Geraden. Figur 6 zeigt das Ergebnis.

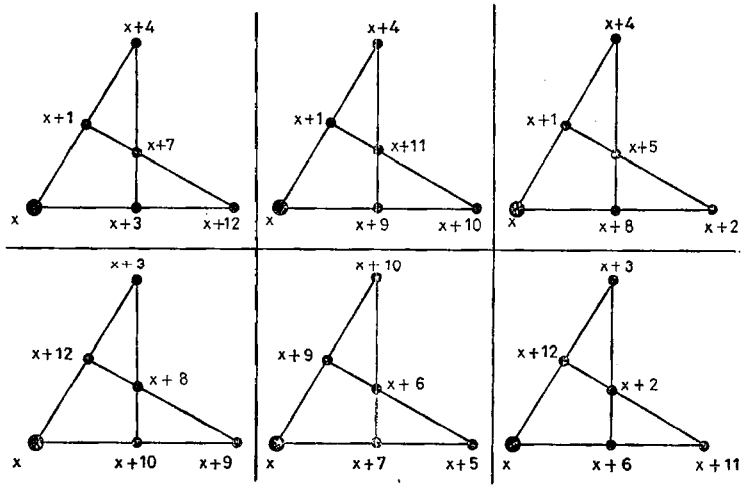

Fig. 6.

LEMMA 3.3.3. *Es gibt genau* 13 *Zentralgeraden.*

Dabei verstehen wir unter Zentralgeraden die durch die beiden in-Punkte eines Ovals bestimmte Gerade. Diese Zentralgerade enthält also auch das Zentrum Z.

BEWEIS. Der Figur 6 entnehmen wir die einzige Zentralgerade $\{x, x+5, x+7\}$. Mit $x \in \{0, 1, \ldots, 12\}$ ist dann das Lemma bewiesen.

LEMMA 3.3.4.

(a) *Zu jeder Zentralgeraden z gibt es genau* 3 *P-Konfigurationen.*

(b) *Diese* 3 *zu z gehörenden P-Konfigurationen haben genau einen, nicht auf z liegenden Punkt gemeinsam. Wir sprechen vom Pol der Geraden z.*

(c) *Unter den* 4 *Geraden jeder P-Konfiguration befindet sich genau eine Zentralgerade.*

BEWEIS. Nach 3.3.3 ist $z = \{x, x+5, x+7\}$ die Zentralgerade.

Der Figur 6 entnehmen wir, daß es zu z genau 3 P-Konfigurationen gibt. Sie sind in Figur 7 nochmals dargestellt. Die um Zentralgerade und Zentrum erweiterten Konfigurationen wurden mit A_x, B_x und C_x bezeichnet.

Figur 7 läßt sofort erkennen, daß die 3 P-Konfigurationen genau einen Punkt, nämlich $x+4$, gemeinsam haben.

Jede der in Figur 7 gezeichneten P-Konfigurationen besteht aus 4 Geraden. Vergleich mit 3.3.3 zeigt, daß jeweils genau eine davon Zentralgerade ist. Wir geben sie explizit an $\{x, x+2, x+8\}$, $\{x+1, x+7, x+12\}$, $\{x+3, x+5, x+11\}$. In Figur 7 sind diese 3 Geraden besonders gekennzeichnet.

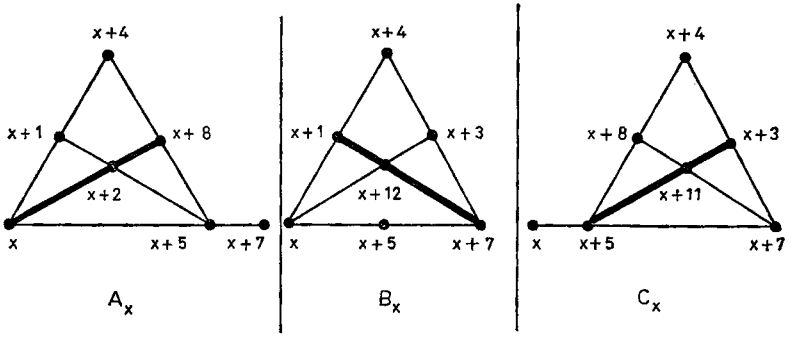

Fig. 7.

SATZ 3.3.5. *Es gibt genau* 39 *gemischte und keine regulären Ovale* [3].

BEWEIS. Nach 3.3.4 (a) gibt es zu jeder Zentralgeraden genau 3 P-.nfigurationen mit festgelegtem Zentrum, also nach der Konstruktion in genau 3 gemischte Ovale. Mit 3.3.3 ist damit der erste Teil des Satzes .wiesen. Die Nichtexistenz regulärer Ovale wird in [9] gezeigt.

LEMMA 3.3.6. *Bei jedem Automorphismus werden gemischte Ovale auf .ischte Ovale und jede Zentralgerade mit Pol auf eine Zentralgerade mit Pol :ldet.*

BEWEIS. Das Lemma folgt mit der Bijektivität und der Inzidenztreue ; den Definitionen des gemischten Ovals, der Zentralgeraden und des ɔls.

SATZ 3.3.7. *Es gibt genau 3 Automorphismen, welche die Zentralgerade*), 5, 7} *mit Pol* 4 *auf sich abbilden. Diese drei Abbildungen bilden eine zy-''ıche Gruppe* G_3.

BEWEIS. Zum Beweis betrachten wir die 3 zur Zentralgeraden {0, 5, 7 ' Pol 4 gehörenden Konfigurationen A_0, B_0, C_0 aus Figur 7. Soll A_0 au} ɔ abgebildet werden, so müssen 7 in 5, 0 in 7 und 5 in 0 übergehen (3.3.4 (c)

und 3.3.6). Der Punkt 4 ist Fixpunkt. Die restlichen Zuordnungen ergeben sich zwangsläufig aus der Forderung der Geradentreue. So kennt man z.B. von der Originalgeraden $\{0, 1, 4\}$ bereits die Bildpunkte 7 und 4. Also ist die Bildgerade $\{7, 3, 4\}$. Dem Punkt 1 muß der Punkt 3 zugeordnet werden. Auf diese Weise fortschreitend ergibt sich in Zykelschreibweise die Abbildung

$$a = (0, 7, 5) \quad (1, 3, 8) \quad (2, 12, 11) \quad (4) \quad (6, 9, 10).$$

Diese Abbildung bildet auch noch die Konfiguration B_0 auf C_0 ab. Soll nun durch eine zweite Abbildung A_0 auf C_0 abgebildet werden, so müssen 7 in 0, 0 in 5 und 5 in 7 übergehen. Der Punkt 4 ist wieder Fixpunkt. Wie bei der ersten Abbildung erhalten wir schließlich in Zykelschreibweise

$$(0, 5, 7) \quad (1, 8, 3) \quad (4) \quad (6, 10, 9) \quad (11, 12, 2).$$

Es zeigt sich, daß dies gerade a^2 ist und weiter, daß gilt $a^3 = \mathrm{id}$.

SATZ 3.3.8. *Es gibt genau einen Automorphismus, welcher die Zentralgerade* $\{0, 5, 7\}$ *mit Pol* 4 *und Zentrum* 7 *auf die Zentralgerade* $\{x, x+5, x+7\}$ *mit Pol* $x+4$ *und Zentrum* $x+7$, *also die Konfiguration* A_0 *auf* A_x *abbildet. Mit derselben Abbildung werden auch die Konfigurationen* B_0 *auf* B_x *und* C_0 *auf* C_x *abgebildet.*

Die sich für $x \in \{0, 1, \ldots, 12\}$ *so ergebenden* 13 *Abbildungen bilden eine zyklische Gruppe* G_{13}.

BEWEIS. Soll A_0 auf A_x abgebildet werden, so müssen zunächst (3.3.4 (c) und 3.3.6) wieder 7 in $x+7$, 0 in x, 5 in $x+5$ und 4 in $x+4$ abgebildet werden. In der Manier von 3.3.7 ergänzt, erhalten wir für $x = 1$ in Zykelschreibweise die Abbildung

$$b_1 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

und weiter $b_x = b_1^x$ für $x \in \{1, 2, \ldots, 12\}$. Mit $b_0 = b_{13} = \mathrm{id}$ sind damit die erste und dritte Aussage des Satzes bewiesen.

Die Anwendung der 13 Abbildungen auf B_0 und auf C_0 liefert selbstverständlich B_x und C_x.

SATZ 3.3.9. *Die Automorphismengruppe von* STS_2 *enthält genau alle Abbildungen aus* $G_3 \times G_{13}$, *hat also die Ordnung* 39.

BEWEIS. Mit den Abbildungen aus $G_3 \times G_{13}$ können wir nach 3.3.7 und 3.3.8 jede der 3 Konfigurationen A_0, B_0, C_0 eindeutig aufeinander und auf jede Konfiguration A_x, B_x bzw. C_x abbilden.

Nach 3.3.6 sind damit alle Automorphismen erfaßt.

4. Ein Problem von ERDÖS

4.1. Das Problem

In einem STS(v) werden Punktmengen E betrachtet, die keine Gerade enthalten. Wie groß ist die maximale Mächtigkeit von E? Für welche $v \in$ STS existieren STS (v) mit Teilmengen E dieser maximalen Mächtigkeit? Dieses auf ERDÖS zurückgehendes Problem wurde im Jahre 1969 [7] gelöst.

Frage: Ist es möglich, bei der Lösung des ERDÖS-Problems den Ovalbegriff zu verwenden?

SATZ 4.2. *Die gesuchte maximale Mächtigkeit beträgt* $r+1$ *für alle* $v \in$ HSTS *und* r *für alle* $v \in$ RSTS. *Für alle genannten Ordnungen, existieren* STS (v) *mit solchen Punktmengen maximaler Mächtigkeit. Im ersten Fall handelt es sich dabei um Hyperovale, im zweiten um reguläre Ovale.*

BEWEIS. In einem Punkt einer Menge E, die keine Gerade enthält, gibt es höchstens r Sekanten. Dies bedeutet $|E| \leqq r+1$. Wird der Maximalwert $r+1$ angenommen, so existieren bezüglich E nur Sekanten. Es liegt also ein Hyperoval vor. Genau für alle $v \in$ HSTS existieren STS (v) mit solchen Punktmengen maximaler Mächtigkeit.

Für alle anderen Steinerzahlen existieren STS (v) mit solchen maximalen Punktmengen nicht. In diesen Fällen muß für alle Punktmengen E, die keine Geraden enthalten, gelten $|E| \leqq r$. Wir wissen, daß für alle $v \in$ RSTS Systeme STS (v) mit Mengen E der maximalen Mächtigkeit r tatsächlich existieren. Diese Mengen sind reguläre Ovale.

5. Schluß

Ziel der vorliegenden Arbeit war es, wenigstens drei der bei den erwähnten Treffen aufgeworfenen Fragen zu beantworten. Dabei kam es nicht darauf an, neue Resultate mitzuteilen. Statt dessen sollte die Tragfähigkeit des Ovalbegriffs bei der Untersuchung von Steiner-Tripel-Systemen verdeutlicht werden.

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AN EQUICONVERGENCE THEOREM

By

M. HORVÁTH, I. JOÓ and V. KOMORNIK
Department for Analysis of the L. Eötvös University, Budapest

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The equiconvergence theorems play an important role in the theory ·pansions. One of the first results of this type was proved by A. HAAR ·ı 1910–11. Later on, general equiconvergence theorems were proved ·ıe self-adjoint Schrödinger operator [2]. However, many problems of ·tical interest require the investigation of the non-selfadjoint case. Under ·al conditions there do not exist complete orthonormal systems of eigen- ·ions. However, introducing the eigenfunctions of higher order a notion ·ar to what is known from the Jordan theorem in linear algebra, the exist- of a Riesz basis consisting of eigenfunctions of higher order was proved ·ıch cases by different authors [6].

The aim of the present paper is to prove a general equiconvergence ·em, using the well-known method of V. A. IL'IN, for a situation more ·al than in [3]. We improve for second order operators the result of V. L'IN [15], namely we drop the condition of the boundedness of v_α, see ·N.

Let G be an arbitrary open interval on the real line, $q \in L_p^{\mathrm{loc}}(G)$ $p > 1$ ı arbitrary complex function, and consider the formal Schrödinger ope- · $Lu = u'' + qu$. Given a complex number λ, the function $u: G \to \mathbf{C}$, $u \equiv 0$ ed an eigenfunction of order -1 of the operator L with the eigen- λ. Furthermore, a function $u: G \to \mathbf{C}$, $u \neq 0$ is called an eigenfunction ·der $m (m = 0, 1, \ldots)$ of the operator L with the eigenvalue λ if u and its ·ative u' are locally absolutely continuous on G and $Lu = \lambda u + u^*$ ·st everywhere on G, where u^* is an eigenfunction of order $m-1$ of the ·ator L with the eigenvalue λ.

Let us now be given a complete system $(u_k) \subset L_2(G)$ of eigenfunctions of ·perator L, denote by λ_k resp. o_k the eigenvalue resp. order of u_k, and ·ne

$$m = \sup_k o_k < \infty,$$

$$\text{in case } o_k > 0,\ \lambda_k u_k - L u_k = u_{k-1}.$$

Denote by μ_k a square root of λ_k and put $\nu_k = |\mathrm{Im}\,\mu_k|$, $\varrho_k = |\mathrm{Re}\,\mu_k|$. Introduce the notation:

$$\sigma_\nu(f, x) := \sum_{\nu_\alpha < \nu} \langle f, v_\alpha \rangle u_\alpha(x) + \sum_{\nu_\alpha = \nu}^{*} \langle f, v_\alpha \rangle u_\alpha(x), \tag{3}$$

$$(f \in L_2(G),\ \nu > 0,\ x \in G)$$

where (v_k) is the dual system of (u_k) (assume its existence), i. e. $(v_k) \subset L_2(G)$ and $\langle v_k, u_j \rangle = \delta_{k,j}$ Let

$$S_\nu(f, x) := \int_{x-R}^{x+R} \frac{\sin \nu(y-x)}{\pi(y-x)} f(y)dy$$

$$(f \in L_2(G),\ \nu > 0,\ x \pm R \in G). \tag{4}$$

We shall prove the following

THEOREM. *Assume that* (1) *and*

$$\sup_{t>0} \sum_{t \le \nu_\alpha + t \le 1} 1 < \infty$$

are fulfilled. Then the following three statements are equivalent:

a) *For any compact interval* $K \subset G$

$$\sup_\alpha \|v_\alpha\|_{L^2(G)} \cdot \|u_\alpha\|_{L^2(K)} < \infty .$$

b) *For any compact interval* $K \subset G$

$$\lim_{\nu \to \infty} \sup_{x \in K} |S_\nu(f, x) - \sigma_\nu(f, x)| = 0$$

for every $f \in L^2(G)$ *and every* $0 < R <$ dist $(K, \partial G)$.

c) *For every compact interval* $K \subset G$

$$\lim_{\nu \to \infty} \|f - \sigma_\nu(f)\|_{L^2(K)} = 0$$

for every $f \in L^2(G)$.

REMARK 1. Our theorem is closely related to many investigations. For example, in the case when $G = (0, \infty)$, $q \equiv 0$ and the eigenfunctions u_α are exponential functions $e^{\sqrt{\lambda_\alpha} x}$, $\mathrm{Re}\,\sqrt{\lambda_\alpha} < 0$, then the condition (of uniform minimality)

$$\sup \|v_\alpha\|_{L^2(K)} \|u_\alpha\|_{L^2(G)} < \infty$$

is equivalent to the Riesz basis property of the system $(u_\alpha) = (e^{\sqrt{\lambda_\alpha} x})$, hence to c) (cf. [13]). Another theorem of type „$a/ \Leftrightarrow c/$" was obtained by V. A. IL'IN [10]. He considered the operator

$$L_y = y'' + p_1(x)y' + p_2(x)y \quad p_1 \in W_1^1(G),\ p_2 \in L^1(G)$$

on a finite interval G, and a complete and minimal system (y_k) of eigenfunctions (usual and of higher order) of L. He proved that if the biorthogonal system (z_k) consists of eigenfunctions (of order ≥ 0) of the adjoint operator L^* and if some conditions about the distribution of (λ_k) are fulfilled, then the uniform minimality is equivalent to the Riesz basis property of (y_k).

In the case of exponentials in a finite interval, the question of the equiconvergence with the trigonometric Fourier series was investigated by R. PALEY and N. WIENER [8]. They proved that if

$$|\lambda_n - n| \leq D < \frac{1}{\pi^2}, \quad n \in \mathbf{Z}$$

then $(e^{i\lambda_n x})$ is complete and minimal in $L^2(0, 2\pi)$ and if we denote by (h_n) its biorthogonal system, the series

$$(*) \qquad \sum_{n=-\infty}^{\infty} \left[e^{inx} \frac{1}{2\pi} \int_0^{2\pi} f(\xi) e^{-in\xi} d\xi - e^{i\lambda_n x} \int_0^{2\pi} f(\xi) h_n(\xi) d\xi \right]$$

converges uniformly to zero over any interval $(\delta, 2\pi - \delta)$, $\delta > 0$. N. LEVINSON [9] obtained the same result for

$$|\lambda_n - n| \leq D < \frac{1}{4}, \quad n \in \mathbf{Z}.$$

In fact this inequality implies the Riesz basis property of $(e^{i\lambda_n x})$ (see [13]) so these are „c)⇒b)" type theorems.

A. M. SEDLETZ'KII ([12], p. 57, see also M. HORVÁTH [14]) generalized these results to the case of any Riesz basis consisting of higher order exponentials $x^k e^{i\lambda_n x}$ if the distribution of the set (λ_n) is regular in some sense. He estimated also the rate of the convergence of the series near the boundary of the interval.

Next we mention the work [11] of V. A. IL'IN. He considered the operator

$$Lu = u^{(n)} + p_1(x) u^{(n-1)} + \ldots + p_n(x) u$$

and a system (u_k), complete and minimal in $L^1(G)$ of eigenfunctions of order ≥ 0 of L and he proved, roughly speaking, the following „a)⇔b)"-type theorem.

The condition

$$\|u_k\|_{L^1(K)} \cdot \|v_k\|_{L^\infty(G)} \leq C(K) < \infty$$

for all compact intervals $K \subset G$ is equivalent to the property that the difference of the trigonometric Fourier series and the series $\Sigma \langle f, v_n \rangle u_n$ of an arbitrary function $f \in L^1(G)$ tends to zero locally uniformly.

REMARK 2. The condition $|\operatorname{Im} \sqrt{\lambda_n}| \leq C$ used by SEDLETZ'KII and IL'IN is not necessary for the system $(e^{i\sqrt{\lambda_n} x})$ to be a Riesz basis in $L^2(0, a)$ for some

$a<\infty$. Theorem 3.4 in [13] gives a large variety of exponential bases which do not satisfy the condition $|\operatorname{Im}\sqrt{\lambda_n}|\leq C$.

REMARK 3. As in [3], we prove the equiconvergence for L^2-functions using only some basis property of (u_k). In [7] an equiconvergence theorem is proved for L^1-functions, there the Riesz basis property of (u_k) had to be completed by the assumption that the biorthogonal system (v_k) consists of eigenfunction of the adjoint operator.

For the proof of the Theorem we need some lemmas.

Let us introduce for any $\mu\in\mathbf{C}$, $t>0$ the functions

$$f_1(\mu,t):=t\frac{\sin\mu t}{\mu},\quad f_i(\mu,t):=\int_0^t\frac{\sin\mu(t-\tau)}{\mu}f_{i-1}(\mu,\tau)d\tau$$

and for any $u\in L_\infty^{\mathrm{loc}}(G)$, $\mu\in\mathbf{C}$, $x\pm t\in G$, $t>0$

$$g_0(u,\mu,x,t):=\int_{x-t}^{x+t}\frac{\sin\mu(t-|x-\xi|)}{\mu}q(\xi)\,u(\xi)\,d\xi,$$

$$g_i(u,\mu,x,t):=\int_0^t\frac{\sin\mu(t-\tau)}{\mu}g_{i-1}(u,\mu,x,\tau)\,d\tau,$$

$$(i=1,2,\ldots).$$

LEMMA 1. [3]. Let u_m be an eigenfunction of order $\leq m$ of the operator L with the eigenvalue $\lambda=\mu^2$ and put $u_{j-1}:=\lambda u_j-Lu_j$ $(j=1,2,\ldots,m)$. Then for $x+t\in G$, $t>0$

$$u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t=$$

$$=\sum_{i=1}^m f_i(\mu,t)u_{m-i}(x)+\sum_{i=0}^m g_i(u_{m-i},\mu,x,t). \tag{6}$$

For any fixed compact interval $K\subset G$, $x\in K$ and $R\in(0,\operatorname{dist}(K,\partial G))$ define the functions

$$w_R(x+t,\nu):=\begin{cases}\sin\nu t/\pi t, & |t|\leq R\\ 0, & |t|>R\end{cases};\quad w(t+x,\nu):=D_{R_0}w_R$$

where

$$D_{R_0}f:=\frac{2}{R_0}\int_{R_0/2}^{R_0}f(R)\,dR,\quad 0<R_0<\operatorname{dist}(K,\partial G).$$

Jsing (6) with the notation

$$\delta(\nu, \nu_a) := \begin{cases} 1 & \nu_a < \nu \\ 0{,}5 & \nu_a = \nu \\ 0 & \nu_a > \nu \end{cases}$$

we get

$$(7) \qquad \langle u_a, w\rangle - \delta(\nu, \nu_a)u_a(x) = D_{R_0}\left(\int_0^R \frac{\sin \nu t}{\pi t}[u_a(x+t)+u_a(x-t)]\,dt\right) -$$

$$-\delta(\nu, \nu_a)u_a(x) = D_{R_0}\left(\int_0^R \frac{\sin \nu t}{\pi t}[u_a(x+t)+u_a(x-t)-2u_a(x)\operatorname{ch}\mu_a t]\,dt\right) +$$

$$+D_{R_0}\left(\int_0^R \left[\frac{2\sin \nu t \operatorname{ch}\mu_a t}{\pi t} - \delta(\nu, \nu_a)\right]dt\right)u_a(x) =$$

$$= \sum_{j=1}^m D_{R_0}\left(\int_0^R \frac{\sin \nu t}{\pi t} f_j(\mu_a, t)dt\right)u_{a-j}(x) +$$

$$+\sum_{j=0}^m D_{R_0}\left(\int_0^R \frac{\sin \nu t}{\pi t} g_j(u_{a-j}, \mu_a, x, t)\,dt\right) +$$

$$+D_{R_0}\left(\int_0^R \left[\frac{2\sin \nu t \operatorname{ch}\mu_a t}{\pi t} - \delta(\nu, \nu_a)\right]dt\right)u_a(x) =$$

$$= \sum_{j=1}^m A_j + \sum_{j\ 0}^m B_j + D,\ 0 < R_0 < \operatorname{dist}(K, \partial G).$$

In [7] the following estimates are proved:

$$(8) \qquad |A_j| \le C\frac{e^{\varrho_a R_0}}{1+(\nu-\nu_a)^2}\bullet\frac{|u_{a-j}(x)|}{(1+|\mu_a|)^j},\ (j = 1, \ldots, m);$$

$$|B_j| \le C\min\left\{\frac{1}{\nu}, \frac{\nu}{(1+|\mu_a|)^2}\right\}\int_{x-R_0}^{x+R_0}\left|q(\xi)\ln\frac{1}{|x-\xi|}\right|d\xi\,\cdot$$

$$\cdot\frac{\|u_{a-j}\|_{L_\infty(x-R_0,\,x+R_0)}\,e^{\varrho_a R_0}}{(1+|\mu_a|)^j},\ (j = 0, 1, \ldots, m);$$

$$|D| \leq C \frac{(1+\varrho_\alpha)e^{\varrho_\alpha R_0}}{1+(\nu-\nu_\alpha)^2} |u_\alpha(x)|.$$

LEMMA 2. ([4]). Let u be an arbitrary eigenfunction of order $\leq m$ of the operator L with the eigenvalue $\lambda = \mu^2$. Then for any compact intervals K, $K' \subset G$, $K' \subset \operatorname{int} K$, there exists a constant C depending on K, K' and m, but independent of u and λ such that:

$$\|u\|_{L_\infty(K)} \leq C \|u\|_{L_\infty(K')}, \tag{9}$$

$$\|u^*\|_{L_\infty(K)} \leq C(1+|\mu|)\|u\|_{L_\infty(K')}. \tag{10}$$

LEMMA 3. ([5]). For any compact intervals K, $K' \subset G$, $K \subset \operatorname{int} K'$ there exist constants R, $C > 0$ depending on K, K' and m, but independent of u, and λ such that for any eigenfunction u of order $\leq m$ of L with the eigenvalue $\lambda = \mu^2$ we have

$$\|u\|_{L_\infty(K)} \leq Ce^{-\varrho R}\|u\|_{L_\infty(K')}, \ (\varrho = |\operatorname{Re} \mu|). \tag{11}$$

PROOF OF THE THEOREM.

a)$\Rightarrow$ b): Fix a compact interval $K \subset G$ and a number $R_0 > 0$ small enough with $0 < R_0 < \operatorname{dist}(K, \partial G)$. By Lemmas 2 and 3 there exists $C > 0$ and a compact interval K' such that

$$\|u_\alpha\|_{L_\infty(K_{R_0})} \leq Ce^{-2\varrho_\alpha R_0}\|u_\alpha\|_{L_2(K')} \ (\forall \alpha) \tag{12}$$

where

$$K_{R_0} := \{x \in G: \operatorname{dist}(x, K) \leq R_0\}.$$

First we prove

$$\sup_{\nu > 0} \sup_{x \in K} \sum_{\alpha=1}^{\infty} |\langle u_\alpha, w\rangle - \delta(\nu, \nu_\alpha) u_\alpha(x)| \cdot \|v_\alpha\|_{L_2(G)} < \infty. \tag{13}$$

Fix the index α and the number $\nu > 0$ arbitrarily, and let $x \in K$. We shall denote by C any constant, independent of α, ν and x, but not necessarily the same at different occurrences. Assume that the eigenfunctions are numerated such that

$$u_j = u^*_{j-1} (\alpha - m \leq j < \alpha).$$

By (12) and by Lemma 2 we may suppose

$$\frac{\|u_{\alpha-j}\|_{L_\infty(K_{R_0})}}{(1+|\mu_\alpha|)^j} \leq Ce^{-2\varrho_\alpha R_0}\|u_\alpha\|_{L_2(K')} \tag{14}$$

$$(j = 0, 1, \ldots, m).$$

By (7), (8). (14) we get

$$|\langle u_\alpha, w\rangle - \delta(\nu, \nu_\alpha)u_\alpha(x)| \leq C\left[\frac{1}{1+(\nu-\nu_\alpha)^2} + \min\left\{\frac{1}{\nu}, \frac{\nu}{1+|\mu_\alpha|^2}\right\}\right]\cdot$$

$$\cdot \|u_\alpha\|_{L_2(K')},$$

and hence, taking into consideration a) and (5) the desired estimate (13) follows, because

$$\sum_{\alpha=1}^{\infty}\left[1+(\nu-\nu_\alpha)^2\right]^{-1} \leq C\sum_{k=1}^{\infty} k^{-2}$$

and

$$\sum_{\alpha=1}^{\infty}\min\left\{\frac{1}{\nu}, \frac{\nu}{1+|\mu_\alpha|^2}\right\} \leq \sum_{\alpha=1}^{\infty}\min\left\{\frac{1}{\nu}, \frac{\nu}{1+\nu_\alpha^2}\right\} \leq$$

$$\leq \sum_{\nu_\alpha<\nu}\frac{1}{\nu} + \sum_{\nu_\alpha\geq\nu}\nu/(1+\nu_\alpha^2) \leq C.$$

Now we are in the position to prove the implication a)$\Rightarrow$b). According to (13), the series $\sum_{\alpha=1}^{\infty}(\langle u_\alpha, w\rangle - \delta(\nu-\nu_\alpha)u_\alpha(x))\cdot\bar{v}_\alpha$ converges absolutely in $L_2(G)$, its sum is equal to

$$\bar{w} - \sum_{\nu_\alpha<\nu} u_\alpha(x)\bar{v}_\alpha - \frac{1}{2}\sum_{\nu_\alpha=\nu} u_\alpha(x)\bar{v}_\alpha,$$

since this function and the sum have the same Fourier coefficients with respect to the complete system (u_α). Consequently we get from (13)

$$\sup_{\nu>0}\sup_{x\in K}\|\bar{w} - \sum_{\nu_\alpha<\nu} u_\alpha(x)\bar{v}_\alpha - \sum_{\nu_\alpha=\nu} u_\alpha(x)\bar{v}_\alpha\|_{L_2(G)} < \infty. \tag{15}$$

It is easy to see that

$$\sup_{\nu>0}\sup_{x\in K}\|\bar{w} - w_R\|_{L_2(G)} > \infty, \tag{16}$$

and finally from (12),

$$\sup_{\alpha}\frac{\|u_\alpha\|_{L_\infty(K)}}{\|u_\alpha\|_{L_2(K')}} < \infty.$$

Hence, according to a) and (5), we obtain

$$\sup_{\nu>0}\sup_{x\in K}\|\sum_{\nu_\alpha=\nu}^{*} u_\alpha(x)\bar{v}_\alpha\|_{L_2(G)} < \infty, \tag{17}$$

i. e.

$$M := \sup_{\nu>0}\sup_{x\in K}\|w_R - \sum_{\nu_\alpha<\nu} u_\alpha(x)v_\alpha - \sum_{\nu_\alpha=\nu}^{*} u_\alpha(x)v_\alpha\|_{L_2(G)} < \infty.$$

Thus for every $f \in L_2(G)$ we have

$$\sup_{\nu>0} \sup_{x \in K} |S_\nu(f, x) - \sigma_\nu(f, x)| \leq M \|f\|_{L_2(G)}. \tag{18}$$

On the other hand, if f is a finite linear combination of the u_α, then

$$\lim_{\nu \to \infty} \sup_{x \in K} |S_\nu(f, x) - \sigma_\nu(f, x)| = 0$$

and the property b) follows by the Banach-Steinhaus theorem. □

b) ⇒ c): It follows from b) that

$$\lim_{\nu \to \infty} \|S_\nu(f) - \sigma_\nu(f)\|_{L_2(K)} = 0.$$

On the other hand it is a classical result that

$$\lim_{\nu \to \infty} \|f - S_\nu(f)\|_{L_2(K)} = 0. \quad \square$$

The proof of c) ⇒ a) is standard. The Theorem is proved. □

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ARRANGEMENTS OF 22 CIRCLES ON A SPHERE

By

JOHN LEECH and TIBOR TARNAI

Department of Computing Science, University of Stirling and Hungarian Institute for Building Science, Budapest

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1. Introduction

A recent paper [8] has presented an account of investigations of the problem of determining the largest angular diameter d_n of n equal circles which can be packed on a sphere without overlapping. That paper surveyed the more recent results for several values of n but did not mention the case $n = 22$, though the packing problem for 22 circles is unusually interesting, with a surprising number of rigid configurations.

A good arrangement of 22 equal circles on a sphere, with circle diameter $d_{22} = 44°24'11.3''$, was published by STROHMAJER [6]. The graph of this packing can be seen in a simplified stereographic projection in Fig. 1. (The vertices of the graph are the centres of the spherical circles and the edges of the graph are great-circle arcs joining the centres of the touching spherical circles.) Later, in an unpublished note, ROBINSON [5] described a better packing for 22 circles, in which the circle diameter is $d_{22} = 44°35'22.8''$. Robinson's

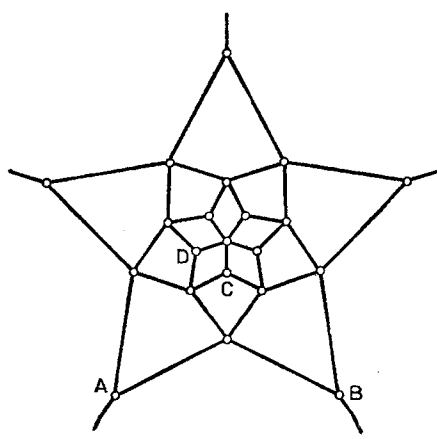

Fig. 1.

arrangement, however, is not unique since it contains hexagons with an inner point having two alternative positions similar to that in Fig. 5(b). (We mention here that Fig. 11 of paper [9] gives Robinson's configuration for 22 circles if the great triangle in it is considered a face of the regular tetrahedron.) Our Fig. 2 (disregarding a reflection in the plane of the equator)

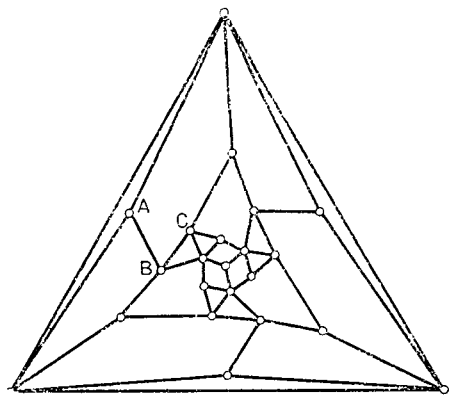

Fig. 2.

shows the graph of the version of Robinson's packing that was presented by GOLDBERG [3]. In the same paper GOLDBERG also gave a new packing with circle diameter $d_{22} = 45°01'$, whose graph can be seen in Fig. 4. This configuration seems to be an improvement of his previous arrangement [2] having fourfold axial symmetry and circle diameter $d_{22} = 43°43'22.7''$ (Fig. 3). We attempted to reproduce Goldberg's configuration in Fig. 4, but we were unsuccessful. This failure convinced us that Goldberg's result is faulty, and that until now the best valid result known was that of ROBINSON [5].

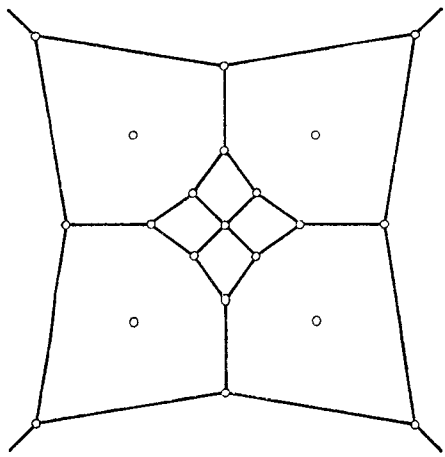

Fig. 3.

The aim of this paper is to point out the mistake in Goldberg's calculations, to correct his result as much as possible, and to present packings of 22 equal circles better than Robinson's packing.

2. Trials for reproduction of Goldberg's figure

It is easy to see that Goldberg's result is erroneous. First, to simplify the calculations, we consider exactly 45° as edge length in Goldberg's configuration in Fig. 4 instead of his value of 45°01′. If we had a centre at (say) the north pole, four centres at latitude 45°N and longitudes (say) 0°, 90°, 180°, 270°, the next four circles, touching pairs of these circles, would have centres at longitudes 45°, 135°, 225°, 315° and latitude arcsin 1/3 = 19° 28′16.4″, considerably different from his figure of 22°02′52″. With similar points in the southern hemisphere, displaced in longitude to maintain the separation, it is not possible to insert the final equatorial set.

A more accurate computation confirms this statement. From the fourfold symmetry of the figure and the fact that the vertex D of triangles lies on the equator, we find that the edge length in the configuration is d_{22} = 45°03′ 11.5″ and that the coordinates of the point B in Fig. 4 are: latitude 44°56′ 48.5″N and longitude 41°30′35.7″E instead of latitude 44°59′00″N and longitude 43°04′46″E published in Table 1 of [3]; while the coordinates of point C in Fig. 4 are: latitude 19°22′15.3″N and longitude 3°29′24.3″W instead of latitude 22°02′52″N and longitude 4°38′51″W published in Table 1 of [3]. However, the distance between the points C and E is only 39°20′30.6″, substantially less than the computed edge length. This means that the circles centred at points C and E overlap, as do their symmetric replicas. Therefore Fig. 4 cannot be the graph of a packing of 22 equal circles on a sphere.

The network in Fig. 4 is overdeterminate; it contains $e = 44$ edges and $n = 22$ vertices. In a general case – disregarding the symmetry of the fig-

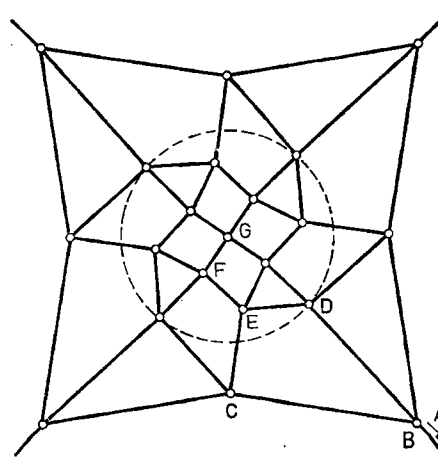

Fig. 4.

ure – a necessary condition of determinacy and rigidity of the graph in Danzerian sense is that the graph contains $e = 2n-2$ edges [9]. In our case it needs removal of two edges from Fig. 4. The question is whether an arrangement better than Robinson's can be obtained by removing two edges, and which two edges should be removed. It is difficult to answer this question directly, so we shall do it indirectly by improvement of Goldberg's basic arrangement in Fig. 3 instead.

Consider Fig. 3, in which a circle is placed at the north pole, another at the south pole, and the north-south axis is an axis of fourfold rotational symmetry. In this case, since the graph is not rigid in Danzerian sense, the northern hemisphere can be rotated relative to the southern one, with an increase

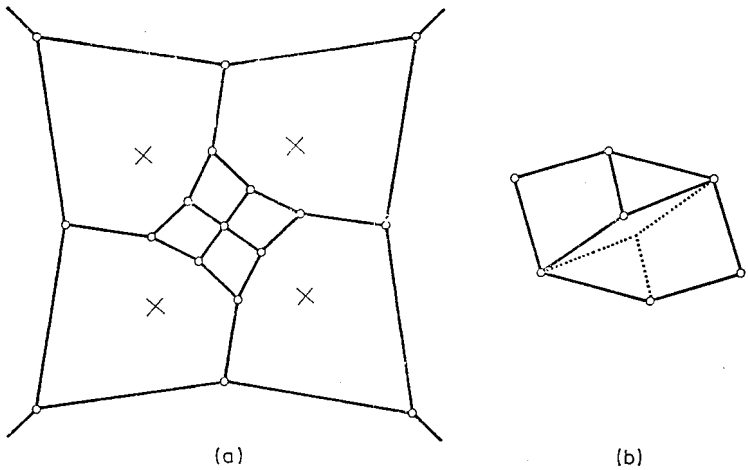

Fig. 5.

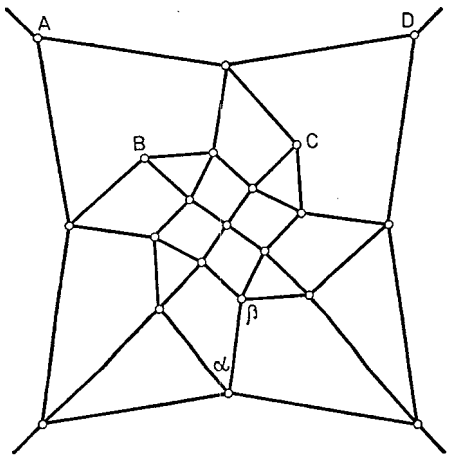

Fig. 6.

in the edge length. This rotation is continued until we reach the arrangement, the main part of the graph of which can be seen in Fig. 5(a). This rotation removes the isolation of the four isolated points, and the edges appearing in the hexagons will be positioned as shown in Fig. 5 (b). (The picture shows the hexagon in equatorial view.) Here the continuous and dotted lines lying inside the hexagon show alternative central symmetric possibilities of locating the 3 new edges which appear in a hexagon. In this way four equally good alternative packings with circle diameter $d_{22} = 44°00'54.2''$ are obtained. Their graphs have $e = 40$ edges, so they are not rigid in Danzerian sense. Therefore, by deforming each graph, these alternatives can be further improved until two additional edges appear in the graph. From the four alternatives, this operation results in altogether three different configurations. The first one in Fig. 6 with circle diameter $d_{22} = 44°06'34.2''$ is unsymmetric. The second one in Fig. 7 with circle diameter $d_{22} = 44°06'34.6''$ and the third one in Fig. 8 with circle diameter $d_{22} = 44°06'32.1''$ have twofold rotational symmetry. The two black spots on edges in Figs 7 and 8, and also in Fig. 9 below, indicate the axis of rotational symmetry. It is easy to see that each edge in Figs 6, 7, 8 is an edge in Fig. 4. However, the graphs in Figs 6, 7, 8 are rigid in Danzerian sense, so distances AB and CD there cannot be decreased to the edge lengths, and so the segments AB and CD, which would complete the figures, cannot belong to the graphs. This fact also demonstrates that the arrangement in Fig. 4 is not possible. These investigations also show that this attempt to produce packings better than Robinson's arrangement is not successful.

3. New arrangements

In order to find new arrangements, we consider first Strohmajer's fivefold symmetric configuration in Fig. 1. Its graph contains no triangles, so it is not rigid in Danzerian sense, and therefore this packing is improvable

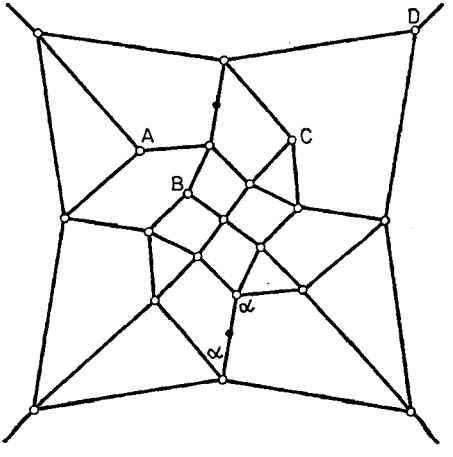

Fig. 7.

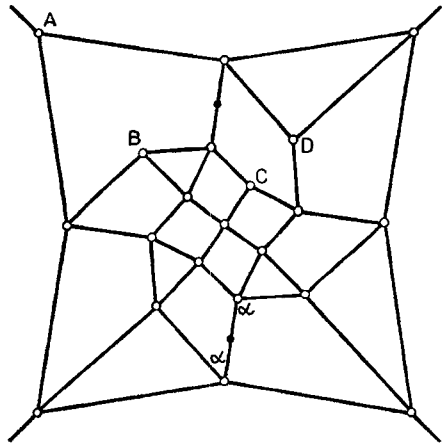

Fig. 8.

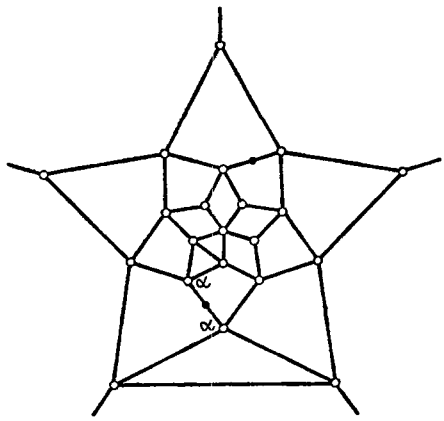

Fig. 9.

by deforming the graph [9]. Displace the 5-valent vertices away from being diametrically opposed until the diagonals AB and CD become joins. In this manner the arrangement in Fig. 9 is obtained, which has twofold symmetry with axis through the points marked $\leftarrow$, and circle diameter $d_{22} = 44°30'$ $51.1''$. This result, however, is inferior to Robinson's result.

Next, we consider another rearrangement of Goldberg's configuration. The graph of Goldberg's basic arrangement in Fig. 3, as stated above, is not rigid in Danzerian sense and so is improvable. The four edges perpendicularly intersecting the equator can be displaced along their lengths, equally but in alternate directions, until two opposite quadrangles at the north pole (and similarly at the south pole) are deformed into pairs of triangles. Since the

four edges have been kept perpendicular to the equator, the northern hemisphere can still be rotated relative to the southern one. This rotation is continued until the isolated points in two opposite hexagons cease to be isolated; the isolation of the points in the other two hexagons is preserved. The new edges appearing in the two opposite hexagons can be located in two central symmetric alternative positions, so we can obtain two different, equally good, Danzerian rigid configurations. The result of this is to enlarge the circle diameter to $d_{22} = 44°36'08.2''$. Fig. 10 shows the one of the two arrangements, which has twofold rotational symmetry. This result is superior to Robinson's result but not so good as the next one.

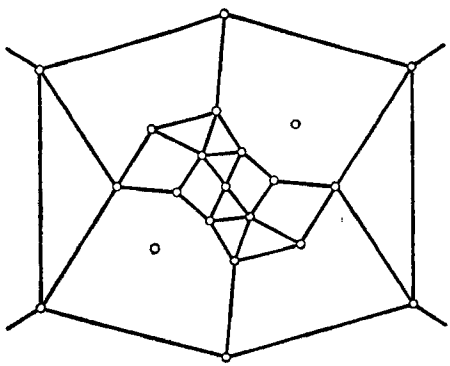

Fig. 10.

We now consider modifying Robinson's arrangement. Robinson's configuration in Fig. 2 is rigid in Danzerian sense but, after removal of edge AB and its symmetric replicas from the graph, it can be improved by deforming it through a slightly worse configuration to a better one. Viewed from the north polar direction we have an equilateral triangle surrounding the pole, with a further equilateral triangle on each side of it. There is one further join from each of these six points. That from any vertex of the first triangle is at a bearing only very slightly west of south; its further end has a longitude about 2°50′ west of its northern end. If the six northern vertices are rotated to the west so as to take these joins through the vertical, we can continue the rotation until the further end is at a longitude about $9°27\frac{2'}{3}$ east of its northern end, and enlarge the separation to $d_{22} = 44°39'16.2''$, before further joins make it rigid again. That is, Fig. 2 is modified by rotation so that, preserving the threefold rotational symmetry of the figure, vertices A and C will be joined by an edge instead of vertices A and B. In this manner the configuration in Fig. 11 is obtained. This result is the best we have found. It should be noted that – since Robinson's configuration is not unique because of the alternative positions of the interior points in the hexagons – the improving procedure described here can also be executed if the position of

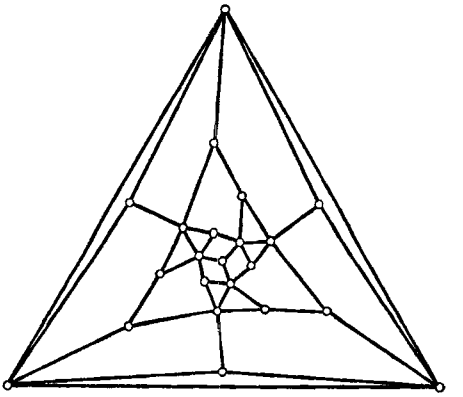

Fig. 11.

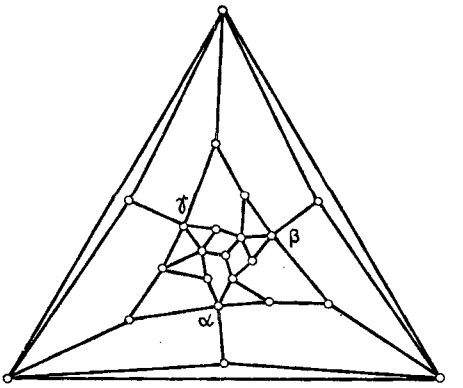

Fig. 12.

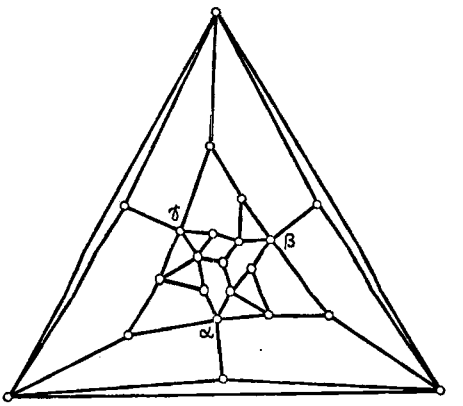

Fig. 13.

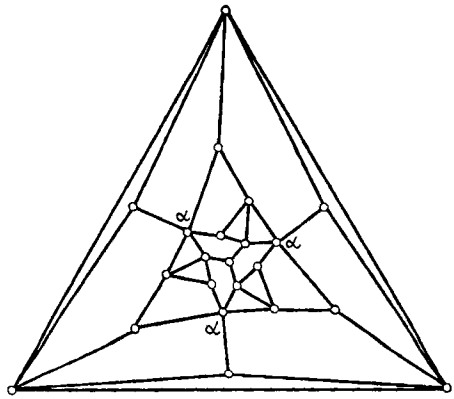

Fig. 14.

the interior points in some of the three hexagons around the south pole in Fig. 2 is changed. If the interior point is changed in one or two or all three hexagons we obtain arrangements in Fig. 12 or Fig. 13 or Fig. 14 with circle diameter $d_{22} = 44°39'15.5''$ or $d_{22} = 44°39'14.9''$ or $d_{22} = 44°39'14.4''$, respectively. One can see that these figures are very close to the figure obtained for the configuration in Fig. 11 but marginally inferior. The last arrangement retains threefold rotational symmetry, the others have no symmetry, but the packings in all of the three cases constitute rigid configurations. Although these results give no improvement to that in Fig. 11 they are worth mentioning since they confirm that there are a surprising number of rigid configurations to be considered.

4. Conclusions

The numerical data of the known and the newly discovered packings of 22 circles are collected in Table I.

Here the number of contacts, that is, the number of the edges of the graph is given in column e (if a graph has isolated points then two edges are reckoned to fix an isolated point). The number e-$2n+3$ characterizing rigidity is also given; it was shown in [9] that if this number is negative or zero the graph is not rigid in Danzerian sense, and such a packing is, in general, improvable by deforming the graph.

The symmetry of the configuration is also presented, but the applied symbols indicating the symmetry of the configuration are not commonly used since, in general, they do not denote the symmetry group of the configuration, but they are useful to reconstruct the configuration. The following notation is used. Consider two axes perpendicularly intersecting each other, one of them is (say) the north-south axis. If the configuration has k-fold rotational symmetry about this axis then it is denoted by C_k, and if moreover it has a plane of symmetry passing through this axis then it is

Table 1.

Packings of $n = 22$ equal circles on the sphere

graph	diameter	e	$e-2n+3$	symmetry	reference or remark
Fig. 1[a]	44°24′11.3″	40	−1	D_5C_2	STROHMAJER [6]
Fig. 2[b]	44°35′22.8″	42	1	C_3	ROBINSON [5]
Fig. 3[a]	43°43′22.7″	36	−5	D_4D_2	GOLDBERG [2]
Fig. 4[c]	(45°01′)	44	3	C_4C_2	GOLDBERG [3]
Fig. 5[ab]	44°00′54.2″	40	−1	C_4, unsymm. C_2, C_2C_2	improvement of Fig. 3
Fig. 6	44°06′34.2″	42	1	unsymm.	derived from Fig. 5
Fig. 7	44°06′34.6″	42	1	C_2	derived from Fig. 5
Fig. 8	44°06′32.1″	42	1	C_2	derived from Fig. 5
Fig. 9	44°30′51.1″	42	1	C_2	improvement of Fig. 1
Fig. 10[b]	44°36′08.2″	42	1	C_2	improvement of Fig. 3
Fig. 11	44°39′16.2″	42	1	C_3	modification of Fig. 2
Fig. 12	44°39′15.5″	42	1	unsymm.	modification of Fig. 2
Fig. 13	44°39′14.9″	42	1	unsymm.	modification of Fig. 2
Fig. 14	44°39′14.4″	42	1	C_3	modification of Fig. 2

[a] The graph is not rigid in Danzerian sense.
[b] The configuration is not unique due to alternative possibilities of locating the 3 new edges which appear in a hexagon.
[c] This result is not correct.

denoted by D_k. If, in addition, the configuration has m-fold rotational symmetry with respect to the axis perpendicular to the north-south axis, then it is denoted by C_kC_m or D_kC_m. If moreover the plane of the equator is a plane of symmetry, then we use notation C_kD_m or D_kD_m.

Table 2.

Angles in the graphs

graph	α	β	γ
Fig. 6	88°02′11.7″	87°08′05.3″	
Fig. 7	87°13′43.1″		
Fig. 8	88°07′46.7″		
Fig. 9	98°49′36.9″		
Fig. 12	69°09′13.5″	69°07′56.7″	69°07′59.3″
Fig. 13	69°09′12.2″	69°09′19.8″	69°08′08.5″
Fig. 14	69°09′18.5″		

In order to facilitate reproduction and checking of our results, we give in Table 2 one or two or three angles of the graph, whose determination is somewhat difficult.

From the data of Table 1 it can be ascertained that our best arrangement is that of Fig. 11 where the circle diameter is

$$d_{22} = 44°39'16.2''.$$

Finally, in correcting Goldberg's result, we should mention surveys of SZÉKELY [7], MELNYK, KNOP and SMITH [4], CLARE and KEPERT [1] and a density diagram of TARNAI [8], which include Goldberg's result and which therefore need correction also.

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NOTE ADDED IN PROOF. While this paper was in press an arrangement of 22 equal circles on a sphere with circle diameter $d_{22} = 44°44'24.6''$ (better than any of ours) was published by LAZIĆ et al. [10]. That arrangement can be obtained by modification of the graph in our Fig. 10.

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ON THE STABILITY OF A CLASS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECT IN A BANACH SPACE

By

P. P. ZABREIKO – D. D. BAINOV – S. I. KOSTADINOV

Belorussian State University and University of Plovdiv

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1. Introduction

The equations with impulse effect are models of real processes which are subject to disturbances in its evolution acting in time very short compared with the entire duration of the process. Although there are numerous applications of these equations in physics, biology, chemistry etc., their theory is developed relatively slowly because of their difficult nature.

The first paper in this direction is the paper by MILMAN and MYSHKIS [1] (1960). The mathematical theory of ordinary differential equations with impulse effect has been further developed in the papers [2]–[4]. The paper [5] initiates the investigations of the equations with impulse effect in a Banach space.

2. Statement of the problem

Consider the differential equation with impulse effect of the form

$$\frac{dx}{dt} = A(t)x|_{t\neq t_n} \tag{1}$$

$$x(t_n+0) = Q_n x(t_n-0) \qquad (n = 1, 2, \ldots), \tag{2}$$

where $A(t)$ $(t\geq t_0)$ is a continuous operator-function on t, the values of which are linear bounded operators mapping the compex Banach space X into itself. The operators $Q_n:X\to X(n = 1, 2, \ldots)$ are continuous and the moments t_n $(n = 1, 2, \ldots)$ of the impulse effect satisfy the condition

$$t_0<t_1<\ldots, \lim_{n\to\infty} t_n = \infty.$$

DEFINITION 1. We say that $x(t)$ is a solution of the equation with impulse effect (1), (2) if $x(t)$ is a piecewise continuous function with discontinuities of the first type at $t = t_n$ such that

$$\frac{dx}{dt} = A(t)x(t)$$

for $t \neq t_n$ $(t \geq t_0)$ and

$$x(t_n+0) = Q_n x(t_n-0), \quad n = 1, 2, \ldots.$$

At the points of discontinuity $t_n (n = 1, 2, \ldots)$ we assume that the function $x(t)$ is continuous from the left.

For $x_0 \in X$ and $t \geq t_0$ the equation with impulse effect (1), (2) has unique solution $x(t)$ which satisfies the condition

$$x(t_0) = x_0. \tag{3}$$

Then we can consider the family of operators $V(t)$ defined by the formula

$$V(t)x_0 = x(t) \qquad (t_0 \leq t < \infty).$$

Denote by $U(t, \tau)$ the evolution operator of the equation

$$\frac{dx}{dt} = A(t)x. \tag{4}$$

We note that for $t_n < t \leq t_{n+1}$ the following equality holds

$$V(t) = U(t, t_n)\, Q_n U(t_n, t_{n-1})\, Q_{n-1} \ldots U(t_2, t_1)\, Q_1 U(t_1, t_0).$$

DEFINITION 2. The equation with impulse effect (1), (2) is called stable if

$$\sup_{t_0 \leq t < \infty} \|V(t)\| < \infty.$$

DEFINITION 3. The equation with impulse effect (1), (2) is called asymptotically stable if

$$\lim_{t \to \infty} \|V(t)x\| = 0 \quad (x \in X).$$

DEFINITION 4. The equation with impulse effect (1), (2) is called uniformly asymptotically stable if

$$\lim_{t \to \infty} \|V(t)\| = 0.$$

LEMMA 1. Let the following condition hold

$$A(t) \int_{t_0}^{t} A(s)ds = \int_{t_0}^{t} A(s)ds \cdot A(t) \quad (t \geq t_0). \tag{5}$$

Then the solution $x(t)$ of (4) for $t \geq t_0$ with initial condition (3) has the form

$$x(t) = e^{\int_{t_0}^{t} A(s)ds} x_0.$$

The proof of Lemma 1 is standard.

REMARK 1. For small $t \geq t_0$ the condition (5) is sufficient as well for the solution of (4) to have the form

$$x(t) = e^{\int_{t_0}^{t} A(s)ds} x_0.$$

In fact, let $x(t) = e^{\int_{t_0}^{t} A(s)ds} x_0$. The equality

$$\left(e^{\int_{t_0}^{t} A(s)ds}\right)' = A(t)e^{\int_{t_0}^{t} A(s)ds}$$

can be presented as

$$\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)!}\left(\left(\int_{t_0}^{t} A(s)ds\right)^{n+1}\right)' - \frac{1}{n!}A(t)\left(\int_{t_0}^{t} A(s)ds\right)^{n}\right) = 0.$$

We set $\Delta(t) = A(t)\int_{t_0}^{t} A(s)ds - \int_{t_0}^{t} A(s)ds\, A(t)$.

Then the following equalities hold:

$$\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)!}\left(\sum_{k=0}^{n}\left(\int_{t_0}^{t} A(s)ds\right)^{k} A(t)\left(\int_{t_0}^{t} A(s)ds\right)^{n-k} - \right.\right.$$

$$\left.\left. - (n+1)A(t)\left(\int_{t_0}^{t} A(s)ds\right)^{n}\right)\right) =$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{k=1}^{n}\left(\left(\int_{t_0}^{t} A(s)ds\right)^{k} A(t)\left(\int_{t_0}^{t} A(s)ds\right)^{n-k} - \right.$$

$$\left. - A(t)\left(\int_{t_0}^{t} A(s)ds\right)^{n}\right) = \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\sum_{k=1}^{n} k\left(\int_{t_0}^{t} A(s)ds\right)^{n-k}\Delta(t).$$

$\left(\int_{t_0}^{t} A(s)ds\right)^{k-1} = 0$, i. e.

$$\frac{1}{2}\Delta(t) = -\sum_{n=2}^{\infty}\frac{1}{(n+1)!}\sum_{k=1}^{n} k\left(\int_{t_0}^{t} A(s)ds\right)^{n-k}\Delta(t)\left(\int_{t_0}^{t} A(s)ds\right)^{k-1}.$$

Finally we get

$$\frac{1}{2}\|\Delta(t)\| \leq \sum_{n=2}^{\infty} \frac{\|\Delta(t)\|}{(n+1)!} \frac{k(k+1)}{2} \left\| \int_{t_0}^{t} A(s)ds \right\|^{n-1} =$$

$$= \frac{\|\Delta(t)\|}{2} (e^{\|\int_{t_0}^{t} A(s)ds\|} - 1).$$

Then for sufficiently small t

$$|e^{\|\int_{\tau}^{t} A(s)ds\|} - 1| < 1$$

and there exists an interval $[t_0, t^*]$ such that $\Delta(t) = O(t \in [t_0, t^*])$.

Applying the step method we come to the conclusion that the equality $\Delta(t) = 0$ holds as well if t belongs to an interval containing $[t_0, t^*]$.

3. Main result

We say that te condition (A) holds if the following condition is fulfilled:

$$(A) \int_{\tau}^{t} A(s)ds \cdot A(t) = A(t) \int_{\tau}^{t} A(s)ds \quad (t_0 < \tau < t < \infty).$$

REMARK 2. Let the condition (A) hold. Then for $t_0 < \tau < t < \infty$ the following equality holds

$$A(t)\ A(\tau) = A(\tau)\ A(t).$$

We say that the conditions (B) are satisfied if:

B1. Condition (A) holds.

B2. There exists a constant $T > 0$ for which $t_{n+1} - t_n \leq T$ for $n = 1, 2, \ldots$.

B3. $\|e^{\int_{\tau}^{t} A(s)ds}\| \leq e^{\gamma(t-\tau)}$ for $0 \leq t - \tau \leq T$

where γ is a constant.

B4. $\|Q_n x\| \leq q_n \|x\| + h_n$ for $n = 1, 2, \ldots$ and $x \in X$ so that $q_1 q_2 \ldots q_{n(t)} \leq$ $\leq Le^{\delta t}$ where $n(t) = n$ for $t_n \leq t < t_{n+1}$ and L and δ are constants.

If the conditions (B) hold, for the solution $x(t)$ of the equation with impulse effect (1), (2) we obtain the following estimation

$$\|x(t)\| \leq e^{\gamma(t-t_n)}\{q_n \|e^{\int_{t_{n-1}}^{t_n} A(s)ds} Q_{n-1} e^{\int_{t_{n-2}}^{t_{n-1}} A(s)ds} Q_{n-2} \ldots Q_1 e^{\int_{t_0}^{t_1} A(s)ds} x_0\| + h_n\} \leq$$

$$\leq e^{\gamma(t-t_{n-1})}q_n\{q_{n-1}\|e^{\int_{t_{n-2}}^{t_{n-1}} A(s)ds} Q_{n-2}\cdots Q_1 e^{\int_{t_0}^{t_1} A(s)ds} x_0\| + h_{n-1}\} + e^{\gamma(t-t_n)}h_n \leq$$

$$\leq e^{\gamma(t-t_{n-2})}q_n q_{n-1}\|Q_{n-2}\cdots Q_1 e^{\int_{t_0}^{t_1} A(s)ds} x_0\| + e^{\gamma(t-t_{n-1})}q_n h_{n-1} + e^{\gamma(t-t_n)}h_n \leq$$

$$\leq \ldots \leq e^{\gamma(t-t_0)}q_n q_{n-1}\cdots q_1\|x_0\| + \{e^{\gamma(t-t_n)}h_n + e^{\gamma(t-t_{n-1})}q_n h_{n-1} + \tag{6}$$

$$+ e^{\gamma(t-t_{n-2})}q_n q_{n-1} h_{n-2} + \ldots + e^{\gamma(t-t_1)}q_n q_{n-1}\cdots q_2 h_1\}.$$

Using this estimation we will find sufficient conditions under which the solutions of the equation with impulse effect are bounded.

THEOREM 1. *Let the conditions* (B) *hold. Let* $h_1 = h_2 = \ldots = 0$.

Then for $\delta+\gamma<0$, $\lim_{t\to\infty} x(t) = 0$ *and for* $\delta+\gamma = 0$ *the solution* $x(t)$ *is bounded for* $t\geq t_0$.

The proof of Theorem 1 follows immediately from the estimation (6).

THEOREM 2. *Let the conditions* (B) *be satisfied. Suppose additionally that*

$$q = q_1 = q_2 = \ldots;\ h = h_1 = h_2 = \ldots$$

and let

$$t_n = t_0 + nk\ (n = 1, 2, \ldots),\ t = t_0 + \tau k,$$

where $n\leq\tau\leq n+1$, $k\leq T$ *is a constant and the inequality*

$$qe^{\gamma k} < 1$$

holds. Then for $\delta+\gamma\leq 0$ *the solution* $x(t)$ *is bounded on the half axis* $t\geq t_0$.

PROOF. We denote by K the expression in the big brackets in (6) and after certain transformations we get

$$K = e^{\gamma(\tau-n)}\frac{q^n e^{\gamma kn} - 1}{qe^{\gamma k} - 1}h.$$

From the assumtions it follows that K is bounded. This conclusion combined with the condition B4 completes the proof.

THEOREM 3. *Suppose that the following conditions hold:*

1. *The condition* (A) *is fulfilled.*

2. *There exist constants* $p\geq 0$ *and* $\sigma>0$ *such that* $0\leq i(t_0, t)-p(t-t_0)\leq\sigma$ *for* $0\leq t_0\leq t<\infty$ *where* $i(a, b)$ *is the number of the ponts* t_n *which lie in the interval* (a, b) $(t_0<a<b<\infty)$.

3. $Q = Q_1 = Q_2 = \ldots$ *where theo perator* Q *is linear and commutes with* $\int_\tau^t A(s)ds$ *for* $t_0\leq\tau\leq t<\infty$.

4. *The operator* Q *has a logarithm, i. e. there exists a linear and bounded operator* $\ln Q: X\to X$ *such that* $Q = e^{\ln Q}$.

Then the equation with impulse effect is bounded, asymptotically bounded, uniformly asymptotically bounded if and only if respectively

$$\text{a)} \quad \sup_{t \geq t_0} \left\| e^{\int_{t_0}^{t} (A(s)+p \ln Q)ds} \right\| < \infty,$$

$$\text{b)} \quad \lim_{t \to \infty} \left\| e^{\int_{t_0}^{t} (A(s)+p \ln Q)ds} x \right\| = 0 \quad (x \in X),$$

$$\text{c)} \quad \lim_{t \to \infty} \left\| e^{\int_{t_0}^{t} (A(s)+p \ln Q)ds} \right\| = 0.$$

PROOF. The operator-function $V(t)$ has the form

$$V(t) = e^{\int_{t_0}^{t} A(s)ds} Q^{i(t_0, t)} = e^{\int_{t_0}^{t} A(s)ds} Q^{[p(t-t_0)]} Q^{i(t_0, t)-[p(t-t_0)]} =$$

$$= e^{\int_{t_0}^{t} A(s)ds} e^{\ln Q[p(t-t_0)]} Q^{i(t_0, t)-[p(t-t_0)]} =$$

$$(7) \qquad = e^{\int_{t_0}^{t} A(s)ds} e^{\ln Q p(t-t_0)} e^{\ln Q[p(t-t_0)]-\ln Qp(t-t_0)} Q^{i(t_0, t)-[p(t-t_0)]} =$$

$$= e^{\int_{t_0}^{t} (A(s)+\ln Qp)ds} e^{\ln Q\{[p(t-t_0)]-p(t-t_0)\}} Q^{i(t_0, t)-[p(t-t_0)]}.$$

Taking into account the representation (7) the assertions a), b) and c) follows immediately.

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ON STRONGLY PRIME AND SUPERPRIME GAMMA RINGS

By

G. L. BOOTH and N. J. GROENEWALD

Department of Mathematics, Univensity of Zululand and Department of Mathematics, University of Port Elizabeth

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1. Preliminaries

Let M and Γ be additive abelian groups. If for all $a, b, c \in M$, $\gamma, \mu, \nu \in \Gamma$, the following conditions are satisfied:

(i) $a\gamma b \in M$
(ii) $(a+b)\gamma c = a\gamma c + b\gamma c$
$a(\gamma+\mu)b = a\gamma b + a\mu b$
$a\gamma(b+c) = a\gamma b + a\gamma c$
(iii) $a\gamma(b\mu c) = (a\gamma b)\mu c$

then M is called a Γ-ring. If, in addition, we have that:

(i′) $\gamma a\mu \in \Gamma$
(ii′) $\gamma a(\mu+\nu) = \gamma a\mu + \gamma a\nu$
$\gamma(a+b)\mu = \gamma a\mu + \gamma b\mu$
$(\gamma+\mu)a\nu = \gamma a\nu + \mu a\nu$
(iii′) $\gamma a(\mu b\nu) = (\gamma a\mu)b\nu$

then M is called a *weak Γ_N-ring*, or a *weak Nobusawa Γ-ring*. A subgroup I of M is called a *left (right) ideal* of M if $M\Gamma I \subseteq I (I\Gamma M \subseteq I)$. If I is both a left and a right ideal of M then I is called an *ideal* of M, denoted $I \lhd M$. If $I, J \lhd M$, $I\Gamma J = 0$ implies $I = 0$ or $J = 0$, then M is called a *prime Γ-ring*. For any ideal I of M, the factor Γ-ring M/I is defined in the obvious way. If $P \lhd M$, and M/P is a prime Γ-ring, then P is called a *prime ideal* of M. If $I \lhd M$ and $0 \neq J \lhd M$ implies $I \cap J \neq 0$, then I is called an *essential ideal* of M. This will be denoted $I \lhd \cdot M$.

The definitions of a radical class of Γ-rings and of the upper radical determined by a class of Γ-rings are exactly as for the corresponding concepts for rings. Following HEYMAN and ROOS [9], a class $\mathcal{M}$ of Γ-rings is called a *special class* if

(i) $\mathcal{M}$ consists of prime Γ-rings;
(ii) $\mathcal{M}$ is hereditary, i. e. $M \in \mathcal{M}$, $I \lhd M$, implies $I \in \mathcal{M}$;
(iii) M a Γ-ring, $I \lhd \cdot M$, $I \in \mathcal{M}$ implies $M \in \mathcal{M}$.

The upper radical determined by a special class is called a *special radical.* It is easily verified that the intersection of an arbitrary family of special radical classes is again a special radical class.

For all other definitions and elementary properties of Γ-rings, we refer to [4] and [11]. We conclude this introduction by establishing a few lemmas which will be of use later.

LEMMA 1.1. ([6], Lemma 2.1). Let M be a Γ-ring, and let $I \lhd J \lhd M$. Then if $\bar{I}$ denotes the ideal of M generated by I

$$\bar{I}\Gamma\bar{I}\Gamma\bar{I} \subseteq I.$$

If A is a subset of the Γ-ring M, the left annihilator $l(A)$ of A is the subset

$$l(A) = \{x \in \mathcal{M} : x\Gamma A = 0\}.$$

The right annihilator $r(A)$ of A is similarly defined. It is easily seen that $l(A)$ (resp. $r(A)$) is a left (resp. right) ideal of M.

LEMMA 1.2. Let M be a prime Γ-ring and let $0 \neq I \lhd M$. Then $l(I) = r(I) = 0$.

The proof of this lemma is similar to that of the corresponding result for rings, and will be omitted.

It is clear that every ring R is an R-ring with the usual operations on R. Every ideal of the ring R will be an ideal of the R-ring R. The converse is not, in general, true but we do have the following result:

LEMMA 1.3. ([4], Lemma 5.1). Let R be a ring, and let A be an ideal of the R-ring R. Let $\bar{A}$ denote the ideal of the ring R generated by A. Then

$$(\bar{A})^5 \subseteq A.$$

LEMMA 1.4. Let R be a ring. Then a subset P of R is a prime ideal of the ring R if and only if P is a prime ideal of the R-ring R.

PROOF. Suppose that P is a prime ideal of the ring R. Then P is an ideal of the R-ring R. Suppose $ARB \subseteq P$, where A and B are ideals of the R-ring R. Then, by Lemma 1.3, $(\bar{A})^5 R (\bar{B})^5 \subseteq P$, whence by the primeness of P, $\bar{A} \subseteq P$ or $\bar{B} \subseteq P$. It follows that $A \subseteq P$ or $B \subseteq P$, whence P is a prime ideal of the R-ring R.

Conversely, suppose that Q is a prime ideal of the R-ring R. Let $\bar{Q}$ be the ideal of the ring R generated by Q. Then $(\bar{Q})^5 \subseteq Q$. Since $\bar{Q}R \subseteq \bar{Q}$ it follows that

$$(\bar{Q}R)^4 \bar{Q} \subseteq Q.$$

Since Q is a prime ideal of the R-ring R, we have that $\bar{Q} \subseteq Q$. Hence, $Q = \bar{Q}$, i. e. Q is an ideal of the ring R. Now suppose A, B are ideals of the ring R and that $AB \subseteq Q$. Since $AR \subseteq A$, $ARB \subseteq Q$. Hence $A \subseteq Q$ or $B \subseteq Q$, i. e. Q is a prime ideal of the ring R.

LEMMA 1.5. Let $\mathcal{M}$ be a special class of Γ-rings and let R be the upper radical defined by $\mathcal{M}$. Then for an arbitrary Γ-ring M,

$$\mathcal{R}(M) = \cap\{I \lhd M : M/I \in \mathcal{M}\}.$$

The proof is exactly the same as that of the corresponding result for rings. An immediate consequence is that M is $\mathcal{R}$-semisimple if and only if M is a subdirect sum of Γ-rings which are in $\mathcal{M}$.

2. Definitions of strongly prime and superprime Γ-rings

Let R be a ring. Then R is called *strongly prime* (resp. *superprime*) if, for all $0 \neq I \lhd R$, there exists a finite set $F \subseteq I$ (resp. a nonzero $x \in I$) with zero right annihilator. $I \lhd R$ is called strongly prime (resp. superprime) if the factor ring R/I is strongly prime (resp. superprime). In [7] and [14], respectively, it is shown that the classes of strongly prime and superprime rings are special. The upper radicals defined by these classes will be denoted by $\iota(R)$ and $\sigma(R)$, respectively.

Let M be a Γ-ring and let $\alpha \in \Gamma$. Then M is said to be *α-strongly prime* (resp. *α-superprime*) if for all $0 \neq I \lhd M$, there exists a finite subset F of I (resp. an element x of I) such that for $y \in M$, $F\alpha y = 0$ (resp. $x\alpha y = 0$) implies $y = 0$.

As in the case of rings ([12], Proposition 1) we have the following characterization of α-strongly prime.

PROPOSITION 2.1. Let M be a Γ-ring and let $\alpha \in \Gamma$. Then M is α-strongly prime if and only if for each $0 \neq a \in M$ there exists a finite subset F of M and $\gamma \in \Gamma$ such that if $x \in M$ then $a\gamma F\alpha x = 0$ implies $x = 0$.

PROOF. Suppose M satisfies the conditions of the proposition. Let $0 \neq I \lhd M$. Let $0 \neq x \in I$ and F a finite subset of M such that if $y \in M$ then $x\gamma F\alpha y = 0$ implies $y = 0$. Then set $G = x\gamma F$. Clearly, $G \subseteq I$ and if $z \in M$ then $G\alpha z = 0$ implies $z = 0$.

Conversely, let M be α-strongly prime and let $0 \neq a \in M$. Since $\langle a \rangle \neq 0$, there exists a finite $F \subseteq \langle a \rangle$ such that $F\alpha x = 0$ implies $x = 0$ for all $x \in M$. Then there exist $\gamma \in \Gamma$, $y \in M$ such that $a\gamma y \neq 0$ (otherwise $F\alpha M = 0$, since $F \subseteq \langle a \rangle$). Hence, $\langle a\gamma y \rangle \neq 0$. Let $G \subseteq \langle a\gamma y \rangle$ be a finite set such that $G\alpha z = 0$ implies $z = 0$ for all $z \in M$. We may assume G is of the form

$$\{a\gamma y,\ a\gamma y\mu_1 z_1, \ldots,\ a\gamma y\mu_n z_n\},$$

where $\mu_1, \ldots, \mu_n \in \Gamma$ and $z_1, \ldots, z_n \in M$. If $H = \{y,\ y\mu_1 z_1, \ldots,\ y\mu_n z_n\}$, then $a\gamma H\alpha z = 0$ implies $z = 0$ for all $z \in M$.

PROPOSITION 2.2. The class of all α-strongly prime (resp. α-superprime) Γ-rings is special.

PROOF. Let $\mathcal{M}$ denote the class of all α-strongly prime Γ-rings.

(i) Let $M \in \mathcal{M}$. Then, if $0 \neq I, J \lhd M$, there exist a finite subset F of I such that $F\alpha x = 0$ implies $x = 0$, for all $x \in M$. If $I\Gamma J = 0$, it follows that $F\alpha J = 0$. Hence $J = 0$, which contradicts our choice of J. Thus M is a prime Γ-ring.

(ii) Let $M \in \mathcal{M}$ and $I \lhd M$. Then suppose that $0 \neq J \lhd I$. By Lemma 1.1, $\bar{J}\Gamma\bar{J}\Gamma\bar{J} \subseteq J$. Moreover, since M is prime $\bar{J}\Gamma\bar{J}\Gamma\bar{J} \neq 0$. Hence there exists a finite subset F of $J\Gamma J\Gamma J$ such that $F\alpha x = 0 \Rightarrow x = 0$ for all $x \in M$. Since $F \subseteq J$, this implies that $I \in \mathcal{M}$, i. e. $\mathcal{M}$ is hereditary.

(iii) Let M be a Γ-ring, and let $I \lhd \cdot M$, $I \in \mathcal{M}$. Suppose $0 \neq J \lhd M$. Then $I \cap J \neq 0$, and hence there exists a finite subset F of $I \cap J$ such that $F\alpha x = 0$ implies $x = 0$. For all $x \in I$ let $y \in M$ be such that $F\alpha y = 0$. It follows that $F\alpha y\Gamma I = 0$. Since $y\Gamma I \subseteq I$, this implies that $y\Gamma I = 0$, i. e. $y \in l(I)$. Since M is a prime Γ-ring, by Lemma 1.2, $y = 0$. Hence $M \in \mathcal{M}$.

The proof for the α-superprime case is similar.

The upper radicals defined by the classes of α-strongly prime and α-superprime Γ-rings are called the *α-strongly prime* and *α-superprime radicals* of M and denoted $\mathfrak{s}_\alpha(M)$ and $\sigma_\alpha(M)$, respectively. The *strongly prime* and *superprime radicals* of M (denoted $\mathfrak{s}(M)$ and $\sigma(M)$, respectively) are defined by the relations

$$\mathfrak{s}(M) = \bigcap_{\alpha \in \Gamma} \mathfrak{s}_\alpha(M) \text{ and } \sigma(M) = \bigcap_{\alpha \in \Gamma} \sigma_\alpha(M).$$

$I \lhd M$ is called an *α-strongly prime* (resp. *α-superprime*) *ideal* of M if the factor Γ-ring M/I is α-strongly prime (resp. α-superprime). It is easily shown that $I \lhd M$ is α-strongly prime if and only if for every ideal J of M which properly contains I, there exists a finite subset F of J such that $F\alpha x \subseteq I$ implies that $x \in I$, for all $x \in M$. A similar characterization exists for α-superprime ideals of M.

It is clear from Lemma 1.5 that $\mathfrak{s}_\alpha(M)$ (resp. $\sigma_\alpha(M)$) is the intersection of all the α-strongly prime (resp. α-superprime) ideals of M. The next two results show that $\mathfrak{s}(M)$ and $\sigma(M)$ are, in a sense, generalizations of the corresponding radicals of rings.

PROPOSITION 2.3. Let R be a ring, and let R' be the usual extension of R to a ring with identity such that $R \lhd R'$. Let $\mathfrak{s}'(R)$ denote the strongly prime radical of the R'-ring R. Then

$$\mathfrak{s}'(R) = \mathfrak{s}(R).$$

PROOF. It is clearly sufficient to show that $P \subseteq R$ is a strongly prime ideal of the ring R if and only if P is an x-strongly prime ideal of the R'-ring R for some $x \in R$. We note firstly that the ideals of the ring R are precisely the ideals of the R'-ring R. Let P be a strongly prime ideal of the ring R and let I be an ideal of R which properly contains P. Then there exists a finite subset F of I such that $Fx \subseteq P \Rightarrow x \in P$ for all $x \in R$. Let e be the identity of R'. Then $Fex \subseteq P \Rightarrow Fx \subseteq P \Rightarrow x \in P$. Hence P is e-strongly prime in the R'-ring R.

Conversely, suppose Q is x-strongly prime in the R'-ring R. Let I be an ideal of R which properly contains Q. Then there exists a finite set $F \subseteq I$ such that $Fxy \subseteq Q$ implies $y \in Q$, for all $y \in R$. Let $F_0 = \{ax : a \in F\}$. Clearly, $F_0 \subseteq I$ and $F_0 y \subseteq Q$ implies $y \in Q$ for all $y \in R$. Hence, Q is strongly prime in the ring R, and the proof is complete.

REMARKS. (1) Let R be a ring with identity and let $\mathcal{L}''(R)$ denote the strongly prime radical of the R-ring R. Using the techniques of Proposition 2.3, it is easy to show that $\mathcal{L}''(R) = \mathcal{L}(R)$.

(2) The second part of Proposition 2.3 made no use of the fact that R' has an identity. Using similar techniques, it may be shown that for an arbitrary ring R,

$$\mathcal{L}(R) \subseteq \mathcal{L}''(R).$$

We have not been ables to ascertain whether the reverse inclusion is in general true.

(3) An analogy of Proposition 2.3 may easily be proved for the superprime radical.

PROPOSITION 2.4. Let R be an arbitrary ring, and let $\sigma'(R)$ denote the superprime radical of the R-ring R. Then

$$\sigma'(R) = \sigma(R).$$

PROOF. We will show that a subset Q of R is a superprime ideal of R if and only if Q is an x-superprime ideal of the R-ring R for some $x \in R$.

Suppose then that Q is a superprime ideal of the ring R. Let I be an ideal of the R-ring R which properly contains Q. Let $\bar{I}$ be the ideal of the ring R which is generated by I. Then by Lemma 1.3, $(\bar{I})^5 \subseteq I$. Moreover, $(\bar{I})^5$ properly contains Q, for if $(\bar{I})^5 \subseteq Q$, since Q is a prime ideal of R, $\bar{I} \subseteq Q$ whence $I \subseteq Q$. This contradicts our choice of I. Hence, there exists $x \in I$ such that $xy \in Q$ implies $y \in Q$, for all $y \in R$. Suppose that $z \in R$ is such that $xxz \in Q$, i. e. $x(xz) \in Q$. Hence $xz \in Q$, which implies $z \in Q$. Thus Q is an x-superprime ideal of the R-ring R.

Conversely, suppose S is an x-superprime ideal of the R-ring R. It follows from Lemma 1.4 that S is an ideal of the ring R. The proof that S is a superprime ideal of the ring R is similar to the converse part of Proposition 2.3, and will be omitted. This concludes the proof.

3. Relations between the radicals

In this section, we will establish relationships between $\mathcal{L}(M)$, $\sigma(M)$ and the other radicals which have been defined for a Γ-ring M. It is trivially true that every α-superprime ideal of M is α-strongly prime and hence that $\mathcal{L}(M) \subseteq \sigma(M)$. Moreover, the reverse inclusion does not hold in general, in view of the fact that it does not hold for rings [14], Proposition 2,3 of the present paper and the subsequent remarks.

The prime, Levitzki, nil and Jacobson radicals, $\mathcal{P}(M)$, $\mathcal{L}(M)$, $\mathcal{N}(M)$ and $\mathcal{J}(M)$, respectively, of a Γ-ring M were defined by COPPAGE and LUH [5]. We refer to this paper for the definitions and properties of these radicals. For $\alpha \in \Gamma$, a subset S of M is called an *α-system if for all $x, y \in S$, $x\alpha y \in S$*. An ideal I of M is called *s-α-prime* if $M - I$ contains an α-system S such that every ideal of M not contained in I has nonempty intersection with S. In [4], Corollary 3.4, it is shown that $\mathcal{N}(M)$ is equal to the intersection of all the s-α-prime ideals of M, as α runs through Γ.

PROPOSITION 3.1. Let M be an arbitrary Γ-ring. Then

$$\mathcal{N}(M) \subseteq \sigma(M).$$

PROOF. In view of the above discussion, it is sufficient to show that every α-superprime ideal of M is necessarily s-α-prime. So let P be an α-superprime ideal of M. Then, for every proper ideal I of M which properly contains P, there exists $x = x(I)$ in I such that $x\alpha y \in I$ implies $y \in I$, for all $y \in M$. It is clear that $x \notin P$. Let S consist of all elements of M of the form $x_1\alpha x_2\alpha \ldots \alpha x_n$ where each x_i is of the form $x(I)$ for some ideal I of M which properly contains P. Clearly, S is an α-system of M. Moreover if $x_1\alpha\alpha x_2\alpha \ldots \alpha x_n \in S \cap P$ where the x_i's are as above, an inductive argument shows $x_i \in P$ which is clearly impossible. Hence, $S \subseteq M - P$. Now let

$$S_1 = \{x \in M : x - s \in P \text{ for some } s \in S\}.$$

It is easily verified that S_1 is an α-system and that $S_1 \subseteq M - P$. Now let $I \lhd M$ be such that $I \nsubseteq P$. Then P is properly contained in $P + I$. Hence there exists $x \in (P + I) \cap S$. Let $x = p + i$, $p \in P$, $i \in I$. Then $x - i \in P$, whence, since $x \in S$, $i \in S_1$. It follows that P is an s-α-prime ideal of M, as required.

REMARKS. (1) It is known [14] that for the ring R that the nil radical $\mathcal{N}(R)$ satifies the inclusion $\mathcal{N}(R) \subseteq \sigma(R)$ but that the reverse inclusion is not, in general, true. In view of Proposition 2.4 and its analogue for the nil radical ([4], Theorem 5.2), the reverse inclusion is also not in general true for Γ-rings.

(2) For rings, the Jacobson radical and the superprime radical are not comparable (cf. [14]). In view of Proposition 2.4 and [5], Theorem 10.1, this is also the case for Γ-rings.

PROPOSITION 3.2. Let M be an arbitrary Γ-ring. Then

$$\mathcal{L}(M) \subseteq \mathcal{A}(M).$$

PROOF. Let M be an α-strongly prime Γ-ring. Suppose that I is a locally nilpotent ideal of M. If $I \neq 0$, there exists a finite subset F of I such that $F\alpha x = 0$ implies $x = 0$, for all $x \in M$. Since I is locally nilpotent, $(F\alpha)^n F = F\alpha F\alpha \ldots F = 0$ for some positive integer n. Assume that n is the smallest positive integer with this property. Then

$$0 = (F\alpha)^n F = F\alpha(F\alpha)^{n-1} F.$$

By definition of F, it follows that $(F\alpha)^{n-1} F = 0$. This contradicts our choice of n. Hence, M has no nonzero locally nilpotent ideals. It follows that the semisimple class of $\mathcal{A}$ is contained in the semisimple class of $\mathcal{L}$, and hence that $\mathcal{L}(M) \subseteq \mathcal{A}(M)$, as required.

REMARKS. It is known ([12], Theorem 2) that the reverse inclusion does not hold for rings. In view of Proposition 2.3 and its analogue for the Levitzki radical ([2], Theorem 6.1), the same is true for Γ-rings.

There is another notion of strong primeness for Γ-rings due to GROENEWALD [6]. A Γ-ring M is strongly prime in this sense if, for all $0 \neq I \triangle M$, there

exists a finite subset F of I such that, if $x \in M$, $F\Gamma x = 0$ implies $x = 0$. It is shown in [6] that the upper radical defined by the class of rings which are strongly prime in this sense is special, and will be denoted $\mathcal{L}_0(M)$. Clearly, if M is α-strongly prime for any $\alpha \in \Gamma$, it is strongly prime in this sense. Hence, $\mathcal{L}_0(M) \subseteq \mathcal{L}(M)$. It is clear that $\mathcal{P}(M) \subseteq \mathcal{L}_0(M)$.

Let M be a prime weak Γ_N-ring. It is easily shown that, for $x, y \in M$, $x\Gamma y = 0$ implies $x = 0$ or $y = 0$ (cf. [1], Theorem 5). Let $0 \neq I \triangle M$. Then let $0 \neq x \in I$. Clearly, if we let $F = \{x\}$, $F\Gamma y = 0$ implies $y = 0$ for all $y \in M$. Thus, for weak Γ_N-rings the notions of primeness and strong primeness in the sense of Groenewald coincide. We have proved:

PROPOSITION 3.3. Let M be a weak Γ_N-ring. Then

$$\mathcal{P}(M) = \mathcal{L}_0(M).$$

EXAMPLE. Let R be a ring, and let $\Gamma = Z$. For $n \in Z$, $x, y \in \mathcal{R}$, define $xny = n(xy)$. Clearly, R is a Γ-ring with this operation. The following are easy to prove:

1. The ideals of the Γ-ring R are precisely the ideals of the ring R.
2. The ring R is strongly prime if and only if the Γ-ring R is strongly prime in the sense of Groenewald.
3. $I \triangleright R$ is locally nilpotent in the ring R if and only if I is locally nilpotent in the Γ-ring R.

It follows that the strongly prime radical $\mathcal{L}_0(R)$ and the Levitzki radical $\mathcal{L}(R)$ coincide with their counterparts in the ring R. This means that, for this choice of Γ, $\mathcal{L}(R) \subseteq \mathcal{L}_0(R)$. The reverse inclusion of course, will not always hold, and $\mathcal{L}_0(R)$ will not then always coincide with $\mathcal{P}(R)$.

In general, $\mathcal{L}(M) \subsetneqq \mathcal{L}_0(M)$. For let R be any ring for which the prime and Levitzki radicals do not coincide. Then R is a weak R-ring, and by [3], Theorems 4.7 and 4.21, the prime and Levitzki radicals of the R-ring R coincide with the corresponding radicals of the ring R. Hence, in this case, we have that:

$$\mathcal{L}_0(R) = \mathcal{P}(R) \subset \mathcal{L}(R).$$

The radical $\mathcal{L}_0(M)$ does, however, satisfy the following analogy of a well-known result in ring theory (cf. [8], Proposition 1.1).

PROPOSITION 3.4. Let M be a prime Γ-ring which satisfies the descending chain condition on right annihilators. Then M is strongly prime in the sense of Groenewald.

PROOF. Let $0 \neq I \triangle M$. $\mathcal{L} = \{r(F) : r$ is a finite subset of $I\}$. By assumption, $\mathcal{L}$ has a minimal element L, say. Let G be a finite subset of I such that $L = r(G)$. Suppose $L \neq 0$. Then, since M is a prime Γ-ring: by [10], Theorem 4, it follows that $I\Gamma L \neq 0$. Hence, there exists $x \in I$ such that $x\Gamma L \neq 0$. Put $G' = G \cap \{x\}$. Let $L' = r(G')$. Clearly, $L' \in \mathcal{L}$ and $L' \cap L$. This contradicts the minimality of L. Hence $L = 0$. It follows that $r(G) = 0$ and consequently M is strongly prime.

Strongly prime in the sense of Groenewald will henceforth be referred to as strongly prime.

A Γ-ring M is called α-completely prime if for all $x, y \in M$, $x\alpha y = 0$ implies $x = 0$ or $y = 0$. It is very easily shown that every α-completely prime Γ-ring M is α-superprime.

PROPOSITION 3.5. The class $\mathcal{M}$ of a α-completely prime Γ-rings is a special class.

PROOF. $\mathcal{M}$ satisfies (i) and (ii) of the definition of a special class trivially. Suppose that M is a Γ-ring, and $A \lhd \cdot M$ and $A \in \mathcal{M}$. Since A is a prime Γ-ring, by [6], Theorem 2.2, M is prime. Hence, by Lemma 1.2, $l(A) = r(A) = 0$. Let $0 \neq a, b \in M$. Then there exist $e, f \in A$, $\gamma, \mu \in \Gamma$ such that $e\gamma a \neq 0$ and $b\mu f \neq 0$. Clearly, $e\gamma a$, $b\mu f \in A$. Since A is α-completely prime, it follows that $e\gamma a\alpha b\mu f \neq 0$. Hence $a\alpha b \neq 0$, and $M \in \mathcal{M}$, as required.

Let $\mathcal{R}_\alpha$ be the upper radical defined by the special class $\mathcal{M}$. An ideal I of a Γ-ring M is called α-completely prime if the factor Γ-ring M/I is α-completely prime. Then, by Lemma 1.5,

$$\mathcal{R}_\alpha(M) = \cap I,$$

where I runs through the α-completely prime ideals of M. Let

$$\mathcal{N}_g(M) = \bigcap_{\alpha \in \Gamma} \mathcal{R}_\alpha(M).$$

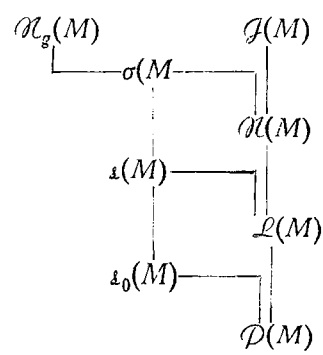
Clearly, $\mathcal{N}_g$ is a special radical, and

$$\sigma(M) \subseteq \mathcal{N}_g(M),$$

for every Γ-ring M. The reverse inclusion is not in general true. For example, let R be the ring of $n \times n$ matrices over a field. Let $M = \Gamma = R$. It is fairly easy to show that, in this case

$$\sigma(M) = 0 \text{ and } \mathcal{N}_g(M) = M.$$

The relationship between the radicals discussed in this paper may be summarised in the diagram:



All inclusions shown are sharp.

4. Other characterizations of the radicals

In [13] VELDSMAN introduced an sp-system to characterize the strongly prime radical of a ring as all those elements with the property that if it is contained in an sp-system, then the zero of the ring must be in the sp-system.

We now define similar systems to give an elementwise characterization of the α-superprime, α-strongly prime and strongly prime radicals.

DEFINITION 4.1. An *α-supersystem* (resp. *α-sp-system*, *sp-system*) in a Γ-ring M is a pair (G, P) where P is an ideal and G is a subset of M such that $G \cap P$ contains no non-zero elements of M and for any, $g \in G$ there is an $e \in \langle g \rangle$ such that $e\alpha z \in G$ for all $z \ddagger P$ (resp. there is a finite subset $F \subseteq \langle g \rangle$ such that $F\alpha z = \{f\alpha z : f \in F\} \subseteq G$ for all $z \notin P$, there is a finite subset $F \subseteq \langle g \rangle$ such that $F\Gamma z \subseteq G$ for all $z \notin P$).

It is easy to show that an ideal of the Γ-ring M is α-superprime (resp. α-strongly prime, strongly prime) if and only if $(M-I, I)$ is an α-supersystem (resp. α-sp-system, sp-system).

PROPOSITION 4.1. For any Γ-ring M
$\sigma_\alpha(M)$ (resp. $\mathfrak{s}_\alpha(M)$, $\mathfrak{s}_0(M)$) $= \{x \in M \mid$ whenever $x \in G$, (G, P) an α-supersystem (resp. α-sp-system, sp-system) for some P, then $0 \in G\}$.

PROOF. Let $x \in \mathfrak{s}_\alpha(M)$ and suppose $x \in G$, where (G, P) is an α-sp-system. If $0 \notin G$, then $G \cap P = \emptyset$. By Zorn's lemma, choose an ideal Q of M with $P \subseteq Q$, and Q maximal with respect to $G \cap Q = \emptyset$. Then $x \notin Q$. We show Q is an α-strongly prime ideal which will contradict $x \in \mathfrak{s}_\alpha(M)$. Let $J \triangle M$ such that $Q \subset J$. Since Q is maximal with respect to $G \cap Q = \emptyset$, we have $G \cap J \neq \emptyset$. If $0 \neq g \in G \cap J$, then there exists a finite set $F \subseteq \langle g \rangle \subseteq J$ such that $F\alpha z \cap G \neq \emptyset$ for all $z \notin P$. Let $F_1 = \{f \in F : f \notin Q\}$. Let $z \in M$ such that $F_1 \gamma z \subseteq Q$. Then $F\gamma z \subseteq Q$ and if $z \notin Q$, then $f\gamma z \cap G \neq \emptyset$ because $P \subseteq Q$. But this contradicts $G \cap Q = \emptyset$. Hence $z \in Q$ must hold. Thus Q a strongly prime ideal follows. The converse inclusion is obvious.

By using the same argument we can show that the theorem holds for the α-superprime radical and the strongly prime radical.

A pair (G, P) will be called a supersystem (resp. a strong sp-system) if (G, P) is an α-supersystem (resp. an α-sp-system) for some $\alpha \in \Gamma$. Then the following result follows immediately from Proposition 4.1.

PROPOSITION 4.2. If M is a Γ-ring, then $\mathfrak{s}(M)$ (resp. $\sigma(M)$) $= \{x \in M$: whenever $x \in G$, (G, P) a supersystem (resp. a strong sp-system), then $0 \in G\}$

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ON A THEOREM OF L. LEINDLER

By

I. N. BRUJ, and I. JOÓ

Department for Mathematics of the University Gomel and Institute for Mathematics of the L. Eötvös University, Budapest

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Suppose $f \in C_{2\pi}$, i. e. f is a 2π – periodic and continuous function on $\mathbf{R}$ and consider the partial sums $s_m(f)$ of its Fourier series

$$(1) \qquad f(x) \sim \sum_{m=-\infty}^{\infty} C_m(f) e^{imx}, \; s_m(f, x) := \sum_{|n| \leq m} C_n(f) e^{inx}.$$

The function f is said to be of power type if its Fourier series has the form

$$(2) \qquad f(x) \sim \sum_{m=0}^{\infty} C_m(f) e^{imx},$$

i. e. $c_m(f) = 0$ for $m < 0$.

S. N. BERNSTEIN ([1], 1912) proved that for the Fejér sums of any function belonging to $C_{2\pi} \cap \mathrm{Lip}\ 1$ the estimate

$$(3) \qquad \left\| \frac{1}{n+1} \sum_{m=0}^{n} (f(x) - s_m(f, x)) \right\|_{C_{2\pi}} = O\left(\frac{\log n}{n}\right), \; (n \geq 2)$$

holds and this estimate is not refinable for the just mentioned class of functions. In 1941 G. ALEXITS proved in [2] that the estimate (3) is refinable if f is also power type and proved for this case the exact estimate

$$(4) \qquad \left\| \frac{1}{n+1} \sum_{m=0}^{n} (f(x) - s_m(f, x)) \right\|_{C_{2\pi}} = O\left(\frac{1}{n}\right), \; (n \geq 1).$$

In 1965 L. LEINDLER (Cf. [3], [5]) investigated the order of the approximation by strong $(C, 1)$ means and proved that for any $f \in C_{2\pi} \cap \mathrm{Lip}\ 1$

$$(5) \qquad \left\| \frac{1}{n+1} \sum_{m=0}^{n} |f(x) - s_m(f, x)| \right\|_{C_{2\pi}} = O\left(\frac{\log n}{n}\right), \; (n \geq 2)$$

holds. It remained open that if we assume additionally, that f is of power type, then can we state $O(1/n)$ or not similarly to the case of $(C, 1)$ means? In the present paper we prove that the estimate of L. LEINDLER is not refinable also in this case, namely we prove the

THEOREM 1. *Suppose $f \in C_{2\pi} \cap \mathrm{Lip}\ 1$ and f is of power type. Then* (5) *holds and the order of this estimate is not refinable.*

The investigations mentioned above have importance also in point of view of approximation on the domains of the complex plane. Namely the generalizations of exponents $\{z^n, n \geq 0\}$ for any compact set $K \cap \mathbf{C}$ with connected complement (for the validity of Riemann's theorem on conformal mapping) $\overline{\mathbf{C}} \setminus K$ instead of the closed unit ball, are the so called Faber polynomials (Cf. [6], [7]). E. g. in the case of $K = [-1, 1]$ we obtained the Tchebichef polynomials. We have the

THEOREM 2. *Let G be a Jordan – domain with smooth boundary Γ. Denote $\Theta(s)$ the angle of the positive real axis and that of the tangent of Γ at the point corresponding to the parameter s (s denotes an arc length parameter of Γ). Suppose, for the modulus of continuity $\omega(\Theta, \delta)$ of Θ the condition*

$$\int_0^1 \frac{\omega(\Theta, \delta)}{\delta} \ln \frac{1}{\delta} d\delta < +\infty$$

holds. Let f be analytic in G, continuous on $\overline{G}$ and $f \in \mathrm{Lip}\ 1(\overline{G})$, i. e.

$$|f(z_1) - f(z_2)| \leq C |z_1 - z_2|, \ (z_1, z_2 \in \overline{G}).$$

Then

$$\max_{z \in \overline{G}} \frac{1}{n+1} \sum_{m=0}^{n} |f(z) - s_m(f, \overline{G}, z)| = O\left(\frac{\ln n}{n}\right), \ (n \geq 2), \tag{6}$$

where $s_m(f, \overline{G}, z)$ denotes the m–th partial sum of f with respect to the Faber system $\{p_n(\overline{G}, z)\}$ generated by $\overline{G}$. The estimate (6) *is not refinable.*

The upper estimate (6) is proved in [8]. The fact that it is not refinable, follows from the Theorem 1.

PROOF. We shall construct a power type function g belonging to the class $C_{2\pi} \cap \mathrm{Lip}\ 1$ such that

$$\frac{1}{n+1} \sum_{m=0}^{n} |g(0) - s_m(g, 0)| \geq C \frac{\log n}{n} (n \geq 1), \tag{7}$$

holds and this proves the exactness of Theorems 1 and 2. Define

$$g(x) := \sum_{k=1}^{\infty} (-1)^{k+1} 2^{-k} t_k(x), \tag{8}$$

where

$$t_k(x) := \sum_{l=2^{k-1}+1}^{2^k} \frac{1}{l} [e^{i(5 \cdot 2^k - l)x} - e^{i(5 \cdot 2^k + l)x}]. \tag{9}$$

The series (8) converges uniformly hence $g(x)$ is a 2π – periodic continuous function and its Fourier series can be given by substituting (9) in (8); so g is power type, too. It is proved in [5] (Lemma 2.5) that $g \in \mathrm{Lip}\ 1$. It remains to prove (7). Observe that $t_k(0) = 0$ and $g(0) = 0$. Suppose that $3 \cdot 2^p \leq n < 3 \cdot 2^{p+1}$. Taking into account that the minimal degree arising in $t_k(x)$ is $9 \cdot 2^{k-1} - 1$ we get that

$$\sum_{m=0}^{n} |g(0) - s_m(g, 0)| \geq \sum_{r=0}^{p-1} \sum_{\nu=9\cdot 2^{r-1}+1}^{5\cdot 2^r} |s_\nu(g, 0)| \geq$$

$$\geq \sum_{r=0}^{p-1} \frac{1}{2^r} \sum_{\nu=9\cdot 2^{r-1}+1}^{5\cdot 2^r} \frac{\nu - 9\cdot 2^{r-1}}{2^r} = \sum_{r=0}^{p-1} \frac{1}{2^{2r}} \sum_{\nu=1}^{2^{r-1}} \nu \geq C \sum_{r=0}^{p-1} 1 = cp \geq c \log n,$$

which proves the Theorem 1.

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О ТОЖДЕСТВАХ ПРОСТЫХ ПРАВОАЛЬТЕРНАТИВНЫХ АЛГЕБР

Ц. ДАШДОРЖ

Государственний Чниверситет, Члан Батор

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Пример простой правоальтернативной алгебры с тождеством $x^3 = 0$ построен И. М. Михеевым в [1]. В работе [2] доказано, что неалтернативные простые правоальтернативные алгебры являются ниль-алгебрами.

В настоящей работе доказано существование некоторого универсального существенного тождественного соотношения, выполняющегося во всех неассоциативных простых правоальтернативных алгебрах над полем $\Phi \ni \frac{1}{2}$.

Пусть A некоторая правоальтернативная алгебра над ассоциативно-коммутативным кольцом $\Phi \ni \frac{1}{2}$, и $\mathcal{R}(A)$ ее универсальная алгебра правых правоальтернативных представлений [6]. Образ элемента $a \in A$ при каноническом вложении в алгебру $\mathcal{R}(A)$ будем обозначать через a'. Рассматривается действие алгебры $\mathcal{R}(A)$ на пространстве $A \oplus 1\Phi$, где 1-внешним образом присоединенная единица к алгебре A. Алгебра $\mathcal{R}(A)$ имеет инволюцию $*$, тождественную на A' [6]. Пусть $(a, b) = a'b' - (ab)'$, $(a, b)^* = b'a' - (ab)'$, $a, b \in A$, символы Смайли.

Пусть $A^{(+)}$-присоединная йорданова алгебра алгебры A с умножением $x \circ y = \frac{1}{2}(xy + yx)$.

Известно, что если I, K-идеалы алгебры $A^{(+)}$ то Φ-модуль, $V_I(K)$ порожденный множеством $\{\{aka\}/a \in I,\ k \in K\}$ также является идеалом в $A^{(+)}$ [3]. Обозначим $I^{\langle 0\rangle} = I$, $I^{\langle m+1\rangle} = V_{I^{\langle m\rangle}}(I^{\langle m\rangle})$.

В работе [3] построены вполне характерстический идеал T алгебры $A^{(+)}$ и функция натурального аргумента $f(k)$, $k \geq 3$, $f(3) = 0$, такие, что для любых элементов $a_1, \ldots, a_k \in A^{(+)}$, $t \in T^{\langle f(k)\rangle}$ имеем для любого i, $1 \leq i \leq k$:

$$\{a_1' \ldots a_i' t' a_{i+1}' \ldots a_k'\} = a_1' \ldots a_i' t' a_{i+1}' \ldots a_k' + a_k' \ldots a_{i+1}' t' a_i' \ldots a_1' \in$$
$$\in \{a'/a \in A\} = A'.$$

Обозначим $T_{m-2} = T^{\langle f(m)\rangle}$

Пусть $r \in \mathcal{R}(A)$. Для фиксированного представления $r = \sum_i a'_{i1} \ldots$ $\ldots a'_{ik_i}$, $a_{ij} \in A$ в виде суммы произведений элементов из A' полагаем $l(r) = \max k_i$.

Пусть

$$B = \{f = (a, b) r_1 t'_s r_2 (c, d)^* \in \mathcal{R}(A)/a, b, c, d \in A,$$
$$r_1, r_2 \in \mathcal{R}(A) + 1 \cdot \Phi, t_s \in T_s, 3l(f) \leq s + 3\}$$

Положим $t_s = t(f)$.

В. Г. Скосырский [6] заметил, что если, $t(f) \in T_s$, $l(f) \leq s + 3$ и $1(f + f^*) = 0$, то $f + f^* = 0$. В самом деле, $f + f^*$-йорданов полином от A'. Следовательно, $f + f^* = (1(f + f))' = 0$

Лемма 1. $B^3 = 0$.

Доказательство. Если $f \in B$, то в силу приведенного замечания В. Г. Скосырского $f^* = -f$. То же самое справедливо для $f_1 f_2 \in B^2$ и $f_1 f_2 f_3 \in B^3$:

$$f_1 f_2 + (f_1 f_2)^*, f_1 f_2 f_3 + (f_1 f_2 f_3)^* \in A'$$

так как $l(f_1 f_2) \leq 2 \max l(fi) \leq s_j + 3$, и $l(f_1 f_2 f_3) \leq 3 \max l(fi) \leq s_j + 3$, где j определяется из равенства $l(f_j) = \max_i l(fi)$.

Опять в силу замечания В. Г. Скосырского

$$\{f_1 f_2\} = 0, \{f_1 f_2 f_3\} = 0, (f_1 f_2)^* = f_2^* f_1^* = (-f_2)(-f_1) =$$
$$= f_2 f_1 = -f_1 f_2, f_1 f_2 f_3 = -(f_1 f_2 f_3)^* = -f_3^* f_2^* f_1^* =$$
$$= -(-f_3)(-f_2)(-f_1) = f_3 f_2 f_1 = -f_3 f_1 f_2 = f_1 f_3 f_2 =$$
$$= -f_1 f_2 f_3; 2 f_1 f_2 f_3 = 0, f_1 f_2 f_3 = 0.$$

Лемма доказана.

Пусть $M \subseteq B$.

Определение 1. Если для всех элементов $(h_1, h_2, h_3) \in M \times M \times M$ выполняется равенство $h_1 w h_2 h_3 = 0 = h_1 h_2 w h_3$ то элемент $w \in \mathcal{R}(A) + 1 \cdot \Phi$ назовем допустимым для M.

Ясно, что элемент $w = id_A = 1$ допустим для B.

Лемма 2. Пусть $w \in \mathcal{R}(A) + 1 \cdot \Phi$ допустим для M и пусть

$$\overline{M} = \{f \in M | t(f) \in T_s, 3(l(f) + l(w) + 2) \leq s + 3,$$
$$f(x, y)^*, (x, y) f \in M, x, y \in A\}.$$

Тогда для любых элементов $x, y \in A$ элементы $w\,[x', y']$, $[x', y']\,w$ $w[x, y]'$ и $[x, y]'w$ допустимы для $\overline{M}$. В частности, элементы $[x', y']$, $[x, y]'$, допустимы для $\overline{B} = \{f \in B | t(f) \in T_s, 3(l(f) + 2) \leq s + 3\}$.

Доказательство. Пусть $f_i \in \overline{M}, t(f_i) = t_{si}$. В силу соотношений

$$(xy)' = x'y' - (x, y), (xy)' = y'x' - (x, y)^*, [x', y'] = (x, y) - (x, y)^*,$$

$$(1) \qquad x'y' = (x' \circ y') + \frac{1}{2}[x', y'] = (x \circ y)' + \frac{1}{2}[(x, y) - (x, y)^*].$$

Имеем

$$f_1[x', y']wf_2 f_3 = f_1(x, y)wf_2 f_3 - f_1(x, y)^* wf_2 f_3.$$

Рассмотрим $f_1(x, y)w\ f_2 f_3 = f_1 \overline{f_2} f_3$, где $\overline{f_2} = (x, y)w\ f_2$. Докажем, что $\overline{f_2} \in B$. Это следует из того, что $3 \cdot l(\overline{f_2}) = 3(l(f_2) + l(w) + 2) \leq s_2 + 3$ и $t(\overline{f_2}) = t(f_2) \in T_{s2}$. Следовательно, согласно лемме 1 имеем $f_1 \overline{f_2} f_3 = 0$.

Рассмотрим $f_1(x, y)^* w\ f_2 f_3 = \overline{f_1} w\ f_2 f_3$, где $\overline{f_1} = f_1(x, y)^*$. Из включения $f_1 \in \overline{M}$ следует, что $f_1(x, y)^* = \overline{f_1} \in M$. Значит, согласно определению 1 имеем $f_1(x, y)^* w\ f_2 f_3 = \overline{f_1} w\ f_2 f_3 = 0$. Этим доказано, что элемент $[x', y']w$ допустим для $\overline{M}$.

Рассмотрим $f_1\ w[x, y]'\ f_2 f_3$. Из соотношений $(xy)' = x'y' - (x, y)$, $(yx)' = y'x' - (y, x)$ следует, что $[x, y]' = [x', y'] - (x, y) + (y, x)$.

Доказано, что элементы $w[x', y']$, $w(x, y)$ допустимы для $\overline{M}$, значит, $w[x, y]'$ тоже допустим для $\overline{M}$.

Рассмотрим $f_1[x, y]' f_2 f_3$. Ясно, что $w = id_A$ допустим для B, отсюда следует, что $1 \cdot [x, y]'$ допустим для

$$\overline{B} = \{f \in B \mid t(f) \in T_s, 3(l(f) + 2) \leq s + 3,$$
$$f(x, y)^*, (x, y)f \in B, x, y \in A\}.$$

Но из $3(l(f) + 2) \leq s + 3$ следует, что $(x, y)f, f(x, y)^* \in B$. Таким образом, $[x, y]'$ допустим для

$$\overline{B} = \{f \in B \mid t(f) \in T_s, 3(l(f) + 2) \leq s + 3\}.$$

Лемма доказана.

Внутреннее дифференцирование $\mathcal{D}_{x, y}$ йордановой алгебры J определяется так:

$$a\mathcal{D}_{x, y} = ax \cdot y - ay \cdot x, a, x, y \in J.$$

Следствие 1. Пусть $b' \in A'$ допустим для подмножества $P \subseteq B$ и

$$\overline{P} = \{f \in P \mid t(f) \in T_s, 3(l(f) + 3) \leq s + 3,$$
$$(x, y)f, f(x, y)^* \in P, x, y \in A\}.$$

Тогда для любых элементов $x, y \in A^{(+)}$ элемент $(b\mathcal{D}_{x, y})'$ допустим для $\overline{P}$.

Доказательство. Согласно определению внутреннего дифференцирования $\mathcal{D}_{x, y}$ имеем $b\mathcal{D}_{x, y} = (b \circ x) \circ y - (b \circ y) \circ x$. Отсюда следует, что в ассоциативной алгебре $\mathcal{R}(A)$ выполняется равенство

$$(b\mathcal{D}_{x, y})' = ((b \circ x) \circ y - (b \circ y) \circ x)' =$$

$$= \frac{1}{4}[b', [x', y']] = \frac{1}{4}(b'[x', y'] - [x', y']b').$$

Ясно, что условия леммы 2 выполняются для подмножества $\overline{P}$ и элемента $b' = w$. Следовательно, для $(f_1, f_2, f_3) \in \overline{P} \times \overline{P} \times \overline{P}$ имеем

$$f_1(b\mathcal{D}_{x,y})' f_2 f_3 = f_1 b'[x', y'] f_2 f_3 - f_1[x', y'] b' f_2 f_3 = 0.$$

Следствие 2. Пусть b', $c' \in A'$ — допустимые элементы для $P \subseteq B$ и $a \in A$ — произвольный элемент. Тогда элемент $(a\mathcal{D}_{b,c})'$ допустим для множеста $\overline{P}$, определенного в условии следствия 1.

Доказательство. Согласно тождеству Якоби в алгебре $\mathcal{R}(A)$ имеем

$$[a', [b', c']] = [[a', b'], c'] - [[a', c'], b'].$$

Следовательно, согласно следствию 1 элементы $[[a', b'], c']$, $[[a', c'], b']$ допустимы для $\overline{P}$. Значит, $[a', [b', c']]$ допустим для $\overline{P}$.

Пусть $b = [b_1, b_2]$, $c = [c_1, c_2]$, $b_i, c_i \in A$. В силу леммы 2 элементы b' и c' допустимы для $\overline{B}$.

Пусть $\mathcal{D} = \mathcal{D}_{b,c}$ внутреннее дифференцирование в $A^{(+)}$, $x \in A^{(+)}$ и $d = (x\mathcal{D}^2)^2\mathcal{D}$ элемент Е. И. Зельманова [4]. Напомним, что идеал T опеределялся вработе [4] как наименьший эндоморфно допустимый идеал свободной йордановой алгебры, содержащий элемент d^{180} и все его частичные линеаризации. Отличие нашего идеала T от идеала из работы [4] состоит в том, что мы начинаем построение не с произвольных элементов b, c, а с элементов b, c вида $b = [b_1, b_2]$, $c = [c_1, c_2]$.

Пусть

$$T = T_1 \supseteq T_2 \supseteq \ldots$$

последовательность Зельманова [4], построенная относительно элемента d в алгебре $A^{(+)}$.

Лемма 3. Множество T' состоит из элементов, допустимых для

$$Q_9 = \{f \in B \mid t(f) \in T_s, 3(l(f) + 36) \leq s + 3\}.$$

Доказательство. Из [4] следует, что для любых элементов z, $y \in A^{\#}$ элементы $(d^{180} \circ y) \circ z$, $d^{180} \circ y$ лежат в $\langle A^{(+)}\mathcal{D}\rangle$, где $\langle A^{(+)}\mathcal{D}\rangle$ подалгебра алгебры $A^{(+)}$ порожденная элементами $x\mathcal{D}, x \in A^{(+)}$ и

$$I = id_{A^{(+)}}(d^{180}) =$$

$$= \{\sum (d^{180} \circ y) \circ z\mathcal{D}_{x_1, y_1} \ldots \mathcal{D}_{x_n, y_n} \mid x_i, y_i \in A^{(+)}, y, z \in A^{(+)\#}, n \in N \cup \{0\}\}.$$

Пусть $(d^{180} \circ y) \circ z = j(q_1\mathcal{D}, \ldots, q_r\mathcal{D}) \in \langle A^{(+)}\mathcal{D}\rangle$ и пусть $м$ — степень йорданова многочлена $j(x_1, \ldots, x_r)$. Используя тождества (1) и доказательство леммы 3 из [4], можно показать, что $м = 9$. т.е. $(d^{180} \circ y) \circ z = \sum_{n \leq 9} q_1\mathcal{D} \ldots q_n\mathcal{D}$.

Докажем, что элемент $(q_1\ \mathcal{D})'$ допустим для $Q_1 = \{f \in B \mid 3(l(f) + 4) \leq s + 3\}$. Действительно, в силу леммы 2 элементы b' и c' допустимы для $P = \overline{B}$. Согласно следствию 2 элемент $(q_1\ \mathcal{D})'$ допустим для

$$\overline{P} = \{f \in P \mid t(f) \in T_s, 3(l(f) + 3) \leq s + 3, (x, y)f, f(x, y)^* \in P\}.$$

Пусть $f \in \overline{P}$. Из включения $\overline{f} = (x,y)f \in P$ следует, что $s+3 \geq 3(l(\overline{f})+2) = 3(l(f)+4)$ т.е. $\overline{P} \subseteq Q_1$. Пусть $f \in Q_1$. Ясно, что $f \in P = \overline{B}$, так как $3(l(f)+2) \leq 3(l(f)+4) \leq s+3$. Из $3(l(\overline{f})+2) = 3(l(f)+4) \leq s+3$ следует, что $\overline{f} = (x,y)f \in P = \overline{B}$, т.е. $Q_1 \subseteq \overline{P}$. Таким образом, $Q_1 = \overline{P}$.

Докажем индукцией по n, что элемент $(q_1\mathcal{D})' \ldots (q_n\mathcal{D})'$ допустим для множества $Q_n = \{f \in B \mid 3(l(f)+4n) \leq s+3\}$. Рассмотрим элемент $(q_1\mathcal{D})' \ldots \ldots (q_{n+1}\mathcal{D})'$. Очевидно, что

$$(q_{n+1}\mathcal{D})' = \frac{1}{4}[[q'_{n+1}, [b', c']] =$$

$$= \frac{1}{4}[[q'_{n+1}, b'], c'] - \frac{1}{4}[[q'_{n+1}, c'], b'].$$

В силу леммы 2 элемент $(q_1\mathcal{D})' \ldots (q_n\mathcal{D})'\ c' = (q_1\mathcal{D})' \ldots (q_n\mathcal{D})'\ [c_1, c_2]'$ допустим, для

$$\overline{Q}_n = \{f \in Q_n \mid t(f) \in T_s, 3(l(f)+n+2) \leq s+3,$$
$$(x,y)f \in Q_n, f(x,y)^* \in Q_n, x, y \in A\}.$$

Докажем, что $\overline{Q}_n = \{f \in B \mid t(f) \in T_s, 3(l(f)+4n+2) \leq s+3\}$. Пусть $f \in \overline{Q}_n$. Из включения $\overline{f} = (x,y)f \in Q_n$ следует, что

$$s+3 \geq 3(l(\overline{f})+4n) = 3(l(f)+4n+2).$$

Пусть $f \in B$ и $3(l(f)+4n+2) \leq s+3$. Из $3(l(f)+4n) \leq 3(l(f)+4n+2) \leq \leq s+3$ следует, что $f \in Q_n$. Положим $\overline{f} = (x,y)f$. Из

$$3(l(\overline{f})+4n) = 3(l(f)+4n+2) \leq s+3$$

следует, что $\overline{f} \in Q_n$.

В силу леммы 2 элемент

$$(q_1\mathcal{D})' \ldots (q_n\mathcal{D})' c' [q'_{n+1}, b']$$

допустим для

$$\overline{\overline{Q}}_n = \{f \in \overline{Q}_n \mid t(f) \in T_s, 3(l(f)+n+3) \leq s+3,\ (x,y)f, f(x,y)^* \in \overline{Q}_n\}.$$

Докажем, что $Q_{n+1} = \overline{\overline{Q}}_n$.

Пусть $f \in \overline{\overline{Q}}_n$. Из $(x,y)f = \overline{f} \in \overline{Q}_n$ следует, что $3(l(\overline{f})+4n+2) = = 3(l(f)+4(n+1)) \leq s+3$, т.е. $\overline{\overline{Q}}_n \subseteq Q_{n+1}$. Пусть $f \in Q_{n+1}$, т.е. $f \in B$ и $3(l(f)+4(n+1)) \leq s+3$. Из $3(l(f)+4n+2) \leq 3(l(f)+4(n+1)) \leq s+3$ следует, что $f \in \overline{Q}_n$. Из $3(l(\overline{f})+4n+2) = 3(l(f)+4(n+1)) \leq s+3$ следует, что $\overline{f} = (x,y)f \in \overline{Q}_n$, т.е. $Q_{n+1} \subseteq \overline{\overline{Q}}_n$.

Таким образом, доказано, что $(k^{180} \circ y) \circ z$ допустим для, $Q_9 = = \{f \in B \mid t(f) \in T_s, 3(l(f)+36) \leq s+3\}$.

Пусть $\mathcal{D}_1 = \mathcal{D}_{x_1, y_1}$, $x_1, y_1 \in A$. Имеем

$$((d^{180} \circ y) \circ z)\mathcal{D}_1 = j(q_1 \mathcal{D}, \ldots, q_n \mathcal{D})\mathcal{D}_1 =$$

$$= \sum_{i=1}^{n} j(q_1 \mathcal{D}, \ldots, q_i \mathcal{D}_1 \mathcal{D}, \ldots, q_n \mathcal{D}) +$$

$$+ \sum_{i=1}^{n} j(q_1 \mathcal{D}, \ldots, q_i \mathcal{D}_{b\mathcal{D}_1, c}, \ldots, q_n \mathcal{D}) +$$

$$+ \sum_{i=1}^{n} j(q_1 \mathcal{D}, \ldots, q_i \mathcal{D}_{b, c\mathcal{D}_1}, \ldots, q_n \mathcal{D}).$$

Ясно, что здесь каждое слагаемое допустимо для Q_9. Мы доказали, что равенство

$$(af_1 i)f_2 f_3 = 0$$

где $a \in A, f_i \in Q_9, i \in I$, является тождеством в многообразии $\mathcal{R}A\,lt$ всех правоальтернативных алгебр.

Пусть $i = i(x_k)$, $\Delta = \Delta_k^i(y)$ — оператор линеаризации и образующий x_k не участвует в записи элементов a, f_1, f_2 и f_3. В силу сильной однородности многообразия $\mathcal{R}A\,lt$ имеем

$$(af_1 i)f_2 f_3 \Delta = (af_1 i\Delta)f_2 f_3 = 0.$$

Откуда следует, что $(af_1 t)f_2 f_3 = 0$, $t \in T$, или $f_1 T' f_2 f_3 = 0$ тоже является тождеством в $\mathcal{R}A\,lt$. Лемма доказана.

Следствие 3. $f_1 T'^2 f_2 f_3 = f_1 f_2 T'^2 f_3 = 0$ для $f_i \in Q_9$.

Доказательство. Пусть $t_i \in T$. Согласно (1) имеем

$$f_1 t_1' t_2' f_2 f_3 = f_1 \left[(t_1 \circ t_2)' + \frac{1}{2}(t_1, t_2) - \frac{1}{2}(t_1, t_2)^* \right] f_2 f_3.$$

Следовательно, в силу леммы 1 и 3 имеем $f_1 t_1' t_2' f_2 f_3 = 0$, если $f_i \in Q_9$, $\bar{f}_1 = f_1(t_1, t_2)^*$, $\bar{f}_2 = (t_1, t_2) f_2 \in B$, т.е. $f_i \in Q_9$.

Лемма 4. Пусть w такой элемент из $\mathcal{R}(A) + 1 \cdot \Phi$, что для всех элементов $f_i \in M \subseteq Q_9$ выполняется $f_1 T' f_2 w f_3 = 0$ и $\overline{M} = \{f \in M \mid t(f) \in T_s, 3(l(f) + l(w) + 2 + 36) \leq s + 3$ и $(x, y)f, f(x, y)^* \in M, x, y \in A\}$.

Тогда для любых элементов $x, y \in A$, $f_i \in \overline{M}$ выполняются равенства

$$f_1 T' f_2 w[x, y]' f_3 = 0, f_1 T' f_2 [x, y]' w f_3 = 0,$$

$$f_1 T' f_2 w[x', y'] f_3 = 0, f_1 T' f_2 [x', y'] w f_3 = 0.$$

В частности, для любых

$$f_i \in \overline{Q}_9 = \{f \in B \mid t(f) \in T_s, 3(l(f) + 38) \leq s + 3\}$$

выполняются равенства:

$$f_1T'f_2[x',y']f_3 = 0, f_1T'f_2[x,y]'f_3 = 0.$$

Доказательство. Беря Q_9 вместо B и действуя точно так же, как и в доказательстве леммы 2, получаем требуемое утверждение.

Лемма 5. Для всех элементов

$$f_i \in Q_{18} = \{f \in B \mid t(f) \in T_s, 3(l(f)+72) \leq s+3\}$$

выполняется равенство $f_1T'f_2T'f_3 = 0$.

Доказательство аналогично доказательству леммы 3. Рассмотрим элемент $f_1T'f_2(q_1\mathcal{D})'f_3$. Известно, что

$$(q_1\mathcal{D})' = \frac{1}{4}[q_1', [b', c']] = \frac{1}{4}[[q_1', b'], c'] - \frac{1}{4}[[q_1', c'], b']$$

и в силу леммы 4 соотношения $f_1T'f_2[x',y']f_3 = 0$ и $f_1T'f_2[x,y]'f_3 = 0$ выполняются для всех $f_i \in \overline{Q}_9$.

Следовательно, согласно лемме 4 равенство $f_1T'f_2[q'_1, b']c'f_3 = 0$ выполняется для всех

$$f_i \in \overline{\overline{Q}}_9 = \{f \in \overline{Q}_9 \mid t(f) \in T_s, 3(l(f)+40) \leq s+3, (x,y)f, f(x,y)^* \in \overline{Q}_9\}.$$

Докажем, что $\overline{\overline{Q}}_9 = Q_{10}$. Ясно, что $\overline{\overline{Q}}_9 \subseteq Q_{10}$. Пусть $f \in Q_{10}$, т.е. $f \in B$ и $3(l(f)+40) \leq s+3$. Из $3(l(f)+38) \leq 3(l(f)+40) \leq s+3$ следует, что $f \in \overline{Q}_9$. Положим $\overline{f} = f(x, y)^*$. Ясно, что из $3(l(\overline{f})+38) = 3(l(f)+40) \leq s+3$ следует $\overline{f} \in \overline{Q}_9$. Следовательно, $f \in \overline{\overline{Q}}_9$. Продолжая рассуж дение дальше точно так же, как и в доказательстве леммы 3. получим, что для всех

$$f_i \in Q_{18} = \{f \in B \mid t(f) \in T_s, 3(l(f)+72) \leq s+3\}$$

выполняется равенство $f_1T'f_2T'f_3 = 0$.

Лемма доказана.

Следствие 4. Для всех элементов $f_i \in Q_{18}$ выполняются равенства

$$f_1T'^2f_2T'f_3 = f_1T'f_2T'^2f_3 = 0, f_1T'^2f_2T'^2f_3 = 0.$$

Доказательство. В силу (1), лемм 1,3 и 5 имеем

$$f_1t_1't_2'f_2t_3'f_3 = f_1\left[(t_1 \circ t_2)' + \frac{1}{2}(t_1, t_2) - \frac{1}{2}(t_1, t_2)^*\right]f_2t_3'f_3 = 0$$

если $f_i \in Q_{18}$, $\overline{f}_1 = f_1(t_1,t_2)^* \in Q_9$, $\overline{f}_2 = (t_1,t_2)f_2 \in Q_9$, т.е. $f_i \in Q_{18}$, $i = 1, 2, 3$. Отсюда в силу следствия 3 имеем $f_1t_1't_2'f_2t_3't_4'f_3 = 0$ если $f_i \in Q_{18}$, $\overline{f}_1 = f_1(t_1,t_2)^* \in Q_9$, $\overline{f}_2 = (t_1,t_2)f_2 \in Q_9$, т.е. $f_i \in Q_{18}$, $i = 1, 2, 3$.

Теорема. *Пусть A неассоциативная простая правоальтернативная алгебра характеристики, отличной от двух. Тогда $T^3_{240} = 0$, где T^3_{240} – куб идеала $T_{240}(A^{(+)})$ в йордановой алгебре $A^{(+)}$.*

Доказательство. В следствии 4 доказано, что для всех элементов $f_i \in Q_{18}$ выполняется равенство $f_1 T'^2 f_2 T'^2 f_3 = 0$. Рассмотрии выражение

$$F = g_1 T'^2 g_2 T'^2 g_3,$$

где $g_i = (a, b, c)' r_1 t'_{s_i} r_2(x, y, z)'$, если $f_i = (a_i, b_i) r_1 t'_{s_i} r_2(x_i, y_i)^* \in Q_{18}$.

Из тождества Тейхмюллера

$$x(y, z, t) - (xy, z, t) + (x, yz, t) - (x, y, zt) + (x, y, z)t = 0$$

следует, что

$$(y, z, t)' = y'(z, t) - (yz, t) + (y, zt) - (y, t)t' \tag{2}$$

$$(y, z, t)' = (z, t)^* y' - (yz, t)^* + (y, zt)^* - t'(y, z)^* \tag{3}$$

В каждом g_i для первого вхождения символа $(y, z, t)'$ возьмем его выражение (2), а для последнего вхождения символа (y, z, t) — его выражение (3). Тогда

$$F = \sum a'_1 h_1 b'_1 T'^2 a'_2 h_2 b'_2 T'^2 a'_3 h_3 b'_3,$$

где $a'_i, b'_i \in A' \cup 1 \cdot \Phi, h_i \in B$ и $l(f_i) \leq l(h_i) \leq l(f_i) + 2, t(f_i) = t(h_i)$, причем из $l(h_i) = l(f_i) + 2$ следует, что $a'_i, b'_i \in 1 \cdot \Phi$.

На основании (1) имеем

$$h_1 b'_1 t'_1 t'_2 a'_2 h_2 b'_2 t'_3 t'_4 a'_3 h_3 = h_1 \left[(b_1 \circ t_1)' + \frac{1}{2}(b_1, t_1) - \frac{1}{2}(b_1, t_1)^* \right] \cdot$$

$$\cdot \left[(t_2 \circ a_2)' + \frac{1}{2}(t_2, a_2) - \frac{1}{2}(t_2, a_2)^* \right] \cdot h_2 \left[(b_2 \circ t_3)' + \frac{1}{2}(b_2, t_3) - \frac{1}{2}(b_2, t_3)^* \right] \cdot$$

$$\cdot \left[(t_4 \circ a_3)' + \frac{1}{2}(t_4, a_3) - \frac{1}{2}(t_4, a_3)^* \right] h_3,$$

где $t_i \in T$. После раскрытия скобок каждое слагаемое представляется в одной из следующих форм:

$$\bar{h}_1 \bar{h}_2 \bar{h}_3,\ \bar{h}_1 t'_1 \bar{h}_2 \bar{h}_3,\ \bar{h}_1 \bar{h}_2 t'_1 \bar{h}_3,\ \bar{h}_1 t'_1 t'_2 \bar{h}_2 \bar{h}_3,\ \bar{h}_1 \bar{h}_2 t'_1 t'_2 \bar{h}_3,\ \bar{h}_1 t'_1 \bar{h}_2 t'_2 \bar{h}_3,$$

$$\bar{h}_1 t'_1 t'_2 \bar{h}_2 t'_3 \bar{h}_3,\ \bar{h}_1 t'_1 \bar{h}_2 t'_2 t'_3 \bar{h}_3,\ \bar{h}_1 t'_1 t'_2 \bar{h}_2 t'_3 t'_4 \bar{h}_3 \tag{4}$$

где $\bar{h}_i \in B, t_i \in T$.

Согласно леммам 1, 3 и следствию 3 слагаемые, имеющие вид первых 5 выражений из (4), равны нулю, когда $\bar{h}_i \in Q_9$. Это условие автоматически выполняется, так как $f_i \in Q_{18}$ и

$$3(l(\bar{h}_i) + 36) \leq 3(l(h_i) + 40) \leq 3(l(f_i) + 42) \leq$$
$$\leq 3(l(f_i) + 72) \leq s_i + 3,\ i = 1, 2, 3.$$

В силу леммы 4 и следствия 5 слагаемые, имеющие вид последних 4 выражений из (4), равны нулю, когда $\bar{h}_i \in Q_{18}$, $i = 1, 2, 3$. Рассмотрим длины $l(\bar{h}_2)$ элементов $\bar{h}_2$, входящих в эти четыре выражения из (4). Нетрудно заметить, что элемент $\bar{h}_2$ смаксимальной длиной $l(\bar{h}_2)$ входит в выражение вида $\bar{h}_1 t' \bar{h}_2 t_2' \bar{h}_3$, когда оно имеет вид

$$\bar{h}_1 t_1' h_2 t_2' \bar{h}_3 = h_1 t_1'(a_2, t_2) h_2(b_2, t_3)^* t_4' h_3 = H_1.$$

С другой стороны самые длинные элементы $\bar{h}_1$, $\bar{h}_3$ у этих же четырех выражений входят в слова $\bar{h}_1 t_1' \bar{h}_2 t_2' \bar{h}_3$ когда они имеют вид:

$$\bar{h}_1 t_1' \bar{h}_2 t_2' \bar{h}_3 = h_1(b_1, t_1)^* t_2' h_2 t_3'(t_4, a_3) h_3 = H_2,$$

$$\bar{h}_1 t_1' \bar{h}_2 t_2' \bar{h}_3 = h_1(b_1, t_1)^* t_2' h_2(b_2, t_3)^* t_4' h_3 = H_3,$$

$$\bar{h}_1 t_1' \bar{h}_2 t_2' \bar{h}_3 = h_1 t_1'(a_2, t_2) h_2 t_3'(a_3, t_4) h_3 = H_4.$$

Легко видеть, что $H_1 = 0$ если

$$h_1, \bar{h}_2 = (a_2, t_2) h_2(b_2, t_3)^*, h_3 \in Q_{18}.$$

Далее, $H_i = 0$, $i \geq 2$, если $\bar{h}_i \in Q_{18}$, $i = 1, 2, 3$. Из $a_2, b_2 \notin 1 \cdot \Phi$ следует, что $l(h_2) = l(f_2)$ а из $b_1 \notin 1 \cdot \Phi$ и $a_3 \notin 1 \cdot \Phi$ следует, что $l(h_1) \leq l(f_1) + 1$ и $l(h_3) \leq \leq l(f_3) + 1$. Значит если

$$3(l(\bar{h}_2) + 72) = 3(l(f_2) + 76) \leq s_2 + 3$$

$$3(l(\bar{h}_i) + 72) \leq 3(l(f_i) + 75) \leq s_i + 3, i = 1, 3,$$

то $H_1 = 0 = H_i$, $i \geq 2$. Таким образом, стало ясно, что если

$$f_1, f_3 \in \{f \in B \mid t(f) \in T_s, 3(l(f) + 75) \leq s + 3\}, f_2 \in Q_{19}.$$

то $F = 0$. Из $5 \leq l(f)$ следует, что

$$243 = 3(5 + 76) \leq 3(l(f_2) + 76) \leq s_2 + 3,$$

$$240 = 3(5 + 75) \leq 3(l(fi) + 75) \leq s_i + 3, \ i = 1, 3.$$

Пусть $f_i = (a_i, b_i) t_i'(c_i, d_i)^*$, $a_i, b_i, c_i, d_i \in A$, $t_1, t_3 \in T_{237}$, $t_2 \in T_{240}$ и $g_i = (x_i, y_i z_i)' t_i'(p_i, q_i, r_i)' x_i, \ldots, r_i \in A$, $i = 1, 2, 3$. Тогда по доказанному нами имеет место

$$g_1 T'^2 g_2 T'^2 g_3 = 0.$$

Для неальтернативных простых правоальтернативных алгебр верно, что $(A, A, A) = A$ [5]. Значит,

$$F_1 = A' T_{237}' A' T'^2 A' T_{240}' A' T'^2 A' T_{237}' A' = 0.$$

Из включений

$$T'^2 \supseteq (T \circ T)' \supseteq (V_T(T))',$$

$$A'T'^2A' \supseteq (V_T(T))'^3 \supseteq (V_{V_T(T)}(V_T(T)))' \supseteq T'_3,$$
$$A'T'_{237}T'_3 \supseteq T'^3_{237} \supseteq (V_{T237}(T_{237}))' \supseteq T'_{238} \supseteq T'_{240}$$

и $T'_3T'_{237}A' \supseteq T'_{240}$ следует, что $0 = F_1 \supseteq T'^3_{240} = 0$. Следовательно, $1\,T'^3_{240} = T^2_{240}T_{240} = 0$, $(T_{240} \circ T_{240}) \circ T_{240} = T^3_{240} = 0$. Для алгебр Кэли-Диксона известно, что $T(\mathbf{C}) = 0$. Теорема доказана.

Следствие. Существует универсальное существенное тождественное соотношение, выполняющееся во всех неассоциативных простых правоальтернативных алгебрах характеристики отличной от двух.

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ON THE GROWTH OF THE EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR I

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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Consider the Schrödinger operator

$$-u''+qu = \lambda u \tag{1}$$

on the interval (0,1), where $\lambda \in \mathbf{C}$ and $q \in C\,(0,1]$ is such that

$$q(x) = O(x^{-\alpha}) \tag{2}$$

holds. We shall study the following question. Which is the maximal growth of the eigenfunction u near $x = 0$ if the potential satisfies (2)? More precisely we shall determine the maximal rate of growth of the function

$$m(t) := \max_{t \le x \le \frac{1}{2}} |u(x)|, \; 0 < t \le \frac{1}{2}, \tag{3}$$

while $t \to 0$. We shall prove the

THEOREM 1. *Suppose that* (2) *is satisfied and* $\alpha > 2$. *Then there exists a constant* $c = c\,(\alpha, \lambda) > 0$ *such that*

$$m(t) = O(e^{ct^{1-\alpha/2}}), \; (t > 0). \tag{4}$$

PROOF. Let $x, x+t \in (0,1)$ and $f \in C^2[x, x+t]$. Then by twofold integration by parts we obtain

$$[uf'-u'f]_x^{x+t} = \int_x^{x+t} u(f''+\lambda f-qf). \tag{5}$$

Let x, $x+2\delta \in (0,1)$, $\delta > 0$ and $f(\xi) := \delta - |x+\delta-\xi|$. Applying (5) first in $[x, x+\delta]$ then in $[x+\delta, x+2\delta]$ we get

$$2u(x+\delta)-u(x)-u(x+2\delta) = \int_x^{x+2\delta} u(\lambda f-qf). \tag{6}$$

Now let $x \in [t, t+\delta]$, $0<t$, δ; $t+2\delta<1/2$. Then from (6) we see that

$$|u(x)| \leq 3\|u\|_{L^\infty\left(t+\delta, \frac{1}{2}\right)} + \delta^2|\lambda| \cdot \|u\|_{L^\infty\left(t, \frac{1}{2}\right)} +$$

$$+\delta\left(\int_x^{x+2\delta} |q|\right) \cdot \|u\|_{L^\infty\left(t, \frac{1}{2}\right)}.$$

If δ is specified such that

$$\delta\left(\int_x^{x+2\delta} |q|\right) < \frac{1}{4}, (x \in [t, t+\delta]); \delta^2|\lambda| < \frac{1}{4}, \tag{7}$$

then we get

$$\|u\|_{L^\infty(t, t+\delta)} \leq 3\|u\|_{L^\infty\left(t+\delta, \frac{1}{2}\right)} + \frac{1}{2}\|u\|_{L^\infty\left(t, \frac{1}{2}\right)},$$

consequently

$$m(t) \leq 8\, m(t+\delta). \tag{8}$$

From (3) we know that

$$\int_x^{x+2\delta} |q| \leq c\int_x^{x+2\delta} r^{-\alpha}dr \leq c\int_t^{t+2\delta} r^{-\alpha}dr =$$

$$= \frac{c}{\alpha-1}\left[t^{1-\alpha} - (t+2\delta)^{1-\alpha}\right].$$

Since $t^{1-\alpha}-(t+2\delta)^{1-\alpha} \leq 2\delta\,(\alpha-1)t^{-\alpha}$ we get $\delta\int_x^{x+2\delta} |q| \leq c\delta^2 t^{-\alpha}$, so if $\delta := c_0 t^{\frac{\alpha}{2}}$ with sufficiently small $c_0>0$ then $\delta\int_x^{x+2\delta} |q| < \frac{1}{4}$. On the other hand, if $t<t_0 := |\lambda|^{-\alpha}$, then $\delta^2|\lambda| < 1/4$ so (7) and then (8) is satisfied. Now define for $t>0$

$$f_1(t) := t,\ f_{n+1}(t) := f_n(t) + c_0[f_n(t)]^{\frac{\alpha}{2}}. \tag{9}$$

The inequality (8) gives

$$m(t) \leq 8^n m(f_n(t)). \tag{8'}$$

We need the following

SUBLEMMA. For sufficiently large $c_1 = c_1(\alpha, \lambda) > 0$ the sequence $\{f_n(c_1 n^{\frac{-2}{\alpha-2}})\}$ is monotone increasing (so is bounded below).

PROOF. We prove by induction on k that

$$f_k(c_1 n^{\frac{-2}{\alpha-2}}) \leq f_{k+1}(c_1(n+1)^{\frac{-2}{\alpha-2}}), (n = 1, 2, \ldots). \tag{10}$$

For $k = 1$ this means that $c_1(n+1)^{\frac{-2}{\alpha-2}} + c_0 c_1^{\frac{\alpha}{2}}(n+1)^{\frac{-2}{\alpha-2}} \geq c_1 n^{\frac{-2}{\alpha-2}}$,

$$c_0 c_1^{\frac{\alpha}{2}-1}(n+1)^{\frac{-2}{\alpha-2}} \geq n^{\frac{-2}{\alpha-2}} - (n+1)^{\frac{-2}{\alpha-2}} \geq c(\alpha)(n+1)^{\frac{-2}{\alpha-2}-1}$$

and this is reduced to $c_1^{\frac{\alpha}{2}-1} \geq c(\alpha)/c_0$, which is guaranteed for large c_1. Now the induction step in (10) from k to $k+1$ being trivial, the Sublemma is proved.

Returning to the proof of the Theorem 1, we see that if

$$c_1(n-1)^{\frac{-2}{\alpha-2}} \geq t > c_1 n^{\frac{-2}{\alpha-2}} \text{ i. e. } n > \left(\frac{t}{c_1}\right)^{\frac{\alpha}{2}-1} \geq n-1,$$

then $m(t) \leq c_2 \cdot 8^n$, hence $m(t) \leq c_2 e^{c_3 t^{1-\alpha/2}}$ as it was to be proved.

REMARK. The estimation of the Theorem 1 can not be improved in the sense that the function $u(t) := e^{ct^{1-\alpha/2}}$ satisfies (1) with a potential $q(t)$ satisfying (2). The problem of estimation of $m(t)$ was raised in [1].

Now consider the following problem. Let $(\lambda_n)_1^\infty \subset \mathbf{C}$ be a sequence for which the series $\Sigma\lambda_n$ is conditionally convergent, i.e. $\Sigma|\lambda_n| = \infty$. A function $f: \mathbf{C} \to \mathbf{C}$ is called additive if $x = \Sigma\varepsilon_n \lambda_n$, $\varepsilon_n = \{^1_0$ implies $f(x) = \Sigma\varepsilon_n f(\lambda_n)$. For later use introduce the following symbol. If (a_n), and (b_n) are real sequences then denote $a_n \prec * b_n$ that there is a sequence (γ_k) such that $a_n < b_n + \gamma_n$ $n = 1, 2, \ldots$ and $\Sigma|\gamma_n| < \infty$. We shall prove the

THEOREM 2. *If f is additive and analytic (entire) function then $f(z) = cz$ for some $c \in \mathbf{C}$.*

For the proof we need the following

LEMMA. If f is additive, analytic and $f \not\equiv 0$ then there are constants $0 < c < C < \infty$ such that

$$c|\lambda_n| \prec * f(\lambda_n) \prec * C|\lambda_n|. \tag{11}$$

PROOF. If $c|\lambda_n| \prec * f(\lambda_n)$ is not true for some $c > 0$ then the sum of the positive members of the sequence $(c|\lambda_n| - |f(\lambda_n)|)$ is infinite. Hence there exists a sequence (λ_{n_k}) such that

$$c|\lambda_{n_k}| > |f(\lambda_{n_k})|, \sum_k |\lambda_{n_k}| = \infty \tag{12}$$

and taking a subsequence we can also suppose that

$$\arg \lambda_{n_k} \to \alpha_0 (k \to \infty). \tag{13}$$

In the terminology of GY. PÓLYA, G. SZEGŐ: *Problems and Theorems in Analysis*, Springer 1972, Vol. I, Part III, 52., α_0 is a "cluster direction" of the series $\Sigma\lambda_n$. Consider together the set of all cluster directions α_0 corresponding to c: it is compact. Letting $c = \epsilon_N \to 0$ these sets diminish and hence their intersection is nonempty. Any element β_0 of this intersection enjoy the following property: there is a sequence (λ_{m_k}) such that

$$\sum_k |\lambda_{m_k}| = \infty, \arg \lambda_{m_k} \to \beta_0, \frac{|f(\lambda_{m_k})|}{|\lambda_{m_k}|} \to 0 \, (k \to \infty). \tag{14}$$

But then for any $\varepsilon > 0$ and for any point of the half-line $xe^{i\beta_0}$ $x > 0$, any neighbourhood $x\, e^{i\beta_0} \in V$ contains a point z such that $|f(z) < \varepsilon$. The continuity of f implies that it vanishes along this half-line and analycity gives then $f \equiv 0$. The contradiction shows the first inequality in (11). In proving the second one, the indirect assumption gives as above the existence of a direction γ_0 and a subsequence (λ_{n_k}) such that

$$\sum_k |f(\lambda_{n_k})| = \infty, \arg \lambda_{n_k} \to \gamma_0, \frac{|f(\lambda_{n_k})|}{|\lambda_{n_k}|} \to \infty \, (k \to \infty). \tag{15}$$

Passing to subsequence we can additionally assume that

$$\sum_k |\lambda_{n_k}| < \infty, \arg \lambda_{n_k} \to \delta_0 \in \mathbf{R} \, (k \to \infty). \tag{16}$$

But then $\Sigma\lambda_{n_k}$ converges and $\Sigma f(\lambda_{n_k})$ diverges, contrarily to the additive property of f. The Lemma is proved.

PROOF OF THE THEOREM 2. Define $c^* := \sup\{c > 0 : c|\lambda_n| < * \; f(\lambda_n)|\}$. The Lemma shows that $0 < c^* < \infty$. By the definition of c^* for all $\varepsilon > 0$ there is a sequence (λ_{n_k}) such that

$$(c^* + \varepsilon)|(\lambda_{n_k}| > |f(\lambda_{n_k})| \; \forall \, k; \; \sum |\lambda_{n_k}| = \infty. \tag{17}$$

Using a compactness argument, analogous to that in the Lemma, we can give a further subsequence (λ_{n_k}) with the property

$$\begin{gathered} \varepsilon_k |\lambda_{n_k}| > |f(\lambda_{n_k})| - c^* |\lambda_{n_k}|, \\ \varepsilon_k \to 0, \arg \lambda_{n_k} \to \alpha_0, \arg f(\lambda_{n_k}) \to \beta_0 \, (k \to \infty). \end{gathered} \tag{18}$$

Consider the function $f^*(z) := f(z) - c^* e^{i(\beta_0 - \alpha_0)} z)$; it is additive and analytic. We have to prove that $f^* \equiv 0$. From $\arg \lambda_{n_k} \to \alpha_0$, $\arg f(\lambda_{n_k}) \to \beta_0$ it follows that

$$\begin{gathered} |f(\lambda_{n_k}) - c^* e^{i(\beta_0 - \alpha_0)} \lambda_{n_k}| \leq |f(\lambda_{n_k})| - c^* |\lambda_{n_k}| + \delta_k |\lambda_{n_k}|, \\ \delta_k \to 0 \, (k \to \infty). \end{gathered} \tag{19}$$

With (18) this yields that

(20) $\quad (\varepsilon_k+\delta_k)|\lambda_{n_k}| \geq |f^*(\lambda_{n_k})|, \sum |\lambda_{n_k}| = \infty, \varepsilon_k+\delta_k \to 0 \, (k \to \infty).$

If $f^* \not\equiv 0$ then $c_0|\lambda_{n_k}| < * |f^*(\lambda_{n_k})|$ by the Lemma and then $c_0|\lambda_{n_k}| < *$ $< *(\varepsilon_k+\delta_k)|\lambda_{n_k}|$ a contradiction. So $f^* \equiv 0$ and the Theorem 2 is proved

References

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ON THE GROWTH OF THE EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR II

By

I. JOÓ

Department for Anlysis of the L. Eötvös University, Budapest

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1. In this paper we continue the investigations of [5]. Let $G = (a, b)$ be any finite interval, $q \in L^1_{\mathrm{loc}}(G)$ a complex function (i. e. q is Lebesgue integrable on every compact subinterval of G) and consider the formal Schrüdinger operator with "potential" $q: Lu := -u'' + q \cdot u$. A function $u_i: G \to \mathbf{C}$, $u_i \not\equiv 0$ ($i = 0, 1, \ldots$) is said to be an eigenfunction of order i (of the operator L) with the eigenvalue $\lambda \in \mathbf{C}$, if it is absolutely continuous together with its derivative on every compact subinterval of G and for almost all $x \in G$ the equation

$$-u_i''(x) + q(x)u_i(x) = \lambda u_i(x) + u_{i-1}(x) \tag{1}$$

holds, where $u_i \overset{\mathrm{def}}{\equiv} 0$ for $i = -1$ and u_{i-1} is an eigenfunction of the operator L of order $i-1$, with the eigenvalue λ, for $i \geq 1$.

It is known (see [1], pp. 167–169) that if $q \in L^1(G)$, then u_i and u_i' can be continuously extended to the closed interval $\bar{G}$ and the extended functions are absolutely continuous on $\bar{G}$. (A more general theorem and a new method is given in [4]). In [3] it was investigated the dependence of $\|u_i\|_\infty / \|u_i\|_2$ on λ for any fixed $q \in L^1(G)$ and it was proved, among others, the estimate

$$\frac{\|u_0\|_\infty}{\|u_0\|_2} \leq 12\left(\frac{1}{b-a} + \|q\|_1\right)^{1/2}, \ (\lambda \geq 0). \tag{2}$$

For the sake of brevity we use the notation $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(G)}$. V. Komornik [6] proved that the estimates of [3] are not refinable.

The aim of the present paper is to investigate the dependence of $u_i(x)$ at the ends of G on the properties of $q \in C(G)$ for any fixed $\lambda \in \mathbf{C}$. We shall assume throughout this paper that

$$G = (0, 1), q \in C(0, 1), q(x) = O\left(\frac{1}{x^\alpha}\right), \alpha > 0. \tag{3}$$

Define further the function

$$m_i(t) := \|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} \quad \left(0<t<\frac{1}{2}\right). \tag{4}$$

We shall prove the

THEOREM 1.

a) *If $\alpha>2$ then there exists a constant $c_1>0$ such that*

$$m_i(t) = O(e^{c_1 t^{1-\alpha/2}}) \quad \left(0<t<\frac{1}{2};\ i = 0, 1, \ldots\right). \tag{5}$$

b) *If $\alpha = 2$ then there exist a contant $c_2>0$ such that*

$$m_i(t) = O(t^{-c_2}) \quad \left(0<t<\frac{1}{2};\ i = 0, 1, \ldots\right). \tag{6}$$

c) *If $1<\alpha<2$ then for $c_3 := 1/\ln\frac{2}{\alpha}$ we have*

$$m_i(t) = O\left(\left[\ln\frac{1}{t}\right]^{c_3}\right) \quad \left(0<t<\frac{1}{2};\ i = 0, 1, \ldots\right). \tag{7}$$

The constants c_1, c_2, c_3 do not depend on i.

For the proof first we show that if $\delta := c_0 t^{\alpha/2}$ and $c_0 = c_0(\alpha, \lambda)$ is small enough then

$$m_i(t) \le 8m_i(t+\delta) + \delta^2 m_{i-1}(t) \quad (i = 0, 1, \ldots). \tag{8}$$

Using (1) and integrating by parts we get that for any $f\in C^2[x, x+t]$

$$[u_i f' - u_i' f]_x^{x+t} = \int_x^{x+t} \{u_i f'' + \lambda u_i f - q u_i f + u_{i-1} f\}. \tag{9}$$

Let $f(\xi) := \delta - |x+\delta-\xi|$ and apply (9) in $[x, x+\delta]$ and $[x+\delta, x+2\delta]$; then subtraction gives

$$2u_i(x+\delta) - u_i(x) + u_i(x+2\delta) = \int_x^{x+2\delta} u_i(\lambda f - qf) + \int_x^{x+2\delta} u_{i-1} f. \tag{10}$$

Now let $x\in[t, t+\delta]$, $t+2\delta<1/2$, then we get from (10)

$$|u_i(x)| \le 3\|u_i\|_{L^\infty\left(t+\delta,\frac{1}{2}\right)} + \delta^2|\lambda|\cdot\|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} +$$

$$+\delta\left(\int_x^{x+2\delta} |q|\right)\|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} + \delta^2\|u_{i-1}\|_{L^\infty\left(t,\frac{1}{2}\right)}.$$

This is valid for all $x \in [t, t+\delta]$, so

$$\|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} \leq 4\|u_i\|_{L^\infty\left(t+\delta,\frac{1}{2}\right)} + \delta^2|\lambda|\|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} +$$

$$+\delta\left(\int_x^{x+2\delta} \|q\|\right)\|u_i\|_{L^\infty\left(t,\frac{1}{2}\right)} + \delta^2\|u_{i-1}\|_{L^\infty\left(t,\frac{1}{2}\right)}.$$

In [5] is proved (and it is easy to verify) that for sufficiently small c_0 we have

$$\delta\left(\int_x^{x+2\delta} |q|\right) < \frac{1}{4}, \quad \delta^2|\lambda| < \frac{1}{4}$$

and the desired inequality (8) follows.

Now we prove the Theorem 1.

a) The estimate (5) is proved in [5] for $i = 0$. The proof of the general case is similar.

b) We prove (6) only for $i = 0$ because that of $i \in \{1, 2, \ldots\}$ is similar. Iterating (8) we get $m_0(t) \leq 8^n m(t(1+c_0)^n)$, hence for $(1+c_0)^{-n-1} < t < (1+c_0)^{-n}$ we have $m_0(t) \leq \text{const} \cdot 8^n \leq \text{cont} \cdot 8^{\ln\frac{1}{t}/\ln(1+c_0)} = \text{const} \cdot t^{-c_2}$.

c) Define for $t > 0$

$$f_1(t) := t, \; f_{n+1}(t) := c_0[f_n(t)]^{\frac{\alpha}{2}}. \tag{11}$$

It is easy to see that

$$f_{n+1}(t) = c_0^{1+\frac{\alpha}{2}+\ldots+\left(\frac{\alpha}{2}\right)^n} t^{\left(\frac{\alpha}{2}\right)^{n+1}}. \tag{12}$$

It follows from $\delta = c_0 t^{\frac{\alpha}{2}}$ that there exists a constant $c_4 > 0$ such that

$$8^k m_i(f_k(t)) \leq 8^{k+1} m_i(f_{k+1}(t)) + c_4 8^k f_{k+1}^2(t)\left[\ln\frac{1}{f_k(t)}\right]^{c_3}.$$

Summing up these inequalities in k from 1 to n and using (12) we get

$$8m_i(t) \leq 8^{n+1} m_i(c_0^{1+\frac{\alpha}{2}+\ldots+\left(\frac{\alpha}{2}\right)^n} t^{\left(\frac{\alpha}{2}\right)^{n+1}}) +$$

$$+ c_4 \sum_{k=1}^{n} 8^k c_0^{2\left[1+\frac{\alpha}{2}+\ldots+\left(\frac{\alpha}{2}\right)^k\right]} t^{2\left(\frac{\alpha}{2}\right)^{k+1}} \left[\ln\frac{1}{c_0^{1+\frac{\alpha}{2}+\ldots+\left(\frac{\alpha}{2}\right)^{k+1}} t^{\left(\frac{\alpha}{2}\right)^k}}\right]^{c_3}.$$

We can assume $c_0 < 1$; then the last estimate can be continued to obtain

$$8m_i(t) \leq 8^{n+1} m_i(c_0^{\frac{1}{1-\alpha/2}} t^{\left(\frac{\alpha}{2}\right)^{n+1}}) +$$

$$+ c_4 \sum_{k=1}^{n} 8^k t^{2\left(\frac{\alpha}{2}\right)^{k+1}} \left[\frac{1}{1-\frac{\alpha}{2}} \ln\frac{1}{c_0} + \left(\frac{\alpha}{2}\right)^k \ln\frac{1}{t}\right]. \tag{13}$$

Let

$$2^{-\left(\frac{\alpha}{2}\right)^{-n}} \leq t \leq 2^{-\left(\frac{\alpha}{2}\right)^{-n-1}}. \tag{14}$$

We shall prove that

$$\sum_{k=1}^{n} 8^k t^{2\left(\frac{\alpha}{2}\right)^{k+1}} \left(\frac{\alpha}{2}\right)^{c_3 k} \left(\ln \frac{1}{t}\right)^{c_3} \leq c_5 8^n. \tag{15}$$

Indeed, it follows from (14) that

$$\left(\frac{\alpha}{2}\right)^{c_3 k} \left(\ln \frac{1}{t}\right)^{c_3} \leq c_6 \left(\frac{\alpha}{2}\right)^{c_3(k-n)}, \quad t^{2\left(\frac{\alpha}{2}\right)^{k+1}} \leq 2^{-2\left(\frac{\alpha}{2}\right)^{k-n}},$$

hence

$$\sum_{k=1}^{n} 8^k t^{2\left(\frac{\alpha}{2}\right)^{k+1}} \left(\frac{\alpha}{2}\right)^{c_3 k} \left(\ln \frac{1}{t}\right)^{c_3} \leq c_6 8^n \sum_{k=1}^{n} \left(\frac{\alpha}{2}\right)^{c_3(k-n)} 2^{-2\left(\frac{\alpha}{2}\right)^{k-n}}.$$

Since

$$\left(\frac{\alpha}{2}\right)^{c_3(k-n)} 2^{-\left(\frac{\alpha}{2}\right)^{k-n}} \leq c_7, \quad \sum_{k=1}^{n} 2^{-\left(\frac{\alpha}{2}\right)^{k-n}} \leq c_8,$$

hence (15) is proved for any t in (14). Now (15) and (13) show that $m_i(t) \leq c_9 8^n$, which implies by (14) the desired inequality (7). The Theorem 1 is proved.

REMARK. The estimates (5) and (6) are not refinable in the sense that the functions on the right hand sides of (5) and (6) satisfies (1) with a potential $q(t)$ satisfying (3). It is open whether (7) is refinable or not. It would be important to investigate the case $\alpha \leq 1$ too.

2. Now we investigate the second problem raised in [5]. Namely, let $\Sigma\lambda_n$ be a conditionally convergent complex series and let the real-valued function f: $\mathbf{C} \to \mathbf{R}$ be additive in the sense that $z = \Sigma\varepsilon_n\lambda_n$, $\varepsilon_n = \{^0_1$ implies $f(z) = \Sigma\varepsilon_n f(\lambda_n)$. We need the

LEMMA 2. For any complex-valued conditionally convergent series the following alternative holds.

a) There exists a line containing $0 \in \mathbf{C}$ such that the sum of the components of λ_n, orthogonal to this line is absolute convergent.

b) $$\{z : z = \Sigma\varepsilon_n\lambda_n, \ \varepsilon_n = \{^0_1\} = \mathbf{C}.$$

PROOF. As usual we call cluster direction of the series $\Sigma\lambda_n$ the number $\alpha \in [0. 2\pi]$ if there is a subsequence (λ_{n_k}) for which $\sum |\lambda_{n_k}| = \infty$, $\arg \lambda_{n_k} \to \alpha$ $(k \to \infty)$. The set of all cluster directions is obviously compact. It can not be contained in a segment of length $< \pi$ because in an angular domain of angle $< \pi$ the convergence of a series implies its absolute convergence. So there is two possibility:

(i) There exist three cluster directions α, β, γ whose half – lines are not contained in a closed half-plane (angular domain with angle).

(ii) There exist two opposite cluster directions (say 0 and π) and the other cluster directions (if exist) are in $(0, \pi)$. In case of (i) any point $z \in \mathbf{C}$ can be given as

$$z = ae^{i\alpha} + be^{i\beta} + ce^{i\gamma};\ a, b, c \geq 0 \tag{1}$$

and $a = 0$ if $\beta \leq \arg z \leq \gamma \pmod{2\pi}$, $b = 0$ if $\gamma \leq \arg z \leq \alpha \pmod{2\pi}$ and $c = 0$ if $\alpha \leq \arg z \leq \beta \pmod{2\pi}$.

Let

$$\sum_{k=1}^{\infty} |\lambda_{n_k}^{(i)}| = \infty,\ \lim_{k\to\infty} \arg \lambda_{n_k}^{(i)} = \begin{cases} \alpha, & i = 1 \\ \beta, & i = 2 \\ \gamma, & i = 3 \end{cases}.$$

Let $z_0 = 0$. Suppose we gave z_N as a finite sum of $\lambda_{n_k}^{(i)}$

$z_N = \sum_{i=1}^{3} \sum_{k=1}^{m_N} \varepsilon_{k,i} \lambda_{n_k}^{(i)}$ and $|z - z_N| < 2^{-N}$. Use the representation (1) for $z - z_N$ instead of z. The components can be arbitrarily approximated by finite sums $\sum_{k=m_N+1}^{m_{N+1}} \varepsilon_{k,i} \lambda_{n_k}^{(i)}$ such that the partial sums of $\sum_{i=1}^{3} \sum_{k=m_N+1}^{m_{N+1}} \varepsilon_{k,i} \lambda_{n_k}^{(i)}$ (where the order of summands is that of the $\lambda_n - s$) do not exceed $C(\alpha, \beta, \gamma)\, 2^{-N}$ in absolute value. This means that any point $z \in \mathbf{C}$ has a representation

$$\Sigma \varepsilon_n \lambda_n = z. \tag{2}$$

(We remark that if e. g. $\alpha < \arg z < \beta$ then there is a representation of type (2) where only the elements $\lambda_{n_k}^{(1)}$, $\lambda_{n_k}^{(2)}$ are used. We present a simple method of construction. To simplify the notation let $\alpha = 0$, $\beta = \frac{\pi}{2}$ and $z = 1 + i$. Define the numbers ε_n using only the indices of $\lambda_{n_k}^{(1)}$, and $\lambda_{n_k}^{(2)}$ such that for some sequence m_N

$$1 - 2^{-N} \leq \operatorname{Im} \sum_{n=1}^{m_{2N}} \varepsilon_n \lambda_n < 1, \tag{iii}$$

$$1 - 2^{-N} \leq \operatorname{Re} \sum_{n=1}^{m_{2N+1}} \varepsilon_n \lambda_n < 1. \tag{iv}$$

This is possible since α and β are cluster directions and hence the mentioned representation is possible.

In case of (ii) distinguish two cases. If

$$\sum_{\operatorname{Im}\lambda_n < 0}{}_n \operatorname{Im} \lambda_n > -\infty, \tag{3}$$

then by the convergence of $\Sigma \lambda_n$ we have

$$\sum_{\operatorname{Im}\lambda_n > 0}{}_n \operatorname{Im} \lambda_n < \infty, \tag{4}$$

and this is the case *a*) of Lemma 2. If (3) is not true then (4) is also false. We can suppose that

$$\text{(5)} \qquad \operatorname{Im}\lambda_{n_k}^{(1)}<0,\ \sum_k \operatorname{Im}\lambda_{n_k}^{(1)} = -\infty,\ \lim_{k\to\infty}\arg\lambda_{n_k}^{(1)} = 0$$

and we know that

$$\text{(6)} \qquad \sum_k |\lambda_{n_k}^{(2)}| = \infty,\ \lim_{k\to\infty}\arg\lambda_{n_k}^{(2)} = \pi.$$

Using (5) and (6) we can give a representation (2) for any z, $\operatorname{Im} z<0$; the construction is obvious and we omit the details. Investigate the half – plane $\operatorname{Im} z<0$. If there is a cluster direction $0<\gamma<\pi$ then the domains $0<\arg z<\gamma$, $\gamma<\arg z<\pi$ can be recovered by representable points as we observed above. The points in bounding half – lines, e. g. any point z with $\arg z = \gamma$ can be shifted into good domains: $\gamma<\arg(z-\lambda_{n_k}^{(2)})<\pi$.

If there is no cluster directions $0<\gamma<\pi$, then we can suppose e. g. that there is a sequence $\lambda_{n_k}^{(3)}$ such that

$$\text{(7)} \qquad \operatorname{Im}\lambda_{n_k}^{(3)}>0,\ \sum_k \operatorname{Im}\lambda_{n_k}^{(3)} = \infty,\ \lim_{k\to\infty}\arg\lambda_{n_k}^{(3)} = 0$$

and then the half – plane $\operatorname{Im} z>0$ can be filled in just as in case $\operatorname{Im} z<0$. The point $\operatorname{Im} z = 0$ can be shifted into good domains, hence all complex numbers has the desired representation (2). The Lemma 2 is proved.

The following theorem generalizes the corresponding result in [1].

THEOREM 3. *Let* $f:\mathbf{C}\to\mathbf{R}$ *be additive and suppose that*

$$\{z:z = \Sigma\varepsilon_n\lambda_n,\ \varepsilon_n = \tbinom{0}{1}\} = \mathbf{C}.$$

Then f *is linear, that is* $f(z) = c_1 \operatorname{Im} z + c_2 \operatorname{Re} z$ $(c_1, c_2\in\mathbf{R})$.

PROOF. By Lemma 2 we have two possibilities, namely (i) and (ii). Investigate first the case (i). Taking appropriate subsequence in the definition of cluster directions, we see that there exist three sequences $(\lambda_{n_k}^{(i)})$; $i = 1, 2, 3$ such that

$$\lim_{k\to\infty}\arg\lambda_{n_k}^{(1)} = \alpha,\ \lim_{k\to\infty}\arg\lambda_{n_k}^{(2)} = \beta,\ \lim_{k\to\infty}\arg\lambda_{n_k}^{(3)} = \gamma,$$

$$\text{(8)} \qquad \sum_k |\lambda_{n_k}^{(i)}| = \infty,\ \lim_{k\to\infty}\frac{f(\lambda_{n_k}^{(i)})}{|\lambda_{n_k}^{(i)}|} = c^{(i)}\ (i = 1, 2, 3)$$

with some $c^{(i)}\in\overline{\mathbf{R}}:=\mathbf{R}\cup\{\pm\infty\}$. If two of them, say $c^{(1)}$ and $c^{(2)}$ are infinite then by the construction given in Lemma 2 we get for any z, $\alpha<\arg z<\beta$ a representation $z = \Sigma\varepsilon_n\lambda_n$ for which $\Sigma\varepsilon f_n(\lambda_n)$ diverges; this contradicts to the additivity of f. If precisely one of $c^{(i)}$, say $c^{(1)}$ is infinite, then $e^{i\alpha}$ can be arbitrarily approximated by finite series $z = \Sigma\varepsilon_{k,1}\lambda_{n_k}^{(1)}$ such that $|f(z)| = |\Sigma\varepsilon_{k,1}f(\lambda_{n_k}^{(1)})|$ is as large as we want. Since $\beta<\arg(-z)<\gamma$ hence

$$-z = \left(\sum_{i=2}^{3}\sum_{k=1}^{\infty}\right)^{*}\varepsilon_{k,i}\,\lambda_{n_k}^{(i)}$$

(the star means that the summands are ordered as in the sequence (λ_n)). The construction described in Lemma 2 shows the estimate $|f(-z)| \leq \leq C(\alpha, \beta, \gamma)(|c^{(2)}| + |c^{(3)}|)$. Hence $f(0) = f(z)+f(-z)$ can be arbitrarily large in absolute value; this contradiction shows that $c^{(i)} \in \mathbf{R}$ $(i = 1, 2, 3)$. This implies a kind of continuity, namely

$$f\left(\left(\sum_{i=1}^{3} \sum_{k=1}^{\infty}\right)^{*} \varepsilon_{k,i}\, \lambda_{n_k}^{(i)}\right) \leq c(\alpha, \beta, \gamma)(|c^{(1)}| + |c^{(2)}| + |c^{(3)}|) \sum_{i=1}^{3} \sum_{k=1}^{\infty} \varepsilon_{k,i}\, |\lambda_{n_k}^{(i)}|.$$

Consequently, taking into account the representation of any z, $\alpha < \arg z < \beta$ described in Lemma 2, we get the linearity of f in this domain: $f(ae^{i\alpha} + be^{i\beta}) = = ac^{(1)} + bc^{(2)}$ $(a, b > 0)$. Here we can write again by continuity. Let $-e^{i\gamma} = = a_0 e^{i\alpha} + b_0 e^{i\beta}$ then $-c^{(3)} = f(-e^{i\gamma}) = a_0 c^{(1)} + b_c c^{(2)}$, hence the linearities of f in the domains $\alpha \leq \arg z \leq \beta$, $\beta \leq \arg z \leq \gamma$, $\gamma \leq \arg z \leq \alpha$ are "compatible" and then f is a linear $\mathbf{R}^2 \to \mathbf{R}$ function in the usual sense.

Now consider the case (ii). Let

$$\lim_{k\to\infty} \arg \lambda_{n_k}^{(1)} = 0, \lim_{k\to\infty} \arg \lambda_{n_k}^{(2)} = \pi,$$

$$\sum |\lambda_{n_k}^{(1)}| = \sum |\lambda_{n_k}^{(2)}| = \infty, \lim_{k\to\infty} f(\lambda_{n_k}^{(j)})/|\lambda_{n_k}^{(j)}| = c^{(j)} \ (j = 1, 2),$$

$$\text{(9)} \qquad \operatorname{Im} \lambda_{n_k}^{(1)} < 0, \sum \operatorname{Im} \lambda_{n_k}^{(1)} = -\infty.$$

This last condition makes possible to write a representation $-i = \Sigma \varepsilon_n \lambda_n$ using only the members $\lambda_{n_k}^{(j)}$ and such that there is a sequence (m_N) with the property

$$\text{(10)} \qquad \operatorname{Re} \sum_{n=1}^{m_{2N}} \varepsilon_n \lambda_n = \frac{1}{N} + O\left(\frac{1}{N^2}\right), \operatorname{Re} \sum_{n=1}^{m_{2N+1}} \varepsilon_n \lambda_n = -\frac{1}{N} + O\left(\frac{1}{N^2}\right)$$

and such that between m_{2N} and m_{2N+1} only the members $\lambda_{n_k}^{(2)}$, between m_{2N-1} and m_{2N} only the members $\lambda_{n_k}^{(1)}$ are used. This yields that the constants $c^{(1)}$ and $c'^{(2)}$ in (9) are finite and opposite: $-c^{(1)} = c^{(2)} =: c_0 \in \mathbf{R}$. Let $f^*(z) := f(z) + c_0 \operatorname{Re} z$, then f^* is also additive and in (9) we have

$$\lim_{k\to\infty} \frac{f^*(\lambda_{n_k}^{(j)})}{|\lambda_{n_k}^{(j)}|} = 0 \ (j = 1, 2).$$

We shall show that $f^*(z) = \text{const} \operatorname{Im} z$. The function f^* will be denoted agani by f. Taking subsequences we can join to (9) the assumption

$$\text{(9')} \qquad \lim_{k\to\infty} \frac{f(\lambda_{n_k}^{(1)})}{\operatorname{Im} \lambda_{n_k}^{(1)}} = c.$$

Since any point z, $\operatorname{Im} z < 0$ has a representation

$$z = \left(\sum_{j=1}^{2} \sum_{k=1}^{\infty}\right)^{*} \varepsilon_{k,j}\, \lambda_{n_k}^{(j)},$$

where the imaginary part of z is approximated by the aid of $(\lambda_{n_k}^{(1)})$ and the oscillation around Re z is obtained using both $(\lambda_{n_k}^{(1)})$ and $(\lambda_{n_k}^{(2)})$, we easily see that c is finite and

$$f(z) = c \operatorname{Im} z \quad (\operatorname{Im} z<0). \tag{11}$$

Now if there exist a third cluster direction $0<\gamma<\pi$, then there exists the sequence $(\lambda_{n_k}^{(3)})$ such that

$$\sum |\lambda_{n_k}^{(3)}| = \infty, \lim_{k\to\infty} \arg \lambda_{n_k}^{(n)} = \gamma, \lim_{k\to\infty} \frac{f(\lambda_{n_k}^{(3)})}{\operatorname{Im} \lambda_{n_k}^{(3)}} = c'. \tag{12}$$

Then our earlier considerations imply that

$$f(z) = c' \operatorname{Im} z \qquad (\operatorname{Im} z\geq 0). \tag{13}$$

Taking the representation of $-e^{i\gamma}$ we get as earlier that $c = c'$, hence f is indeed linear. It remains the case when only 0 and π are cluster directions. Then we can assume e. g. that there exist $(\lambda_{n_k}^{(3)})$ with

$$\operatorname{Im} \lambda_{n_k}^{(3)} > 0, \ \sum \operatorname{Im} \lambda_{n_k}^{(3)} = \infty, \ \lim_{k\to\infty} \arg \lambda_{n_k}^{(3)} = 0, \tag{14}$$

$$\lim_{k\to\infty} \frac{f(\lambda_{n_k}^{(3)})}{\operatorname{Im} \lambda_{n_k}^{(3)}} = c'.$$

As we showed this implies

$$f(z) = c' \operatorname{Im} z \quad (\operatorname{Im} z\geq 0). \tag{15}$$

If we converge to the points x of the real axis, the ambiguity of the resulting values $f(x)$ can be eliminated only when $c = c'$. and in this case $f(z) = cz$ $(z\in \mathbf{C})$. The Theorem 3 is proved.

REMARK. The method used in the proof can be easily applied to show the following analogon of Theorem 3: if $\Sigma\lambda_n$ is a real conditionally convergent series and f: $\mathbf{R}\to\mathbf{R}$ is additive, then $f(x) = cx$. On the other hand it is easy to prove that for any $\varepsilon>0$ there exists a sequence $(\lambda_n)_1^\infty \subset \mathbf{C}$ such that $\Sigma|\lambda_n| = 1$ and $\{\Sigma\varepsilon_n\lambda_n\colon \varepsilon_n = 0 \text{ or } 1 \text{ for all } n\}$ covers $C\left(0, \frac{1}{\pi}-\varepsilon\right)$, the open disk with origin at 0 and with radius $\frac{1}{\pi}-\varepsilon$. It is not hard to prove that this is the best possible in the sense that there is no $(\lambda_n)_1^\infty \subset \mathbf{C}$ such that

$$\sum |\lambda_n| = 1 \text{ and } C\left(0, \frac{1}{\pi}\right) \subset \{\sum \varepsilon_n \lambda_n : \varepsilon_n = 0 \text{ or } 1 \ \forall n\}.$$

(It would be interesting to characterize the the linear functions in the set of additive ones on sums of just describted type). Indeed, let $N_0 \in \mathbf{N}$ and

$$\lambda_n := \frac{1}{2^{N_0}} \exp\left\{i \frac{2\pi n}{N_0}\right\} \qquad (1 \leq n \leq 2^{N_0}). \tag{16}$$

Since

$$2 \sum_{n=1}^{2^{N_0-2}} \cos \frac{n\pi}{2^{N_0-1}} = 2 \frac{\cos\left(\frac{2^{N_0-2}+1}{2} \frac{\pi}{2^{N_0-1}}\right) \cos\left(2^{N_0-2} \frac{\pi}{2^{N_0-1}}\right)}{\sin \frac{\pi}{2^{N_0}}},$$

hence

$$\frac{1}{2^{N_0+1}} 2 \sum_{n=1}^{2^{N_0-2}} \cos \frac{n\pi}{2^{N_0-1}} = \sum_{n=1}^{2^{N_0-2}} \lambda_n + \sum_{n=3\cdot 2^{N_0-}}^{2^{N_0-1}} \lambda_n \to \frac{1}{\pi} \qquad (N_0 \to \infty). \tag{17}$$

The sums

$$\exp\left\{i \frac{k\pi}{2^{N_0-1}}\right\} \frac{1}{2^{N_0}} \cos \frac{\pi}{2^{N_0-1}};$$

$$\exp\left\{i \frac{k\pi}{2^{N_0-1}}\right\} \frac{1}{2^{N_0}} \left(\cos \frac{\pi}{2^{N_0-1}} + \cos \frac{2\pi}{2^{N_0-1}}\right) \ldots;$$

$$\exp\left\{i \frac{k\pi}{2^{N_0-1}}\right\} \frac{1}{2^{N_0}} \sum_{n=1}^{2^{N_0-2}} \cos \frac{n\pi}{2^{N_0-1}} \qquad (k = 1, \ldots, 2^{N_0})$$

have all the form

$$\sum_{n=1}^{2^{N_0}} \varepsilon_n \lambda_n, \quad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}. \tag{18}$$

Taking (17) into account we see that for any sufficiently large $N_0 = N_0(\varepsilon)$ any point in $C\left(0, \frac{1}{\pi} - \varepsilon\right)$ is at a distance not greater than $\frac{1}{2}\left(\frac{1}{\pi} - \varepsilon\right)$ from a point of the form (18). Now define

$$\lambda_{2^{N_0}+n} = \frac{1}{2^{N_1+2}} \exp\left\{i \frac{2n\pi}{2^{N_1}}\right\} \qquad (1 \leq n \leq 2^{N_1}). \tag{19}$$

The above arguments can be repeated to obtain that the union of the disks of radius $\frac{1}{4}\left(\frac{1}{\pi} - \varepsilon\right)$ with centers of the form

$$\sum_{n=1}^{2^{N_0}+2^{N_1}} \varepsilon_n, \lambda_n, \qquad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}$$

contains $C\left(0, \frac{1}{\pi}-\varepsilon\right)$. In the third step we define $\lambda_{2^{N_0}+2^{N_1}+n}$, $1 \leq n \leq 2^{N_2}$ such that the numbers

$$\sum_{n=1}^{2^{N_0}+2^{N}+2^{N_2}} \varepsilon_n \lambda_n, \qquad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}$$

approximate any points of $C\left(0, \frac{1}{\pi}-\varepsilon\right)$ at a distance $\leq \frac{1}{8}\left(\frac{1}{\pi}-\varepsilon\right)\ldots$ etc. Since

$$\sum_{n=1}^{2^{N_0}} |\lambda_n| = \frac{1}{2}, \qquad \sum_{n=2^{N_0}+1}^{2^{N_0}+2^{N_1}} |\lambda_n| = \frac{1}{4} \ldots \text{ etc,}$$

we have $\sum_1^\infty |\lambda_n| = 1$ and the required property of $(\lambda_n)_1^\infty$ follows from the construction. Now suppose indirectly that there exists a sequence $(\lambda_n)_1^\infty \subset \mathbf{C}$ such that $\sum_1^\infty |\lambda_n| = 1$ and $C\left(0, \frac{1}{\pi}\right) \subset \{\varepsilon_n \lambda_n, \varepsilon_n = \binom{0}{1}\}$. Then for any $\varphi \in [0, 2\pi]$

$$\sum_{n \in N(\varphi)} (a_n \cos\varphi + b_n \sin\varphi) \geq \frac{1}{\pi}, \tag{20}$$

where

$$N(\varphi) := \{n : a_n \cos\varphi + b_n \sin\varphi > 0\}; \quad \lambda_n = a_n + ib_n; \quad a_n, b_n \in \mathbf{R}.$$

Consequently

$$\sum_{n=1}^{\infty} |a_n \cos\varphi + b_n \sin\varphi| \geq \frac{2}{\pi}. \tag{21}$$

If we set $a_n + ib_n = r_n e^{i\varphi_n}$, $r_n = \sqrt{a_n^2 + b_n^2}$, then $a_n \cos\varphi + b_n \sin\varphi = r_n \cos(\varphi - \varphi_n)$ implies

$$\sum_{n=1}^{\infty} r_n |\cos(\varphi - \varphi_n)| \geq \frac{2}{\pi} \qquad (0 \leq \varphi < 2\pi). \tag{22}$$

From $\int_0^{2\pi} |\cos(\varphi - \varphi_n)| d\varphi = 4$ we see that

$$\int_0^{2\pi} \sum_{n=1}^{\infty} r_n |\cos(\varphi - \varphi_n)| d\varphi = 4 = \int_0^{2\pi} \frac{2}{\pi} d\varphi,$$

consequently

$$\sum_{n=1}^{\infty} |a_n \cos\varphi + b_n \sin\varphi| \equiv \frac{2}{\pi} \qquad (0 \leq \varphi < 2\pi), \tag{23}$$

and then

$$\sum_{n\in N(\varphi)}{}_n (a_n \cos\varphi + b_n \sin\varphi) \equiv \frac{1}{\pi} \qquad (0 \leq \varphi < 2\pi). \tag{24}$$

Consider any φ, $\varphi \pm \pi \neq \varphi_n \; \forall_n$. Property $C\left(0, \frac{1}{\pi}\right) \subset \{\sum \varepsilon_n \lambda_n \colon \varepsilon_n = 0 \text{ or } 1 \; \forall_n\}$ and (24) imply that

$$\sum_{n\in N(\varphi)}{}_n \lambda_n = \frac{1}{\pi} e^{i\varphi} \tag{25}$$

even if $C\left(0, \frac{1}{\pi}\right)$ denotes the open disk. Now let $\varphi + \pi < \varphi_{n_0} < \varphi' + \pi$ and $\varphi \pm \pi \neq \neq \varphi_n$, $\varphi' \pm \pi \neq \varphi_n \; \forall\, n$. If $\varphi' - \varphi < \frac{\pi}{2}$ then

$$\left| \sum_{n\in N(\varphi')}{}_n \lambda_n - \sum_{n\in N(\varphi)}{}_n \lambda_n \right| \geq |\lambda_{n_0}|.$$

On the other hand, $\varphi' - \varphi$ can be supposed arbitrarily small, hence, $\frac{1}{\pi}(e^{i\varphi'} - e^{i\varphi})$ is also arbitrarily small; this contradicts to (25).

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ON ADDITIVE FUNCTIONS

By

M. HORVÁTH

Department for Analysis of the L. Eötvös University, Budapest

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This paper continues the investigations of [1], [2] taking instead of $\sum_{n=1}^{\infty} q^{-n}$, $1<q<2$ a general series $\sum \lambda_n$ satisfying some conditions. While the main tool applied in [1], [2] is the solution of a difference equation of infinite order, in the present paper the difficulty is a measure theoretic problem.

Consider a sequence

$$\lambda_n>0, \quad \sum_{n=1}^{\infty} \lambda_n = 1$$

with the property

$$\lambda_n \leq \sum_{k=n+1}^{\infty} \lambda_k \quad (n = 1, 2, \ldots). \tag{1}$$

Define for $0 \leq x \leq 1$

$$\varepsilon_n(x) := \begin{cases} 1 \text{ for } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leq x \\ 0 \text{ for } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x. \end{cases} \tag{2}$$

The property (1) is equivalent to the assertion

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n \quad (0 \leq x \leq 1). \tag{3}$$

A function

$$F:[0, 1] \to \mathbf{R}$$

is called additive if

$$F(x) = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n \quad (0 \leq x \leq 1), \tag{4}$$

$$F(x) = \sum_{n=1}^{\infty} \tilde{\varepsilon}_n(x)\lambda_n, \quad \tilde{\varepsilon}_n(x) := 1 - \varepsilon_n(1-x) \quad (0 \leq x \leq 1). \tag{5}$$

We shall see that further conditions on the sequence (λ_n) imply that any additive and nonnegative function F must be linear, $F(x) = cx$. In [1] and [2] analogous results are obtained using only (4), not (5). But we use (5) only to prove continuity of F which do not follow from (4), see [1]. So the main result of this paper establishes Theorem 5.2. of [2] for general series $\Sigma \lambda_n$. We remark that I. Joó [3] considered the following related problem. Let $\Sigma \lambda_n$ be a conditionally convergent complex-valued series and $F: \mathbf{C} \to \mathbf{C}$ be an analytic (entire) function additive in the sense that

$$z = \sum \varepsilon_n \lambda_n, \quad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases} \text{ implies } F(z) = \sum \varepsilon_n F(\lambda_n).$$

He proved that F is linear, $F(z) = cz$.

First we investigate the properties of the expansions (3).

LEMMA 1. The stepfunctions $\varepsilon_n(x)$ are continuous from the right.

PROOF. By induction on n. The case $n = 1$ follows from (2). Let the proof be ready for $n-1$ and take the index n. Let $0 \leq x < 1$, $x < x_k \leq 1$, $x_k \to x$ $(k \to \infty)$. Then there exists $\delta_x > 0$ such that $x + \delta_x \geq x_k > x$ implies

$$\sum_{i=1}^{n-1} \varepsilon_i(x_k)\lambda_i = \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i.$$

If $\varepsilon_n(x) = 0$ then for sufficiently large k $\varepsilon_n(x_k) = 0$; if $\varepsilon_n(x) = 1$ then for all $x < x_k \leq x + \delta_x$ we have $\varepsilon_n(x_k) = 1$ as it can be seen from (2). The proof is complete.

COROLLARY. An additive and nonnegative function $F: [0, 1] \to \mathbf{R}$ is continuous.

PROOF. Since $\sum F(\lambda_n) = F(1) < \infty$ hence the series

$$F(x) = \sum \varepsilon_n(x)\lambda_n$$

is uniformly convergent and its partial sums are continuous from the right, hence F has the same property. On the other hand, $\tilde{\varepsilon}_n(x)$ is left-continuous and then F is left-continuous, too.

LEMMA 2. If $F \geq 0$ is additive then it is monotone increasing.

PROOF. It is indirect. Suppose that there exists $0 \leq x < y \leq 1$ such that

$$F(x) > F(y).$$

Let

$$F(z) = \min_{[y,\, x]} F \qquad (z \in [x, y]); \tag{6}$$

then $z \neq x$. If the regular expansion of z, i. e.

$$z = \sum_{n=1}^{\infty} \varepsilon_n(z)\lambda_n$$

is infinite then for large N

$$x < z_N = \sum_{n=1}^{N} \varepsilon_n(z)\lambda_n < z.$$

Since $\sum_{n=1}^{N} \varepsilon_n(z)\lambda_n$ is the regular expansion of z_N, hence

$$F(z_N) = \sum_{n=1}^{N} \varepsilon_n(z)F(\lambda_n) < F(z)$$

in contradiction with (6). If

$$z = \sum_{n=1}^{N_0} \varepsilon_n(z)\lambda_n, \quad \varepsilon_{N_0}(z) = 1$$

then write λ_{N_0} by the "greedy" algorithm just as in (2), but using the sequence $(\lambda_{N_0+n})_{n=1}^{\infty}$

$$\lambda_{N_0} = \sum_{n=N_0+1}^{\infty} \varepsilon_n\lambda_n. \tag{7}$$

If the expansion (7) is infinite, then

$$z_{1,\,N} = \sum_{n=1}^{N_0-1} \varepsilon_n(z)\lambda_n + \sum_{n=N_0+1}^{N} \varepsilon_n\lambda_n \tag{8}$$

is the regular expansion of its sum for any $N > N_0$. Consequently

$$F(z_{1,\,N}) = F(z) - F(\lambda_{N_0}) + \sum_{n=N_0+1}^{N} \varepsilon_n F(\lambda_n)$$

hence it is monotone increasing. Since $z_{1,\,N} \to z$ implies $F(z_{1,\,N}) \to F(z)$ by Lemma 1., we get

$$F(z_{1,\,N}) < F(z)$$

in contradiction with (6) for N large enough. If the expansion (7) is finite,

$$\lambda_{N_0} = \sum_{n=N_0+1}^{N_1} \varepsilon_n\lambda_n, \qquad \varepsilon_{N_1} = 1 \tag{9}$$

then take the greedy expansion of λ_{N_1} with respect to $(\lambda_{N_1+N})_{n=1}^{\infty}$. If it is infinite then a contradiction can be given as above. If it is finite then we iterate the process with its last member. Suppose that the process was iterated k times and the expansions of type (7) were all finite

$$\lambda_{N_1} = \sum_{n=N_1+1}^{N_2} \varepsilon_n \lambda_n, \ldots, \lambda_{N_k} = \sum_{n=N_1+1}^{N_{k+1}} \varepsilon_n \lambda_n. \tag{10}$$

In this case the expansion

$$z^{(k)} = \sum_{n=1}^{N_{k+1}-1} \varepsilon_n \lambda_n$$

is regular. Now $z - z^{(k)} = \lambda_{N_{k+1}}$ implies $z^{(k)} \to z$. On the other hand, the sequence

$$F(z^{(k)}) = \sum_{n=1}^{N_{k+1}-1} \varepsilon_n F(\lambda_g)$$

is monotone increasing and tends to $F(z)$, hence

$$F(z^{(k)}) < F(z);$$

contradiction for large k. The Lemma 2 is proved.

Consider again the greedy expansions

$$\lambda_n = \sum_{k=n+1}^{\infty} \varepsilon_{n,k} \lambda_k. \tag{11}$$

Define the following two properties:

$$\lambda_1 \geq \lambda_2 \geq \ldots \tag{12}$$

(13) There exists a number $N \in \mathbf{N}$ such that for any n the expansion (11) does not contain $N+1$ consecutive zero coefficients between two coefficients 1.

LEMMA 3. Suppose (1), (12), (13). Then for any index sequence the set

$$H := \{x \in [0, 1] : \varepsilon_{n_k}(x) = 0 \text{ for all } k\}$$

has (Lebesgue-) measure 0.

PROOF. As in [1] define the finite set

$$V_n = \{x \in [0, 1] : \varepsilon_{n+1}(x) = \varepsilon_{n+2}(x) = \ldots = 0\}, \qquad n \in \mathbf{N}.$$

If $0 < x \in V_n$ then the largest number $\hat{x} \in V_n$, $\hat{x} < x$ is called the left neighbour of x in V_n. Clearly $\hat{x}$ can be expressed in the following manner, already described in the proof of Lemma 2. Let $k \leq n$ be the largest index such that $\varepsilon_k(x) = 1$. If the expansion (11) of λ_k contains a nonvanishing coefficient of index larger than n, then

$$\hat{x} = \sum_{i=1}^{k-1} \varepsilon_i(x) \lambda_i + \sum_{i=k+1}^{n} \varepsilon_{k,i} \lambda_i$$

and it is the regular expansion (3) of $\hat{x}$. If the expansion (11) of λ_k has the form

$$\lambda_k = \sum_{i=k+1}^{k_1} \varepsilon_{k,\,i}\lambda_i, \qquad \varepsilon_{k,\,k_2} = 1$$

and $k_1 \leq n$ then take the expansion of λ_{k_1}. If it contains a nonvanishing coefficient of index larger than n then the regular expansion of $\hat{x}$ is

$$\hat{x} = \sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i + \sum_{i=1}^{k_1-1} \varepsilon_{k,\,i}\lambda_i + \sum_{i=k+1}^{n} \varepsilon_{k_1,\,i}\lambda_i.$$

If

$$\lambda_{k_1} = \sum_{i=k_1+1}^{k_1-1} \varepsilon_{k_1,\,i}\lambda_i + \lambda_{k_2}$$

and $k_2 \leq n$ then take the expansion of λ_{k_2} .... and so on. Hence we can construct the left neighbour of x. It is obvious from the construction that

$$x - \hat{x} \leq \lambda_n \text{ and } x - \hat{x} = \lambda_n \text{ if } \varepsilon_n(x) = 1. \tag{14}$$

Now observe that if $0 < x \in V_n$ and $\varepsilon_n(x) = 0$ then between its $N+1$ consecutive left-neighbours there exists an x_0 with $\varepsilon_n(x_0) = 1$. Indeed, by (12) and (13) there exists a largest index $n \geq k \geq n-N$ such that $\varepsilon_k(\hat{x}) = 1$; then by (12) $\varepsilon_{k+1}(\hat{\hat{x}}) = 1$ so there exists a largest index $n \geq k_1 \geq n-N+1$ such that $\varepsilon_{k_1}(\hat{\hat{x}}) = 1$ etc. Since

$$\hat{x} < y \leq x \text{ implies } \varepsilon_k(y) = \varepsilon_k(x) \qquad 1 \leq k \leq n$$

hence the above considerations and (14) show that the set

$$H_n = \{x \in [0, 1] : \varepsilon_n(x) = 0\}$$

avoids in any segment of length $(N+2)\lambda_n$ a subsegment of length λ_n. Now

$$H = \bigcap_{k=1}^{\infty} H_{n_k}$$

and the assertion mes $H = 0$ follows (analogously to the usual proof of the fact that e. g. the Cantor set has zero measure.) The Lemma 3 is proved.

Consider now the condition

$$\lambda_n \geq c_0 \sum_{k=n+1}^{\infty} \lambda_k \qquad (n \in \mathbf{N}). \tag{1'}$$

LEMMA 4. Suppose (1), (1′), (12), (13). Then there exists a number $c > 0$ such that

$$\lim_{n\to\infty} \frac{F(\lambda_n)}{\lambda_n} = c. \tag{15}$$

PROOF. F is monotone increasing by Lemma 2, hence it is differentiable a.e. Take a number $x \in [0, 1]$ with infinitely many nonzero coefficients ($\varepsilon_{n_k}(x) = 1$) such that $F'(x)$ exists; this is obviously possible. Define

$$S_n(x) := \sum_{k=1}^{n} \varepsilon_k(x)\lambda_k;$$

this is the regular expansion of its sum. Consequently

$$\frac{F(S_{n_k}(x)) - F(S_{n_k-1}(x))}{S_{n_k}(x) - S_{n_k-1}(x)} = \frac{F(\lambda_{n_k})}{\lambda_{n_k}}$$

and the left hand side converges by (1′) to

$$c = F'(x) = \lim_{k\to\infty} \frac{F(\lambda_{n_k})}{\lambda_{n_k}}. \tag{16}$$

If there is another sequence (n'_k) such that

$$\lim_{k\to\infty} \frac{F(\lambda_{n'_k})}{\lambda_{n'_k}} = c' \tag{17}$$

for some $0 \leq c'$ or $c' = +\infty$ then argue as follows. By Lemma 3 the set of points x having only finitely many nonzero coefficients $\varepsilon_{n_k}(x) = 1$ has measure 0. Hence the set of points $\tilde{x}$ for which $\varepsilon_{n'_k}(\tilde{x}) = 1$ and $\varepsilon_{n'_i}(\tilde{x}) = 1$ for infinitely many k and i has measure 1 so it contains a differentiability point. The considerations leading to (16) show that

$$c = F'(\tilde{x}) = c'$$

and this proves the Lemma 4.

THEOREM. *Suppose* (1), (1′), (12), (13). *Then any nonnegative and additive function F must be linear*:

$$F(x) = cx \qquad (x \in [0, 1]).$$

PROOF. We showed in proving Lemma 4. that

$$F'(x) = c \quad \text{for a. e.} \quad x \in [0, 1];$$

it remains to verify that F is absolutely continuous. It follows from (15) that there is a number $C > 0$ such that

$$F(\lambda)_n \leq C\lambda_n \qquad (n \in \mathbf{N}).$$

The function

$$F^*(x) := Cx - F(x)$$

is also additive and $F^*(\lambda_n) \geq 0$ implies $F^*(x) \geq 0$. Now Lemma 2 asserts that F^* is monotone increasing, hence $x < y$ implies

$$0 \leq F(y) - F(x) \leq C(y - x);$$

F belongs to the Lipschitz class, and then it is absolutely continuous so it must be linear. The Theorem is proved.

Since above we used (5) only to show that F is continuous, so we have the following

COROLLARY. Suppose (1), (1′), (12), (13). Then any nonnegative and continuous function F satisfying (4) is linear,

$$F(x) = cx \qquad (x \in [0, 1]).$$

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НЕРАВЕНСТВО ТИПА ХАУСДОРФА–ЮНГА ДЛЯ СОБСТВЕННЫХ ФУНКЦИЙ ОПЕРАТОРА ЛАПЛАСА В ТРЁХМЕРНОЙ ОБЛАСТИ

В. А. ИЛЬИН, и И. ЙО

Московский Университет им. Ломоносова, Москва и Математический Институт Университета им. Этвеша, Будапешт

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Настоящая статья является развитием исследований её авторов, ранее опубликованных в работе [1]. Как и в работе [1] мы изучаем разложения по собственным функциям, вообще говоря, несамосопряжённого оператора Лапласа в произвольной трёхмерной области G для функции, зависящей только от сферического радиуса.

Пусть $\{u_n\}$ — система собственных функций оператора Лапласа в области G. Под этим мы понимаем только следующее: каждая комплекснозначная функция $u_n(x)$ принадлежит классу $C^2(G)$ и для некоторого, вообще говоря, комплексного числа λ_n является в открытой области G решением уравнения

$$\Delta u_n + \lambda_n u_n = 0.$$

Ни полнота, ни минимальность, ни ортогональность системы $\{u_n\}$ не требуется. Более того, система $\{u_n\}$ не является ограниченной в области G в совокупности. Это показывает пример собственных функций оператора Лапласа в трёхмерном шаре с однородным краевым условием первого рода на границе этого шара: для обладающих сферической симметрией собственных функций указанной задачи существует такая положительная постаянная α, что в центре x_0 указанного шара справедливо неравенство:

$$|u_n(x_0)| \geq \alpha |\lambda_n|^{1/2}.$$

Тем не менее, нам удалось доказать, что при выполнении двух естественных условий

$$|\operatorname{Im} \sqrt{\lambda_n}| \leq C_1, \tag{1}$$

$$\sum_{t \leq \sqrt{\lambda_n} \leq t+1} |u_n(y)|^2 \leq C_2 (t+1)^2 \tag{2}$$

(t — любое неотрицательное вещественное число, y — любая внутренняя точка области G) для всякой функции, зависящей только от расстояния переменной точки x области G от фиксированной точки этой же области y и имеющей в G компактный носитель, справедливо неравенство типа Хаусдорфа — Юнга.

Полученный нами результат является новым даже для случая, когда $\{u_n\}$ представляет собой систему собственных функций произвольного самосопряженного расширения оператора Лапласа в области G с точечным спектром (при этом оба условия (1) и (2) всегда выполнены).

Переходим к развёрнутой формулировке и к доказательству полученных результатов.

Определение. Будем говорить, что комплекснозначная функция $f(x)$ принадлежит в области G классу радиальных функций, если $f(x)$ зависит только от расстояния точки x от фиксированной точки y области G и обращается в нуль вне некоторого шара радиуса R целиком лежащего в области G.

Теорема. *Если функция $f(x)$ принадлежит в области G классу радиальных функций, зависящих от расстояния r до фиксированной точки y этой области, и для некоторого p из полуинтервала $1<p\leq 2$ допускает существование интеграла $\int\limits_0^R r^p|f(r)|^p\,dr$ и если выполнены условия* (1) *и* (2), *причём выполнение условия* (2) *требуется только в данной фиксированной точке y, то для функции $f(x)$ справедливо неравенство типа Хаусдорфа — Юнга*

$$\left\{\sum_{n=1}^{\infty}\left|\int\limits_G \overline{f}(x)u_n(x)\,dx\right|^q\right\}^{1/q}\leq C\left\{\int\limits_0^R r^p|f(r)|^p\,dr\right\}^{1/p}, \tag{3}$$

в котором q связано с p условием $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Замечание. Тривиально проверяется, что для существования интеграла, стоящего в правой части (3), достаточно чтобы принадлежащая классу радиальных функций $f(x)$ допускала существование интеграла $\int\limits_G |f(x)|^{p'}\,dx$ при любом p' удовлетворяющем условию $p'>\dfrac{3p}{p+1}$. В самом деле, записывая интеграл, стоящий в правой части (3), в виде ($p'>p$)

$$\int\limits_0^R\left[r^{\frac{2p}{p'}}\cdot|f(r)|^p\right]\cdot r^{p-\frac{2p}{p'}}\,dr$$

и затем применяя к этому интегралу неравенство Гёльдера

$$\int_0^R |\varphi(r)|\,|\Psi(r)|\,dr \leq \left(\int_0^R |\varphi(r)|^{\varrho}\,dr\right)^{1/q} \cdot \left(\int_0^R |\Psi(r)|^{\varrho_1}\,dr\right)^{1/\varrho_1}$$

при

$$\varrho = \frac{p'}{p} > 1,\ \varrho_1 = \frac{p'}{p'-p},\ \varphi(r) = r^{\frac{2p}{p'}}|f(r)|^p,\ \Psi(r) = r^{p-\frac{2p}{p'}}$$

и учитывая существование интеграла

$$\int_0^R r^{p-\frac{2p}{p'}}\,dr$$

мы убедимся в справедливости сформулированного замечания.

Перейдём к доказательству сформулированной теоремы. Пусть $f(r)$ — произвольная функция, зависящая только от расстояния r переменной точки x от фиксированной точки y области G, обращающаяся в нуль при $r \geq R$ и допускающая интеграл, стоящий в правой части (3).

Как и в работе [1] с помощью формулы среднего значения Вебера для регулярных решений уравнения $\Delta u_n + \lambda_n u_n = 0$, мы убедимся в том, что коэффициент Фурье f_n функции f по системе $\{u_n\}$ имеет вид

$$f_n = \int_G u_n(x)\,\bar{f}(x)dx = \frac{4\pi\, u_n(y)}{\sqrt{\lambda_n}} \int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr. \tag{4}$$

Теорема будет доказана, если мы установим, что сумма по всем номерам n

$$\left(\sum_{n=1}^{\infty} |f_n|^q\right)^{1/q} \equiv 4\pi\left\{\sum_{n=1}^{\infty} \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}}\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^q\right\}^{1/q} \tag{5}$$

мажорируется величиной, стоящей в правой части (3). Представим сумму, стоящую в фигурных скобках в (5) в следующем виде

$$\sum_{n=1}^{\infty} |f_n|^q = (4\pi)^q \sum_{k=1}^{\infty}\left[\sum_{\frac{2\pi}{R}(k-1) \leq |\sqrt{\lambda_n}| \leq \frac{2\pi}{R}k} \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}}\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^p\right]. \tag{6}$$

Учитывая, что для каждого номера n, фигурующего во внутренней (заключённой в квадратные скобки) сумме справедливо представление $\sqrt{\lambda_n} = \pm\frac{2\pi}{R}k + \delta_n$ с одним из двух знаков $+$ или $-$ и с вытекающей из неравенства (2) оценкой

$$\delta_n = O(1) \tag{7}$$

и повторяя рассуждениия, проведенные в работе [1], мы установим для интеграла, стаящего в (6), следующее соотношение

$$\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr = \pm\int_0^R \varrho f(\varrho)\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho \pm$$

$$\pm\int_0^R \delta_n \sin(\delta_n r)\left(\int_r^R \varrho\bar{f}(\varrho)\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr +$$

$$+\int_0^R \delta_n \cos(\delta_n r)\left(\int_r^R \varrho\bar{f}(\varrho)\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right)dr. \tag{8}$$

Из (8) с учётом оценки (7) и вытекающих из неё оценок $|\sin(\delta_n r)|\leq C_1$ $\cos(\delta_n r)|\leq C_2$ ми получим, применяя неравенство Гёльдера, что существуют постаянные C_3, C_4, C_5 такие, что

$$\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^q \leq C_3\left|\int_0^R \varrho\bar{f}(\varrho)\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^q +$$

$$+C_4\int_0^R\left|\int_r^R \varrho\bar{f}(\varrho)\sin\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^q dr + C_5\int_0^R\left|\int_r^R \varrho\bar{f}(\varrho)\cos\left(\frac{2\pi}{R}k\varrho\right)d\varrho\right|^q dr, \tag{9}$$

причём (9) справедливо с одним и теми же постоянныи C_3, C_4, C_5 для всех номеров n и для каждого k.

Разбиваем сумму (6) на две суммы

$$\sum_{n=1}^{\infty}|f_n|^q = (4\pi)^q\left\{\sum_{0\leq|\sqrt{\lambda_n}|<\frac{2\pi}{R}}\frac{|u_n(y)|^q}{|\lambda_n|^{q/2}}\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^q +\right.$$

$$\left.+\sum_{n=2}^{\infty}\left[\sum_{\frac{2\pi}{R}(k-1)\leq|\sqrt{\lambda_n}|\leq\frac{2\pi}{R}k}\frac{|u_n(y)|^q}{|\lambda_n|^{q/2}}\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^q\right]\right\}. \tag{10}$$

Первую из сумм в правой части (10) оцениваем, учитывая, что в силу неравенства (1) для всех λ_n, удовлетворяющих неравенству $0\leq|\sqrt{\lambda_n}|\leq$ $\leq 2\pi/R$, справедливо неравенство

$$\left|\frac{\sin(r\sqrt{\lambda_n})}{r\sqrt{\lambda_n}}\right|\leq C_6.$$

Применяя к интегралу в этой сумме неравенство Гёльдера, получим

$$\sum_{0\le|\sqrt{\lambda_n}|<\frac{2\pi}{R}} \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}} \left| \int_0^R r\bar{f}(r) \sin(r\sqrt{\lambda_n})\, dr \right|^q \le$$

$$\le \sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}} |u_n(y)|^q \left[\int_0^R |rf(r)|^p\, dr \right]^{q/p} \left[\int_0^R \left| \frac{\sin(r\sqrt{\lambda_n}}{r\sqrt{\lambda_n}} \right|^q r^q\, dr \right] \le$$

$$\le C_7 \left[\int_0^R |rf(r)|^p\, dr \right]^{q/p} \sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}} |u_n(y)|^q. \tag{11}$$

Заметим, что в силу оценки (2) для всех номеров n, для которых $0\le$ $\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}$, справедливо неравенство $|u_n(y)|\le C_8$, из которого вытекает (с учётом того, что $q\ge 2$), что

$$\sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}}{}_n |u_n(y)|^q \equiv \sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}}{}_n |u_n(y)|^2 |u_n(y)|^{q-2} \le$$

$$\le C_8^{q-2} \sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}}{}_n |u_n(y)|^2 \le C_9. \tag{12}$$

Из (11) и (12) вытекает, что

$$\sum_{0\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}}{}_n \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}} \left| \int_0^R r\bar{f}(r) \sin(r\sqrt{\lambda_n})\, dr \right|^q \le C_{10} \left[\int_0^R r^p |f(r)|^p\, dr \right]^{q/p}. \tag{13}$$

Перейдём к оценке второй из сумм, стоящих в правой части (10). Используя оценку (9), мы перепишем эту вторую из сумм в следующем виде

$$\sum_{k=2}^{\infty} \left[\sum_{\frac{2\pi}{R}(k-1)\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}k}{}_n \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}} \left| \int_0^R r\bar{f}(r) \sin(r\sqrt{\lambda_n})\, dr \right|^q \right] \le$$

$$\le \left\{ C_3 \left| \int_0^R \varrho\bar{f}(\varrho) \sin\left(\frac{2\pi}{R}k\varrho\right) d\varrho \right|^q + C_4 \int_0^R \left| \int_r^R \varrho\bar{f}(\varrho) \sin\left(\frac{2\pi}{R}k\varrho\right) d\varrho \right|^q dr + \right.$$

$$\left. + C_5 \int_0^R \left| \int_r^R \varrho\bar{f}(\varrho) \cos\left(\frac{2\pi}{R}k\varrho\right) d\varrho \right|^q dr \right\} \sum_{\frac{2\pi}{R}(k-1)\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}k}{}_n \left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right|^q. \tag{14}$$

Из неравенства (2) сразу же заключаем, что при всех $|\sqrt{\lambda_n}| \geq \frac{2\pi}{R}$ справедлива оценка

$$\left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right| \leq C_{11}, \tag{15}$$

из которой в силу того, что $q \leq 2$, с помощью того же неравенства (2) получим

$$\sum_{\frac{2\pi}{R}(k-1) \leq |\sqrt{\lambda_n}| \leq \frac{2\pi}{R}k}{}_n \left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right|^q \equiv \sum_{\frac{2\mu}{R}(k-1) \leq |\sqrt{\lambda_n}| \leq \frac{2\pi}{R}k}{}_n \left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right|^2 \cdot \left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right|^{q-2} \leq$$

$$\leq C_{11}^{q-2} \sum_{\frac{2\pi}{R}(k-1) \leq |\sqrt{\lambda_n}| \leq \frac{2\pi}{R}k}{}_n \left| \frac{u_n(y)}{\sqrt{\lambda_n}} \right|^2 \leq C_{12}. \tag{16}$$

Используя в правой части (14) оценку (16), придём к неравенству

$$\sum_{k=2}^{\infty} \left[\sum_{\frac{2\pi}{R}(k-1) \leq |\sqrt{\lambda_n}| \leq \frac{2\pi}{R}k}{}_n \frac{|u_n(y)|^q}{|\lambda_n|^{q/2}} \left| \int_0^R r\bar{f}(r) \sin(r\sqrt{\lambda_n})\, dr \right|^q \right] \leq$$

$$\leq \sum_{k=2}^{\infty} \left\{ C_{13} \left| \int_0^R \varrho \bar{f}(\varrho) \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^q + C_{14} \int_0^R \left| \int_r^R \varrho \bar{f}(\varrho) \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^q dr + \right.$$

$$\left. + C_{15} \int_0^R \left| \int_r^R \varrho \bar{f}(\varrho) \cos\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^p dr \right\}. \tag{17}$$

Остаётся заметить, что по предположению существует интеграл, стоящий в правой части (3), а это обеспечивает справедливость неравенства Хаусдорфа – Юнга по тригонометрической системе для функции одной переменной $r\bar{f}(r)$ на сегмента $0 \leq r \leq R$ (см. [2]). На основании этого мы получим

$$\sum_{k=2}^{\infty} \left| \int_0^R \varrho \bar{f}(\varrho) \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^q \leq C_{16} \left(\int_0^R |rf(r)|^p\, dr \right)^{q/p}, \tag{18}$$

$$\sum_{k=2}^{\infty} \int_0^R \left| \int_0^R \bar{f}\varrho(\varrho) \sin\left(\frac{2\pi}{R} k\varrho\right) d\varrho \right|^q dr \leq$$

$$\leq C_{16} \int_0^R \left(\int_r^R |\varrho \bar{f}(\varrho)|^p\, d\varrho \right)^{q/p} dr \leq C_{16} R \left(\int_0^R |rf(r)|^p\, dr \right)^{q/p}, \tag{19}$$

и совершенно аналогично

$$(20)\qquad \sum_{k=2}^{\infty}\int_0^R\left|\int_r^R \varrho\bar{f}(\varrho)\cos\left(\frac{2\pi}{R}k\,\varrho\right)d\,\varrho\right|^q dr\le C_{16}R\left(\int_0^R|rf(r)|^p\,dr\right)^{q/p}.$$

Из (17) – (20) вытекает, что

$$\sum_{k=2}^{\infty}\left[\sum_{\frac{2\pi}{R}(k-1)\le|\sqrt{\lambda_n}|\le\frac{2\pi}{R}k}\frac{|u_n(y)|^q}{|\lambda_n|^{q/2}}\left|\int_0^R r\bar{f}(r)\sin(r\sqrt{\lambda_n})\,dr\right|^q\right]\le$$

$$(21)\qquad \le C_{17}\left(\int_0^R r^p|\bar{f}(r)^p\,dr\right)^{q/p}.$$

Вставляя (13) и (21) в правую часть (6), мы получим неравенство

$$\sum_{n=1}^{\infty}|f_n|^q\le(4\pi)^qC_{18}\left(\int_0^R r^p|f(r)|^p\,dr\right)^{q/p}$$

эквивалентное неравенству Хаусдорфа–Юнга (3). Тем самым, доказательство теоремы завершено.

Ещё раз подчеркнём, что полученный нами результат является новым даже для таких хорошо изученных ситуаций как разложение по собственным функциям произвольного самосопряжённого расширения оператора Лапласа с точечным спектром и разложение по системе кратных экспонент.

Тот факт, что мы устанавливаем неравенство Хаусдорфа–Юнга для функций из класса радиальных, имеет большое значение в спекральной теории ибо при изучении асимптотики спектральной функции и при доказательстве сходимости спектральных разложений приходится иметь дело именно с функциями из класса радиальных функций (см, например, [3]).

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A NOTE ON WALD'S IDENTITY

By

N. L. BASSILY and S. ISHAK

Ain Shams University of Cairo

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Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and consider a random variable X. We shall use the following terminology. The expectation of X is *finite* if both X^+ and X^- have finite expectation and in this case we let

$$E(X) = E(X^+) - E(X^-).$$

Further, the expectation of X *exists* if exactly one of $E(X^+)$ and $E(X^-)$ is equal to $+\infty$. In this case we also let

$$E(X) = E(X^+) - E(X^-).$$

Finally, we say that the expectation of X *does not exist*, if both $E(X^+)$ and $E(X^-)$ are equal to $+\infty$.

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables and let ν be a stopping time with respect to the σ-fields $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, $n \geqq 1$. Suppose that $P(\nu < +\infty) = 1$, $a = E(Y_1)$ is finite and not equal to 0. Let $S_n = Y_1 + \ldots + Y_n$, $n \geqq 1$. Consider the random variable

$$S_\nu = \sum_{n=1}^{\infty} S_n \chi(\nu = n),$$

which equals 0 on the event $\{\nu = +\infty\}$. Wald's identity says that under the above conditions $E(S_\nu)$ is finite if and only if $E(\nu) < +\infty$ and in this case

$$E(S_\nu) = aE(\nu).$$

(That the finiteness of $E(S_\nu)$ implies the finiteness of $E(\nu)$ is treated e. g. in the book [1] by A. A. Borovkov.)

It follows that the conditions $E(Y_1) = a \neq 0$ and $E(S\nu) = \pm\infty$, or $E(S\nu)$ does not exist, imply that $E(\nu) = +\infty$. Conversely, if $E(Y_1) = a \neq 0$ and $E(\nu) = +\infty$, then necessarily $E(S_\nu)$ is not finite (in other words, $E(S_\nu) = \pm\infty$ or $E(S\nu)$ does not exist). In fact, if $E(S_\nu)$ is finite then by the above identity of Wald we would have $E(\nu) < +\infty$.

Therefore it is interesting to give conditions which ensure that if $E(Y_1) = a > 0$ and $E(\nu) = +\infty$ then $E(S\nu)$ exists (and equals $+\infty$). This would lead to a kind of Wald's identity. Namely, under the conditions which we formulate below we shall have

$$E(S_\nu) = +\infty = a\, E(\nu).$$

We prove the following

THEOREM. *Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with $E(Y_1) = a > 0$ and finite $E(Y_1^2)$. Let ν be a stopping time with respect to the sequence $\{\mathcal{F}_n\}_{n=1}^\infty$ of σ-fields with $P(\nu < +\infty) = 1$ and $E(\nu) = +\infty$. Then $E(S\nu) = +\infty$.*

PROOF. In the light of the preceding considerations we only have to show that

$$E(S_\nu^-) < +\infty.$$

Since

$$S_\nu = \sum_{n=1}^\infty S_n \chi(\nu = n) \geq \inf_{n \geq 1} S_n,$$

we have that

$$S_\nu^- \leq (\inf_{n \geq 1} S_n)^-$$

Consequently, it suffices to show that

$$E((\inf_{n \geq 1} S_n)^-) < +\infty.$$

We shall use the following well known assertion.

If ν is a stopping time with respect to $\{\mathcal{F}_n\}_{n=1}^\infty$ such that $P(\nu < +\infty) = 1$ then the σ-field generated by the random variables $\nu, Y_1, \ldots, Y_\nu$ and the σ-field $\sigma(Y_{\nu+1}, Y_{\nu+2}, \ldots)$ are independent. The sequence $\{Y_{\nu+1}, Y_{\nu+2}, \ldots\}$ has the same distribution as $\{Y_1, Y_2, \ldots\}$. Especially, the random variables $Y_{\nu+j}$ have the same distribution as Y_1. (See e. g. [1] and [2].)

To prove the finiteness of $E((\inf_{n \geq 1} S_n)^-)$ let ν_1 be the first k such that $S_k \geq 1$. Then ν_1 is a stopping time with respect to $\{\mathcal{F}_n\}_{n=1}^\infty$ and it is known that $E(\nu_1) < +\infty$ and that $E(\nu_1^2) < +\infty$ ([2]). Consider now the new sequence $Y_k^* = Y_{\nu+k}$, $k = 1, 2, \ldots$. Then by the preceding assertion this is independent of ν_1 and of $Y_1, \ldots, Y_{\nu_1}$. Let now ν_2 be the first k such that $S_k^* = Y_1^* + \ldots + Y_k^* \geq 1$. Then ν_2 has the same properties as ν_1 and ν_1 and ν_2 are identically distributed and independent. This being, we consider the sequence $\{Y_k^{**}\} = \{Y_{\nu_1+\nu_2+k}, k = 1, 2, \ldots\}$ and construct ν_3 in the same way as ν_1 and ν_2.

As a result we obtain the infinite sequence $\nu_1, \nu_2, \ldots$ of stopping times which are independent and identically distributed. We also have $E(\nu_i) < +\infty$ and $E(\nu_i^2) < +\infty$, $i = 1, 2, \ldots$.

Introduce the notations

$$Z_1 = \inf_{k<\nu_1} S_k,\ Z_2 = \inf_{k<\nu_2} S_k^*,\ Z_3 = \inf_{k<\nu_3} S_k^{**},\ \ldots$$

Then it is clear that

$$\inf_{n\geq 1} S_n = \inf(Z_1+S_{\nu_0}, Z_2+S_{\nu_0+\nu_1}, Z_3+S_{\nu_0+\nu_1+\nu_2}, \ldots),$$

where by definition $\nu_0 = 0$, $S_0 = 0$, $S_{\nu_0+\nu_1}\geq 1$, $S_{\nu_0+\nu_1+\nu_2}\geq 2, \ldots$.

From this

$$\{\inf_{n\geq 1} S_n < -N\} = \bigcup_{k=0}^{\infty} \{S_{\nu_0+\ldots+\nu_k}+Z_{k+1} < -N\}.$$

It follows that

$$\{\inf_{n\geq 1} S_n < -N\} \subset \bigcup_{k=0}^{\infty} \{Z_{k+1}+k < -N\}.$$

Consequently,

$$P\{\inf_{n\geq 1} S_n < -N) \leq \sum_{k=0}^{\infty} P(Z_{k+1}+k < -N) = \sum_{j=N}^{\infty} P(Z_1 < -j),$$

since the Z_k's are identically distributed. If X is a random variable with the distribution function $F(x)$ then, as it is well known,

$$E((X)^-) = \int_0^{+\infty} F(-x)\,dx \text{ and } E(X^2) \geq \sum_{m=1}^{\infty} mP(|X|\geq m).$$

Using these formulas we have

$$E((\inf_{n\geq 1} S_n)^-) = \sum_{N=0}^{\infty} \int_N^{N+1} P(\inf_{n\geq 1} S_n < -x)\,dx \leq \sum_{N=0}^{\infty} P(\inf_{n\geq 1} S_n < -N) \leq$$

$$\leq \sum_{N=0}^{\infty}\sum_{j=N}^{\infty} P(Z_1 < -j) = \sum_{j=0}^{\infty} (j+1)P(Z_1 < -j) =$$

$$= \sum_{l=1}^{\infty} lP(|Z_1| > l-1) = \sum_{m=1}^{\infty} mP(|Z_1|\geq m) \leq E(Z_1^2).$$

We also trivially see that

$$E(Z_1^2) \leq F\left(\left(\sum_{i=1}^{\nu_1} |Y_i|\right)^2\right).$$

Now by the C_2-inequality

$$E\left(\left(\sum_{i=1}^{\nu_1} |Y_i|\right)^2\right) \leq 2\left[E\left(\left(\sum_{i=1}^{\nu_1} |Y_i| - \nu_1 E(|Y_1|)\right)^2\right) + E(\nu_1^2)\, E^2(|Y_1|)\right].$$

By the second Wald identity we obtain (cf. [3])

$$E\left(\left(\sum_{i=1}^{\nu_1} |Y_1| - \nu_1 E(|Y)_1|\right)^2\right) = E(\nu_1)\ \mathrm{Var}\ (|Y_1|).$$

This finally implies that

$$E((\inf_{n \geq 1} S_n)^-) \leq 2[E(\nu_1)\ \mathrm{Var}\ (|Y_1|) + E(\nu_1^2)\, E^2(|Y_1|)],$$

which proves the assertion.

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ОБРАЩЕНИЕ В НУЛЬ В ОСОБОЙ ТОЧКЕ СОБСТВЕННЫХ ФУНКЦИЙ ОПЕРАТОРА ШРЕДИНГЕРА С СИЛЬНО СИНГУЛЯРНЫМ ПОТЕНЦИАЛОМ

А. Р. ХАЛМУХАМЕДОВ

Ташкентский Государственный Университет им. В. И. Ленина, Ташкент

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Рассмотрим уравнение

$$-\Delta u(x)+q(x)u(x) = \mu^2 u(x), \quad x\in\Omega \tag{1}$$

где $\Omega\subset R^N$, $N\geq 1$ произвольная ограниченная область, Δ — оператор Лапласа в R^N, μ — некоторое комплексное число. Кроме того

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|^2} \tag{2}$$

и $a(t)\in C[0, \infty)$ — произвольная неотрицательная функция удовлетворяющая условию Дини т.е.

$$\int_0^1 |a(t)-a(0)|\frac{dt}{t} < \infty. \tag{3}$$

Для любых комплексных чисел a, b и $r\geq 0$ положим

$$\alpha = \frac{N-2}{2}, \quad v = \sqrt{\alpha^2+a(0)}, \tag{4}$$

$$W(a, b) = Jv(a)Yv(b)-Jv(b)Yv(a), \tag{5}$$

$$I_\mu g(r) = \frac{\pi}{2} r^{-v}\int_0^r W(t\mu, r\mu)[a(t)-a(0)]t^{v-1}g(t)dt, \tag{6}$$

$$S_r g(x) = \int_\Theta g(x+r\Theta)d\Theta. \tag{7}$$

В случае, когда $a(t)\equiv a$ в работе [1] И. Йо доказал, что ограниченные решения уравнения (1) обращаются в нуль в особой точке x_0.

Справедлива следующая теорема:

Теорема. *Пусть для решения $u(x)\in C(\Omega\setminus\{x_0\})$ уравнения* (1) *справедлива оценка*

$$u(x) = o(|x-x_0|^{v-3\alpha}. \tag{8}$$

Тогда найдется такая функция $\varphi(x)\in C(\Omega)$, $\varphi(x_0)\neq 0$, что решение $u(x)$ представимо в виде

$$u(x) = |x-x_0|^{v-\alpha}\varphi(x)$$

и для функции $\varphi(x)$ при $0\leq r<\operatorname{dist}(x_0, \partial\Omega)$ справедлива следующая формула (9)

$$S_r\varphi(x_0) = 2^{\nu+1}\pi^{N/2}\frac{\Gamma(1+2v)}{\Gamma\left(\frac{N}{2}\right)}\cdot\frac{Jv(2\mu)}{(2\mu)^{\nu}}\varphi(x_0)+I\mu(S_r\varphi)(x_0). \tag{9}$$

Прежде чем доказать теорему докажем несколько лемм.

Лемма 1. Уравнение

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right)+\left(2\mu^2-\frac{v^2}{r}\right)y+\frac{a(r)-a(0)}{2}y = 0, r\in[0,1] \tag{10}$$

имеет решение $y_1(r)\in C[0, 1]$, представимое в виде $y_1 = r^v\Psi_1(r)$, где

$$\Psi_1(r)\in C[0, 1] \text{ и } \Psi_1(0)\neq 0.$$

Доказательство леммы 1. Докажем, что решение y_1 интегрального уравнения

$$y_1(r) = \mu^{-\nu}J_\nu(r\mu)+\frac{\pi}{2}\int_0^r W(t\mu, r\mu)[a(t)-a(0)]y_1(t)\frac{dt}{t} \tag{11}$$

удовлетворяет всем условиям леммы.

Сначала, с помощью метода последовательных приближений, решим интегральное уравнения (11).

При $k = 1, 2, \ldots$ положим

$$F_0(r) = \mu^{-\nu}J_\nu(r\mu),$$

$$F_k(r) = \frac{\pi}{2}\int_0^r W(t\mu, r\mu)[a(t)-a(0)]F_{k-1}(t)\frac{dt}{t},$$

$$h(r) = \int_0^r |a(t)-a(0)|\frac{dt}{t}.$$

Методом математической индукции докажем, что справедливо неравенство

$$|F_k(r)| \leq Cr^{\nu}(C_0h(r))^k \tag{12}$$

где C, C_0 — некоторые положительные постоянные.

Для $k = 0$ справедливость неравенства (12) очевидна. Предположим теперь, что неравенство (12) имеет место при $k-1$ и докажем его справедливость при k. Для этого при $r|\mu| \leq 1$ воспользуемся неравенством [2]:

$$|W(t\mu, r\mu)| \leq C\left(\frac{r}{t}\right)^{\nu}, \quad t \leq r \leq |\mu|^{-1}. \tag{13}$$

Тогда

$$|F_k(r)| \leq Cr^{\nu} \cdot \int_0^r (C_0h(t))^{k-1}|a(t)-a(0)|\frac{dt}{t} \leq Cr^{\nu}(C_0h(r))^k.$$

В случае, когда $r|\mu| > 1$, в силу неравенств [2]:

$$|W(t\mu, r\mu)| \leq C|\mu|^{-\frac{1}{2}}t^{-\frac{1}{2}}, \quad |\mu|^{-1} \leq t \leq r,$$

$$|W(t\mu, r\mu)| \leq C(t|\mu|)^{-\nu}, \quad t \leq |\mu|^{-1} \leq r$$

получим

$$|F_k(r)| \leq C|\mu|^{-\nu}\int_0^{|\mu|^{-1}} (C_0h(t))^{k-1}|a(t)-a(0)|\frac{dt}{t} + C|\mu|^{-\frac{1}{2}} \times$$

$$\times \int_0^r t^{\nu-\frac{1}{2}}(C_0h(t))^{k-2}|a(t)-a(0)|\frac{dt}{t} \leq Cr^{\nu}(C_0h(r))^k.$$

Таким образом, справедливость неравенства (12) доказана. В силу этого неравенства, при $C_0h(r) < \frac{1}{2}$ решения y_1 уравнения (11) представляется в виде $y_1 = \sum_{k=0}^{\infty} F_k(r)$, причем ряд в последнем равенстве сходится абсолютно и равномерно по r, $0 \leq r < r_0$, где r_0 точная верхняя грань тех r, для которых $C_0h(r) < \frac{1}{2}$.

Следовательно, в силу непрерывности функции $h(r)$ и неравенства (12) функция $\Psi_1(r) = r^{-\nu}\sum_{k=0}^{\infty} F_k$ также будет непрерывной на отрезке $[0, r_0)$ и $\Psi_1(0) \neq 0$. Кроме того $y_1(r) = r^{\nu}\Psi_1(r)$.

С помощью подстановки легко убедиться, что найденная функция $y_1(r)$ на отрезке $[0, r_0)$ удовлетворяет уравнению (10). В случае, когда $r_0 < 1$ функция $y_1(r)$ является искомой функцией.

В случае, когда $r_0 < 1$, остается заметить, что функция $y_1(r)$ допускает единственное продолжение, удовлетворяющая уравнению (10). Это следует из непрерывности коэффициентов уравнения (10) при $r > 0$.

Лемма 1 доказана.

Лемма 2. Общее решение уравнения (10) представляется в следующем виде

$$y(r) = C_1 r^{\nu}\Psi_1(r) + C_2 r^{-\nu}\Psi_2(r) \tag{14}$$

где C_1, C_2 — постоянные, $\Psi_1, \Psi_2 \in C[0, 1]$, $\Psi_1(0) \neq 0$, $\Psi_2(0) \neq 0$.

Доказательство леммы 2. В силу леммы 1 для доказательства леммы 2 достаточно построить решение $y_2(r)$ уравнения (10) представляемое в виде $y_2 = r^{-\nu}\Psi_2(r)$

Пусть $y_1(r)$ решение уравнения (10) из леммы 1. С помощью подстановки нетрудно убедиться, что функция, определяемая равенством

$$y_2(r) = y_1(r)\int_r^1 \frac{dt}{t y_1^2(t)}$$

удовлетворяет уравнению (10).

Положим $\Psi_2 = r^{\nu} y_2(r)$. Тогда в силу леммы 1 получим

$$|\Psi_2(r)| \leq c r^{2\nu}\int_r^1 t^{-2\nu-1}dt \leq c r^{2\nu}[r^{-2\nu} - 1] \leq c.$$

Поэтому $\Psi_2 \in C[0, 1]$ и $\Psi_2(0) \neq 0$.

Кроме того, решения $y_1 = r^{\nu}\Psi_1$ и $y_2 = r^{-\nu}\Psi_2$ линейно независимы. Поэтому общее решение уравнения (10) определяется с помощью равенства (14).

Лемма 2 доказана.

Непосредственно из леммы 2 вытекает следующая лемма:

Лемма 3. Пусть для решения $y \in C\,(0,1]$ уравнения (10) справедлива оценка $y = o(r^{-\nu})$. Тогда $y \in C\,[0,1]$ и представимо в следующем виде $y = r^{\nu}\Psi(r)$ где $\Psi \in C[0, 1]$ и $\Psi(0) \neq 0$.

Из равенства (6) и оценки (13) вытекает следующая лемма:

Лемма 4. Пусть $g(r) \in C[0,1]$ — произвольная функция. Тогда справедливо равенство

$$\lim_{r\to 0} I_{\mu} g(r) = 0. \tag{15}$$

Доказательство теоремы: Пусть для решения u уравнения (1) справедлива оценка (8). Положим

$$\varphi(x) = |x-x_0|^{\alpha-\nu}u(x),\ x\in\Omega\backslash\{x_0\}, \tag{16}$$

$$z(r) = S_r\Psi(x_0),\ r = |x-x_0|. \tag{17}$$

Тогда в силу оценки (8) имеем $z = o(r^{-2\alpha})$.

Так как u удовлетворяет уравнению (1), то в силу определения (4) чисел α и ν функция $\varphi(x)$ будет решением уравнения

$$\Delta\varphi+\frac{2(v-\alpha)}{r}\frac{\partial\varphi}{\partial r}+\mu^2\varphi = \frac{a(r)-a(0)}{r}\varphi. \tag{18}$$

Докажем теперь оценку

$$\frac{d}{d\varepsilon}S_\varepsilon\varphi(x_0) = o(\varepsilon^{1-N}),\ 0<\varepsilon<1. \tag{19}$$

Для того чтобы доказать это воспользуемся оценкой (8) для функции $u(x)$. Заметим, что в силу этой оценки получим

$$\int\limits_{|x-x_0|\le\varepsilon}\Delta u(x)dx = \int\limits_{|x-x_0|\le\varepsilon}[q(x)-\mu^2]u(x)\,dx =$$

$$= o\Big(\int\limits_{|x-x_0|\le\varepsilon}(x-x_0)^{\nu-3\alpha-2}dx\Big) = o(\varepsilon^{\nu-\alpha}).$$

С другой стороны, в силу формулы Грина имеем

$$\int\limits_{|x-x_0|\le\varepsilon}\Delta u(x)\,dx = \varepsilon^{N-1}\int\limits_{\Theta}\left[\varepsilon^{\nu-\alpha}\frac{\partial\varphi}{\partial\varepsilon}+(\nu-\alpha)\varepsilon^{\nu-\alpha-1}\psi\right]d\Theta =$$

$$= \varepsilon^{N-1+\nu-\alpha}\frac{d}{d\varepsilon}S_\varepsilon\varphi(x_0)+o(\varepsilon^{\nu-\alpha}).$$

Поэтому справедлива оценка (19). Применим теперь формулу Грина по области $\{y:\varepsilon\le|y-x_0|\le R\}$, $0<R<1$, к функциям $\varphi(y)$ и $g(y)\equiv 1$. Получим

$$\int\limits_{|y-x_0|\le R}\Delta\varphi\,dy = R^{N-1}\frac{d}{dR}S_R\varphi(x_0)-\varepsilon^{N-1}\frac{d}{d\varepsilon}S_\varepsilon\varphi(x_0).$$

Отсюда при $\varepsilon\to 0$ в силу оценки (19) вытекает следующее равенство:

$$\int\limits_{\varepsilon\le|y-x_0|\le R}\Delta\varphi\,dy = R^{N-1}\frac{d}{dR}S_R\varphi(x_0).$$

Следовательно

$$\int_0^R [a(r)-a(0)]S_r\,\varphi r^{N-3}\,dr = \mu^2\int_0^R S_r\,\varphi r^{N-1}\,dr +$$

$$+2(\nu-\alpha)\int_0^R \frac{d}{dr}(S_r\,\varphi)\,r^{N-2}\,dr + R^{N-1}\frac{d}{dR}S_R\,\varphi.$$

Дифференцируя последнее равенство по R, в силу (17) находим, что

$$R^{N-1}\frac{d^2z}{d_Rz}+(1+2\nu)R^{N-2}\frac{dz}{dR}+\mu^2R^{N-1}z = [a(R)-a(0)]R^{N-3}z.$$

Разделив последнее равенство на R^{N-1} и заменив R на r, получим

$$\frac{d^2z}{dr^2}+\frac{(1+2\nu)}{r}\frac{dz}{dr}+\left[\mu^2-\frac{a(r)-a(0)}{r^2}\right]z = 0\cdot \tag{20}$$

Положим $y = r^\nu z(r)$. Тогда нетрудно убедиться, что y будет решением уравнения (10) и справедлива оценка $y = o(r^{-\nu})$. Поэтому и в силу леммы 3 функция $y(r)$ представима в виде $y = r^\nu\Psi(r)$, где $\Psi(r)\in C[0, 1]$, $\Psi(0)\neq 0$.

Следовательно по определению функции $y\,(r)$ получим, что $z\in C[0, 1]$, $z(0)\neq 0$.

С другой сороны, функция $z(r)$ выражается через общее решение уравнения (20) в следующем виде

$$z(r) = Ar^{-\nu}J_\nu(r\mu)+Br^{-\nu}Y_\nu(r\mu)+I_\mu(z)(r)$$

где A и B — некоторые постоянные.

Так как $z(r)$ ограничена в нуле, получим, что $B = 0$. В силу леммы 4 и в силу очевидного равенства

$$\lim_{r\to 0}\frac{J_\nu(r\mu)}{r_\nu} = \frac{2^{-\nu}\mu^\nu}{\Gamma(1+2\nu)}$$

получим, что

$$A = 2^\nu\mu^{-\nu}\Gamma(1+2\nu)z(0).$$

Таким образом, для функции $z(r)$ справедлива формула

$$z(r) = 2^\nu\Gamma(1+2\nu)\frac{J_\nu(r\mu)}{(r\mu)^\nu}z(0)+I_\mu(z)(r).$$

Положим $\varphi(x_0) = \Gamma\left(\frac{N}{2}\right)\frac{\pi^{-\frac{N}{2}}}{2}z(0)$. Отсюда следует, что функция $\psi(x)$ непрерывна во всей области Ω и $\varphi(x_0)\neq 0$. Поэтому в силу равенства (21) и определения функции $z(r)$ получим равенство (9).

Теорема доказана.

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INTERSECTION POINTS

By

K. BEZDEK and R. CONNELLY*

Department of Geometry, L. Eötvös University, Budapest
Department of Mathematics, Cornell University, Ithaca

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1. Introduction

In a recent note [1] K. BEZDEK, J. PACH gave a short proof of the following theorem, which was conjectured by L. FEJES TÓTH. Taket wo points O_1, O_2 in the Euclidean plane having mutual distance at most 2, and set $S_1 = \{O_1, O_2\}$. If S_i $(i \geq 1)$ has already been determined, then add to the picture all unit circles whose centers belong to S_i and which have not been drawn before. Further, let S_{i+1} be defined as the union of S_i and the set of all intersection points of unit circles (i.e. circles with radius one) whose centers are in S_i (An intersection point is a common point of at least two circles). Then $\bigcup_{i=1}^{\infty} S_i$ is either identical with the set of vertices of a regular triangle-lattice of side length 1, or it is everywhere dense in the plane. (An infinite point set S is said to be everywhere dense in the plane if every (circular) disk contains at least one element of S. In this article a circle means a closed curve and a disk is a closed region enclosed by a corresponding circle with positive radius).

Obviously the set of unit circles of this construction has the property that if a unit circle contains two intersection points $\left(\text{i.e., two points of } \bigcup_{i=1}^{\infty} S_i\right)$, then it belongs to the system of unit circles. The following question arises naturally. Let $\mathcal{C}$ be a system of unit disks covering the Euclidean plane such that any unit circle which contains two intersection points of unit circles of $\mathcal{C}$ belongs to $\mathcal{C}$. (A collection of disks covers the plane when any point of the Euclidean plane belongs to at least one disk.) What can we say about the intersection points of $\mathcal{C}$? This problem generalizes the previous question because, for example, the system of intersection points of face-circumcircles of the regular tiling $\{4,4\}$ with edge length $\sqrt{2}$ cannot be generated by two points O_1, O_2. We shall prove the following.

THEOREM 1. *Suppose that a system $\mathcal{C}$ of unit disks covers the Euclidean plane such that any unit circle which contains two intersection points of $\mathcal{C}$ belongs to $\mathcal{C}$. Then the intersection points of $\mathcal{C}$ are either identical with the vertices of a rectangular lattice, where the diagonals of the rectangles are equal to 2 and the shortest side of the rectangles is > 1, or they are identical with the vertices of a regular triangular lattice of side length 1, or they are everywhere dense in the plane (Fig. 1).*

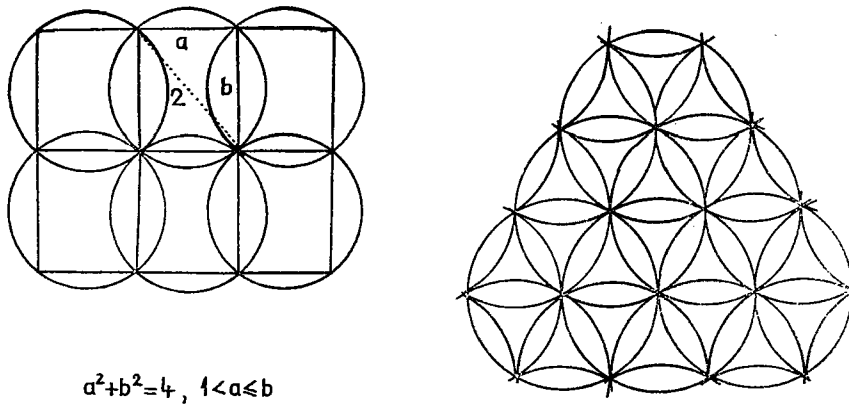

Fig. 1.

An open problem is the higher dimensional case. As a special case it is easy to prove the following. Let $O_1, O_2, \ldots, O_n$ be *n points* in n-dimensional Euclidean space such that the unit spheres centred at these points have a point in common. Set $S_1 = \{O_1, O_2, \ldots, O_n\}$. If S_i $(i \geq 1)$ has already been determined then add to the configuration all unit spheres whose centers belong to S_i and which have not been drawn in before. Further, let S_{i+1} be defined as the union of S_i and the set of all intersection points of unit spheres whose centers are in S_i. (A point is an intersection point if it belongs to at least n unit spheres.) Then $\bigcup_{i=1}^{\infty} S_i$ is everywhere dense in n-space $\left(\text{i.e., any neighborhood of any point contains at least one point from } \bigcup_{i=1}^{\infty} S_i\right)$.

Now let $\mathcal{C}$ be an arrangement of unit disks in the plane, where we require that any two unit disks of $\mathcal{C}$ are distinct sets of the plane. We call the arrangement consisting of unit disks centred at the intersection points of $\mathcal{C}$ the dual arrangement $\mathcal{C}^*$ of $\mathcal{C}$. We shall compare the density of $\mathcal{C}^*$ with the density of $\mathcal{C}$. (The density may be interpreted roughly as the total area of the disks divided by the area of the whole plane. A rigorous definition of the density can be found in [4].) In fact, we have the following.

THEOREM 2. *Let $\mathcal{C}$ be an arrangement of unit disks in the plane with density $d(\mathcal{C})$ such that any unit disk of $\mathcal{C}$ is intersected by at least one unit disk of $\mathcal{C}$ but no two of the unit disks of $\mathcal{C}$ are tangent. Let $d(\mathcal{C}^*)$ denote the (lower) density of the dual arranguement $\mathcal{C}^*$ of $\mathcal{C}$. Then $d(\mathcal{C}^*) \geq d(\mathcal{C})$.*

REMARK. An open question is whether $d(\mathcal{C}^*) \geq d(\mathcal{C})$ for any covering $\mathcal{C}$ consisting of unit spheres of n-dimensional Euclidean space. If $n = 2$, then, using the proof of Theorem 2, we get $d(\mathcal{C}^*) \geq \frac{3}{4} d(\mathcal{C})$.

Finally let $f(n)$ denote the maximum of those positipe integers k for which there exist k homothetic copies of a convex body in n-dimensional Euclidean space such that any two of them touch each other. (A convex body is defined as a compact convex subset of n-dimensional Euclidean space with nonempty interior. We say that two convex bodies touch each other if they have at least one common point, but their interiors are disjoint. Finally, two sets X, Y of n-dimensional Euclidean space are said to be homothetic if either X is a translate of Y or there exists a point O and a positive real number λ such that $X = \{P \mid \overrightarrow{OP} = \lambda \overrightarrow{OP^*}$ where $P^* \in Y\}$.) We ask the following question. Is it true that $f(n) = 2^n$? In the special case when only translates occur, L. DANZER and B. GRÜNBAUM [5] proved $f(n) = 2^n$ which was conjectured by P. ERDŐS [2, 3] and V. KLEE [7]. Here we claim the following as a joint observation with J. PACH.

PROPOSITION. $2^n \leq f(n) < 3^n$ for any integer $n \geq 2$.

REMARK. Also, we know the following fact. The maximum number of (distinct) n-dimensional ($n \geq 2$) balls such that any two touch each other is $n+2$. Here is a simple proof.

First, if we have congruent n-dimensional balls ($n \geq 1$) such that any two touch each other, then there are at most $n+1$ of them. Second, suppose that we have a system of n-dimensional (distinct, not necessarily congruent) balls ($n \geq 2$) in which any two of them touch each other. Then we choose an inversion whose center is an intersection point of two balls. The images of the spheres of our two balls are parallel hyperplanes and the images of the other balls are again balls. Of course they not only touch each other pairwise but the two parallel hyperplanes are supporting hyperplanes for all of the balls. Thus in our second system, the centers of the congruent balls lie in a hyperplane. By our first observation, the total number of the balls is at most $(n-1)+1 = n$. Hence the number of balls in the original system is at most $n+2$.

2. Proof of Theorem 1.

A set is called locally dense if there exists an open disk in which the set is (everywhere) dense. Let $\mathcal{L}$ be the set of intersection points of $\mathcal{C}$ as in Theorem 1.

LEMMA 1. If $\mathcal{L}$ is locally dense, then it is everywhere dense in the plane.

PROOF. Let B be the open disk with center O in which $\mathcal{L}$ is everywhere dense. We shall show that $\mathcal{L}$ is everywhere dense in the open disk B' of radius 2 with center O (Fig. 2). Choose an arbitrary interior point P of B'. If P is an interior point of B, then, of course, we are done. Otherwise, let us denote the arbitrary open disk centered at P by B_p.

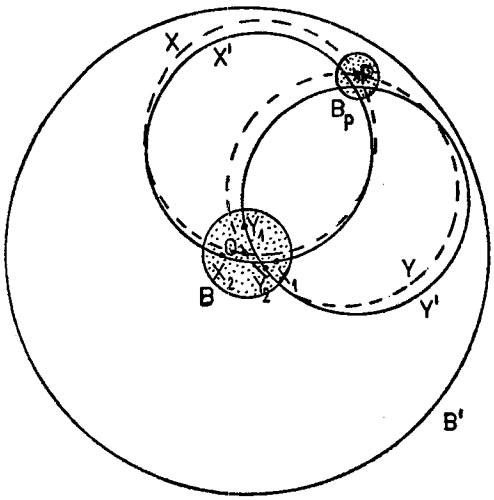

Fig. 2.

We have to show that there exists a point of $\mathcal{L}$ in B_p. In order to find this, let us connect the points O, P by two unit circles X and Y. So we can choose the points X_1, X_2, Y_1, Y_2 of $B \cap \mathcal{L}$ such that the unit circles X', Y' going through the points X_1, X_2 and Y_1, Y_2, are very close to the unit circles X an Y and, therefore, one intersection point of X', Y' will be in B_p. This yields that $\mathcal{L} \cap B_p \neq 0$ because the circles X', Y' belong to $\mathcal{C}$ by definition. After this, if O is any interior point of B', and we continue our procedure, then we get Lemma 1.

We shall need the following notion. An empty disk (or an empty circle) of $\mathcal{L}$ is a disk which does not contain any point of $\mathcal{L}$ in its interior, but we can find at least three points of $\mathcal{L}$ on its boundary.

LEMMA 2. If E is an empty circle of $\mathcal{L}$, then the radius of E is equal to either 1 or $\frac{1}{\sqrt{3}}$ and the points of $\mathcal{L}$ belonging to this empty circle form either the vertices of a rectangle or a regular triangle.

PROOF. The radius of E cannot be greater than 1 because then there are additional intersection points.

(i) Suppose that the radius of E is smaller than 1. Denote M_1, M_2, $\ldots$, M_k the points of $\mathcal{L}$ lying on the circle E. We know that $k \geq 3$. If $k \geq 4$, then suppose that M_1, M_2, M_3, M_4 are points lying in counterclockwise order on the circle E. But then two unit circles, connecting the points M_1,

M_3 and M_2, M_4, belong to $\mathcal{C}$ and intersect at a point belonging to the interior of the disk E, which is a contradiction. Hence $k = 3$.

Let us denote by C_i the unit circle going through the points M_j, M_l such that the intersection point M_i lies outside the disk C_i ($1 \leq i \leq 3$, $j \neq i$, $l \neq i$, $j \neq l$ and $1 \leq j \leq 3$, $1 \leq l \leq 3$). We shall show that any two of the disks C_1, C_2, C_3 touch each other. It is enough to prove this for the disks C_1, C_3. Suppose on the contrary, that the unit circles C_1, C_3 intersect at the points M_2, M_4 (Fig. 3). Obsviosuly M_4 lies outside the empty disk E. We claim that the points M_1, M_3, M_4 cannot lie on a unit circle. In order to prove this, we assume that there exists a unit circle C^* containing the points M_1, M_3, M_4. (Fig. 4)

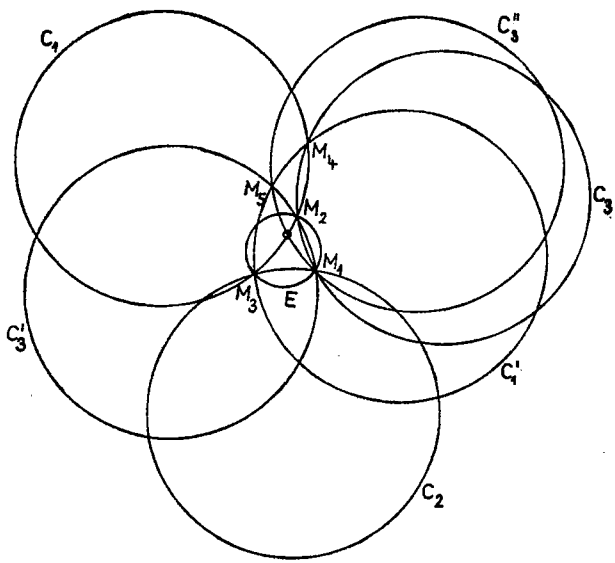

Fig. 3.

So the point M_2 lies inside the triangle $\triangle\ M_1M_3M_4$ and it is easy to see that if the angles of this triangle at the vertices M_1, M_3 are α and β, then $180° - \alpha = \sphericalangle M_3M_2M_4$ and $180° - \beta = \sphericalangle M_1M_2M_4$. Hence, $\sphericalangle M_1M_2M_3 = \alpha + \beta$ and $\sphericalangle M_1M_4M_3 = 180° - (\alpha + \beta)$ which yields that the reflected unit circle of C^* about the line M_1M_3 contains M_2, i.e., E is a unit circle which is a contradiction.

Now let us consider Fig. 3. If C_1', denotes the unit disk which is different from C_1 and which contains on its boundary the points M_3, M_4, then M_1 lies inside C_1'. This is true because of the previous observation that M_1 cannot lie on the unit circle C_1', and if M_1 is outside C_1', then the arc $\widehat{M_1M_2} < \pi$, belonging to the unit circle C_3, will intersect the unit circle C_1', (M_2 is inside the unit disk C_1') at a point of $\mathcal{L}$ lying inside the E, which is a contradiction. (Do not forget that the unit disks C_1, C_2, C_3, C_1', belong to $\mathcal{C}$.) Obviously, if C_3', is the unit circle different from C_3 going through

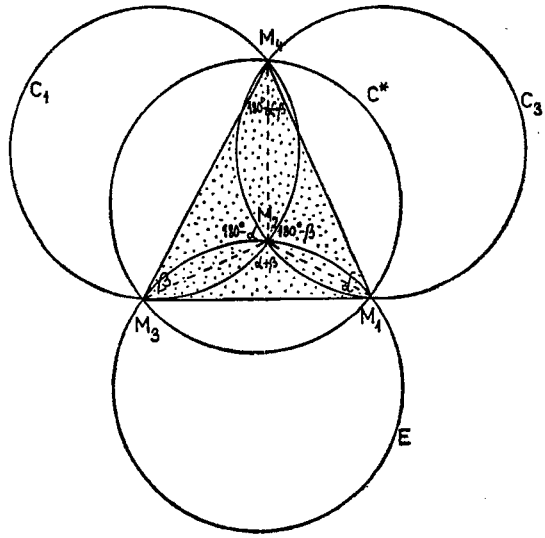

Fig. 4.

the points M_1, M_2, then M_3 lies inside the unit disk C_3'. But then there exists an intersection point M_5 of the unit circles C_1', C_3', inside the disk C_1. Of course, $M_5 \in \mathcal{L}$. Moreover, if C_3'' is the unit circle different from, C_3' going through the points M_1, M_5, then M_2 is inside the disk C_3'', but M_3 is outside the disk C_3''. Consequently, the arc $\widehat{M_2M_3} < \pi$, belonging to the unit circle C_1, intersects the unit circle C_3'', at a point of $\mathcal{L}$ belonging to the interior of E, which is a contradiction. That is, the unit disks C_1, C_3 touch each other and this is true for any pair of the unit disks C_1, C_2, C_3. Hence, $\triangle\ M_1M_2M_3$ is a regular triangle of side length 1 and E has radius $\frac{1}{\sqrt{3}}$ (Fig. 5).

(ii) Suppose that the radius of E is 1. Let us denote the points of $\mathcal{L}$ lying on the empty circle E by $M_1, M_2, \ldots, M_k$. As before it is easy to see that k cannot be greater than 4. Hence, $3 \leq k \leq 4$. Let us consider the convex hull conv $\{M_1, M_2, \ldots, M_k\}$ of the points $M_1, M_2, \ldots, M_k$. We claim that the center point O of E lies in the interior of conv $\{M_1, M, \ldots, M_k\}$. Suppose on the contrary that O lies on the boundary of conv $\{M_1, \ldots, M_k\}$ or outside (Fig. 6).

Of course now $E \in \mathcal{C}$. Without loss of generality we may suppose that the line M_1M_k separates the conv$\{M_1, \ldots, M_k\}$ from the center O. Then let $\widehat{M_1M_k}$ be the arc of unit circle E separated from conv$\{M_1, \ldots, M_k\}$ by the line M_1M_k and let us denote the midpoint of $\widehat{M_1M_k}$ by N. $\mathcal{C}$ forms a covering of the plane; therefore, there exists a disk C^* of $\mathcal{C}$ different from E which covers N. But then this unit disk C^* cannot cover both of the points M_1, M_k.

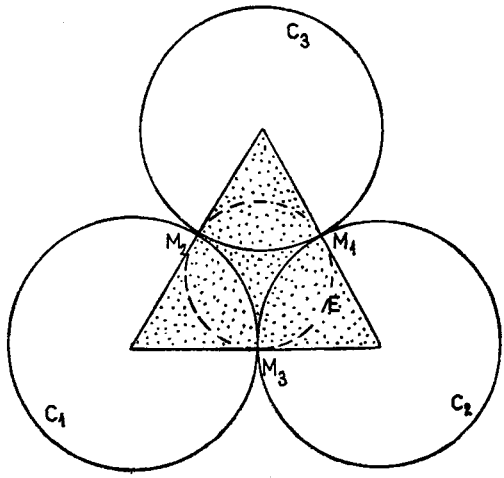

Fig. 5.

I.e., there is an intersection point on $\widehat{M_1M_k}$ different from M_1 and M_k which is a contradiction. Hence, $O \in \operatorname{int} \operatorname{conv}\{M_1, \ldots, M_k\}$. Using this we get that any angle of the convex polygon $\operatorname{conv}\{M_1, \ldots, M_k\}$ must be at least a right angle, otherwise, we could find an intersection point of the unit circles of C in the interior of E. Consequently, since $3 \leq k \leq 4$ the only possibility is that $k=4$, and $\operatorname{conv}\{M_1, M_2, M_3, M_4\}$ is a rectangle. This completes the proof of Lemma 2.

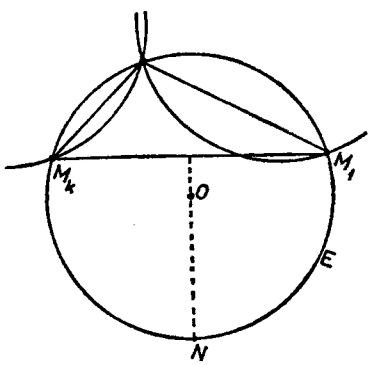

Fig. 6.

LEMMA 3. The empty circles of $\mathcal{L}$ are congruent.

PROOF. Using Lemma 2, there exist two possibilities. Either the radius of an empty circle of $\mathcal{L}$ equals 1 or $\frac{1}{\sqrt{3}}$. Therefore, it is enough to show

that if we have an empty circle with radius $\frac{1}{\sqrt{3}}$, then any other empty circle has the same radius. Let E denote our empty circle of radius $\frac{1}{\sqrt{3}}$. We know by Lemma 2 that $E \cap \mathcal{L}$ is a regular triangle with vertices M_1, M_2, M_3 (Fig. 7).

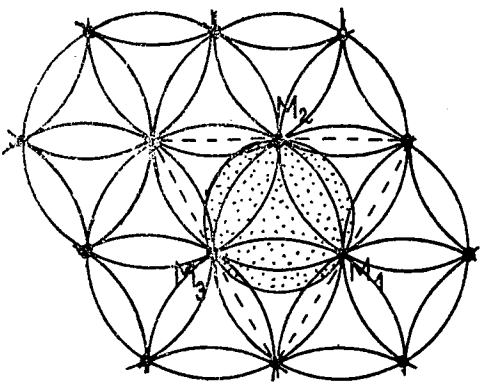

Fig. 7.

Let us consider the system of six unit circles going through two of the points M_1, M_2, M_3. We get 12 intersection points (are shown by Fig. 7) which, of course, belong to $\mathcal{L}$ since the unit circles belong to $\mathcal{C}$ and, on the other hand, they form some of the vertices of a regular triangle-lattice Γ generated by the triangle $\triangle M_1M_2M_3$. If we continue this procedure using another three points which form a regular triangle of side length 1 we find that the vertices of Γ belong to $\mathcal{L}$. But this means immediately that any empty circle of $\mathcal{L}$ cannot have a radius greater than $\frac{1}{\sqrt{3}}$. Consequently, using again Lemma 2, the radius of any empty circle of $\mathcal{L}$ must be equal to $\frac{1}{\sqrt{3}}$, and we are done.

LEMMA 4. If $\mathcal{L}$ is not locally dense, then the empty disks of $\mathcal{L}$ cover the plane.

PROOF. First of all, we shall prove that if $\mathcal{L}$ is not locally dense then $\mathcal{L}$ does not have accumulation points.

Suppose on the contrary, that $\mathcal{L}$ has an accumulation point A. This means that there exists a sequence $a_1, a_2, \ldots, a_n, \ldots$ of intersection points (i.e., of points of $\mathcal{L}$) such that $\lim_{n\to\infty} a_n = A$.We suppose without loss o generality that the points $A, a_1, a_2, \ldots, a_n, \ldots$ lie on a unit circle C of $\mathcal{C}$.

We claim then that any point of $\mathcal{L} \cap C$ is an accumulation point of $\mathcal{L}$. In order to prove this, let $P \in \mathcal{L} \cap C$ be an arbitrary intersection point. Without loss of generality, we may suppose that the points A, a_1, a_2, ... a_n, ... are consecutive points clockwise on C (Fig. 8) and P is different from a_1.

After this, let M_n be that intersection point of the unit circles different from C and going through the points a_1, a_n and a_{n-1}, a_{n+1} which lies inside C ($n \geq 3$). Obviously, $\lim_{n\to\infty} M_n = A$. Let us consider the unit circle going through the points a_1, M_n which does not contain a_n and denote it by C_n. We know that $C_n \in \mathcal{C}$ for any $n \geq 3$ and $\lim_{n\to\infty} C_n = C$. Consequently, if C_p denotes the unit circle which intersects C at P, then there exists a sequence $P_3, \ldots, P_n, \ldots$ of points such that $P_n \in C_n \cap C_p$ ($n \geq 3$) and $\lim_{n\to\infty} P_n = P$. That is, P is an accumulation point of $\mathcal{L}$, and this is what we wanted to prove.

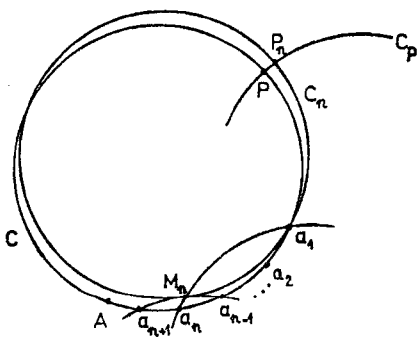

Fig. 9.

But now we can prove more. Namely any point of $\mathcal{L} \cap C$ is an accumulation point of $\mathcal{L} \cap C$. In order to prove this, choose again the arbitrary point $P \in \mathcal{L} \cap C$. We know, using the previous observation, that there exists a sequence $P_3, \ldots, P_n, \ldots$ of intersection points such that $\lim_{n\to\infty} P_n = P$. Obviously we can find a point Q inside C such that Q is an intersection point (i.e., a point of $\mathcal{L}$) and does not lie on the unit circle C_p. So the unit circle connecting the points Q and P_n ($n \geq 3$) cannot contain more than two points of the sequence $P_3, \ldots, P_n, \ldots$. (The points $P_3, \ldots, P_n, \ldots$ lie on C_p!) Consequently there exists a subsequence $P_{i_1}, P_{i_2}, \ldots, P_{i_n}, \ldots$ of the sequence $P_3, \ldots, P_n, \ldots$ such that we can construct unit circles $C_{i_1}, C_{i_2}, \ldots, C_{i_n}, \ldots$ going through the points Q, P_{i_1}; Q, P_{i_2}; ...; Q, P_{i_n}; ... having the property that there exists a system $P'_{i_1}, P'_{i_2}, \ldots, P'_{i_n}, \ldots$ of distinct intersection points with $P'_{i_1} \in C_{i_1} \cap C$, $P'_{i_2} \in C_{i_2} \cap C, \ldots, P'_{i_n} \in C_{i_n} \cap C, \ldots$, and $\lim_{n\to\infty} P'_{i_n} = P$. Since $P'_{i_n} \in \mathcal{L}$ we are done.

Now consider the set $\mathcal{L} \cap C$. If $\overline{\mathcal{L} \cap C}$ donotes the closed hull of the set $\mathcal{L} \cap C$, then we claim that $\overline{\mathcal{L} \cap C} = C$. Otherwise, there exists an open arc $\widehat{AB}$ of C such that $\widehat{AB} \cap \overline{\mathcal{L} \cap C} = \emptyset$ but $A \in \overline{\mathcal{L} \cap C}$ and $B \in \overline{\mathcal{L} \cap C}$. (Of course the length of $\widehat{AB}$ is smaller than π.) We proved before that any point of $\mathcal{L} \cap C$ is an accumulation point of $\mathcal{L} \cap C$; therefore, there exist two sequences $A_1, A_2, \ldots, A_n, \ldots$ and $B_1, B_2, \ldots, B_n, \ldots$ of the points belonging to $\mathcal{L} \cap C$ such that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$. Without loss of generality, we may suppose that the points $A_1, A_2, \ldots, A_n, \ldots$ are consecutive points clockwise and the points $B_1, B_2, \ldots, B_n, \ldots$ are consecutive points counterclockwise on C (Fig. 9).

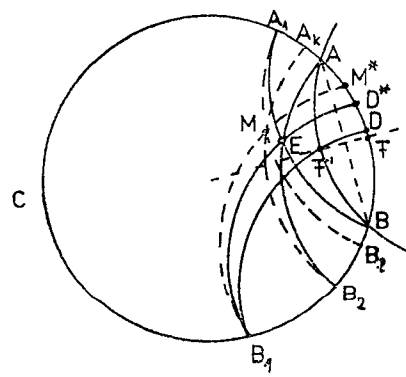

Fig. 9.

Let F be the midpoint of the open arc $\widehat{AB}$ and reflect F in the line AB getting the point F'. Without loss of generality we may suppose that B_1 (and so the other points of the sequence $B_1, B_2, \ldots, B_n, \ldots$ too) is (are) close to B. I.e., the unit circle which is a rotated copy of C through an angle $<\pi$ about the point B_1 clockwise connecting the points B_1, F' intersects C at a point D belonging to the open arc $\widehat{AF}$ $(<\pi)$. Again without loss of generality, we may suppose that $\widehat{A_1A} < \widehat{B_2B}$ (where $\widehat{A_1A}$ is an arc on C with length smaller than π and $\widehat{B_1B}$ is another arc whose length is smaller than π too). Consequently the unit circles different from C connecting the points A_1, B and B_2, A will intersect each other at a point E lying on the same side of the line F, F' as the points $D, A, A_1, A_2, \ldots, A_n, \ldots$ Therefore, if we rotate the unit circle C clockwise through an angle $<\pi$ about the point B_1 such that it contains E, then the new unit circle will intersect C, at a point D^* belonging to the open arc $\widehat{AD}$ $(<\pi)$.

After this, it is clear what we have to do. Let us choose a point A_k close enough to A and a point B_l close enough to B such that we get an intersection point M of the unit circles connecting the points A_1, B_l and

B_2, A_k, close to E. This shows that if we rotate C through an angle $<\pi$ about B_1 in a clockwise manner such that M is on it, then this new unit circle intersects C at a point M^* belonging to the open arc $\widehat{AD}$. But this is a contradiction because $M^* \in \mathcal{L} \cap C$ and so $\widehat{AB} \cap \overline{\mathcal{L} \cap C} = \emptyset$ although we supposed that $\widehat{AB} \cap \overline{\mathcal{L} \cap C} = \emptyset$.

Hence $\overline{\mathcal{L} \cap C} = C$. I.e., the point set $\mathcal{L} \cap C$ is everywhere dense on C. But from this we get immediately that $\mathcal{L}$ is everywhere dense inside the unit disk C, which is a contradiction because we knew that in our case $\mathcal{L}$ is not locally dense. This means that we could prove that if $\mathcal{L}$ is not locally dense, then $\mathcal{L}$ cannot have accumulation points.

In order to complete the proof of Lemma 4 we have to show that using the above observation, the empty disks of $\mathcal{L}$ cover the plane.

Let O be any point of the plane. If $O \notin \mathcal{L}$ then because $\mathcal{L}$ cannot have accumulation points, there exists a disk which does not contain any point of $\mathcal{L}$ in its interior, but whose interior contains O. Increasing this disk, we shall reach a position when the boundary will cover at least three intersection points (i.e., at least three points of $\mathcal{L}$) and no point of $\mathcal{L}$ will be inside this disk, i.e., it is an empty disk whose interior covers the point O (Fig. 10). If $O \in \mathcal{L}$ then a similar argument shows that there exists an empty circle containing the point O. (Actually in both cases we used the fact that the radius of an empty circle cannot be arbitrary large; in our case this means that it cannot be greater then 1.) This completes the proof of Lemma 4.

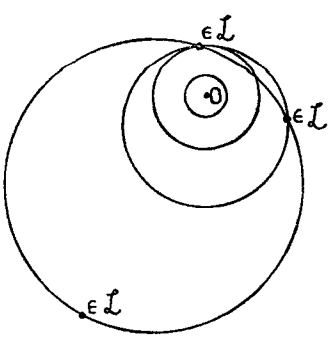

Fig. 10.

Now let us complete the proof of Theorem 1. If $\mathcal{L}$ is locally dense, then by Lemma 1 it is everywhere dense in the plane. If $\mathcal{L}$ is not locally dense, then we know using Lemma 4 that the empty disks of $\mathcal{L}$ cover the plane. So because of Lemma 3, these empty circles are congruent. And by Lemma 2, $\mathcal{L}$ is identical either with the vertices of a rectangular-lattice with diagonals 2 or with the vertices of a regular triangle-lattice of side length 1. In addition, it is obvious that the smallest sides of the rectangles must be greater than 1.

3. Proof of Theorem 2.

If a point of the plane is covered by infinitly many unit disks, then the density $d(\mathcal{C})$ and also the density $d(\mathcal{C}^*)$, are infinite and we do not have to prove anything. Therefore, we may suppose that $\mathcal{C} = \{C_i | C_i$ is a unit disk, $i = 1,2, \ldots\}$ and any intersection point of $\mathcal{C}$ belongs to finitly many unit circles of $\mathcal{C}$; moreover any unit circle of $\mathcal{C}$ contains finitly many intersection points of $\mathcal{C}$. Let M be an arbitrary intersection point contained by m unit circles of $\mathcal{C}$. We assign the real number $\frac{1}{m}$ to each center point of a unit circle of $\mathcal{C}$ which contains the point M. Compute this for every intersection point of $\mathcal{C}$. Then choose an arbitrary unit circle C_i of $\mathcal{C}$ with center point O_i. Let us denote the intersection points of $\mathcal{C}$ belonging to C_i by $M_{i_1}, M_{i_2}, \ldots, M_{i_k}$, assigning the real numbers $\frac{1}{m_{i_1}}, \frac{1}{m_{i_2}}, \ldots, \frac{1}{m_{i_k}}$ to the center point O_i. In order to prove our theorem, it is enough to show that

$$\frac{1}{m_{i_1}} + \frac{1}{m_{i_2}} + \ldots + \frac{1}{m_{i_k}} \geq 1.$$

But since any unit disk of $\mathcal{C}$ is intersected by at least one other, and no two of the unit disks of $\mathcal{C}$ touch each other, we know that $k \geq 2$ and $m_{i_j} \leq k$ for any $1 \leq j \leq k$. Consequently

$$\frac{1}{m_{i_1}} + \frac{1}{m_{i_2}} + \ldots + \frac{1}{m_{i_k}} \geq \frac{1}{k} + \frac{1}{k} + \ldots + \frac{1}{k} = k\frac{1}{k} = 1$$

which completes the proof of Theorem 2.

4. Proof of the Proposition

The case of the cube shows that $f(n) \geq 2^n$ $(n \geq 2)$. On the other hand, suppose that $K_1, K_2, \ldots, K_k$ is a system of homothetic convex bodies such that any two of them touch each other. Without loss of generality, we may suppose that K_1 is the smallest. Then in each convex body K_i, $(i \geq 2)$ we can find a translate K_i^* of K_1 which touches K_1. In this way we construct $(k-1)$ translates $K_2^*, K_3^*, \ldots, K_k^*$ of K_1 which touch K_1 and whose interiors are pairwise disjoint. But then the theorem of H. Hadwiger [6] shows that $k-1 \leq 3^n - 1$ with equality if and only if K_1 is a parallelotope. This completes the proof.

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ON SOME PAIRS OF MULTIPLICATIVE FUNCTIONS

By

K. H. INDLEKOFER and I. KÁTAI*

Department for Mathematics – Informatics of the University Paderborn,
Computer Center of the L. Eötvös University, Budapest

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Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) denote the class of complex-valued multiplicative (completely multiplicative) functions. The subclass of the arithmetical functions f satisfying the relation

$$\sum_{n\leq x} f(n) = O(x) \quad (x\to\infty) \tag{1}$$

will be denoted by L_1.

Our purpose in this paper is to give a complete characterization of the functions $f, g\in\mathcal{M}^*$ for which the relation

$$\sum_{n\leq x} |g(n+K)-f(n)| = O(x) \tag{2}$$

holds. Here K is an arbitrary positive integer.

The following assertions are obvious. If $f, g\in L_1\cap\mathcal{M}^*$, then (2) is satisfied. If f, g is a solution of (2) and one of f, g belongs to $L_1\cap\mathcal{M}^*$, then so does the other. Let $f(n) = n^s U(n)$, $g(n) = n^s V(n)$, $Re\, s\leq 1$, $V(n+K) = U(n)$ $(\forall n\in\mathbf{N})$. Then the module of $U(n)$ may take only the values 0 and 1, as it follows immediately from Theorem 1 in [4] consequently (2) is true for f, g as well.

THEOREM. *Assume that $f, g\in\mathcal{M}^*$, satisfy* (2). *If $f\notin L_1$ then*

$$f(n) = n^s U(n),\ g(n) = n^s V(n), \tag{3}$$

$0\leq \mathrm{Re}\, s\leq 1$, *furthermore*

$$V(n+K) = U(n) \quad (\forall n\in\mathbf{N}). \tag{4}$$

REMARKS. (1) All solutions of (4) for $U, V\in\mathcal{M}^*$ have been determined in [4], Theorem 1.

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(2) The basic idea of the proof is to reduce the problem to an equation $\sum \frac{1}{n}|g_1(n+K)-f_1(n)|<\infty$, and apply Theorem 4 in [4]

(3) The special case $f(n)=g(n)$ has been treated in [6].

LEMMA 1. Assume that $f, g\in\mathcal{M}^*$, and there exists a constant $C\neq 0$ such that

$$\frac{1}{x}\sum_{n\leq x}|g(n+K)-Cf(n)|\ll 1. \tag{5}$$

If there exists a prime p, $p\nmid K$ for which $f(p)=0$ or $g(p)=0$, then $f, g\in L_1$.

PROOF. It is enough to prove Lemma 1 under the condition that K is "minimal", i.e. that (5) does not hold for any proper divisor K_1 of K.

1. We may assume that $f(q)=g(q)=0$ for all $q|K$. If there exists a prime q such that $f(q)\neq 0$, $g(q)\neq 0$, then by $K=qK_1$ we have

$$\frac{1}{x}\sum_{n\leq x}\left|g(n+K_1)-C\frac{f(q)}{g(q)}f(n)\right|\ll 1, \tag{6}$$

that contradicts to the minimality of K.

If $f(q)=0$ and $g(q)\neq 0$, then summing over the integers of form $n=mq$ in (2) we find

$$\frac{1}{x}\sum_{m\leq x}|g(m+K_1)|\ll 1,$$

which implies that $g\in L_1$. In the case $f(q)\neq 0$, $g(q)=0$ we get similarly that $f\in L_1$.

2. We may assume that $f, g\geq 0$, $C>0$. It is enough to observe that (5) implies the inequality

$$\frac{1}{x}\sum_{n\leq x}\big||g(n+K)|-|C|\cdot|f(n)|\big|\ll 1.$$

We shall assume that $f, g\geq 0$, $C>0$.

3. Let $\mathcal{P}=\{p,\ (p,K)=1,\ f(p)=0\}$, $\mathcal{R}=\{q,\ (q,K)=1, g(q)=0\}$,

$\mathcal{P}_1=\{p\in\mathcal{P}\,|\,\exists n_0\equiv K \pmod p$ such that $g(n_0)\neq 0\}$,

$\mathcal{R}_1=\{q\in\mathcal{R}\,|\,\exists n_0\equiv -K \pmod q$ such that $f(n_0)\neq 0\}$.

Assume that $\mathcal{P}\cup\mathcal{R}$ is not empty. We prove that $f, g\in L_1$.

Case I.: $\mathcal{P}_1\neq\emptyset$. Let $p\in\mathcal{P}_1$, and let $n_0\equiv K \pmod p$ be such that $g(n_0)\neq 0$. Let ν run over the integers $\nu\equiv 1 \pmod p$. Then $p\nmid \nu n_0-K$, consequently from (2) we conclude

$$\sum_{\substack{\nu\equiv 1 \pmod p\\ \nu<x}} g(\nu n_0)\ll x, \quad\Rightarrow\quad \sum_{\substack{\nu\equiv 1 \pmod p\\ \nu<x}} g(\nu)\ll x.$$

Let now ν run over the arithmetic progression $\nu \equiv l \pmod p$, where $(l, p) = 1$. We shall estimate the sum

$$\sum_{\substack{\nu \equiv l (\mathrm{mod}\ p) \\ \nu < x}} g(\nu).$$

This is zero if $g(\nu) = 0$ for every ν. Let $g(\nu_0) \neq 0$, $\nu_0 \equiv l \pmod p$.

Then $\nu_0^{p-2} \equiv l^{-1} \pmod p$, and $k_\nu = \nu_0^{p-2}\, \nu \equiv 1 (\mathrm{mod}\ p)$.

So we have

$$\sum_{\substack{\nu \equiv 1 (\mathrm{mod}\ p) \\ \nu < x}} g(k_\nu) \ll \sum_{\substack{\mu \equiv 1 (\mathrm{mod}\ p) \\ \mu < \nu_0^{p-2} x}} g(\mu) \ll x.$$

Doing this for each l, $(l, p) = 1$, we find

$$\sum_{\substack{(n, p) = 1 \\ n \leq x}} g(n) \ll x. \tag{7}$$

If $g(p) = 0$ then $g \in L_1$, and we are ready. So we assume that $g(p) \neq 0$.

Let us assume first that there exists an η, $f(\eta) \neq 0$, $\eta \not\equiv 1 (\mathrm{mod}\ p)$. Let $s_m = pm\eta - K\eta + K$. Then $(s_m, p) = (K((1-\eta), p) = 1$, furthermore

$$\sum_{m \leq x} g(pm) \ll x + \Sigma f(pm - K) \ll x + \Sigma f(\eta(pm - K)) \ll x + \Sigma g(s_m) \ll x.$$

In this case we are ready as well.

Remains the case: $f(\eta) = 0$ for each η, $\eta \not\equiv 1 \pmod p$. We have $pm - K \equiv -K (\mathrm{mod}\ p)$. If $1 \not\equiv -K (\mathrm{mod}\ p)$, then $f(pm - K) = 0$ for every m, consequenty $\sum_{m \leq x} g(pm) = O(x)$, $g \in L_1$. Assume now that $1 \equiv -K \pmod p$ i.e. $p | K + 1$. Let $K = -1 + p^\beta s$, $(s, p) = 1$. We shall prove that

$$\sum_{\substack{n \leq x \\ (n, p) = 1}} f(n) \ll x. \tag{8}$$

If $f(n) = 0$ n, $(n, p) = 1$, then (8) is true.

Let now n run over the integers n for which $p^\delta \| n - 1$, $\delta \neq \beta$. Then $n + K = p^\delta m + p^\beta s$, $p^\delta \| n + K$ if $\delta < \beta$ and $p^\beta \| n + K$ if $\delta > \beta$. The contribution of these terms in (8), can be majorated by

$$x + \sum g(n + K) \ll x + \sum_{\delta \leq \beta} \Big(\sum_{\substack{\nu < x \\ (\nu, p) = 1}} f(\nu) \Big) g(p^\alpha) \ll x.$$

If $f(n_0) \neq 0$, $(n_0, p) = 1$, then $f(n_0^k) \neq 0$ $(k \in \mathbf{N})$, therefore for an arbitrary γ there exists n_1 such that $f(n_1) \neq 0$, $n_1 \not\equiv 1 \pmod{p^\gamma}$. Now we estimate the contribution of integers n for which $p^\beta \| n - 1$. Then we shall choose two integers P_1, P_2 so that $P_1 = 1 + t_1 p^{\gamma_1}$, $P_2 = 1 + t_2 p^{\gamma_2}$, $\beta < \gamma_1 < \gamma_2$, $f(P_1) \neq 0$, $f(P_2) \neq 0$, $(t_1 t_2, p) = 1$ and put $n = 1 + mp^\beta$, $(m, p) = 1$. Since $nP_i = (1 + t_i p^{\gamma_i})\ (1 + p^\beta m) = 1 + p^\beta m + t_i p^{\gamma_i} + t_i m p^{\beta + \gamma_i}$, therefore $nP_i + K = p^\beta (m + s) + t_i p^{\gamma_i} + t_i m p^{\beta + \gamma_i}$. Let $p^\lambda \| m + s$. If $\lambda \neq \gamma_i - \beta$ then the exponent

σ of p defined by $p^\sigma \| nP_i + K$ is determined by λ, β, γ_i and $\sigma \leq \gamma_i$. Since $\lambda = \gamma_i - \beta$ holds at most for one of the γ_1, γ_2, therefore

$$\Sigma f(n) \ll \Sigma f(P_1 n) + \Sigma f(P_2 n),$$

where in the first sum $p^{\gamma_1 - \beta} \| (m+s)$ and in the second $p^{\gamma_2 - \beta} \| m+s$. Since

$$\sum f(P_i n) \ll x + \sum g(P_i n + K) \ll x + g(p^\sigma) \sum_{(\nu, p)=1} g(\nu) \ll x,$$

we find that

$$\sum_{p^\beta \| n-1} f(n) \ll x.$$

So we have proved that

$$\sum_{n \equiv 1 (\mathrm{mod}\, p)} f(n) \ll x.$$

But in our case $f(\eta) = 0$ for every η, $(\eta, K) = 1$, $p \nmid \eta - 1$ Consequently (8) is true, and $f(p) = 0$ implies that $f \in L_1$.

Case II. $\mathcal{R}_1 \neq \emptyset$. Hence one can deduce that f, $g \in L_1$ by the same method as was used in Case I., so we omit the proof.

Case III. $\mathcal{P}_1 = \emptyset$, $\mathcal{R}_1 = \emptyset$, $\mathcal{P} \cup \mathcal{R}$ is not empty. If $p \in \mathcal{P}$, then $g(n) = 0$ $\forall n$, $n \equiv K (\mathrm{mod}\, p)$, consequently $\mathcal{R}$ is an infinite set, and so is $\mathcal{P}$.

Now we shall see that the following assertion is true. (COND): $f(n) = 0$ if and only if $g(n+K) = 0$.

This assertion is true if $(n, K) > 1$. If $p \in \mathcal{P}$, then $g(n) = 0$ $\forall n$, $n \equiv K$ (mod p), since $\mathcal{P}_1 = \emptyset$. Let $f(n_0) = 0$, $(n_0, K) = 1$. Then there exists $p | n_0$, $p \in \mathcal{P}$, therefore $g(n) = 0$ for $n = n_0 + K$. So the implication $f(n) = 0 \Rightarrow$ $\Rightarrow g(n+K)$ is true. Assume now that $g(n+K) = 0$, $(n, K) = 1$. Then there exists $q | n+K$, $q \in \mathcal{R}$. Since $q \notin \mathcal{R}_1$, therefore $f(n) = 0$. By this (COND) is true.

In [3] (see p. 262–264) it was proved that $\mathcal{P} \cup \mathcal{R} \neq \emptyset$, (COND) imply that $f(n) = g(n) = 0$ $\forall n \geq 2$.

By this Lemma 1 has been proved.

LEMMA 2. Let f, $g \in \mathcal{M}^*$, $f(n) = g(n) = 0$ if $(n, K) > 1$ and $f(n) \neq 0$, $g(n) \neq 0$ if $(n, K) = 1$. Assume that (2) holds and f, $g \notin L_1$. Then $|f(n)| = |g(n)|$ $\forall n \geq 1$.

PROOF. Let $H(n) = \dfrac{f(n)}{g(n)}$ defined on the set $(n, K) = 1$. Then $H(n) \neq 0$.

Let λ be an integer that has exactly the same set of prime factors as K. Then clearly $(\lambda + 1, K) = 1$. Let ϱ be an integer coprime to K and put $C = \lambda\varrho + 1$; we have $(C, K) = 1$.

We shall prove that

$$H(\lambda + 1) = H(C). \tag{9}$$

From (2) we have the inequalities:

$$\sum_{N \leq x} |g(C) f((\lambda+1)\varrho N) - g(C) g((\lambda+1)\varrho N + K)| \ll x,$$

$$\sum_{N\leq x} |g((\lambda+1)\varrho NC+KC)-f((\lambda+1)\varrho NC+K\lambda\varrho)| \ll x,$$

$$\sum_{N\leq x} |f(\varrho)f((\lambda+1)NC+K\lambda)-f(\varrho)g((\lambda+1)CN+K(\lambda+1))| \ll x,$$

$$\sum_{N\leq x} |f(\varrho)g(\lambda+1)g(CN+K)-f(\varrho)g(\lambda+1)f(CN)| \ll x.$$

From the triangle-inequality we get

$$\sum_{N\leq x} |g(C)f(\lambda+1)f(\varrho)-f(\varrho)f(C)g(\lambda+1)|\cdot|f(N)| \ll x.$$

Since $f\notin L_1$, $f(\varrho)\neq 0$, this inequality implies (9).

Let K^* be any positive integer all prime divisor of which divides K, and which contains all prime divisor of K at least on the first power. Let ϱ be coprime to K, $C = 1+K^*\varrho$. Then $(K^*+1, K) = 1$ and from (9) we have

$$H(1+K^*\varrho) = H(K^*+1). \tag{10}$$

Since for $(\nu, K) = 1$, $(1+K^*)^\nu = 1+\nu K^*+\ldots = 1+\varrho K^*$, and $(\varrho, K) = 1$, therefore $H((1+K^*)^\nu) = H(1+K^*)$. Since $H((1+K^*))\neq 0$, we have

$$H((1+K^*))^{\nu-1} = 1 \text{ if } (\nu, K) = 1. \tag{11}$$

Let $T(n) := |H(n)|$. From (10) and (11) we get that

$$T(1+K^*) = T(1+K^*\varrho) = 1 \text{ if } (\varrho, K) = 1. \tag{12}$$

Let now $n\equiv 1 (\mathrm{mod}\ K)$. Then n can be written in the form $n = 1+K^*\varrho$, where $(\varrho, K) = 1$, $K|K^*$, and all prime divisors of K^* divide K. Consequently $T(n)=1$. Let now $(n, K)=1$, n be an arbitray integer. Then $n^{\varphi(K)} = 1 (\mathrm{mod}\ K)$, consequently $T(n^{\varphi(K)}) = 1$. Since $T\in\mathcal{M}^*$ and $T\geq 0$ therefore $T(n) = 1$. So we have proved that $T(n) = 1$ $\forall n$, $(n, K) = 1$. i.e. that $|f(n)| = |g(n)|$ $\forall n$, $(n, K) = 1$.

Let $h(n) = |f(n)|$. Then from (2) we have

$$\sum_{n\leq x} |h(n+K)-h(n)| = O(x). \tag{13}$$

Since $h\notin L_1$, from the Theorem in [6] we infer that $|h(n)| = n^\sigma = |f(n)| = |g(n)|$ with a suitable σ, $0<\delta\leq 1$, $(n, K) = 1$.

Let $f_1(n) = f(n)n^{-\sigma}$, $g_1(n) = g(n)n^{-\sigma}$.

Then

$$g_1(n+K)-f_1(n) = f(n+K)\left(\frac{1}{(n+K)^\sigma}-\frac{1}{n^\sigma}\right)+\frac{g(n+K)-f(n)}{n^\sigma},$$

$$|g_1(n+K)-f_1(n)| \leq (n+K)^\sigma\left(\frac{1}{n^\sigma}-\frac{1}{(n+K)^\sigma}\right)+\frac{1}{n^\sigma}|g(n+K)-f(n)|$$

Summing up for $n \in [x, 2x]$ we get that

$$\sum_{x \le n < 2x} \frac{1}{n} |g_1(n+K) - f_1(n)| \le \frac{(2x+K)^\sigma}{x^{1+\sigma}} + \frac{1}{x^{1+\sigma}} \sum_{x \le n \le 2x} |g(n+K) - f(n)|,$$

and by (2) the right hand side is less than $O(x^{-\sigma})$.

Forming these inequalities for $x = 2^\nu$ ($\nu = 1,2, \ldots$) and summing, we get that

$$\sum_{n=1}^{\infty} \frac{1}{n} |g_1(n+K) - f_1(n)| < \infty. \tag{14}$$

In [4] it was proved that (14) with $|f_1(n)| = |g_1(n)| = 1 \ \forall n$, $(n, K) = 1$ implies that $f_1(n) = n^{i\tau} U(n)$, $g_1(n) = n^{i\tau} V(n)$, and U, V satistfy (4).

By this our theorem is proved.

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SOME REMARKS ON ARITHMETICAL FUNCTIONS SATISFYING LINEAR RECURSIONS IN SHORT INTERVALS

By

I. KÁTAI

Computer Center of the L. Eötvös University, Budapest

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1. Let $\mathcal{A}^*$ ($\mathcal{M}^*$) denote the class of complex-valued completely additive (multiplicative) functions. Let $\mathcal{M}_0^*$ be the subclass of $\mathcal{M}^*$ containing all those $f \in \mathcal{M}^*$ that are nowhere zero.

Our purpose in this paper is to determine all functions in $\mathcal{A}^*$ and $\mathcal{M}_0^*$ which satisfy some linear recursions in a suitable sequence of relatively small intervals.

The exact assertions will be formulated in sections **2** and **3.**

LEMMA 1. Let $f \in \mathcal{A}^*$, or let $f \in \mathcal{M}_0^*$. Assume that there exists an $\varepsilon > 0$ and a suitable sequence $N_1 < N_2 < \dots$ of integers, such that

$$f(n) = A_j \; \forall \, n \in [N_j, N_j + (2+\varepsilon)\sqrt{N_j}]$$

($j = 1,2, \dots$). Then $f(n) = 0$ if $f \in \mathcal{A}^*$ and $f(n) = 1$ if $f \in \mathcal{M}_0^*$ for each $n \in \mathbf{N}$.

This assertion for $f \in \mathcal{A}^*$ has been proved in [1]. The case $f \in \mathcal{M}_0^*$ can be considered in the same way, so we omit its proof.

For some generalizations see [2], [3], [4]

LEMMA 2. Let $g \in \mathcal{M}_0^*$. Assume that there exist $s \in \mathbf{C}$, $\varepsilon > 0$ and a sequence of integers $N_1 < N_2 < \dots$ such that

$$(1.1) \qquad g(n) = n^s A_j \quad N_j \leq n \leq N_j + (2+\varepsilon)\sqrt{N_j} \; (j = 1, 2, \dots).$$

Then $g(n) = n^s$ for every $n \in \mathbf{N}$.

PROOF. Let $f(n) := g(n) n^{-s}$ and apply Lemma 1. □

In the space of the sequences $\{x_1, x_2, \dots\}$ let E, I, Δ denote the operators defined by the following relations: $Ex_n = x_{n+1}$, $\Delta x_n = x_{n+1} - x_n$, $Ix_n = x_n$. Let $\mathbf{C}[x]$ denote the ring of polynomials over $\mathbf{C}$. For $P \in \mathbf{C}[x]$, $P(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k$ we shall write $P(E)x_n = \alpha_0 x_n + \alpha_1 x_{n+1} + \dots + \alpha_k x_{n+k}$ ($n \in \mathbf{N}$).

LEMMA 3. Let $g \in \mathcal{M}_0^*$. Assume that there exists a nonnegative integer h, $\varepsilon > 0$, and a suitable sequence of integers $N_1 < N_2 < \dots$ such that

$$(1.2) \qquad \Delta^h g(n) = 0 \quad n \in [N_j, N_j + 2(1+h+\varepsilon)\sqrt{N_j}] \ (j = 1, 2, \dots).$$

Then $g(n) = 1$, identically.

PROOF. If (1.2) holds with $h = 0$ or 1, then Lemma 1 contains the assertion. We may assume that $h > 1$ and that the condition (1.2) does not hold for $h-1$ instead of h, for any j. From (1.2) we have that $g(n) = Q_j(n)$ in $n \in [N_j, N_j + 2(1+h+\varepsilon)\sqrt{N_j}]$, where $Q_j \in \mathbf{C}[x]$, deg $Q_j = h-1$. If $Q_j(x) = A_j x^{h-1}$ for an infinite subsequence of the integers j_l, then the conditions of Lemma 2 are satisfied and we are ready.

Let now $N = N_j$, $M = 2(1+h+\varepsilon)\sqrt{N_j}$, $g(n) = Q(n)$ in $n \in [N, N+M] = J$, $Q(x) = B_{h-1}x^{h-1} + B_k x^k + \dots$, where $B_{h-1} \neq 0$, $B_k \neq 0$, and $B_\nu = 0$ if $k < \nu < h-1$. Let p, q, m be positive integers and

$$(1.3) \qquad R_{p,q}(m) := Q(pm)\ Q(q(m+1)) - Q(p(m+1))\ Q(qm).$$

Then $R_{p,q}(m)$ is a polynomial of m of degree $\leq 2h-3$. If pm, qm, $p(m+1)$, $q(m+1) \in J$, then Q can be replaced by g, and by using its complete multiplicativity, we get that $R_{p,q}(m) = 0$. If this happens for $2h-2$ distinct values of m, then this implies that the polynomial $R_{p,q}$ is identically zero. But this cannot occur if $p \neq q$, since by an easy computation one can get that $R_{p,q}$ is a polynomial of exact degree $h-2+k$.

Let $p = [\sqrt{N}] + 1 = m_0$, $q = p+1$, $M^* = 2(1+h)\sqrt{N} + 4h$. Then $N \leq pm_0 < q(m_0 + 2h - 1) \leq N + M^*$, and $M^* < M$ if j is large enough. Consequently pm, qm, $p(m+1)$, $q(m+1) \in J$ hold for $m = m_0 + s$ $(s = 0, 1, \dots, 2h-2)$. This is a contradiction. We are ready. □

LEMMA 4. Let $f \in \mathcal{A}^*$. Assume that there exists a nonnegative integer h, $\varepsilon > 0$ and a suitable sequence of integers $N_1 < N_2 < \dots$ such that

$$(1.4) \qquad \Delta^h f(n) = 0 \quad n \in [N_j, N_j + (2+h+\varepsilon)\sqrt{N_j}] \ (j = 1, 2, \dots).$$

Then $f(n) = 0$, identically.

PROOF. If $h = 0$ or 1, then the assertion follows from Lemma 1. Let now $h \geq 2$, and assume that (1.4) does not hold with $h-1$ instead of h. Then $f(n) = Q_j(n)$ with a suitable polynomial $Q_j \in \mathbf{C}[x]$, deg $Q_j = h-1$. Choosing a typical $N = N_j$, $M = (2+h+\varepsilon)\sqrt{N_j}$ we take $f(n) = Q(n)$ in $n \in [N, N+M]$. Now we define

$$S_{p,q}(m) := Q(p(m+1)) + Q(qm) - Q(pm) - Q(q(m+1)).$$

If pm, qm, $p(m+1)$, $q(m+1) \in [N, N+M]$, then the right hand side can be written as $f(p(m+1)) + f(qm) - f(pm) - f(q(m+1))$, and this is zero, since $f \in \mathcal{A}^*$. In the other hand, $S_{p,q}(m)$ is a polynomial in m of exact degree $h-2$, if $p \neq q$.

By choosing $p = m_0 = [\sqrt{N}]+1$, $q = p+1$, we get that $n \le pm_0 \le \le (p+1)(m_0+h-1) \le N+M$, which implies that $S_{p,q}(m) = 0$ $m = m_0, \ldots, m_0+h-2$, that is a contradiction. □

2. Definition 1. We shall say that an arithmetical function f satisfies condition $C_{h,\gamma}$, if there exists a sequence $N_1 < N_2 < \ldots$ of integers, a suitable sequence b_j of real numbers, $b_j \to \infty$, and a suitable sequence $\Phi_j(z)$ of non-zero polynomials over $\mathbf{C}$, of deg $\Phi_j \le h$, such that

$$\Phi_j(E)f(n) = \text{constant} = A_j \text{ in } n \in [N_j, N_j + b_j N_j^{\gamma}]. \tag{2.1}$$

Let
$$\alpha_0 = \frac{1}{2}, \alpha_s = \frac{s+1}{s+2} \ (s \ge 1).$$

Theorem 1. *If $f \in \mathcal{A}^*$ and f satisfies C_{h,α_h} then $f(n) = 0$, identically.*

Proof. We may assume that Φ_j are monic polynomials, furthermore that $\Phi_j(0) \ne 0$.

If $h = 0$, then Lemma 1 implies our assertion. We may assume that $h \ge 1$ and that h is the minimal value for which C_{h,α_h} is satisfied.

Let j be a large value, $N = N_j$, $M = b_j N_j^{\alpha_h}$, $\Phi_j(z) = \Phi(z) = \xi_0 + \xi_1 z + \ldots + \xi_h z^h$, $\xi_h = 1$. Let the roots of Φ be $\Theta_1, \ldots, \Theta_h$, $\Phi(z) = (z-\Theta_1) \ldots (z-\Theta_h)$.

If $\Theta_1 = \ldots = \Theta_h = 1$, then $\Phi(z) = (z-1)^h$. If this occurs for infinitely many j, then $\Delta^h f(n) = A_j$, and so $\Delta^{h+1} f(n) = 0$ in the corresponding interval $[N_j, N_j + b_j N^{\alpha_h}]$. In this case, by Lemma 4, we are ready. Now we assume that $\Phi_j(z) \ne (z-1)^h$, $\Phi(z) \ne (z-1)^h$. Since

$$\Phi(E)f(n) = \text{constant} = A \quad \text{in } n \in [N, N+M]$$

therefore for every $r \in \mathbf{C}[x]$

$$r(E)\Phi(E)f(n) = \text{constant} = r(1)A \tag{2.2}$$

in $n \in [N, M+M\text{-}s]$, $s = \deg r$. For an arbitrary integer $m > 0$ let $k_m(z) = \prod_{j=1}^{h} (z - \Theta_j^m)$. Since $k_m(z^m)$ is a multiple of $\Phi(z)$, deg $k_m(z^m) = mh$, therefore $k_m(z^m) = u_m(z)\Phi(z)$, deg $u_m = (m-1)h$, consequently

$$k_m(E^m)f(n) = A_m \quad \text{if} \quad n \in [N, N+M-(m-1)h]. \tag{2.3}$$

Let $k_m(z) = \eta_0 + \eta_1 z + \ldots + \eta_h z^h$, $\eta_h = 1$, and write $n = m\nu$, $\nu \in \mathbf{N}$ in (2.3). Then for

$$\nu \in \left[\frac{N}{m}, \frac{N+M}{m} - h\right] \tag{2.4}$$

we have $\eta_0 f(m\nu) + \eta_1 f(m(\nu+1)) + \ldots + \eta_h f(m(\nu+h)) = A_m$, i.e. that

$$k_m(1)f(m) + k_m(E)f(\nu) = A_m,$$

and so that

$$k_m(E)f(\nu) = A'_m;\ A'_m = A_m - k_m(1)f(m). \tag{2.5}$$

Let $\lambda_m(z) = k_{m+1}(z) - k_m(z)$. Then either $\lambda_m(z)$ is the zeropolynomial, or $\deg \lambda_m < h$. Furthermore in

$$\nu \in \left[\frac{N}{m}, \frac{N+M}{m+1} - h\right] \tag{2.6}$$

the relation (2.5) holds with $m+1$ instead of m as well, consequently

$$\lambda_m(E)f(\nu) = A'_m - A''_m. \tag{2.7}$$

We shall choose such a set of integers m for which the length of the interval is large, i.e. satisfies

$$\frac{N+M}{m+1} - h - \frac{N}{m} > b_j^*\left(\frac{N}{m}\right)^{\alpha_{h-1}} \tag{2.8}$$

with a suitable sequence $b_j^* \to \infty$. We have $M = b_j N^{\alpha_h}$. If for a subsequence j_ν one can find an $m = m_\nu$ satisfying (2.8) such that $\lambda_{m_\nu}(z) \neq 0$, then f satisfies $C_{h-1,\alpha_{h-1}}$ contrary to our assumption. So we may assume that (2.8) implies that $\lambda_m(z) = 0$. The inequality (2.8) holds with $b_j^* = \sqrt{b_j}$ if

$$\frac{2}{b_j} N^{1-\alpha_h} \leq m \leq \left(\frac{1}{4}\sqrt{b_j}\right)^{1/(1-\alpha_{h-1})} N^{1-\alpha_h}, \tag{2.9}$$

which is easy to see. Consequently $\lambda_m(z) \equiv 0$ for these values of m.

Let U be the least and V be the largest m satisfying (2.9). Let D be a large constant depending only on h, which we shall determine later. Since $b_j \to \infty$, therefore $V \geq DU$ if j is large enough. We shall prove that if

$$\{\Theta_1^m, \ldots, \Theta_h^m\} = \{\Theta_1^{m+1}, \ldots, \Theta_h^{m+1}\} \tag{2.10}$$

for each m, $U \leq m \leq DU$, then there exists such an $m_1 \in [U, DU]$ for which $\{\Theta_1^{m_1}, \ldots, \Theta_t^{m_1}, \ldots, \Theta_h^{m_1}\} = \{1, \ldots, 1\}$. But in this case $k_{m_1}(z) = (z-1)^h$, and by (2.5), (2.4) we have $\Delta^h f(\nu) =$ constant in (2.4). If this occurs for infinitely many $N = N_j$, then we shall get our theorem by Lemma 4 applying it with $h+1$ instead of h.

Let $E_m = \{\Theta_1^m, \ldots, \Theta_h^m\}$ listing the elements with multiplicity. The subset $E_m^{(r)}$ denote those elements Θ_i^m which occur with multiplicity r in E_m. Let r be such an integer for which $E_m^{(r)} \neq \emptyset$. Let $F_m^{(r)}$ be the same set as $E_m^{(r)}$ but in $F_m^{(r)}$ we list each elements only once. Let $F_m^{(r)} = \{\varrho_1^m, \ldots, \varrho_t^m\}$. From (2.10) it is clear that

$$F_m^{(r)} = F_{m+1}^{(r)}. \tag{2.11}$$

Since $\Phi(0) \neq 0$, therefore $\varrho_\nu \neq 0$. From (2.11) we get that $F_{sU}^{(r)} = F_U^{(r)} (s = 1, 2, \ldots, D)$.

Let σ denote the permutation $\sigma\varrho_i^U = \varrho_i^{2U}$. Let $i_1, i_2, \ldots, i_\mu, i_1$ be a cycle of the permutation, such that

$$\varrho_{i_1}^U = \varrho_{i_2}^{2U}, \varrho_{i_2}^U = \varrho_{i_3}^{2U}, \ldots, \varrho_{i_{\mu-1}}^U = \varrho_{i_\mu}^{2U}, \varrho_{i_\mu}^U = \varrho_{i_1}^{2U}.$$

Hence we get that

$$\varrho_{i_l} = \varrho_{i_l}^{2U}, \text{ i. e. that } \varrho_{i_l}^{(2^l U)} = 1$$

($l = 1,2, \ldots, \mu$). The length of the largest cycle that can be occur is $\leq h$. Therefore by choosing d to be the least common multiple of the quantities $2^l - 1$ ($l = 1, \ldots, h$), we have $\Theta_i^{Ud} = 1$ ($i = 1, \ldots, h$), if $d \leq D$.

This finishes the proof of our Theorem 1. □

3. THEOREM 2. Let $f \in \mathcal{M}_0^*$, and let it satisfy condition C_{h,α_h}.

Then $f(n) = n^s$ for some integer $s \in [1, h]$.

PROOF. The proof is similar to that of Theorem 1. If $h = 0$ then Lemma 1 implies our assertion. We assume that $h \geq 1$ and that h is the minimal value for which C_{h,α_h} is satisfied.

Let j be a large number, $N, M, \Phi, \Theta_1, \ldots, \Phi_h$ be the same quantities, as in the proof of Theorem 1. The formulas (2.2), (2.3) are satisfied. Since $\eta_0 f(m\nu) + \ldots + \eta_h f(m(\nu+h)) = f(m)[\eta_0 f(\nu) + \ldots + \eta_h f(\nu+h)]$, using that $f \in \mathcal{M}^*$ we have $f(m)k_m(E)f(\nu) = A_m$, that by $f(m) \neq 0$ implies that

$$k_m(E)f(\nu) = A'_m, \; A'_m = A_m \cdot f^{-1}(m). \tag{3.1}$$

We may assume that $\Phi_j(z) \neq (z-1)^h$, for each large j, since in the opposite case we can apply Lemma 3 with $h+1$ instead of h, and this leads to our assertion immediately. We can continue the proof in the same way as was done in Theorem 1. □

REMARK. The multiplicative functions satisfying a linear recursion on the whole set of positive integers has been considered by A. SÁRKÖZY in [5].

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ON RIESZ BASES

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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1. The system $(\varphi_n)_{n\in\mathbf{Z}}$ in a separable Hilbert space H is called Riesz basis if there exists an isomorphism $A: H\to H$ of H onto itself such that $(A\,\varphi_n)_{n\in\mathbf{Z}}$ is an orthonormal basis of H. A system $(\varphi_n)_{n\in\mathbf{Z}}$ is said to be an L – basis in H if it is a Riesz basis in the closure of its linear hull.

Let $T_q := 2(q-1)/q$, $\mathbf{Z}_q = \mathbf{Z}\backslash q\mathbf{Z}$

$$\Psi_q := \{e^{i\pi kt} : k\in\mathbf{Z}_q\},$$

where $q\in\mathbf{N}$ (natural number) and $\mathbf{Z}$ denotes the set of integers.

The aim of the present paper is to investigate the Riesz bases from exponents and some related problem. First we are looking for the properties of Riesz bases of $L^2(a, b)$ over the subintervals of (a, b). We shall prove the

THEOREM 1.

(a) *Ψ_q is a Riesz basis in $L^2(0, T_q)$;*

(b) *Ψ_q is an $\mathcal{L}$ – basis in $L^2(0, T)$ if $T>T_q$;*

(c) *There exists $\Psi_q^T\subset\Psi_q$ such that Ψ_q^T is a Riesz basis in $L^2(0, T)$ if $T<T_q$.*

The proof is based on the theory of entire functions so some notions will be introduced below.

Consider the more general case $\varphi_n = e^{i\lambda_n x}$, $H = L^2(0, 2\pi)$, $\lambda_n\in\mathbf{C}$. First, an entire function $f(z)$ is called of exponential type if

$$\limsup_{r\to\infty}\frac{\ln M(r)}{r}<\infty,\ M(r) := \max_{|z|=r}|f(z)|.$$

In this case the indicator function

$$h(\Theta) := \limsup_{r\to\infty}\frac{\ln|f(re^{i\Theta})|}{r}$$

can also be defined. It is known (Cf. [11]) that for any exponential type entire function there exists a unique compact and convex set $G\subset\mathbf{C}$ such that

$$h(\Theta)=\sup_{x+iy\in G}(x\cos\Theta+y\sin\Theta)$$

The set G is called the indicator diagram of f. After B. JA. LEVIN [12] we call sine type function an exponential type entire function f whose zeros lie in a strip parallel to the real axis (i.e. the imaginary part of its zeros is bounded) and if $\ln|f(x+iy_0)|$, $x\in\mathbf{R}$ is bounded for an appropriate $y_0\in\mathbf{R}$.

Recently several necessary and sufficient conditions for the system $\{e^{i\lambda_n x}: \lambda_n\in\mathbf{C}\}$ to be a Riesz basis have been given in the terminology of entire functions (Cf. [2]). Roughly speaking, we take the entire function with zeros λ_n and indicator diagram $[-ia, 0]\subset\mathbf{C}$; then the Riesz basis property of $(e^{i\lambda_n x})$ in $L^2(0, a)$ is characterized by the growth properties of this function along the real line. The statements of this type are based on deep results from the theory of Hardy spaces. The following sufficient condition, very useful in the applications, was found formely by Š. A. AVDONIN; its proof requires only the study of some infinite products:

THEOREM A ([8], see also in [2]). Let $(\lambda_n)_{n\in\mathbf{Z}}$ be the set of zeros of a sine type function with indicator diagram $[-ia, 0]$, all λ_n being simple zeros. Take a bounded sequence $(\delta_n)_{n\in\mathbf{Z}}$ of complex numbers. Define

$$\Delta_x(R):=\sum_{x-R\le \operatorname{Re}\lambda_n\le x+R}\operatorname{Re}\delta_n$$

and suppose that

$$\inf_{n\neq m}|(\lambda_n+\delta_n)-(\lambda_m+\delta_m)|>0, \tag{1}$$

$$\limsup_{R\to\infty}\frac{|\Delta_x(R)|}{2R}<\frac{1}{4}. \tag{2}$$

Then $(e^{i(\lambda_n+\delta_n)x})_{n\in\mathbf{Z}}$ forms a Riesz basis in $L^2(0, a)$.

The following theorem of Š. A. AVDONIN and the author [10] shows that the exponents λ_n of some Riesz basis are precisely the sequences which are close to the sequence of the zeros of a sine type function in some sense.

THEOREM B ([10]). Suppose that the system $(\lambda_n)_{n\in\mathbf{Z}}$ satisfies the separability condition: $\inf_{n\neq m}|\lambda_n-\lambda_m|>0$. Then the following two statements are equivalent for any $a>0$.

(a) $(e^{i\lambda_n x})_{n\in\mathbf{Z}}$ is a Riesz basis in $L^2(0, a)$.

(b) There exists a sine type function with indicator diagram $[-ia, 0]$ and with real zeros $(\mu_n)_{n\in\mathbf{Z}}$ such that

$$d\operatorname{Re}(\lambda_{n-1}-\lambda_n)\le\operatorname{Re}(\mu_n-\lambda_n)\le d\operatorname{Re}(\lambda_{n+1}-\lambda_n) \tag{3}$$

holds with some $0<d<1/4$ (the sequence (μ_n) is indexed increasingly and (λ_n) in the order of the real parts).

For the sake of brevity we rewrite (2) in an equivalent form. Let $\Lambda = \{\lambda_n : n \in \mathbf{Z}\}$ be a separable set (i.e. $\inf\limits_{n \neq m} |\lambda_n - \lambda_m| > 0$) lying in some strip $|\operatorname{Im} z| \leq C$. A partition $\Lambda = \bigcup\limits_{j \in \mathbf{Z}} \Lambda_j$ is said to be A – partition if we give a partition of the strip by vertical lines such that the lenghts l_j of the domain defining Λ_j form a bounded sequence. Now (2) is equivalent to the existence of an A–partition $\Lambda = \bigcup\limits_j \Lambda_j$ such that

$$\left| \sum_{\lambda_n \in \Lambda_j}{}_n \operatorname{Re} \delta_n \right| \leq d l_j \left(d < \frac{1}{4}, j \in \mathbf{Z} \right). \tag{2'}$$

PROOF OF THE THEOREM 1.

(a) Using the easily verificable identities

$$\sin k\pi x + \sin k\pi\left(x + \frac{2}{q}\right) + \ldots + \sin k\pi\left(x + 2\frac{q-1}{q}\right) \equiv 0, \tag{4}$$

$$\cos k\pi x + \cos k\pi\left(x + \frac{2}{q}\right) + \ldots + \cos k\pi\left(x + 2\frac{q-1}{q}\right) \equiv 0, \tag{5}$$

we get

$$\left| \sum_{q+k} \left[a_k \cos k\pi\left(x + 2\frac{q-1}{q}\right) + b_k \sin k\pi\left(x + 2\frac{q-1}{q}\right) \right] \right| \leq$$

$$\leq \left| \sum_{q+k} [a_k \cos k\pi x + b_k \sin k\pi x] \right| + \ldots +$$

$$+ \left| \sum_{q+k} \left[a_k \cos k\pi\left(x + 2\frac{q-2}{q}\right) + b_k \sin k\pi\left(x + 2\frac{q-2}{q}\right) \right] \right| \leq$$

$$\leq \sqrt{q-1} \left\{ \left| \sum_{q+k} [a_k \cos k\pi x + b_k \sin k\pi x] \right|^2 + \ldots + \right.$$

$$\left. + \left| \sum_{q+k} \left[a_k \cos k\pi\left(x + 2\frac{q-2}{q}\right) + b_k \sin k\pi\left(x + 2\frac{q-2}{q}\right) \right] \right|^2 \right\}$$

and taking infinite sums it follows

$$\sum_{k=1}^{\infty} C_k^2 = \int_0^2 u(x)^2 dx \leq q \int_0^{2\frac{1-q}{q}} \left| \sum_{q+k} [a_k \cos k\pi x + b_k \sin k\pi x] \right|^2 dx$$

which proves the Riesz basis property of Ψ_q.

(b) This statement follows from the fact thet the canonical projection $L^2(O, T_q) \to L^2(O, T)$ is isometric.

(c) Use the Theorem A. Let $\lambda_n = 2\pi\frac{n}{T}$. Then the conditions of the Theorem A are fulfilled for the system $\{\lambda_n\}$. We have to define the numbers $\{\delta_n\}$ and an A-partition of $\Lambda = \bigcup_{j\in\mathbf{Z}} \Lambda_j$ for which the conditions (1) and (2') hold. We construct them in two step. Before we remark the

STATEMENT. In the sequence $\{[2n/T]\}$ there is no q consecutive integers and there exists an $m_0 \in \mathbf{N}$ such that in any sequence of qm_0 consecutive integers at least m_0+1 numbers do not belong to the series $\{[2n/T]\}$.

PROOF. This follows immediately from $T < T_q$.

First step. Consider the sequence $kq, kq+1, \ldots, kq+q-1$. It contains $<q$ members from the sequence $\{[2n/T]\}$. If $kq \notin \{[2n/T]\}$ then we define $(2n/T)_*: = [2n/T]$ $(kq+1 \leq [2n/T] \leq kq+q-1)$. If $kq = [2n_0/T]$, then we put $(2n_0/T)_*: = kq+1$ and we add 1 to every member of the sequence $\{[2n/T]\}$ until the first member, until we get $[2n/T]+1 = kq+l \neq [2(n+1/T)]$. By the first step we have the property $(2n/T)_* \in \mathbf{Z}\backslash q\mathbf{Z}$.

Second step. Denote $\{c_k : k \in \mathbf{Z}\} = \mathbf{Z}\backslash q\mathbf{Z}\backslash\{(2n/T)_* : n \in \mathbf{Z}\}$.

We know $c_{k+1} - c_k < qm_0$. Now let

$$A_j : = \{n : 100\, jqm_0 \leq 2n/T < 100(j+1)qm_0\}.$$

Suppose $c_{k_j}, c_{k_j+1}, \ldots, c_{k_{j+1}-1} \in A_j$.

Let $(2n/T)_{**}: = (2n/T)_*$ $(n \in A_j, (2n/T)_* < c_k)$ and $(2n/T)_{**}: = (2n/T)_* - 1$ for n with $c_{k_j} < (2n/T)_* < c_{k_j+1}$, $q \nmid (2n/T)_* - 1$; ;$(2n/T)_{**}: = (2n/T)_* - 2$ for n with $c_{k_j} < (2n/T)_* < c_{k_j+1}$, $q \mid (2n/T)_* - 1$.

Consider the sum

$$\sum_{\substack{n \in A_j \\ (2n/T)_* < c_{k_j+1}}} \left(\frac{2n}{T} - \left(\frac{2n}{T}\right)_{**}\right).$$

If this sum is negative, then we take $(2n/T)_{**}: = (2n/T)_*$ if $c_{k_j+1} < (2n/T)_* < c_{k_j+2}$. If this sum is non-negative, then set $(2n/T)_{**}: = (2n/T)_* - 1$ for $c_{k_j+1} < (2n/T)_* < c_{k_j+2}, q \nmid (2n/T)_* - 1$, $(2n/T)_{**}: = (2n/T)_* - 2$ for $c_{k_j+1} < (2n/T)_* < c_{k_j+1}$, $q \mid (2n/T)_* - 1$. Then investigate the sum

$$\sum_{\substack{n \in A_j \\ (2n/T)_* < c_{k_j+2}}} \left(\frac{2n}{T} - \left(\frac{2n}{T}\right)_{**}\right).$$

Use the same process for the definition of $(2n/T)_{**}$ as above. We continue this process until $c_{k_{j+1}}$. It follows from the construction of $(2n/T)_{**}$ that

$$\left|\sum_{n \in A_j} \left(\frac{2n}{T} - \left(\frac{2n}{T}\right)_{**}\right)\right| \leq 4qm_0.$$

Now define an A-partition of $\Lambda = \{\lambda_n\}$ as follows: $\Lambda_j := \{\lambda_n : n \in A_j\}$. The length of Λ_j is (according to $\lambda_n = 2\pi n/T$ and by the definition of A_j) $l_j \approx 100\pi\, qm_0$ and therefore $4qm_0 \leq l_j\, d$ for some $d < 1/4$. Applying the Theorem A, the proof of the Theorem 1 is complete.

REMARKS. 1. In the above Theorem 1 the distribution of the exponents is very regular: they form a linear sequence, omitting from it a linear subsequence. When we apply e.g. the Fourier method in the theory of partial differential equetions (Cf. [21]) we need the Riesz basis property of a system of exponentials which are close to some regularly distributed system. For example, the square roots of the eigenvalues of the Schrödinger operator are at a distance belonging to l_2 from a linear sequence. We can easliy prove in general that if $(e^{i\lambda_n x})_{-\infty}^{\infty}$ is a Riesz basis in $L^2(0, a)$, if $\sum |\delta_n|^2 < \infty$ and if the numbers $\lambda_n + \delta_n$ are all distinct then $(e^{i(\lambda_n+\delta_n)x})_{-\infty}^{\infty}$ is also a Riesz basis in $L^2(0,a)$. We refer first to the following theorems of N. K. BARI [7]. Let $(\varphi_n)_{-\infty}^{\infty}$ be a Riesz basis in a Hilbert space H and suppose $\sum \|\varphi_n - \Psi_n\|^2 < \infty$. Then the system $(\Psi_n)_{-\infty}^{\infty}$ is minimal iff it is complete and in this case (Ψ_n) is in fact Riesz basis. On the other hand if (φ_n) is a Riesz basis and $\sum \|\varphi_n - \Psi_n\| \leq K$ when $K > 0$ depends on (φ_n), then (Ψ_n) is also a Riesz basis. Applying this for $\varphi_n = e^{i\lambda_n x}$, $\Psi_n = e^{i(\lambda_n+\delta_n)x}$ we see that for large $N \in \mathbf{N}$ the system

$$(e^{i\lambda_n x})_{-N}^{N} \cup (e^{i(\lambda_n+\delta_n)x})_{|n|>N}$$

is a Riesz basis in $L^2(0, a)$. Now a lemma of Levinson [22] is needed which states that if a system $(e^{i\lambda_n x})_{-\infty}^{\infty}$ is complete in $L^2(0, a)$ and $\lambda_0' \neq \lambda_n$ $(n \neq 0)$ then the system $\{e^{i\lambda_0' x}\} \cup \{e^{i\lambda_n x}\}_{n\neq 0}$ is also complete. Unifying this with the above mentioned Bari theorem we get the following interesting corollary. Let $(e^{i\lambda_n x})_{-\infty}^{\infty}$ be a Riesz basis in $L^2(0, a)$ and change finitely many exponentials of the system to anothers. If the new system consists of different elements, it will be a Riesz basis in $L^2(0, a)$. Hence, we can change the elements $e^{i\lambda_n x}$, $|n| \leq N$ in the Riesz basis

$$(e^{i\lambda_n x})_{-N}^{N} \cup (e^{i(\lambda_n+\delta_n)x})_{|n|>N}$$

to $e^{i(\lambda_n+\delta_n)x}$ to get the Riesz basis $(e^{i(\lambda_n+\delta_n)x})_{-\infty}^{\infty}$.

2. Now suppose only that $(e^{i\lambda_n x})_{-\infty}^{\infty}$ is Riesz basis and $\delta_n \to 0$. If the numbers $\lambda_n + \delta_n$ are distinct, the Riesz basis property of $(e^{i(\lambda_n+\sigma_n)x})_{-\infty}^{\infty}$ follows. To see this we use instead the Bari theorem on quadratically close systems in general Hilbert space ($\sum \|\varphi_n - \Psi_n\|^2 < \infty$) a theorem of DUFFIN and SCHAFFER [23] on exponential bases stating that if $(e^{i\lambda_n x})_{-\infty}^{\infty}$ is Riesz basis then there exists an $\varepsilon > 0$ such that $|\delta_n| < \varepsilon\, \forall_n$ implies that $(e^{i(\lambda_n+\delta_n)x})_{-\infty}^{\infty}$ is a Riesz basis. Applying this theorem for the case $\delta_n \to 0$ and changing finitely many elements $e^{i\lambda_n x}$ to $e^{i(\lambda_n+\delta_n)x}$ (this is justified in **1.**) we see that $\delta_n \to 0$ also ensures the Riesz basis property of $(e^{i(\lambda_n+\delta_n)x})$ consisting of distinct elements. In Theorem B below there is given

a characterization of exponential bases. This theorem generalized the sufficient condition of KADEČ [24] to a necessary and sufficient condition, further the above mentioned theorem of Duffin and Schaeffer follows from the Theorem B.

3. In connection with the point (c) of Theorem 1 introduce the following terminology. The system $(\lambda_n)_{-\infty}^{\infty} \subset \mathbf{C}$, $\sup_n |\operatorname{Im} \lambda_n| < \infty$ has uniform density d if for $\varepsilon > 0$ there exists $T > 0$ such that

$$\left| \frac{1}{T} \operatorname{card} \{n : x \leq \operatorname{Re} \lambda_n \leq x + T\} - d \right| < \varepsilon \quad (x \in \mathbf{R}).$$

By the method applied to prove the point (c) it can be easily seen that if $(\lambda_n)_{-\infty}^{\infty}$ has uniform density $d > 0$ then for any $0 < a < 2\pi d$ we can extract from the system $(e^{i\lambda_n x})_{-\infty}^{\infty}$ a subsystem forming a Riesz basis in $L^2(0, a)$. Here the question arises: which are the Riesz bases $(e^{i\lambda_n x})_{-\infty}^{\infty}$ of $L^2(0, 2\pi d)$ where (λ_n) has uniform density d? Recently we have proved with M. HORVÁTH in a joint paper that all exponential bases have this property.

2. We can also consider Riesz bases from exponentials of higher order, i.e. systems of the form

$$e(\Lambda) = (e^{i\lambda_n x}, xe^{i\lambda_n x}, \ldots, x^{m_n - 1} e^{i\lambda_n x})_{n \in \mathbf{Z}} \subset L^2(0, a).$$

In what follows we consider basis criteria for $e(\Lambda)$ and some of its relatives. Recall that a non – negative function $\omega \in L^1_{\mathrm{loc}}(\mathbf{R})$ satisfies the Muckenhoupt condition [13] if

$$(A_2) \qquad \sup_{I \in J} \left(\frac{1}{|I|} \int_I \omega \right) \left(\frac{1}{|I|} \int_I \frac{1}{\omega} \right) < \infty,$$

where J denotes the family of all finite intervals of $\mathbf{R}$. Definition: Denote $\Lambda := \{(\lambda_n, m_n) := n \in \mathbf{Z}\}$. Then the generating function of the pair (Λ, a) is an exponential type entire function $F_{\Lambda, a}(z)$ with indicator diagram $[-ia, 0]$ and whose zeros are λ_n with multiplicity m_n. It is known ([11, Theorem 7.8.3]) that the generating function is unique.

A. M. SEDLECKII proved the

THEOREM C ([14]). Let $0 < \delta \leq \operatorname{Im} \lambda_n \leq K < \infty$, $\sup m_n = m < \infty$ and suppose that

(a) the system (λ_n) is separable,
(b) there is a generating function $F = F_{\Lambda, a}$ satisfying

$$|F|^2|_{\mathbf{R}} \in (A_2).$$

Then the system $e(\Lambda)$ forms a Riesz basis in $L^2\left(-\frac{a}{2}, \frac{a}{2}\right)$.

Recently M. HORVÁTH proved the converse statement.

THEOREM D ([9]). Let $\sup m_n < \infty$, $\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$, $0 < \delta \leq \operatorname{Im} \lambda_n \leq$ $\leq K < \infty$.

Then the following assertions are equivalent

(a) $e(\Lambda)$ forms a Riesz basis in $L^2(0, a)$ (or in any interval of length a).
(b) The generating function of (Λ, a) exists and

$$|F_{\Lambda, a}|^2|_{\mathbf{R}} \in (A_2).$$

The proof is based on the following result of B. S. PAVLOV [2]: the condition (b) is equivalent to the statement that the canonical projection of $L^2(0, \infty)$ onto $L^2(0, a)$ maps isomorphically the closed linear hull of $e(\Lambda)$ in $L^2(0, \infty)$ onto $L^2(0, a)$.

This result makes possible the generalization of the theorems established for the case of eigenfunctions of order o(i.e. $m_n = 1$), Cf. [2]. First introduce the following terminology: the function $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ is called a function of bounded mean oscillation (in symbols, $f \in \mathrm{BMO}(\mathbf{R})$) if

$$\sup_{I \in J} \frac{1}{|I|} \int_I |f - f_I| < \infty, \text{ where } f_I := \frac{1}{|I|} \int_I f.$$

Define further the following operator of conjugation

$$f_{**}(x) := \frac{1}{\pi} \text{p. v.} \int_{\mathbf{R}} \left[\frac{1}{x-t} + \frac{t}{t^2+1} \right] f(t)\, dt,$$

where the function $f(t)$ satisfies

$$\int_{\mathbf{R}} \frac{|f(t)|}{1+t^2} dt < \infty, \tag{7}$$

and the notation p.v. means the Cauchy principal value of the integral.

For $\gamma > 0$ denote by P_γ the set of all $f \in \mathrm{BMO}(\mathbf{R})$ for which there are $y_0 > 0$, $c \in \mathbf{R}$ and $u, v \in L^\infty(\mathbf{R})$ such that

$$\frac{1}{\pi} \int_{\mathbf{R}} \frac{y_0}{|x-t|^2 + y_0^2} f(t)\, dt = c + u_{**}(x) + v(x) \text{ and } \|v\|_\infty \leq \gamma.$$

Then the following generalization of Theorem 8 of, [2] holds.

THEOREM E. Let $(\lambda_n) \subset \mathbf{R}$ and denote

$$N_\Lambda(x) := \begin{cases} \sum_{\lambda_n \in [0, x]} m_n, & x \geq 0 \\ -\sum_{\lambda_n \in [x, 0]} m_n, & x < 0. \end{cases}$$

Suppose $\sup_n m_n < \infty$. Then the family $e(\Lambda)$ forms a Riesz basis in $L^2(0, a)$ if and only if

(a) (λ_n) is separated

(b) $N_\Lambda(x) - \frac{a}{2\pi}x \in P_{1/4}$.

Now we prove the following corollary of Theorem D:

THEOREM 2. *Let $(\lambda_n) \subset \mathbf{R}$ be separated, $(\delta_n) \subset \mathbf{R}$ be bounded and suppose that $\sup_n m_n < \infty$. Then for $\Lambda^* := \{(\lambda_n + i\delta_n, m_n) : n \in \mathbf{Z}\}$ the family $e(\Lambda^*)$ forms a Riesz basis in L^2 (0, a) if and only if e (Λ) does.*

PROOF. We know from [1, p, 323] that the generating function has the form

$$F(z) = ce^{i\frac{a}{2}z} \text{p. v.} \prod \left(1 - \frac{z}{\lambda_n}\right)^{m_n} := ce^{i\frac{a}{2}z} \lim_{r\to\infty} \prod_{\substack{n \\ |\lambda| \le r}} \left(1 - \frac{z}{\lambda_n}\right)^{m_n}.$$

So we have

$$\frac{F_{\Lambda^*+iy}(x)}{F_{\Lambda+iy}(x)} = \prod \left|\frac{\lambda_n + iy}{\lambda_n^* + iy}\right|^{2m_n} \prod \left|\frac{\lambda_n + i\delta_n + iy - x}{\lambda_n + iy - x}\right|^{2m_n}.$$

Now

$$\left|\frac{\lambda_n + i\delta_n + iy - x}{\lambda_n + iy - x}\right|^{2m_n} = \left(1 + \frac{\delta_n(2y + \delta_n)}{(\lambda_n - x)^2 + y^2}\right)^{m_n} < \frac{3}{2}$$

if y is large enough. Further let λ_n be the point of Λ nearest to the fixed $x \in \mathbf{R}$ and let $d = \inf_{k \ne n} |\lambda_k - \lambda_n|$. Then

$$\sum_{n\in\mathbf{Z}} \frac{\delta_n(2y + \delta_n)}{(\lambda_n - x)^2 + y^2} \le \frac{1}{2} + \text{const} \sum_{m \ne n} |\lambda_m - x|^{-2} \le \frac{1}{2} + \text{const}\, d < \infty$$

which shows that $\log |F_{\Lambda^*+iy}(x)|^2 - \log |F_{\Lambda+iy}(x)|^2 \in L^\infty(\mathbf{R})$.

By Theorem D the proof is complete.

THEOREM 3. *Let $0 < \delta \le \operatorname{Im} \lambda_n$, $\sup_n m_n < \infty$ and let e (Λ) be a Riesz basis in L^2 (0, a). Then there exists an $\varepsilon > 0$ such that taking any λ_n^*, $|\lambda_n^* - \lambda_n| < \varepsilon$, the system e (Λ^*) remains Riesz basis in $L^2(0, a)$.*

The proof repeats only the ideas of [2, Theorem 2.4] so we omit the details.

In the above considerations the condition

$$|\operatorname{Im} \lambda_n| \le \text{const} \tag{8}$$

was naturally implied by the normality condition

$$0 < c \le \|e^{i\lambda_n x}\|_{L^2(0,\, a)} \le C < \infty$$

(or shortly $\|e^{i\lambda_n x}\|_{L^2(0,\, a)} \asymp 1$.

But if we allow systems $(c_n e^{i\lambda_n x})$ as exponential basis, the condition (8) can not be proved by the above way; in fact (8) is not necessary for the Riesz basis property. More precisely, in [2] is proved the following

THEOREM F. Suppose that $0<\delta\leq \operatorname{Im} a_n$, $\lim\limits_{n\to\pm\infty} a_n = +\infty$ and that $(a_n : n\in\mathbf{Z})$ satisfies the Carleson condition

$$(C) \qquad \inf_n \prod_{\substack{k\\ k\neq n}} \left| \frac{a_n - a_k}{a_n - \bar{a}_k} \right| > 0.$$

Then for any $a>0$ the family $(a_n : n\in\mathbf{Z})$ can be complemented up to such a family $(\lambda_k : k\in\mathbf{Z})$ that the system $(c_k e^{i\lambda_k x} : k\in\mathbf{Z})$ form a Riesz basis in $L^2(0, a)$ for appropriate constants $c_k>0$.

Here and in the whole theory of exponential bases developed in [2] a fundamental role is played by the fact that the exponentials $e^{i\lambda_n x}$ have large oscillation. This means that the real parts of the exponents, $\operatorname{Re}\lambda_n$ must be large in some sense. The Carleson condition ensures this property: it means that the sequence (a_n) is distributed "near" the real axis. In connection with these remarks we pose the following

PROBLEM 1. Is there a Riesz basis of the form $(c_n e^{i\lambda_n x})$ in some $L^2(0, a)$, $a>0$ such that $|\operatorname{Re}\lambda_n|\leq$ const ? More directly: is there a Riesz basis with $\operatorname{Re}\lambda_n \equiv 0$ for all n?

We mention that $\mathcal{L}$ – basis with $\operatorname{Re}\lambda_n \equiv 0$ can be easily found:

THEOREM 4. *Let $0<a<\infty$ be arbitrary and $\lambda_n = 2^{n^2}$. Then for some $N_0 = N_0(a)$ the system $(e^{i\lambda_n x} : n = N_0, \ldots)$ is an $\mathcal{L}$ – basis in $L^2(0, a)$.*

PROOF. We shall use the following known fact, proved by N. K. BARI (see [7] or [20, p. 172]): the system (φ_n), $\|\varphi_n\|\asymp 1$ in a Hilbert space H forms an $\mathcal{L}$ – basis if and only if the Gram matrix $\Gamma = (\langle\varphi_n, \varphi_k\rangle)_{n,k}$ defines an isomorphism of l_2 onto itself. We have obviously

$$\|e^{\lambda_n x}\|^2 = \int_0^a e^{2\lambda_n t}\,dt = \frac{e^{2\lambda_n a}-1}{2\lambda_n} \asymp \frac{e^{2\lambda_n a}}{\lambda_n}$$

so the general element of the Gram matrix is

$$\gamma_{n,m} := \left\langle \frac{e^{\lambda_n x}}{\|e^{\lambda_n x}\|}, \frac{e^{\lambda_m x}}{\|e^{\lambda_m x}\|} \right\rangle \asymp \frac{e^{(\lambda_n+\lambda_m)a}}{\lambda_n+\lambda_m} \cdot \frac{\sqrt{\lambda_n}}{e^{\lambda_n a}} \cdot \frac{\sqrt{\lambda_m}}{e^{\lambda_m a}} = \frac{\sqrt{\lambda_n\lambda_m}}{\lambda_n+\lambda_m}.$$

We shall prove that for some $N_0 = N_0(a)$

$$\Gamma = (\gamma_{n,m})_{n,\, m\geq N_0} = I + K,$$

where K is a compact operator with $\|K\|<1$; it will be enough to show that

$$\sum_{n\geq N_0} \sum_{m>n} |\gamma_{n,m}|^2 < \frac{1}{2}$$

or, what is the same

$$\sum_{n\geq N_0}\sum_{m>n}|\gamma_{n,m}|^2<C(a). \tag{9}$$

Since $n<m$ implies $\lambda_n<\lambda_m$ and then $\frac{\sqrt{\lambda_n\lambda_m}}{\lambda_n+\lambda_m}<\sqrt{\frac{\lambda_n}{\lambda_m}}$, so it is enough to prove that

$$\sum_{n>N_0}\lambda_n\sum_{n<m}\frac{1}{\lambda_m}<C(a). \tag{9'}$$

But using the inequality $\lambda_n/_{n+k}\leq(2^{2n+1})^{-k}$ the estimate (9′) canh be easily guaranteed and the proof is complete.

PROBLEM 2. Let $1<q<2$ be arbitrary and consider the expansion by gready algoritm: $1=\sum_{n=1}^{\infty}a_n(q)q^{-n}$. What can we say on the length of 0–sequences in $\{a_n(q)\}$ for almost every q? What is true for the length of 1–sequences? It is easy to check that for a. e. q they are unbounded,

$$\left\{\sum_0^n\varepsilon_iq^i,\ \varepsilon_i=\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right.\right\}=\{y_n\}\nearrow\infty,$$

then $y_{n+1}-y_{nn}\to 0$?

3. In the final part of this paper we shall deal with function theoretical problems. In [18] J. A. HAIGHT proved the following interesting

THEOREM G. There exists a measurable set $D\subset[1,2]$, $|D|>0$ suc that

$$\left|(0,\infty)\setminus\bigcup_{n=1}^{\infty}nD\right|=\infty.$$

The construction proving Theorem G played an important role in the investigations of Z. MAGYAR [17] and M. HORVÁTH [15] concerning a problem for functions not belonging to L^2.

In connection with Theorem G we pose the problem of the speed of divergence of $\left|(0,n)\setminus\bigcup_{k=1}^{n}kD\right|$. We shall prove the

THEOREM 5.

(a) *There is no set $D\subset(1,2)$ with positive measure and such that $\left|(0,n)\setminus\bigcup_{k=1}^{n}kD\right|\geq cn$ $(n\in\mathbf{N})$, namely, this quantity is of $o(n)$ for any measurable $D\subset(1,2)$.*

(b) *For any sequence $a_n\to\infty$ $(n\to\infty)$ there exists a set $D\subset(1,2)$ with positive measure satisfying the property*

$$\left|(0,n)\setminus\bigcup_{k=1}^{n}kD\right|\geq n/a_n\quad(n\in\mathbf{N}).$$

PROOF. The statement (a) easily follows from the fact that there exists a Lebesgue density point of D. The proof of (b) is a modification of the proof

of Theorem G. We shall construct the set D by giving the system of intervals avoiding any set $k\,D$. Introduce the following notation of HAIGHT:

$$A(x_1, x_2, q, \varepsilon) = [x_1, x_2) \cap \cup \{(e^{t/q}, e^{(t+\varepsilon)/q}) : t \in \mathbf{Z}\}.$$

It is proved in [18] that for

$$q > 16(x_2/x_1) \Big/ \frac{x_2}{x_1} - 1$$

we have

$$\frac{\varepsilon}{4}(x_2 - x_1) < |A(x_1, x_2, q, \varepsilon)| < 3\varepsilon(x_2 - x_1). \tag{10}$$

Let (n_k) be a monotone increasing sequence whose rate of growth is quick in a sense to be specified later. Define $B_k := A(n_k, n_{k+1}, q_k, 2^{-k})$, where $q_k \in \mathbf{N}$ is a number satisfying

$$q_k > \max\left\{\frac{16 n_{k+1}/n_k}{(n_{k+1}/n_k) - 1}, 2^{k n_{k+1}}\right\}$$

and such that the inequalities

$$\left|\ln i - \frac{p_k^{(i)}}{q_k}\right| < \frac{1}{q_k^{1+\frac{1}{n_{k+1}}}} < \frac{1}{2^k} \cdot \frac{1}{q_k}, \quad (i = 1, \ldots, n_{k+1}) \tag{11}$$

hold sor some $p_k^{(i)} \in \mathbf{N}$; this is possible by the well-known Dirichlet theorem on simultaneous approximation, see [19, p. 14]. Define further $C_k := [1, 2) \cap \bigcup_{n=1}^{\infty} \frac{1}{n} B_k$ In order to ensure that the set B_k avoid all multiples of a set D we must suppose $C_k \cap D = \emptyset$ so it is plausible to define D as $D := [1, 2) \setminus \bigcup_{k=k_0}^{\infty} C_k$ where k_0 will be given later. We estimate the Lebesgue measure of C_k as follows. Let $x \in C_k$ then $nx \in B_k$ for some n, hence $e^{t/q_k} < nx < e^{t+2^{-k}/q_k}$ for some t. From $1 \le x$, $nx \in B_k$ it follows that $n \le n_{k+1}$ and then by (11) and

$$e^{\frac{t}{q_k} - \ln n} < x < e^{\frac{t+2^{-k}}{q_k} - \ln n}$$

we get that for some integer s

$$e^{s-2^{-k}/q_k} < x < e^{s+2^{-k+1}/q_k}.$$

In other words, $x \in C_k$ implies that

$$x e^{1/2^k q_k} \in A(e^{1/2^k q_k}, 2e^{1/2^k q_k}, q_k, 3/2^k)$$

which means by (10) the estimate

$$|C_k| \le e^{-1/2^k q_k} |A(e^{1/2^k q_k}, 2e^{1/2^k q_k}, q_k\, 3/2^k)| < 9/2^k.$$

So if k_0 is large enough ($k_0>4$) then $|D|>0$. Taking into account the definition of the sets B_k we see that for

$$n_k \leq n < n_{k+1} : |(0, n) \setminus \bigcup_{r=1}^{n} rD| > \frac{c}{2^k} n$$

with some $c>0$ independent of k and n. If the sequence n_k tends rapidly to infinity then

$$\frac{c}{2^k} n > \frac{n}{a_n} (n_k \leq n < n_{k+1})$$

which completes the proof of Theorem 5.

PROBLEM 3. Are there disjoint intervals $I_k \subset (0, \infty)$ with the properties

$$\sum_{k=1}^{\infty} |I_k|^2 = \infty \text{ and } \left| (1, 2) \setminus \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \frac{1}{n} I_k \right| > 0?$$

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SOME PROPERTIES OF THE GOLOD-EXAMPLE*

By

ERZSÉBET HORVÁTH

Department of Mathematics, Faculty of Mechanical Engineering,
University of Technology, Budapest

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The aim of this paper is to prove certain properties of Golod's following example [1]

For any natural number $d>1$ there exists an associative algebra R_d over a field K for which R_d is generated by d elements, R_d is not nilpotent, but every $(n-1)$-generated subalgebra of R_d is nilpotent.

Throughout the paper we denote by $I \lhd R$ that I is an ideal of the algebra R, $L(R)$ stands for the locally nilpotent radical of R. The right annihilator of a subset $H \subseteq R$ will be denoted by $\mathrm{Ann}_R(H)$.

Let us recall that an algebra R (not necessarily with unity) is said to be *(right) strongly prime* if for every nonzero element $x \in R$ there is a finite subset $F \subseteq R$ such that $\mathrm{Ann}_R(xF) = 0$. It can be shown, similarly as in [6] Prop. 1. that an algebra R is (right) strongly prime if and only if for any nonzero $I \lhd R$ there is a finite subset $F \subseteq I$ such that $\mathrm{Ann}_R(F) = 0$. An algebra R is said to be *bounded (right) strongly prime of bound n or less* if for every element $x \in R$ there is an n-element subset $F \subseteq R$ such that $\mathrm{Ann}_R(xF) = 0$.

In the above cases F is called an *insulator of x in R* or an *insulator in I* respectively. In the following strongly prime will always mean right strongly prime.

PROPERTY 1. a) *Suitable homomorphic image S_d of R_d has the property that every $(d-1)$-element subset of S_d has a nonzero right annihilator, but there is a d-element subset in S_d the right annihilator of which is zero.*

b) *S_d can be constructed to be strongly prime as well.*

PROOF. Let R_d be the Golod-example mentioned above. As R_d is finitely generated, by Zorn's lemma there exists an ideal $I \lhd R_d$ maximal with the

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property that $R_d^n \not\subseteq I$ for any natural number n. As in [2] or [3] one can show that $S_d = R_d/I$ is strongly prime. In R_d every $(d-1)$-element subset generates a nilpotent subalgebra, so this holds in S_d as well. Thus every $(d-1)$-element subset of S_d has a nonzero right annihilator. Now let $H = \{\bar{x}_1, \ldots, \bar{x}_d\}$ be the image of generators of R_d in S_d. Then if $\mathrm{Ann}_{S_d}(\bar{x}_1, \ldots, \bar{x}_d) = J/I \neq 0$ for some ideal $J \lhd R$ then by our assumption $R_d^n \subseteq J$ for some n. So $R_d^{n+1} \subseteq I$, contradiction. So we are done.

REMARK 1. If we would not need our example to be strongly prime we could have used a simpler construction just taking the nonzero factor $S_d = R_d/L(R_d)$. Then we would get similarly as above that every $(d-1)$-element subset has a nonzero right annihilator. This time if $H = \{\bar{x}_1, \ldots, \bar{x}_d\}$ is the set of images of generators of H then $\mathrm{Ann}_{S_d}(H) \lhd S_d$ and $\mathrm{Ann}_{S_d}(H)^2 = 0$, so $\mathrm{Ann}_{S_d}(H) = 0$ as well.

Let R be an algebra (not necessarily with unity), $I \lhd R$. We say that R is *relatively k-strongly prime with respect to the ideal I* if for every $J \lhd R$ properly containing I there is an insulator in J consisting of k elements.

REMARK 2. a) If an algebra R is bounded strongly prime of bound k or less then it is relatively k-strongly prime with respect to every ideal $I \lhd R$.

b) If an algebra R is relatively k-strongly prime with respect to the ideal I then R/I is not necessarily relatively k-strongly prime with respect to the ideal 0. e.g. if $R = \langle x \rangle$ and $I = \langle x^2 \rangle$ then R is bounded strongly prime of bound 1 but R/I has no insulator

c) If R is relatively k-strongly prime with respect to the ideal 0 then R is strongly prime as well.

LEMMA 1. *Let S_d be a ring satisfying the conditions of Property* 1. a) *Let M_d be a maximal element of the set*

$$\left\{ I \lhd S_d \;\middle|\; \begin{array}{l} \textit{every d-element subset of } I \\ \textit{has a nonzero right annihilator} \end{array} \right\}.$$

Then $M_d \neq S_d$ and S_d is relatively d-strongly prime with respect to ideal M_d, but it is not relatively $(d-1)$-strongly prime with respect to any ideal in S_d.

PROOF. Easy to see.

REMARK 3. In [4] it is proved that for any division ring D the algebra $\mathcal{M}_d(D)$ of $d \times d$ matrices over D is bounded strongly prime of bound exactly d. However this algebra is relatively 1-strongly prime with respect to the ideal 0 as the unit matrix is an insulator. Now we have an infinite chain of algebras $\{S_d\}$ which are relatively d-strongly prime but not relatively $(d-1)$-strongly prime.

LEMMA 2. *Let R be a relatively k-strongly prime algebra with respect to the ideal $I \lhd R$. Let us suppose that there is no insulator in I. Then R/I is relatively k-strongly prime with respect to ideal 0.*

PROOF. The proof is similar to that of [5] Th. 2.1 and therefore we omit it.

PROPERTY 2. *Let S_d and M_d be the same as in Lemma 1. Then either there is a natural number $n>d$ such that the ideal M_d has an insulator consisting of n elements or every nonzero ideal in S_d/M_d has an insulator consisting of d elements.*

PROOF. By its definition M_d cannot have an insulator of at most d elements. Let us suppose that M_d does not have an insulator of more then d elements. Since by Lemma 1 S_d is relatively d-strongly prime with respect to the ideal M_d, by Lemma 2 we have that S_d/M_d is relatively d-strongly prime with respect to the ideal 0.

REMARK 4. Let S_d be defined to be R_d/I as in the proof of Property 1. If the first case does not hold in Property 2 then $M_d = 0$, otherwise the generators of $S_d^m \leq M_d$ would be an insulator ih M_d.

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A COMPACT HYPERBOLIC SPACE FORM DERIVED FROM A PLANE FORM

By

MARICA ŠARAC

Faculty of Science, University of Belgrade

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1. Introduction

Following Poincaré's geometric method, we shall construct an isometry group G, acting discontinuously and freely on the hyperbolic 3-space $\mathbf{H}^3$. This group is given by a compact fundamental Dirichlet polyhedron $\mathcal{D}$ and by pairing its faces via isometries satisfying certain conditions. These isometries generate G and we obtain a hyperbolic space form $\tilde{\mathcal{D}} = \mathbf{H}^3/G$. Since the group G contains only orientation preserving isometries, the manifold $\mathbf{H}^3/G$ becomes orientable compact hyperbolic space form.

This example seems to be not mentioned in the literature up to this time. It was surprising fact that the construction can be derived from a compact non-orientable hyperbolic plane form of genus 3, by choosing a special fundamental dodecagon. We shall also exploit the existence of a supergroup C of G, generated by reflections, for describing the metric data of $\tilde{\mathcal{D}} = \mathbf{H}^3/G$, [9, 10, 11].

Concerning the Euclidean and the spherical space forms, we refer to the monograph [15] of J. A. Wolf. In the appendix to its Russian translation Y. D. Burago has given an account of the hyperbolic space forms (of constant negative curvature $K = -1/k^2$), [3]. We mention the report [13] of W. P. Thurston on his remarkable results. For each dimension there are infinitely many mutually non-homeomorphic compact hyperbolic space forms. The construction of the dodecahedron space of C. Weber and H. Seifert (1933), [14], has been extended by L. A. Best [2] for the space-filler regular hyperbolic polyhedra. In this manner he obtained 11 hyperbolic space forms. V. S. Makarov and I. S. Gutsul constructed infinite series of compact and non-compact (of finite volume) hyperbolic space forms [19], [17]. All these space forms are orientable and their constructions are based on Poincaré's theorem for fundamental polyhedra of discontinuous isometry groups. N. K. Al-Jubouri [1] gave the first example for a non-orientable compact hyperbolic space form, by means

of algebraic construction, by taking a normal subgroup of the dodecahedral reflection group. Independently of this, MOLNÁR E. constructed two infinite series of non-orientable compact hyperbolic space forms [7], [8]. F. L. DAMIAN and V. V. BALKAN constructed non-compact, non-orientable hyperbolic space forms of finite volume [18]. These papers also contain numerous references to other publications.

I offer cordial thanks to my mentor Docent EMIL MOLNÁR, who guides my studies in Budapest at the Department of Geometry of Eötvös Loránd University, for helping me in preparing the manuscript.

2. Dirichlet polyhedra for discontinuous isometry groups acting freely on $\mathbf{H}^3$

Let G be an isometry group acting discontinuously and freely on the 3-dimensional Bolyai-Lobachevskyan hyperbolic space $\mathbf{H}^3$, that means for any point $P \in \mathbf{H}^3$ its stabilizer in G contains only the identity map 1, and its orbit under G, i.e.

$$P^G = \{P^g \in \mathbf{H}^3 : g \in G\}, \tag{1}$$

is a discrete point set in $\mathbf{H}^3$. Now consider, with the metric d of $\mathbf{H}^3$, the Dirichlet polyhedron

$$\mathcal{D}_P = \{X \in \mathbf{H}^3 : d(X, P) < d(X, P^g), \text{ for every } g \in G \setminus \{1\}\}. \tag{2}$$

Then G acts discontinuously on $\mathbf{H}^3$ and $\mathcal{D}_P$ is a fundamental domain for G, i.e. the following conditions hold:

(i) $\mathcal{D}_P$ is a domain, i.e. a non empty connected open set;
(ii) No two distinct points of $\mathcal{D}_P$ are equivalent under G;
(iii) Every point is equivalent under G to some point of $\overline{\mathcal{D}}_P$.

Here $\overline{\mathcal{D}}_P$ denotes the closure of $\mathcal{D}_P$ and $\partial D_P := \overline{\mathcal{D}}_P \setminus D_P$ will denote the boundary of $\mathcal{D}_P$.

A face of $\partial \mathcal{D}_P$ will be denoted by $f_{g^{-1}}$ if it lies on the symmetry plane of the points P and $P^{g^{-1}}$ ($g \in G$, $g^{-1} \neq g$). As a polygon, $f_{g^{-1}}$ is equivalent to f_g of $\partial \mathcal{D}_P$: f_g lies on the symmetry plane of P and P^g. Then the isometry $g \in G$ transforms $f_{g^{-1}}$ onto f_g and the domain $\mathcal{D}_P$ maps onto $\mathcal{D}_{P^g} := (\mathcal{D}_P)^g$, a domain that is a neighbour of $\mathcal{D}_P$ along the common face f_g. The roles of g and g^{-1} can be changed. Now, $g \neq g^{-1}$ because of free action of G.

The isometries attached to the above pairing of the faces of $\mathcal{D}_P$ generate the group G. For any isometry $\alpha \in G$, $(\mathcal{D}_P)^\alpha = \mathcal{D}_{P^\alpha}$ is a domain in the Dirichlet tessellation according to the orbit P^G. This domain has a neighbour $(\mathcal{D}_P)^{g^\alpha} = \mathcal{D}_{P^{g^\alpha}}$ along the face (f_g).

The pairing of the faces of $\mathcal{D}_P$ eventually induces a subdivision of the edges into oriented segment such that a segment does not contain two equivalent points in its interior. Consider an arbitrary equivalence class consisting of edge segments $e_1, e_2, \ldots, e_r$ with face angles $\varepsilon(e_1), \varepsilon(e_2), \ldots, \varepsilon(e_r)$, respectively, defined as follows.

We choose an edge segment, say e_1, and consider one of the faces denoted by $f_{g_1^{-1}}$ whose boundary contains e_1. The isometry g_1 maps e_1 and $f_{g_1^{-1}}$ onto e_2 and f_{g_1}. There exists exactly one other face $f_{g_2^{-1}}$ with e_2 on its boundary, furthermore an isometry g_2, which maps e_2 and $f_{g_2^{-1}}$ onto e_3 and f_{g_2}, and so on. We obtain a cycle of isometries $g_1, g_2, \ldots, g_r$ according to the scheme

$$(e_1, f_{g_1^{-1}}) \xrightarrow{g_1} (e_2, f_{g_1});\ (e_2, f_{g_2^{-1}}) \xrightarrow{g_2} (e_3, f_{g_2});\ \ldots, (e_r, f_{g_r^{-1}}) \xrightarrow{g_r} (e_1, f_{g_r}).$$

We remark that the transformations $g_1, g_2, \ldots, g_r$ are not necessarily distinct. The above facts can also be formulated as follows. The edge segment e_1 is successively surrounded by domains

$$\mathcal{D}_P^{g_1^{-1}}, \mathcal{D}_P^{g_2^{-1}g_1^{-1}}, \ldots, \mathcal{D}_P^{g_i^{-1}\ldots g_2^{-1}g_1^{-1}}, \ldots, \mathcal{D}_P^{g_r^{-1}\ldots g_2^{-1}g_1^{-1}} \tag{4}$$

which fill an angular region of measure

$$\varepsilon(e_2) + \varepsilon(e_3) + \ldots + \varepsilon(e_r) + \varepsilon(e_1) = 2\pi \tag{5}$$

because of free action of G. Finally, the cycle transformation $g_1, g_2, \ldots, g_r$, belonging to the edge segment e_1, is the identity, thus we have the cycle relation

$$g_1 g_2 \cdots g_r = 1. \tag{6}$$

Choosing now the face f_{g_r} to e_1 as the first step of the process above, we obtain the cycle transformation

$$g_r^{-1} \cdots g_2^{-1} g_1^{-1} = (g_1 g_2 \cdots g_r)^{-1}. \tag{7}$$

Starting with e_i and $f_{g_i^{-1}}$, we arrive at

$$g_i \cdots g_r g_1 g_2 \cdots g_{i-1} = (g_i \cdots g_r) g_1 g_2 \cdots g_r (g_i \cdots g_r)^{-1} \tag{8}$$

as cycle transformation. These lead to relations equivalent to (6). The cycle relations (6) form a complete set of relations for G, i.e. any relation in G can be algebraically derived from these.

The converse problem of POINCARÉ [12], i.e. the question whether a polyhedron $\mathcal{D}$, with a system of identifying isometries, is a fundamental domain for a discontinuous, freely acting isometry group G generated by the identifications, will be considered only in a restricted form for bounded polyhedra and freely acting groups generated by orientation preserving isometries [12], [4].

3. Construction of discontinuous freely acting isometry groups generated by orientation preserving isometries

(i) A bounded polyhedron $\mathcal{D}$ is an open connected bounded subset of the hyperbolic 3-space $\mathbf{H}^3$, where the boundary $\partial\mathcal{D}$ is the union of finitely many faces $\{f_i\}$, defined as follows. Each face f_i is a closed polygon in a hyperbolic plane. The sides of the polygon f_i are called edges of $\mathcal{D}$, and the end points of these edges are called vertices of $\mathcal{D}$. We require that for

each edge e there are exactly two faces f_i and f_j (they may lie on a common plane) with $e = f_i \cap f_j$. Any two faces are either disjoint, or they intersect in a common edge, or intersect in a common vertex. An edge is either a subset of a face, or meets the face in a common vertex, or is disjoint from the face. Two edges are disjoint or meet in a common vertex.

(ii) Identifications on the polyhedron $\mathcal{D}$ are face pairings by orientation preserving isometries, satisfying conditions (a)–(b) below.

(a) For each face $f_{g^{-1}}$ of $\mathcal{D}$ there is another face f_g and an identifying isometry g of the space $\mathbf{H}^3$, which maps $f_{g^{-1}}$ onto f_g and $\mathcal{D}$ onto $\mathcal{D}^g \neq \mathcal{D}$. $\mathcal{D}^g$ is called the neighbour of $\mathcal{D}$ along f_g.

(b) The isometry g^{-1} maps the face f_g onto $f_{g^{-1}}$ and $\mathcal{D}$ onto $\mathcal{D}^{g^{-1}}$, joining the polyhedron $\mathcal{D}$ along $f_{g^{-1}}$.

(iii) The identifications of faces of $\mathcal{D}$ generate an isometry group G. We have to impose edge conditions on $\mathcal{D}$ so that it is a fundamental polyhedron for the group G acting freely on the space $\mathbf{H}^3$.

The face pairing induces a classification of the oriented edge segments so that no segment contains G-equivalent points in its interior. Take an arbitray edge segment class $e_1, e_2, \ldots, e_r$ with angles $\varepsilon(e_i)$ between faces at e_i, and perform the procedure described in section 2. Consider a segment, say e_1, and choose one of the faces, denoted by $f_{g^{-1}}$, whose boundary contains e_1. An isometry g_1 maps e_1 and $f_{g_1^{-1}}$ onto e_2 and f_{g_1}. There is exactly one face $f_{g_2^{-1}}$ with e_2 on its boundary, furthermore, an isometry g_2, which maps e_2 and $f_{g_2^{-1}}$ onto e_3 and f_{g_2}, and so on. We get a cycle of isometries $g_1, g_2, \ldots, g_r$ according to scheme (3).

(c) Assume that (5) holds for the face angles at e_i in each segment equivalence class. This condition guarantees so-called cycle relation (6).

Now we state the specified Poincaré theorem, without proof [4].

THEOREM 1. *Let $\mathcal{D}$ be a bounded polyhedron satisfying conditions* (i)–(iii) *and* (a)–(c). *Let G be the group generated by the orientation preserving identification of the faces. Then G is discontinuous freely acting group, $\mathcal{D}$ is a fundamental polyhedron for G and the cycle relations, of type* (6) *for every equivalence class of edge segments, form a complete set of relations for G.*

The identified polyhedron denoted by $\tilde{\mathcal{D}}$ geometrically characterizes the transformation group G, but a given group G can have more fundamental polyhedra.

4. The construction

In Figure 1 we present the graph of the polyhedron $\tilde{\mathcal{D}}$ to be constructed together with its symbolic picture. Take on a plane α_3 in the hyperbolic 3-space $\mathbf{H}^3$ a regular dodecagon $\mathcal{D}$ with angles of measure $\frac{2\pi}{3}$. Let O denote its centre. We place congruent Lambert quadrangles $ON\bar{A}_1\bar{A}_2$

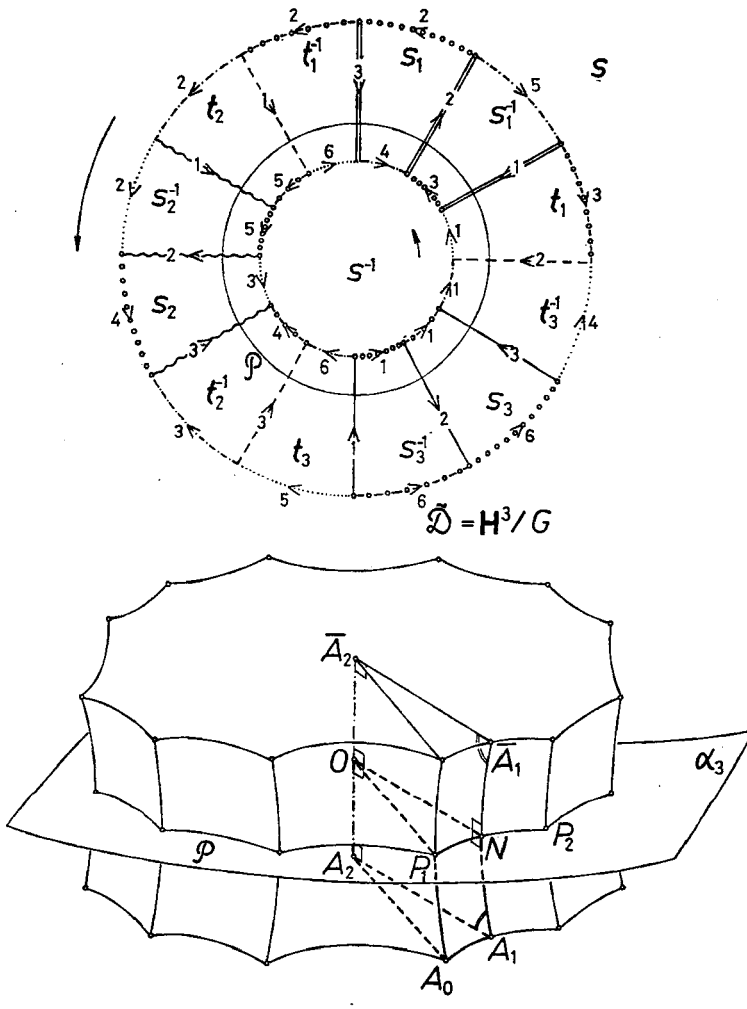

Fig. 1.

and ONA_1A_2 joining symmetrically each other and perpendicularly to α_3, plane of $\mathcal{P}$. We assume the angles, distinct from rectangles, to be

$$\sphericalangle \bar{A}_2\bar{A}_1N = \frac{\pi}{3} = \sphericalangle A_2A_1N. \tag{9}$$

We have $O\bar{A}_2 = OA_2$ and $\bar{A}_2\bar{A}_1 = A_2A_1$, by symmetry.

Now we can construct the regular dodecagons: f_{s-1} as the lower base, with centre A_2 and with radius A_2A_1 of the inscribed circle, perpendicularly to $A_2\bar{A}_2$; f_s as the upper base, with centre $\bar{A}_2$ and with radius $\bar{A}_2\bar{A}_1$ of the

inscribed circle, perpendicularly to $A_2\bar{A}_2$. Moreover, we can construct the side faces which are orthogonal to the plane α_3 of dodecagon $\mathcal{P}$, so that all they form, with both of base faces, angles of measure $\frac{\pi}{3}$. Finally, we get the polyhedron $\mathcal{D}$ with centre O. Such a polyhedron $\mathcal{D}$ is also called a lens.

The metric data of $\mathcal{D}$ will be described in section 5 (see table).

Now, we turn to identify the faces of $\mathcal{D}$. The result is symbolized in the Schlegel diagram of $\tilde{\mathcal{D}}$ where we omit the „f" 's from the face symbols. The "lower" base $f_{s^{-1}}$ of $\tilde{\mathcal{D}}$ is paired to its "upper" base f_s by the screw motion s of screw rotation angle $\frac{5\pi}{6}$; the pairs of side faces $f_{s_i^{-1}}$, f_{s_i} ($i = 1, 2, 3$) are identified by screw motion s_i of screw rotation angle π; the pairs of side faces $f_{t_i^{-1}}$, f_{t_i} ($i = 1, 2, 3$) are identified by translations t_i.

The identifications of the side faces above can be induced by those of the sides of $\mathcal{P}$ in the midplane α_3, and vice versa. The action of s_i will be glide reflection, that of t_i will be translation in α_3. These observations motivated us to construct the group G, as it was mentioned in the introduction.

We list now the edge equivalence classes of $\tilde{\mathcal{D}}$, induced by previous identifications, giving for each class the cycle transformation, in conformity with (3). We establish incidentally that the condition (c) also holds.

There are 12 side edges, divided into 4 classes:

(10) $s_1 s_1 t_1$ ═══⟩═══

(11) $s_2 s_2 t_2$ ~~~⟩~~~

(12) $s_3 s_3 t_3$ ——⟩——

(13) $t_1 t_3 t_2$ - - - -⟩- - - -

each angle is $\frac{2\pi}{3}$, so $3 \cdot \frac{2\pi}{3} = 2\pi$.

There are 24 base edges, divided into 4 classes:

(14) $s \quad s_2 \quad s \quad t_3 \quad s^{-1} \quad t_1$ · · · ·⟩· · · ·

(15) $s \quad t_2^{-1} \quad s^{-1} \quad s_1^{-1} \quad s^{-1} \quad t_3^{-1}$ – · – ·⟩· – · –

(16) $s \quad t_1 \quad s^{-1} \quad t_2 \quad s \quad s_3$ ∘ – ∘ –⟩∘ – ∘ –

(17) $s \quad s_1^{-1} \quad s \quad s_2^{-1} \quad s \quad s_3^{-1}$ ∘ ∘ ∘ ∘⟩∘ ∘ ∘ ∘

each angle is $\frac{\pi}{3}$, so $6 \cdot \frac{\pi}{3} = 2\pi$.

We conclude that $\tilde{\mathcal{D}}$ is a fundamental polyhedron for the group G with presentation

$$G = (s, s_1, s_2, s_3, t_1, t_2, t_3 - s_1^2 t_1 = s_2^2 t_2 = s_3^2 t_3 = t_1 t_3 t_2 = s s_2 s t_3 s^{-1} t_1 =$$

$$(18) \qquad = s t_2^{-1} s^{-1} s_1^{-1} s^{-1} t_3^{-1} = s t_1 s^{-1} t_2 s s_3 = s s_1^{-1} s s_2^{-1} s s_3^{-1} = 1).$$

Forming the orbit O^G, where O is the centre of $\mathcal{D}$, we obtain, by a symmetry argument, that $\mathcal{D} = \mathcal{D}_0$ is a Dirichlet polyhedron.

Note that in terms of the three generators s, s_1 and s_2 the presentation of the group G have been written as the minimal presentation

$$G = (s, s_1, s_2 - s_1^{-2} s^{-1} s_2^{-2} s^2 s_1^{-1} s s_2^{-1} s^2 = s^2 s_2^2 s^{-1} s_1^{-2} s s_2^{-1} s^2 s_1^{-1} s s_2^{-1} =$$

$$(19) \qquad = s_2^2 s^{-1} s_1 s^{-2} s_2 s^{-1} s_1 s^{-2} s_1^{-2} s = s_2^2 s s_1^{-1} s s_2^{-1} s^2 s_1^{-1} s s_2^{-1} s s_1^2 = 1)$$

The effect of adding new relations

$$(20) \qquad s s_1 s^{-1} s_1^{-1} = s s_2 s^{-1} s_2^{-1} = s_1 s_2 s_1^{-1} s_2^{-1} = 1$$

to the presentation (20) of G is to form a new group, which is a factor group of G, so-called commutator quotient group of G. This group, conveniently denoted by $G/[G, G]$ is the largest Abelian factor group of G. The presentation of $G/[G, G]$ may be simplified so that (apart from (20))

$$(21) \qquad G/[G, G] = (s, s_1, s_2 - s^6 = 1, s_1^3 = s_2^3 = s^{-1}).$$

Introducing a new generator $s_0 = s_2 s_1^{-1}$, we obtain

$$(22) \qquad G/[G, G] = (s_1, s_0 - s_1^{18} = s_0^3 = s_1 s_0 s_1^{-1} s_0^{-1} = 1).$$

Hence, the Abelian group $G/[G, G]$ is $C_{18} \times C_3$, the direct product of cyclic groups C_{18} and C_3 i.e., as it is well known, the group

$$(23) \qquad G/[G, G] = C_{18} \times C_3$$

is just the first homology group of the manifold $\tilde{\mathcal{D}} = \mathbf{H}^3/G$.

5. Metric construction of $\mathcal{D}$ in vector model of $\mathbf{H}^3$

We consider the 4-dimensional real vector space $\mathbf{V}^4$, whose dual space, i.e. the space of its linears forms, is denoted by $\mathcal{V}_4^*$. In the usual way the projective 3-space $\mathbf{P}^3$ ($\mathbf{V}^4$, $\mathcal{V}_4^*$) can be introduced. The 1-dimensional subspaces of $\mathbf{V}^4$ (or the 3-spaces of $\mathcal{V}_4^*$) represent the points of $\mathbf{P}^3$ and the 1-subspaces of $\mathcal{V}_4^*$ (or the 3-subspaces of $\mathbf{V}^4$) represent the planes of $\mathbf{P}^3$. The point X ($\mathbf{x}$) and the plane a (a) are incident iff $\mathbf{x} \cdot a = 0$, i.e., the value of the linear form a on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \dot{\mathbf{V}}^4$, $a \in \dot{\mathcal{V}}_4^*$, the dot denoted that the zero vector and the zero form are excluded). The straight lines of $\mathbf{P}^3$ are characterized by 2-subspaces of $\mathbf{V}^4$ or of $\mathcal{V}_4^*$, respectively. If $\{\mathbf{e}_i\}$ is a basis in $\mathbf{V}^4$ and $\{e^j\}$ is its dual basis in $\mathcal{V}_4^*$, i.e., $\mathbf{e}_i \; e^j = \delta_i^j$ (the Kronecker symbol), then the form $a = e^j \, a_j$ takes the value $\mathbf{x} \; a = x^i a_i$ on the vector $\mathbf{x} = x^i \mathbf{e}_i$. We use the summation convention for the same upper and lower indices.

We can introduce a hyperbolic projective metric in $\mathbf{P}^3$, by giving a bilinear form (see [9, 16])

$$\langle ; \rangle : \mathcal{V}_4^* \times \mathcal{V}_4^* \to \mathbf{R}, \quad \langle b^i u_i ; b^j v_j \rangle = u_i b^{ij} v_j,$$

by means of the Schläfli matrix

$$(24) \quad (\langle b^i ; b^j \rangle) = (b^{ij}) = \begin{bmatrix} 1 & -\cos\frac{\pi}{12} & 0 & 0 \\ -\cos\frac{\pi}{12} & 1 & -\cos\frac{\pi}{3} & 0 \\ 0 & -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{3} \\ 0 & 0 & -\cos\frac{\pi}{3} & 1 \end{bmatrix}$$

where the basis $\{b^i\}$ in $\mathcal{V}_4^*$ represents some planes m_i, being in connection with the polyhedron $\mathcal{D}$ $(i = 0, 1, 2, 3)$. The planes $m_0 = A_1A_2ON$ and $m_1 = A_0A_2OP_1$ will be the symmetry ones of $\mathcal{D}$ (Fig. 2), $m_3 = A_0A_1A_2$ will be a base plane and $m_2 = A_0A_1NP_1$ will be a side plane of $\mathcal{D}$.

We can simply compute that

$$(25) \qquad B = \det(b^{ij}) = \frac{2 - 3\sqrt{3}}{16} < 0.$$

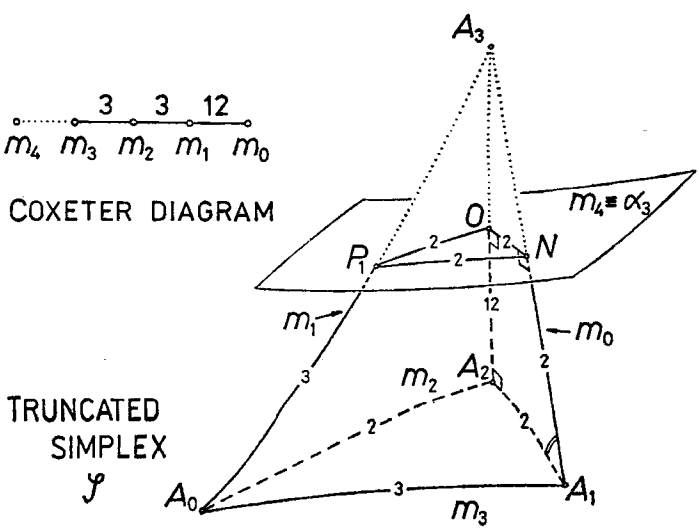

Fig. 2.

We shall see that the inverse matrix (a_{ij}) has great importance. The equation $b^{ij}a_{jk} = \delta^i_k$ holds, iff

$$
\begin{aligned}
a_{00} &= \frac{1}{B}\left(1-2\cos^2\frac{\pi}{3}\right)<0,\\
a_{01} &= \frac{1}{B}\left(\frac{3}{4}\cos\frac{\pi}{12}\right) = a_{10}<0,\\
a_{02} &= \frac{1}{B}\left(\cos\frac{\pi}{12}\right)\left(\cos\frac{\pi}{3}\right) = a_{20}<0,\\
a_{03} &= \frac{1}{B}\left(\cos\frac{\pi}{12}\right)\left(\cos^2\frac{\pi}{3}\right) = a_{30}<0,\\
a_{11} &= \frac{1}{B}\left(1-\cos^2\frac{\pi}{3}\right)<0,\\
a_{12} &= \frac{1}{B}\left(\cos\frac{\pi}{3}\right) = a_{21}<0,\\
a_{13} &= \frac{1}{B}\left(\cos^2\frac{\pi}{3}\right) = a_{31}<0,\\
a_{22} &= \frac{1}{B}\left(1-\cos^2\frac{\pi}{12}\right)<0,\\
a_{23} &= \frac{1}{B}\left[\cos\frac{\pi}{3}-\left(\cos\frac{\pi}{3}\right)\left(\cos^2\frac{\pi}{12}\right)\right] = a_{32}<0,\\
a_{33} &= \frac{1}{B}\left(\frac{3}{4}-\cos^2\frac{\pi}{12}\right)>0.
\end{aligned}
\tag{26}
$$

Now, we take the dual basis $\{\mathbf{a}_j\}$ in $\mathbf{V}^4$, defined by $\mathbf{a}_j\beta^i = \delta^i_j$. Geometrically, the vectors $\mathbf{a}_j$ represent the vertices of the simplex whose side planes are described by the forms β^i. The induced bilinear form

$$\langle;\rangle: \mathbf{V}^4\times\mathbf{V}^4\to\mathbf{R},\ \langle x^i\mathbf{a}_i; y^j\mathbf{a}_j\rangle = x^i a_{ij} y^j$$

is defined just by the matrix

$$(\langle\mathbf{a}_i;\mathbf{a}_j\rangle) = (a_{ij}) \tag{27}$$

in (26).

From the chain of the minor determinants

$$b^{33} = 1 > 0, \begin{vmatrix} b^{22} & b^{23} \\ b^{32} & b^{33} \end{vmatrix} = 1 - \cos^2\frac{\pi}{3} > 0,$$

$$\begin{vmatrix} b^{11} & b^{12} & b^{13} \\ b^{21} & b^{22} & b^{23} \\ b^{31} & b^{32} & b^{33} \end{vmatrix} = 1 - 2\cos^2\frac{\pi}{3} > 0, \; B < 0 \tag{28}$$

we see that the signature of the bilinear form $\langle\,;\,\rangle$ is $(+, +, +, -)$. This means that the projective metric in $\mathbf{P}^3(\mathbf{V}^4, \mathcal{V}_4^*)$ is hyperbolic, indeed. The vectors $\mathbf{x}$ in the cone

$$\mathcal{K} = (\mathbf{x} \in \mathbf{V}^4 : \langle \mathbf{x}; \mathbf{x}\rangle < 0\} \tag{29}$$

define the ordinary points $(\mathbf{x})$ of the Bolyai-Lobachevskyan 3-space, the corresponding polar forms $\mathbf{x}^* = : \varkappa$ describe their polar planes $(\varkappa)$, for which $\langle \varkappa, \varkappa\rangle = \langle \mathbf{x}, \mathbf{x}\rangle < 0$.

The forms u, defining planes which intersect the cone $\mathcal{K}$, describe the ordinary planes (u) of the Bolyai-Lobachevskyan 3-space, their poles (u_*) are defined by the polar vectors $u_* = \mathbf{u}$, and $\langle u; u\rangle = \langle \mathbf{u}; \mathbf{u}\rangle > 0$ holds for them.

The boundary points and their polar planes are defined by isotropic vectors $\mathbf{p}$ and forms φ, i.e. $\langle \mathbf{p}; \mathbf{p}\rangle = \langle \varphi; \varphi\rangle = 0$.

Hence, we can state that the planes m_i to $\mathcal{D}$, represented by the forms b^i, are indeed ordinary planes of the hyperbolic 3-space, modelled in $\mathbf{P}^3$ $(\mathbf{V}^4; \mathcal{V}_4^*)$. From (26) we see that the vertices $A_0(\mathbf{a}_0)$, $A_1(\mathbf{a}_1)$ and $A_2(\mathbf{a}_2)$ of a simplex described by the vectors $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, are ordinary points, but this is not true for the point $A_3(\mathbf{a}_3)$, described by the vector $\mathbf{a}_3$ (Fig. 2.) Therefore, the polar plane (a_3) represents the ordinary plane α_3, the midplane of $\mathcal{D}$. This means that we have a once truncated simplex $\mathcal{S}$ with faces m_i (b^i) $(i = 0, 1, 2, 3,)$ and with truncating face α_3 (a_3).

It is well-known that the bilinear form induces the distance and the angle measure of the hyperbolic 3-space. If $(\mathbf{x})$ and $(\mathbf{y})$ are two ordinary points, with $\langle \mathbf{x}; \mathbf{y}\rangle < 0$, then their distance $d(\mathbf{x}, \mathbf{y})$ is determined by

$$\operatorname{ch}\frac{d}{k} = \frac{-\langle \mathbf{x}; \mathbf{y}\rangle}{\sqrt{\langle \mathbf{x}; \mathbf{x}\rangle \cdot \langle \mathbf{y}; \mathbf{y}\rangle}} \geqq 1. \tag{30}$$

Here $k = \sqrt{-\frac{1}{K}}$ is the metric constant of $\mathbf{H}^3$, where $K < 0$ is the sectional curvature.

Let (u) and (v) be ordinary planes. They intersect in an ordinary line iff $\langle u; u\rangle \langle v; v\rangle - \langle u; v\rangle^2 > 0$. Their angle α (u, v) in this case is determined by

$$\sin\alpha = \sqrt{\frac{\langle u; u\rangle \cdot \langle v; v\rangle - \langle u; v\rangle^2}{\langle u; u\rangle \cdot \langle v; v\rangle}} \geqq 0, \left(0 \leqq \alpha \leqq \frac{\pi}{2}\right). \tag{31}$$

Thus we can check that the Coxeter diagram (Fig. 2) correctly describes the situation. For angle data this is clear from (24) and from the definition of $\alpha_3(a_3)$. For the distance $d(m_3, \alpha_3)$ we can use the extended formula of (26)

$$\text{(32)} \qquad \operatorname{ch}\frac{d}{k} = \frac{\langle a_3; b^3\rangle}{\sqrt{\langle a_3; a_3\rangle \cdot \langle b^3; b^3\rangle}} = \frac{1}{\sqrt{a_{33} \cdot 1}} = \frac{1}{\sqrt{a_{33}}} > 1.$$

We apply (26) and get $d = O\overline{A}_2 = OA_2 = 0{,}4246029\, k$.

Other data of the truncated simplex $\mathcal{S}$ or of $\mathcal{D}$ can be computed also in this way (see table 1).

The Coxeter group C, generated by reflections in mirrors m_0, m_1, m_2, m_3, α_3 of the truncated simplex $\mathcal{S}$, is a supergroup of our group G of finite index 48 since $\mathcal{D}$ is the union of 48 congruent copies of $\mathcal{S}$. Therefore, we could express the generators of G also by matrices with respect to bases $\{b^i\}$ or $\{\mathbf{a}_i\}$.

Data	Simbol	Applied formula	Result
Distance $OA_2 = O\overline{A}_2$	d	$\operatorname{ch}\frac{d}{k} = \frac{\langle a_3; b^3\rangle}{\sqrt{\langle a_3; a_3\rangle\langle b^3; b^3\rangle}} = \frac{1}{\sqrt{a_{33}}}$	$d = k \cdot 0{,}4246029$
In-radius ON	ϱ	$\operatorname{sh}\frac{\varrho}{k} = \frac{-b^{23}}{\sqrt{\frac{1}{a_{33}} - 1}}$	$\varrho = k \cdot 1{,}276687$
In-radius A_2A_1	r	$\operatorname{ch}\frac{r}{k} = \frac{-a_{12}}{\sqrt{a_{11} \cdot a_{22}}}$	$r = k \cdot 1{,}4409526$

Table 1.

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ON CANONICAL QUASI-IDEALS IN RINGS*

By

O. STEINFELD

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

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Dedicated to Professors HANS-JÜRGEN HOEHNKE and HANNS J. WEINERT on their 60th birthdays

1. Introduction

An additive subgroup (a non-empty subset) Q of a ring (a semigroup) A is called a *quasi-ideal* of A if $QA \cap AQ \subseteq Q$. It is known that a non-empty subset Q of a semigroup S with 0 is a quasi-ideal of S if and only if it is an intersection of a right ideal R and a left ideal L of S, that is, $Q = R \cap L$. Example 2.1 in [8] due to A. H. CLIFFORD shows that the analogous result for rings does not hold in general. In his paper [12], H. J. WEINERT has given a nice generalization of Cliffordi's example. A non-zero quasi-ideal Q of a ring (semigroup with 0) A is called *minimal* (0-*minimal*) if Q does not properly contain any non-zero quasi-ideal of A. If R is a minimal (0-minimal) right ideal, L a minimal (0-minimal) left ideal of a ring (semigroup with 0) A such that the intersection $R \cap L = Q$ is not zero, then Q is a minimal (0-minimal) quasi-ideal of A. Example 7.1 (a) in [8] shows that the converse of this result does not hold in general.

A non-zero quasi-ideal Q of a semigroup with 0 (a ring) A is called *canonical* if Q is the intersection of a 0-minimal (minimal) right ideal R and a 0-minimal (minimal) left ideal L of A, that is, $0 \neq Q = R \cap L$. This important notion was introduced by A. H. CLIFFORD [1] for semigroups with 0, and he has proved in [1] many interesting results concerning this notion.

A ring (semigroup with 0) A is called *semiprime*, if A has no non-zero nilpotent ideals. By Theorems 7.2, 7.7 and Propositions 7.6a, 7.6b in [8], the following results hold:

(A) Every minimal (0-minimal) quasi-ideal of a *semiprime* ring (semigroup) is canonical.

(B) Let Q be a canonical quasi-ideal of a *semiprime* ring (semigroup) A and let $a \in A$, then Qa (aQ) is either 0 or a canonical quasi-ideal of A.

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(C) The product Q_1Q_2 of any two canonical quasi-ideals Q_1 and Q_2 of a *semiprime* ring (semigroup) A is either 0 or a canonical quasi-ideal of A.

In our paper [9], we have sharpened the results (B) and (C) for *any semigroups with* 0. The main results (Theorems 2.2 and 2.8) of section 2 are analogous to Theorems 2.3 and 2.9 of [9] and they sharpen the results (B) and (C) for *any rings*. To the proof of the mentioned theorems in [9], we have used essentially a well-known Green's Lemma (see e.g. Lemma 2.2 in [2]). In this paper, we need to define the Green's relations in rings and to prove an analogue of this Green's Lemma for rings (see Lemma 2.4 and Corollary 2.5).

It is known that a simple ring has a minimal left ideal if and only if it has a minimal right ideal. Thus the completely 0-simple semigroups and the simple rings containing at least one minimal one-sided ideal can be considered as analogous notions. We shall use the following terminology. A simple ring A is called *completely simple* if A has at least one minimal right or left ideal. A two-sided ideal M of a ring A is said to be a *completely simple ideal* of A if M is a completely simple ring. In the paper L. M. GLUSKIN and O. STEINFELD [4], it is given some equivalent conditions to the property that a ring A has at least one completely simple ideal. In Theorem 3.2, we have proved that this property is equivalent to the condition that a ring A has at least one *idempotent* canonical quasi-ideal. By Theorems 8.1 and 10.1 of our book [8], the primitive regular semigroups and the semiprime rings satisfying the minimum condition for principal left ideals are analogous notions. Analogously to Theorem 3.2 in [9], Theorem 3.3 characterizes this class of rings by means of idempotent canonical quasi-ideals. In the paper [9], we have used an interesting characterization of the canonical quasi-ideals due to A. H. CLIFFORD to the proof of Theorem 3.2, the analogous Theorem 3.3 in this paper is proved by means of two known results of L. M. GLUSKIN and O. STEINFELD [4].

2. Canonical quasi-ideals

First we give a sufficient condition for the existence of a canonical quasi-ideal in a ring.

THEOREM 2.1. (Cf. O. STEINFELD [6] and [8, Theorem 6.7a]). *If the product RL of a minimal right ideal R and a minimal left ideal L of a ring A is not zero, then $RL = R \cap L$, that is, RL is a canonical quasi-ideal of A.*

The main result of this section is the following

THEOREM 2.2. *Let Q be a canonical quasi-ideal of a ring A, and let $a \in A$. Then Qa (aQ) is either 0 or a canonical quasi-ideal of A.*

REMARKS. 1. This result is an essential sharpening of Proposition 7.6a in [8] and it is analogous to Theorem 2.3 in [9].

2. Example 5 of P. N. STEWART [10] proves the existence of a ring A such that A has a minimal quasi-ideal Q and an element a such that Qa is neither 0 nor a minimal quasi-ideal of A. The example of Stewart

solves the ring theoretic part of Problem 7.3 in [8]. (Independently from P. N. STEWART, G. POLLÁK has given two examples, which solve the ring and semigroup theoretic parts of this problem.)

To the proof we need the following known

LEMMA 2.3. Let L be a minimal left ideal of a ring A, and let $a \in A$, then La is either 0 or a minimal left ideal of A.

We also need some notions and notations which are analogous to that of Green's relations defined in semigroups. Let A be a ring and let $a \in A$. The *principal left (principal right) ideal* $L(a)$ $(R(a))$ *generated by* a is $Ia + Aa = L(a)$ $(Ia + aA = R(a))$, where I denotes the ring of all integers. We define $a \mathcal{L} b$ $(a, b \in A)$ to mean that a and b generate the same principal left ideal of A, that is, $a \mathcal{L} b \Leftrightarrow L(a) = L(b)$. Clearly $\mathcal{L}$ is an equivalence relation on A; let L_a denote the equivalence class of A mod $\mathcal{L}$ containing the element $a (\in A)$.

Dually we define $a \mathcal{R} b$ to mean that $R(a) = R(b)$ $(a, b \in A)$, and $\mathcal{R}$ is an equivalence relation on A. By R_a we mean the equivalence class of A mod $\mathcal{R}$ containing the element a of A.

By definition of $\mathcal{L}$, $a \mathcal{L} b$ $(a, b \in A)$ if and only if there exist k, k' in I and u, u' in A such that $a = kb + ub$ and $b = k'a + u'a$. For the sake of simplicity we introduce the following notation $kx + ux = (k, u)x$ $(k \in I, x, u \in A)$. Dually $a \mathcal{R} b$ $(a, b \in A)$ if and only if there exist m, m' in I and v, v' in A such that $a = mb + bv = b(m, v)$ and $b = m'a + av' = a(m', v')$.

LEMMA 2.4. (Cf. [2, Lemma 2.2] and Lemma 2.5 in [9]). Let $a \mathcal{L} b$ in a ring A. Then there exist a one-to-one mapping Φ of $R(a)$ onto $R(b)$ and a one-to-one mapping Ψ of $R(b)$ onto $R(a)$ such that Φ maps R_a onto R_b, Ψ maps R_b onto R_a and $\Psi = \Phi^{-1}$ holds.

COROLLARY 2.5. (Cf. Corollary 2.6 in [9]). Let $a \mathcal{L} b$ in a ring A, then $R(a)$ is minimal if and only if $R(b)$ is minimal.

PROOF OF LEMMA 2.4. From $a \mathcal{L} b$ we have $a = kb + ub = (k, u)b$ and $b = k'a + u'a = (k', u')a$ $(k, k' \in I, u, u' \in A)$. Let $ma + av = a(m, v)$ $(m \in I, v \in A)$ denote an arbitrary element of $R(a)$, and let $nb + bw = b(n, w)$ $(n \in I, w \in A)$ be an arbitrary element of $R(b)$. Consider the mappings Φ: $a(m, v) \to \Phi(a(m, v)) = (k', u')(a(m, v)) = ((k', u')a)(m, v) = b(m, v) \in R(b)$ and

$$\Psi : b(n, w) \to \Psi(b(n, w)) = (k, u)(b(n, w)) = ((k, u)b)(n, w) = a(n, w) \in R(a),$$

then $\Psi(\Phi(a(m, v))) = \Psi(b(m, v)) = a(m, v)$ and $\Phi(\Psi(b(n, w))) = \Phi(a(n, w)) = b(n, w)$, that is, $\Psi = \Phi^{-1}$.

If $qb + bz = b(q, z)$ $(q \in I, z \in A)$ is an element of $R(b)$, then for the element $qa + az = a(q, z)$ of $R(a)$ it holds $\Phi(a(q, z)) = b(q, z)$, that is, Φ maps $R(a)$ onto $R(b)$. Dually Ψ maps $R(b)$ onto $R(a)$.

Let $x \in R_a$, then $x \mathcal{R} a$, that is, $R(x) = R(a)$. By definition of $\mathcal{R}$ there exist $r, r' \in I$ and $d, d' \in A$ such that $a = rx + xd = x(r, d)$ and $x = r'a + ad' = a(r', d')$, whence

$$k'x + u'x = (k', u')x = ((k', u')a)(r', d')$$

and

$$k'a+u'a = (k', u')a = ((k', u')x)(r, d).$$

Again by definition of $\mathcal{R}$ we have $R((k', u')x) = R((k', u')a)$. This fact and the relations

$$(k', u')a = b,\ (k', u')x = (k', u')(a(r', d')) = ((k', u')a)(r', d') = b(r', d')$$

imply $R(b) = R(b(r', d'))$. Since $\Phi(x) = \Phi(a(r', d')) = b(r', d')$, therefore Φ maps R_a into R_b. Now consider an arbitrary element y of R_b. By definition of $\mathcal{R}$, there exist $s, s' \in I$ and $c, c' \in A$ such that $y = b(s, c)$ and $b = y(s', c')$. Hence $(k, u)y = ((k, u)b)(s, c)$ and $(k, u)b = ((k, u)y)(s', c')$. By definition of $\mathcal{R}$ this means that $R((k, u)b) = R((k, u)y)$. On the other hand, since $(k, u)\, b = a$ and $(k, u)y = (k, u)b(s, c) = a(s, c)$ we get $R(a) = R(a(s, c))$. These facts and $\Phi^{-1}(y) = \Psi(y) = \Psi(b(s, c)) = a(s, c)$ imply that Φ maps R_a onto R_b.

Evidently the mapping Ψ has also the properties given in Lemma 2.4.

From Lemma 2.4 it follows

$$x \in R(a) \Leftrightarrow (k', u')x \in R(b) \qquad (k' \in I,\ u' \in A),$$

$$y \in R_a \Leftrightarrow (k', u')y \in R_b \qquad (k' \in I,\ u' \in A),$$

whence $R(a) = R_a \cup 0 \Leftrightarrow R(b) = R_b \cup 0$.

Thus Lemma 2.4 implies Corollary 2.5.

Proof of Theorem 2.2. By definition, Q has the form $Q = R \cap L$, where R and L are minimal right and minimal left ideals of A, respectively. If $Qa \neq 0$, then $0 \neq Qa \subseteq R \cap La$. This implies $La \neq 0$; by Lemma 2.3, La is a minimal left ideal of A. So it suffices to show that $R \cap La \subseteq Qa$. Let $0 \neq c \in R \cap La$, then there exists a non-zero element b in L such that $c = ba$. Hence $R = R(c) = R(ba) = Iba + baA \subseteq bA \subseteq R(b)$ and $L = Ib + Ab = L(b)$. Now we consider a non-zero element q of $Q = R \cap L$. So we have $R = R(q)$ and $L = L(q) = L(b)$, whence $q \mathcal{L} b$. By Corollary 2.5, the right ideal $R(b)$ is minimal. From these facts it follows that $R = R(b)$ and $b \in R$. This means that $b \in R \cap L = Q$, thus $c = ba \in Qa$, that is, $R \cap La \subseteq Qa$.

The relation $aQ = aR \cap L$ can be shown by the duals of Lemmas 2.3, 2.4 and Corollary 2.5.

We say that the ring A is *the sum of its subrings* B_λ $(\lambda \in \lambda)$ if every element a of A can be written in the form $a = \sum_{\text{finite}} b_\lambda\ (b_\lambda \in B_\lambda)$ with finitely many elements b_λ of B_λ $(\lambda \in \Lambda)$, that is, $\sum_{\lambda \in \Lambda} B_\lambda = A$.

Theorem 2.2 implies

Corollary 2.6. If the ring A contains at least one canonical quasi-ideal, then the sum U of all canonical quasi-ideals of A is a two-sided ideal of A.

PROOF. Let $u = \sum_{i=1}^{k} u_i \in U$ and $a \in A$. By definition of U, u_i $(i = 1, \ldots, k)$ is an element of a canonical quasi-ideal Q_i $(i = 1, \ldots, k)$ of A. In view of Theorem 2.2, the product $Q_i a$ $(i = 1, \ldots, k)$ is either 0 or a canonical quasi-ideal of A, whence $ua = \sum_{i=1}^{k} u_i a \in \sum_{i=1}^{k} Q_i a \subseteqq U$, that is, U is a right ideal of A. Dually one can show that U is a left ideal of A.

By means of Corollary 2.6 one gets

COROLLARY 2.7. Let K be a minimal left (right or two-sided) ideals of a ring A. If K contains at least one canonical quasi-ideal of A, then K is the sum of all the canonical quasi-ideals of A contained in K.

Now we are going to prove the following sharpening of [8, Theorem 7.7].

THEOREM 2.8. *Let A be a ring and let Q_1, Q_2 be any two canonical quasi-ideals of A, then the product Q_1Q_2 is either* 0 *or a canonical quasi-ideal of A.*

PROOF. By the definition, we have $Q_1 = R_1 \cap L_1$ and $Q_2 = R_2 \cap L_2$, where R_1, R_2 and L_1, L_2 are minimal right and minimal left ideals of A, respectively. Assume that $Q_1Q_2 \neq 0$, then $0 \neq Q_1Q_2 = (R_1 \cap L_1)(R_2 \cap L_2) \subseteqq R_1L_2 \subseteqq R_1 \cap L_2$. Since $Q_1Q_2 \neq 0$, there exists a non-zero element q_2 in Q_2 such that $Q_1q_2 \neq 0$. By Theorem 2.2, Q_1q_2 is a canonical quasi-ideal of A such that $Q_1q_2 = R_1 \cap L_1q_2$. As q_2 is an element of L_2, we have $0 \neq L_1q_2 \subseteqq L_2$, thus Lemma 2.3 implies $L_1q_2 = L_2$. Hence $Q_1q_2 = R_1 \cap L_2$. From the relation

$$R_1 \cap L_2 = Q_1q_2 \subseteqq Q_1Q_2 \subseteqq R_1L_2 \subseteqq R_1 \cap L_2,$$

it follows $Q_1Q_2 = R_1 \cap L_2$; *q. e. d.*

REMARKS. 1. It is evident that the product Q_1Q_2 of *any two quasi-ideals* Q_1 and Q_2 of a ring A is a subring of A, but Q_1Q_2 is not a quasi-ideal of A, in general. (See Example 1 in STEWART [10] and see Propositions 6.1 and 6.2 in WEINERT [12].)

2. It is not difficult to show that the associative law $(Q_1Q_2)Q_3 = Q_1(Q_2Q_3)$ holds for the products of the quasi-ideals Q_1, Q_2 and Q_3 of A. This fact and Theorem 2.8 imply:

COROLLARY 2.9. (Cf. Corollary 2.10 in [9]). Assume that the ring A has at least one canonical quasi-ideal. Let V denote the set of all canonical quasi-ideals of A, then $V \cup 0$ is a multiplicative semigroup with 0.

PROBLEM. Does there exist a ring (a semigroup with zero) A with two minimal (0-minimal) quasi-ideals, the product of which is not a quasi-ideal of A?

3. Idempotent canonical quasi-ideals

In order to sharpen Theorem 2.1 we need some preliminaries.

LEMMA 3.1. (See e.g. L. M. GLUSKIN and O. STEINFELD [4, Corollary 1.6]). A simple ring contains at least one minimal left ideal if and only if it contains at least one minimal right ideal.

We shall use the terminology given in Introduction.

THEOREM 3.2. (Cf. L. M. GLUSKIN and O. STEINFELD [4]). *The following conditions on a ring A are equivalent:*

(i) *A has at least one minimal right ideal R and at least one minimal left ideal L such that $(LR)^2 \neq 0$;*

(ii) *A has at least one minimal right ideal R and at least one minimal left ideal L such that $RL \neq 0$;*

(iii) *A has at least one idempotent canonical quasi-ideal Q (that is, Q is canonical and $Q^2 = Q$);*

(iv) *A has at least one completely simple ideal.*

REMARKS. 1. The equivalence of conditions (i), (ii) and (iv) was proved in [4].

2. This result is an essential sharpening of Theorem 2.1.

PROOF OF THEOREM 3.2. (i)$\Rightarrow$(iii). By condition (i), there exist a minimal left ideal L and at least a minimal right ideal R of A such that $(LR)^2 = LRLR \neq 0$, then $RL \neq 0$. By Theorem 2.1, $RL = Q$ is a canonical quasi-ideal of A. Since $(LR)^2 = LRLR \neq 0$, therefore $LRL \neq 0$, whence $LRL = L$. Thus $Q^2 = RLRL = RL = Q$.

Now assume that condition (iii) holds. Then there exist a minimal right ideal R and a minimal left ideal L of A such that $Q = R \cap L (\neq 0)$ and $(R \cap L)^2 = R \cap L$. Hence $0 \neq R \cap L = (R \cap L)(R \cap L) \subseteq RL$. This means that (iii)$\Rightarrow$(ii).

REMARK. Let A be a ring containing at least one completely simple ideal. Then by Theorem 3.2, A has the following properties:

$(\mathrm{i})_0$ A has at least one minimal right ideal R and at least one minimal left ideal L such that $LR \neq 0$;

$(\mathrm{iii})_1$ A has at least one canonical quasi-ideal;

$(\mathrm{iii})_2$ A has at least one idempotent minimal quasi-ideal.

The following example shows that these three conditions together do not imply condition (iv) in Theorem 3.2.

EXAMPLE 3.1. (See Example 4.3 in [8]). Let D be a division ring and let T denote the ring of all triangular matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} (a_{11}, a_{12}, a_{22} \in D).$$

Consider the matrices

$$E_{11} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \text{ and } E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix},$$

where e denotes the identity element of D. Evidently TE_{11}, TE_{12} are minimal left ideals of T and $E_{12}T$ is a minimal right ideal of T such that

$$TE_{11}\cdot E_{12}T\neq 0 \text{ (but } (TE_{11}\cdot E_{12}T)^2 = 0),$$

$$E_{12}T\cap TE_{12}\neq 0 \text{ (but } E_{12}T\cdot TE_{12} = 0).$$

Furthermore $E_{11}TE_{11}$ is an idempotent minimal quasi-ideal of A, but it is not a canonical quasi-ideal of A.

These facts mean that T has the properties $(\mathrm{i})_0$, $(\mathrm{iii})_1$ and $(\mathrm{iii})_2$, but evidently T contains no completely simple ideals.

In order to give the main result of this section we need the following notion. The ring A is said to be the *discrete direct sum* of its two-sided ideals $B_\gamma(\gamma\in\Gamma)$, if $A = \sum_{\gamma\in\Gamma} B_\gamma$ and $B_\gamma\cap(\sum_{\gamma\neq\delta\in\Gamma} B_\delta) = 0$.

It is known that the primitive regular semigroups and the semiprime rings satisfying the minimum condition for principal left ideals can be considered as analogous notions. (See e.g. Theorems 8.1 and 10.1 in [8].) We shall call these rings as *semiprime MPL-rings*.

The following result is analogous to Theorem 3.2 in [9].

THEOREM 3.3. (Cf. O. STEINFELD [7] and [8, Theorem 8.1]). *The following conditions on a ring A are equivalent:*

(a) *A has idempotent canonical quasi-ideals Q_γ $(\gamma\in\Gamma)$ such that A is the discrete direct sum of the ideals $AQ_\gamma A(\gamma\in\Gamma)$;*

(b) *A is the discrete direct sum of two-sided ideals which are completely simple ideals of A;*

(c) *A is a semiprime MPL-ring.*

REMARK. The equivalence of conditions (b) and (c) is known (see e.g. C. C. FAITH [3] and F. SZÁSZ [11]).

To the proof of Theorem 3.3 we need the following known results:

THEOREM 3.4. (See L. M. GLUSKIN and O. STEINFELD [4]). *Let R and L be minimal right and minimal left ideals of a ring A, respectively, such that $RL\neq 0$, then $ARLA$ is a completely simple ideal of A.*

THEOREM 3.5. (See L. M. GLUSKIN and O. STEINFELD [4], and cf. R. P. RICH [5]). *If M is a completely simple ideal of a ring A, then $M = LR$, where L and R are minimal left and minimal right ideals of A, respectively, furthermore $RL\neq 0$.*

PROOF OF THEOREM 3.3. (a)$\Rightarrow$(b). As Q_γ $(\gamma\in\Gamma)$ is an idempotent quasi-ideal of A, we have $(AQ_\gamma A)^2\neq 0$. On the other hand, Q_γ is an idempotent canonical quasi-ideal of A, so there exist a minimal right ideal R_γ and a minimal left ideal L_γ of A such that $Q_\gamma = R_\gamma\cap L_\gamma$ and $(R_\gamma\cap L_\gamma)^2 = R_\gamma\cap L_\gamma$. Hence $0\neq R_\gamma\cap L_\gamma = (R_\gamma\cap L_\gamma)^2\subseteq R_\gamma L_\gamma\subseteq R_\gamma\cap L_\gamma = Q_\gamma$, that is, $R_\gamma L_\gamma\neq 0$ and $Q_\gamma = R_\gamma L_\gamma$. Theorem 3.4 implies that the ideals $AQ_\gamma A$ $(\gamma\in\Gamma)$ are completely simple ideals of A.

(b)⇒(a). In view of condition (b), A is the discrete direct sum of the ideals A_γ ($\gamma \in \Gamma$) and the A_γ are completely simple ideals of A. From Theorem 3.5, it follows that every $A_\gamma (\gamma \in \Gamma)$ has the form $A_\gamma = L_\gamma R_\gamma$, where L_γ and R_γ are minimal left and minimal right ideals of A, respectively, and $(L_\gamma R_\gamma)^2 = A_\gamma^2 \neq 0$, $R_\gamma L_\gamma \neq 0$. (From the proof of Theorem 3.5 it follows that L_γ and R_γ are also minimal left and minimal right ideals of A_γ, respectively.) These facts and Theorem 2.1 imply that $R_\gamma L_\gamma$ is an idempotent canonical quasi-ideal of A, furthermore $A_\gamma = AR_\gamma L_\gamma A$.

Theorem 3.3 implies

COROLLARY 3.6. A ring A is completely simple if and only if A contains an idempotent canonical quasi-ideal Q such that $A = AQA$.

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WEAKLY COMPOSITE LUCAS NUMBERS*

By

P. KISS and BUI MINH PHONG

Teacher's Training College, Eger and
Computer Center, L. Eötvös University, Budapest

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A natural number m is called weakly composite if the reciprocal sum of its distinct prime divisors is not greater than 2, i.e.

$$\sum_{p|m} \frac{1}{p} \leqq 2$$

Proving conjecture of I. KÁTAI, J. GALAMBOS [2] showed that for any sufficiently large n there is a weakly composite number between n and $n+\log\log\log n$.

In this note we prove a similar result for Lucas numbers.

The Lucas numbers R_n ($n = 0, 1, 2, \ldots$) with paramterers A, B are defined by the recursion

$$R_n = A \cdot R_{n-1} + B \cdot R_{n-2}$$

for $n > 1$ and by the initial terms $R_0 = 0$, $R_1 = 1$. Denoting by α and β the roots of the characteristic polynomial $x^2 - Ax - B$, we say the sequence of Lucas numbers is non-degenerate if A, B are non-zero integers and α/β is not a root of unity. In this case $R_n \neq$ for any $n > 0$.

Throughout the paper we assume that A, B are fixed integers and the sequence is non-degenerate, furthermore we shall denote by c_1, c_2, ... positive constants which are absolute ones or they depend only on the parameters A and B.

If p is a prime and $p \nmid B$, then there are Lucas numbers divisible by p. The least positive index of these numbers is called the rank of apparation of p in the sequence and we denote it by $r(p)$. Thus $r(p) = n$ if $p | R_n$ but $p \nmid R_m$ for $0 < m < n$. We say p is a primitive prime divisor of $R_{r(p)}$. We note that if $p \nmid B$ and $p | A$ for a prime p, then $r(p) = 2$ and $p | R_n$ for any $n \geqq 2$;

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furthermore if $p|B$ but $p\nmid A$, then there is no term in the sequence divisible by p (see Theorem 2.1 in [3]). In this case we assume that $r(p) = \infty$ or we say $r(p)$ does not exist.

For the reciprocal sum of prime divisors of Lucas numbers in [4] we proved that

$$\sum_{p|R_m} \frac{1}{p} < \log\log\log m + c_1$$

for any sufficiently large m. It is an extension of a result of P. Erdős who proved a similar estimation in the special case $R_n = 2^n - 1$, i.e. in the case $A = 3$, $B = -2$. In this case P. Erdős [1] and C. Pomerance [6] obtained some interesting results concerning the reciprocal sum of the divisors (not only primes) of the terms, as well.

Now we show that the reciprocal sum of the prime divisors of the terms in Lucas sequences can be less than a constant for infinitely many indices. In this case we may say R_n is weakly composite, since the essential point in the definition of weakly compositeness does not require that the value of the constant should be just 2 as in [2].

Theorem. *For any $n>3$ there is a Lucas number R_m such that*

$$\sum_{p|R_m} \frac{1}{p} < c$$

and $n<m\leqq n+\log\log n$, where c is a constant depending only on A and B.

First we prove two auxiliary results. In the proofs we shall use some elementary properties of Lucas numbers which can be found e.g. in [5].

Lemma 1. Let $m>3$ be an integer and let $Q_m = R_1\cdot R_2\cdot\ldots R_m$. Then for the distinct prime p divisors of Q_m we have

$$\sum_{p|Q_m} \frac{1}{p} = \sum_{r(p)\leqq m} \frac{1}{p} = \log\log m + O(1).$$

Proof. It is known that $r(p)\leqq p+1$ for any p for which $r(p)$ exists, and so

$$\sum_{r(p)\leqq m} \frac{1}{p} \geqq \sum_{p\leqq m} \frac{1}{p} + O(1) = \log\log m + O(1). \tag{1}$$

On the other hand

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for any $n \geqq 0$ and from it

$$|R_n| < e^{c_2 n} \tag{2}$$

follows. By (2), R_n has at most $c_3 n$ distinct prime divisors since the product of $c_3 n$ distinct primes is greater than $e^{c_2 n}$ for some c_3. It implies that the

number of distinct prime divisors of Q_m is less than c_4m^2. But each of the first c_4m^2 primes is less than c_5m^3, so we have

$$\sum_{r(p)\leqq m} \frac{1}{p} < \sum_{p<c_5m^3} \frac{1}{p} = \log\log m + O(1). \tag{3}$$

From (1) and (3) the lemma follows.

LEMMA 2. For any real $x>1$ we have

$$\sum_{r(p)\leqq x} \frac{1}{p\cdot r(p)} < c_6.$$

PROOF. If p is a prime and $p|R_n$, then naturally $p\leqq R_n$ and by (2) it follows that $n>c_7 \log p$. Thus $r(p)>c_7\log p$ by the definition of $r(p)$ and so

$$\sum_{r(p)\leqq x} \frac{1}{p\cdot r(p)} < \frac{1}{c_7}\sum_{p\leqq N} \frac{1}{p\log p},$$

where N is a sufficiently large integer (e.g. $N>e^{c_2x}$). But using some summation formulas, e.g. Abel's identity, one can easily see that

$$\sum_{p\leqq N} \frac{1}{p\log p} = O(1)$$

for any N, from which the assertion follows.

Now we prove our theorem.

PROOF OF THE THEOREM. Let $n>3$ be an integer and

$$k = \log\log n.$$

We give an upper estimation for the sum

$$S_n = \sum_{n<m\leqq n+k} \left(\sum_{p|R_m} \frac{1}{p}\right). \tag{4}$$

It is known that $p|R_m$ if and only if $r(p)|m$ in case $p\nmid B$ and, as we have seen above, $p|R_m$ for any $m\geqq 2$ in case $p|(A, B)$. Let

$$\sum_{p|(A,B)} \frac{1}{p} = c_8,$$

where $c_8 = 0$ if A and B are relatively prime. Then

$$S_n = \sum_{n<m\leqq n+k} \left(\sum_{d|m}\left(\sum_{\substack{r(p)=d\\ p\nmid B}} \frac{1}{p}\right)\right) + c_8[k] \leqq$$

$$\leqq \sum_{d\leqq n+k}\left(\left(\left[\frac{n+k}{d}\right]-\left[\frac{n}{d}\right]\right)\cdot \sum_{\substack{r(p)=d\\ p\nmid B}} \frac{1}{p}\right) + c_8k.$$

From this, using Lemma 1, Lemma 2, the definition of k and the fact

$$\left[\frac{n+k}{d}\right]-\left[\frac{n}{d}\right]\leqq\frac{k}{d}+1,$$

we get

$$S_n\leqq\sum_{d\leqq n+k}\left(\left(\frac{k}{d}+1\right)\cdot\sum_{\substack{r(p)=d\\ p\nmid B}}\frac{1}{p}\right)+c_8k\leqq$$

$$\leqq k\sum_{r(p)\leqq n+k}\frac{1}{pr(p)}+\sum_{r(p)\leqq n+k}\frac{1}{p}+c_8k\leqq$$

$$\leqq c_9k+\log\log(n+k)+O(1)<c_{10}\log\log n.$$

But there are $[k]=[\log\log n]$ summands in the outside sum of (4), so there is an integer m such that $n<m\leqq n+\log\log n$ and

$$\sum_{p|R_m}\frac{1}{p}\leqq\frac{S_n}{[\log\log n]}<c_{11}$$

which proves the theorem with $c=c_{11}$.

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ON SOME PROBLEMS RAISED IN CONNECTION WITH EIGENFUNCTION EXPANSIONS*

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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1. In this paper we investigate three problems of function theoretical character the origin of which lies in spectral theoretical works. To introduce the first one, consider a function $\omega(t)\in C[0, \infty)$ satisfying

(i) $\omega(0) = 0$, $\omega(t)>0$ $(t>0)$,
(ii) $\omega(2t)\leq C\omega(t)$ $(t>0)$,
(iii) $\omega(t)$ is monotone increasing,
(iv) $t^{-1}\omega(t)$ is monotone decreasing.

Introduce further the usual L^2-modulus of continuity

$$\omega(f, \delta)_2: = \sup_{|h|\leq\delta}\left[\int_0^{1-h} |f(x+h)-f(x)|^2\, dx\right]^{1/2}$$

and the $n-$th best approximation of a function $f\in L^2(0,1)$ by trigonometric polynomials of order $\leq n$

$$E_n(f)_2: = \inf_{t_n\in T_n} \|f-t_n\|_2.$$

In [1] is proved that for a class of functions, the continuity modulus of which have the order $O(\omega(\delta))$, the Fourier coefficients and the equiconvergence between two eigenfunction expansions of these functions have also the order $O(\omega(\delta))$. Here the question arises, what is the connection in general between the order of the Fourier coefficients and that of the modulus of the function. The statement that the two mentioned order is equal, is not true in general! In what follows we give a transformation $f\to\hat{f}$ such that the order of the Fourier coefficients of f equals to the order of the modulus of $\hat{f}$.

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THEOREM 1. *There exists a (nonlinear) mapping* $F: L^1(0,1) \to L^2(0,1)$ *satisfying the following property. Let* $\omega(t) \in C[0, \infty)$ *be a function satisfying* (i)–(iii) *and*

(iv′) $\qquad t^{\varepsilon-1}\omega(t)$ *is monotone decreasing for some* $\varepsilon > 0$.

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos 2k\pi x + b_k \sin 2k\pi x)$$

be the Fourier series of $f(x)$. *Then we have*

$$a_k, b_k = O\left(\omega\left(\frac{1}{k}\right)\right) \Leftrightarrow \omega(Ff, \delta)_2 = O(\omega(\delta)). \tag{1}$$

PROOF. It is widely known that for any $g \in L^2(0,1)$

$$E_n(g)_2 \le C\omega\left(g, \frac{1}{n}\right)_2 \le C\frac{1}{n}\sum_{k=1}^{n} E_k(g)_2 \ (n \in \mathbf{N}). \tag{2}$$

We shall prove that

$$E_n(g)_2 = O\left(\omega\left(\frac{1}{n}\right)\right) \Leftrightarrow \omega(g, \delta)_2 = O(\omega(\delta)). \tag{3}$$

Indeed, the "if" part ($\Leftarrow$) follows from

$$E_n(g)_2 \le C\omega\left(g, \frac{1}{n}\right)_2 \le C\omega\left(\frac{1}{n}\right).$$

Consider the "only if" part ($\Rightarrow$). Let $2^N \le n < 2^{N+1}$, then

$$\omega\left(g, \frac{1}{n}\right)_2 \le \frac{C}{n}\sum_{k=1}^{n} E_k(g)_2 \le \frac{C}{n}\sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \le$$

$$\le \frac{C}{n}\sum_{j=0}^{N-1}\sum_{k=2^j}^{2^{j+1}-1} \omega\left(\frac{1}{k}\right) + \frac{C}{n}\sum_{k=2^N}^{n} \omega\left(\frac{1}{k}\right) \le$$

$$\le \frac{C}{n}\sum_{j=0}^{N} 2^j\omega\left(\frac{1}{2^j}\right) \le \frac{C(\varepsilon)}{n} 2^N\omega\left(\frac{1}{2^N}\right) \le C(\varepsilon)\omega\left(\frac{1}{2^N}\right) \le$$

$$\le C(\varepsilon)\omega\left(\frac{1}{2^{N+1}}\right) \le C(\varepsilon)\omega\left(\frac{1}{n}\right),$$

since by (iv′):

$$2^j\omega\left(\frac{1}{2^j}\right) \le 2^{-\varepsilon}2^{j+1}\omega\left(\frac{1}{2^{j+1}}\right) \le 2^{-\varepsilon(N-j)}2^N\omega\left(\frac{1}{2^N}\right).$$

The assertion (3) being proved we shall give the construction of Ff as follows. Let

$$\bar{d}_n := \max_{k\geq n} |a_k|, \bar{\bar{d}}_n := \max_{k\geq n} |b_k|.$$

Then

$$a_k, b_k = O\left(\omega\left(\frac{1}{k}\right)\right) \Leftrightarrow \bar{d}_n, \bar{\bar{d}}_n = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{4}$$

because $\omega(t)$ is monotone increasing. Define

$$\tilde{d}_n := (\bar{d}_n^2 - \bar{d}_{n+1}^2)^{1/2}, \tilde{\tilde{d}}_n := (\bar{\bar{d}}_n^2 - \bar{\bar{d}}_{n+1}^2)^{1/2}$$

and let

$$Ff(x) := \frac{\tilde{d}_0}{2} + \sum_{k=1}^{\infty} (d_k \cos 2k\pi x + \tilde{\tilde{d}}_k \sin 2k\pi x).$$

Since $\sum (\tilde{d}_n^2 + \tilde{\tilde{d}}_n^2) < \infty$, this series converes in $L^2(0,1)$. Obviously

$$E_n(Ff)_2 = \left[2 \sum_{k=n+1}^{\infty} (\tilde{d}_k^2 + \tilde{\tilde{d}}_k^2)\right]^{1/2} = \left[2(\bar{d}_{n+1}^2 + \bar{\bar{d}}_{n+1}^2)\right]^{1/2}$$

and hence

$$a_n, b_n = O\left(\omega\left(\frac{1}{n}\right)\right) \Leftrightarrow E_n(Ff) = O\left(\omega\left(\frac{1}{n}\right)\right), \tag{5}$$

which proves (1) by (3). □

We can formulate here three problems.

PROBLEM 1. Is the mapping F continuous from L^1 to L^2? (or from L^2 to L^2?).

PROBLEM 2. Is there a continuous linear mapping $F: L^1 \to L^2$ (or $L^2 \to L^2$) satisfying property (1)?

PROBLEM 3. Is there a multiplier type mapping $F: L^2 \to L^2$ satisfying (1) for any $\omega(t)$ (or for a fixed $\omega(t)$)?

REMARK. In Theorem 1 the condition (iv′) can not be substituted by (iv), i.e. (1) does not hold for Ff constructed in the proof. Indeed, let $a_k = b_k = 1/k$, then

$$\tilde{d}_k = \tilde{\tilde{d}}_k = \frac{\sqrt{2k+1}}{k(k+1)}$$

and

$$[\omega(Ff, \delta)_2]^2 \geq \|Ff(x+\delta) - Ff(x)\|_2^2 =$$

$$= C \sum_{k=1}^{\infty} \{[\tilde{d}_k(\cos k\delta - 1) + \tilde{\tilde{d}}_k \sin k\delta]^2 + [\tilde{\tilde{d}}_k(\cos k\delta - 1) + \tilde{d}_k \sin k\delta]^2\} =$$

$$= C \sum_{k=1}^{\infty} (\tilde{d}_k^2 + \tilde{\tilde{d}}_k^2)(2 - 2\cos k\delta) \geq$$

$$\geq C \sum_{k=1}^{\sqrt{\frac{1}{\delta}}} \frac{1}{k}(2 - 2\cos k\delta) = C \sum_{k=1}^{\sqrt{\frac{1}{\delta}}} \frac{1}{k^3}(k^2\delta^2 + O(k^4\delta^4)) \geq$$

$$\geq C\delta^2 \ln \frac{1}{\delta} - C\delta^4 \frac{1}{\delta} \geq C\delta^2 \ln \frac{1}{\delta}$$

so $\omega(Ff, \delta)_2 \not\leq c\delta$ indeed. Here we can formulate the

PROBLEM 4. Does Theorem 1 remain valid (with another construction of Ff) if we replace instead of (iv′) the original condition (iv)?

2. To show the summability of a series is much easier in general than to prove the convergence (see [4], [5]). Sometimes it is useful to prove the convergence thorough summability and some further assumptions; the results of this type are called Tauberian theorems. In what follows we present a Tauber type theorem related to Riesz summability of entire order s.

THEOREM 2. *Fix a positive sequence* λ_k $(k = 1,2, \ldots)$ *and the natural numbers* m, $s \in \mathbf{N}$. *Take an arbitrary sequence* $(c_k)_1^\infty \subset \mathbf{R}$ *such that*

(a) $$\sum_{\lambda_k<\lambda}{}_k c_k \left(1 - \frac{\lambda_k}{\lambda}\right)^s = O(1) \ (\lambda > 0),$$

(b) $$\sum_{\lambda_k<\lambda} c_k \left(\frac{\lambda_k}{\lambda}\right)^m = O(1) \ (\lambda > 0),$$

(c) $$\sum_{\lambda_k<\lambda}{}_k |c_k| < \infty \ \textit{for all } \lambda > 0.$$

Then

(d) $$\sum_{\lambda_k<\lambda}{}_k c_k = O(1) \ (\lambda > 0).$$

PROOF. The inequality (b) can be written in the form

$$-ct^m \leq \sum_{\lambda_k<t} c_k \lambda_k^m \leq ct^m \ (t > 0)$$

for some $c > 0$. Integrate here in t from 0 to λ:

$$-c\frac{\lambda^{m+1}}{m+1} \leq \sum_{\lambda_k<\lambda} \int_{\lambda_k}^{\lambda} c_k \lambda_k^m dt = \lambda \sum_{\lambda_k<\lambda} c_k \lambda_k^m - \sum_{\lambda_k<\lambda} c_k \lambda_k^{m+1} \leq c\frac{\lambda^{m+1}}{m+1}.$$

Dividing by λ^m we obtain

$$\sum_{\lambda_k<\lambda} c_k \lambda_k^{m+1} / \lambda^{m+1} = O(1).$$

Hence if (c) is valid and (b) holds for m then it holds for $m+1$, too. On the other hand we can integrate (b) in its original form; then we obtain

$$O(\lambda) = \int_0^\lambda \sum_{\lambda_k<\lambda} c_k\left(\frac{\lambda_k}{t}\right)^m dt = \sum_{\lambda_k<\lambda} \int_{\lambda_k}^\lambda c_k\left(\frac{\lambda_k}{t}\right)^m dt =$$

$$= \sum_{\lambda_k<\lambda} c_k \lambda_k^m \frac{\lambda^{1-m} - \lambda_k^{1-m}}{1-m};$$

dividing by λ we get

$$\sum_{\lambda_k<\lambda} c_k \frac{\lambda_k}{\lambda} = O(1),$$

hence if (b) holds for some m then it holds for $m = 1$ too. Consequently (b) holds for all values $m = 1, 2, \ldots$.

Taking the binomial expansion of $\left(1-\frac{\lambda_k}{\lambda}\right)^s$ we get

$$\sum_{\lambda_k<\lambda} c_k = \sum_{\lambda_k<\lambda} c_k\left(1-\frac{\lambda_k}{\lambda}\right)^s - \sum_{j=1}^{s} (+1)^j \sum_{\lambda_k<\lambda}\left(\frac{\lambda_k}{\lambda}\right)^j = O(1).$$

Theorem 2 is proved.

REMARK. A more careful analysis shows that the Theorem 2 remains valid for arbitrary real $s>0$. We shall give applications of Theorem 2 in subsequent papers.

3. At last we estimate the diameter in C-norm of the set of Lip (α, M) functions belonging to a ball of radius δ in the space $L^2(0,1)$.

THEOREM 3. *Let* $0 \le \alpha \le 1$.

$$A_M^\alpha(\delta) := \sup\{\|f\|_{C[0,1]} : f \in \mathrm{Lip}(\alpha, M), \|f\|_{L^2(0,1)} \le \delta\} \asymp c(\alpha, M)\delta^{\frac{2\alpha}{2\alpha+1}}. \tag{6}$$

PROOF. We shall show that if $f \in$ Lip (α, M) with $\|f\|_\infty = \delta$ then $\|f\|_2 \ge c(\alpha+M)\delta^{1+\frac{1}{2\alpha}}$ and for any $\delta>0$ there exists $f_0 \in$ Lip. (α, M) with $\|f_0\|_\infty = \delta$ and $\|f_0\| = c(\alpha, M)\delta^{1+\frac{1}{2\alpha}}$. Indeed, let $f \in$ Lip (α, M), $\|f\|_\infty = \delta$. We can suppose $f(0) = \delta$. Define

$$f_0(t) := \begin{cases} \delta - Mt^\alpha & \text{for } t<t_0, \\ 0 & \text{for } t \ge t_0, \end{cases}$$

where $t_0 = \left(\frac{\delta}{M}\right)^{\frac{1}{\alpha}}$. Now $f \in$ Lip (α, M) implies $|f(t)| \geq f_0(t)$ for all t and hence, as an easy computation shows

$$\|f\|_2^2 \geq \|f_0\|_2^2 = \int_0^{t_0} (\delta - Mt^\alpha)\, dt = \delta^{2+\frac{1}{\alpha}} M^{-\frac{1}{\alpha}} \frac{2\alpha^2}{(1+2\alpha)(1+\alpha)}.$$

Cearly $f_0 \in$ Lip(α, M) and hence the Theorem 3 is proved. □

REMARK 1. Denote for $f_0 \in L^2(0,1)$

$$A_M^\alpha(f_0, \delta) := \sup\{\|f-g\|_{C[0,1]} : f, g \in \text{Lip}(\alpha, M), \|f-f_0\|_2 \leq \delta, \|g-f_0\|_2 \leq \delta\}.$$

Then for large M (depending on f_0) we have

$$0 < A_M^\alpha(f_0, \delta) \asymp c(\alpha, M)\delta^{\frac{2\alpha+1}{2\alpha}}.$$

Indeed, for large M there exists $\tilde{f} \in$ Lip(α, M) with $\|\tilde{f} - f_0\|_2 \leq \delta$ and then by shifting the elements of Lip (α, M) in the neighbourhood $\|f - f_0\|_2 \leq \delta$ by $\tilde{f}$ we get

$$A_M^\alpha(f_0, \delta) \leq A_{2M}^\alpha(2\delta) \leq b(\alpha, M)\delta^{\frac{2\alpha}{2\alpha+1}}.$$

The exactness of this estimate also can be seen just as in case $f_0 = 0$.

REMARK 2. At last we prove that if k man lose n type object and receive (each) n object randomly so that no one get more than $1 \leq l \leq n$ of the same, type, then the probability that noone get his owen tends to e^{-n} $(k \to \infty)$. It seems that this statement is new. B. D. McKAY gave a different proof (privat communication). Denote A the event that one man get his own, $A(I)$ that the i-th man gets own for all $i \in I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, k\}$, then obviously $A = \sum_{r=1}^{k} (-r)^{r-1} \sum_{|I|=r} A(I)$. Denote $\sum_s^*$ resp. $\sum_t^{**}$ the summation over all (s_i^j), $1 \leq i \leq n$, $1 \leq j \leq k$, $0 \leq s_i^j \leq l$, $\sum_k s_k^j = n$, $j = 1, \ldots, k$ resp. (t_i), $l \leq t_i \leq n$, $i = 1, \ldots, r$ further let

$$B := \sum_s^* \sum_t^{**} (k-l_1)! \ldots (k-l_n)! / \prod_{i=1}^{r} (s_{t_i}^{j_i} - 1)! \prod_{\substack{(h,j) \neq (t_i, j_i) \\ (h,j)}} s_h^j!,$$

where the (ordered) system (t_i) contains l_1 times the number $1, \ldots, l_n$ times the number n. Let

$$C = \sum_s^* n^{3r+1} (k!)^n / k \ldots (k-r) \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j!$$

for $r<k$ and $C = 0$ for $r = k$. It is easy to see that $B - C \leq A \leq B$

$$B \geq \sum_s{}^* (k!)^n / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j! \sum_t{}^{**}{}_{k^{-r}} \prod_{i=1}^{r} s_{t_i}^{j_i} = \sum_s{}^* (k!)^n / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j! n^r k^{-r}$$

and

$$B \leq \left[\sum_s{}^* (k!)^n / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j! \right] \cdot \left[(k(k-1)\ldots(k-r+1))^{-1} \cdot \prod_{i=1}^{r} s_{t_i}^{j_i} \right] =$$

$$= \left[\sum_s{}^* (k!)^n n^r / \prod_{j=1}^{k} \prod_{i=1}^{n} s^{i}! \right] \cdot [k(k-1)\ldots(k-r+1)]^{-1},$$

hence

$$\sum_s{}^* (k!)^n / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j! [n^r k^{-r} - n^{3r+1}(k(k-1)\ldots(k-r))^{-1}] \leq A(I) \leq$$

$$\leq \sum_s{}^* (k!)^n n^r / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j! \cdot k(k-1)\ldots(k-r+1).$$

Taking into account that the number of all redistributions is

$$\sum_s{}^* (k!)^n / \prod_{j=1}^{k} \prod_{i=1}^{n} s_i^j!,$$

we obtain

$$\sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \cdot [n^r k^{-r} - n^{3r+1}(k(k-1)\ldots(k-r))^{-1}] \leq 1 - p \leq$$

$$\leq \sum_{r=1}^{k} (-1)^r \binom{k}{r} n^r / k(k-1)\ldots(k-r+1).$$

Since for small r the r-th member of both sums is nearly $(-1)^{r-1} n^r / r!$, hence $p \sim 1 + \sum_{r=1}^{k} (-1)^r n^r / r! \sim e^{-n}$, the sign becomes equality as $k \to \infty$. □

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ON THE UNIFORM RIESZ SUMMABILITY OF EGIENFUNCTION EXPANSIONS I*

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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Let Ω be an arbitrary domain in $\mathbf{R}^N (N>3)$ having C^∞ –smooth boundary. Let q be an arbitrary "potential function from the class $L_2(\Omega)$ such that $q \geq 0$ in Ω and

$$|D^\alpha q(x)| \leq C_\alpha \omega(|x-x_0|)/|x-x_0|^{2+|\alpha|} \quad (x_0 \in \Omega) \tag{1}$$

for some ω, satisfying

$$\int_{+0} \frac{\omega(t)}{t}\, dt < \infty. \tag{2}$$

Let $1 \leq m \leq N/4$ be arbitrary natural number and consider the operator

$$L = L(x, D) = (-\Delta + q(x)\cdot)^m.$$

Denote L an arbitrary non–negative selfadjoint extension of L from the domain $C_0^\infty(\Omega)$ with discrete spectrum. According to a theorem of K. O. FRIEDRICHS [1], [2] there exists such a selfadjoint extension. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \leq \ldots$ denote the sequence of the eigenvalues of the operator L and let $(U_n)_1^\infty$ be the complete orthonormal system of the corresponding eigenfunctions in $L_2(\Omega)$. In the case $m \geq 2$ assume that

$$\sum_{\lambda_k < \lambda}{}_k\, 1 = (O\lambda^\gamma) \tag{3}$$

holds with some $\gamma = \gamma(m) > 0$.

For any $s \geq 0$ and $f \in L_2(\Omega)$ consider the $s-th$ Riesz means of the spectral expansion of f:

$$E_\lambda^s f(x) := \sum_{\lambda_k < \lambda}{}_k \left(1 - \frac{\lambda_k}{\lambda}\right)^s (f, U_k) U_k(x).$$

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We shall prove the following theorems.

THEOREM 1. *Let* $p \geq 1$, $s \geq 0$, $l > 0$, $l+s \geq (N-1)/2$, $pl > N$. *Then for any* $f \in \mathring{H}_p^l(\Omega)$ $\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x)$ *uniformly in* x *on every compact subset of* Ω.

THEOREM 2. *Let* $s \geq 0$, $l \geq 0$, $l+a \geq (N-1)/2$. *Then for any*

$$f \in \mathring{H}_2^l(\Omega), \ \lim_{\lambda \to \infty} E_\lambda^s f(x) = 0$$

uniformly on every compact subset of $\Omega \setminus \operatorname{supp} f$.

These theorems generalize that of [9]–[14].

We need some premilinary results given in next sections.

1. Estimation of the Green function

Define

$$Z := \left\{ z \in C : \left| \arg z - \frac{\pi}{2} \right| \leq \frac{\pi}{2} - \varepsilon, \ \varepsilon > 0 \text{ fixed} \right\},$$

$$B := \{ x \in \mathbf{R}^N : |x - x_0| < R \},$$

where $0 < R < \frac{1}{3} \operatorname{dist}(x_0, \partial\Omega) =: R^*$.

Denote $G := G(x, y, \mu)$ the Green function of $\hat{L}_\mu := \hat{L} - \mu^2 I$, i.e. the kernel function of the operator $(\hat{L} - \mu^2 I)^{-1}$. We shall prove the

THEOREM 1.1. *We have*

$$|G(x, y, \mu)| \leq C |x-y|^{2m-N} e^{-\alpha|\mu||x-y|}, \tag{1.0}$$

uniformly in $\mu \in Z$ *and* $x, y \in B$. *Here* $C, \alpha > 0$ *are some constants independent of* x, y, *and* μ.

PROOF. It is easy to see that

$$(-\Delta + q(x))^m = (-\Delta)^m \sum_{|\alpha| \leq 2m-2} C_{m,\alpha}(x) D^\alpha, \tag{1.1}$$

where (assume $x_0 = 0$)

$$|D^\beta C_{m,\alpha}(x)| \leq C_1 \omega(|x|) |x|^{|\alpha|-2m-|\beta|}, \ (0 \leq |\beta| \leq N). \tag{1.2}$$

It is known (HÖRMANDER [6]) that for the fundamental solution $E_0(x, y, \mu)$ of the operator $(-\Delta)^m - \mu^{2m} I$. which decreases exponentially, the estimate

$$|D_x^\alpha E_0(x, y, \mu)| \leq C_2 |x-y|^{2m-N-|\alpha|} e^{-2\gamma|x-y||\mu|}, \tag{1.3}$$

$$(x, y \in B, \ \mu \in Z)$$

holds. Hence we are looking for $E(x, y, \mu)$ in the form

$$E(x, y, \mu) = E_0(x, y, \mu) - \int_B E_0(x, u, \mu) h(u, y, \mu) \omega(|u|) |u|^{-2m} du. \tag{1.4}$$

Then for the function $h(x, y, \mu)$ we get the integral equation:

$$(1.5)\quad h(x, y, \mu) = F_0(x, y, \mu) + \int_\Omega F_0(x, u, \mu) h(u, y, \mu) \omega(|u|) |u|^{-2m} du,$$

where

$$F_0(x, y, \mu) = -\frac{|x|^{2m}}{\omega(|x|)} \sum_{|\alpha| \leq 2m-2} D_x^\alpha [C_{m,\alpha} E_0(x, y, \mu)].$$

Indeed, if we define

$$(\hat{\Phi} f)(x) := \int_B \Phi(x, y) f(y) dy,$$

then

$$(\hat{E} f, L_\mu \varphi) = \int_B \int_B f(y) \Big[E_0(x, y, \mu) -$$

$$- \int_B E_0(x, u, \mu) h(u, y, \mu) \omega(|u|) |u|^{-2m} du \Big] L_\mu \varphi(x) dx,$$

$$L_\mu \varphi = [(-\Delta + q)^m - \mu^{2m}] \varphi = ((-\Delta)^m - \mu^{2m}) \varphi + \sum_{|\alpha| \leq 2m-2} C_{m,\alpha} D^\alpha,$$

$$[(-\Delta)^m - \mu^{2m}] E_0(x, y, \mu) = \delta(x - y),$$

$$(\hat{E} f, L_\mu \varphi) = (f, \varphi) + \int_B \int_B f(y) E_0(x, y, \mu) \sum_{|\alpha| \leq 2m-2} C_{m,\alpha} D^\alpha \varphi(x) dy dx -$$

$$- \int_B \int_B h(x, y, \mu) \omega(|x|) |x|^{-2m} f(y) \varphi(x) dx dy -$$

$$- \int_B \int_B \int_B E_0(x, u, \mu) h(u, y, \mu) \omega(|u|) |u|^{-2m} du \cdot$$

$$\cdot \sum_{|\alpha| \leq 2m-2} C_{m,\alpha} D^\alpha \varphi dx f(y) dy =$$

$$= (f, \varphi) + \int_B \int_B f(y) \varphi(x) \{ \sum_{|\alpha| \leq 2m-2} D_x^\alpha (C_{m,\alpha} E_0(x, y, \mu)) -$$

$$- h(x, y, \mu) \omega(|x|) |x|^{-2m} - \int_B [\sum_{|\alpha| \leq 2m-2} D_x^\alpha (C_{m,\alpha} E_0(x, u, \mu))] \cdot$$

$$\cdot h(u, y, \mu) \omega(|u|) |u|^{-2m} du \}.$$

Hence the equation (1.5) for $h(x, y, \mu)$ follows.

Define for $k = 1, 2, \ldots$

$$(1.6)\qquad F_k(x, y, \mu) := \int_B F_0(x, u, \mu) F_{k-1}(u, y, \mu) \omega(|u|) |u|^{-2m} du.$$

Obviously

$$h(x, y, \mu) = \sum_{k=0}^{\infty} F_k(x, y, \mu), \tag{1.7}$$

if the series converges uniformly. For the proof of the uniform convergence of the series in (1.7) we need some lemmas.

Define

$$a(R) := \int_B \frac{\omega(|u|)}{|u|^N} du,$$

$$I := \int_B |x-y|^{2m-N} |u-y|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du,$$

$$I^* := \int_B |x-u|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du,$$

$$r := |x-y|.$$

LEMMA 1.1. We have

$$I \leq 4a(R) r^{2m-N}, \tag{1.8}$$

$$I^* \leq 2a(R), \tag{1.9}$$

uniformly in $x, y \in B$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$.

PROOF. Consider the sets

$$B(x, y) := B \cap \{u \in B : |x-u| \leq |y-u|\}.$$

Obviously

$$B(x, 0) \cup B(y, x) = B, \qquad B(0, x) \cup B(x, 0) = B.$$

Now estimate I^*. We have

$$\int_{B(0,\, x)} |x-u|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du \leq \int_B \frac{\omega(|u|)}{|u|^N} du = a(R),$$

further, obviously,

$$\frac{\omega(|u|)}{|u|} \leq \frac{\omega(|x-u|)}{|x-u|} \qquad (u \in B(x, 0)),$$

$$B(x, 0) \subset \{u \in \mathbf{R}^N : |x-u| < R\} =: B',$$

consequently

$$\int_{B(x,0)} |x-u|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du \leq \int_{B(x,0)} \frac{\omega(|x-u|)}{|x-u|^N} du \leq$$

$$\leq \int_{B'} \frac{\omega(|x-u|)}{|x-u|^N} du = a(R),$$

and hence (1.9) follows.

Estimate I. Obviously

$$r = |x-y| \leq |x-u| + |u-y| \leq 2|u-y| \quad (u \in B(x,y)),$$

$$r = |x-y| \leq |x-u| + |u-y| \leq 2|x-u| \quad (u \in B(y,x)),$$

consequently, applying (1.9) (which was proved above) we get

$$\int_{B(x,0)} |x-u|^{2m-N}|y-u|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du \leq \left(\frac{2}{r}\right)^{N-2m} I^*(x) \leq 2\left(\frac{2}{r}\right)^{N-2m} a(R),$$

$$\int_{B(y,x)} |x-u|^{2m-N}|y-u|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du \leq$$

$$\leq \left(\frac{2}{r}\right)^{N-2m} I^*(y) \leq 2\left(\frac{2}{r}\right)^{N-2m} a(R),$$

and hence (1.8) follows.

Now we prove that the series in (1.7) converges uniformly in $x, y \in B$ and $\mu \in Z$. For this we must estimate the functions F_k. Define

$$f_k(x,y,\mu) := e^{|\operatorname{Im}\mu|\cdot|x-y|} F_k(x,y,\mu).$$

Obviously

$$|f_k(x,y,\mu)| \leq C \int_B |x-u|^{2m-N} |f_{k-1}(u,y,\mu)| \frac{\omega(|u|)}{|u|^{2m}} du. \tag{1.10}$$

Using (1.3) we prove by induction

$$|f_k(x,y,\mu)| \leq [c_0 a(R)]^k |x-y|^{2m-N} (x,y \in B,\ \mu \in Z). \tag{1.11}$$

For $k = 0$ (1.11) follows from the definition of the functions f_0 and F_0, taking into account (1.3). Now using (1.10) and the induction hypothesis we get

$$|f_k(x,y,\mu)| \leq C \int_B |x-u|^{2m-N} |f_{k-1}(x,y,\mu)| \frac{\omega(|u|)}{|u|^{2m}} du \leq$$

$$\leq C[c_0 a(R)]^{k-1} \int_B |x-u|^{2m-N} |u-y|^{2m-N} \frac{\omega(|u|)}{|u|^{2m}} du.$$

Hence (1.11) follows from (1.8).

It follows from (1.11) that the series in (1.7) converges absolutely and uniformly in $x, y \in B$ and $\mu \in Z$ if R (the radius of B) is choosen so that $C_0 a(R) < 1/2$.

Let $R_0 = \sup\{R : C_0 a(R) > 1/2\}$, $R_1 := \min\{R_0, \operatorname{dist}(0, \partial\Omega)\}$ From (1.7) and (1.11) we obtain the

LEMMA 1.2. We have

$$|E(x, u, \mu)| \leq C|x-y|^{2m-N} e^{-|\operatorname{Im}\mu|\cdot|x-y|} \tag{1.12}$$

uniformly in $x, y \in B$ and $\mu \in Z$, $0 < R < R_1$.

Now estimate the grandient of the fundamental solution.

LEMMA 1.3. We have

$$|\nabla E(x, y, \mu)| \leq C e^{-|\operatorname{Im}\mu|\cdot|x-y|} |x-y|^{2m-N} \left\{ \frac{1}{|x-y|} + \frac{\omega^{\varepsilon}(|x|) + \omega^*(|x|)}{|x|} \right\}, \tag{1.13}$$

$$0 \leq \varepsilon < 1,$$

where

$$\omega^*(t) = \int_0^t \frac{\omega(\tau)}{\tau} d\tau.$$

PROOF. According to (1.4).

$$e^{|\operatorname{Im}\mu||x-y|} |\nabla E(x, y, \mu| \leq C|x-y|^{2m-N} +$$

$$+ \int_{|u| \leq R} \frac{1}{|x-u|^{N-2m+1}} \cdot \frac{1}{|u-y|^{N-2m}} \cdot \frac{\omega(|u|)}{|u|^{2m}} du.$$

Denote I the last integral and estimate it:

$$I = \int_{\substack{|\mu| \leq R \\ |x-u| \geq \frac{1}{2}|x-y|}} + \int_{\substack{|u| \leq R \\ |x-u| \leq \frac{1}{2}|x-y|}} = I_1 + I_2.$$

Obviously

$$I_1 \leq |x-y|^{2m-1-N} \int_{|u| \leq R} \frac{1}{|u-y|^{N-2m}} \cdot \frac{\omega(|u|)}{|u|^{2m}} du \leq C|x-y|^{2m-1-N}.$$

Now estimate I_2. If $|x-u| \leq \frac{1}{2}|x-y|$, then $|u-y| \geq |x-y| - |u-x| \geq$ $\geq \frac{1}{2}|x-u|$, henceforth

$$I_2 \leq |x-y|^{2m-N} \int_{|u| \leq R} |x-u|^{2m-1-N} \frac{\omega(|u|)}{|u|^{2m}} du =$$

$$= |x-y|^{2m-N}\left[\int\limits_{|u|\leq\frac{1}{2}|x|} + \int\limits_{\frac{1}{2}|x|\leq|u|\leq R}\right] = |x-y|^{2m-N}[I_2'+I_2''].$$

If $|u|\leq\frac{1}{2}|x|$, then $|x-u|\geq|x|-|u|\geq\frac{1}{2}|x|$, hence

$$I_2'\leq c\frac{1}{|x|}\int\limits_{|u|\leq\frac{1}{2}|x|}\frac{1}{|x-u|^{N-2m}}\cdot\frac{\omega(|u|)}{|u|^{2m}}du\leq C|x|^{-1}\int\limits_0^{|x|}\frac{\omega(t)}{t}dt.$$

Further we have

$$I_2''\leq\int\limits_{\frac{|x|}{2}\leq|u|\leq R}\frac{1}{|x-u|^{N-2m+1}}\cdot\frac{\omega(|u|)}{|u|^{2m}}du\leq$$

$$\leq\frac{\omega^\varepsilon(|x|)}{|x|^\varepsilon}\int\limits_{\frac{|x|}{2}\leq|u|\leq R}\frac{du}{|x-u|^{N-2m+1}|u|^{2m-\varepsilon}}\leq$$

$$\leq\frac{\omega^\varepsilon(|x|)}{|x|^\varepsilon}\left[\int\limits_{\frac{|x|}{2}\leq|u|\leq R}\frac{du}{|u|^{N+1-\varepsilon}}+\int\limits_{\substack{\frac{|x|}{2}\leq|u|\leq R\\|u|\geq|x-u|}}\frac{du}{|x-u|^{N+2m-1}\cdot|u|^{2-\varepsilon}}\right]\leq$$

$$\leq\frac{\omega^\varepsilon(|x|)}{|x|^\varepsilon}\left\{\int\limits_{\frac{|x|}{2}}^{R}t^{-2+\varepsilon}dt+\int\limits_{\frac{|x|}{2}\leq|x-u|\leq R}|x-u|^{-N-1+\varepsilon}du+\right.$$

$$\left.+|x|^{\varepsilon-2}\int\limits_{|x-u|\leq\frac{|x|}{2}}|x-u|^{1-N}du\right\}\leq C\frac{\omega^\varepsilon(|x|)}{|x|^\varepsilon}\left\{1+|x|^{-1+\varepsilon}+|x|^{\varepsilon-2}\int\limits_0^{|x|}dt\right\}=$$

$$=C[\omega^\varepsilon(|x|)/|x|^\varepsilon][1+|x|^{\varepsilon-1}]\leq C\omega^\varepsilon(|x|)/|x|,$$

where $0\leq\varepsilon<1$ is an arbitrary fixed number.

Define

$$E^*(x,y,\mu):=E(y,x,\mu).$$

The equation

$$\int\limits_\Omega E^*(x,y,\mu)\hat{L}_\mu(y,D)f(y)dy=f(x)\qquad(x\in B)\tag{1.14}$$

holds for every $f\in\mathcal{D}(\hat{L})$ with supp $f\subset B$. Now let $\eta\in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ for $x\in\Omega_0\subset B$.

Let

$$H(x,y,\mu) = E^*(x,y,\mu)\cdot\eta(y),$$

$$K(x,y,\mu) = \sum_{0\le|\alpha|\le 2m}\frac{1}{\alpha!}D^\alpha\eta(y)L^\alpha(y,D)E^*(x,y,\mu),$$

where $\alpha!:=\alpha_1!\ldots\alpha_N!$,

We have

$$(L(y,D)-\mu^{2m})H(x,y,\mu) = \eta(y)[L(y,D)-\mu^{2m}]E^*(x,y,\mu)+K(x,y,\mu).$$

If $y\in\Omega_0$, then $K(x,y,\mu)=0$, hence

$$|K(x,y,\mu)|\le Ce^{-\alpha|\mu|} \tag{1.15}$$

uniformly in $x\in\Omega_0$, $y\in\Omega$, $\mu\in Z$.

It follows from (1.14) that for every $f\in\mathcal{D}(\hat{L})$

$$\hat{H}\,\hat{L}_\mu f(x) = [f-\hat{K}f](x), \quad (x\in\Omega_0).$$

On the other hand, for any $f\in C_0^\infty(\Omega)$

$$(\hat{L}-\mu^2 I)^{-1}f = \hat{L}_\mu^{-1}f\in\mathcal{D}(\hat{L}).$$

Consequently

$$\hat{H}f(x) = \hat{G}_\mu f(x)-\hat{K}\,\hat{G}_\mu f(x), \quad f\in C_0^\infty(\Omega), \quad (x\in\Omega_0).$$

Extending this equality for every $f\in L_2(\Omega)$ we obtain the following:

LEMMA 1.4. For every $f\in L_2(\Omega)$ and $x\in\Omega_0$

$$\hat{G}_\mu f(x)-\hat{H}f(x) = \hat{K}\,\hat{G}_\mu f(x). \tag{1.16}$$

Proof of the Theorem 1.1. The Theorem 1.1 follows from Lemma 1.4, 1.2 and from (1.15) immediately.

2. The mean value formula

Suppose the function $\varphi(x,\mu)$ satisfies the equation

$$[-\Delta+q(x)]^m\varphi(x,\mu)-\mu^{2m}\varphi(x,\mu) = 0 \tag{2.1}$$

where $q(x)$ satisfies (1), (2), $\mu>0$, m a natural number.

DEFINITION. Let l be an arbitrary natural number. We say that $g(x,\mu)\in M^l(\Omega)$ if

1. $g(x,\mu)$ is measurable in $\mu\ge 1$ and $x\in\Omega$,
2. for every compact set $K\subset\Omega$ there exists a constant $C(K)>0$ such that $|g(x,\mu)|\le C(K)\mu^l$ $(x\in K,\ \mu\ge 1)$.

Let $M^{\infty}: = \bigcup_{l} M^{l}(\Omega)$.

DEFINITION. We say $g(x, \mu) \in E(\Omega)$ if

1. $g(x, \mu)$ is measurable for $\mu \geq 1$ and $x \in \Omega$.
2. for every compact set $K \subset \Omega$ there exists $C(K) > 0$ and $\beta = \beta(K)$, $0 < \beta < \operatorname{dist}(\partial k, \partial\Omega)$ such that

$$|g(x, \mu)| \leq C(K) e^{-\beta\mu} \quad (x \in K, \ \mu \geq 1).$$

For arbitrary complex numbers a, b and for any p, $r \geq 0$ set

$$W_p(a, b): = J_\nu(a) Y_p(b) - J_p(b) Y_\nu(a), \qquad \nu = \frac{N-2}{2}; \tag{2.2}$$

$$S_r g(x, \mu): = \int_{\Theta} g(x + r\Theta, \mu) d\Theta; \ \tilde{S}_r g(x, \mu): = \int_0^r t^{N-1} S_t g(x, \mu) dt; \tag{2.3}$$

$$I_\mu^{r, p} g(x, \mu): = \int_0^r W_p(t\mu, r\mu) S_t g(x, \mu) t^{N/2} dt. \tag{2.4}$$

We shall prove the

THEOREM 2.1. *Let $x \in K \subset \Omega$ and $0 < r < \operatorname{dist} x(, \partial\Omega)$. Then for any normed in $L_2(\Omega)$ solution $\varphi(x, \mu)$ of the equation* (2.1)

$$S_r \varphi(x, \mu) = (2\pi)^{N/2} \frac{J_\nu(r\mu)}{(r\mu)^\nu} \varphi(x, \mu) + \frac{\pi}{2} r^{-\nu} I_\mu^r (q\varphi + \psi)(x, \mu) \tag{2.5}$$

holds, where $\Psi(x, \mu) \in E(\Omega)$ and $\Psi(x, \mu) = 0$ if $m = 1$.

PROOF. Fix an arbitrary compact set K and a number $R > 0$ such that $0 < 2R < \operatorname{dist}(\partial k, \partial\Omega)$. First we prove the theorem for the case $m = 1$. To this consider the function

$$F_p(x, y): = \frac{J_p(|x-y|\sqrt{\lambda})}{|x-y|^P} Y_\nu(R\sqrt{\lambda}) - \frac{Y_P(|x-y|\sqrt{\lambda})}{|x-y|^P} J_\nu(R\sqrt{\lambda}).$$

An easy calculation shows that

$$(\Delta + \lambda) F_\nu = 0, \quad (x \neq y); \tag{2.6}$$

$$\frac{\partial F_\nu}{\partial r} = -\sqrt{\lambda} r F_{\nu+1}(r);$$

further

$$\lim_{r \to 0} r^{2\nu+1} \frac{\partial F_\nu}{\partial r} = -\lim_{r \to 0} \sqrt{\lambda} r^{\nu+1} Y_{\nu+1}(r\sqrt{\lambda}) J_\nu(R\sqrt{\lambda}) =$$

$$= \sqrt{\lambda} J_\nu(R\sqrt{\lambda}) \lim_{r \to 0} r^{\nu+1} Y_{\nu+1}(r\sqrt{\lambda}) = C \frac{J_\nu(R\sqrt{\lambda})}{(\sqrt{\lambda})^\nu}.$$

First consider the case $x \neq 0$. Then

$$\int\limits_{\substack{|x-y|\leq R\\ |y|\geq\varepsilon_1\\ |x-y|\geq\varepsilon_2}} (\varphi\Delta F_\nu - F_\nu\Delta\varphi)dy = \int\limits_{|x-y|=R} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds -$$

$$- \int\limits_{|y|=\varepsilon_1} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds - \int\limits_{|x-y|=\varepsilon_2} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds =$$

$$= \frac{\partial F_\nu}{\partial r}\bigg|_{r=R} S_R\varphi - \int\limits_{|x-y|=\varepsilon_2} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds -$$

$$- \int\limits_{|y|=\varepsilon_1} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds.$$

Taking into consideration

$$\left|\frac{\partial\varphi}{\partial n}\right| \leq C\frac{\omega(|x|)}{|x|}, \qquad (x\neq 0),$$

we have

$$\lim_{\varepsilon_2\to 0} \int\limits_{|x-y|=\varepsilon_2} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds = C\frac{J_\nu(R\sqrt{\lambda})}{(\sqrt{\lambda})^\nu}\varphi(x)\omega \tag{2.8}$$

further

$$\lim_{\varepsilon_1\to 0} \int\limits_{|y|=\varepsilon_1} \left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds = 0, \tag{2.7}$$

because

$$\left|\frac{\partial F_\nu}{\partial n}\right| \leq C, \quad |F_\nu| \leq C, \quad \left|\frac{\partial\varphi}{\partial n}\right| \leq C\frac{\omega(|x|)}{|x|}.$$

Taking the limits $\varepsilon_1, \varepsilon_2 \to 0$ and using (2.1) we get for $m = 1$

$$\int\limits_{|x-y|\leq R} F_\nu(x, y)q(y)\varphi(y, \mu)dy = \tag{2.9}$$

$$= -\sqrt{\lambda}\,RF_{\nu+1}(R)R^{2\nu+1}C\frac{J_\nu(R\sqrt{\lambda})}{(\sqrt{\lambda})^\nu}\varphi(x, \mu).$$

On the other hand

$$\sqrt{\lambda}R^{2\nu+2}F_{\nu+1}(R) =$$

$$= [J_{\nu+1}(R\sqrt{\lambda})Y_\nu(R\sqrt{\lambda}) - Y_{\nu+1}(R\sqrt{\lambda})J_\nu(R\sqrt{\lambda})]\sqrt{\lambda}R^{2\nu+2-\nu-1} = \frac{2}{\pi}R^\nu,$$

consequently

$$\int_\Theta \varphi(x+R\Theta)d\Theta = -\omega_N C\frac{\pi}{2}\frac{J_\nu(R\sqrt{\lambda})}{(R\sqrt{\lambda})^\nu}\varphi(x,\mu) + \tag{2.10}$$

$$+\frac{\pi}{2}R^{-\nu}\int_0^R W_\nu(t\sqrt{\lambda}, R\sqrt{\lambda})t^{N/2}\cdot\int_\Theta q(x+t\Theta)\varphi(x+t\Theta,\mu)d\Theta dt.$$

Using

$$J_{\nu+1}(Z)Y_\nu(Z) - Y_{\nu+1}(Z)J_\nu(Z) = \frac{2}{\pi Z},$$

we get

$$C = \lim_{Z\to 0} Z^{\nu+1}Y_{\nu+1}(Z) = \lim_{Z\to 0}\left\{-\frac{Z}{J_\nu(Z)}\cdot\frac{2}{\pi} + \frac{Z^{\nu+1}J_{\nu+1}(Z)}{J_\nu(Z)}Y_\nu(Z)\right\} =$$

$$= -\frac{\Gamma(\nu+1)}{2^{-\nu}}\cdot\frac{\pi}{2} = \frac{2^{\nu+1}\Gamma(\nu+1)}{\pi},$$

hence $\frac{1}{2}\omega_N C\pi = (2\pi)^{N/2}$.

Thus (2.5) is proved for $m = 1$ and $x \neq 0$ (when $\Psi = 0$). Now let $x = 0$. In this case the method used above gives the equality

$$\int_{\varepsilon\leq|y|\leq R}(\varphi\Delta F_\nu - F_\nu\Delta\varphi)dy = \int_{|y|=R}\left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds -$$

$$-\int_{|y|=\varepsilon}\left(\varphi\frac{\partial F_\nu}{\partial n} - F_\nu\frac{\partial\varphi}{\partial n}\right)ds.$$

Obviously,

$$\int_{|y|=R}\left(\varphi\frac{\partial F\nu}{Fn} - F\nu\frac{\partial\varphi}{\partial n}\right)ds = -\frac{2}{\pi}R^\nu\int_\Theta\varphi,$$

further

$$\left|\frac{\partial\varphi}{\partial n}\right| \leq \frac{C}{|y|}, \quad |F_\nu| \leq \frac{C}{|y|^{2\nu}},$$

henceforth

$$\lim_{\varepsilon\to 0} \int\limits_{|y|=\varepsilon} F_\nu \frac{\partial\varphi}{\partial n} ds = 0.$$

According to the continuity of $\varphi(x, \mu)$ we have

$$\lim_{\varepsilon\to 0} \int\limits_{|y|=\varepsilon} \varphi \frac{\partial F_\nu}{\partial n} ds = (2\pi)^{N/2} \frac{2}{\pi} \varphi(0)$$

and the Theorem 2.1 is proved in the case of $m = 1$. Now we prove it for $m \geq 2$. To this we first prove some lemmas.

Let α be any complex number such that $|\alpha| = 1$ and $\operatorname{Im} \alpha \neq 0$ and set

$$K_\mu(x, y) := \frac{(\mu\alpha)^\nu W_{\nu+1}(|x-y|\mu\alpha, R\mu\alpha) q(y)}{2^{\nu+2}\pi^\nu J_{\nu+1}(R\mu\alpha)|x-y|^\nu}, \tag{2.11}$$

$$T(\mu) := \max_{x\in K} \int\limits_{|x-y|\leq R} |K_\mu(x, y)| dy, \quad \mu > 0. \tag{2.12}$$

We shall prove the

LEMMA 2.1. There exists a positive constant C such that for $\mu > R^{-1}$

$$T(\mu) \leq C[\omega(\mu^{-1}) + \omega^*(\mu^{-1})]$$

holds, where ω^* is defined in the Lemma 2.3.

PROOF. According to the definition of q and that of K_μ in (2.11) we have

$$\int\limits_{|x-y|\leq R} |K_\mu(x, y)| dy \leq C\mu^\nu |J_{\nu+1}(R\mu\alpha)|^{-1} \cdot$$

$$\cdot \int\limits_{|x-y|\leq R} |W_{\nu+1}(|x-y|\mu\alpha, R\mu\alpha)| \cdot |x-y|^{-\nu} q(y) dy =$$

$$= C\mu^\nu |J_{\nu+1}(R\mu\alpha)|^{-1} \left[\int\limits_{|x-y|\leq\mu^{-1}} + \int\limits_{\mu^{-1}\leq|x-y|\leq R} \right] =$$

$$= C\mu^\nu |_{\nu+1}(R\mu\alpha)|^{-1} [I_1 + I_2].$$

Using

$$|J_{\nu+1}(R\mu\alpha)| = \frac{1}{\sqrt{2\pi R\mu}} e^{|\operatorname{Im}\alpha|R\mu}\left[1+O\left(\frac{1}{R\mu}\right)\right], \tag{2.13}$$

$$|Y_{\nu+1}(R\mu\alpha)| = \frac{1}{\sqrt{2\pi R\mu}} e^{|\operatorname{Im}\alpha|R\mu}\left[1+O\left(\frac{1}{R\mu}\right)\right], \tag{2.14}$$

we obtain

$$I_1 \leq C\mu^{-1/2}e^{|\operatorname{Im}\alpha|R\mu} \int\limits_{|x-y|\leq\mu^{-1}} [\mu^{\nu}|y|^{-2}\omega(y)+\mu^{-\nu}|x-y|^{-2\nu}|y|^{-2}\omega(y)]dy \leq$$

$$\leq C\mu^{-1/2}e^{|\operatorname{Im}\alpha|R\mu}\int\limits_0^{\mu^{-1}} [\mu^{\nu}t^{2\nu-1}\omega(t)+\mu^{-1}t^{-1}\omega(t)dt \leq$$

$$\leq C\mu^{-1/2-\nu}[\omega(\mu^{-1})+\omega^*(\mu^{-1})]e^{|\operatorname{Im}\alpha|R\mu}.$$

Now estimate I_2. Using formulas (7), (8) on p. 12 and (1), (2) on p. 98 of the book [5] we obtain

$$|W_{\nu+1}(|x-y|\mu\alpha, R\mu\alpha)| \leq C\mu^{-1}|x-y|^{-1/2}e^{(R-|x-y|)|\operatorname{Im}\alpha|\mu}, \tag{2.15}$$
$$(\mu^{-1}\leq|x-y|\leq R).$$

Hence, taking into account the equality

$$\int\limits_0^{\infty} e^{-at}t^{b-1}dt = \Gamma(b)a^{-b} \quad (a>0,\ b>0)$$

($\Gamma(b)$ – is the Gamma function of Euler) we get

$$I_2 \leq C\mu^{-1}e^{|\operatorname{Im}\alpha|R\mu} \int\limits_{\mu^{-1}\leq|x-y|\leq R} e^{-|x-y||\operatorname{Im}\alpha|\mu}|x-y|^{-\nu-1/2}|y|^{-2}\omega(y)dy \leq$$

$$\leq C\mu^{-1/2}e^{|\operatorname{Im}\alpha|R\mu}\omega(\mu^{-1})\int\limits_0^{\infty} e^{-|\operatorname{Im}\alpha|\mu t}t^{\nu-1}dt \leq C\mu^{-1/2-\nu}\omega(\mu^{-1})e^{|\operatorname{Im}\alpha|R\mu}$$

i.e. for $R\mu>1$ we have

$$|I_1+I_2| \leq C\mu^{-1/2-\nu}[\omega(\mu^{-1})+\omega^*(\mu^{-1})]e^{|\operatorname{Im}\alpha|R\mu}. \tag{2.16}$$

Using (2.13) and (2.16)

$$T(\mu) \leq C[\omega(\mu^{-1})+\omega^*(\mu^{-1})]$$

follows.

LEMMA 2.2. Let α be an arbitrary complex number such that $|\alpha| = 1$ $\operatorname{Im} \alpha \neq 0$. Let $\varphi(x, \mu)$ be a solution of

$$[-\Delta + q(x) \cdot + \alpha^2 \mu^2 \cdot]\varphi(x, \mu) = \Psi(x, \mu) \ (x \in \Omega) \tag{2.17}$$

in the space $L_2^{\mathrm{loc}}(\Omega)$, where $\Psi(x, \mu) \in E(\Omega)$. Suppose for every compact set $K \subset \Omega$ there exists a number $l \geq 0$ such that

$$\int_K |\varphi(x, \mu)|^2 dx = O(\mu^l) \quad (\mu \geq 0). \tag{2.18}$$

Then $\varphi(x, \mu) \in E(\Omega)$.

PROOF. Fix arbitrary compact sets $K_0 \subset \Omega$, $K_2 \subset \Omega$ with $K_0 \subset\subset K_2$ and a number $R > 0$ and define

$$K_1 := \{y \in \Omega : \operatorname{dist}(y, K_0) \leq 2R\} \subset K_0 \subset \Omega.$$

Then for any solution $\varphi(x, \mu)$ of the equation (2.17)

$$\tilde{S}_R \varphi(x, \mu) = (2\pi)^{N/2} \frac{J_{\nu+1}(R\mu\alpha) R^{\nu+1}}{(\mu\alpha)^{\nu+1}} \varphi(x, \mu) +$$

$$+ \frac{\pi}{2} \cdot \frac{R^{\nu+1}}{\mu\alpha} I_{\mu\alpha}^{R, \nu+1}(q\varphi + \psi)(x, \mu),$$

i.e.

$$\varphi(x, \mu) = f(x, \mu) + \hat{P}\varphi(x' \mu), \ (x \in K_0), \tag{2.19}$$

where

$$f(x, \mu) := \left(\frac{\mu\alpha}{2\pi R}\right)^{\nu+1} [J_{\nu+1}(R\mu\alpha)]^{-1} \left[\tilde{S}_R \varphi - \frac{\pi}{2} \frac{R^{\nu+1}}{\mu\alpha} I_{\mu\alpha}^{R, \nu+1}(\psi)\right](x, \mu),$$

$$\hat{P}\varphi(x, \mu) := \int_{K_1} P(x, y)\varphi(y, \mu) dy,$$

$$P(x, y) := \begin{cases} K_\mu(x, y) & \text{if } x \in K_0 \cap y \in V_R(x) \\ 0 & \text{if } x \notin K_0 \cup y \notin V_R(x) \end{cases},$$

$$V_R(x) := \{y : |x - y| \leq R\}$$

and $K_\mu(x, y)$ is defined in (2.11).

It is easy to see that for the integral operator $\hat{P}$: $\hat{P}(C(K_1)) \subset C(K_0)$, hence for any $f \in C(\Omega)$ and for $j = 0, 1, \ldots$

$$\|\hat{P}^j f\|_{C(K_0)} \leq T^j(\mu) \|f\|_{C(K_1)}.$$

It follows from Lemma 2.1 that $T(\mu)<1/2$ if $\mu\geq\mu_0$, hence for the solutoin $\varphi(x, \mu)$ of the Fredholm type integral equation (of second order)

$$\varphi(x, \mu) = \sum_{j=0}^{\infty} (\hat{P}^j f)(x, \mu)$$

absolutely and uniformly in $x\in K_0$, consequently

$$\|\varphi\|_{C(K_0)}\leq\|f\|_{C(K_1)}. \tag{2.20}$$

According to $\Psi\in E(\Omega)$, there exist positive constants $C(K_2)$ and β, with $0<\beta<R|\operatorname{Im}\alpha|<\operatorname{dist}(\partial K_0, \partial\Omega)$ such that

$$\|\psi\|_{C(K_2)}\leq C(K_2)e^{-\beta\mu}.$$

Hence, similarly to the proof of (2.16) we obtain

$$\|R^{\nu+1}I^{R,\nu+1}_{\alpha\mu}(\psi)\|_{C(K_1)}\leq C\mu^{-\nu-1/2}e^{|\operatorname{Im}\alpha|R\mu-\beta\mu}.$$

Consequently, according to the assumption (2.18) and (2.13) there exists $l\geq 0$ such that

$$\|f\|_{C(K_1)}\leq C\mu^{\nu+3/2}e^{-|\operatorname{Im}\alpha|R\mu}[\mu^l+\mu^{-\nu-3/2}e^{|\operatorname{Im}\alpha|R\mu-\beta\mu}]\leq Ce^{-\beta\mu}.$$

Hence, taking into account (2.20) we get

$$\|\varphi\|_{C(K_0)}\leq C(K)e^{-\beta\mu}, \text{ i.e. } \varphi(x. \mu)\in E(\Omega). \quad \square$$

LEMMA 2.3. Let α, $|\alpha|\leq 2m-2$ be an arbitrary multiindex and $\varphi(x, \mu)\in L_2(\Omega)$ be an arbitrary solution of (2.1). Then there exists $l>0$ such that

$$\int_K |D^\alpha_x\varphi(x, \mu)|^2dx = O(\mu^l) \quad (\mu\geq 1)$$

uniformly on every compact set $K\subset\Omega$.

PROOF. Fix any compact sets $K_0\subset K\subset\Omega$ such that $0\in K_0$ and let $\omega\in C^\infty_0(\Omega)$, supp $\omega\subset K$ such that $\omega(x) = 1$ on K_0. Let $z\in Z$ and consider the solution $E(x, y, z)$ of the equation

$$(-\Delta+q(x))^mE(x, y, z)-Z^{2m}E(x, y, Z) = \delta(x-y).$$

Applying the Green's formula for the functions $\varphi(x, \mu)$ and $\omega(y)E(x, y, z)$ we obtain

$$\varphi(x, \mu) = \int_K E_1(x, y, \mu)\varphi(y, \mu)dy, \tag{2.21}$$

where

$$E_1(x, y, \mu) := [-\Delta+q(y)]^m(1-\omega(y))E(x, y, z)+ \\ +(\mu^{2m}-Z^{2m})\omega(y)E(x, y, z).$$

Iterating (2.21) some times, we get a kernel belonging to L_2, and then applying the Cauchy–Schwartz inequality, we obtain:

$$\|\varphi\|_{L^\infty(K)} \leq C\mu^{l_1}$$

with some $l_1 > 0$. Hence, taking into consideration the Lemma 1.3 and the condition $|\alpha| \leq 2m-2 < N/2$ we get

$$\int_K |D^\alpha \varphi(x, \mu)|^2 dx \leq$$

$$\leq C\mu^{l_1} \int_K \left[\int_\Omega \left(\sum_{|\beta| \leq |\alpha|} \mu^{2m} |x-y|^{2m-N-|\beta|} |y|^{-2m-(|\alpha|-|\beta|)} \omega(|y|) dx \right) \right]^2 dx \leq C\mu^l,$$

where $l = l_1 + 4m$.

PROOF OF THE THEOREM 2.1. Denote α_j, $j = 1, \ldots, m$ the m-th roots of α, and let $\alpha_1 = 1$. Define

$$A := -\Delta + q(x), \quad \varphi_1(x, \mu) = \varphi(x, \mu^{2m}),$$

$$\varphi_{j+1}(x, \mu) := [A - \alpha_j^2 \mu^2] \varphi_j(x, \mu), \quad (j = 1, \ldots, m).$$

Obviously, $\varphi_{m+1} \equiv 0, q(x) \in L_2(\Omega)$ and

$$(A^m - \mu^{2m})\varphi = \prod_{j=1}^{m} (A - \alpha_j^2 \mu^2)\varphi.$$

Applying Lemmas 2.2 and 2.3 we obtain $\varphi_j \in E(\Omega)$, $j = m, m-1, \ldots, 2$,. Because $\varphi(x, \mu)$ fulfills the equation

$$[-\Delta + q(x) - \mu^2]\varphi(x, \mu) = \varphi_2(x, \mu)$$

and $\varphi_2(x, \mu) \in E(\Omega)$ if $\Psi(x, \mu) = \varphi_2(x, \mu)$, we get the desired (2.5).

3. Estimate of the sum of square of eigenfunctions

We shall prove the

THEOREM 3.1. *For any* $\lambda \geq 1$

$$\sum_{|\mu_n - \lambda| \geq 1} U_n^2(x) = O(\lambda^{N-1}) \tag{3.1}$$

uniformly on every compact set $K \subset \Omega$. *Here* $\mu_n = \sqrt[1/2m]{\lambda_n}$

PROOF. Fix any compact set $K \subset \Omega$ and any $x \in K$. Consider the function

$$v(r) = \begin{cases} \dfrac{1}{(2\pi)^{N/2}} \lambda^{N/2} \dfrac{J_\nu(r\lambda)}{r^\nu}, & R \leq r \leq 2R, \\ 0, & \text{othervise}, \end{cases} \tag{3.2}$$

where $\nu = \dfrac{N-2}{2}$ and $0 < R < \dfrac{1}{2}$ dist $(x, \partial\Omega)$, $r = |x-y|$, $y \in K$.

For the eigenfunctions U_n, according to the Theorem 2.1

$$S_r U_n(x) = (2\pi)^{N/2} \frac{J_\nu(r\mu_n)}{(r\mu_n)^\nu} U_n(x) +$$

$$(3.3) \qquad + \frac{\pi}{2} r^{-\nu} \int_0^r \{J_\nu(t\mu_n) Y_\nu(r\mu_n) - J_\nu(r\mu_n) Y_\nu(t\mu_n)\} t^{\nu+1} S_t(qU_n + \psi) dt$$

where

$$S_r g(x) = \int_\Theta g(x + r\Theta) d\Theta.$$

Using 3.3 calculate the Fourier coefficients of $v(r)$ with respect to the system $\{U_n\}$:

$$(3.4) \qquad v_n = \int_\Omega v(r) U_n(y) dy =$$

$$= \int_{R \le |x-y| \le 2R} (2\pi)^{-N/2} \lambda^{N/2} \frac{J_\nu(|x-y|\lambda)}{|x-y|^\nu} U_n(y) dy =$$

$$= \frac{\lambda^{N/2}}{(2\pi)^{N/2}} \int_R^{2R} J_\nu(r\lambda) r^{\nu+1} S_r U_n(x) dr =$$

$$= \frac{\lambda^{N/2}}{\mu^\nu_n} U_n(x) \int_R^{2R} J_\nu(r\lambda) J_\nu(r\mu_n) r dr +$$

$$+ \frac{\pi \lambda^{N/2}}{2(2\pi)^{N/2}} \int_R^{2R} J_\nu(r\lambda) r \int_0^r \{J_\nu(t\mu_n) Y_\nu(r\mu_n) - J_\nu(r\mu_n) Y_\nu(t\mu_n)\} t^{\nu+1} S_t(qu_n + \psi) dt.$$

Denote

$$(3.5) \qquad I_{1,n}(g) := \sqrt{\lambda} \int_R^{2R} J_\nu(\lambda r) Y_\nu(\mu_n r) r \int_0^r J_\nu(t\mu_n) t^{\nu+1} S_t(g) dt dr,$$

$$(3.6) \qquad I_{2,n}(g) := \sqrt{\lambda} \int_R^{2R} J_\nu(\lambda r) J_\nu(\mu_n r) r \int_0^r Y_\nu(t\mu_n) t^{\nu+1} S_t(g) dt dr.$$

First prove

$$A := \mu_n \int_R^{2R} J_\nu(r\lambda) J_\nu(r\mu_n) r dr > c > 0, \tag{3.7}$$

$$(|\mu_n - \lambda| \leq 1, \lambda R > 1).$$

Using

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z^{3/2}}\right),$$

we have

$$A = \frac{2}{\pi} \mu_n \int_R^{2R} \left[\frac{1}{\sqrt{\lambda r}} \cos\left(r\lambda - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{(\lambda r)^{3/2}}\right)\right] \cdot$$

$$\cdot \left[\frac{1}{\sqrt{\mu_n r}} \cos\left(r\mu_n - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{(r\mu_n)^{3/2}}\right)\right].$$

Obviously

$$A = \frac{2}{\pi} \frac{\mu_n}{\sqrt{\lambda\mu_n}} \int_R^{2R} \cos\left(r\lambda - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cos\left(r\mu_n - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) dr + O\left(\frac{1}{\lambda}\right).$$

Using $|\mu_n - \lambda| < 1$, $\cos\varphi \cos\psi = \frac{1}{2}[\cos(\varphi+\psi) + \cos(\varphi-\psi)]$ we get

$$A = \frac{1}{\pi} \sqrt{\frac{\mu_n}{\lambda}} \left[\int_R^{2R} \cos\left(r(\lambda+\mu_n) - \nu\pi - \frac{\pi}{2}\right) dr + \right.$$

$$\left. + \int_R^{2R} \cos\left(r(\lambda-\mu_n)\right) dr\right] + O\left(\frac{1}{\lambda}\right) =$$

$$= \frac{1}{\pi} \sqrt{\frac{\mu_n}{\lambda}} \left[\frac{\sin\left(r(\lambda+\mu_n) - \nu\pi - \frac{\pi}{2}\right)}{\lambda+\mu_n}\Bigg|_R^{2R} + \right.$$

$$\left. + \frac{\sin(2R(\lambda-\mu_n)) - \sin(R(\lambda-\mu_n))}{\lambda-\mu_n}\right] + O\left(\frac{1}{\lambda}\right)$$

hence, taking into account $\sin\varphi - \sin\psi = 2\cos\frac{\varphi+\psi}{2}\sin\frac{\varphi-\psi}{2}$

we obtain

$$A = \frac{1}{\pi}\,\frac{\cos\left[\frac{3R}{2}(\lambda-\mu_n)\right]\sin\left[\frac{R}{2}(\lambda-\mu_n)\right]}{\lambda-\mu_n} + O\left(\frac{1}{\lambda}\right).$$

We can assume $R<\frac{\pi}{6}$. Hence, using

$$\cos\varphi \geq \frac{\sqrt{2}}{2}, \qquad \frac{\sin\varphi}{\varphi} \geq \frac{1}{2}, \qquad |\varphi| \leq \frac{\pi}{4}$$

we get

$$A \geq \frac{1}{2}\,\frac{1}{\pi}\,\frac{R}{2}\,\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{R}{8\sqrt{2}\pi} = C>0$$

(3.7) is proved. It follows from (3.4) and (3.7):

$$|U_n(x)| \leq C|v_n| + C\lambda^{\frac{N-1}{2}}\left[|I_{1n}(qu_n+\psi)| + |I_{2n}(qU_n+\psi)|\right].$$

For the proof of the Theorem 3.1 it is enough to prove the following estimates

$$\sum_{n=1}^{\infty} |v_n|^2 \leq C\lambda^{N-1}, \tag{3.8}$$

$$\sum_{|\mu_n-\lambda|\leq 1} |I_{1n}(qu_n|^2 \leq C, \tag{3.9}$$

$$\sum_{|\mu_n-\lambda|\leq 1} |I_{2n}(qu_n|^2 \leq C, \tag{3.10}$$

$$\sum_{|\mu_n-\lambda|\leq 1} |I_{1n}(\psi)|^2 \leq C, \tag{3.11}$$

$$\sum_{|\mu_n-\lambda|\leq 1} |I_{2n}(\psi)|^2 \leq C. \tag{3.12}$$

The estimates (3.8), (3.9), (3.10) one can prove similarly as in [10]. For the proof of (3.11) change the integrals in (3.5). It follows:

$$I_{1,n} = \sqrt{\lambda}\int_R^{2R} J_\nu(\lambda r)Y_\nu(\mu_n r)r\int_0^r J_\nu(t\mu_n)t^{\nu+1}S_t(\psi)dtdr =$$

$$= \sqrt{\lambda}\int_0^R J_\nu(t\mu_n)t^{\nu+1}S_t(\psi)dt\int_R^{2R} J_\nu(\lambda r)Y_\nu(\mu_n r)rdr +$$

$$+\sqrt{\lambda}\int_R^{2R} J_\nu(t\mu_n)t^{\nu+1}S_t(\psi)dt\int_t^{2R} J_\nu(\lambda r)Y_\nu(\mu_n r)rdr.$$

If $|\mu_n-\lambda|\leq 1$, then

$$\left|\int_t^{2R} J_\nu(\lambda r)Y_\nu(\mu_n r)rdr\right|\leq C\lambda^{-1},\quad (R\leq t\leq 2R). \tag{3.13}$$

This follows immediately from the asymptotic of $J_\nu(x)$ and $Y_\nu(x)$ for $x\geq 1$.

According to the Theorem 2.1 $\psi\in E(\Omega)$, hence there exist positive constants C, β such that

$$|\psi(x,\mu_n)|\leq Ce^{-\beta\mu_n},\quad (x\in K), \tag{3.14}$$

hence

$$\int_0^{2R} |J_\nu(t\mu_n)|t^{\nu+1}S_t|\psi|dt\leq Ce^{-\beta\mu_n},$$

consequently, according to (3)

$$\sum_{|\mu_n-\lambda|\leq 1}{}_n |I_{1,n}|^2\leq C\lambda^{-1}\sum_{|\mu_n-\lambda|\leq 1}{}_n\int_0^{2R} |J_\nu(t\mu_n)|t^{\nu+1}S_t|\psi|dt\leq$$

$$\leq C\lambda^{-1}e^{-2\beta\lambda}\sum_{\mu_n<\lambda}{}_n 1\leq C\lambda^{\gamma-1}e^{-2\beta\lambda}\leq C.$$

The estimate (3.11) is proved. Now prove (3.12). Changing the order of integrals in $I_{2,n}$ we get

$$I_{2,n}=\sqrt{\lambda}\int_R^{2R} J_\nu(\lambda r)J_\nu(\mu_n r)r\int_0^r Y_\nu(t\mu_n)t^{\nu+1}S_t\psi dtdr=$$

$$=\sqrt{\lambda}\int_0^R Y_\nu(t\mu_n)t^{\nu+1}S_t\psi dt\int_R^{2R} J_\nu(\lambda r)J_\nu(\mu_n r)rdr+$$

$$+\sqrt{\lambda}\int_R^{2R} Y_\nu(t\mu_n)t^{\nu+1}S_t\psi dt\int_t^{2R} J_\nu(\lambda r)J_\nu(\mu_n r)rdr=I_{2,n}^1+I_{2,n}^2.$$

On the order hand the symptotic of $J_\nu(x)$ for $x\geq 1$ gives

$$\left|\int_t^{2R} J_\nu(\lambda r)J_\nu(\mu_n r)rdr\right|\leq C\lambda^{-1}, \tag{3.16}$$

further, according to (3.14) we have

$$\int_0^{2R} |Y_\nu(t\mu_n)|t^{\nu+1}S_t|\psi|dt\leq C\int_0^{1/\mu_n}\mu_n^{-\nu}te^{-\beta\mu_n}dt+$$

$$+C\int_{1/\mu_n}^{2R}\mu_n^{-1/2}t^{\nu+1/2}e^{-\beta\mu_n}dt\leq Ce^{-\beta\mu_n}[\mu_n^{2-\nu}+\mu_n^{-1/2}]\leq Ce^{-\beta\mu_n},$$

consequently, taking into consideration (3.15) and (3.16) we obtain

$$\sum_{|\mu_n-\lambda|\le 1}{}_n |I_{2,n}|^2 \le C\lambda^{-1} \sum_{|\mu_n-\lambda|\le 1}{}_n \left(\int_0^{2R} |Y_\nu(t\mu_n)| t^{\nu+1} S_t |\psi| dt\right)^2 \le$$

$$\le C\lambda^{-1} e^{-\beta\lambda} \sum_{\mu_n<\lambda}{}_n 1 \le C\lambda^{\nu-1} e^{-2\beta\lambda} \le C$$

which proves (3.12). □

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THE NUMBER OF SOLUTIONS OF THE GENERALIZED PSEUDOPRIME CONGRUENCE $a^{n-k} \equiv b^{n-k} \pmod{n}$

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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In 1972 A. ROTKIEWICZ asked in his book the following question. "Let a, $k>1$ be fixed positive integers. Do there exist infinitely many composite integers n such that

$$a^{n-k} \equiv 1 \pmod{n} \qquad ?" \tag{1}$$

(Cf. [10], Problem 18). A composite solution n of (1) is called a pseudoprime with respect to a if $k = 1$. It is known that infinitely many pseudoprimes with respect to a exist for all a. In [10] A. ROTKIEWICZ established several necessary conditions for the congruence (1) to be true for an arbitrary pair (a, k). Other investigations have shown that (1) has infinitely many composite solutions when (a, k) is subject to various restrictions (see e.g. [5], [9], [10], [11] and [12]). Finally P. KISS and B. M. PHONG in [4] have given the positive answer for the problem mentioned above, proving that for any fixed integers a, $k>1$ there exist infinitely many composite integers n satisfying the congruence (1).

Recently, W. L. MC DANIEL [6], [7] and B. M. PHONG [3] investigated a composite solution n of the congruence

$$a^{n-k} \equiv b^{n-k} \pmod{n}, \tag{2}$$

where a, b and k are fixed positive integers. Their result shows that if $(a, b, k) \neq (2^s+1, 2^s-1, 3)$ for $s \geq 2$ or either of $(c+1, c, 2)$ and $(c+3, c, 2)$ for $c \geq 2$ then the congruence (2) also has infinitely many composite solutions n. In [8] W. L. MC DANIEL proved that there exist infinitely many triples (a, b, k) of each of the $(c+1, c, 2)$, $(c+3, c, 2)$ and $(2^s+1, 2^s-1, 3)$ such that (2) has infinite many composite solutions.

On the other hand, R. BAILLIE and S. S. WAGTAFF, JR [1] proved that for any given odd positive integer n there exist exactly

$$\prod_{p|n} (n-1, p-1) \tag{3}$$

distinct values of a modulo n, for which the congruence $a^{n-1} \equiv 1 \pmod{n}$ holds.

The aim of the present paper is to extend the result of R. BAILLIE and S. S. WAGTAFF, JR [1] mentioned above for the congruences (1) and (2). We shall use the following notation. If n is a positive integer, then we define n^* as follows:

$$1^*: = 1 \text{ and } n^*: = \frac{n}{\prod\limits_{p|n} p} \text{ if } n > 1,$$

where p runs through all prime divisors of n.

First we prove the

THEOREM 1. *Let n and k be fixed positive integers with $2 \nmid n$ and $n \geq k$. Then there exist exactly*

$$(n^*, k) \prod_{p|n} (n-k, p-1) \tag{4}$$

distinct values of a modulo n, for which the congruence (1) *holds.*

We note that in case $k = 1$ the formula (4) gives that of (3).

PROOF. Consider first the congruence

$$f(x) = x^{n-k} - 1 \equiv 0 \pmod{p^\alpha}, \tag{5}$$

where p is an odd prime number and $\alpha \geq 1$ is an integer. It is well known that for the moduli p^α there exist primitive roots. Let g be a primitive root modulo p^α. Then the congruence (5) is equivalent to that of

$$(n-k) \operatorname{ind}_g x \equiv 0 \pmod{\varphi(p^\alpha)}. \tag{6}$$

Since $(n-k, \varphi(p^\alpha)) | 0$, the congruence (6), and so (5) has exactly $(n-k, \varphi(p^\alpha))$ solutions.

Let $n = \prod\limits_{p^\alpha \| n} p^\alpha$, where p's are odd primes and hence $p^\alpha \| n$ denotes that $p^\alpha | n$, but $p^{\alpha+1} \nmid n$. Consider the congruence

$$f(x) = x^{n-k} - 1 \equiv 0 \pmod{n}. \tag{7}$$

Using the number of solutions of the congruence (5) and the Chinese Remainder Theorem we obtain that the congruence (7) has exactly

$$\prod_{p^\alpha \| n} (n-k, \varphi(p^\alpha)) \tag{8}$$

distinct solutions (mod n). On the other hand, we have

$$\prod_{p^\alpha \| n} (n-k, \varphi(p^\alpha)) = \prod_{\substack{p^\alpha \| n \\ p^a \| (n-k)}} \left(p^a \frac{n-k}{p^a}, p^{\alpha-1}(p-1)\right) = \tag{9}$$

$$= \prod_{\substack{p^\alpha \| n \\ p^a \| (n-k)}} p^{\min(a, p-1)} \left(\frac{n-k}{p^a}, p-1\right) =$$

$$= \prod_{\substack{p^\beta \| n^* \\ p^\alpha \| (n-k)}} p^{\min(\alpha,\beta)}(n-k, p-1) = (n^*, n-k) \prod_{p|n} (n-k, p-1) =$$

$$= (n^*, k) \prod_{p|n} (n-k, p-1),$$

because $(p, p-1) = (p, (p-k)/p^\alpha) = 1$ and $n^*|n$. Thus (8) and (9) give (4). This completes the proof.

Another question one might ask is: for given positive integers n, k and t, how many pairs (a, b) satisfy (2) and $a-b = t$? The answer is given in the

THEOREM 2. *Let n, k and t be arbitrary given positive integers with $(n, 2t) = 1$. and $n \geq k$. Then there exist exactly*

$$(n^*, k) \prod_{p|n} \{(n-k, p-1)-1\} \tag{10}$$

distinct values of a modulo n, for which there is an integer b such that $a-b = t$ and (2) holds.

For the proof of the Theorem 2 we need some lemmas. For given integers $a > b \geq 1$ put $E_i(a, b) := a^i - b^i$ $(i \geq 0)$. If a positive integer n is a divisor of one term in $E(a, b) := \{E_i(a, b)\}_{i \geq 0}$, then we denote by $e(n : a, b)$ the least positive integer m such that $n|E_m(a, b)$. The following result is well known.

LEMMA 1. ([2], [13]) We have

(i) $n|E_m(a, b)$ if and only if $e(n; a, b)|m$,
(ii) $e(p; a, b)|p-1$,
(iii) if $p|E_m(a,b)$, then $p^\beta \| E_{p^\beta m}(a,b)/E_m(a,b)$,

where $(n, ab) = 1$ and p is an odd prime with $(p, ab) = 1$.

LEMMA 2. Let t, m and α be given positive integers and let p an odd prime number with $(p, t) = 1$. Then there exist exactly

$$(p^{\alpha-1}, m)\{(m, p-1)-1\} \tag{11}$$

distinct values of a modulo p^α, for which there is a positive integer b such that $a-b = t$ and

$$E_m(a, b) \equiv 0 \pmod{p^\alpha}. \tag{12}$$

PROOF. For given integer t we put

$$F_m(x) := (x+t)^m - x^m = \sum_{i=0}^{m-1} \binom{m}{i} t^{m-i} x^i. \tag{13}$$

Then

$$F_m(a) = E_m(a, b). \tag{14}$$

Consider first the congruence

$$F_m(x) \equiv 0 \pmod{p}. \tag{15}$$

Since $(p,t) = 1$, any solution x of (15) satisfy $(x+t, p) = (x, p) = 1$. Taking into account the Lemma 1 (i) and (14) it is easy to see that the congruence (15) is equivalent to that of

$$F_d(x) \equiv 0 \pmod{p}, \tag{16}$$

where $d = (m, p-1)$. Since $d|p-1$, hence, using (13) we get

$$F_{p-1}(x) = F_d(x) \cdot G(x), \tag{17}$$

where $G(x)$ is a polynomial in x with integer coefficients and $\deg F_d(x) = d-1$, $\deg G(x) = p-d-1$. The congruence $F_{p-1}(x) \equiv 0 \pmod{p}$ has exactly $p-2$ solutions (mod p) and so by (17) it follows that (16) has exactly $d-1$ solutions (mod p). Thus the congruence (15) has exactly $(m, p-1)-1$ solutions (mod p).

Now consider the congruence

$$F_m(x) \equiv 0 \pmod{p}. \tag{18}$$

Using (13) we have $(F_m(x))' = mF_{m-1}(x)$. Thus, if $p \nmid m$ then for any solution x of (15) $(F_m(x))' \not\equiv 0 \pmod{p}$. This proves that the congruence (18) has exactly $(m, p-1) - 1$ solutions (mod p). Since $(p, m) = 1$ we have $(p^{\alpha-1}, m) = 1$ and so we get (11).

Let now $p|m$ and we write $m = p^\beta m'$, where $\beta \geq 1$ and $(p, m') = 1$. In this case for any integer x we have $F'_m(x) \equiv 0 \pmod{p}$. Since (15) has $(m, p-1)-1$ solutions (mod p), the congruence (18) has $p^{\alpha-1}\{(m, p-1)-1\}$ solutions (mod p^α) if $\alpha \leq \beta+1$. From this we also get (11) because in our case $(p^{\alpha-1}, m) = p^{\alpha-1}$.

Finally, assume $\alpha > \beta+1$, where $m = p^\beta m'$, $\beta \geq 1$ and $(p, m') = 1$. We have

$$E_m(a, b) = \frac{E_{p^\beta m'}(a, b)}{E_{m'}(a, b)} E_{m'}(a, b)$$

and so, using the Lemma 1 (iii) and (14), the congruence (18) is equivalent to the congruence

$$F_{m'}(x) \equiv 0 (mod\ p^{\alpha-\beta}). \tag{19}$$

As we have shown above, since $p \nmid m'$, hence the congruence (19) has $(m', p-1)-1 = (m, p-1) - 1$ sulution (mod $p^{\alpha-\beta}$). Since (18) and (19) are equivalent, hence we obtain that the congruence (18) has $p^\beta\{(m, p-1)-1\}$ solutions (mod p^α). In this case, we have $\alpha > \beta+1$, $(p^{\alpha-1}, m) = p^a$ and so we obtain also the formula (11). This completes the proof of Lemma 2.

PROOF OF THE THEOREM 2.

Let $n = \prod_{p^\alpha \| n} p^\alpha$ and consider the congruences

$$E_{n-k}(a, b) = a^{n-k} - b^{n-k} \equiv 0 \pmod{n}$$

and

$$E_{n-k}(a, b) = a^{n-k} - b^{n-k} \equiv 0 \pmod{p^\alpha}. \tag{21}$$

Applying Lemma 2 with $m = n-k$ for the congruence (21) and using the Chinese Remainder Theorem we obtain that there exist exactly

$$\prod_{p^\alpha \| n} (p^{\alpha-1}, n-k)\{(m, p-1)-1\}$$

distinct values of a modulo n, for which there is an integer b such that $a-b = t$ and (20) holds. On the other hand we have

$$\prod_{p^\alpha \| n} (p^{\alpha-1}, n-k)\{(m, p-1)-1\} = \prod_{p^\beta \| n^*} (p^\beta, n-k) \prod_{p|n} \{(m, p-1)-1\} =$$

$$= (n^*, n-k) \prod_{p|n} \{(m, p-1)-1\} = (n^*, k) \prod_{p|n} \{(m, p-1)-1\} =$$

$$= (n^*, k) \prod_{p/n} \{(n-k, p-1)-1\},$$

and this proves the Theorem 2.

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ON A THEOREM OF M. FEDER

By

ADAM P. BOSZNAY

Department for Mathematics of University of Technology, Budapest

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Introduction

Let $(X, \|\ \|)$ be a Banach space. An $\mathcal{M} \subset X$ subset is proximinal, if for all $x \in X$, there is an element $m(x) \in M$ such that

$$\|x - m(x)\| = \inf_{m \in M} \|x - m\| \equiv d(x, M).$$

A proximinal set is trivially closed.

E. W. CHENEY [1] raised the following question. Let be U, V proximinal subspaces of a Banach space X. Assuming that $U+V$ is closed, what is the situation with the proximinality of $U+V$?

M. FEDER [2] gave an example with the above assumptions such that $U+V$ is not proximinal. On the other hand, he has shown that $U+V$ is proximinal in the case when U is proximinal, V is reflexive, $U \cap V$ is finite dimensional and $U+V$ is closed.

In this remark – modifying the proof of FEDER – we show that the finite dimensionality of $U \cap V$ is not necessary.

The result

THEOREM. *Let $(X, \|\ \|, \|\ \|)$ be a Banach space, $U \subset X$ a proximinal, $V \subset X$ a reflexive subspace of it. Let us assume that $U+V$ is closed. Then $U+V$ is proximinal.*

PROOF. Let us start with the following (probably well-known)

LEMMA. Let $A: Y \to Z$ a linear operator, where Y, Z are Banach spaces. Then there exists an $Y' \subset Y$ dense subset such that $A|Y'$ is continuous.

PROOF OF THE LEMMA. We define

$$Y_n = \{y \in Y;\ \|Ay\| \leq n\}.$$

Clearly $\bigcup_{n=1}^{\infty} Y_n = Y$. Using the category argument, there exists an n^* and a ball centered at some y^* with radius $\varepsilon > 0$ with the property

$$B(y^*, \varepsilon) \subset \overline{Y}_{n^*}.$$

This implies the existence of the desired Y' subset using a shift of y^* into 0. The Lemma is proved.

Returning to the proof of the Theorem, let be $x_0 \in Y$. Clearly, there exist $u_1, u_2, \ldots, \in U$, $v_1, V_2, \ldots \in V$ such that

$$\lim_{K \to \infty} \|x_0 - (u_k + v_k)\| = d(x_0, u + v).$$

Let be U^* a (non necessarily closed) subspace of U such that $U + V = U^* + V$ with the property $U^* \cap V = \{0\}$.

We define the linear map $P : U + V \to V$ on the following way:

$$P(u^* + v) = v \text{ for all } u^* \in U^*, v \in V.$$

By the Lemma, P is continuous on a dense subset $W \subset U + V$.

Now, we can modify the sequences $u_1, u_2, u_3, \ldots, v_1, v_2, v_3, \ldots$ so, that $u_1', u_2', \ldots, \in U$, $v_1', v_2', v_3', \ldots \in V$ and $u_1' + V_1' \in W$, $u_2' + V_2' \in W, \ldots$ with the property

$$\lim_{K \to \infty} \|x_0 - (u_K' + v_K')\| = d(x_0, U + V).$$

Clearly, the sequence $u_1' + v_1'$, $u_2' + v_2'$, $u_3' + v_3'$, ... is bounded, so, the same is true for the sequences

$$g_K = P(u_K' + v_k') \qquad (k = 1, 2, \ldots)$$

and

$$u_K' + v_K' - g_K \qquad (k = 1, 2, \ldots).$$

Here $g_K \in V$, $u_K' + v_K' - g_K \in U^*$, so, $u_K' + v_K' - g_K \in U$.

After this point we use (almost) the proof of Feder. For the sake of completeness, we reproduce it here.

Being g_1, g_2, g_3, a bounded sequence of the reflexive space V, it has ([3], p. 58. Th. 1) a weakly convergent subsequence. Without loss of generality, we assume that $g_1, g_2, g_3, \ldots$ weakly converges to an element $g_0 \in V$.

Using now a well-known result of Mazur there exists a sequence $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{g}_3, \ldots$

$$\tilde{g}_n = \sum_{p_n < i \leq p_{n+1}} \lambda_i g_i$$

with properties

$$\lambda_i \geq 0, \sum_{p_n < i \leq p_n + 1} \lambda_i = 1,$$

and

$$\lim_{n\to\infty} \|\tilde{g}_n - g_0\| = 0. \tag{1}$$

Introduce

$$\tilde{f}_n = \sum_{p_n<i\le p_{n+1}} \lambda_i(u_i' + v_i' - g_i),$$

$$\|x_0 - \tilde{f}_n - \tilde{g}_n\| = \|\sum_{p<i\le p_{n+1}} (x_0 - u_i' - v_i' + g_i - g_i)\| \le \sum_{p_n<i\le p_{n+1}} \lambda_i \|x_0 - u_i' - v_i'\|$$

so

$$\lim_{n\to\infty} \|x_0 - \tilde{f}_n - \tilde{g}_n\| = d(x_0, U+V),$$

applying (1),

$$\lim_{n\to\infty} \|x_0 - \tilde{f}_n - g_0\| = d(x_0, U+V). \tag{2}$$

Let be f_0 a nearest element to $x_0 - g_0$ in U (here we have used the proximinality of U). Being $\tilde{f}_n \in U$,

$$\|x_0 - g_0 - f_0\| \le \|x_0 - g_0 - \tilde{f}_n\|,$$

so,

$$\|x_0 - g_0 - f_0\| = d(x_0, U+V)$$

clearly.

The Theorem is proved.

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GENERALIZATION OF A THEOREM OF G. ALEXITS

By

A. BOGMÉR

Directorate of Posts for the Environs of Budapest

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The notion of trigonometric conjugate function played an important role in the harmonic analysis and in the approximation theory. In 1941 G. ALEXITS [2], [3] proved that for any 2π-periodic continuous function $\|f-\sigma_n(f)\|_{C[0,2\pi]} = O\ (1/n)$ if and only if the conjugate function $\tilde{f}$ of f belongs to Lip 1. Later on from the ideas and results of G. Alexits developped a fruitful branch of Fourier analysis, the theory of "saturation". Recently E. M. STEIN and B. MUCKENHOUPT [4] generalized the notion of conjugate function for ultraspherical case and using this they developed a theory analogous to classical harmonic analysis, e.g. they proved the analogon of the Riesz Marcel theorem etc. B. MUCKENHOUPT defined analogously the Laguerre and Hermite conjugate and showed in [5], [6], [7] that they are also suitable for developing a corresponding harmonic analysis.

On the basis of the notion of Hermite-conjugate I. Joó developed the analogon of the theory of Fejér's means for the Hermite case, i.e. he showed that this notion is very useful and suitable also in approximation theory. Among others he generalized Alexits above mentioned classical result for the Hermite case [8], [9], [10].

Our aim in this paper is to prove an analoguous theorem for the Fejér-means of the Jacobi-Fourier expansions.

Using our result, M. HORVÁTH [12] recently proved a saturation theorem for Abel-Poisson means.

Denote $j_n(\alpha, \beta, x)$ $(\alpha, \beta > -1)$ the orthonormal system of Jacobi polynomials, i.e. j_n is a polynomial of degree n for every $n \in \{0, 1, 2, \ldots\}$ further

$$\int_{-1}^{1} j_n(\alpha, \beta, x) j_m(\alpha, \beta, x)(1-x)^\alpha(1+x)^\beta dx = \delta_{n,m},$$

$$\delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}; \quad n, m \in \mathbf{N}.$$

Denote

$$j_{\alpha,\beta}: = \{j_n(\alpha, \beta, \cos\Theta\}_{n=0}^{\infty},$$

$$I_{\alpha,\beta}: = \{\sin\Theta j_n(\alpha, \beta, \cos\Theta)\}_{n=1}^{\infty}$$

$$w_{\alpha,\beta}(x): = (1-x)^{\alpha}(1+x)^{\beta}.$$

For any orthonormal system $\Phi = \{\varphi_n\}$ denote $\mathcal{D}(\Phi)$ the set of measurable functions for which the Fourier coefficients with respect to Φ exist.

DEFINITION (after Stein-Muckenhoupt [4]).
Let $f \in \mathcal{D}(j_{\alpha,\beta})$ and

$$f(\Theta) \sim \sum_{k=0}^{\infty} a_k(f) j_k(\alpha, \beta, \cos\Theta),$$

$$a_k(f): = \int_{-1}^{1} f(x) j_k(\alpha, \beta, x)(1-x)^{\alpha}(1+x)^{\beta} dx.$$

If there exists an $\tilde{f} \in \mathcal{D}(I_{\alpha,\beta})$ with Jacobi-Fourier series

$$\tilde{f}(\Theta) \sim \sum_{k=1}^{\infty} -a_k(f) j_{k-1}(\alpha+1, \beta+1, \cos\Theta)\sin\Theta$$

$$-a_k(f) = \int_0^{\Pi} \tilde{f}(\Theta)\sin\Theta j_{k-1}(\alpha+1, \beta+1, \cos\Theta)\varrho^2(\Theta) d\Theta$$

then we call $\tilde{f}$ the *Jacobi conjugate function of* f of parameter α, β.
Introduce the notation

$$\varrho(\Theta): = 2^{\frac{\alpha+\beta+1}{2}}\left(\sin\frac{\Theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\Theta}{2}\right)^{\beta+\frac{1}{2}}, 0<\Theta<\Pi$$

and let

(1) $$L^p(\varrho): = \{f:[0, \Pi] \to \mathbf{R}: f\varrho \in L^p(0, \pi)\}.$$

We shall prove the following statement, seeming new also for $\alpha = \beta$

THEOREM. *Let* $\alpha, \beta \geq -\frac{1}{2}$, $1 < p < \infty$ *and* $f \in L^p(\varrho)$. *Suppose that there exists* $\tilde{f} \in L^p(\varrho)$. *Then the following assertions are equivalent*

$$\|\varrho(\Theta)[\sigma_n(f, \Theta) - f(\Theta)]\|_{L^p(0,\Pi)} = O(1/n)$$

(2) $\tilde{f}$ *is locally absolutely continuous on* $(0, \pi)$ *and*

$$\lim_{\substack{\Theta\to 0\\ \Theta\to\Pi}} \varrho^2\tilde{f} = 0, [\varrho^2\tilde{f}]'\varrho^{-1} \in L^p(0, \Pi).$$

REMARK. We hope to return to the cases $\alpha < -\frac{1}{2}$ or $\beta < -\frac{1}{2}$ in a subsequent paper.

The proof uses the ideas of the paper [9].

We need some lemmas.

LEMMA 1.

$$[\varrho^2(\Theta)\sin\Theta j_{n-1}(\alpha+1,\beta+1,\cos\Theta)]'\varrho^{-2}(\Theta) =$$

$$= [n(n+\alpha+\beta+1)]^{\frac{1}{2}} j_n(\alpha,\beta,\cos\Theta). \tag{3}$$

PROOF. Using the notation of the SZEGŐ's book [11], we can write the Rodrigues formula

$$(1-x)^\alpha(1+x)^\beta p_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}[(1-x)^{n+\alpha}(1+x)^{n+\beta}]^{(n)}.$$

Consequently

$$[(-1)^{n-1}2^{n-1}(n-1)!(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)]' =$$

$$= [(1-x)^{n+\alpha}(1+x)^{n+\beta}]^{(n)} = (-1)^n 2^n n!(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)$$

Using the fact that for the normalizational constants $h_n^{(\alpha,\beta)}$, defined by

$$P_n^{(\alpha,\beta)}(x) = (h_n^{(\alpha,\beta)})^{\frac{1}{2}} j_n(\alpha,\beta,x)$$

we have

$$\left(\frac{h_n^{(\alpha,\beta)}}{h_{n-1}^{(\alpha+1,\beta+1)}}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{n+\alpha+\beta+1}{n}\right)^{\frac{1}{2}}$$

we obtain

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}j_{n-1}(\alpha+1,\beta+1,x)]' =$$

$$= -[n(n+\alpha+\beta+1)]^{\frac{1}{2}}(1-x)^\alpha(1+x)^\beta j_n(\alpha,\beta,x).$$

Now the substitution $x=\cos\Theta$, $1-x=2\sin^2\frac{\Theta}{2}$, $1+x=2\cos^2\frac{\Theta}{2}$ gives (3).

LEMMA 2. Let

$$\sum_{k=0}^{\infty} a_k j_k(\alpha,\beta,\cos\Theta), \tag{4}$$

any formal Jacobi series such that its Fejér means σ_n are bounded, $\|\varrho\sigma_n\|_p = O(1)$ for some $1<p<\infty$. Then there exists an f with $\varrho f \in L^p(0,\pi)$ such that (4) is the Jacobi expansion of f.

The proof can be given as that of the corresponding Lemma 3. in [9]; we omit the details.

LEMMA 3 [13, 14, 15, 16]. Let $1 \leq p \leq \infty$, $\alpha, \beta \geq -\frac{1}{2}$ and $\varrho f \in L^p(0, \pi)$. then

$$\|\varrho \sigma_n f\|_p \leq c(p) \|\varrho f\|_p. \tag{5}$$

LEMMA 4. [10]. Let (φ_n) be a sequence in a Banach space B and let $\lambda_n = [n(n+\alpha+\beta+1)]^{\frac{1}{2}}$. Then the following statements are equivalent for the Fejér means σ_n:

$$\|\sigma_n(\sum \lambda_k \varphi_k)\| = O(1), \tag{A}$$

$$\|\sigma_n(\sum \varphi_k) - \sigma\| = O(1/\lambda_n) \text{ for some } \sigma \in B. \tag{B}$$

Now we can prove our Theorem. (1)$\Rightarrow$(2).

We can suppose $a_0(f) = 0$, i.e.

$$f(\Theta) \sim \sum_{k=1}^{\infty} a_k j_k(\alpha, \beta, \cos \Theta). \tag{6}$$

Consider the series

$$\sum_{k=1}^{\infty} [k(k+\alpha+\beta+1)]^{\frac{1}{2}} a_k j_k(\alpha, \beta, \cos \Theta). \tag{7}$$

From (1) and Lemma 4. it follows that

$$\|\varrho \sigma_n((7))\|_p = O(1)$$

and it means by Lemma 2 that there exists g, $\varrho g \in L^p(0, \pi)$ with Jacobi series (7).

From (3) we get

$$\varrho \sigma_n(g) = [\varrho^2 \sigma_n(\tilde{f})]' \varrho^{-1}.$$

Using Lemma 3. we obtain

$$\|\varrho[\sigma_n(g) - g]\|_p \to 0 \quad (n \to \infty)$$

or

$$\|\varrho^{-1}([\varrho^2 \sigma_n \tilde{f}]' - \varrho^2 g)\|_p \to 0 \tag{8}$$

and again from Lemma 3

$$\|\varrho^{-1}(\varrho^2 \sigma_n \tilde{f} - \varrho^2 f)\|_p \to 0. \tag{9}$$

The comparison of (8) and (9) shows (see [9]) that $\varrho^2 \tilde{f}$ is locally absolutely continuous and $(\varrho^2 \tilde{f})' = \varrho^2 g$ and hence $\varrho^{-1}(\varrho^2 f)' \in L^p(0, \pi)$. The relation

$\lim\limits_{\substack{\Theta\to 0\\ \Theta\to\pi}} \varrho^2\tilde{f} = 0$ follows from the fact that $\varrho^2(\Theta)\sigma_n\tilde{f}(\Theta) = \int\limits_0^{\Theta} \varrho^2\sigma_n g$ tends uniformly to $\int\limits_0^{\Theta} \varrho^2 g = \varrho^2(\Theta)\tilde{f}(\Theta)$.

(2)⇒(1).

We know that

$$\varphi: = [\varrho^2\tilde{f}]'\varrho^{-1} \in L^p(0, \pi).$$

Let

$$\varphi(\Theta) \sim \sum_{k=0}^{\infty} d_k j_k(\alpha, \beta, \cos\Theta). \tag{10}$$

The coefficients will be counted as follows

$$d_k = \int\limits_0^{\pi} \varphi(\Theta) j_k(\alpha, \beta, \cos\Theta)\varrho(\Theta)d\Theta = \int\limits_0^{\pi} [\tilde{f}\varrho^2]' j_k(\alpha, \beta, \cos\Theta)d\Theta =$$

$$= [k(k+\alpha+\beta+1)]^{\frac{1}{2}} \int\limits_0^{\pi} \tilde{f}(\Theta)\varrho^2(\Theta)\sin\Theta j_{k-1}(\alpha+1, \beta+1, \cos\Theta)d\Theta =$$

$$= \begin{cases} [k(k+\alpha+\beta+1)]^{\frac{1}{2}} a_{k-1}(\tilde{f}) = [k(k+\alpha+\beta+1)]^{\frac{1}{2}} a_k(f) & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases}$$

By Lemma 3.

$$\|\varrho\sigma_n(\varphi)\|_p = O(1)$$

and then Lemma 4. states that there exists $\sigma \in L^p(0, \pi)$ such that

$$\|\varrho(\sigma_n(f) - \sigma)\|_p = O(1/n).$$

Since $\|\varrho(\sigma_n(f) - f)\|_p \to 0$, hence

$$f = \sigma$$

hence (1) holds.

The Theorem is proved.

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ANSWER TO A PROBLEM OF PAUL TURÁN

By

I. JOÓ and NGUYEN XUAN KY

Department for Analysis of the L. Eötvös University, Budapest

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Dedicated to the memory of GÉZA FREUD

The aim of the present paper is to prove an L_2-convergence theorem with order for Lagrange interpolation based on the roots of Laguerre polynomials $L_n^{(\alpha)}\, x)$, $\alpha \geq 0$. Our result is continuation of the classical investigations of J. BALÁZS, P. ERDŐS and P. TURÁN [1]–[4]; see also the further references.

1. Introduction

In 1911 D. JACKSON proved the following inequalitiy

$$E_n(f) \leq \frac{C}{n} \|f'\|_\infty, \tag{1}$$

where

$$E_n(f) := \inf_{t_n \in T_n} \|f - t_n\|_{C[0,2\pi]}, f \in C[0,2\pi]$$

T_n denotes the set of all trigonometric polynomials of order at most n, $C[0,2\pi]$ denotes the class of 2π-periodic continuous functions. The inequality (1) plays an important role the theory of trigonometric approximations.

Later on many authors gave the analogoues of (1) for different approximations. First result of this type was proved by G. FREUD [5], namely, he proved the following inequality:

$$E_n(\varrho, f)_p \leq \frac{c(p)}{\sqrt{n}} \|\varrho f'\|_{L^\infty(-\infty,\infty)}, \; 1 \leq p \leq \infty, \tag{2}$$

where

$$\varrho(x) := e^{-x^2/2}, E_n(\varrho, f)_p := \inf_{p_n \in \mathcal{P}_n} \|\varrho(f - p_n)\|_{L^p(-\infty,\infty)},$$

$\mathcal{P}_n$ denotes the set of algebraic polynomials of degree at most n. This inequality was further generalized by G. FREUD in [6] for more general weight functions. The analogous problem for the Laguerre weight was solved by the first author of the persent paper in 1975, but it remained unpublished. Recently the same result was found independently by the second of us, and we have recognized that this result can be applied to solve a problem of P. TURÁN. We present all this in the paper. Remark that Jackson type theorem was proved also for Jacobi weight in [21]. The first step in the direction of solution of Turán's problem is due to K. BALÁZS [15] by giving another approximative representation for the Fourier transform and by estimating the order of approximation. MERCER [16] condinued these considerations.

The authors are indebted to professor J. BALÁZS for setting the problem and for his valuable discussions.

2. The Jackson-type inequality

Let

$$u_\alpha(x):=x^\alpha e^{-x}(\alpha>-1, x>0),\ 1\le p\le\infty$$

and define

$$E_n(u_\alpha,f)_p:=\inf_{p_n\in\mathcal{P}_n}\|u_\alpha(f-p_n)\|_p, \|\cdot\|_p:=\left(\int_0^\infty|\cdot|^p\right)^{1/p}.$$

Introduce the following function classes

$$W_p^0(\alpha):=\{f: u_\alpha f\in L^p(0,\infty)\},$$

$W_p^k(\alpha):=\{f: f, f', \dots, f^{(k-1)}$ are locally absolutely continuous and $u_{\alpha+1/2}f^{(l)}\in L^p(0,\infty)$ for $l=0,1,\dots,k\}$.

We shall prove the following Jackson type inequality.

THEOREM. 1. *Suppose,* $1\le k\in\mathbf{N}$, $f\in W_p^k(\alpha)$, $1\le p<\infty$ *and* $\alpha\ge 0$. *Then*

$$E_n(u_\alpha,f)_p\le\frac{c(p,k)}{n^{k/2}}\|u_{\alpha+k/2}f^{(k)}\|_p, \tag{1}$$

with a constant $c(p,k)$ *depending only on* p *and* k.

For the proof we need a lemma.

LEMMA 1. Let $\alpha>-1$ and

$$\mathcal{P}_n^\perp(u_\alpha):=\left\{g\in L^\infty(0,\infty): \int_0^\infty gp_nu_\alpha=0\quad\forall p_n\in\mathcal{P}_n\right\}.$$

Then for any $g \in \mathcal{P}_n^{\perp}(u_\alpha$ we have

$$\left|\int_0^x g u_\alpha\right| \leq \frac{c(\alpha)}{\sqrt{n}} \|g\|_\infty u_{\alpha+1/2}(x). \tag{2}$$

PROOF.

Denote $\Gamma x(t)$ the characteristic function of the segment $[0, x]$; then

$$\left|\int_0^x g u_\alpha\right| = \left|\int_0^\infty g \Gamma_x u_\alpha\right| = \left|\int_0^\infty g(\Gamma_x - p_n) u_\alpha\right| \leq \|g\|_\infty \int_0^\infty |\Gamma_x - p_n| u_\alpha$$

implies

$$\left|\int_0^x g u_\alpha\right| \leq \|g\|_\infty E_n(\Gamma_x, u_\alpha)_1,$$

hence it will be enough to prove that

$$E_n(\Gamma_x, u_\alpha)_1 \leq \frac{c(\alpha)}{\sqrt{n}} u_{\alpha+1/2}(x). \tag{3}$$

We know the Freud's estimate ([27], p. 72)

$$E_{2n-2}(\Gamma_x, u_\alpha)_1 \leq \lambda_n(x) \tag{4}$$

where

$$\lambda_n(x) := \left(\sum_{k=0}^n p_k^2(u_\alpha, x)\right)^{-1}$$

is the Christoffel-function. It is known that (Cf.[7], [30])

$$\lambda_n(x) \leq \frac{c(\alpha)}{\sqrt{n}} u_{\alpha+1/2}(x) \text{ if } \frac{c_1(\alpha)}{n} \leq x \leq c_2(\alpha) n. \tag{5}$$

Now for $\frac{c_1(\alpha)}{n} \leq x \leq c_2(\alpha) n$ (3) follows from (4) and (5).

If $0 \leq x \leq C_1(\alpha)/n$ then approximate Γ_x by the constant function 0:

$$E_n(\Gamma_x, u_\alpha) \leq \int_0^x u_\alpha \leq c(\alpha) x^{\alpha+1} \leq \frac{c(\alpha)}{\sqrt{n}} x^{\alpha+1/2} \leq \frac{c(\alpha)}{\sqrt{n}} u_{\alpha+1/2}(x).$$

If $X \geq C_2(\alpha) n$ then approximate Γ_x by 1:

$$E_n(\Gamma_x, u_\alpha) \leq \int_x^\infty u_\alpha \leq \frac{c(\alpha)}{\sqrt{n}} u_{\alpha+1/2}(x).$$

The last inequality can be checked by differentiating both sides. So (3) and Lemma 1 is proved.

PROOF of Theorem 1 for $p = 1$ (we prove this case for $\alpha > -1$).

If X is a normed space, $X_0 \subset X$ is a closed subspace and $x_1 \notin X_0$ then

$$\operatorname{dist}(x_1, X_0) := \sup_{\substack{f \in X_0^{\perp} \\ \|f\| \leq 1}} |\langle f, x_1 \rangle|$$

(The supremum runs over the functionals $f \in X^*$ with norm ≤ 1 which are are vanishing on X_0). This is an easy application of the Hahn-Banach theorem. As a special case we get

$$E_n(f, u_\alpha)_1 = \sup_{\substack{g \in \mathcal{P}_n^{\perp}(u_\alpha) \\ \|g\|_\infty \leq 1}} \int_0^\infty f g u_\alpha. \tag{6}$$

Integrating by parts we obtain

$$\int_0^\infty f g u_\alpha = \left[f(x) \int_0^x g u_\alpha \right]_0^\infty - \int_0^\infty f'(x) \left(\int_0^x g u_\alpha \right) dx. \tag{7}$$

Now

$$\left| \int_0^\infty f'(x) \left(\int_0^x g u_\alpha \right) dx \right| \leq \|f' u_{\alpha+1/2}\|_1 \cdot \left\| \frac{1}{u_{\alpha+1/2}} \int_0^x g u_\alpha \right\|_\infty \leq \frac{c(\alpha)}{\sqrt{n}} \|f' u_{\alpha+1/2}\|_1,$$

hence Theorem 1 will be proved for $p = 1$ if we show

$$\left[f(x) \int_0^x g u_\alpha \right]_0^\infty = 0. \tag{8}$$

Consider first the case $X \to 0+$. Then

$$\left| \int_0^x g u_\alpha \right| \leq c \int_0^x t^\alpha dt \leq c(\alpha) x^{\alpha+1},$$

hence

$$\left| f(x) \int_0^x g u_\alpha \right| \leq c(\alpha) |f(x) - f(1)| \, x^{\alpha+1} + c(\alpha) |f(1)| x^{\alpha+1}.$$

The second number here tends to zero by $\alpha > -1$, and the first one can be estimated as follows:

$$\left| x^{\alpha+1} \int_x^1 f' \right| \leq \sqrt{x} \int_x^1 |f'(t)| t^{\alpha+1/2} dt \leq c\sqrt{x} \int_x^1 |f'| u_{\alpha+1/2} \to 0,$$

$$\left| x^{\alpha+1} \int_x^1 f' \right| \leq x^{\alpha+1} \|f'\|_{L^1(0,1)} \to 0.$$

Now consider the case $x \to \infty$. If follows from (7) that there exists the limit

$$A := \lim_{x\to\infty} f(x) \int_0^x g u_\alpha.$$

Suppose indirectly that $|A| > 0$; then for large x we have

$$\frac{1}{2}|A| \leq \left| f(x) \int_0^x g u_\alpha \right| \leq c(\alpha)|f(x) u_{\alpha+1/2}(x)|,$$

$$\frac{A}{2\sqrt{x}} \leq c(\alpha)|f(x) u_\alpha(x)|$$

which contradicts to $fU_\alpha \in L^1(0,\infty)$. This contradiction proves (8) and hence Theorem 1 for $p = 1$.

Now let $p = \infty$. In order to avoid technical difficulties we apply the substitution $f^*(t) := f(t^2/2)$. Denote for $\beta > -1$ by $W_p[\beta]$ the space of all oddfunctions $g: \mathbf{R} \to \mathbf{R}$, locally absolutely continuous on $(0, \infty)$ and satisfying

$$U_\beta g, U_\beta g' \in L^p(\mathbf{R}), \; U_\beta = |x|^\beta e^{-x^2}.$$

If can be simply checked that the transform $f \to f^*$ is a one-to-one correspondance between $W_p^1(\alpha)$ and $W_p[\beta]$ further

(9) $\|u_\alpha f\|_p = c_1(p, \alpha) U\|_{2\alpha+1/p} f^*\|_p; \|u_{\alpha+1/2} f'\|_p = c_2(p, \alpha)\|U_{2\alpha+1/p} f^{*\prime}\|_p.$

Introduce the notation

$$E_n(U_\beta, \varphi)_p := \inf_{p_n \in \mathcal{P}_n} \|U_\beta(\varphi - p_n)\|_p.$$

Since φ is odd, the best approximating polynomial must be also odd hence we can apply the substitution $x = t^2/2$ to obtain

(10) $$E_n(u_\alpha, f)_p = c_3(p, \alpha) E_{2n}(U_{2\alpha+1'/p} f^*)_p.$$

The above considerations show that (for any fixed $1 \leq p \leq \infty$). Theorem 1 is equivalent to

(11) $$E_n(U_\beta, g)_p \leq \frac{c(p, \beta)}{\sqrt{n}} \|U_\beta g'\|_p$$

for any $\beta \geq \frac{1}{p}$ and $g \in W_p[\beta]$.

In what follows we prove (11) instead of (1). We showed (11) for $p = 1$ and $\beta > 0$. (The case $\beta = 0$, $p = 1$ was proved by FREUD [5]). Take the case $p = \infty$, $\beta \geq 0$. Use the device of G. FREUD [5]. First we have for $\beta \geq 0$ and $1 \leq p \leq \infty$

(12) $$\|U_\beta \sigma_n g\|_p \leq c(p) \|U_\beta g\|_p.$$

(see [18]). Define for $x>0$ the function

$$\varphi_x(t):=\begin{cases} U_\beta^{-2}(t)\operatorname{sgn}(x-1), & t\in\langle 1,x\rangle,\\ 0 & ,\ t\in[0,\infty)\setminus\langle 1,x\rangle,\\ \varphi_x(-t) & ,\ t<0\end{cases}$$

where

$$\langle 1,x\rangle:=\begin{cases}[1,x], & x>1,\\ [x,1], & x<1.\end{cases}$$

Then for the function

$$q_{n+1}(x):=g(1)+\int_1^x \vartheta_n(g',t)dt,$$

where

$$\vartheta_n:=\frac{s_n+s_{n+1}+\ldots+s_{2n-1}}{n}$$

is the de la Valleé-Poussin mean, we get

$$|g(x)-q_{n+1}(x)|=\left|\int_1^x (g'-\vartheta_n(g'))\right|=$$

$$=\left|\int_0^\infty [g'-\vartheta_n(g')]\varphi_x U_\beta^2\right|=\left|\int_0^\infty [g'-\vartheta_n(g')][\varphi_x-p_n]U_\beta^2\right|\leq$$

$$\leq c\|U_\beta g'\|_\infty\int_0^\infty |\varphi_x-p_n|U_\beta$$

and hence

$$|g(x)-q_{n+1}(x)|\leq cE_n(U_\beta,\varphi_x)_1\|U_\beta g'\|_\infty. \tag{13}$$

We know that

$$E_{2n}(U_\beta,\varphi_x)_1=cE_n(u_\alpha,\Psi_x)=\sup_{\substack{g\in\mathcal{P}_n^\perp(u_\alpha)\\ \|g\|_\infty\leq 1}}\int_0^\infty \psi_x g u_\alpha$$

where $2\alpha+1=\beta$ and

$$\psi_x(z)=\varphi_x(\sqrt{2z})=\begin{cases}(\sqrt{2z})^{-2\beta}e^{2z}\operatorname{sgn}(x-1), & t\in\left[\frac{1}{2},\frac{x^2}{2}\right],\\ 0 & ,t\in[0,\infty]\setminus\left[\frac{1}{2},\frac{x^2}{2}\right].\end{cases}$$

Integrate by parts

$$\int_0^\infty \psi_x g u_\alpha = \left[\psi_x(z) \int_0^z g u_\alpha \right]_{1/2}^{x^2/2} - \int_{1/2}^{x^2/2} \left(\int_0^z g u_\alpha \right) \psi_x'(z) dz$$

hence by Lemma I

$$\left| \int_0^\infty \psi_x g u_\alpha \right| \leq \frac{c(\alpha)}{\sqrt{n}} \left\{ u_{\alpha+1/2}\left(\frac{x^2}{2}\right) \left| \psi_x\left(\frac{x^2}{2}\right) \right| + u_{\alpha+1/2}\left(\frac{1}{2}\right) \left| \psi_x\left(\frac{1}{2}\right) \right| + \right.$$

$$\left. + \int_{1/2}^{x^2/2} u_{\alpha+1/2}(z) |\psi_x'(z)| dz \right\} \leq \frac{c(\alpha)}{\sqrt{n}} U_\beta^{-1}(x)$$

as it can be easily seen. Now we get from (13)

$$E_n(U_\beta, g)_\infty \leq \| U_\beta(x) |g(x) - q_{n+1}(x)| \|_\infty \leq \frac{c(\alpha)}{\sqrt{n}} \| U_\beta g' \|_\infty .$$

So Theorem 1 is proved for $p = 1$ and $p = \infty$. In case $1 < p < \infty$ we apply the usual interpolation arguments. It follows from (10) that

$$E_{2n}(U_\beta, g)_p \leq \| U_\beta(g - \vartheta_n g) \|_p \leq c(\beta, p) E_n(U_\beta, g)_p$$

so (11) is equivalent to

$$\| U_\beta(g - \vartheta_n g) \|_p \leq \frac{c(\beta, p)}{\sqrt{n}} \| U_\beta g' \|_p$$

which is proved for $\beta \geq 0$ in case $p = 1$ and hence it remains valid for $1 < p < \infty$ by the Marcinkiewicz interpolation theorem. More precisely: consider first the mapping

$$U_\beta h \to U_\beta(H - \vartheta_n H)$$

where $H(x) := \int_0^x h$ and h is a simple function in the terminology of the Zygmund-book [28]. In this case

$$U_\beta h, U_\beta H \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$$

and

$$\| U_\beta(H - \vartheta_n H) \|_\infty \leq \frac{c(\beta)}{\sqrt{n}} \| U_\beta h \|_\infty ,$$

$$\| U_\beta(H - \vartheta_n H) \|_1 \leq \frac{c(\beta)}{\sqrt{n}} \| U_\beta h \|_1 ,$$

Then the Marcinkiewicz interpolation theorem states that for $1<p<\infty$

$$\|U_\beta(H-\vartheta_n H)\|_p \leq \frac{c(p,\beta)}{\sqrt{n}}\|U_\beta h\|_p, \tag{14}$$

for any simple function h. Now let g be an arbitrary function with $U_\beta g \in L^p(\mathbf{R})$ and let $G(x) := \int_0^x g$.

There exists simple functions h_k such that $U_\beta h_k \to U_\beta g$ in $L^p(\mathbf{R})$. Now it follows from (14) that $U_\beta H_k$ is a Cauchy sequence, so it has a limit $U_\beta H$ in $L^p(\mathbf{R})$. On the other hand $U_\beta H_k$ tends to $U_\beta G$ pointwise; so $U_\beta G \in L^p(\mathbf{R})$. We get analogously that $U_\beta(H_k-\vartheta_n H_k)$ tends in L^p to $U_\beta(G-\vartheta_n G)$ and then

$$\|U_\beta(G-\vartheta_n G)\|_p \leq \frac{c(p,\beta)}{\sqrt{n}}\|U_\beta g\|_p.$$

The Theorem 1 is proved for $k=1$. In case $k>0$ we use a simple induction. Since

$$E_n(u_\alpha, f)_p = E_n(u_\alpha, f-p_n)_p \leq \frac{c(\alpha,p)}{\sqrt{n}}\|f'-p_u')u_{\alpha+1/2}\|_p$$

implies

$$E_n(u_\alpha f)_p \leq \frac{c(\alpha,p)}{\sqrt{n}} E_{n-1}(u_{\alpha+1/2}, f')_p,$$

hence the induction on k shows (1). The proof of Theorem 1 is complete.

REMARK. The above proof shows by (9) that

$$W_p^k(\alpha) = \{f: f, f', \ldots, f^{(k-1)} \text{ are locally absolutely continuous and } u_{\alpha+k/2} f^{(k)} \in L^p(0,\infty)\}. \tag{15}$$

Define the following modulus of continuity:

$$\omega(g, U_\beta\delta)_p = \omega(g, U_\beta, a, b, \delta) := \sup_{0\leq h\leq\delta}\|[g(x+h)-g(x)]U_\beta(x)\|_{L^p(0,b)} + \sup_{0\leq h\leq\delta}\|[g(x-h)-g(x)]U_\beta(x)\|_{L^p(b,\infty)},$$

where $\beta\geq 0$, $gU_\beta \in L^p(\mathbf{R})$ and $0<a<b<\infty$.

We know that

$$U_\beta(x) \leq c(a,b)U_\beta(x+h) \quad (a<x<b,\ 0<h<a),$$
$$U_\beta(x) \leq c(a,b)U_\beta(x-h) \quad (a<x<\infty,\ 0<h<a).$$

Consequently $U_\beta g \in L^p(\mathbf{R})$ implies $U_\beta(x)g(x+h) \in L^p(0,b)$ and $U_\beta(x)g(x-h) \in L^p(a,\infty)$, so the above modulus of continuity is finite. We shall prove the

THEOREM 2. *Let* $1 \le p \le \infty$, $\alpha \ge 0$, $u_\alpha f \in L^p(0, \infty)$. *Then we have*

$$E_n(u_\alpha, f)_p \le c\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right),$$

where $\beta = 2\alpha + \frac{1}{p}$, $f^*(t) = f(t^2/2)$ *and the constant* c *does not depend of* n, f.

PROOF. As in [5] use the Minkowski inequality to obtain

$$\left\{\int_b^b \left| 2n^{1/2} \int_{n^{-1/2}/2}^{n^{-1/2}} [f^*(\Theta+t) - f^*(\Theta-t)]dt \right|^p d\Theta\right\}^{1/p} \le$$

$$\le 2n^{1/2} \int_{n^{-1/2}/2}^{n^{-1/2}} \left\{\int_a^b |f^*(\Theta+t) - f^*(\Theta-t)|^p d\Theta\right\}^{1/p} dt \le$$

$$\le cn^{1/2} \int_{n^{-1/2}/2}^{n^{-1/2}} \left\{\int_a^b |[f^*(\Theta+t) - f^*(\Theta-t)]U_\beta(\Theta)|^p d\Theta\right\}^{1/p} dt \le$$

$$\le c\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right).$$

Consequently there exist $a \le \Theta_n \le b$ such that

$$\left| 2n^{1/2} \int_{n^{-1/2}/2}^{n^{-1/2}} [f^*(\Theta_n+t) - f^*(\Theta_n-t]\, dt \right| \le c\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right). \tag{16}$$

Define the function

$$\varphi_n(\Theta) := \begin{cases} 2\sqrt{n} \displaystyle\int_{n^{-1/2}/2}^{n^{-1/2}} f^*(t+\Theta)dt \quad (0 < \Theta \le \Theta_n), \\ 2\sqrt{n} \displaystyle\int_{n^{-1/2}/2}^{n^{-1/2}} [f^*(\Theta-t) + f^*(\Theta_n+t) - f^*(\Theta_n-t)] (\Theta_n \le \Theta < \infty), \end{cases}$$

and let $\varphi_n(-\Theta) = \varphi_n(\Theta)$. Using again the Minkowski inequality and (16) we obtain

$$\|[\varphi_n - f^*]U_\beta\|_p \le c\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right)_p.$$

Since

$$\varphi_n'(\Theta) = \begin{cases} 2\sqrt{n}\left[f^*(n^{-1/2}+\Theta) - f^*\left(\frac{n^{-1/2}}{2} + \Theta\right)\right] \quad (0 < \Theta \le \Theta_n), \\ 2\sqrt{n}\left[f^*(\Theta - n^{-1/2}) - f^*\left(\Theta - \frac{n^{-1/2}}{2}\right)\right] \quad (\Theta_n \le \Theta < \infty) \end{cases}$$

hence

$$\|\varphi_n' U_\beta\|_p \le c\sqrt{n}\,\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right)$$

is immediate. Now

$$E_n(u_\alpha, f)_p = cE_{2n}(U_\beta, f^*)_p \le c\{\|[\varphi_n - f^*]U_\beta\|_p + E_{2n}(U_\beta, \varphi_n)_p\} \le$$

$$\le c\left\{\|[\varphi_n - f^*]U_\beta\|_p + \frac{1}{\sqrt{n}}\|\varphi_n' U_\beta\|_p\right\} \le c\omega\left(f^*, U_\beta, \frac{1}{\sqrt{n}}\right)$$

which completes the proof.

2. On the Turán's problem

Let $L_n(f, x)$ be the n-th Lagrange interpolation polynomial of $f \in C[0, \infty$ based on the roots of the Laguerre polynomials L_n^λ of parameter λ.

Let

$$0 < \varepsilon < \frac{1}{2},\quad 0 < \delta < \frac{\lambda+1}{2},\ \alpha := \frac{\lambda+1}{2} - \delta,\ \hat{f}(x) := f\left(\frac{x}{\frac{1}{2}\varepsilon}\right).$$

We shall prove the

THEOREM 3. *Let* $\lambda > -1$. *If* $\hat{f} \in W_\infty^0(\alpha)$ *then*

$$\int_0^\infty [f(x) - L_n(f, x)]^2 x^\lambda e^{-x} dx = o(1)\ (n \to \infty). \tag{17}$$

If $\hat{f} \in W_\infty^k(\alpha)$ *for some* $k \in \{1, 2, \ldots\}$, *then*

$$\int_0^\infty [f(x) - L_n(f, x)]^2 x^\lambda e^{-x} dx \le \frac{c(\lambda, \varepsilon, \delta, k)}{n^{k/2}} \|u_{\alpha+k/2}\hat{f}^{(k)}\|_\infty. \tag{18}$$

PROOF. We argue as in [2] of BALÁZS and TURÁN. Define the quantities

$$p(x) := x^\lambda e^{-x},\ h(x) := x^{-1+2\delta} e^{-2\varepsilon x},\ g(x) := x^{-2\alpha} e^{(1-2\varepsilon)x}.$$

It is proved in [17] (see also [29]) that

$$g^{(2n)}(x) > 0 \qquad (x > 0;\ n = 0, 1, 2, \ldots).$$

Hence it can be proved as in [2] that

$$\sum_{\nu=1}^{n} g(x_\nu) \int_0^\infty l_\nu^2(x) p(x) dx \le \int_0^\infty h(x) dx = c(\varepsilon, \delta) < \infty,$$

where l_ν are the fundamental polynomials of the Lagrange interpolation.

There exists a polynomial of the form $\hat{p}_{n-1}(x) = p_{n-1}\left(\frac{x}{\frac{1}{2}-\varepsilon}\right)$ such that

$$(19)\quad E_{n-1}(u_\alpha, \hat{f})_\infty = \|(\hat{f}-\hat{p}_{n-1})u_\alpha\|_\infty = c\|[f(x)-p_{n-1}(x)]x^\alpha e^{-x\left(\frac{1}{2}-\varepsilon\right)}\|_\infty.$$

Using the orthogonality relations

$$\int_0^\infty l_k(x)l_m(x)x^\lambda e^{-x}dx = 0 \quad k\neq m)$$

we get that

$$\int_0^\infty [f(x)-L_n(f,x)]^2x^\lambda e^{-x}dx \leq 2\int_0^\infty |f(x)-p_{n-1}(x)|^2x^\lambda e^{-x}dx +$$

$$+2\int_0^\infty |L_n(f-p_{n-1},x)|^2x^\lambda e^{-x}dx \leq 2\|(f-p_{n-1})x^\alpha e^{-\left(\frac{1}{2}-\varepsilon\right)x}\|_\infty^2 \int_0^\infty x^{2\delta-1}e^{-2\varepsilon x}dx +$$

$$+\sum_{k=1}^n |f(x_k)-p_{n-1}(x_k)|^2 \int_0^\infty l_k^2(x)x^\lambda e^{-x}dx \leq$$

$$\leq \|(f-p_{n-1})x^\alpha e^{-\left(\frac{1}{2}-\varepsilon\right)x}\|_\infty^2 \left\{\int_0^\infty x^{2\delta-1}e^{-2\varepsilon x}dx + \sum_{k=1}^n g(x_k)\int_0^\infty l_k^2(x)x^\lambda e^{-x}dx\right\} \leq$$

$$\leq c\|(f-p_{n-1})x^\alpha e^{-\left(\frac{1}{2}-\varepsilon\right)x}\|_\infty^2 \leq cn^{-k}\|u_{\alpha+k/2}\hat{f}^{(k)}\|_\infty^2 \quad (k=0,1,\ldots)$$

as (19) and Theorem 1 shows; this proves (18).

It is known [9] that the polynomials are dense the L^2 spaces weighted by $x^\lambda\ e^{-x}$ and for them (17) is obvions, hence (17) follows from the Banach-Steinhaus theorem.

Theorem 3 is proved.

Now we come to the Turán's problem. BALÁZS and TURÁN considered in [3] the following problem. Consider a function

$$\varphi \in C[0, \infty)$$

such that

$$(20)\qquad \lim_{t\to+\infty} \varphi(t)e^{-ct} = 0.$$

Suppose that we are given only finitely many discrete values of φ. What can we say on the Fourier transform of $\varphi(t)e^{-t}$?

They chose the nodes the zeros of the n-th Laguerre polynomial $L_n^{(0)}(x)$ and took the known Lagrange interpolation polynomial $L_n\varphi$ instead of the (unknown) function φ.

Let

$$\Phi(\varphi, x) := \mathcal{F}(\varphi(t)e^{-t})(x) = \int_0^\infty \varphi(t)e^{-t+ixt}dt,$$

$$\Phi_n(\varphi, x) := \mathcal{F}(L_n(\varphi, t)e^{-t})(x) = \int_0^\infty L_n(\varphi, t)e^{-t+ixt}dt.$$

In [3] is proved that if $\varphi(t)$ is a polynomial of degree $< n$ then $\Phi_n(\varphi, x) = \Phi(\varphi, x)$, futher for any φ salisfying (20)

$$\Phi_n(\varphi, x) \to \Phi(\varphi, x) \quad (n \to \infty) \tag{21}$$

uniformly for $x > 0$. We have

$$|\Phi(\varphi, x) - \Phi_n(\varphi, x)| \leq \int_0^\infty |\varphi(t) - L_n(\varphi, t)|e^{-t}dt \leq$$

$$\leq \left\{\int_0^\infty |\varphi(t) - L_n(\varphi, t|^2 e^{-t}dt\right\}^{1/2} \left\{\int_0^\infty e^{-t}dt\right\}^{1/2}; \tag{22}$$

in [3] the relation

$$\int_0^\infty |\varphi(t) - L_n(\varphi, t)|^2 e^{-t}dt \to 0 \quad (n \to \infty) \tag{23}$$

is in fact proved. In connection with these investigations P. TURÁN raised in [4] the following question (Problem XIII.): What can we say about the speed of convergence in (21) or (23) if the continuity modulus of φ is known? Our following theorem gives an answer.

THEOREM 4. *Use the notations of Theorem 3. and let* $\lambda > -1$, $\hat{f} \in W_\infty^0(\alpha)$. *Then we have*

$$\left(\int_0^\infty |f - L_n f|^2 t^\lambda e^{-t}dt\right)^{1/2} \leq c\omega\left(f^*, U_{2\alpha}, \frac{1}{\sqrt{n}}\right)_\infty. \tag{24}$$

PROOF. In proving Theorem 3 we showed that

$$\left(\int_0^\infty |f - L_n f|^2 t^\lambda e^{-t}dt\right)^{1/2} \leq cE_{n-1}(u_\alpha, \hat{f})_\infty.$$

Combining this with Theorem 2 we get (24) at once.

REMARK. Obviously

$$\omega(f^*, U_{2\alpha}, \delta)_\infty \leq 2\omega(f^*, \delta)_\infty := 2 \sup_{0 \leq h \leq \delta} \|f^*(x+h) - f^*(x)\|_{C(\mathbf{R})}.$$

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ON SOME PROBLEMS OF M. HORVÁTH*
(Saturation theorems for Walsh-Fourier expansions)

By

I. JOÓ

Department for Analysis of the L. Eötvös University, Budapest

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The classical saturation theorem of ALEXITS [7], p. 43 and 64, concerning the order of the approximation of the Fejér means, have been generalized for classical orthogonal expansions by many authors: I. Joó investigated the Hermite case [5], [16], [17], [18], A. BOGMÉR [19] and M. HORVÁTH [20] the Jacobi case finally M. HORVÁTH [21] the Laguerre case. In the papers [18], [20], [21] a saturation theorem is proved for the Abel-poisson means of resp. the Hermite, Jacobi and Laguerre expansions. In [21] M. HORVÁTH raised the problem: is there a known expansion, different from the classical ones, for which these saturation theorems hold?

In what follows we prove both saturation theorems for the Walsh expansions. In this case the conjugation operator is a simple multiplier operator hence the proofs are easier than in the above mentioned papers.

In the second part of this paper we prove estimates for the square sums of Laguerre polynomials. As it is known, in the investigation of eigenfunction expansions the estimates of the square sums of the eigenfunctions play an inportant role. E.g. we know that given a finite interval G, a potential $q\in L^1(G)$ and a complete orthonormal system $(U_n)\subset L^2(G)$ from the eigenfunctions of the operator $Lu=-u''+qu$ with eigenvalues $\{\lambda_k>0\}$, the eigenfunctions are necessarily uniformly bounded and moreover

$$(A)\qquad \sum_{|\mu-\sqrt{\lambda_n}|} |u_n(x)|^2 \le C(x\in G, \mu\ge 1)$$

(see [30]). This estimate can be locally inverted if $q\in L^p(G)$ for some $p>1$. Namely for any compact set $K\subset G$ there exists $M>0$ and $\alpha>0$ satisfying ([31])

$$(B)\qquad \sum_{|\mu-\sqrt{\lambda_n}|\le M} |u_n(x)|^2 \ge \alpha>0\ (x\in K, \mu>0).$$

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The estimates (A) and (B) do not remain necessarily true for infinite interval or for $q \in L^1_{\mathrm{loc}}(G)$. In the special case of Hermite functions

$$u_n(x) := e^{-\frac{x^2}{2}} h_n(x)$$

satisfying the equation ([27]).

$$-u_n'' + qu_n = \lambda_n u_n, \lambda_n = 2n, q(x) = x^2 - 1.$$

I proved in [26] the two sided estimate

$$(C) \qquad 0 < c_1\sqrt{k} \leq \max_{x \in \mathbf{R}} \sum_{k \leq \sqrt{n} \leq k+1} u_n^2(x) \leq c_2\sqrt{k} \quad (k \geq 1).$$

M. Horváth in [25] investigated Laguerre expansions. Through an equiconverge theorem he proved the Fejér–Lebesgue theorem corresponding to Laguerre series, namely: the Riesz means of order $\geq \frac{1}{4}$ of any function f from the (weighted) L^1 converge to f a.e. The Laguerre functions

$$v_n(x) = \sqrt{2}x^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}} l_n^{(\alpha)}(x^2)$$

satisfies ([27])

$$-v_n' + qv_n = \lambda_n v_n$$

$$q = x^2 - 2\alpha - 2 + \frac{\alpha^2 - \frac{1}{4}}{x^2}, \lambda_n = 4n.$$

In [25] is proved that

$$(D) \qquad \sum_{k \leq \sqrt{n} \leq k+1} v_n^2(x) \leq c\sqrt{k} \left(x > 0, \alpha \geq -\frac{1}{2} \right).$$

This is a generalization of the right inequality of (C). Intruduce the weight

$$U_\beta(x) = |x|^\beta e^{-\frac{x^2}{2}} \ (x \in \mathbf{R})$$

and denote $p_n(U_\beta^2, x)$ the polynomials orthonormal with respect to the weight U_β^2:

$$\int_{-\infty}^{\infty} p_n(U_\beta^2, x) p_m(U_\beta^2, x)\, U_\beta^2(x)\, dx = \delta_{n,m}.$$

By the substitution we see that

$$p_{2n}(U^2_{\alpha-\frac{1}{2}}, t) = l_n^{(\alpha)}(t^2)$$

$$p_{2n+1}(U^2_{\alpha+\frac{1}{2}}, t) = t\, l_n^{(\alpha+1)}(t^2)$$

hence *(D)* implies for $\beta \geq 0$ the estimate

$$(D') \qquad U_\beta^2(x) \sum_{k \leq \sqrt{n} \leq k+1} p_n^2(U_\beta^2, x) \leq c\sqrt{k} \; (k \geq 1, x \in \mathbf{R}).$$

Since for $\beta = 0$, $p_n(U_\beta^2, x)$ is the n-th Hermite polynomial $h_n(x)$, $(D)'$ indeed generalizes the right inequality of *(C)*. Here the following question arises naturally: can also the left hand inequality of *(C)* be generalized for $U_\beta(x)$ weight? More precisely : does the inequality

$$(E) \qquad 0 < c\sqrt{k} \leq \max_{x \geq 0} \sum_{k \leq \sqrt{n} \leq k+1} \nu_n^2(x)$$

hold for all $\alpha \geq -1/2$ with some constant $c > 0$? In the second part of this paper we shall prove an analogous statement, namely the lower estimate of the sum $\sum_{k=0}^{n} \nu_k^2(x)$.

Introduce the Walsh system as follows. Let

$$2^\omega = \{(\varepsilon_n)_0^\infty : \varepsilon_n = 0 \text{ or } 1\}$$

be the Walsh group with the addition

$$(1) \qquad (\varepsilon_n) \mathbin{\dot{+}} (\delta_n) := (|\varepsilon_n - \delta_n|).$$

We can define a mapping

$$\mathcal{A} : 2^\omega \to [0, 1],$$

$$(\varepsilon_n) \to \sum_{n=0}^{\infty} \varepsilon_n 2^{-n-1}$$

which is onto and the inverse mapping is well-defined for $x \in [0, 1]$ which are not dyadic rationals. Define for $x, t \in [0, 1]$, not dyadic rationals the sum

$$(2) \qquad x \mathbin{\dot{+}} t := \mathcal{A}(\mathcal{A}^{-1}x \mathbin{\dot{+}} \mathcal{A}^{-1}t)$$

(which can be not dyadic rational).

The Rademacher system is defined by

$$(3) \qquad r_0(t) := \begin{cases} 1, 0 < t < 1/2 \\ -1, 1/2 < t < 1 \end{cases}; \; r_0(t+1) := r_0(t)$$

and for $n \geq 1$

$$(4) \qquad r_n(t) := r_0(2^n t).$$

It is easy to see that $x = \sum_{j=0}^{\infty} \varepsilon_j 2^{-j-1}$ implies

$$(5) \qquad r_n(x) = (-1)^{\varepsilon_n}$$

consequently

$$(6) \qquad r_n(x \mathbin{\dot{+}} t) = r_n(x) r_n(t).$$

The Walsh system $(w_n)_0^\infty$ is defined as

$$w_n := \prod_{j=0}^{\infty} r_j^{\varepsilon_j} = \prod_{\varepsilon_j=1} r_j \text{ if } n = \sum_{j=0}^{\infty} \varepsilon_j 2^j. \tag{7}$$

We see from (1) and (5) that

$$w_n w_{n'} = w_{n+n'} \tag{8}$$

and in particular

$$w_{l2^n+s} = w_{l2^n} w_s. \tag{8'}$$

From (6) it follows

$$w_n(x \dot{+} t) = w_n(x)\, w_n(t), \tag{9}$$

further $\int_0^1 r_{n_1} r_{n_2} \cdot \ldots \cdot r_{n_k} = 0\ (n_i \neq n_j)$ implies

$$\int_0^1 w_n w_m = \delta_{n,m}. \tag{10}$$

So the Walsh system is a uniformly bounded orthonormal system on (0,1) and then the theorem of F. Riesz [1] implies that

$$\left\| \sum_{i=0}^{n} \gamma_i w_i \right\|_p \leq c(p) \left(\sum_{i=0}^{n} |\gamma_i|^q \right)^{1/q} \tag{11}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The Walsh – Fourier series of a function $f \in L^1(0, 1)$ is

$$f(x) \sim \sum_{j=0}^{\infty} c_j w_j(x),\ c_j := \int_0^1 f w_j$$

and the partial sums are

$$s_n[f, x] = \sum_{j=0}^{n-1} c_j w_j(x) = \int_0^1 f(t) D_n(x \dot{+} t)\, dt = (f * D_n)(x),$$

$$D_n(t) := \sum_{j=0}^{n-1} w_j(t). \tag{12}$$

Looking for a closed form for the kernel $D_n(t)$ we get

$$\sum_{j=2^k}^{2^{k+1}-1} w_j = \sum_{j=0}^{2^k-1} w_{2^k} w_j = r_k \sum_{j=0}^{2^k-1} w_j = r_k D_{2^k}$$

i.e. $D_{2^{k+1}}-D_{2^k}=r_kD_{2^k}$, whence

$$D_{2^k}(t)=\prod_{\nu=0}^{k-1}(1+r_\nu(t))=\begin{cases}2^k \text{ if } 0<t<2^{-k},\\ 0 \text{ if } 2^{-k}<t<1.\end{cases} \tag{13}$$

Unifying (12) and (13) we obtain

$$S_{2^N}[f,x]=\frac{1}{|\omega|}\int_\omega f \text{ if } x\in\omega,\ |\omega|=2^{-N} \tag{14}$$

(that is ω is the dyadic interval of length 2^{-N} containing x).

Consequently

$$\lim_{N\to\infty} S_{2^N}[f,x]=f(x) \quad \text{a.e.}; \tag{15}$$

this is false for ordinary Fourier series, see [1]. The relation (15) shows also the completeness of $(w_n)_0^\infty$ in $L^1(0,1)$.

Introduce the conjugate partial sums ([8]):

$$\tilde{S}_n[f,x]:=w_n(x)\,S_n[w_nf,x]. \tag{16}$$

In another form

$$\tilde{S}_n[f,x]=\int_0^1 f(t)\tilde{D}_n(x\dot{+}t)\,dt=(f*\tilde{D}_n)(x),$$

$$\tilde{D}_n(t):=w_n(t)\,D_n(t). \tag{16'}$$

We can easily obtain

$$\tilde{D}_n(t)=\sum_{j=0}^{\infty}\varepsilon_j\left(\sum_{\nu=2^j}^{2^{j+1}-1}w_\nu\right) \text{ if } n=\sum_{j=0}^{\infty}\varepsilon_j2^j. \tag{17}$$

Indeed, this is a consequence of

$$\{n,n\dot{+}1,\ldots,n\dot{+}(n-1)\}=\bigcup_{\varepsilon_j=1}\{2^j,2^j+1,\ldots,2^{j+1}-1\}, \tag{18}$$

which can be proved as follows. Denote H the right hand side of (18). Now $\nu\in H$ implies $n\dot{+}\nu<n$, hence

$$n\dot{+}H\subset\{0,1,\ldots,n-1\}.$$

Since H (and then $n+H$) has exactly n elements, the inclusion is in fact equality; so (18) and (17) hold indeed. From (13) we derive

$$\sum_{j=2^k}^{2^{k+1}-1}w_j(t)=\left\{\begin{array}{l}2^k \text{ if } 0<t<2^{-k-1},\\ -2^k \text{ if } 2^{-k-1}<t<2^{-k},\\ 0 \text{ if } 2^{-k}<t<1.\end{array}\right\}=r_k(t)\,D_{2^k}(t) \tag{19}$$

and then (17) gives that

(20) $$\tilde{D}_n(x \mathbin{\dot{+}} t) \text{ is constant in each } \omega \text{ if } x \notin \omega.$$

We shall prove

LEMMA 1.

a) If $k = 2^{k_1} + \ldots + 2^{k_s}$ $(k_1 < \ldots < k_s)$

then

$$|\{n < 2^{k_s+1} : n \mathbin{\dot{+}} k < n\}| = 2^{k_s} - 1.$$

b) Define

$$\tilde{\sigma}_n f := \frac{\tilde{S}_0 f + \ldots + \tilde{S}_{n-1} f}{n}$$

then

$$\tilde{\sigma}_n w_k \to \frac{1}{2}\left(1 - \frac{1}{2^{k_s}}\right) w_k \quad (n \to \infty).$$

PROOF.

a) If $n < 2^{k_s}$ then $n \mathbin{\dot{+}} k \geq 2^{k_s} > n$ and if $2^{k_s} \leq n < 2^{k_s+1}$ then $n \mathbin{\dot{+}} k < 2^{k_s} \leq n$ so a) follows.

b) We know

(21) $$\tilde{S}_n w_k = \begin{cases} w_k & \text{if } n \mathbin{\dot{+}} k < n, \\ 0 & \text{if } n \mathbin{\dot{+}} k \geq n. \end{cases}$$

Let for $j \in \mathbf{N}$

$$j = r \cdot 2^{k_s+1} + j', \; 0 \leq j' < 2^{k_s+1},$$

then, of course,

$$j \mathbin{\dot{+}} k < j \Leftrightarrow j' \mathbin{\dot{+}} k < j'.$$

Since for any fixed r the proportion of "good" j'-s (i.e. for which $j' \mathbin{\dot{+}} k < j'$) is

$$\frac{2^{k_s} - 1}{2^{k_s+1}} = \frac{1}{2}(1 - 2^{-k_s})$$

by a), hence b) follows. □

DEFINITION ([8]). The *conjugate function* of $f \in L^1(0,1)$ with expansion

$$f \sim \Sigma c_k w_k$$

is a function $\tilde{f} \in L^1(0,1)$ with expansion

(22) $$\tilde{f} \sim \frac{1}{2} \Sigma (1 - 2^{-k_s}) c_k w_k$$

(if such an $\tilde{f}$ exists in L^1).

We need the following norm estimate, proved by Paley

THEOREM 1 ([32] or [2], p. 90). *If* $1<p<\infty$ *and* $f\in L^p(0,1)$ *then*

$$||S_n f||_p \leq A_p ||f||_p$$

where the constant A_p *depends only on* p.

It is not hard to derive from this the

THEOREM 2. *For* $1<p<\infty$ *the conjugation operator maps continuously the whole* $L^p(0,1)$ *into itself:*

$$||\tilde{f}||_p \leq C(p)||f||_p.$$

PROOF. The Banach–Steinhaus theorem implies by Theorem 1 that

$$\|S_n f - f\|_p \to 0 \ (n\to\infty, f\in L^p). \tag{23}$$

The conjugation operator is a multiplier operator; this means that the operator multiplies the coefficients by fixed numbers, the k-th one by $\frac{1}{2}(1-2^{-k_s})$. The sequence of multipliers

$$\left\{\frac{1}{2}(1-2^{-k_s}): k = 0, 1, \ldots\right\}$$

is a bounded and monotone increasing sequence. As in the multiplier theorem of STEČKIN ([3] or [12], p. 105) we can prove that arbitrary sequence (α_n) bounded variation

$$\sum_{k=0}^{\infty} |\alpha_k - \alpha_{k+1}| < \infty$$

is a multiplier sequence, i.e. (for $1<p<\infty$)

$$\left\|\sum_{k=0}^{n} a_k\alpha_k w_k\right\|_p \leq C\cdot\left\|\sum_{k=0}^{n} a_k w_k\right\|_p$$

with C depending only on (α_k). Indeed, by Abel transform we see from Theorem 1

$$\left\|\sum_{k=0}^{n} a_k\alpha_k w_k\right\| = \left\|\alpha_n S_n + \sum_{k=0}^{n-1}(\alpha_k-\alpha_{k+1})S_k\right\|_p \leq$$

$$\leq |\alpha_n|\|S_n\|_p + \sum_{k=0}^{n-1}|\alpha_k-\alpha_{k+1}|\cdot\|S_k\|_p \leq c\|S_n\|_p.$$

This means in the special case $\alpha_n = \frac{1}{2}(1-2^{-k_s})$ that the partial sums of the formal series $\sum \frac{1}{2}(1-2^{-k_s})a_k w_k$ are bounded, i.e. the sequence of operators

$$T_n: L^p(0,1)\to L^p(0,1),$$

$$f \to \sum_{k=0}^{n} \frac{1}{2}(1-2^{-k_s})a_k w_k$$

are uniformly bounded. Using the same device as in [4], p. 23 (see also in [5]) we see that $T_n f$ converge in L^p to an $\tilde{f} \in L^p$

$$T_n f \to \tilde{f}.$$

Indeed, let for a fixed $f \in L^p(0, 1)$

$$\hat{T}_n: L^p(0, 1) \to \mathbf{R},$$
$$g \to \langle T_n f, g \rangle.$$

Now

$$|\hat{T}_n g| \leq \|T_n f\|_p \|g\|_q \leq c\|g\|_q,$$

hence $(\hat{T}_n)$ is uniformly bounded. It converges in a dense set of Walsh–polynomials

$$g = \sum_{j=1}^{n} c_k w_k,$$

hence there exists a continuous functional $\hat{T}$ with

$$\hat{T}g := \lim_{n\to\infty} \hat{T}_n g, \; g \in L^q(0, 1).$$

Since $\hat{T}g$ has the form $\langle h, g \rangle$ for some $h \in L^p(0, 1)$ and

$$a_k(h) = \langle h, w_k \rangle = \hat{T}w_k = \lim_{n\to\infty} \hat{T}_n w_k = \frac{1}{2}(1-2^{-k_s})a_k,$$

hence $h = \tilde{f}$ and by (23) we get $\|\tilde{f}\|_P \leq C(p)\|f\|_p$.

Theorem 2 is proved.

DEFINITION ([6]). Consider the difference quotient operators

$$(d_n f)(x) := \sum_{k=0}^{n-1} 2^{k-1}[f(x \dotplus 2^{-k-1}) - f(x)].$$

a) We say that a function has a *dyadic derivative* at the point $x \in [0,1)$ if there exists the limit

$$f^{[1]}(x) := \lim_{n\to\infty} (d_n f)(x);$$

$f[1](x)$ is the dyadic derivative of f.

b) If for $f \in L^p(0,1)$, $1 \leq p < \infty$ the functions $d_n f$ converge in L^p to some $g \in L^p$,

$$\lim_{n\to\infty} \|d_n f - f\|_p = 0,$$

then g is called the *strong dyadic derivative* of f and we denote

$$Df := g.$$

THEOREM 3 ([6]).

a) $Dw_k = kw_k$,

b) $D : L^p(0,1) \to L^p(0,1)$

is a linear and closed operator whose inverse is absolutely continuous. Namely

c) *There exists a function* $W \in L^1(0,1)$ *with expansion*

$$W \sim \sum_{k=1}^{\infty} \frac{1}{k} w_k.$$

d) *The transformation*

$$If := W * f = \int_0^1 f(t)\, W(x \mathbin{\dot{+}} t)\, dt$$

is the inverse of D : *if* $f \in L^1(0,1)$ *and*

$$a_0(f) = \int_0^1 f = 0$$

then

$$DIf = IDf = f.$$

The function If is called the dyadic integral of f.

Some questions about the dyadic derivative and integral were investigated by F. SCHIPP, P. SIMON, J. PÁL, cf. [2], [9], [10], [11].

Now we prove an Alexits type theorem

THEOREM 4. *Let* $1 \le p \le \infty$ *and* $f \in L^p(0,1)$. *Then*

$$\|\sigma_n f - f\|_p = O\left(\frac{1}{n}\right). \tag{24}$$

if and only if

$$Df \in L^p. \tag{25}$$

Further we have

$$\|\sigma_n f - f\| = o\left(\frac{1}{n}\right) \Leftrightarrow f \equiv \text{const}. \tag{24'}$$

PROOF. First prove (24′). Let $f \sim \Sigma a_k w_k$, then for fixed $k \le n$ we have

$$\frac{k}{n+1}|a_k| = \left|\int_0^1 [\sigma_n f - f] w_k\right| \le \|\sigma_n f - f\|_p \cdot \|w_k\|_q = o\left(\frac{1}{n}\right)$$

which implies for $k \neq 0$ that $a_k = 0$, so $f = c$. Conversely if $f = c$ then $\sigma_n f = f$. Now prove that (24)$\Leftrightarrow$(25). By the Alexits lemma ([7], p. 62–63)

$$\|\sigma_n f - f\|_p = O\left(\frac{1}{n}\right) \Leftrightarrow \|\sigma_n(\sum ka_k w_k)\|_p = O(1). \tag{26}$$

We know that the Fejér means of a function from $L^p(0,1)$ are bounded in L^p-norm provided $1 < p \leq \infty$. In case $p < \infty$ this follows from Theorem 1, in case $p = \infty$ the boundedness of the strong Fejér means is already known [13]. Using this and the proof of Lemma 3 of [5] we get that

$$\|\sigma_n f - f\|_p = O\left(\frac{1}{n}\right) \Leftrightarrow \exists g \in L^p,\ g \sim \sum ka_k w_k. \tag{27}$$

Now take $g \in L^p$ with the expansion indicated in (27) and count the coefficients of Ig:

$$\int_0^1 (I_g)(x) w_k(x)\, dx = \int_0^1 \left(\int_0^1 g(t)\, w(x \dot{+} t)\, dt \right) w_k(x)\, dx =$$

$$= \int_0^1 g(t) \left(\int_0^1 w(x \dot{+} t)\, w_k(x)\, dx \right) dt =$$

$$= \int_0^1 g(t) \left(\int_0^1 w(x) w_k(x \dot{+} t) dx \right) dt =$$

$$= \frac{1}{k} \int_0^1 g(t)\, w_k(t)\, dt = a_k \ (k \geq 1).$$

Since $S_{2^N} f \to f$ a.e. for any $f \in L^1(0,1)$, hence the Walsh system is complete in $L^1(0,1)$, consequently

$$Ig = f + c$$

and then

$$g = DIg = Df$$

so $Df \in L^p(0,1)$. Conversely suppose that $Df \in L^p(0,1)$; its coefficients are

$$\int_0^1 (Df) w_k = \lim_{n \to \infty} \int_0^1 (d_n f) w_k =$$

$$= \lim_{n \to \infty} \int_0^1 f d_n w_k = \int_0^1 f D w_k = k \int_0^1 f w_k = k a_k$$

i.e. $g = Df \in L^p$ has the expansions figuring in (27). Theorem 4. is proved.

THEOREM 4′. *Denote*

$$R_n^r(\sum \varrho_k) := \sum_{k=0}^{n} \left(1 - \frac{k^r}{(n+1)^r}\right) \varrho_k$$

n-th Riesz mean of parameter r. Let $r \geq 1$ be entire and denote D^r the r-th ated strong derivation. Then for $1 < p \leq \infty$ we have

$$D^r f \in L^p(0, 1) \Leftrightarrow \|R_n^r f - f\|_p = O\left(\frac{1}{n^r}\right)$$

$$f = c \Leftrightarrow \|R_n^r f - f\|_p = o\left(\frac{1}{n^r}\right). \tag{29}$$

PROOF. This theorem is analogous to KRÁLIK [14], Korollar, p. 369. Using the Alexits- type lemma, (Satz 1 in [14]) asserting that

$$\|R_n^r(\sum \varphi_k)\| = O\left(\frac{1}{n^r}\right) \Leftrightarrow \|R_n^r(\sum k^r \varphi_k)\| = O(1)$$

valid in arbitrary Banach spaces we can give same proof as in Theorem 4. Remark that the Riesz means R_n^r, $r > 1$ are convex linear combinations of the Fejér means σ_k, $k \leq n$ hence we have

$$\|R_n^r f\|_p \leq c(p) \|f\|_p, \ 1 < p \leq \infty$$

as for the Fejér means. The remained part of the proof is immediate.

Consider the Abel – Poisson means

$$T_x f(y) = f(x, y) := \int_0^1 f(z) P(x, y, z)\, dz \ (x > 0) \tag{30}$$

where

$$P(x, y, z) = \sum_{k=0}^{\infty} e^{-kx} w_k(y \mathbin{\dot{+}} z) = \sum_{k=0}^{\infty} e^{-kx} w_k(y) w_k(z). \tag{31}$$

The series in (31) converge uniformly for fixed $x > 0$ hence for $f \in L^1(0,1)$, $f \sim \Sigma a_k w_k$ we have

$$f(x, y) = \sum a_k e^{-kx} w_k(y) \tag{32}$$

again with uniform convergence since $|a_k| \leq \|f\|_1$.

THEOREM 5. *Let $1 < p < \infty$ and $f \in L^p(0,1)$. Then*

$$\|f(x, \cdot) - f\| = \bar{\bar{o}}(x) \ (x \to 0+) \Leftrightarrow f = 0, \tag{33}$$

$$\|f(x, \cdot) - f\| = O(x) \Leftrightarrow Df \in L^p. \tag{34}$$

PROOF. The semigroup property

$$T_{x_1} T_{x_2} f = T_{x_1 + x_2} f \tag{35}$$

can be proved as follows

$$T_{x_1}T_{x_2}f(y) = \int_0^1 (T_{x_2}f)(z)\,P(x_1, y, z)\,dz =$$

$$= \int_0^1 \left(\int_0^1 f(t)\,P(x_2, z, t)\,dt \right) P(x_1, y, z)\,dz =$$

$$= \int_0^1 f(t) \int_0^1 P(x_1, y, z)\,P(x_2, z, t)\,dz\,dt =$$

$$= \int_0^1 f(t)\,P(x_1 + x_2, y, t)\,dt = T_{x_1+x_2}f(y).$$

Since the partial sums $S_n f$ converge to f in L^p-norm, the Abel means must also converge:

(36) $$\|f(x, \cdot) - f\|_p \to O(1 < p < \infty,\ f \in L^p(0, 1)).$$

Consequently the operators $\{T_x : x > 0\}$ with $T_0 f := f$ form a continuous operator semigroup in the reflexive Banach space $L^p(0,1)$. So we have to prove that

(37) $$f \in D(A) \Leftrightarrow Df \in L^p,$$

where A is the infinitesimal generator of the semigroup $\{T_x : x \geq 0\}$ and $D(A)$ is its domain (see [15], p. 90.). As we have seen above, (37) states the same as

(38) $$f \in D(A) \Leftrightarrow \exists g \in L^p,\ g \sim \Sigma k a_k w_k.$$

Prove (38). Let first $f \in D(A)$, then

$$\int_0^1 (Af) w_k = \lim_{x \to 0+} \int_0^1 \frac{f(x, y) - f(y)}{x} w_k(y)\,dy =$$

$$= \lim_{x \to 0+} \frac{e^{-kx} - 1}{x} a_k = -k a_k,$$

hence

$$-Af \sim \Sigma k a_k w_k$$

so the only if part of (38) is proved. To prove the if part, take a function, $g \in L^p$, $g \sim \Sigma k a_k w_k$. It is easy see that

(39) $$-A(\sigma_n f) = \sigma_n g$$

and by Theorem 1

$$\|\sigma_n f - f\|_p \to 0, \|\sigma_n g - g\|_p \to 0 \ (n \to \infty). \tag{40}$$

Since the operator A is closed ([15]), (39), and (40) imply that $f \in D(A)$ and $Af = -g$. The proof is complete.

REMARK. If $1 < p < \infty$ and $f \in L^p(0,1)$ then

$$Df \in L^p \Leftrightarrow D\tilde{f} \in L^p \tag{41}$$

and

$$\|\sigma_n f - f\|_p = O\left(\frac{1}{n}\right) \Leftrightarrow \|\sigma_n \tilde{f} - \tilde{f}\|_p = O\left(\frac{1}{n}\right). \tag{42}$$

Indeed, the sequences

$$\alpha_k = 1 - 2^{-k_s}, \beta_k = (1 - 2^{-k_s})^{-1}$$

satisfy

$$\sum |\alpha_{k+1} - \alpha_k| < \infty, \sum |\beta_{k+1} - \beta_k| < \infty$$

hence using the proof of Theorems 2 and 4 we get

$$Df \in L^p \Leftrightarrow \exists g \in L^p, g \sim \sum k a_k w_k \Leftrightarrow$$

$$\Leftrightarrow \|S_n(\sum k a_k w_k)\|_p = O(1) \Leftrightarrow \left\| S_n\left(\sum \frac{1}{2}(1 - 2^{-k_s}) k a_k w_k \right\| =$$

$$= O(1) \Leftrightarrow \exists h \in L^p, h \sim \sum \frac{1}{2}(1 - 2^{-k_s}) k a_k w_k \Leftrightarrow D\tilde{f} \in L^p.$$

(41) follows. By Theorems 4 and 2 we obtain (42) from (41).

We formulate here the following

PROBLEM. Does the statements (41) or (42) remain true for $p = \infty$?

In what follows we shall deal with lower estimates for the square sum of Laguerre polynomials. Recall some notions as given in [24] Let $d\alpha$ be an m-distribution function on $\mathbf{R}$, $p_n(x) = p_n(d\alpha, x)$ the n-th polynomial orthonormal to $d\alpha$

$$\int_{\mathbf{R}} p_n p_m \, d\alpha = \delta_{n,m}.$$

Denote by $X_{i,n} = X_{i,n}(d\alpha)$,

$$X_{1,n} > X_{2,n} > \ldots > X_{n,n}$$

the roots of $p_n(d\alpha, x)$. The Christoffel function of the m-distribution $d\alpha$ is

$$\lambda_n(d\alpha, x) := \left(\sum_{k=0}^{n-1} p_k(d\alpha, x) \right)^{-1}.$$

Consider further the denominator of the Christoffel—Darboux formula

$$\Psi_n(x, \xi) = p_{n-1}(\xi)p_n(x) - p_n(\xi)p_{n-1}(x).$$

In x, the polynomial Ψ_n has degree

$$n^*: = \begin{cases} n & \text{if } p_{n-1}(\xi) \neq 0 \\ n-1 & \text{if } p_{n-1}(\xi) = 0 \end{cases}$$

and denote its zeros by

$$\xi_1 > \xi_2 > \ldots > \xi_j = \xi > \xi_{j+1} > \ldots > \xi_{n^*}.$$

The following inequality of Posse is known ([24], p. 33). Let $n \geq 3$ and $f \in C^{2n^*-1}(-\infty, \xi)$ with

$$f > 0, f' > 0, \ldots, f^{(2n^*-1)} > 0 \text{ on } (-\infty, \xi).$$

Then

$$\sum_{\xi_k < \xi} \lambda_n(d\alpha, \xi_k) f(\xi_k) \leq \int_{-\infty}^{\xi} f d\alpha \leq \sum_{\xi_k \leq \xi} \lambda_n(d\alpha, \xi_k) f(\xi_k),$$

which implies by $\xi = \xi_k$ that

$$\lambda_n(d\alpha, \xi) f(\xi) \leq \int_{\xi_{j+1}}^{\xi_{j-1}} f d\alpha.$$

We shall need the Laguerre weight

$$u_\alpha(x) = x^\alpha e^{-x} \quad (x > 0, \alpha > -1).$$

THEOREM 6. *Let $\alpha > -1$. If C_1 is large enough, C_2 is small enough (depending on α) then*

$$\lambda_n(u_\alpha, x) \leq \frac{c(x)}{\sqrt{n}} x^{\alpha + \frac{1}{2}} e^{-x}, \frac{c_1}{n} \leq x \leq c_2 n.$$

REMARK. This statement was proved by FREUD [23] for $\alpha \geq 0$. In [29] the authors stated its validity for any $\alpha > -1$; the proof will be given here.

PROOF. Use the Posse inequality with $f(x) = e^x$:

$$\lambda_n(\xi) e^\xi \leq \int_{\xi_{j+1}}^{\xi_{j-1}} e^x x^\alpha e^{-x} dx = \frac{\xi_{j-1}^{\alpha+1} - \xi_{j+1}^{\alpha+1}}{\alpha+1} = (\xi_{j-1} - \xi_{j+1}) \tilde{\xi}_j \tag{43}$$

with an appropriate

$$\xi_{j+1} < \tilde{\xi}_j < \xi_{j-1}.$$

We know [24] that the sequences ξ_k and $X_{k,n}$ separate each other. More precisely in case $n^* = n-1$ we have

$$x_{n,n} < \xi_{n^*} < x_{n-1,n} < \xi_{n^*-1} < \ldots < x_{2,n} < \xi < x_{1,n} \tag{44}$$

and in case $n^* = n$ one of the statements

$$\xi_{n^*} < x_{n,n} < \xi_{n^*-1} < \ldots < x_{2,n} < \xi_1 < x_{1,n} \tag{45}$$

$$x_{n,n} < \xi_{n^*} < \ldots < \xi_2 < x_{1,n} < \xi_1 \tag{46}$$

holds. The following estimates are known [27] for the smallest and largest zeros of Laguerre polynomials

$$\frac{c'}{n} > x_{n,n} > \frac{\left(\frac{j_1}{2}\right)^2}{n+\frac{\alpha+1}{2}} \quad (n > n_0) \tag{47}$$

where j_1 is the smallest zero of the Bessel function $J_\alpha(x)$;

$$c''n < \frac{\left(\frac{j_n}{2}\right)^2}{n+\frac{\alpha+1}{2}} < x_{1,n} < 2n+\alpha+1+\sqrt{(2n+\alpha+1)^2+\frac{1}{4}-\alpha^2} =$$

$$= 4n+O(1), \tag{48}$$

where j_n is the n-th zero of $J_\alpha(x)$. Using (44)–(48) we see that if c_1 is large enough, c_2 is small enough and $\frac{c_1}{n} \leq \xi \leq c_2 n$ then (for $n > n_0$)

$$3 \leq j \leq n^* - 3$$

and hence (43) implies that

$$\lambda_n(\xi)e^\xi \leq (\xi_{j-1}-\xi_{j+1})\tilde{\xi}_j^\alpha \leq (x_{l-2,n}-x_{i+2,n})x^\alpha_{i\pm 2,n} \tag{49}$$

where the sign $\pm$ depends on the sign of α. To estimate (49) further we use the following known theorem of STURM ([27]). Suppose that the function $U(t)$ satisfies

$$U''+Ru = 0 \tag{50}$$

where the potential R is positive and continuous in $[t_1, t_2]$ further $U(t_1) = U(t_2) = 0$, $U(t) \neq 0$ for $t \in (t_1, t_2)$.

Then

$$\min_{[t_1,t_2]} R \leq \frac{\pi^2}{(t_2-t_1)^2} \leq \max_{[t_1,t_2]} R.$$

We shall apply this with

$$u(t) = e^{-\frac{t^2}{2}} t^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(t^2)$$

since in this case ([27]) (50) holds with

$$R(t) = 4n - t^2 + 2\alpha + 2 + \frac{1-4\alpha^2}{4x^2}.$$

Introdice the notation

$$a \asymp b,$$

this means that there exist positive constants c, C which may depend only on α satisfying

$$ca \leq b \leq Ca.$$

The Sturm theorem implies as in [23] that

$$\varphi_n(x_{k,n}) \asymp x_{k,n} - x_{k+1,n} \tag{51}$$

where the function φ_n is defined by

$$\varphi_n(x) = \begin{cases} \dfrac{1}{n} & , 0 \leq x < \dfrac{1}{n}, \\ \sqrt{\dfrac{x}{4n-x}}, & \dfrac{1}{n} \leq x \leq 4n - 4n^{1/3}, \\ n^{1/3} & , 4n - n^{1/3} \leq x. \end{cases}$$

So (49) can be further estimated by

$$\lambda_n(\xi)e^{\xi} \leq c_4 \sqrt{\frac{x_{j\pm 2,n}}{4n - x_{j\pm 2,n}}} x_{j\pm 2,n}^{\alpha} \leq c \frac{1}{\sqrt{n}} x_{j\pm 2,n}^{\alpha+\frac{1}{2}}.$$

Finally we know from [27], (6.31.11) that

$$C\frac{(j_{n-i})^2}{n} \leq x_{i,n} \leq c\frac{(n-i)^2}{n}$$

and it follows from [27], (1.71.7) that

$$j_k \asymp k \quad (k > k_0)$$

So if the index i is $< n - n_0$ with large n_0 then

$$x_{i,n} \asymp \frac{(n-i)^2}{n}, \; x_{i+1} \asymp x_i.$$

Taking c_1 large enough we can obtain that

$$j < n - n_0 - 2$$

and hence

$$\lambda_n(\xi)e^{\xi} \leq c\frac{1}{\sqrt{n}}x_{j\pm 2,n}^{\alpha+\frac{1}{2}} \leq c\frac{1}{\sqrt{n}}\xi^{\alpha+\frac{1}{2}}.$$

Theorem 6. is proved. □

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INTERPOLATION BETWEEN DYADIC HARDY SPACES H^p: THE REAL METHOD

By

V. ECHANDIA

Central University of Venezuela

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1. Preliminaries

In what follow L^p will denote $L^p[0,1]$, $0<p\leq\infty$. In this section following [6], we shall introduce the dyadic Hardy spaces.

DEFINITION 1. For each $f\in L^1$, let

$$E_n f = S_{2^n} f \quad (n\in\mathbf{N}:=\{0,1,2,\ldots\})$$

where $S_{2^n}f$ is the 2^nth partial sum of the Walsh series of f.

In [6] it is proved that

$$\|E_n f\|_p \leq \|f\|_p \ (f\in L^p;\ 1\leq p\leq\infty)\ \square. \tag{1}$$

DEFINITION 2. The dyadic maximal operator $f^* = E^*f$ is defined by

$$f^* = E^*f = \sup_{n\in\mathbf{N}} |E_n f| \ (f\in L^1).$$

For $f\in L^1$, set

$$\tilde{E}_0 f = |E_0 f|,$$

and for $n\in\mathbf{P}$, also define

$$\tilde{E}_n f = \sup_{0\leq m\leq n} (|E_m f| + |E_m(f r_m)|),$$

where r_m is the m'th Rademacher function.

For each $f\in L^1$ set

$$\tilde{E} f = \sup_{n\in\mathbf{N}} \tilde{E}_n f.$$

Is it easy to see that

$$E^*f \leq \tilde{E}f \leq 3E^*f. \tag{2}$$

The following lemma is proved in [6].

Denote $\chi(A)$ the characteristic function of the set A.

LEMMA 1. If $f \in L^1$ has zero mean and we define $f_n = E_n f$, and for $k \in Z$ we put

$$f^{(k)} = \sum_{n=0}^{\infty} \chi\{2^k < \tilde{E}_n f \leq 2^{k+1}\}(f_{n+1} - f_n);$$

then we have

$$E^*(f^k) \leq 2^{k+2} \chi\{\tilde{E} f > 2^k\};$$

and

$$f = \sum_{k \in Z} f^{(k)}$$

a.e. on $[0,1]$. Moreover, if $E^* f \in L^1$, then this series converge to f in L^1 norm.

This series will be called the canonical decomposition of f.

DEFINITION 3. The dyadic Hardy spaces is defined as

$$\mathbf{H}^p := \left\{ f \in L^1 : \|f\|_{\mathbf{H}^p} := \left(\int_0^1 (E^* f)^p \, dx \right)^{1/p} < \infty \right\} \quad (0 < p < \infty).$$

Notice that $\|\ldots\|_{\mathbf{H}^p}$ is a norm on $\mathbf{H}^p$, when $1 \leq p < \infty$, and a quasinorm when $0 < p < 1$. For a detailed study of these spaces see [6].

Based on the idea of [4]; $\mathbf{H}^{p,q}$ is defined as follow:

$$f \in \mathbf{H}^{p,q} \text{ if and only if, } f^* \in L^{p,q}[0,1],$$

and

$$\|f\|_{\mathbf{H}^{p,q}} = \|f^*\|_{L^{p,q}},$$

where $L^{p,q}$ is the Lorentz space on $[0,1]$.

Notice that for $p > 1$ $\mathbf{H}^{p,q} = L^{p,q}$.

2. Characterization of intermediate spaces between $\mathbf{H}^p$ $(0 < p < \infty)$ and L^∞

The following lemma will give a decomposition of $\mathbf{H}^p$ functions into "good" and "bad" parts.

LEMMA 2. Let $f \in L^1$ be of zero mean, $y > 0$ and $0 < p < \infty$. Then f can be written as the sum of two functions g and h which satisfy the following inequalities with some absolute constante $c > 0$,

i) $$\|g\|_\infty < cy$$

ii) $$\|h\|_{\mathbf{H}^p} \leq c \int_{\{f^* > y\}} (f^*(t))^p dt.$$

PROOF. Let $k_0 \in Z$ such that $\frac{2^{k_0-1}}{3} < y \leq \frac{2^{k_0}}{3}$; and take the canonical decomposition of f:

$$f = \sum_{k \in Z} f^{(k)} = \sum_{k<k_0} f^{(k)} + \sum_{k \geq k_0} f^{(k)} = g + h;$$

since

$$|f^{(k)}| \leq |E^* f^{(k)}| \leq 4\chi\{\tilde{E}f > 2^k\}2^k$$

therefore

$$|g| \leq \sum_{k<k_0} |f^{(k)}| \leq 4 \sum_{k<k_0} 2^k < c_1 2^{k_0-1} < cy,$$

consequently,

$$\|g\|_\infty < cy.$$

To prove the other inequality, we have that

$$h^* \leq \sum_{k \geq k_0} E^* f^{(k)} \leq 4 \sum_{k \geq k_0} \chi\{\tilde{E}f > 2^k\}2^k.$$

Applying Abel's transformation, we obtain

$$h^* \leq 4 \sum_{k=k_0}^{\infty} \chi\{2^k < \tilde{E}f \leq 2^{k+1}\}2^k + 4\, 2^{k_0}\chi\{\tilde{E}f > 2^{k_0}\}.$$

Therefore

$$|h^*|^p \leq c_p \left(\sum_{k=k_0}^{\infty} \chi\{2^k < \tilde{E}f \leq 2^{k+1}\}2^{kp} + 2^{k_0 p}\chi\{\tilde{E}f > 2^{k_0}\} \right),$$

where c_p is constant depending only on p. Hence we obtain that

$$\int_0^1 |h^*|^p dt \leq c_p \sum_{k=k_0}^{\infty} \int_{\{2^k < \tilde{E}f \leq 2^{k+1}\}} 2^{kp} dt + c_p \int_{\{E^* f > 2^{k_0}\}} 2^{k_0 p} dt \leq$$

$$\leq c_p \sum_{k=k_0}^{\infty} \int_{\{2^k < \tilde{E}f \leq 2^{k+1}\}} |\tilde{E}f|^p dt + c_p \int_{\{\tilde{E}f > 2^{k_0}\}} |\tilde{E}f|^p dt.$$

From the fact that $\{\tilde{E}f > 2^{k_0}\} = \bigcup_{k=k_0} \{2^k < \tilde{E}f \leq 2^{k+1}\}$, and using (2), we obtain that

$$\int_0^1 |h^*|^p dt \leq 2c_p \int_{\{\tilde{E}f > 2^{k_0}\}} |\tilde{E}f|^p dt \leq 2c_p \int_{\left\{E^* f > \frac{2^{k_0}}{3}\right\}} |3E^* f|^p dt \leq$$

$$\leq \tilde{c}_p \int_{\{f^* > y\}} |f^*|^p dt.$$

The proof is complete.

Now we prove the following

THEOREM 1. *If* $0<p<\infty$; $0<\Theta<1$; $0<q<\infty$; *then*

$$(\mathbf{H}^{p_0}, L^\infty)_{\Theta, q} = \mathbf{H}^{p, q};$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0}.$$

PROOF. Set $T(f) = f^*$, clearly T is a sublinear operator. The equality

$$\|f^*\|_{p_0} = \|f\|_{\mathbf{H}^{p_0}},$$

and (1) imply that T is a bounded operator form H^{p_0} into L^{p_0}, and from L^∞ into L^∞. By interpolation we can conclude that T is bounded from $(\mathbf{H}^{p_0}, L^\infty)_{\Theta, q}$ into $(L^{p_0}, L^\infty)_{\Theta, q} = L^{p, q}$. Hence if $f \in (\mathbf{H}^{p_0}, L^\infty)_{\Theta, q}$, then $f^* \in L^{p, q}$ and therefore $f \in \mathbf{H}^{p, q}$; i.e. $(\mathbf{H}^{p_0}, L^\infty)_{\Theta, q} \subset \mathbf{H}^{p, q}$.

For the inverse inclusion, take $f \in \mathbf{H}^{p, q}$. Denote by $\overline{f^*}$ the non increasing rearragement of f^*. Fix $t>0$, and take $y = \overline{f^*}(t^{p_0})$. Let $f = g_t + h_t$ be the decomposition in Lemma 1. Then we have

$$\|h_t\|_{\mathbf{H}^{p_0}} = \left(\int_0^1 |h_t^*(x)|^{p_0} dx\right)^{1/p} \leq c\left(\int_{\{f^*>\overline{f^*}(t^{p_0})\}} (f^*(x))^{p_0} dx\right)^{1/p_0} \leq$$

$$\leq c\left(\int_0^{t^{p_0}} (\overline{f^*}(s))^{p_0} ds\right)^{1/p_0}.$$

Therefore

$$\int_0^\infty (t^{-\Theta}(\|h_t\|_{\mathbf{H}^{p_0}}))^q \frac{dt}{t} \leq c \int_0^\infty t^{-\Theta q}\left(\int_0^{t^{p_0}} \overline{f^*}(s) ds\right)^{q/p_0} \frac{dt}{t} =$$

$$= c \int_0^\infty t^{-\Theta q/p_0}\left(\int_0^t (\overline{f^*}(s))^{p_0} ds\right)^{q/p_0} \frac{dt}{t} =$$

$$= c \int_0^\infty \left(t^{-\Theta} \int_0^t (\overline{f^*}(s))^{p_0} ds\right)^{q/p_0} \frac{dt}{t} = c \int_0^\infty \left(\frac{1}{t} \int_0^t (\overline{f^*}(s))^{p_0} ds\right)^{q/p_0} t^{\frac{q(1-\Theta)}{p_0}} \frac{dt}{t}.$$

Applying theorem 8 of [5] if $q<p_0$, or Hardy's inequality if $q \geq p_0$, we obtain

$$\int_0^\infty (t^{-\Theta}\|h_t\|_{\mathbf{H}^{p_0}})^q \frac{dt}{t} \leq c \int_0^\infty t^{\frac{q(1-\Theta)}{p_0}} (\overline{f^*}(t))^q \frac{dt}{t} = c\|f^*\|^p_{L^{p,q}}.$$

Furthermore

$$\int_0^\infty (t^{(1-\Theta)}\|g_t\|_{L^\infty})^q \frac{dt}{t} \leq c \int_0^\infty (t^{(1-\Theta)} f^*(t^{p_0}))^q \frac{dt}{t} =$$

$$= c \int_0^\infty (t^{p_0/p} \overline{f^*}(t^{p_0}))^q \frac{dt}{t} = c \int_0^\infty (u^{1/p} \overline{f^*}(u))^q \frac{du}{u} = c\|f^*\|_{L^{p,q}}.$$

The inequality

$$K(t, f; \mathbf{H}^{p_0}, L^\infty) \leq \|h_t\|_{\mathbf{H}^{p_0}} + t\|g_t\|_{L^\infty},$$

implies

$$\int_0^\infty \left[(t^{-\Theta} K(t, f))^q \frac{dt}{t} \right]^{1/q} \leq c\|f^*\|_{L^{p,q}} = c\|f\|_{\mathbf{H}^{p,q}}$$

Hence

$$\mathbf{H}^{p,q} \subset (\mathbf{H}^{p_0}, L^\infty)_{\Theta,q}$$

which complete the proof.

By reiteration we get the following Corollary to Theorem 1.

COROLLARY 1. If $0 < \Theta < 1$; $0 < q \leq \infty$; $0 < p_i < \infty$ $(i = 0,1)$, and

$$\frac{1}{p} = \frac{1-\Theta}{p_1} + \frac{\Theta}{p_0},$$

then

$$(\mathbf{H}^{p_0,q_0}, \mathbf{H}^{p_1,q_1})_{\Theta,q} = \mathbf{H}^{p,q}.$$

We have the following result analogous to [5].

COROLLARY 2. If $1 \leq p_1 < \infty$, $0 < q_1 < \infty$, $0 < q < \infty$, $0 < \Theta < 1$, and

$$\frac{1}{p} = 1 - \Theta + \frac{\Theta}{p_1};$$

then

$$(\mathbf{H}^1, L^{p_1,q_1})_{\Theta,q} = L^{p,q}$$

in particular, if $q_1 = p_1$, $p = q$, one gets

$$(\mathbf{H}^1, L^{p_1})_{\Theta,p} = L^p.$$

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Address:
MATHEMATICAL INSTITUTE, L. EÖTVÖS UNIVERSITY
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Technikai szerkesztő:
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