ANNALES
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SECTIO MATHEMATICA

TOMUS XXXII.

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ON THE DIVERGENCE OF EIGENFUNCTION EXPANSIONS

By

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Generalizing some previous results of DU BOIS REYMOND and L. FEJÉR [4], [5], A. N. KOLMOGOROFF [2] proved the

THEOREM A. *There exists $f \in L_1(0,2\pi)$ such that for its trigonometrical partial sums $S_n(f, x)$*

$$\sup_{n \geq 1} |S_n(f, x)| = \infty \tag{1}$$

holds in almost every $x \in (0,2\pi)$ (in every $x \in [0,2\pi]$).

After, this divergence theorem was extended to different directions by E. M. STEIN, K. TANDORI, and many others (Cf. monographies).

The divergence problem for eigenfunction expansions connected with the Laplace operator was solved by V. A. IL'IN [3] and it was strenghtened in [9].

S. V. BOCKAREV [6] proved the

THEOREM B. *Let $\{\phi_n\}_1^\infty$ be any orthonormal system on* (0,1) *with the property*

$$|\phi_n(x)| \leq M, \ ((0 \leq x \leq); \ n = 1, 2, \ldots). \tag{2}$$

The there exists $f \in L_1(0,1)$ such that the Fourier series of f with respect to the system $\{\phi_n\}$ diverges on some set $H \subset (0,1)$; $|H| > 0$.

A different divergence phenomenon was obtained by J. MARCINKIEWICZ in [12], namely he proved the

THEOREM C. *There exists $f \in C[0,1]$ and $\lambda_n > 0$ such that for any $g \in L_0(0,1)$ there exists a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ for which*

$$\lim_{k \to \infty} \frac{f(x+\lambda_{n_k}) - f(x)}{\lambda_{n_k}} = g(x) \text{ a.e. } x \in (0,1). \tag{3}$$

The analogon (in some sense — but this is not clear at this moment) of this result for trigonometric series was proved by D. MENCHOFF [10] and it was generalized by A. A. TALALYAN [11] for arbitrary orthonormal systems, namely in [11] is proved the

THEOREM D. *If $\{\phi_n\}_1^\infty$ denotes an arbitrary complete in L_2 (0,1) orthonormal system, then it is possible to construct from it a series $\Sigma\, a_n \phi_n$ $(a_n \to 0)$ such that the sequence $\{\Sigma_1^N a_n \phi_n\}_{N=1}^\infty$ is dense in L_o (0,1) = S (0,1) (S is a complete metric topological vector space if it is endowed with the topology defined by the convergence in measure).*

The aim of the present paper is to generalize and strenghten the just mentioned results and to state some open problems.

1. On universal functions

THEOREM I. *Denote $\Omega \subset \mathbf{R}^N (N \geq 1)$ any bounded domain and $\lambda_n \to 0$ be any sequence in $\mathbf{R}^N (\lambda_n \neq 0, n = 1, 2, \ldots)$. Then for every $f \in L_1(\Omega)$, except a set of first category, the sequence*

$$\left\{ \frac{f(x+\lambda_n)-f(x)}{|\lambda_n|} \right\}_{n=1}^{\infty} \tag{1.1}$$

is dense in $L_p(\Omega)$ for every $p \in [0,1)$.

PROMBEM 1. Does Theorem I hold for $p \geq 1$? What can we say on weak density? What can we say for arbitrary $\{\lambda_n\} \subset \mathbf{R}^N$ or on infinite interval?

PROOF. We need two lemmas.

Fix the direction $e_0 := (0, 0, \ldots, 0, 1)$ in $\mathbf{R}^N$. For brevity we call a stepfunction $f \in L_p(\Omega)$ to be "good" if it has compact support in Ω, its values are rational numbers and the set $\{f = c\}$ is empty or is the finite union of cubes with diadic rational vertices for every $c \in \mathbf{R}$. It is well known that the good stepfunctions are dense in $L_p(\Omega)$ for $1 \leq p < \infty$.

LEMMA 1.1. *Let $0 < p < 1$; $f, g \in L_p(\Omega)$ be arbitrary good stepfunctions; $\epsilon > 0$, $\delta > 0$ be arbitrary (small) numbers. Then there exists a good stepfunction $h \in L_p(\Omega)$ and $\lambda \in \mathbf{R}^N$, $|\lambda| < \delta$ such that*

$$\int_\Omega |h|^p < \epsilon, \tag{1.2}$$

$$\int_{\Omega_{-|\lambda|}} \left| \frac{(f+h)(x+\lambda)-(f+h)(x)}{|\lambda|} - g(x) \right|^p dx < \epsilon, \tag{1.3}$$

where

$$\Omega_{-|\lambda|} := \{x\in\Omega: \operatorname{dist}(x,\partial\Omega)\geq |\lambda|\}.$$

PROOF. If $\lambda\to 0$, then, according to $p<1$, we have

$$\int_{\Omega_{-|\lambda|}} \left| \frac{f(x+\lambda)-f(x)}{|\lambda|} \right|^p dx\to 0,$$

hence, it is enough to construct a function h for which (1.2) holds and for sufficiently small $|\lambda|$

$$\int_{\Omega_{-|\lambda|}} \left| \frac{h(x+\lambda)-h(x)}{|\lambda|} - g(x) \right|^p dx<\epsilon,$$

is fulfilled.

Consider the lattice of cubes on each of which g is constant and denote d the maximal length whose entire mutiples give all the vertical lengths of these cubes. Divide the lattice into a lattice with lengths of edge $2^{-n}\cdot d$. Denote c the value of g on a fixed original cube K with edge-length d. Denote by K_1, $K_2,\ldots, K_{2^n}$ *any sequence of small cubes in* K lying along the direction e_0.

Define

$$h\Big|_{K_i} := \left\{\frac{i}{2^{n/2}}\right\}\cdot\frac{c}{d2^{n/2}},$$

where $\{x\} := x-[x]$. Because $\operatorname{supp} h\subset\operatorname{supp} g$, hence $\int_\Omega |h|^p\to 0$ $(n\to\infty)$. On the other hand for $\lambda_n := d2^{-n}e_0$ the function

$$\Delta(x) := \frac{h(x+\lambda_n)-h(x)}{\lambda_n} - g(x)\neq 0$$

only on a set of measure $\leq \text{const}\cdot 2^{-n/2}$, further $|\Delta(x)|\leq\text{const}\cdot 2^{n/2}$. Hence, taking into account $p<1$, we have

$$\int_{\Omega_{-|\lambda|}} \left| \frac{h(x+\lambda_n)-(h+x)}{|\lambda_n|} - g(x) \right|^p dx\to 0\ (n\to\infty). \ \square$$

LEMMA 1.2. *Let E be a complete metric space, F be a separable metric space, $L_n{:}E\to F$ be continuous mappings* ($n = 1, 2, \ldots$). *Then, if $M := \{x\in E: (L_n(x))_{n=1}^\infty$ is dense in $F\}$, is dense in E, then $E\backslash M$ is of first category in E.*

PROOF. *Let* $\{y_n\}_1^\infty$ be any dense set in F and

$$H_{k,l,n} := \{x \in E : \rho_F(L_n x, y_l) \geq k^{-1}\},$$

$$F_{k,l} := \bigcap_n H_{k,l,n}.$$

Obviously, $E \setminus M = \bigcup_k \bigcup_l F_{k,l}$, *further* $F_{k,l}$ is closed and it has not inner point (because M is dense in E). □

PROOF OF THE THEOREM I. According to Lemma 1.2 it is enough to show that for every good stepfunction f_0 and for every $\epsilon > 0$ there exists $f_1 \in L_p(\Omega)$ such that

$$\int_\Omega |f_0 - f_1|^p < \epsilon \quad \text{and} \quad \left\{ \frac{f_1(x+\lambda_n) - f_1(x)}{|\lambda_n|} \right\}$$

is dense in $L_p(\Omega)$. Now count the good stepfunctions and denote the resulting sequence $\{\Psi_k\}$. The method of the proof of Lemma 1.1 gives : there exists $n_1 > 1$ and a stepfunction h_1 such that $\int_\Omega |h_1|^p < \epsilon$ and

$$\int_{\Omega_{-|\lambda_{n_1}|}} \left| \frac{(f_0+h_1)(x+\lambda_{n_1}) - (f_0+h_1)(x)}{|\lambda_{n_1}|} - \psi_1(g) \right|^p dx < \frac{\epsilon}{2}.$$

If $h_1, h_2, \ldots, h_k$ are defined, then let $n_k > n_{k-1}$ be so large that by the construction of the proof of Lemma 1.1 there exists a stepfunction h_k such that

$$\int_\Omega |h_k|^p < \frac{\epsilon}{2^k} \min\{1, |\lambda_{n_1}|, \ldots, |\lambda_{n_{k-1}}|\}$$

and

$$\int_{\Omega_{-|\lambda_{n_k}|}} \left| \frac{(f_0+h_1+\ldots+h_k)(x+\lambda_{n_k}) - (f_0+h_1+\ldots+h_k)(x)}{|\lambda_{n_k}|} - \right.$$

$$\left. - \psi_k(x) \right|^p dx < \frac{\epsilon}{2^k}.$$

Let $f_1 := f_0 + \sum_{k=1}^{\infty} h_k$. Then, obviously,

$$f_1 \in L_p(\Omega), \int_{\Omega} |f_0 - f_1|^p \leq \sum_{k=1}^{\infty} \int_{\Omega} |h_k|^p < \epsilon$$

and

$$\int_{\Omega_{-|\lambda_{n_k}|}} \left| \frac{f_1(x+\lambda_{n_k}) - f_1(x)}{|\lambda_{n_k}|} - \psi_k(x) \right|^p dx \leq$$

$$\leq \int_{\Omega_{-|\lambda_{n_1}|}} \left| \frac{(f_0+h_1+\ldots+h_k)(x+\lambda_{n_k}) - (f_0+h_1+\ldots+h_k)(x)}{|\lambda_{n_k}|} - \psi_k(x) \right|^p dx +$$

$$+ \int_{\Omega_{-|\lambda_{n_k}|}} \sum_{i=1}^{\infty} \left| \frac{h_{k+i}(x+\lambda_{n_k}) - h_{k+i}(x)}{|\lambda_{n_k}|} \right|^p dx \leq$$

$$\leq \frac{\epsilon}{2^k} + \sum_{i=1}^{\infty} \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2^{k-1}},$$

i.e. $\{(f_1(x+\lambda_n) - f_1(x))/|\lambda_n|\}$ is dense in $L_p(\Omega)$. □

REMARK 1.1. If Ω is an arbitrary (not necessarily bounded) domain, then the proof of the Theorem I gives the same result for $f \in L_p(\Omega)$ instead of $f \in L_1(\Omega)$.

REMARK 1.2. Recently the first part of the Problem 1 was solved by A. BOGMÉR, Z. BUCZOLICH, M. HORVÁTH and A SÖVEGJÁRTÓ independently on each other [20]—[22], at the same time. The negative answer for the Problem 1 follows immediately from the following theorem of M. HORVÁTH [21] : *If $\Omega \subset \mathbf{R}^N$ is any (not necessarily bounded) domain and $|\lambda_n| \to 0$, $\{\lambda_n\} \subset \mathbf{R}^N$, then for every $f \in L_1(\Omega)$ either the estimate*

$$\int_{\Omega_{-|\lambda|}} \left| \frac{f(x+\lambda) - f(x)}{|\lambda|} \right| dx \geq c|\lambda|$$

holds for every $|\lambda| < \delta$, (where $\delta = \delta(f)$ and $c = c(f)$ are suitable positive numbers, independent on λ), or there is a direction e in which f is constant, i.e. for every $t \in \mathbf{R}$ $f(x+t\,e) = f(x)$ holds for almost every $x \in \Omega_{-|t|}$.

PROBLEM 2. Does the Theorem I remain valid if we replace the norm — convergence of (1.1) by a weaker convergence concept, e.g. by the Cesaro or Abel summation processes?

2. On universal series

THEOREM II. *Let* $\{\phi_n\}$ *be an arbitrary complete in* L_2 (0, 1) *orthonormal system. Then for every* $(c_n)\in C_o$ *(*C_o *denotes the Banach space of null-sequences) except a set of first category, the partial sums*

$$\{\sum_1^N c_n\phi_n : N = 1, 2, \ldots\}$$

form a dense set in L_p (0,1) *for every* $p\in[0,1)$. *In case* $p\geq 1$, *if* $\{\phi_n\}$ *is uniformly bounded, there is no* $(c_n)\in C_o$ *such that* $\{\sum_1^N c_n\phi_n : N = 1, 2, \ldots\}$ *is dense in* $L_p(\Omega)$.

PROBLEM 3. Does the last statement hold for any (not necessarily uniformly bounded) orthonormal system ?

PROBLEM 4. How to generalize the Theorem II for infinite interval and for multidimension ?

PROBLEM 5. Generalize the Theorem II for summation processes.

PROBLEM 6. Are there effective summation processes for universal series ? (e.g. for every universal series is there a process by which we get a convergent series) ?

PROOF. The proof of the Theorem D is based on the following

LEMMA E. *Let* $H\subset L_2$ (0,1) *be any orthonormal system and* f, $g\in L_2$ (0,1). *Then there exists a sequence* $\{F_n\}\subset L_2$ (0,1) *for which*

a) $F_n\to f$ *in measure*
b) $\sup_{h\in H} |\langle F_n, h\rangle-\langle g, h\rangle| \to 0$, $(n\to\infty)$.

This means that the convergence in measure and that of the weak one are independent. For the proof of the Theorem II we must to generalize the Lemma E for $0<p<1$ (i.e. set a') $F_n\to f$ *in* L_p (0,1) instead of a)).

To this we need the

LEMMA 2.1. *Let* $0<p<1$; $0<\delta,\eta<1$; g, $\phi_1, \ldots, \phi_n\in L_2$ (0,1). *Then there exists a stepfunction* $F\in L_2$ (0,1) *for which*

1. $\int_0^1 |F|^p < c\delta^{1-p}$, $c = (\int_0^1 |g|^2)^{p/2}$,

2. $\int_0^1 |F|^2 \leq c\delta^{-1} \int_0^1 |g|^2$,

3. $|\langle F-g, \phi_k\rangle| < \eta$; $k = 1, 2, \ldots, n$.

PROOF OF THE LEMMA 2.1. Consider a partition of the interval (0,1) into intervals $I_1, I_2, \ldots, I_r$ and let for $g \in L_2$ (0,1), $J_1 \subset I_1, J_2 \subset I_2, \ldots, J_r \subset I_r$ be concentrical with $I_1, I_2, \ldots, I_r$ resp. of lengths $\delta |I_s|$ $(s = 1, 2, \ldots, r)$ further

$$F(x) := \begin{cases} \dfrac{1}{\delta |I_s|} \displaystyle\int_{I_s} g & \text{for } x \in J_s,\ s \in \{1, 2, \ldots, r\}, \\ 0 & \text{if } x \notin \displaystyle\bigcup_{s=1}^{r} J_s. \end{cases}$$

We show that if the partition

$$(0,1) = \bigcup_{s=1}^{r} I_s$$

is fine enough, then $F(x)$ fulfills the conditions 1., 2. and 3. Indeed,

$$\int_0^1 |F|^p = \sum_{s=1}^{r} \delta |I_s| \left| \frac{1}{\delta |I_s|} \int_{I_s} g \right|^p \leq$$

$$\leq \delta^{1-p} \sum_{s=1}^{r} |I_s| \left(\frac{1}{|I_s|} \int_{I_s} |g|^2 \right)^{p/2} \leq$$

$$\leq \delta^{1-p} \left(\sum_{s=1}^{r} |I_s| \frac{1}{|I_s|} \int_{I_s} |g|^2 \right)^{p/2} = \delta^{1-p} \left(\int_0^1 |g|^2 \right)^{p/2}.$$

(We have used the concavity of $x^{p/2}$!).

$$\int_0^1 |F|^2 = \sum_{s=1}^{r} \delta |I_s| \left(\frac{1}{\delta |I_s|} \int_{I_s} g \right)^2 \leq$$

$$\leq \frac{1}{\delta} \sum_{s=1}^{r} |I_s| \frac{1}{|I_s|} \int_{I_s} |g|^2 = \frac{1}{\delta} \int_0^1 |g|^2.$$

Denote for $x \in I_j$ $(j=1, 2, \ldots, r)$

$$g^{(r)}(x) := \frac{1}{|I_j|}\int_{I_j} g$$

and define the functions $\phi_k^{(r)}$ analogously. For the proof of 3 it is enough to remark that $\langle F, \phi_k^{(r)}\rangle = \langle g^{(r)}, \phi_k^{(r)}\rangle$.

If the partition $I_1, \ldots, I_r$ is fine enough, then

$$|\langle g^{(r)}, \phi_k^{(r)}\rangle - \langle g, \phi_k\rangle| < \frac{\eta}{2}, \ (k = 1, 2, \ldots, n),$$

further we can assume $\|\phi_k^{(r)} - \phi_k\|_{L_2} \leq \sqrt{\delta}\eta/2\|g\|_{L_2}$, whence

$$|\langle F, \phi_k^{(r)}\rangle - \langle F, \phi_k\rangle| \leq \|F\|_{L_2}\|\phi_k^{(r)} - \phi_k\|_{L_2} \leq$$

$$\leq \delta^{-\frac{1}{2}}\|g\|_{L_2}\delta^{\frac{1}{2}}\eta(2\|g\|_{L_2})^{-1} = \frac{\eta}{2}, \text{ i. e.}$$

$$|\langle F - g, \phi_k\rangle| < \eta. \quad \square$$

Proof of the modified form of the Lemma E. It is enough to show that if $\{\phi_n\}$ is an orthonormal system in $L_2(0, 1)$, $g \in L_2(0, 1)$, $0 < p < 1$, $\epsilon > 0$ then there exists $F \in L_2(0, 1)$ such that

$$\int_0^1 |F|^p < \epsilon$$

and

$$|\langle F - g, \phi_k\rangle| < \epsilon, \ (k = 1, 2, \ldots).$$

Consider a partition $(0, 1) = \bigcup_{S=1}^{m} I_S$ into disjoint intervals such that $\int_{I_S} |g|^2 < \epsilon$.

We are looking for F in the form $F = \sum_{S=1}^{m} F_S$, where $F_S(x) = 0$ for $x \in I_S$. By Lemma 2.1 there exists F_1 defined on I_1 and such that

$$\int_{I_1} |F_1|^p < \epsilon^{1-p}|I_1|\left(\int_{I_1} |g|^2\right)^{p/2} < |I_1|\epsilon$$

and $|\langle F_1 - g, \phi_1\rangle| \leq \epsilon/2$.

Let $n_1 := 1$. If $F_1, F_2, \ldots, F_{k-1}$ are defined, then let n_k large enough, such that $|\langle F_1+\ldots+F_{k-1}-g, \phi_n\rangle| < \epsilon/2$ for $n > n_k$. By Lemma 2.1 there exists F_k defined on I_k for which

$$\int_{I_k} |F_k|^p < |I_k|\epsilon, \quad \int_{I_k} |F_k|^2 \leq \frac{1}{\epsilon}\int_{I_k} |g|^2 < \epsilon.$$

and $|\langle F_1+F_2+\ldots+F_k-g, \phi_n\rangle| < \epsilon$ for $n \leq n_k$.

For the function $F := F_1+F_2+\ldots+F_m$ we have

$$\int_0^1 |F|^p = \sum_{S=1}^{m} \int_{I_S} |F_S|^p < \epsilon$$

and

$$|\langle F_1+\ldots+F_m-g, \phi_n\rangle| < \epsilon \text{ for } n \leq n_m.$$

We know $|\langle F_1+\ldots+F_{m-1}-g, \phi_n\rangle| < \epsilon/2$ for $n > n_m$ and $\int_{I_m} |F_m|^2 < \epsilon$ hence

$$|\langle F_1+\ldots+F_m-g, \phi_n\rangle| < \frac{\epsilon}{2}+\sqrt{\epsilon} \text{ for } n > n_m. \square$$

REMARK 2.1. In the construction of the proof of Lemma E and Lemma 2.1 the number $p \in (0, 1)$ does not play any role, i.e. the functions F and $\{F_n\}$ are satisfactory for any $0 < p < 1$, hence, also for $p = 0$. On the other hand we can prove Lemma 2.1 also for unbounded interval, i.e. for $\mathbf{R}$. In this case choose $A > 0$ such that

$$|(\int_{-\infty}^{-A} + \int_{A}^{\infty})g, \phi_k| < \epsilon$$

(for $k = 1, 2, \ldots$), after construct F on $(-A, A)$ so that $\int_{-A}^{A} |F|^p < \epsilon$ and

$$|\langle F-g, \phi_k\rangle_{L_2(-A, A)}| < \epsilon, \ (k = 1, 2, \ldots).$$

We have proved the

LEMMA 2.2. *Let $\{\phi_k\}$ be any orthonormal system in $L_2(\mathbf{R})$, f, $g \in L_2(\mathbf{R})$. Then there exists $\{F_n\} \subset L_2(\mathbf{R})$ and $\epsilon_n \to 0$ such that*

a) $F_n \in L_p(\mathbf{R})$, $F_n \xrightarrow{L_p(\mathbf{R})} f\ (n \to \infty)$ *for every* $p \in [0, 1)$,

b) $|\langle F_n - g, \phi_k \rangle| < \epsilon_n$ *for* $k = 1, 2, \ldots$.

After, the proof of the first part of the Theorem II goes by the method of [13].

For the proof of the second part, assume

$$\sum_{i=1}^{n_k} c_i \phi_i \to 0$$

in $L_p(0, 1)$ (we can suppose $c_1 \neq 0$), then

$$|c_1| = \left| \langle \sum_{i=1}^{n_k} c_i \phi_i, \phi_1 \rangle \right| \leq \|\phi_1\|_{L_p} \left\| \sum_{i=1}^{n_k} c_i \phi_i \right\|_{L_p} \to 0,$$

which is a contradiction.

3. On Sturm—Liouville expansions

Let G be an arbitrary open interval on the real line, $q \in L_1^{\mathrm{loc}}(G)$ be an arbitrary complex function and consider the formal Schrödinger operator $Lu = -u'' + qu$. Given a complex number λ, the function $u: G \to \mathbf{C}$, $u = 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. Furthermore, a function $u: G \to \mathbf{C}$, $u \not\equiv 0$ is called an eigenfunction of oder m ($m = 1, 2, \ldots$) of the operator L with the eigenvalue λ if u and its derivative u' are locally absolutely continuous on G and $Lu = \lambda u - u^*$ almost everywhere on G, where u^* is an eigenfunction of order $m-1$ of the operator L with the eigenvalue λ.

Let us introduce for any $\mu \in \mathbf{C}$, $t > 0$ the functions

$$f_1(\mu, t) := t \frac{\sin \mu t}{\mu}, \quad f_i(\mu, t) := \int_0^t \frac{\sin \mu(t-\tau)}{\mu} f_{i-1}(\mu, \tau) d\tau,$$

and for any $u \in L_\infty^{\mathrm{loc}}(G)$, $\mu \in \mathbf{C}$, $x \pm t \in G$, $t > 0$ the functions

$$g_0(u, \mu, x, t) := \int_{x-t}^{x+t} \frac{\sin \mu(t - |x - \xi|)}{\mu} q(\xi) u(\xi) d\xi,$$

$$g_i(u, \mu, x, t) := \int_0^t \frac{\sin \mu(t-\tau)}{\mu} g_{i-1}(u, \mu, x, \tau) d\tau,$$

$$(i = 1, 2, \ldots).$$

LEMMA 3.1 ([18]). *Let u_m be an eigenfunction of order $\leq m$ of the operator L with the eigenvalue $\lambda = \mu^2$ and put $u_{j-1} := \lambda u_j - Lu_j$, $j = 1, 2, \ldots, m$. Then for $x \pm t \in G$, $t > 0$*

$$u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t = \sum_{i=1}^{m} f_i(\mu, t)u_{m-i}(x)+$$

$$+\sum_{i=0}^{m} g_i(u_{m-i}, \mu, x, t). \tag{3.1}$$

Given a compact interval $K = [a, b] \subset G$, fix $R>0$ such that $K_R := [a-R, b+R] \subset G$. Fixing $\mu>0$ and $x \in K$ arbitrarily, define $w: G \to R$ by

$$w_R(x+t) := \begin{cases} \dfrac{\sin \mu t}{\pi t} & \text{if } |t| \leq R. \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Let us now given a system $(u_k) \subset L_2(G)$ of eigenfunctions of the operator L. Let λ_k (resp. o_k) denote the eigenvalue (resp. the order) of u_k and assume that the following conditions are satisfied:

(3.3) $m := \sup o_k < \infty$,

(3.4) in case $o_k > 0$, $\lambda_k u_k - Lu_k = u_{k-1}$,

(3.5) (u_k) is a Bessel system (Cf. [23]) i.e. for any

$$f \in L_2(G), \sum_k |\langle u_k, f\rangle|^2 \leq c_0 \|f\|_L^2$$

where c_0 is a constant independent of f.

Now let us introduce the notation

$$\sigma_\mu(f, x) := \sum_{\operatorname{Re}\mu_k < \mu} \langle f, v_k\rangle u_k(x), \tag{3.6}$$

where (v_k) is the dual system of (u_k), i.e. $(v_k) \subset L_2(G)$ and $\langle v_k, u_j\rangle = \delta_{kj}$, further for brevity, μ_k denote a square root of λ_k for which $\operatorname{Re}\mu_k \geq 0$.

Assume (v_k) consists of the eigenfunctions of the operator[1] $L^*v := -v'' + \overline{q(x)}\, v$ with eigenvalues $\bar{\lambda}_k$.

[1]Suppose (v_k) is Bessel system.

Let l_k is the largest order such that u_k occurs in the succesive definition of an eigenfunction of order l_k. Denote the corresponding for v_k by l_k^* and o_k^*.

REMARK 3.1. If the functions u_k satisfy some boundary conditions and the functions v_k the adjoint boundary conditions, then these assumptions are fulfilled automatically.

Denote

$$S_\mu(f, x) = S_\mu(f, x, R) := \int_{x-R}^{x+R} \frac{\sin \mu(y-x)}{\pi(y-x)} f(y)dy, \tag{3.7}$$

$$(\mu > 0,\ x \pm R \in G).$$

THEOREM III. *If $q \in L_p(G)$, $p > 1$, then for any compact interval $K \subset G$, for any sufficiently small $R > 0$, and for all $f \in L_1(G)$ we have*

$$\lim_{\mu \to \infty} \sup_{x \in K} |S_\mu(f, x) - \sigma_\mu(f, x)| = 0. \tag{3.8}$$

If $q \in L_1(G)$, then for any sufficiently small $R > 0$ and for all $f \in L_1(G)$ we have

$$\lim_{\mu \to \infty} [S_\mu(f, x) - \sigma_\mu(f, x)] = 0, \text{ a.e. } x \in G. \tag{3.9}$$

PROBLEM 7. Does (3.8) hold for $q \in L_1(G)$?

REMARK 3.2. If G is a finite interval and $q \in L_1(G)$ (resp. $q \in L_p(G)$ for some $p > 1$) then there exists an $f \in L_1(G)$ such that $\sup |\sigma_\mu(f, x)| = \infty$ in almost every $x \in G$ (resp. in every $x \in G$). This statement follows from the Theorem III immediately, taking into account the Theorem A of KOLMOGOROFF. According to the fact that in the case of a complete orthonormal system with property (2) we can not to state almost everywhere divergence on (0,1) in the Theorem B, the statement above does not follows from Bockarev's Theorem.

For the proof of the Theorem III we need some lemmas.

Denote S_{R_0} the operation

$$f \to S_{R_0}[f] := \frac{2}{R_0} \int_{R_0/2}^{R_0} f(R)dR$$

and

$$\Delta_k := S_{R_0}\left[\frac{2}{\pi} \int_0^R \frac{\sin \mu t}{t} \cos \mu_k t dt\right] - \delta(\mu, \rho_k),\ \delta(\alpha, \beta) := \begin{cases} 1, & \alpha > \beta \\ \frac{1}{2}, & \alpha = \beta \\ 0, & \alpha < \beta \end{cases}$$

LEMMA 3.2. *For any $R_0>0$ there exists $C(R_0)>0$ such that*

$$|\Delta_k| \le C(R_0)\frac{1+\nu_k^2}{1+|\mu-\rho_k|^2}\operatorname{ch}\nu_k R_0, \tag{3.10}$$

where $\rho_k := \operatorname{Re}\mu_k$, $\nu_k := |\operatorname{Im}\mu_k|$.

PROOF. We develop the ideas of [18]. Use the decomposition (Cf. [18], (48))

$$\frac{2}{\pi}\int_0^R \frac{\sin\mu t}{t}\cos\mu_k t dt = I_1+I_2+I_3+I_4,$$

$$I_1 := \frac{2}{\pi}\int_0^\infty \frac{\sin\mu t\cos\rho_k t}{t}dt,$$

$$I_2 := \frac{2}{\pi}\int_R^\infty \frac{\sin\mu t\cos\rho_k t}{t}dt,$$

$$I_3 := \frac{2}{\pi}\int_0^R \sin\mu t\cos\rho_k t\frac{\operatorname{ch}\nu_k t-1}{t}dt,$$

$$I_4 := -\frac{2i}{\pi}\int_0^R \sin\mu t\sin\rho_k t\frac{\operatorname{sh}\nu_k t}{t}dt.$$

We prove the inequality

$$\left|\frac{2}{R_0}\int_{R_0/2}^{R_0} I_3\,dR\right| \le C(R_0)\frac{1+\nu_k^2}{1+|\mu-\rho_k|^2}\operatorname{ch}\nu_k R_0.$$

Twofold integration by parts gives:

$$I_3 = \frac{1}{\pi}\left[\left(\frac{-\cos(\mu+\rho_k)t}{\mu+\rho_k}+\frac{-\cos(\mu-\rho_k)t}{\mu-\rho_k}\right)\frac{\operatorname{ch}\nu_k t-1}{t}\right]_0^R +$$

$$+\frac{1}{\pi}\left[\left(\frac{\sin(\mu+\rho_k)t}{\mu+\rho_k}+\frac{\sin(\nu-\rho_k)t}{\mu-\rho_k}\right)\left(\frac{\operatorname{ch}\mu_k t-1}{t}\right)'\right]_0^R -$$

$$-\frac{1}{\pi}\int_0^R\left(\frac{\sin(\mu+\rho_k)t}{(\mu+\rho_k)^2}+\frac{\sin(\mu-\rho_k)t}{(\mu-\rho_k)^2}\right)\left(\frac{\operatorname{ch}\nu_k t-1}{t}\right)''dt.$$

Now

$$\left|\frac{2}{R_0}\int_{R_0/2}^{R_0}\left(\frac{\cos(\mu+\rho_k)R}{\mu+\rho_k}+\frac{\cos(\mu-\rho_k)R}{\mu-\rho_k}\right)\frac{\operatorname{ch}\nu_k R-1}{R}dR\right|\leq$$

$$\leq\frac{2}{R_0}\left|\left[\left(\frac{\sin(\mu+\rho_k)R}{(\mu+\rho_k)^2}+\frac{\sin(\mu-\rho_k)R}{(\mu-\rho_k)^2}\right)\frac{\operatorname{ch}\nu_k R-1}{R}\right]_{R_0/2}^{R_0}\right|+$$

$$+\frac{2}{R_0}\left|\int_{R_0/2}^{R_0}\left(\frac{\sin(\mu+\rho_k)R}{(\mu+\rho_k)^2}+\frac{\sin(\mu-\rho_k)R}{(\mu-\rho_k)^2}\right)\left(\frac{\operatorname{ch}\nu_k R-1}{R}\right)'dR\right|.$$

Taking into consideration that the functions $[(\operatorname{ch}\nu_k t-1)/t]'$, $[(\operatorname{ch}\nu_k t-1)/t]''$ don't change sign, we obtain the desired estimate. Using the analogous property of the functions $(\operatorname{sh}\nu_k t/t)'$ and $(\operatorname{sh}\nu_k t/t)''$ the same proof gives the estimate

$$\left|\frac{2}{R_0}\int_{R_0/2}^{R_0}I_4dR\right|\leq C(R_0)\frac{1+\nu_k^2}{1+|\mu-\rho_k|^2}\operatorname{ch}\nu_k R_0$$

and using the functions $(1/t)'$, $(1/t)''$, we get

$$\left|\frac{2}{R_0}\int_{R_0/2}^{R_0}I_2dR\right|\leq C(R_0)\frac{1+\nu_k^2}{1+|\mu-\rho_k|^2}\operatorname{ch}\nu_k R_0.$$

At last it is well known (and one can easily prove)

$$I_1=\delta(\mu,\rho_k)$$

which completes the proof. □

Now estimate the function

$$A_j(\mu, \nu) := \frac{2}{R_0} \int_{R_0/2}^{R_0} \int_0^R \frac{\sin \nu t}{\pi t} f_j(\mu, t) dt dR.$$

LEMMA 3.3. *We have* for $\nu > 0$, $\mu_k \in \mathbf{C}$

$$|A_j(\mu_k, \nu)| \leq \frac{C(R_0) e^{\rho_k R_0}}{1 + |\nu - \rho_k|^2} (1 + |\mu_k|)^{-j}, \ (j = 1, 2, \ldots). \tag{3.11}$$

PROOF. First suppose $|\mu_k| \leq 1$ and define F_j by

$$f_j(\mu_k, t) = t F_j(\mu_k, t). \tag{3.12}$$

For $|\mu_k| \leq 1$ we have $|\sin \mu_k t / \mu_k t| \leq C(R_0)$, $|\cos \mu_k t| \leq C(R_0)$, hence, according to the definition of f_j

$$|f_j^{(r)}(\mu_k, t)| \leq C(j, R_0) t^{4-r}, \ (r = 0, 1, 2), \tag{3.13}$$

holds, therefore

$$|F_j''(\mu_k, t)| \leq C(j, R_0), j \geq 1, \ \left(' = \frac{\partial}{\partial t} \right). \tag{3.14}$$

On the other hand

$$\int_0^R \sin \nu t F_j(\mu_k, t) dt = \left[-\frac{\cos \nu t}{\nu} F_j(\mu_k, t) \right]_0^R + \int_0^R \frac{\cos \nu t}{\nu} F_j^1(\mu_k, t) \, dt =$$

$$= -\frac{\cos \nu R}{\nu} F_j(\mu_k, R) + \left[\frac{\sin \nu t}{\nu^2} F_j'(\mu_k, t) \right]_0^R - \int_0^R \frac{\sin \nu t}{\nu^2} F_j''(\mu_k, t) dt.$$

Obviously

$$\int_{R_0/2}^{R_0} \frac{\cos \nu R}{\nu} F_j(\mu_k, R) dR = \left[\frac{\sin \nu R}{\nu^2} F_j(\mu_k, R) \right]_{R_0/2}^{R_0} -$$

$$- \int_{R_0/2}^{R_0} \frac{\sin \nu R}{\nu^2} F_j'(\mu_k, R) dR,$$

hence

$$|A_j(\mu_k, \nu|) \leq C(j, R_0) \cdot \frac{1}{\nu^2}, \ (|\mu_k|) \leq 1),$$

further it is trivial from (3.14) that $|A_j(\mu_k, \nu)| \leq C(j, R_0)$, consequently we have proved

$$|A_j(\mu_k, \nu)| \leq C(j, R_0)\frac{1}{1+\nu^2}, \ (|\mu_k| \leq 1). \tag{3.15}$$

Now return to the case $|\mu_k| \geq 1$.

To this we prove by induction that there exist polynomials P_{j-1}, Q_{j-1} of degree at most $j-1$ such that P_{j-1} contains only terms of even order, Q_{j-1} contains only terms of odd order and

$$f_j(\mu, t) = \frac{t}{\mu^{2j-1}}[P_{j-1}(\mu t) \sin \mu t + Q_{j-1}(\mu t) \cos \mu t] \tag{3.16}$$

holds. (For brevity we omit the index k below). Indeed, for $j = 1$ set $P_0(z) \equiv \equiv 1$, $Q_0(z) \equiv 0$. After consider

$$f_{j+1}(\mu, t) = \int_0^t \frac{\sin \mu(t-\tau)}{\mu} \cdot \frac{\tau}{\mu^{2j-1}}[P_{j-1}(\mu t) \sin \mu t + Q_{j-1}(\mu t) \cos \mu t].$$

Use the relations

$$\sin \mu(t-\tau) \sin \mu\tau = \frac{1}{2}[\cos \mu(t-2\tau) - \cos \mu t],$$

$$\sin \mu(t-\tau) \cos \mu\tau = \frac{1}{2}[\sin \mu t + \sin \mu(t-2\tau)].$$

Now

$$\frac{\mu^{2j+1}}{t} \int_0^t \frac{\tau}{\mu^{2j}} P_{j-1}(\mu\tau)d\tau, \ \frac{\mu^{2j+1}}{t} \int_0^t \frac{\tau}{\mu^{2j}} Q_{j-1}(\mu\tau)d\tau$$

are polynomials of (μt) of degree at most j and the first contains only odd, the second only even terms. The expansion of f_{j+1} contains also the integrals:

$$\frac{1}{2} \int_0^t \frac{\cos \mu(t-2\tau)}{\mu^{2j}} \tau P_{j-1}(\mu\tau)d\tau, \ \frac{1}{2} \int_0^t \frac{\sin \mu(t-2\tau)}{\mu^{2j}} \tau Q_{j-1}(\mu\tau)d\tau.$$

One can see by partial integration that these can be transformed to the form

$$\frac{1}{\mu^{2j+2}}[R_j(\mu t) \sin \mu t + S_j(\mu t) \cos \mu t]$$

where R_j and S_j are polynomials of (μt), R_j contains only odd, S_j only even terms. (We used that $f_{j+1}(0) = 0$). Henceforth (3.16) follows by induction.

Now return to the estimation of A_j in the case of $|\mu_k| \geq 1$. Integrating by parth, using (3.16) and

$$\sin \nu t \sin \mu t = \frac{1}{2}[\cos (\nu-\mu)t - \cos (\nu+\mu)t], \tag{3.17}$$

we get

$$A_j = \frac{2}{R_0} \int_{R_0/2}^{R_0} \int_0^R \frac{\sin \nu t}{\pi} \frac{1}{\mu^j} \left[\frac{P_{j-1}(\mu t)}{\mu^{j-1}} \sin \mu t + \right. \tag{3.18}$$

$$\left. + \frac{Q_{j-1}(\mu t)}{\mu^{j-1}} \cos \mu t \right] dt dR.$$

Consider only the term containing P_{j-1} (the case of the term Q_{j-1} is similar). According to (3.17) it is enough to investigate only the term (one can estimate the others similarly);

$$\int_0^R \cos (\nu-\mu)t \frac{P_{j-1}(\mu t)}{\mu^{j-1}} dt = \left[\frac{\sin (\nu-\mu)\, t}{\nu-\mu} \cdot \frac{P_{j-1}(\mu t)}{\mu^{j-1}} \right]_0^R -$$

$$- \int_0^R \frac{\sin (\nu-\mu)t}{\nu-\mu} \cdot \frac{P'_{j-1}(\mu t)}{\mu^{j-2}} dt = \frac{\sin (\nu-\mu)R}{\nu-\mu} \cdot \frac{P_{j-1}(\mu R)}{\mu^{j-1}} +$$

$$+ \left[\frac{\cos (\nu-\mu)t}{(\nu-\mu^2)} \cdot \frac{P'_{j-1}(\mu,\ t)}{\mu^{j-2}} \right]_0^R - \int_0^R \frac{\cos (\nu-\mu)t}{(\nu-\mu)^2} \cdot \frac{P''_{j-1}(\mu t)}{\mu^{j-3}} dt.$$

If we apply to each term the operation $\frac{2}{R_0} \int_{R_0/2}^{R_0}$ and integrate by parth in the first term of the right hand side, we obtain for each term the estimate $O(|\nu-\mu|^{-2})$. Hence because of

$$|\sin z| \leq \operatorname{ch} \operatorname{Im} z, \quad |\cos z| \leq \operatorname{ch} \operatorname{Im} z, \tag{3.19}$$

we get (using the index k again):

$$|A_j(\mu_k, \nu)| \leq \frac{C(j, R_0)}{|\nu-\rho_k|^2} \cdot \frac{e^{\rho_k R_0}}{(1+|\mu_k|)^j}.$$

On the other hand it is trivial from (3.18) and (3.19) that

$$|A_j(\mu_k, \nu)| \leq \frac{C(j, R_0)}{(1+|\mu_k|)^j} e^{\rho_k R_0}$$

and (3.11) is proved in the case $|\mu_k| \geq 1$ too. □

Define

$$B_j := \frac{2}{R_0} \int_{R_0/2}^{R_0} \int_0^R \frac{\sin \nu t}{\pi t} g_j(u_{k-j}, \mu_k, x, t) dt dR,$$

$$G_j(\mu_k, t-a) := \int_0^{t-a} \int_0^{t_1} \cdots \int_0^{t_{j-1}} \frac{\sin \mu_k(t-a-t_1)}{\mu_k} \cdot \frac{\sin \mu_k(t_1-t_2)}{\mu_k} \cdots$$

$$\cdots \frac{\sin \mu_k(t_{j-1}-t_j)}{\mu_k} \cdot \frac{\sin \mu_k t_j}{\mu_k} dt_j \cdots dt_1,$$

$$I_j := \int_a^R \sin \nu t \frac{G_j(\mu_k, t-a)}{t} dt = I_j(\mu_k, a, R, \nu).$$

Obviously

$$B_j = \frac{2}{R_0} \int_{R_0/2}^{R_0} \left[\frac{1}{\pi} \int_{x-R}^{x+R} q(\xi) U_{k-j}(\xi) I_j(\mu_k, |x-\xi|, R, \nu) d\xi \right] dR. \tag{3.20}$$

LEMMA 3.4. *We have for any* $\epsilon > 0$ *and* $(R_0/2) \leq R \leq R_0$, $\nu > 0$, $\mu_k \in \mathbf{C}$

(3.21)

$$|I_j(\mu_k, a, R, \nu)| \leq C(j, R_0) \left(\frac{1}{\nu(1+|\mu_k|)^\epsilon} + \frac{|\mu_k|}{\nu^2} \right) \cdot \frac{e^{\rho_k R}}{(1+|\mu_k|)^j} \cdot \frac{1}{a^\epsilon}$$

further

(3.22)
$$|I_j(\mu_k, a, R, \nu)| \leq C(j, R_0) \frac{\nu}{\nu(1+|\mu_k|)^{\epsilon}} \cdot \frac{e^{\rho_k R}}{(1+|\mu_k|)^j} \left| \ln \frac{1}{a} \right|.$$

PROOF. It is similar to that of in [15], hence we omit it.

LEMMA 3.5. *We have for any Bessel system* (u_k) *of eigenfunctions of order* $\leq m > \infty$ *of the operator* $Lu = -u'' + qu$, $(q \in L^1(G),\ |G| < \infty)$ *with eigenvalues* $\{\lambda_k\} \subset \mathbf{C}$

(3.23)
$$\sup_{\mu \geq 1} \sum_{|\mu - \rho_k| \leq 1} \|u_k\|^2_{L^\infty(G)} < \infty.$$

PROOF. It is proved in [18] that for any compact interval $K \subset G$ there exists $R > 0$ with

(3.24)
$$\sup_{\mu > 0} \sum_{|\mu - \rho_k| \leq 1} (\|u_k\|_{L^\infty(K)} \operatorname{ch} \nu_k R)^2 < \infty.$$

V. V. TICHOMIROFF proved ([23], Theorem 4): If $G = (a, b)$ is a finite interval, u is an eigenfunction of the operator $Lu = -u'' + qu$, $q \in L^1(G)$ of order $\leq m$, with eigenvalue $\lambda \in \mathbf{C}$, then there exists a constant c depending only on R and m such that

(3.25)
$$\|u\|_{L^\infty(G)} \leq C \|u_k\|_{L^\infty(G_{-R})} \exp(|\operatorname{Im} \sqrt{\lambda}| R),$$

where $G_{-R} := [a+R, b-R]$, $0 < R < (b-a)/2$.

Now let $R \in \left(0, \frac{b-a}{2}\right)$ be arbitrary and apply (3.24) for $K^* := G_{-R}$ denote $\bar{R}$ the number R stated in (3.24). Set $R^* := (\min \bar{R}, R)$. We get from (3.24) and (3.25)

$$\sum_{|\mu - \rho_k| \leq 1} \|u_k\|^2_{L^\infty(G)} \leq C \sum_{|\mu - \rho_k| \leq 1} (\|u_k\|_{L^\infty(K^*)} \operatorname{ch} \nu_k R^*)^2 < \infty. \quad \square$$

REMARK 3.3 Using 3.24 and 3.25 by the method of [31] we get for some $M > 0$ and $\alpha > 0$:

$$\sum_{|\mu - \rho_k| \leq M} \|u_k\|^2_{L^2(G)} \geq \alpha > 0 \ (\mu \geq 1).$$

Hence: $\rho_{k+1} - \rho_k = O(1)$.

LEMMA 3.6. *At the assumptions of the Theorem III we have*

$$\sup_{\mu>0}\sup_{x\in K}\sum_{k=1}^{\infty}|(u_k, w)-\delta(\mu, \rho_k)u_k(x)|\,\|v_k\|_{L^\infty(G)}<\infty, \tag{3.26}$$

where $w:=S_{R_0}[w_R]$ *and* w_R *is defined in* (3.2).

PROOF. Apply the Lemma 3.1 for u_k. We obtain for $x\in K$ and $0<R_0<$ $<\frac{1}{2}$ dist $(K, \partial G)$:

$$(u_k, w)-\delta(\mu, \rho_k)u_k(x)= \tag{3.27}$$

$$=S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t}(u_k(x-t)+u_k(x+t))dt\right]-\delta(\mu, \rho_k)u_k(x)=$$

$$=S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t}(u_k(x-t)+u_k(x+t)-2u_k(x)\cos \mu_k t)dt\right]+$$

$$+S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t}\cos \mu_k t-\delta(\mu, \rho_k)\right]u_k(x)=$$

$$=\sum_{j=1}^{m} S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t}f_j(\mu_k, t)dt\right]u_{k-j}(x)+$$

$$+\sum_{j=0}^{m} S_{R_0}\left[\int_0^R \frac{\sin \mu t}{\pi t}g_j(u_{k-j}, \mu_k, x, t)dt\right]+$$

$$+\Delta_k u_k(x)=\sum_{j=1}^{m} A_j u_{k-j}(x)+\sum_{j=0}^{m} B_j+\Delta_k u_k(x).$$

We shall need the following two estimates of V. KOMORNIK [24], [25]:

$$\|u_{k-j}\|_{L^\infty(K)}\le C(K, K')(1+|\mu_k|)^j\|u_k\|_{L^\infty(K')} \tag{3.28}$$

for any compact intervals $K, K' \subset G$, $K \subset \operatorname{int} K'$;

$$\|u_k\|_{L^\infty(K')} \leq C(K', K'')e^{-\epsilon \nu_k}\|u_k\|_{L^\infty(K'')}, \tag{3.29}$$

where $K', K'' \subset G$ are arbitrary compact intervals such that $K' \subset \operatorname{int} K''$ and $\epsilon = \epsilon(K', K'') > 0$.

Now for any fixed $K \subset G$ choose K', K'' such that $K \subset \operatorname{int} K'$, $K' \subset \operatorname{int} K''$, $K'' \subset G$ and after, choose R_0 so that $R_0 < \epsilon$ be fulfilled (ϵ is given in (3.29)).

Using (3.28) and (3.29) we obtain for any $x \in K$

$$\left|\sum_{j=1}^{m} A_j u_{k-j}(x)\right| \leq \sum_{j=1}^{m} |A_j| \cdot |u_{k-j}(x)| \leq$$

$$\leq C(R_0)\frac{e^{R_0 \nu_k}}{1+|\mu-\rho_k|^2}\sum_{j=1}^{m}(1+|\mu_k|)^{-j}\|u_{k-j}\|_{L^\infty(K)} \leq$$

$$\leq C(R_0)mC(K, K')\frac{e^{\nu_k(R_0-\epsilon)}}{1+|\mu-\rho_k|^2}\|u_k\|_{L^\infty(K'')} \leq \frac{C}{1+|\mu-\rho_k|^2}\|u_k\|_{L^\infty(K'')}.$$

Using (3.20), (3.21), (3.22), (3.28), (3.29) we get

$$\left|\sum_{j=0}^{m} B_j\right| \leq \sum_{j=0}^{m} |B_j| \leq \sum_{j=0}^{m} C(\epsilon, j, R_0)\|u_{k-j}\|_{L^\infty(K_{R_0})} \cdot$$

$$\cdot \frac{e^{R_0 \nu_k}}{(1+|\mu_k|)^j} \cdot \min\left\{\frac{\mu}{(1+|\mu_k|)^2}, \frac{1}{\mu(1+|\mu_k|)^\epsilon} + \frac{|\mu_k|}{\mu^2}\right\} \leq$$

$$\leq C(\epsilon, m, R_0, K, K', K'')\alpha(\mu, \mu_k, \epsilon)\|u_k\|_{L^\infty(K'')}.$$

At last, using (3.10), (3.28), (3.29) it follows

$$|\Delta_k u_k(x)| \leq C(R_0, K, K'')\frac{1}{1+|\mu-\rho_k|^2}\|u_k\|_{L^\infty(K'')}.$$

Summarising our estimates we have

$$|\langle u_k, w\rangle - \delta(\mu, \rho_k)u_k(x)| \le C\left(\frac{1}{1+|\mu-\rho_k|^2} + \alpha(\mu, \mu_k, \epsilon)\right)\|u_k\|_{L^\infty(K^\epsilon)}.$$

Hence

$$\sum_{k=1}^{\infty} |\langle u_k, w\rangle - \delta(\mu, \rho_k)u_k(x)|\,\|v_k\|_{L^\infty(G)} \le$$

$$\le C\sum_{k=1}^{\infty}\left[\frac{1}{1+|\mu-\rho_k|^2} + \alpha(\mu, \mu_k, \epsilon)\right]\|u_k\|_{L^\infty(G)}\|v_k\|_{L^\infty(G)} \le$$

$$\le C\sum_{n=1}^{\infty}\sum_{n\le\rho_k\le n+1}{}_k\left[\frac{1}{1+(\mu-\rho_k)^2} + \right.$$

$$\left. + \min\left(\frac{1}{\mu}, \frac{\mu}{1+\rho_k^2}\right)\right]\|u_k\|_{L^\infty(G)}\|v_k\|_{L^\infty(G)} \le$$

$$\le C\sum_{n=1}^{\infty}\left[\frac{1}{1+(\mu-n)^2} + \min\left(\frac{1}{\mu}, \frac{\mu}{1+n^2}\right)\right]\cdot$$

$$\cdot\Big(\sum_{n\le\rho_k\le n+1}{}_k\|u_k\|^2_{L^\infty(G)}\Big)^{1/2}\Big(\sum_{n\le\rho_k\le n+1}\|v_k\|^2_{L^\infty(G)}\Big)^{1/2} \le$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{1+(\mu-n)^2} + C\sum_{n=1}^{\infty}\min\left(\frac{1}{\mu}, \frac{\mu}{1+n^2}\right) \le$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{n^2} + C\sum_{n<\mu}{}_n\frac{1}{\mu} + C\sum_{n>\mu}{}_n\frac{\mu}{1+n^2} \le C. \quad \square$$

Proof of the Theorem III. By Lemma 3.6 the series

$$\sum_{k=1}^{\infty}(\langle u_k, w\rangle - \delta(\mu, \rho_k)u_k(x))\,\overline{v}_k$$

converges in $L^2(G)$ for any fixed $x\in K$ and is equal to

$$\Theta_\mu(x) := \overline{w} - \sum_{\rho_k<\mu}{}_k\, u_k(x)\overline{v}_k - \frac{1}{2}\sum_{\rho_k=\mu} u_k(x)\overline{v}_k \tag{3.30}$$

(because the sum and the just defined function $\Theta_\mu(x)$ have the same Fourier coefficients with respect to the complete in $L^2(G)$ system $\{u_k\}$). Consequently, by Lemma 3.6

$$\sup_{\mu>0}\sup_{x\in K}\|\Theta_\mu(x)\|_{L^\infty(G)}<\infty. \tag{3.31}$$

On the other hand, it is easy to see that for any $0<R<\operatorname{dist}(K,\partial G)$

$$\sup_{\mu>0}\sup_{x\in K}\|\bar{w}-w_R\|_{L^\infty(G)}<\infty. \tag{3.32}$$

and

$$\sup_{\mu>0}\sup_{x\in K}\|\sum_{\rho_k=\mu}u_k(x)\bar{v}_k\|_{L^\infty(G)}<\infty. \tag{3.33}$$

From (3.31), (3.32), (3.33) we obtain

$$M:=\sup_{\mu>0}\sup_{x\in K}\|w_R-\sum_{\rho_k<\mu}u_k(x)\bar{v}\|_{L^\infty(G)}<\infty, \tag{3.34}$$

hence

$$\sup_{\mu>0}\sup_{x\in K}|S_\mu(f,x,R)-\sigma_\mu(f,x)|\le M\|f\|_{L^\infty(G)}.$$

On the other hand the linear hull $\mathcal{L}\{u_k\}$ form a dense set in $L^1(G)$ and because they are also in $L^2(G)$, by Joó—Komornik's theorem [18] we have for any $f\in\mathcal{L}\{u_k\}$

$$\lim_{\mu\to\infty}\sup_{x\in K}|S_\mu(f,x,R)-\sigma_\mu(f,x)|=0. \qquad \square$$

PROBLEM 8. Is there a function $f\in C[0,\infty)$, $0<f(x)<1 (x\in\mathbf{R})$ such that

$$\int_0^\infty f^2=\infty \text{ and}$$

$$\sum_{k=1}^\infty f(x_k(c))<\infty \text{ (a. e. } c>0),$$

where

$$\int_0^{x_k(c)} f=kc \quad (k=1,2,\ldots)?$$

Is there such a function of bounded variation or with bounded derivative?

REMARK 3.4. For special potentials we get the classical orthogonal functions (Jacobi, Laguerre and Hermite functions). Occurs the question of existence of universal series on infinite interval (e.g. of universal Hermite series).

This problem is still open (as we know) and it led us to the the investigation of the Fejér means of Hermite-Fourier series. We got in this question a positive result, and it is given in **4** below. After we give an open question concerning this.

4. On Fejér means of Hermite-Fourier series

It is well known that the trigonometric conjugate function play an important role in the theory of trigonometric Fourier series. A classical result concerning this is the following theorem of G. ALEXITS [26]:

THEOREM E. *For the Fejér sums $\sigma_n(f)$ of the trigonometric Fourier series of the 2π -periodic function f the estimate $\|\sigma_n(f)-f\|_p=O\left(\frac{1}{n}\right)$ holds if and only if the conjugate function $\tilde{f}$ of f belongs to the class* Lip $(1, p)$, $1\leq p\leq\infty$.

Our aim is to prove one direction of this theorem for Hermite-Fourier series (the other direction remains an open problem). To this we need a modified but equivalent form of Hermite-conjugate function introduced by B. MUCKENHOUPT [28]: Denote $H=\{h_n\}$ the complete in $L^2(\mathbf{R})$ orthonormal system of Hermite functions, and $\mathfrak{F}(H)$ the set of all functions which have Hermite-Fourier series.

DEFINITION: *Let $f\in\mathfrak{F}(H)$ and*

$$f\sim\sum_{k=0}^{\infty} a_k(f)h_k\left(=\sum_{0}^{\infty} a_k h_k\right). \tag{4.1}$$

If there exists $\tilde{f}\in\mathfrak{F}(H)$ such that

$$\tilde{f}\sim\sum_{k=1}^{\infty} a_k\, h_{k-1}, \tag{4.2}$$

then $\tilde{f}$ is called to be the Hermite-conjugate function of f.

THEOREM IV. *Let $1<p<\infty$. Suppose there exists $\tilde{f}$ and $e^{-x^2/2}f(x)$, $e^{-x^2/2}\tilde{f}(x)\in L^p(\mathbf{R})$, $\sum_{n=1}^{\infty}\frac{1}{n\epsilon_n}<\infty$,*

$$\|e^{-x^2/2}[\sigma_n(f,x)-f(x)]\|_{L^p(\mathbf{R})}=O\left(\frac{1}{\sqrt{n}\,\epsilon_n}\right) \tag{4.3}$$

are fulfilled. Then $\tilde{f}$ is absolutely continuous on every finite interval and $[e^{-x^2}\tilde{f}(x)]' e^{x^2/2} \in L^p(\mathbf{R})$.

For the proof we need some lemmas.

LEMMA 4.1. *Let $\{\phi_n\}$ be any sequence in a Banach space B, denote σ_n^* and σ_n the n-th Fejér sum of the series $\Sigma\phi_n$ and $\Sigma\phi_n/\sqrt{n}$ respectively. Then, if there exists $\sigma \in \beta$ such that*

$$\|\sigma_n - \sigma\|_\beta = O\left(\frac{1}{\sqrt{n}\epsilon_n}\right)$$

it imply $\|\sigma_n^*\|_\beta = O(1)$.

PROOF. The statement and the proof is a modification that of used by G. ALEXITS in proving the Theorem E.

We can not replace $\epsilon_n \equiv 1$ as shown by numerical examples, however we can prove by simple application of Abel's transformation the

LEMMA 4.1* *If $\lambda_n \to \infty$, $\lambda_{n+1} = O(\lambda_n)$ and σ_n, σ_n^* are the n-th Riesz means of parameter 1/2 of $\Sigma\phi_n$, $\Sigma\phi_n/\sqrt{\lambda_n}$ then with absolute C_1 and C_2:*

$$\|\sigma_n^*\|_\beta \le C_1\sqrt{\lambda_n}\|\sigma_n - \sigma\|_\beta, \quad \|\sigma_n - \sigma\|_\beta \le C_1 \frac{\|\sigma_n^*\|}{\sqrt{\lambda_n}}. \tag{4.4}$$

LEMMA 4.2. *Let $1 \le p < \infty$, $g, h \in L^p(\mathbf{R})$. Suppose there exists a sequence $\{\phi_n\} \subset C^1(\mathbf{R})$ such that*

$$\|[\phi_n(x) - g(x)]e^{x^2/2}\|_{L^p(\mathbf{R})} \to 0 \ (n \to \infty), \tag{4.5}$$

$$\|[\phi_n'(x) - h(x)]e^{x^2/2}\|_{L^p(\mathbf{R})} \to 0 (n \to \infty). \tag{4.6}$$

Then g is absolutely continuous on every finite interval and $g'(x) = h(x)$ for a.e. $x \in \mathbf{R}$.

PROOF. It is a simple calculation, hence we omit it.

PROOF OF THE THEOREM IV. Introduce the formal series

$$\sum_{k=0}^{\infty} a_k h_k(x) e^{-x^2/2}, \tag{4.7}$$

$$\sum_{k=1}^{\infty} \sqrt{k} a_k h_k(x) e^{-x^2/2}, \tag{4.8}$$

$$\sum_{k=1}^{\infty} \sqrt{k} a_k h_k(x) e^{-x^2}, \tag{4.9}$$

$$\sum_{k=1}^{\infty} a_k h_{k-1}(x) e^{-x^2}. \tag{4.10}$$

Denote $\sigma_n(i)$ the n-th Fejér mean of the series (i). We can write (4.3) in the form:

$$\|\sigma_n(4.7) e^{-x^2/2} f(x)\|_{L^p(\mathbf{R})} = O\left(\frac{1}{\sqrt{n}\epsilon_n}\right),$$

and hence, using Lemma 4.1 we get

$$\|\sigma_n\ (4.8)\|_{L^p(\mathbf{R})} = O(1). \tag{4.11}$$

Now we prove

$$\sigma_n\ (4.8) = \sigma_n\ (4.9) e^{x^2/2} = -\frac{1}{\sqrt{2}}\ \sigma_n'\ (4.10)\ e^{-x^2/2}. \tag{4.12}$$

By Szegő [27] p. 105 and 106 we have

$$h_k(x) = H_k(x)/\sqrt{\sqrt{\pi}\cdot 2^k \cdot k!}$$

and

$$H_k(x) = 2xH_{k-1}\ (x) - H_{k-1}'\ (x).$$

Using these relations, we obtain

$$[h_{k-1}\ (x) e^{-x^2}]' = e^{-x^2} h_{k-1}'(x) - 2xh_{k-1}(x) e^{-x^2} =$$

$$= -e^{-x^2} \frac{1}{\sqrt{\sqrt{\pi} 2^{k-1} (k-1)!}}\ [2xH_{k-1}(x) - H_{k-1}'(x)] =$$

$$= -e^{-x^2} \frac{\sqrt{\sqrt{\pi} 2^k k!}}{\sqrt{\sqrt{\pi} 2^{k-1} (k-1)!}}\ h_k(x) = -\sqrt{2k} h_k(x) e^{-x^2},$$

hence (4.12) follows and we can write (4.11) in the form

$$\|\sigma_n'(4.10) e^{x^2/2}\|_{L^p(\mathbf{R})} = O(1). \tag{4.13}$$

On the other hand it is easy to deduce from (4.12): there exists $h(x) \in L^p(\mathbf{R})$ such that

$$\sqrt{k}a_k = \int_{-\infty}^{\infty} h(x)h_k(x)e^{-x^2}dx \ (k = 1, 2, \ldots) .$$

Hence G. FREUD's result [29] gives

$$\|[\sigma_k(4.9)e^{x^2/2} - h(x)]\|_{L^p(\mathbf{R})} \to 0 \ (n \to \infty),$$

and taking into account also (4.11) we have

$$\|[\sigma_n'(4.10) - h(x)e^{-x^2/2}]e^{x^2/2}\|_{L^p(\mathbf{R})} \to 0 \ (n \to \infty). \tag{4.14}$$

Now apply G. FREUD's result [29] for the function $\tilde{f}(x)e^{-x^2/2}$. We get

$$\|[\sigma_n(4.2) - \tilde{f}(x)]e^{-x^2/2}\|_{L^p(\mathbf{R})} \to 0 \ (n \to \infty),$$

i.e. taking into consideration $\sigma_n(4.2)e^{-x^2/2} = \sigma_n(4.10)e^{x^2/2}$ it follows

$$\|[\sigma_n(4.10) - \tilde{f}(x)e^{-x^2}]e^{x^2/2}\|_{L^p(\mathbf{R})} \to 0 \ (n \to \infty) . \tag{4.15}$$

Using Lemma 4.2 we obtain from (4.14) and (4.15)

$$[e^{-x^2}\tilde{f}(x)]' = h(x)e^{-x^2/2},$$

i.e.

$$[e^{-x^2}\tilde{f}(x)]'e^{x^2/2} \in L^p(\mathbf{R}). \quad \square$$

PROBLEM 9. *Suppose* $[e^{-x^2}\tilde{f}(x)]'e^{x^2/2} \in L^p(\mathbf{R})$. *Does it follows* (4.3) (*using also the natural assumptions* $e^{-x^2/2}f(x)$, $e^{-x^2/2}\tilde{f}(x) \in L^p(x)$)?

5. Divergence in three dimension

Now we consider the divergence problem for the multidimensional case. For the sake of simplicity we concern ourselves for $N = 3$ and $L = -\Delta$ (i.e. consider only the Laplace operator). However all our results remains valid for $N \geq 2$, and for the Schrödinger operator instead of the Laplace one, but we must assume much more on the potential function, than in the one dimensional case. We have the

THEOREM V. *Suppose* $1 \leq p < 3$. *Then there exists an* $f \in L_p(\Omega)$ *such that* $\overline{\lim}_{\mu \geq 1} |\sigma_\mu(f, x)| = +\infty$ *for a. e.* $x \in \Omega$.

For the proof it is enough to prove the lemmas 1—3 below, and after Theorem V follows by the metod of [9].

LEMMA 5.1. *For any compact set* $K \subset \Omega$ *there exists* $C_1 > 0$ *and* $B > 0$ *such that*

$$\sum_{\substack{|\mu - \rho_n| \leq 1 \\ \rho_n \geq B\nu_n \geq B^2}} \left[|u_n(x)| \operatorname{ch}\left(\frac{R\nu_n}{2}\right)\right]^2 \leq C_1 \mu^2 \quad (x \in K,\ \mu \geq 1) \tag{5.1}$$

where $\rho_n = |\operatorname{Re}\sqrt{\lambda_n}|$, $\nu_n = |\operatorname{Im}\sqrt{\lambda_n}|$ *as well as in* **3**.

PROOF. Define for $0 < R < \frac{1}{2}$ dist $(K, \partial\Omega)$ the function

$$w = w_R(r, \mu) = \begin{cases} \frac{\mu}{r} \sin r\mu & \text{if } R \leq r \leq 2R \\ 0 & \text{if } r \notin (R, 2R) \end{cases}$$

and using the Titchmarsh formula [30]

$$\int_\Theta u_n(x + r\Theta)\, d\Theta = \frac{\sin r\mu}{r\mu} u_n(x), \tag{5.2}$$

calculate the Fourier coefficients of w with respect to the system $\{u_n\}$:

$$w_n := \int_\Omega w_R(|x-y|, \mu) u_n(y)\, dy = \int_R^{2R} r^2 \frac{\mu}{r} \sin r\mu \left(\int_\Theta u_n(xr + \Theta)\, d\Theta \right) dr =$$

$$= \int_R^{2R} r\mu \sin r\mu \frac{\sin r\mu_n}{r\mu_n} u_n(x)\, dr = u_n(x) \frac{\mu}{\mu_n} \int_R^{2R} \sin r\mu \sin r\mu\, dr =$$

$$= u_n(x)\,\frac{\mu}{\mu_n}\int_R^{2R}(\cos r\,(\mu-\mu_n)-\cos r\,(\mu+\mu_n))dr =$$

$$= u_n(x)\,\frac{\mu}{\mu_n}\left(\frac{\sin 2R(\mu-\mu_n)-\sin R(\mu-\mu_n)}{\mu-\mu_n} - \right.$$

$$\left. - \frac{\sin 2R(\mu+\mu_n)-\sin R(\mu+\mu_n)}{\mu+\mu_n}\right) =$$

$$= u_n(x)\,\frac{\mu}{\mu_n}\left(\frac{\sin 2R(\mu-\mu_n)(2\cos R(\mu-\mu_n)-1)}{\mu-\mu_n} - \right.$$

$$\left. - \frac{\sin R(\mu+\mu_n)(2\cos R(\mu+\mu_n)-1)}{\mu-\mu_n}\right).$$

Hence

(5.3)

$$|w_n| \geq |u_n(x)|\frac{\mu}{|\mu_n|}\left(\frac{\operatorname{sh} R\nu_n(2\operatorname{sh} R\nu_n-1)}{\sqrt{(\mu-\rho_n)^2-\nu_n^2}} - \frac{\operatorname{ch} R\nu_n(2\operatorname{ch} R\nu_n-1)}{\sqrt{(\mu-\rho_n)^2-\nu_n^2}}\right).$$

Let $B^*>0$ be such that $\nu_n \geq B^*$ implies $\operatorname{sh} R\nu_n \geq 1$. Then we have $2\operatorname{sh} R\nu_n \geq$ $\geq 1+\operatorname{sh} R\nu_n \geq \operatorname{ch} R\nu_n$, i.e.

$$\operatorname{cth} R\nu_n \leq 2,\ (\nu_n \geq B^*),$$

$$2\operatorname{sh} R\nu_n - 1 \geq \operatorname{sh} R\nu_n \geq 1,\ (\nu_n \geq B^*),$$

$$\frac{2\operatorname{ch} R\nu_n - 1}{2\operatorname{sh} R\nu_n - 1} \leq 2\frac{\operatorname{ch} R\nu_n}{\operatorname{sh} R\nu_n} \leq 4.$$

Choose $B := \max\{B^*, 16\}$ and suppose $\rho_n \geq B\nu_n \geq B^2$, $|\mu-\rho_n| \leq 1$. Then obviously:

$$\frac{(\mu+\rho_n)^2+\nu_n^2}{(\mu-\rho_n)^2+\nu_n^2} = \frac{B^2(\mu+\rho_n)^2+B^2\nu_n^2}{B^2(\mu-\rho_n)^2+B^2\nu_n^2} \geq \frac{B^2\rho_n^2+B^4}{B^2+\rho_n^2} =$$

$$= \frac{B^2(\rho_n^2+B^2)}{B^2+\rho_n^2} = B^2 \geq 16^2, \text{ i.e.}$$

$$\frac{\sqrt{(\mu+\rho_n)^2+\nu_n^2}}{\sqrt{(\mu-\rho_n)^2+\nu_n^2}} \geq 16 \geq 2\frac{\operatorname{ch} R\nu_n(2 \operatorname{ch} R\nu_n-1)}{\operatorname{sh} R\nu_n(2 \operatorname{sh} R\nu_n-1)},$$

hence

$$-\frac{\operatorname{ch} R\nu_n(2 \operatorname{ch} R\nu_n-1)}{\sqrt{(\mu+\rho_n)^2+\nu_n^2}} \geq -\frac{1}{2}\frac{\operatorname{sh} R\nu_n(2 \operatorname{sh} R\nu_n-1)}{\sqrt{(\mu-\rho_n)^2+\nu_n^2}},$$

Consequently, it follows from (5.3) the estimate

$$|w_n| \geq |u_n(x)|\frac{\mu}{|\mu_n|}\frac{1}{2}\frac{\operatorname{sh} R\nu_n(2 \operatorname{sh} R\nu_n-1)}{\sqrt{(\mu-\rho_n)^2+\nu_n^2}} \geq \tag{5.4}$$

$$\geq |u_n(x)|\frac{\mu}{|\mu_n|}\frac{1}{2}\frac{\operatorname{sh} R\nu_n}{\sqrt{1+\nu_n^2}} \geq |u_n(x)|\frac{\mu}{|\mu_n|}\frac{\operatorname{ch} R\nu_n}{1+\nu_n} \geq$$

$$\geq |u_n(x)|\frac{\mu}{|\mu_n|}\frac{\operatorname{ch} R\nu_n}{2\nu_n}. \text{ Because } \frac{\operatorname{ch} R\nu_n}{R\nu_n} \geq \operatorname{ch}\frac{R\nu_n}{2},$$

using (5.4) and the Bessel inequality we get:

$$\sum_{\substack{|\mu-\rho|\leq 1\\ \rho_n\geq B\nu_n\geq B^2}}\left[|u_n(x)|\operatorname{ch}\frac{R\nu_n}{2}\right]^2 \leq \sum_{\substack{|\mu-\rho|\leq 1\\ \rho_n\geq B\nu_n\geq B^2}}\left[|u_n(x)|\frac{\operatorname{ch} R\nu_n}{R\nu_n}\right]^2 =$$

$$= \left(\frac{|\mu_n|}{\mu}\frac{2}{R}\right)^2 \sum_{\substack{|\mu-\rho_n|\leq 1\\ \rho_n\geq B\nu_n\geq B^2}}\left[|u_n(x)|\frac{u}{|u_n|}\frac{1}{2}\frac{\operatorname{ch} R\nu_n}{\nu_n}\right]^2 \leq$$

$$\leq C\sum_{n=1}^{\infty}|w_n|^2 \leq C\mu^2,$$

because $\quad |\nu_n| = \sqrt{\rho_n^2+\nu_n^2} \leq \rho_n\sqrt{1+\frac{1}{B^2}} \leq \mu+1.$

LEMMA 5.2. *For every compact set $K\subset\Omega$ there exists $c_2>0$ such that*

$$\sum_{\substack{|\mu-\rho_n|\leq 1\\ \rho_n\leq B\nu_n\\ \nu_n\geq B}}\left[|u_n(x)|\operatorname{ch}\left(\frac{R\nu_n}{2}\right)\right]^2 \leq C_2\mu^2, \ (x\in K,\ \mu\geq 1). \tag{5.5}$$

$B>0$ is the same constant (independent on K and μ) as in Lemma 5.1.

PROOF. Define

$$w_R^* := (r, \mu) := \begin{cases} \dfrac{\mu}{r} & \text{if} \quad R \le r \le 2R, \\ 0 & \text{if} \quad r \notin (R, 2R) \end{cases}$$

and calculate the Fourier coefficiets of w_n^* with respect to the system (u_n); by Titchmarsch's formula (5.2) we get:

$$w_n^* := \int_\Omega w_n^*(|x-y|, \mu) u_n(y) dy = \int_R^{2R} r^2 \frac{\mu}{r} \left(\int_\Omega u_n(x+r\Theta) \right) dr =$$

$$= \int_R^{2R} r\mu \frac{\sin \mu_n r}{\mu_n r} u_n(x) dr = u_n(x) \frac{\mu}{\mu_n} \int_R^{2R} \sin \mu_n r dr =$$

$$= u_n(x) \frac{\mu}{\mu_n} \frac{\cos R\mu_n - \cos 2R\mu_n}{\mu_n},$$

hence

$$|w_n^*| = |u_n(x)| \frac{\mu}{|\mu_n|} \left| \frac{\cos 2R\mu_n - \cos R\mu_n}{\mu_n} \right| \ge$$

$$\ge |u_n(x)| \frac{\mu}{|\mu_n|} \cdot \frac{\operatorname{sh} 2R\nu_n - \operatorname{ch} R\nu_n}{|\mu_n|} =$$

$$= |u_n(x)| \frac{\mu}{|\mu_n|} \cdot \frac{\operatorname{ch} R\nu_n (2 \operatorname{sh} R\nu_n - 1)}{|\mu_n|} \ge |u_n(x)| \frac{\mu}{|\mu_n|} \frac{\operatorname{ch} R\nu_n}{|\mu_n|}.$$

Now $\rho_n \le B\nu_n$ and $\nu_n \ge B$, hence

$$|\mu_n|^2 = \rho_n^2 + \nu_n^2 \le \nu_n^2 (B^2+1) \text{ i.e. } \frac{1}{\nu_n^2} \le \frac{B^2+1}{|\mu_n|^2}.$$

Taking into account also $\nu_n \operatorname{ch}\left(\dfrac{R\nu_n}{2}\right) \le C \dfrac{\operatorname{ch}(R\nu_n)}{R\nu_n}$ we get

$$\sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B\nu_n\\ \nu_n\ge B}} \left[|U_n(x)|\,\mathrm{ch}\left(\frac{R\nu_n}{2}\right)\right]^2 \le \sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B\nu_n\\ \nu_n\ge B}} \left[\frac{1}{\nu_n}|U_n(x)|\frac{\mathrm{ch}\,(R\nu_n)}{R\nu_n}\right]^2 \le$$

$$\le \frac{B^2+1}{R^2} \sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B\nu_n\\ \nu_n\ge B}} \left[|U_n(x)|\frac{\mathrm{ch}\,(R\nu_n)}{|\mu_n|}\cdot\frac{1}{\nu_n}\right]^2 \le$$

$$\le \left(\frac{B+1}{R}\cdot\frac{|\mu_n|}{\mu}\cdot\frac{1}{\nu_n}\right)^2 \sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B\nu_n\\ \nu_n\ge B}} \left[|U_n(x)|\frac{\mu}{|\mu_n|}\cdot\frac{\mathrm{ch}\,(R\nu_n)}{|\mu_n|}\right]^2 \le$$

$$\le C \sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B\nu_n\\ \nu_n\ge B}} \left[|U_n(x)|\frac{\mu}{|\mu_n|}\cdot\frac{\mathrm{ch}\,(R\nu_n)}{|\mu_n|}\right]^2 \le C\sum_{n=1}^{\infty} |w_n^*|^2 \le C\mu^2.$$

By the method of [3] we get

LEMMA 5.3. *For any compact set $K\subset\Omega$ there exists $C_3>0$ with*

$$\sum_{\substack{|\mu-\rho_n|\le 1\\ \rho_n\le B}} |U_n(x)|^2 \le C_3\mu^2,\ (x\in K,\ \mu\ge 1). \tag{5.6}$$

After this preparation our theorem follows by Il'in's method.

REMARK 5.1 In [31] there were proved convergence and lokalization theorems for the singular Schrödinger operator, in the points of the domain considered. The locally uniform variant of these theorems remained open. However, a carefull analyse of the proof given in [31] gives also this variant. Namely, we have proved in [31] uniform estimate for the Green function of the corresponding operator, and we show now that applying the well known method of V. A. IL'IN, this estimate impliy the uniform estimate for the sum of square (type Lemmas 5.1—5.3). To this it is enough to prove that for $\delta = \frac{N}{4}+\sigma$, and for

$$f(x,\lambda_n,y)=f(y):=\phi(\lambda_n,|x-y|)\,|x-y|^{-P}Y_p(\sqrt{\lambda_n}\,|x-y|)q(y)U_n(y)$$

we have

$$L^{-\sigma}f \in L_2(\Omega)\left(\phi \in C_0^\infty(0, \infty),\ \phi(t) = 1 \text{ if } \sqrt{\lambda_n}t < \frac{1}{2} \text{ and } 0 \text{ for } \sqrt{\lambda_n}t > 1\right);$$

we use the notations and the results of [31].
To this let $f_\xi(y) := f(y)\cdot\chi_{[0,\xi]}(|x-y|)$. For any $g \in L_2(\Omega)$ we have $(L^{-\sigma}f_\xi, g) = (f_\xi, L^{-\sigma}g)$, further, obviously $\|L^{-\sigma}g\|_\infty \le \|g\|_2$, hence $|(L^{-\sigma}f_\xi, g)| \le |(f_\xi, L^{-\sigma}g)| \le \|f_\xi\|_1 \|L^{-\sigma}g\|_\infty \le C\|f_\xi\|_1 \|g\|_2$. Here $g \in L_2(\Omega)$ is arbitrary, so $\|L^{-\sigma}f_\xi\|_2 \le C\|f\|_1$ and taking $\xi \to 0$: $\|L^{-\sigma}f\|_2 \le C\|f\|_1$.

REMARK 5.2 (in July 7, 1987). Some problems of the present paper are solved. We give here some references on related papers.

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ZUR THEORIE LIMESFORMER RÄUME I.

Von

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0. Einleitung

Lösungen „topologischer Probleme" sind bis heute im wesentlichen unter zwei Aspekten behandelt worden.

Zum einen steht die Einführung und Untersuchung von topologischen Kategorien im Vordergrund, so daß sich die neue Disziplin „Categorical topology" entwickeln konnte (vergl. [7], [9], [16], [17], [20]). Beim anderen wurden spezielle Strukturen untersucht, in denen man „topologische Fragestellungen" geeignet darstellen konnte.

Ausgehend von der gemeinsamen „Wurzel" R_0—*Top,* der schwach symmetrischen topologischen Räume (vergl. [6]) kann man vier verschiedene Stränge einer Entwicklung zu allgemeineren Strukturen nachzeichnen, die sich wie folgt benennen lassen:

1. Hüllenkategorien (vergl. [5], [6], [12]);
2. Konvergenzkategorien (vergl. [4], [10], [11], [17], [21]);
3. Nearkategorien (vergl. [1], [6], [9], [10], [21]);
4. Proximitätskategorien (vergl. [13], [14], [15], [18]).

In dieser Arbeit wird die axiomatisch definierte Kategorie *Limform* der limesformen Räume mit den limesformen Abbildungen als gemeinsame „Krone" dieser Stränge vorgestellt und untersucht. Die genannten Kategorien lassen sich dabei als Unterkategorien auffassen, wobei in den jeweiligen Spezialfällen sowohl geeignete Bireflektionen als auch Bikoreflektionen angegeben werden. Darüberhinaus besitzt *Limform* eine Unterkategorie, die sowohl *Top* als auch *LE-Prox* (vergl. [15]) enthält und bessere Eigenschaften als diese besitzt, wie z.B. die Tatsache, daß die Produktbildung zweier Quotientenabbildungen in jener wieder eine Quotientenabbildung ist.

Der grundlegende Begriff der Kompaktheit eines topologischen Raumes resultiert in zwei fundamentalen Verallgemeinerungen für limesforme Räume, die im ausgehenden Spezialfall übereinstimmen.

Der Begriff „epikompakt" läßt sich dabei durch surjektive Morphismen übertragen.

Die vorliegende Arbeit entstand im Rahmen des Forschungsschwerpunktes „Kategorielle Methoden in Topologie und Informatik" an der FU Berlin.

1. Einige Symbole

X sei eine Menge; $\mathfrak{P}^1 X = \mathfrak{P}X$ bezeichne die Potenzmenge von X; $\mathfrak{P}^{n+1}X$ bezeichne die Potenzmenge von $\mathfrak{P}^n X$;

mit kleinen lateinischen Buchstaben $x, y, z, \ldots$ werden i.a. Elemente von X bezeichnet;

mit großen lateinischen Buchstaben $A, B, C, \ldots$ werden i.a. Teilmengen von X bezeichnet;

mit Frakturbuchstaben $\mathfrak{M}, \mathfrak{N}, \mathfrak{R}, \ldots$ werden i.a. Teilmengen von $\mathfrak{P}X$ bezeichnet, die dann Mengensysteme über X heißen;

$x \in X$ bezeichne die Aussage: x ist Element von X;

$x \notin X$ bezeichne die Aussage: x ist kein Element von X;

„$\subset$" bezeichne die Inklusion;

für $A \subset X$ bezeichne $X \setminus A$ das Komplement von A bezüglich X;

$\emptyset$ bezeichne die Leere Menge;

$\forall$ bezeichne den Quantor „für alle";

$\exists$ bezeichne den Quantor „es existiert";

$\cap$ bezeichne die Operation „Schnitt";

$\cup$ bezeichne die Operation „Vereinigung";

$\dot{\cup}$ bezeichne die Operation „disjunkte Vereinigung";

für $\mathfrak{M} \subset \mathfrak{P}X$ *setze:* $\cap \mathfrak{M} = \bigcap_{F \in \mathfrak{M}} F$;

$\cup \mathfrak{M} = \bigcup_{F \in \mathfrak{M}} F$;

stack $\mathfrak{M} = \{C \subset X \mid \exists F \in \mathfrak{M} F \subset C\}$:

sec $\mathfrak{M} = \{C \subset X \mid \forall F \in \mathfrak{M}\ F \cap C \neq \emptyset\}$;

$\mathfrak{M} \ll \mathfrak{N}: \Leftrightarrow \forall F \in \mathfrak{M} \exists G \in \mathfrak{N} F \supset G$;

$\mathfrak{M} \vee \mathfrak{R} = \{F \cup G \mid F \in \mathfrak{M},\ G \in \mathfrak{N}\}$;

für $\mathfrak{M} \subset \mathfrak{P}X$ und eine Funktion f setze: $f\mathfrak{M} := \{f[F] \mid F \in \mathfrak{M}\}$;

$f^{-1}\mathfrak{M} := \{f^{-1}[F] \mid F \in \mathfrak{M}\}$;

$f[F]$ bezeichne dabei das Bild von F unter der Abbildung f;

$f^{-1}[F]$ bezeichne dabei das Urbild von F unter der Abbildung f;

$X \times X$ bezeichne das kartesische Produkt von X mit sich selbst;
(x, y) bezeichne ein geordnetes Paar;
mit kleinen griechischen Buchstaben $\alpha, \beta, \delta, \ldots$ werden i.a. Teilmengen von $\mathfrak{P}X \times \mathfrak{P}X$ bezeichnet;

$$\triangle_X := \{(x, x) \mid x \in X\};$$

für $U, V \subset X \times X$ setze: $U^{-1} := \{(x, y) \in X \times X \mid (y, x) \in U\}$;
$U \circ V := \{(x, y) \in X \times X \mid \exists z \in X((x, z) \in V \text{ und } (z, y) \in U)\}$;
für $\mathfrak{U}, \mathfrak{V} \subset \mathfrak{P}(X \times X)$ setze: $\mathfrak{U}^{-1} := \{U^{-1} \mid U \in \mathfrak{U}\}$;
$\mathfrak{U} \circ \mathfrak{V} := \{R \subset X \times X \mid \exists U \in \mathfrak{U} \exists V \in \mathfrak{V} R \supset U \circ V\}$;
mit großen griechischen Buchstaben $\Gamma, \Omega, \Lambda, \ldots$ werden i.a. Teilmengen von $\mathfrak{P}^2X$ bezeichnet;
für Abbildungen L_1, L_2: $\mathfrak{P}X \to \mathfrak{P}^3X$ setze:

$$L_1 \leq L_2 : \Leftrightarrow \forall A \in \mathfrak{P}X \; L_1(A) \subset L_2(A).$$

2. Limesformitäten

2.1 Definition: Für eine Menge X heißt eine Abbildung L: $\mathfrak{P}X \to \mathfrak{P}^3X$ *Limesformität (limesformer Operator)* auf X, *wenn gilt:*
(Lf_1) $A \in \mathfrak{P}X$ und $\mathfrak{R}_1 \in L(A)$ mit $\mathfrak{R}_1 \ll \mathfrak{R}_2 \in \mathfrak{P}^2X$ implizieren $\mathfrak{R}_2 \in L(A)$,
(Lf_2) $A_1, A_2 \in \mathfrak{P}X$ und $A_1 \subset A_2$ implizieren $L(A_1) \subset L(A_2)$,
(Lf_3) $L(\emptyset) \neq \emptyset$ und $\emptyset \notin L(X)$,
(Lf_4) $x \in X$ impliziert $\{\{x\}\} \in L(\{x\})$;
das Paar (X,L) heißt dann *limesformer Raum.* Für $\mathfrak{R} \in L(A)$ heißt A *Limes* von $\mathfrak{R}$ und $\mathfrak{R}$ heißt *A—limitiert.*

2.2. Beispiele: X sei eine Menge;

(i) h: $\mathfrak{P}X \to \mathfrak{P}X$ bezeichne eine *"extended topology"* (e.t.) auf X, d.h. h erfüllt die Axiome:
(et_1) $A_1 \subset A_2$ impliziert $h(A_1) \subset h(A_2)$,
(et_2) $A \in \mathfrak{P}X$ impliziert $A \subset h(A)$,
für $A \in \mathfrak{P}X$ setze:

$$L_h(A) = \{\mathfrak{R} \in \mathfrak{P}^2X \mid h(A) \in \text{ stack } \mathfrak{R}\}.$$

(ii) $\delta \subset \mathfrak{P}X \times \mathfrak{P}X$ bezeichne eine *LE-Proximität* auf X, d.h. δ erfüllt die Axiome:
(P_1) $A\,\delta(B \cup C)$ ist äquivalent zu $A\delta B$ oder $A\delta C$,
(P_2) $(A \cup B)\,\delta C$ ist äquivalent zu $A\delta C$ oder $B\delta C$,
(P_3) $A\delta B$ impliziert $A \neq \emptyset \neq B$,
(P_4) $A \cap B \neq \emptyset$ impliziert $A\delta B$,
(P_5) $A\delta B$ und $B \subset \mathrm{cl}_\delta(C)$ implizieren $A\delta C$, wobei

$$\mathrm{cl}_\delta(C) = \{x \in X \mid \{x\} \delta C\};$$

für $A \in \mathfrak{P}X$ setze:

$$L_\delta(A) = \{\mathfrak{R} \in \mathfrak{P}^2 X \mid \delta(A) \ll \mathfrak{R}\}, \text{ wobei}$$
$$\delta(A) = \{F \subset X \mid (X \setminus F, A) \notin \delta\}.$$

Beachte, daß folgende äquivalente Schreibweisen verwandt werden:

$$A \delta B \text{ genau dann, wenn } (A, B) \in \delta.$$

(iii) $\Gamma \subset \mathfrak{P}^2 X$ bezeichne eine prämerotopische Struktur auf X, d.h. Γ erfüllt die Axiome:

(M_1) $\mathfrak{M}_1 \in \Gamma$ und $\mathfrak{M}_1 \ll \mathfrak{M}_2 \in \mathfrak{P}^2 X$ implizieren $\mathfrak{M}_2 \in \Gamma$,
(M_2) $x \in X$ impliziert $\{\{x\}\} \in \Gamma$,
(M_3) $\emptyset \neq \Gamma \neq \mathfrak{P}^2 X$;
für $A \in \mathfrak{P}X$ setze: $L_\Gamma(A) = \Gamma$.

(iv) $\mathfrak{K} : X \to \mathfrak{S}(X)$ bezeichne eine *Stapelkonvergenzstruktur* auf X, wobei $\mathfrak{S} \in \mathfrak{S}(X) :\Leftrightarrow \emptyset \neq \mathfrak{S} \subset \mathfrak{P}X$ mit stack $\mathfrak{S} = \mathfrak{S}$ und $\mathfrak{K}$ erfüllt die Axiome:

(K_1) $x \in X$ impliziert $x \in \mathfrak{K}(x)$,
(K_2) $x \in X$ und $\mathfrak{S}_1 \in \mathfrak{K}(x)$ und $\mathfrak{S}_1 \subset \mathfrak{S}_2 \in \mathfrak{S}(X)$ implizieren $\mathfrak{S}_2 \in \mathfrak{K}(x)$; für $A \in \mathfrak{P}X$ setze:

$$L_{\mathfrak{K}}(A) = \begin{cases} \{\mathfrak{K} \in \mathfrak{P}^2 X \mid \exists x \in A \exists \mathfrak{S} \in \mathfrak{K}(x) \mathfrak{S} \ll \mathfrak{K}\} & \text{falls } A \neq \emptyset \\ \{\mathfrak{K} \in \mathfrak{P}^2 X \mid \emptyset \in \mathfrak{K}\} & \text{falls } A = \emptyset. \end{cases}$$

(v) $\mathfrak{X} \subset \mathfrak{P}X$ sei eine *Topologie* auf X; d.h. $\mathfrak{X}$ bezeichne das System der offenen Mengen; für $A \in \mathfrak{P}X$ setze:
$L_{\mathfrak{X}}(A) = \{\mathfrak{R} \in \mathfrak{P}^2 X \mid \mathfrak{X}(A) \ll \mathfrak{R}\}$, wobei

$$\mathfrak{X}(A) = \{F \subset X \mid \exists O \in \mathfrak{X} A \subset O \subset F\}.$$

2.3. Definition: Für limesforme Räume (X, L), (Y, T) heißt eine Abbildung $f: X \to Y$ *limesforme Abbildung* (von (X, L) nach (Y, T)), wenn gilt:

$A \in \mathfrak{P}X$ und $\mathfrak{R} \in L(A)$ implizieren $f\mathfrak{R} \in T(f[A])$. Für eine bijektive Abbildung $f: X \to Y$ bezeichne f^{-1} die Umkehrabbildung, so heißt f *homöoforme Abbildung* (von (X, L) nach (Y, T)), wenn f und f^{-1} limesforme Abbildungen sind.

2.4. Definition: Die (konkrete) Kategorie der limesformen Räume und limesformen Abbildungen wird mit *Limform* bezeichnet. Die Isomorphismen in der Kategorie *Limform* sind also gerade die homöoformen Abbildungen.

3. Merotope Räume

3.1. DEFINITION: Für einen limesformen Raum (X, L) heißt L *Meroformität* auf X, und das Paar (X, L) heißt *meroformer Raum*, wenn L die Bedingung (M) erfüllt, wobei:

$$\text{(M)}\ L(X) \subset L(\emptyset).$$

3.2. SATZ: *Die volle Unterkategorie Meroform der Kategorie Limform, deren Objekte die meroformen Räume sind, ist*

(a) *bireflektiv in Limform;*

(b) *isomorph zur Kategorie der prämerotopen Räume und prämerotopen Abbildungen.*

BEWEIS:

zu (a): Für einen limesformen Raum (X, L) setze für $A \in \mathfrak{P}X$: $L_m(A) = L(X)$, so ist (X, L_m) ein meroformer Raum, und $1_X : (X, L) \to (X, L_m)$ ist die *Meroform* Bireflektion von (X, L);

zu (b): Für einen prämerotopen Raum (X, Γ) ist (X, T_Γ) ein meroformer Raum, wobei für $A \in \mathfrak{P}X$ gilt: $T_\Gamma(A) = \Gamma$. Dies ergibt offenkundig einen Isomorphismus zwischen der Kategorie der prämerotopen Räume mit den prämerotopen Abbildungen und der Kategorie *Meroform*.

3.3. BEMERKUNG: Vom kategoriellen Standpunkt aus gibt es keinen wesentlichen Unterschied zwischen der Kategorie der prämerotopen Räume und *Meroform*. Da *Meroform* in *Limform* bireflektiv liegt, werden Limites in *Meroform* (bzw. in der Kategorie der prämerotopen Räume) in der Kategorie *Limform* gebildet.

4. Hüllenstrukturen

4.1. DEFINITION: Für einen limesformen Raum (X, L) heißt L *Kernformität* auf X, und das Paar (X, L) heißt *kernformer Raum*, wenn L das Axiom (K) erfüllt, wobei:

$$\text{(K)}\ A \in \mathfrak{P}X \text{ impliziert } VL(A) \in L(A) \text{ mit}$$

$$VL(A) = \left\{ \cup \{F_{\mathfrak{R}} \mid \mathfrak{R} \in L(A)\} \mid F_{\mathfrak{R}} \in \mathfrak{R} \forall \mathfrak{R} \in L(A) \right\};$$

$$VL(A) \text{ heißt dann } \textit{Kern} \text{ von } L(A).$$

4.2. BEMERKUNG: Man beachte, daß die in Beispiel 2.2. (i), (ii), (v) definierten limesformen Operatoren L_h, L_δ und $L_{\mathfrak{X}}$. Kernformitäten auf ihren jeweiligen Trägermengen sind.

4.3. DEFINITION: Eine Kernformität L auf einer Menge X heißt *Hüllenformität*, wenn L das Axiom (H) erfüllt, wobei:

$$(\mathrm{H})A \in \mathfrak{P}X \text{ und } \mathfrak{R} \in L(A) \text{ implizieren } \mathrm{cl}_L(A) \in \text{ stack } \mathfrak{R}$$
$$\text{mit } \mathrm{cl}_L(A) = \{x \in X \mid \exists \mathfrak{M} \in L(A) x \in \bigcap \mathfrak{M}\}.$$

4.4. Satz: *Die Kategorie der e.t. Räume und stetigen Abbildungen ist isomorph zur Kategorie Hüllform der hüllenformen Räume und limes formen Abbildungen.*

Beweis: Für einen e.t. Raum (X, h) ist der für alle $A \in \mathfrak{P}X$ durch $L_h(A) = \{\mathfrak{R} \in \mathfrak{P}^2 X \mid h(A) \in \text{ stack } \mathfrak{R}\}$ definierte limesforme Operator L_h *eine Hüllenformität auf* X. Umgekehrt betrachte für eine Hüllenformität T auf X die für alle $A \in \mathfrak{P}X$ durch $\mathrm{cl}_L(A) = \{x \in X \mid \exists \mathfrak{R} \in T(A) x \in \bigcap \mathfrak{R}\}$ definierte e.t. cl_T auf X.

Wir zeigen:

(i) $\mathrm{cl}_{L_h} = h$;

(ii) $L_{\mathrm{cl}_T} = T$.

Zu (i) „$\leq$": $x \in \mathrm{cl}_{L_h}(A)$ impliziert die Existenz eines Mengensystems $\mathfrak{R} \in L_h(A)$ mit $x \in \bigcap \mathfrak{R}$. Es folgt $h(A) \supset F$ für ein geeignetes $F \in \mathfrak{R}$, mithin gilt $x \in h(A)$.

„$\geq$": Für $x \in h(A)$ setze: $\mathfrak{R} = \{h(A)\}$, so gilt $\mathfrak{R} \in L_h(A)$ und $x \in \bigcap \mathfrak{R}$; also folgt $x \in \mathrm{cl}_{L_h}(A)$.

Zu (ii): „$\leq$": Sei $\mathfrak{R} \in L_{\mathrm{cl}_T}(A)$, so folgt $\mathrm{cl}_T(A) \in \text{ stack } \mathfrak{R}$. Da T insbesondere das Axiom (K) erfüllt, gilt $VT(A) \in \mathrm{T}(A)$.

Wir zeigen $VT(A) \ll \mathfrak{R}$, woraus dann unter Beachtung von (Lf_1) die Behauptung folgt. $F \in VT(A)$ impliziert für jedes $\mathfrak{M} \in T(A)$ die Existenz eines $F_{\mathfrak{M}} \in \mathfrak{M}$ mit der Darstellung $F = \bigcup \{F_{\mathfrak{M}} \mid \mathfrak{M} \in T(A)\}$.

Ziel: $\mathrm{cl}_T(A) \subset F$.

Sei dazu $x \in \mathrm{cl}_T(A)$, so gibt es ein $\mathfrak{M}' \in T(A)$ mit $x \in \bigcap \mathfrak{M}'$; insbesondere gilt $x \in F_{\mathfrak{M}'}$, und da $F_{\mathfrak{M}'} \subset F$ folgt $x \in F$. Nun ist $\mathrm{cl}_T(A) \in \text{ stack } \mathfrak{R}$, also gilt $F \in \text{ stack } \mathfrak{R}$, was zu zeigen war.

„$\geq$": Sei umgekehrt $\mathfrak{R} \in T(A)$, so gilt mit Axiom (H) $\mathrm{cl}_T(A) \in \text{ stack } \mathfrak{R}$ und konsequenterweise ist $\mathfrak{R} \in L_{\mathrm{cl}_T}(A)$.

Abschließend zeigen wir, daß für hüllenforme Räume (X, L_1), (Y, L_2) die folgenden Aussagen äquivalent sind:

(iii) $f\colon (X, L_1) \to (Y, L_2)$ ist limesforme Abbildung;

(iv) $f\colon (X, \mathrm{cl}_{L_1}) \to (X, \mathrm{cl}_{L_2})$ ist stetig.

(iii) impliziert (iv): Sei $y \in f[\mathrm{cl}_{L_1}(A)]$ so gilt $y = f(x)$ für $x \in \mathrm{cl}_{L_1}(A)$, mithin existiert ein Mengensystem $\mathfrak{R}_1 \in L_1(A)$ mit $x \in \bigcap \mathfrak{R}$. Da f nach Voraussetzung limesforme Abbildung ist, gilt $f\mathfrak{R} \in L_2(f[A])$.

Wir zeigen: $y \in \bigcap f\mathfrak{R}$.

Sei dazu $G \in f\mathfrak{R}$ so ist $G = f[F]$ für $F \in \mathfrak{R}$, und damit ist $x \in F$, woraus mit $y = f(x) \in G$ die Behauptung folgt.

(iv) impliziert (iii): Umgekehrt sei $\mathfrak{R}\in L_1(A)$, so folgt $\mathrm{cl}_{L_1}(A)\supset F$ für ein $F\in\mathfrak{R}$ unter Anwendung des Axioms (H). Da f stetig ist, gilt $\mathrm{cl}_{L_2}(f[A])\supset f[\mathrm{cl}_{L_1}(A)]\supset f[F]$. Es folgt die Aussage $VL_2(f[A])\ll f\mathfrak{R}$, und unter Anwendung von (K) und (Lf_1) gilt schließlich $f\mathfrak{R}\in L_2(f[A])$.

4.5. Lemma: Für einen kernformen Raum (X, L) sind die folgenden Aussagen äquivalent:
(i) L ist Hüllenformität auf X;
(ii) $A\in\mathfrak{P}X$ und $\mathfrak{R}\in L(A)$ implizieren die Existenz eines Mengensystems $\mathfrak{M}\in L(A)$ mit $\mathfrak{M}\ll\mathfrak{R}$ und $\bigcap\mathfrak{M}\in\mathfrak{M}$.

Beweis: (ii) impliziert (i): Für $A\in\mathfrak{P}X$ sei $\mathfrak{R}\in L(A)$, so gibt es nach Voraussetzung ein Mengensystem $\mathfrak{M}\in L(A)$ mit $\mathfrak{M}\ll\mathfrak{R}$ und $\bigcap\mathfrak{M}\in\mathfrak{M}$. Es folgt $\mathrm{cl}_L(A)\supset\bigcap\mathfrak{M}$, mithin gilt $\mathrm{cl}_L(A)\in\operatorname{stack}\mathfrak{R}$.

(i) impliziert (ii): Für $A\in\mathfrak{P}X$ sei $\mathfrak{R}\in L(A)$, so gilt nach Voraussetzung $\mathrm{cl}_L(A)\in\operatorname{stack}\mathfrak{R}$. Setze $\mathfrak{M}=\{\mathrm{cl}_L(A)\}$, so ist $\bigcap\mathfrak{M}\in\mathfrak{M}$ und $\mathfrak{M}\ll\mathfrak{R}$.

Wir zeigen: $\operatorname{stack} VL(A)\subset\operatorname{stack}\mathfrak{M}$.

Sei $F\in\operatorname{stack} VL(A)$, so folgt $F\supset\bigcap(\bigcap\{\operatorname{stack}\mathfrak{R}\mid\mathfrak{R}\in L(A)\})\supset\mathrm{cl}_L(A)$, woraus die Behauptung folgt. Da L Kernformität auf X ist, resultiert die Implikation unter Anwendung von (Lf_1). Beachte auch im Folgenden die Gleichung $\operatorname{stack} VL(A)=\bigcap\{\operatorname{stack}\mathfrak{R}\mid\mathfrak{R}\in L(A)\}$.

4.6. Satz: *Die Kategorie Hüllform ist eine bikoreflektive Unterkategorie der Kategorie Kernform der kernformen Räume und limesformen Abbildungen.*

Beweis: Für einen kernformen Raume (X, L) setze für $A\in\mathfrak{P}X$: $L^-(A)=\{\mathfrak{R}\in\mathfrak{P}^2X\mid\exists\mathfrak{M}\in L(A)\cap\mathfrak{M}\in\operatorname{stack}\mathfrak{R}\}$, so ist (X, L^-) ein hüllenformer Raum, und 1_X: $(X, L^-)\to(X, L)$ ist die *Hüllform* Bikoreflektion von (X, L). Trivialerweise erfüllt L^- die Axiome (Lf_1)—(Lf_4). Es bleibt zu zeigen, daß L^- den Bedingungen (K) und (H) genügt.

Zu (K): Sei $A\in\mathfrak{P}X$, so gilt nach Voraussetzung $VL(A)\in L(A)$.

Wir zeigen: $\bigcap VL(A)\in\operatorname{stack} VL^-(A)$.

Wähle für jedes $\mathfrak{R}\in L^-(A)$ ein Mengensystem $\mathfrak{M}^{\mathfrak{R}}\in L(A)$ mit $\bigcap\mathfrak{M}^{\mathfrak{R}}\supset F^{\mathfrak{R}}\in\mathfrak{R}$ und setze: $F=\bigcup\{F^{\mathfrak{R}}\mid\mathfrak{R}\in L^-(A)\}$, so gilt $F\in VL^-(A)$. Es genügt zu zeigen: $\bigcap VL(A)\supset\bigcup\{\bigcap\mathfrak{M}^{\mathfrak{R}}\mid\mathfrak{R}\in L^-(A)\}$; denn dann gilt unter Anwendung obiger Inklusion $\bigcap VL(A)\supset F$. Dazu wiederum reicht folgende Inklusion nachzuweisen:

$$\bigcap(\bigcap\{\operatorname{stack}\mathfrak{R}\mid\mathfrak{R}\in L(A)\})\supset\bigcup\{\bigcap\mathfrak{M}^{\mathfrak{R}}\mid\mathfrak{R}\in L^-(A)\}.$$

$x\notin\bigcap(\bigcap\{\operatorname{stack}\mathfrak{R}\mid\mathfrak{R}\in L(A)\})$ und $\mathfrak{R}'\in L^-(A)$ implizieren die Existenz einer Menge $R\in\bigcap\{\operatorname{stack}\mathfrak{R}\mid\mathfrak{R}\in L(A)\})$ mit $x\notin R$ und $R\in\operatorname{stack}\mathfrak{M}^{\mathfrak{R}'}$. Es folgt $\mathfrak{R}\supset\bigcap\mathfrak{M}^{\mathfrak{R}'}$, mithin gilt $x\notin\mathfrak{M}^{\mathfrak{R}'}$; *d.h.* $x\in\bigcup\{\bigcap\mathfrak{M}^{\mathfrak{P}}\mid\mathfrak{P}\in L^-(A)\}$, weil $\mathfrak{R}'\in L^-(A)$ beliebig war.

Zu (H): Für $A \in \mathfrak{P}X$ sei $\mathfrak{R} \in L^{-}(A)$, so folgt die Existenz eines Mengensystems $\mathfrak{M} \in L(A)$ mit $\bigcap \mathfrak{M} \in$ stack $\mathfrak{R}$. Setze $\mathfrak{M}' = \{\bigcap \mathfrak{M}\}$, so gilt $\mathfrak{M}' \in L^{-}(A)$ mit $\mathfrak{M}' \ll \mathfrak{R}$ und $\bigcap \mathfrak{M}' \in \mathfrak{M}'$. Unter Anwendung von Lemma 4.5. ist dann das Axiom (H) erfüllt.

Trivialerweise ist 1_X: $(X, L^{-}) \to (X, L)$ eine limesforme Abbildung. Sei nun (Y, T) hüllenformer Raum und f: $(Y, T) \to (X, L)$ eine limesforme Abbildung, so bleibt zu zeigen, daß f eine limesforme Abbildung von (Y, T) nach (X, L^{-}) ist.

Für $A \in \mathfrak{P}Y$ sei $\mathfrak{R} \in T(A)$, so folgt mit Lemma 4.5. die Existenz eines Mengensystems $\mathfrak{M} \in T(A)$, so daß die Aussagen $\mathfrak{M} \ll \mathfrak{R}$ und $\bigcap \mathfrak{M} \in \mathfrak{M}$ gelten. Da nach Voraussetzung f limesforme Abbildung von (Y, T) nach (X, L) ist, folgt $f\,\mathfrak{M} \in L(f\,[A])$. Wir zeigen: $\bigcap f\,\mathfrak{M} \in$ stack $f\,\mathfrak{R}$. Nun gilt $\bigcap \mathfrak{M} \supset F \in \mathfrak{R}$, und daraus resultiert $\bigcap f\,\mathfrak{M} \supset f\,[\bigcap \mathfrak{M}] \supset f\,[F]$, also ist $f\,\mathfrak{R} \in L^{-}(f\,[A])$.

4.7. Bemerkung: Da mit dem vorausgehenden Satz *Hüllform* bikoreflektiv in *Kernform* liegt, werden die Kolimites in *Hüllform* (bzw. in der Kategorie der e.t. Räume (vergl. Satz 4.4)) in der Kategorie *Kernform* gebildet.

4.8. Satz: *Die Kategorie Kernform ist eine bireflektive Unterkategorie der Kategorie Limform.*

Beweis: Für einen limesformen Raum (X, L) setze für $A \in \mathfrak{P}X$: $L_k(A) = \{\mathfrak{R} \in \mathfrak{P}^2X \mid VL(A) \ll \mathfrak{R}\}$, so ist (X, L_k) ein kernformer Raum und 1_X: $(X, L) \to (X, L_k)$ ist die *Kernform* Bireflektion von (X, L). Trivialerweise erfüllt L_k die Axiome (Lf_1)—(Lf_4) und das Axiom (K). Weiter gilt, daß die Abbildung 1_X eine limesforme Abbildung von (X, L) nach (X, L_k) ist. Sei nun (Y, T) kernformer Raum und f: $(X, L) \to (Y, T)$ limesforme Abbildung, so zeigen wir, daß f: $(X, L_k) \to (Y, T)$ limesforme Abbildung ist. Sei dazu $A \in \mathfrak{P}X$ und $\mathfrak{R} \in L_k(A)$, so folgt $VL(A) \ll \mathfrak{R}$. Da f nach Voraussetzung limesforme Abbildung von (X, L) nach (Y, T) ist, gilt $f\,\mathfrak{M} \in T(f\,[A])$ für alle $\mathfrak{M} \in L(A)$. Konsequenterweise ergeben sich die Aussagen:

$$VT(f\,[A]) \ll V\{f\mathfrak{M} \mid \mathfrak{M} \in L(A)\} \ll f\,(VL(A)) \ll f\mathfrak{R}.$$

Nun ist aber T eine Kernformität auf Y, so daß unter Anwendung von (Lf_1) die Behauptung folgt.

4.9. Bemerkung: Da *Kernform* in *Limform* bireflektiv liegt, werden in Analogie zur Bemerkung 3.3. Limites in *Kernform* in der Kategorie *Limform* gebildet. Man beachte desweiteren, daß für einen kernformen Raum (X, L) die Aussage $A \in \mathfrak{P}X$ impliziert $\{A\} \in L(A)$ erfüllt ist.

4.10. Hinweis: Liegt bei der im Beispiel 2.2. (i) erwähnten „extended topology" h die Idempotenz vor, d.h. gilt für jedes $A \in \mathfrak{P}X$ $h(h(A)) = h(A)$ wie bei den kuratowskischen Hüllenoperatoren, so erfüllt der entsprechend definierte limesforme Operator L_h das zusätzliche Axiom (U), wobei:

(U) $A \in \mathfrak{P}X$ impliziert $L(\mathrm{cl}_L(A)) \subset L(A)$ mit

$$\mathrm{cl}_L(A) = \{x \in X \mid \exists \mathfrak{R} \in L(A) x \in \cap \mathfrak{R}\}$$

(vergl. auch Definition 4.3.).
Man beachte weiter, daß jede Meroformität L auf einer Menge X *automatisch* das Axiom (U) erfüllt.

4.11. Definition: Für einen limesformen Raum (X, L) heißt L *Ultraformität* auf X, und das Paar (X, L) heißt *ultraformer Raum*, wenn L das Axiom (U) in Hinweis 4.10. erfüllt.

4.12. Korollar: Für einen prämerotopen Raum (X, Γ) ist der in Satz 3.2. definierte meroforme Operator L_m eine Ultraformität auf X.

4.13. Definition: Für einen limesformen Raum (X, L) heißt L *Superformität* auf X, und das Paar (X, L) heißt *superformer Raum*, wenn L die Axiome (U) und (K) erfüllt.

4.14. Bemerkung: Für eine *Čech-topology* h auf einer Menge X (d.h. einer Abbildung h: $\mathfrak{P}X \to \mathfrak{P}X$, welche die Axiome (et_1) und (et_2) in Beispiel 2.2. (i) und darüberhinaus das Axiom (et_3) erfüllt, wobei:

$$(\mathrm{et}_3) A \in \mathfrak{P}X \text{ impliziert } h(h(A)) = h(A)$$

genügt die erklärte Limesformität L_h sowohl den Bedingungen einer Superformität in Definition 4.13. als auch den Axiomen einer Hüllenformität in Definition 4.3.

4.15. Definition: Für einen limesformen Raum (X, L) heißt L *Primformität* auf X, und das Paar (X, L) heißt *primformer Raum*, wenn L die Axiome (U), (K) und (H) erfüllt.

4.16. Satz: *Die volle Unterkategorie Primform der Kategorie Superform, deren Objekte die primformen Räume sind, ist*
(a) *bikoreflektiv in Superform;*
(b) *isomorph zur Kategorie der Čechtopologischen Räume und stetigen Abbildungen.*

Beweis:
Zu (a): Wir zeigen unter Beachtung von Satz 4.6., daß der für einen superformen Raum (X, L) definierte hüllenforme Operator L^- eine Primformität auf X ist.

Für $A \in \mathfrak{P}X$ sei dazu $\mathfrak{R} \in L^-(\mathrm{cl}_{L^-}(A))$, so existiert ein Mengensystem $\mathfrak{M} \in L(\mathrm{cl}_{L^-}(A))$ mit $\cap \mathfrak{M} \in$ stack $\mathfrak{R}$. Wir zeigen: $\mathrm{cl}_{L^-}(A) \subset \mathrm{cl}_L(A)$.

Für $x \in \mathrm{cl}_{L^-}(A)$ gibt es ein Mengensystem $\mathfrak{R} \in L^-(A)$ mit $x \in \bigcap \mathfrak{R}$. Wähle $\mathfrak{P} \in L(A)$, so daß $\bigcap \mathfrak{P} \in$ stack $\mathfrak{R}$ gilt aufgrund der Definition von L^-. Es folgt $\bigcap \mathfrak{P} \supset F \in \mathfrak{R}$, mithin erhält man $x \in \bigcap \mathfrak{P}$, woraus $x \in \mathrm{cl}_L(A)$ resultiert. Unter Anwendung des Axioms (Lf_2) in der Definition der Limesformität L ergeben sich unter weiterer Beachtung der Bedingung (U) die Inklusionen: $L(\mathrm{cl}_{L^-}(A)) \subset \subset L(\mathrm{cl}_L(A)) \subset L(A)$, und damit folgt $\mathfrak{M} \in L(A)$.

Da $\bigcap \mathfrak{M} \in$ stack $\mathfrak{R}$ gilt, ist nach Definition $\mathfrak{R} \in L^-(A)$.

Zu (b): Unter Verwendung von Satz 4.4. genügt es zu zeigen, daß cl_T idempotent ist. Für $A \in \mathfrak{P}X$ sei $x \in \mathrm{cl}_T(\mathrm{cl}_T(A))$, so gibt es ein Mengensystem $\mathfrak{M} \in T(\mathrm{cl}_T(A))$ mit $x \in \bigcap \mathfrak{M}$. Da T nach Voraussetzung (U) erfüllt, folgt $\mathfrak{M} \in T(A)$, mithin ist $x \in \mathrm{cl}_T(A)$.

4.17. Satz: *Die Kategorie Superform ist eine bireflektive Unterkategorie der Kategorie Ultraform.*

Beweis:
Unter Anwendung von Satz 4.8. zeigen wir, daß der für einen ultraformen Raum (X, L) definierte Operator L_k eine Superformität ist. Für $A \in \mathfrak{P}X$ sei dazu $\mathfrak{R} \in L_k(\mathrm{cl}_{L_k}(A))$, so gilt $VL(\mathrm{cl}_{L_k}(A)) \ll \mathfrak{R}$.

Wir zeigen: $VL(A) \ll VL(\mathrm{cl}_{L_k}(A))$.

Dazu ist hinreichend, folgende Inklusion zu prüfen, nämlich: $\bigcap\{\text{stack}\, \mathfrak{R} \mid \mathfrak{R} \in L(A)\} \subset \bigcap\{\text{stack}\, \mathfrak{P} \mid \mathfrak{P} \in L(\mathrm{cl}_{L_k}(A))\}$. Da nun L die Bedingung (U) erfüllt, reicht es wiederum zu untersuchen, ob

$$\bigcap\{\text{stack}\, \mathfrak{R} \mid \mathfrak{R} \in L(\mathrm{cl}_L(A))\} \subset \bigcap\{\text{stack}\, \mathfrak{P} \mid \mathfrak{P} \in L(\mathrm{cl}_{L_k}(A))\}$$

gilt. Dazu genügt es wegen (Lf_2) folgende Inklusion nachzuweisen: $\mathrm{cl}_{L_k}(A) \subset \subset \mathrm{cl}_L(A)$. $x \notin \mathrm{cl}_L(A)$ impliziert für jedes $\mathfrak{M} \in L(A)$ die Existenz einer Menge $F^{\mathfrak{M}} \in \mathfrak{M}$ mit $x \notin F^{\mathfrak{M}}$.

Setze $F = \bigcup\{F^{\mathfrak{M}} \mid \mathfrak{M} \in L(A)\}$, so gilt $F \in VL(A)$ und $x \notin F$. Für $\mathfrak{B} \in L_k(A)$ folgt $VL(A) \ll \mathfrak{B}$, mithin ergibt sich die Inklusionskette $\bigcap \mathfrak{B} \subset \bigcap VL(A) \subset F$, also resultiert $x \notin \bigcap \mathfrak{B}$. Damit gilt aber $VL(A) \ll \mathfrak{R}$, was die Aussage $\mathfrak{R} \in L_k(A)$ zur Folge hat.

4.18. Definition: Für einen limesformen Raum (X, L) heißt L *Sternformität* auf X, und das Paar (X, L) heißt *sternformer* Raum, wenn L Axiom (S) erfüllt, wobei:
(S) $A \in \mathfrak{P}X$ und $\mathfrak{R} \in L(A)$ implizieren die Existenz eines Mengensystems $\mathfrak{R}^* \in \in L(A)$, so daß für alle $F^* \in \mathfrak{R}^*$ eine Menge $C \subset X$ und ein Mengensystem $\mathfrak{R} \in \in L(C)$ existieren mit $C \in$ stack $\mathfrak{R}$ und $F^* \in$ stack $\mathfrak{R}$.

4.19. Satz: *Jeder primforme Raum (X, L) ist ein sternformer Raum.*

Beweis: Für $A \in \mathfrak{P}X$ sei $\mathfrak{R} \in L(A)$; setze $\mathfrak{R}^* = VL(A)$, so gilt wegen Axiom (K), daß $\mathfrak{R}^* \in L(A)$. Sei $F^* \in \mathfrak{R}^*$, so folgt $F^* = \bigcup\{F^{\mathfrak{M}} \mid \mathfrak{M} \in L(A)\}$ mit $F^{\mathfrak{M}} \in \mathfrak{M}$ für alle $\mathfrak{M} \in L(A)$. Setze $C = \mathrm{cl}_L(A)$; so gilt wegen Axiom (H) $C \in$ stack $\mathfrak{R}$. Für $\mathfrak{R} \in L(C)$ folgt unter Beachtung des Axioms $(U)\mathfrak{R} \in L(A)$, mithin ist $F^{\mathfrak{R}} \in \mathfrak{R}$, was die Aussage $F^* \in$ stack $\mathfrak{R}$ zur Folge hat, und die Behauptung beweist.

5. Konvergenzräume

5.1. Definition: Für einen limesformen Raum (X, L) heißt L *Nullformität* auf X, und das Paar (X, L) heißt *nullformer Raum*, wenn L das Axiom (N) erfüllt wobei:

$$(N)\mathfrak{N}\in L(\emptyset) \text{ impliziert } \emptyset\in\mathfrak{N}.$$

5.2. Bemerkung: Man beachte, daß die in Beispiel 2.2. (ii), (iv), (v) definierten limesformen Operatoren L_δ, $L_{\mathfrak{N}}$ und $L_{\mathfrak{N}}$ Nullformitäten auf ihren jeweiligen Trägermengen sind.

5.3. Satz: *Die Kategorie Nullform der nullformen Räume und limesformen Abbildungen ist eine bikoreflektive Unterkategorie der Kategorie Limform.*

Beweis: Für eine limesformen Raum (X, L) setze für $A\in\mathfrak{P}X$:

$$L_\emptyset(A) = \begin{cases} L(A) & \text{falls } A\neq\emptyset \\ \{\mathfrak{N}\in\mathfrak{P}^2X \mid \emptyset\in\mathfrak{N}\} & \text{falls } A=\emptyset, \end{cases}$$

so ist $(X, L_\emptyset)$ ein nullformer Raum und 1_X: $(X, L_\emptyset)\to(X, L)$ ist die *Nullform* Bikoreflektion von (X, L).

5.4. Definition: Eine Nullformität L auf einer Menge X heißt *Konformität*, wenn L das Axiom (C) erfüllt, wobei:

(C) $A\in\mathfrak{P}X\setminus\{\emptyset\}$ impliziert $L(A) = \bigcup_{x\in A} L(\{x\})$.

5.5. Satz: *Die Kategorie S conv der Stapelkonvergenzräume und stetigen Abbildungen ist isomorph zur Kategorie Konform der konformen Räume und limesformen Abbildungen.*

Beweis: Für einen Stapelkonvergenzraum $(X, \mathfrak{K})$ ist der für alle $A\in\mathfrak{P}X$ durch

$$L_{\mathfrak{K}}(A) = \begin{cases} \{\mathfrak{N}\in\mathfrak{P}^2X \mid \exists x\in A\,\exists\mathfrak{S}\in\mathfrak{K}(x)\,\mathfrak{S}\ll\mathfrak{N}\} & \text{falls } A\neq\emptyset \\ \{\mathfrak{N}\in\mathfrak{P}^2X \mid \emptyset\in\mathfrak{N}\} & \text{falls } A=\emptyset \end{cases}$$

definierte limesforme Operator $L_{\mathfrak{K}}$ eine Konformität auf X.

Zu (Lf_1): trivial.

Zu (Lf_2): Für Mengen $A_1, A_2\in\mathfrak{P}X$ sei $A_1\subset A_2$ und o.E. $A_1=\emptyset$ und $A_2\neq\emptyset$, da sonst trivial. Sei $\mathfrak{N}\in L_{\mathfrak{K}}(A_1)$, so folgt nach Definition $\emptyset\in\mathfrak{N}$. Wähle $x\in A_2$, so gilt mit Axiom (K_1) in Beispiel 2.2. (iv) die Aussage $\dot{x}\in\mathfrak{K}(x)$. Wir haben sofort $\dot{x}\ll\mathfrak{N}$, *und somit gilt* $\mathfrak{N}\in L_{\mathfrak{K}}(A_2)$.

Zu (Lf_3): Trivialerweise gilt $\{\emptyset\}\in L_{\mathfrak{K}}(\emptyset)$, und $\emptyset\notin L_{\mathfrak{K}}(X)$.

Zu (Lf_4): trivial.

Zu (N): Dieses Axiom gilt aufgrund der Definition von $L_{\mathfrak{K}}$.

Zu (C): Für $A\in\mathfrak{P}X\setminus\{\emptyset\}$ sei $\mathfrak{N}\in L_{\mathfrak{K}}(A)$, so wähle $x\in A$ und einen Stapel $\mathfrak{S}\in\mathfrak{K}(x)$ mit $\mathfrak{S}\ll\mathfrak{N}$, mithin folgt $\mathfrak{N}\in L_{\mathfrak{K}}(\{x\})$. Die umgekehrte Inklusion gilt wegen des Erfülltseins der Bedingung (Lf_2).

Umgekehrt betrachte für eine Konformität T auf X die für alle $x \in X$ durch

$$\mathfrak{Q}_T(x) = \{\mathfrak{S} \in \mathfrak{S}(X) \mid \exists \mathfrak{M} \in T(\{x\}) \mathfrak{M} \ll \mathfrak{S}\}$$

definierte Stapelkonvergenzstruktur $\mathfrak{Q}_T$ auf X. Trivialerweise sind die Axiome (K_1) und (K_2) erfüllt. Wir zeigen nun:
(i) $\mathfrak{Q}_{L_{\mathfrak{K}}} = \mathfrak{K}$;
(ii) $L_{\mathfrak{Q}_T} = T$.

Zu (i): „$\leq$": Für $x \in X$ sei $\mathfrak{S} \in \mathfrak{Q}_{L_{\mathfrak{K}}}(x)$, so gibt es ein Mengensystem $\mathfrak{M} \in L_{\mathfrak{K}}(\{x\})$ mit $\mathfrak{M} \ll \mathfrak{S}$. Wähle einen Stapel $\mathfrak{S}' \in \mathfrak{K}(x)$, so daß $\mathfrak{S}' \ll \mathfrak{M}$ erfüllt ist. Da $\mathfrak{S}$ insbesondere ein stack ist, folgt $\mathfrak{S}' \subset \mathfrak{S}$, mithin gilt unter Anwendung von Axiom (K_2) im Beispiel 2.2. (iv) die Aussage $\mathfrak{S} \in \mathfrak{K}(x)$.

„$\geq$": Umgekehrt sei $\mathfrak{S} \in \mathfrak{K}(x)$, so gilt trivialerweise die Aussage $\mathfrak{S} \in L_{\mathfrak{K}}(\{x\})$, mithin folgt unter Beachtung, daß $\mathfrak{S}$ ein Stapel ist, die Behauptung.

Zu (ii): *1. Fall:* $A = \emptyset$.

„$\leq$": Für $\mathfrak{N} \in L_{\mathfrak{Q}_T}(A)$ folgt nach Definition $\emptyset \in \mathfrak{N}$, was unter Beachtung der Axiome (Lf_3) und (Lf_1) die Aussage $\mathfrak{N} \in T(\emptyset)$ nach sich zieht.

„$\geq$": Für $\mathfrak{N} \in T(A)$ folgt mit Axiom $(N) \emptyset \in \mathfrak{N}$, und nach Definition gilt dann $\mathfrak{N} \in L_{\mathfrak{Q}_T}(\emptyset)$.

2. Fall: $A \neq \emptyset$.

„$\leq$": Für $\mathfrak{N} \in L_{\mathfrak{Q}_T}(A)$ gibt es ein $x \in A$ und einen Stapel $\mathfrak{S} \in \mathfrak{Q}_T(x)$ mit $\mathfrak{S} \ll \mathfrak{N}$. Wähle $\mathfrak{M} \in T(\{x\})$, so daß $\mathfrak{M} \ll \mathfrak{S}$ gilt. Es folgt $\mathfrak{M} \ll \mathfrak{N}$, woraus $\mathfrak{N} \in T(\{x\})$ resultiert unter Anwendung von (Lf_1). Mit (Lf_2) gilt $T(\{x\}) \subset T(A)$, und somit ist $\mathfrak{N} \in T(A)$.

„$\geq$": Umgekehrt sei $\mathfrak{N} \in T(A)$, so folgt unter Beachtung der Bedingung (C) die Existenz eines Elementes $x \in A$, so daß $\mathfrak{N} \in T(\{x\})$ gilt. Setze $\mathfrak{S} = \text{stack}\, \mathfrak{N}$, so ist unter Berücksichtigung von (Lf_3) und (Lf_1) $\mathfrak{S} \in \mathfrak{Q}_T(x)$ mit $\mathfrak{S} \ll \mathfrak{N}$, woraus $\mathfrak{N} \in L_{\mathfrak{Q}_T}(A)$ folgt.

Abschließend zeigen wir, daß für konforme Räume (X, L_1), (Y, L_2) die folgenden Aussagen äquivalent sind:

(iii) $f: (X, L_1) \to (X, L_2)$ ist limesforme Abbildung;
(iv) $f: (X, \mathfrak{Q}_{L_1}) \to (Y, \mathfrak{Q}_{L_2})$ ist stetig.

(iii) impliziert (iv) : Für $x \in X$ sei $\mathfrak{S} \in \mathfrak{Q}_{L_1}(x)$, so gibt es ein Mengensystem $\mathfrak{M} \in L_1(\{x\})$ mit $\mathfrak{M} \ll \mathfrak{S}$. Da f nach Voraussetzung limesforme Abbildung ist, gilt $f\mathfrak{M} \in L_2(\{f(x)\})$. Es folgt $f\mathfrak{M} \ll f\mathfrak{S} \subset f(\mathfrak{S})$, womit sich $f(\mathfrak{S}) \in \mathfrak{Q}_{L_2}(f(x))$ ergibt unter Beachtung der Definition $f(\mathfrak{S}) = \text{stack}\, f\mathfrak{S}$.

(iv) impliziert (iii) : Sei $A \in \mathfrak{P}X$ und 1. Fall: $A = \emptyset$.

Für $\mathfrak{R}\in L_1(A)$ gilt mit Axiom (N) $\emptyset\in\mathfrak{R}$, und damit ist $\emptyset\in f\,\mathfrak{R}$, woraus die Aussage $f\,\mathfrak{R}\in L_2(f\,[A])$ folgt unter Beachtung der Axiome (Lf_3) und (Lf_1).

2. Fall: $A\neq\emptyset$.

Für $\mathfrak{R}\in L_1(A)$ gibt es nach Axiom (C) ein $x\in A$, so daß $\mathfrak{R}\in L_1(\{x\})$ gilt. Es folgt stack $\mathfrak{R}\in\mathfrak{Q}_{L_1}(x)$. Da f nach Voraussetzung stetig ist, gilt f (stack $\mathfrak{R}$)$\in\mathfrak{Q}_{L_2}(f(x))$. Wähle ein Mengensystem $\mathfrak{M}\in L_2(\{f(x)\})$ mit $\mathfrak{M}\ll f$ (stack $\mathfrak{R}$). Nun gilt f (stack $\mathfrak{R}$)$\ll f$ stack $\mathfrak{R}\ll f\,\mathfrak{R}$, mithin folgt $f\,\mathfrak{R}\in L_2(\{f(x)\})$, und da $x\in A$, resultiert $L_2(\{f(x)\})\subset L_2(f\,[A])$ mit Axiom (Lf_2), mithin ist $f\mathfrak{R}\in L_2(f\,[A])$.

5.6. Bemerkung: Spezielle Stapelkonvergenzräume sind zum Beispiel die Limesräume im Sinne von Fischer oder die pseudotopologischen Räume nach Choquet, welche die Konvergenz von Filtern (spezielle Stapel) gegen Elemente beschreiben und zusätzliche Axiome voraussetzen. Die limesformen Räume umfassen also nach vorangehendem Satz auch diese Strukturen. Filtermerotopien im Sinne von Katětov beschreiben die Konvergenz von Filtern im allgemeinen unabhängig von den Punkten der jeweiligen Trägermengen. All diese Überlegungen führen zur folgenden Definition.

5.7. Definition: Für einen limesformen Raum (X, L) heißt L *Rasterformität* auf X, und das Paar (X, L) heißt *rasterformer Raum*, wenn L das Axiom (R) erfüllt, wobei:

(R) $A\in\mathfrak{P}X$ und $\mathfrak{N}\in L(A)$ implizieren die Existenz eines Rasters $\mathfrak{R}\in(A)$ mit $\mathfrak{R}\ll\mathfrak{N}$.

5.8. Bemerkung: Unter einem *Raster* soll dabei ein Mengensystem über einer Menge verstanden werden, der folgende Axiome erfüllt:

(R_1) $\mathfrak{R}\neq\emptyset$

(R_2) F_1, $F_2\in\mathfrak{R}$ implizieren die Existenz einer Menge $F_3\in\mathfrak{R}$ mit $F_3\subset F_1\cap F_2$.

Wir definieren einen *Präfilter* als einen Raster, der zusätzlich das Axiom (R_3) erfüllt, wobei:

(R_3) $\emptyset\notin\mathfrak{R}$.

Unter Beachtung der vorangehenden Bemerkung lassen sich nun Limesräume, Choqueträume und auch filtermerotopische Räume als spezielle rasterforme Räume auffassen. Unter Berücksichtigung von Lemma 4.5. gilt diese Aussage auch für Hüllenformitäten bzw. den dazu isomorphen „extended topologies". Schließlich sei noch auf die Beispiele 2.2. (ii) bzw. 2.2. (v) hingewiesen.

5.9. Ergänzung: Im Zusammenhang mit den vorher erwähnten Konvergenzstrukturen lassen sich nun in naheliegender Weise zwei zusätzliche Axiome für limesforme Räume formulieren, die kurz vorgestellt werden:

Für einen limesformen Raum (X, L) heißt L *Biformität*, wenn gilt:

(B) $A\in\mathfrak{P}X$ und $\mathfrak{R}_1$, $\mathfrak{R}_2\in L(A)$ implizieren $\mathfrak{R}_1\vee\mathfrak{R}_2\in L(A)$;

Lokalformität, wenn gilt:

(L) $x\in X$ und $\mathfrak{R}\in L(\{x\})$ implizieren $\mathfrak{R}\vee(\{x\})\in L(\{x\})$.

Wir geben dazu abschließend das Beispiel einer Biformität auf einer Menge, die keine Kernformität ist.

N bezeichne die Menge aller natürlichen Zahlen. Setze $X = \mathbf{N} \dot{\cup} \{\infty\}$ und definiere eine Abbildung L^∞: $\mathfrak{P}X \to \mathfrak{P}^3X$ für jedes $A \in \mathfrak{P}X$ durch die folgende Vorschrift:

$$L^\infty(A) = \begin{cases} \{\mathfrak{R} \in \mathfrak{P}^2X \mid A \in \text{ stack } \mathfrak{R}\} & \text{falls } \infty \notin A, \\ \left.\begin{array}{r} \{\mathfrak{R} \in \mathfrak{P}^2X \mid \exists E \subset \mathbf{N} \text{ endlich} \\ A \cup E \in \text{ stack } \mathfrak{R}\} \end{array}\right\} & \text{sonst.} \end{cases}$$

6. Proximitäten

6.1. Definition: Für einen limesformen Raum (X, L) heißt L *Grillformität* auf X, und das Paar (X, L) heißt *grillformer Raum*, wenn L das Axiom (G) erfüllt, wobei:

(G) $A_1, A_2 \in \mathfrak{P}X$ und $\mathfrak{N} \in L(A_1 \cup A_2)$ implizieren die Existenz von A_1-limitierten bzw. A_2-limitierten Mengensystemen $\mathfrak{N}_1$, $\mathfrak{N}_2$ mit der Eigenschaft $\mathfrak{N}_1 \vee \mathfrak{N}_2 \ll \mathfrak{N}$.

6.2. Bemerkung: Man beachte zunächst, daß jede Meroformität auf einer Menge X *automatisch* das Axiom (G) erfüllt.
Desweiteren betrachten wir auf einer Menge X eine proximale Limesstruktur $\mathfrak{P}$ im Sinne von Marny, d.h. eine Menge von Präproximitäten $\delta \subset \mathfrak{P}X \times \mathfrak{P}X$, welche nur die Bedingungen (P_1), (P_2) und (P_3) einer LE-Proximität erfüllen (vergl. dazu Beispiel 2.2. (ii)) und darüberhinaus den folgenden Axiomen genügt:

(PL_a) Die diskrete Präproximität $\emptyset$ auf X ist ein Element von $\mathfrak{P}$,
(PL_b) $\delta_1 \cup \delta_2 \in \mathfrak{P}$ genau dann, wenn $\delta_1, \delta_2 \in \mathfrak{P}$ gilt,
(PL_1) $\delta_X \in \mathfrak{P}$ mit $\delta_X = \{(A, B) \in \mathfrak{P}X \times \mathfrak{P}X \mid A \cap B \neq \emptyset\}$,
(PL_2) $\delta \in \mathfrak{P}$ impliziert $\delta^{-1} \in \mathfrak{P}$, wobei $\delta^{-1} = \{(A, B) \in \mathfrak{P}X \times \mathfrak{P}X \mid (B, A) \in \delta\}$,
(PL_3) $\delta_1, \delta_2 \in \mathfrak{P}$ implizieren $\delta_1 \circ \delta_2 \in \mathfrak{P}$ mit $(A, B) \notin \delta_1 \circ \delta_2$ genau dann, wenn es Mengen $C, D \in \mathfrak{P}X$ gibt, so daß die Aussagen $C \cup D = X, (A, C) \notin \delta_2$ und $(D, B) \notin \delta_1$ gelten.

Für $A \in \mathfrak{P}X$ definieren wir:

$$L_{\mathfrak{P}}(A) = \{\mathfrak{N} \in \mathfrak{P}^2X \mid \exists\ \delta \in \mathfrak{P}\ \delta(A) \ll \mathfrak{N}\}, \text{ wobei } \delta(A) = \{F \subset X \mid (X \setminus F, A) \notin \delta\}$$

(vergl. nochmals Beispiel 2.2. (ii)).

Wir zeigen nun, daß $L_{\mathfrak{P}}$ das Axiom (G) erfüllt.

Zu (G): Für $A_1, A_2 \in \mathfrak{P}X$ sei $\mathfrak{N} \in L_{\mathfrak{P}}(A_1 \cup A_2)$, so gibt es nach Definition von $L_{\mathfrak{P}}$ eine Präproximität $\delta \in \mathfrak{P}$ mit $\delta(A_1 \cup A_2) \ll \mathfrak{N}$. Es folgen die Aussagen $\delta(A_1) \in L_{\mathfrak{P}}(A_1)$ und $\delta(A_2) \in L_{\mathfrak{P}}(A_2)$.

Wir zeigen: $\delta(A_1) \vee \delta(A_2) \subset \delta(A_1 \cup A_2)$.

Für $F \in \delta(A_1) \vee \delta(A_2)$ gilt die Darstellung $F = F_1 \cup F_2$ mit $F_1 \in \delta(A_1)$ und mit

$F_2 \in \delta(A_2)$, mithin resultieren die Aussagen: $(X \backslash F_1, A_1) \notin \delta$ und $(X \backslash F_2, A_2) \notin \delta$. Unter Beachtung der Axiome (P_1)—(P_3) ergibt sich

$$((X \backslash F_1) \cap (X \backslash F_2), A_1 \cup A_2) \notin \delta, \text{ d.h. } (X \backslash (F_1 \cup F_2), A_1 \cup A_2) \notin \delta,$$

mithin gilt $F \in \delta(A_1 \cup A_2)$. Wir weisen darauf hin, daß $L_{\mathfrak{P}}$ außerdem eine Sternformität auf X ist (vergl. Definition 4.18.).

Zu (S): Sei dazu $\mathfrak{N} \in L_{\mathfrak{P}}(A)$, so gibt es eine Präproximität $\delta \in \mathfrak{P}$ mit $\delta(A) \ll \mathfrak{N}$. Mit (PL_3) folgt $\delta \circ \delta \in \mathfrak{P}$. Setze $\mathfrak{N}^* = \{F^* \subset F \mid (X \backslash F^*, A) \notin \delta \circ \delta\}$, so ist $\mathfrak{N}^* \in L_{\mathfrak{P}}(A)$. Für $F^* \in \mathfrak{N}^*$ gilt $(X \backslash F^*, A) \notin \delta \circ \delta$, mithin existieren Mengen $C, D \subset X$ mit den Eigenschaften $C \cup D = X$, $(X \backslash F^*, C) \notin \delta$ und $(D, A) \notin \delta$. Wir setzen: $E = X \backslash D$ und $\mathfrak{R} = \delta(E)$, mithin gelten die Aussagen $\mathfrak{R} \in L_{\mathfrak{P}}(E)$ und $E \in$ stack $\mathfrak{N}$. Angenommen $F^* \notin$ stack $\mathfrak{R}$, sofort gilt $(X \backslash F^*, E) \in \delta$, was die Aussage $(X \backslash F^*, C) \in \delta$ nach sich zieht und zum Widerspruch führt, da $E \subset C$ gilt.

Proximale Limesräume stellen in natürlicher Weise eine Verallgemeinerung der Efremovič—Proximitätsräume dar.

Zum Schluß gehen wir auf eine Symmetriebedingung des limesformen Operators $L_{\mathfrak{P}}$ ein, die durch folgendes Axiom festgelegt wird.

6.3. Definition: Für einen limesformen Raum (X, L) heißt L *Spinformität* auf X, und das Paar (X, L) heißt *spinformer Raum*, wenn L das Axiom (SPIN) erfüllt, wobei:

(SPIN) $A, B \in \mathfrak{P}X$ und $\mathfrak{N} \in L(B)$ mit $A \in$ sec $\mathfrak{N}$ implizieren die Existenz eines A-limitierten Mengensystems $\mathfrak{M}$, so daß $B \in$ sec $\mathfrak{M}$ gilt.

6.4. Proposition: Der limesforme Operator $L_{\mathfrak{P}}$ auf der Menge X ist eine Spinformität auf dieser Menge.

Beweis: Seien $A, B \in \mathfrak{P}X$ *und* $\mathfrak{N} \in L_{\mathfrak{P}}(B)$ mit $A \in$ sec $\mathfrak{N}$, so existiert eine Präproximität $\delta \in \mathfrak{P}$, und es gilt $(A, B) \in \delta$. Mit (PL_2) folgt dann $\delta^{-1} \in \mathfrak{P}$. Setze $\mathfrak{M} = \{F \subset X \mid (X \backslash F, A) \notin \delta^{-1}\}$, so ist $\mathfrak{M} \in L_{\mathfrak{P}}(A)$. Angenommen $B \notin$ sec $\mathfrak{M}$, so gilt $X \backslash B \supset F \in \mathfrak{M}$ und daher folgt $(X \backslash F, A) \notin \delta^{-1}$, d.h. $(A, X \backslash F) \notin \delta$. Da $B \subset X \backslash F$ gilt, haben wir schließlich $(A, B) \notin \delta$ unter Beachtung der Axiome (P_1)—(P_3), was zu einem Widerspruch führt.

6.5. Definition: Für einen limesformen Raum (X, L) heißt L *Filterformität* auf X, und das Paar (X, L) heißt *filterformer Raum*, wenn L die Axiome (R) und (N) erfüllt (wir sagen dann auch L erfüllt (F)).

6.6. Bemerkung: Man beachte, daß die in Beispiel 2.2. (ii), (v) definierten limesformen Operatoren L_δ *bzw.* $L_{\mathfrak{X}}$ Filterformitäten auf ihren jeweiligen Trägermengen sind. Es gilt sogar *allgemeiner*, daß der limesforme Operator $L_{\mathfrak{P}}$ auf X eine Filterformität auf dieser Menge ist. Zusätzlich weisen wir daraufhin, daß ein Čechscher Hüllenoperator auf einer Menge X, d.i. eine "extended topology" h, die zusätzlich das Axiom (et_0) $\emptyset = h(\emptyset)$ erfüllt, ebenfalls eine Filterformität auf ihrer Trägermenge bedingt und daß jede solche

Filterformität in eineindeutiger Weise wieder zu einem Čechschen Hüllenoperator führt unter Beachtung von Satz 4.4..

6.6. Definition: Für einen ultraformen Raum (X, L) heißt L *Proxiformität* auf X, und das Paar (X, L) heißt *proxiformer Raum*, wenn L die Axiome (F), (G) und (K) erfüllt.

6.7. Bemerkung: Man beachte bei dieser Definition, daß jede Proxiformität insbesondere eine Superformität auf ihrer Trägermenge ist.

6.8. Satz: *Die Kategorie LE—PROX der Leader—Proximitätsräume und p-stetigen Abbildungen ist isomorph zur Kategorie PROXFORM der proxiformen Räume und limesformen Abbildungen.*

Beweis: Für einen LE-Proximitätsraum (X, δ) ist der für alle $A \in \mathfrak{P}X$ durch $L_\delta(A) := \{\mathfrak{M} \in \mathfrak{P}^2 X \mid \delta(A) \ll \mathfrak{M}\}$ mit $\delta(A) := \{F \subset X \mid X \setminus F,\ A) \notin \delta\}$ definierte limesforme Operator L_δ eine Proxiformität auf X.

Zu (Lf_1): trivial.

Zu (Lf_2): Für $A_1, A_2 \in \mathfrak{P}X$ gelte $A_1 \subset A_2$, wir zeigen: $\delta(A_2) \subset \delta(A_2)$. $F \in \delta(A_2)$ impliziert $(X \setminus F, A_2) \notin \delta$, woraus mit (P_1) die Aussage $(X \setminus F, A_1) \notin \delta$ folgt.

Zu (Lf_3): trivial unter Beachtung von (P_1)—(P_3).

Zu (Lf_4): Sei $x \in X$, so zeigen wir: $\delta(\{x\}) \ll \{\{x\}\}$. $F \in \delta(\{x\})$ impliziert $(X \setminus F, \{x\}) \notin \delta$, und mit (P_4) folgt $X \setminus F \cap \{x\} = \emptyset$, was zur Aussage $x \in F$ führt.

Zu (U): Für $A \in \mathfrak{P}X$ sei $\mathfrak{M} \in L_\delta(\mathrm{cl}_{L_\delta}(A))$, wir zeigen: $\mathfrak{M} \in L_\delta(A)$, d.h. zeige: $\delta(A) \ll \mathfrak{M}$. Nach Voraussetzung folgt $\delta(\mathrm{cl}_{L_\delta}(A)) \ll \mathfrak{M}$. Es reicht zu beweisen, daß die Aussage $\delta(A) \subset \delta(\mathrm{cl}_{L_\delta}(A))$ erfüllt ist. $F \notin \delta(\mathrm{cl}_{L_\delta}(A))$ impliziert $(X \setminus F, \mathrm{cl}_{L_\delta}(A)) \in \delta$. Wir weisen nach, daß folgende Inklusion gilt: $\mathrm{cl}_{L_\delta}(A) \subset \mathrm{cl}_\delta(A) = \{x \in X \mid \{x\} \delta A\}$, denn dann folgt mit (P_5) die Behauptung. $x \in \mathrm{cl}_{L_\delta}(A)$ impliziert die Existenz von $\mathfrak{R} \in L_\delta(A)$ mit $x \in \bigcap \mathfrak{R}$. Es folgt $X \setminus \{x\} \notin$ stack $\mathfrak{R}$, und da $\delta(A) \ll \mathfrak{R}$ resultiert $X \setminus \{x\} \notin \delta$, was zur Aussage $x \in \mathrm{cl}_\delta(A)$ führt. Nach Konstruktion ist L_δ eine Kernformität auf X.

Zu (F): Zunächst zeigen wir, daß L_δ das Axiom (N) erfüllt. Sei dazu $\mathfrak{M} \in L_\delta(\emptyset)$, so gilt $\delta(\emptyset) \ll \mathfrak{M}$. Da $\emptyset \in \delta(\emptyset)$ wegen (P_1)—(P_3) gilt, folgt $\emptyset \in \mathfrak{M}$.

Zu (R): Für $A \in \mathfrak{P}X$ sei $\mathfrak{M} \in L_\delta(A)$, mithin folgt $\delta(A) \ll \mathfrak{M}$. Trivialerweise ist $\delta(A) \in L_\delta(A)$, mithin bleibt zu zeigen, daß $\delta(A)$ ein Raster ist (vergl. auch Bemerkung 5.8.).

Zu (R_1): $X \in \delta(A)$ unter Anwendung der Axiome (P_1)—(P_3).

Zu (R_2): Seien $F_1, F_2 \in \delta(A)$, so folgen $(X \setminus F_1, A) \notin \delta$ und $(X \setminus F_2, A) \notin \delta$, woraus unter Anwendung von (P_2) die Aussage $((X \setminus F_1) \cup (X \setminus F_2), A) \notin \delta$ resultiert. Da $(X \setminus F_1) \cup (X \setminus F_2) = X \setminus (F_1 \cap F_2)$ gilt, folgt $F_1 \cap F_2 \in \delta(A)$. Das Axiom (G) ist bereits in Bemerkung 6.2. nachgewiesen worden, so daß L_δ die Axiome einer Proxiformität erfüllt. Umgekehrt betrachten wir für einen proxiformen Operator T auf X, die durch

$$\alpha_T = \{(A, B) \in \mathfrak{P}X \times \mathfrak{P}X \mid \exists \mathfrak{R} \in T(B) A \in \sec \mathfrak{R}\}$$

erklärte LE-Proximität α_T auf X.

Zu (P_1): Für $(A, B) \notin \alpha_T$ und $(A, C) \notin \alpha_T$ sei $\mathfrak{R} \in T(B \cup C)$, wir zeigen: $A \notin$ sec $\mathfrak{R}$. Da T nach Voraussetzung das Axiom (G) erfüllt, existieren ein B-limitiertes Mengensystem $\mathfrak{R}_1$ und ein C-limitiertes Mengensystem $\mathfrak{R}_2$, so daß $\mathfrak{R}_1 \vee \mathfrak{R}_2 \ll \mathfrak{R}$ gilt. Es folgen die Aussagen $A \notin$ sec $\mathfrak{R}_1$ und $A \notin \mathfrak{R}_2$, woraus sich $X \setminus A \supset F_1 \in \mathfrak{R}_1$ und $X \setminus A \supset F_2 \in \mathfrak{R}_2$ ergibt. Mithin gilt $X \setminus A \supset F_1 \cup F_2 \supset F \in \mathfrak{R}$, so daß $X \setminus A \in$ stack $\mathfrak{R}$ folgt, was aber die Aussage $A \notin$ sec $\mathfrak{R}$ nach sich zieht und die Behauptung beweist.

Umgekehrt gelte $(A, B) \in \alpha_T$, so folgt die Existenz eines B—limitierten Mengensystems $\mathfrak{R}$ mit $A \in$ sec $\mathfrak{R}$. Mit (Lf_2) gilt $\mathfrak{R} \in T(B \cup C)$, woraus sich $(A, B \cup C) \in \alpha_T$ ergibt. Analog verhält es sich, sofern man den Fall $(A, C) \in \alpha_T$ betrachtet.

Zu (P_2): Für $(A, C) \notin \alpha_T$ und $(B, C) \notin \alpha_T$ sei $\mathfrak{R} \in T(C)$, wir zeigen: $A \cup B \notin$ sec $\mathfrak{R}$. Da T das Axiom (R) erfüllt, existiert ein Raster $\mathfrak{R} \in T(C)$ mit $\mathfrak{R} \ll \mathfrak{R}$, mithin gilt $A, B \notin$ sec $\mathfrak{R}$. Es folgt $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B) \in$ stack $\mathfrak{R}$ unter Anwendung von (R_2). Somit gilt $A \cup B \notin$ sec $\mathfrak{R}$, was die Behauptung beweist.

Zu (P_3): Für $(A, B) \in \alpha_T$ gibt es ein B-limitiertes Mengensystem $\mathfrak{R}$ mit $A \in$ sec $\mathfrak{R}$. Angenommen $B = \emptyset$, so gilt mit Axiom $(N) \emptyset \in \mathfrak{R}$, was zu einem Widerspruch führt. Wegen (Lf_3) gilt trivialerweise $A \neq \emptyset$.

Zu (P_4): Se $A \cap B \neq \emptyset$ mit $x \in A \cap B$, so folgt mit (Lf_4) $\{\{x\}\} \in T(\{x\})$. Mit (Lf_2) gilt dann $T(\{x\}) \subset T(B)$ und $A \in$ sec $\{\{x\}\}$, so daß die Aussage $(A, B) \in \alpha_T$ resultiert.

Zu (P_5): Schließlich gelte $(A, B) \in \alpha_T$ mit $B \subset \mathrm{cl}_{\alpha T}(C)$, wir zeigen: $(A, C) \in \alpha_T$. Zunächst folgt die Existenz eines B-limitierten Mengensystems $\mathfrak{R}$ mit $A \in$ sec $\mathfrak{R}$. Mit (Lf_2) folgt $\mathrm{cl}_{\alpha T}(C)$ ist Limes von $\mathfrak{R}$. Wir beweisen folgende Inklusion: $\mathrm{cl}_{\alpha T}(C) \subset \mathrm{cl}_T(C)$. Die Aussage $x \in \mathrm{cl}_{\alpha T}(C)$ impliziert $\{x\} \alpha_T C$, mithin existiert ein C-limitiertes Mengensystem $\mathfrak{M}$ mit $x \in \cap \mathfrak{M}$, woraus $x \in \mathrm{cl}_T(C)$ folgt. Mit Anwendung von (Lf_2) gilt nun $T(\mathrm{cl}_{\alpha T}(C)) \subset T(\mathrm{cl}_T(C)) \subset T(C)$ unter zusätzlicher Berücksichtigung des Axioms (U). Somit ist $\mathfrak{R} \in T(C)$, und es gilt $(A, C) \in \alpha_T$.

Als Nächstes zeigen wir die Aussagen:

(i) $$\alpha_{L\delta} = \delta;$$

(ii) $$L_{\alpha T} = T.$$

Zu (i): „$\subset$": Für $(A, B) \in \alpha_{L_\delta}$ folgt die Existenz eines Mengensystems $\mathfrak{M} \in L_\delta(B)$ mit $A \in$ sec $\mathfrak{M}$, mithin gilt $X \setminus A \notin$ stack $\mathfrak{M}$. Es folgt $X \setminus A \notin \delta(B)$, so daß $(A, B) \in \delta$ gilt.

„$\supset$": Sei $(A, B) \in \delta$, so folgt $\delta(B) \in L_\delta(B)$. Angenommen $A \notin$ sec $\delta(B)$, so gibt es eine Menge $F \in \delta(B)$ mit $X \setminus A \supset F$. Wir haben dann $(X \setminus F, B) \notin \delta$ woraus mit (P_2) die Aussage $(A, B) \notin \delta$ folgt, die einen Widerspruch nach sich zieht.

Zu (ii): „$\leq$": Für $A \in \mathfrak{P}X$ sei $\mathfrak{M} \in L_{\alpha_T}(A)$, so gilt $\alpha_T(A) \ll \mathfrak{M}$. Da T das Axiom (K) erfüllt, zeigen wir: $VT(A) \ll \alpha_T(A)$. Angenommen es gibt $F \in VT(A)$ mit $F \notin$ stack $\alpha_T(A)$, so gilt $(X \setminus F, A) \in \alpha_T$. Wähle $\mathfrak{R} \in T(A)$ mit $X \setminus F \in$ sec $\mathfrak{R}$, und damit ist $F \notin$ stack $\mathfrak{R}$, was einen Winderspruch darstellt.

„$\geq$": Umgekehrt sei $\mathfrak{M} \in T(A)$, wir zeigen: $\alpha_T(A) \ll \mathfrak{M}$. Für $F \in \alpha_T(A)$ folgt $(X \setminus F, A) \notin \alpha_T$, mithin gilt insbesondere $X \setminus F \notin$ sec $\mathfrak{M}$. Wir haben $F \in$ stack $\mathfrak{M}$, was die Behauptung beweist.

Abschließend zeigen wir, daß für LE-Proximitätsräume (X, δ_1), (Y, δ_2) die folgenden Aussagen äquivalent sind:

(iii) $f: (X, \delta_1) \to (Y, \delta_2)$ ist p-stetig;

(iv) $f: (X, L_{\delta_1}) \to (Y, L_{\delta_2})$ ist limesforme Abbildung.

(iii) impliziert (iv): Für $A \in \mathfrak{P}X$ sei $\mathfrak{M} \in L_{\delta_1}(A)$, so gilt $\delta_1(A) \ll \mathfrak{M}$. Wir zeigen: $\delta_2(f[A]) \ll f\mathfrak{M}$. $F \in \delta_2(f[A])$ impliziert die Aussage $(Y \setminus F, f[A]) \notin \delta_2$. Da f nach Voraussetzung p-stetig ist, folgt $(X \setminus f^{-1}[F], A) \notin \delta_1$, d.h. $f^{-1}[F] \in \delta_1(A)$. $f^{-1}[F] \supset G \in \mathfrak{M}$, was die Inklusionskette $F \supset f[f^{-1}[F]], \supset f[G] \in f\mathfrak{M}$ nach sich zieht.

(iv) impliziert (iii): Für A, $B \in \mathfrak{P}X$ sei $(f[A], f[B]) \notin \delta_2$, so resultiert $Y \setminus f[A] \in \delta_2(f[B])$. Da $\delta_1(B) \in L_{\delta_1}(B)$ und f nach Voraussetzung limesforme Abbildung ist, ergibt die Aussage $f\delta_1(B) \in L_{\delta_2}(f[B])$. Mithin folgt $\delta_2(f[B]) \ll$ $\ll f\delta_1(B)$, und daher existiert eine Menge $F \in \delta_1(B)$ mit $Y \setminus f[A] \supset f[F]$. Daraus ergibt sich $(X \setminus F, B) \notin \delta_1$. Nun gilt

$$F \subset f^{-1}[f[F]] \subset f^{-1}[Y \setminus f[A]] = X \setminus f^{-1}[f[A]] \subset X \setminus A,$$

woraus sich $X \setminus F \supset A$ ergibt. Also gilt $(A, B) \notin \delta_1$, was zu zeigen war.

6.9. Bemerkung: Für einen Lodato-Proximitätsraum (X, δ) erfüllt L_δ *das Axiom (SPIN) in Definition 6.3.. Umgekehrt genügt für einen proxiformen Operator T,* der eine Spinformität ist, die LE-Proximität α_T der symmetriebedingung eines Lodato-Proximitätsraumes.

7. Topologische Räume

7.1. Definition: Für einen limesformen Raum (X, L) heißt L *Vollformität* auf X, und das Paar (X, L) heißt *vollformer Raum,* wenn L das Axiom (V) erfüllt, wobei: (V) $\mathfrak{A} \in \mathfrak{P}^2X$ *und* $\mathfrak{N} \in L(\cup \mathfrak{A})$ implizieren für jedes $A \in \mathfrak{A}$ die Existenz eines A-limitierten Mengensystems $\mathfrak{N}^A$, so daß $V\{\mathfrak{N}^A \mid A \in \mathfrak{U}\} \ll \mathfrak{N}$ gilt.

7.2. Bemerkung: Man beachte zunächst, daß jede Meroformität auf einer Menge X *automatisch* das Axiom (V) erfüllt. Desweiteren betrachten wir auf einer Menge X eine syntopogene Struktur $\mathfrak{S}$, deren topogene Ordnungen perfekt seien. Dabei heißt eine Menge $\mathfrak{S}$ von topogenen Ordnungen $< \subset \mathfrak{P}X \times \mathfrak{P}X$ syntopogene Struktur auf X, wenn die folgenden Axiome erfüllt sind:

(S_1) $\mathfrak{S} \neq \emptyset$,
(S_2) $<_1, <_2 \in \mathfrak{S}$ implizieren die Existenz einer topogenen Ordnung

$$<_3 \in \mathfrak{S} \text{ mit } <_1 \cup <_2 \subset <_3,$$

(S_3) $< \in \mathfrak{S}$ impliziert die Existenz einer topogenen Ordnung $<^* \in \mathfrak{S}$ mit $< \subset <^{*2}$.

Eine topogene Ordnung „$<$" genügt dabei den folgenden Bedingungen:
(T_1) $\emptyset < \emptyset$ und $X < X$,
(T_2) $A \subset A' < B' \subset B$ implizieren $A < B$,
(T_3) $A < B$ impliziert $A \subset B$,
(T_4) $A < B$ und $A' < B'$ implizieren $A \cup A' < B \cup B'$ und $A \cap A' < B \cap B'$. „$<$" heißt perfekt, wenn zusätzlich (P) gilt, wobei:

(P) Aus $A_i < B_i$ $(i \in I)$ folgt $\bigcup_{i \in I} A_i < \bigcup_{i \in I} B_i$ für beliebige Indexmengen I.

Syntopogene Räume sind eine gemeinsame Verallgemeinerung der quasiuniformen Räume, der Pervin-Proximitätsräume und der topologischen Räume.

Wir setzen nun für $A \in \mathfrak{P}X$: $L_{\mathfrak{S}}(A) = \{\mathfrak{N} \in \mathfrak{P}^2 X \mid \exists < \in \mathfrak{S} < (A) \ll \mathfrak{N}\}$, wobei $<(A) = \{B \subset X \mid A < B\}$.

Zu (V): Sei dazu $\mathfrak{N} \in L_{\mathfrak{S}}(\cup \mathfrak{A})$ für $\mathfrak{A} \in \mathfrak{P}^2 X$, so folgt die Existenz einer perfekten topogenen Ordnung $< \in \mathfrak{S}$ mit $<(\cup \mathfrak{A}) \ll \mathfrak{N}$. Für jedes $A \in \mathfrak{A}$ gilt dann nach Konstruktion von $L_{\mathfrak{S}}$ die Aussage $<(A) \in L_{\mathfrak{S}}(A)$.

Wir zeigen: $V\{<(A) \mid A \in \mathfrak{A}\} \ll <(\cup \mathfrak{A})$. Für $F \in V\{<(A) \mid A \in \mathfrak{A}\}$ gilt: $F = \cup\{B_A \mid A \in \mathfrak{A}\}$, so daß $B_A \in <(A)$ für alle $A \in \mathfrak{A}$ folgt. Mithin resultiert $A' < B_A$, für jedes $A' \in \mathfrak{A}$. Da „$<$" nach Voraussetzung perfekt ist, ergibt sich $\cup \mathfrak{A} < \cup\{B_A \mid A \in \mathfrak{A}\} =: F$, und wir haben sogar $F \in <(\cup \mathfrak{A})$. Schließlich weisen wir darauf hin, daß jede Vollformität trivialerweise eine Grillformität ist (vergl. auch Definition 6.1.).

7.3. Definition: Für einen limesformen Raum (X, L) heißt L *Prätopoformität* auf X, und das Paar (X, L) heißt *prätopoformer Raum*, wenn L die Axiome (V) und (F) in Definition 6.5. erfüllt.

7.4. Bemerkung: Wir zeigen, daß der in Bemerkung 7.2. erklärte limesforme Operator $L_{\mathfrak{S}}$ eine Prätopoformität auf der Menge X ist.

Zu (N): Sei $\mathfrak{N} \in L_{\mathfrak{S}}(\emptyset)$, so folgt die Existenz einer perfekten topogenen Ordnung $< \in \mathfrak{S}$ mit $<(\emptyset) \ll \mathfrak{N}$. Mit ($T_1$) gilt $\emptyset \in <(\emptyset)$, woraus $\emptyset \in \mathfrak{N}$ resultiert.

Zu (R): Für $A \in \mathfrak{P}X$ sei $\mathfrak{N} \in L_{\mathfrak{S}}(A)$, so folgt $<(A) \ll \mathfrak{N}$ für ein Element $< \in \mathfrak{S}$, mithin ist $<(A) \in L_{\mathfrak{S}}(A)$. Mit ($T_1$) gilt $X < X$, also folgt $A \subset X < X$, woraus mit (T_2) $A < X$ resultiert, mithin gilt $<(A) \neq \emptyset$. Damit ist (R_1) erfüllt.

Zu (R_2): Seien $B_1, B_2 \in <(A)$, mithin ist $A < B_1$ und $A < B_2$, woraus mit (T_4) $A < B_1 \cap B_2$ resultiert, so daß $B_1 \cap B_2 \in <(A)$ gilt. Das Axiom (V) ist schon in Bemerkung 7.2. nachgeweisen worden, mithin erfüllt $L_{\mathfrak{S}}$ die Bedingungen einer Prätopoformität auf X. Als ein weiteres Beispiel betrachte man den in Beispiel 2.2. (v) erklärten limesformen Operator $L_{\mathfrak{X}}$.

7.5. Lemma: Für einen prätopoformen Raum (X, T) ist das durch $\mathfrak{Y}_T = \{O \subset X \mid O \in \text{stack } VT(O)\}$ definierte Mengensystem $\mathfrak{Y}_T$ eine Topologie auf X.

Beweis:

Zu (Top_1): $\emptyset \in$ stack $VT(\emptyset)$, da T das Axiom (N) erfüllt; $X \in$ stack $VT(X)$, weil nach (Lf_3) $\emptyset \notin T(X)$ ist.

Zu (Top_2): Für O_1, $O_2 \in \mathfrak{Y}_T$ sei $\mathfrak{R} \in T(O_1 \cap O_2)$, so existiert wegen (R) ein Raster $\mathfrak{R} \in T(O_1 \cap O_2)$ mit $\mathfrak{R} \ll \mathfrak{R}$. Unter Anwendung von (Lf_2) folgt $\mathfrak{R} \in T(O_1)$ und $\mathfrak{R} \in T(O_2)$, mithin existieren Mengen F_1, $F_2 \in \mathfrak{R}$, so daß die Aussagen $O_1 \supset F_1$ und $O_2 \supset F_2$ gelten. Es folgt $O_1 \cap O_2 \supset F_1 \cap F_2 \supset F_3 \in \mathfrak{R}$, da $\mathfrak{R}$ insbesondere (R_2) erfüllt. Somit gilt $O_1 \cap O_2 \in$ stack $\mathfrak{R} \subset$ stack $\mathfrak{R}$, was zu zeigen war.

Zu (Top_3): Sei $\mathfrak{A} \subset \mathfrak{Y}_T$ und $\mathfrak{R} \in T(\cup \mathfrak{A})$, so existiert nach (V) für jedes $A \in \mathfrak{A}$ ein A-limitiertes Mengensystem $\mathfrak{R}^A$, so daß die Aussage $V\{\mathfrak{R}^A \mid A \in \mathfrak{A}\} \ll \mathfrak{R}$ gilt. Da jedes $A \in \mathfrak{A}$ zu $\mathfrak{Y}_T$ gehört, existiert eine Menge $F^A \in \mathfrak{R}^A$ *mit* $A \supset F^A$. Wir haben sofort $\cup \mathfrak{A} \supset \cup \{F^A \mid A \in \mathfrak{A}\} =: F \in V\{\mathfrak{R}^A \mid A \in \mathfrak{A}\}$. $F \supset N \in \mathfrak{R}$, woraus schließlich $\cup \mathfrak{A} \in$ stack $\mathfrak{R}$ resultiert.

7.6. Definition: Für einen prätopoformen Raum (X, L) heißt L *Topoformität* auf X, und das Paar (X, L) heißt *topoformer Raum*, wenn L (K) und das Axiom (T) erfüllt, wobei:
(T) $A \in \mathfrak{P}X$ impliziert $VL(A) \ll \mathfrak{Y}_L(A)$ (vergl. dabei insbesondere Beispiel 2.2. (v) bzw. Lemma 7.5.)

7.7. Satz: *Die volle Unterkategorie Topform der Kategorie Prätopform der prätopoformen Räume und limesformen Abbildungen, deren Objekte die topoformen Räume sind, ist*
(a) *bireflektiv in Prätopform;*
(b) *ismorph zur Kategorie der topologischen Räume und stetigen Abbildungen.*

Beweis:

Zu (a): Für einen prätopoformen Raum (X, T) setze für $A \in \mathfrak{P}X$: $T_t(A) = \{\mathfrak{R} \in \mathfrak{P}^2 X \mid \mathfrak{Y}_T(A) \ll \mathfrak{R}\}$, so ist (X, T_t) ein topoformer Raum, und 1_X: $(X, T) \to (X, T_t)$ ist die *Topform* Bireflektion von (X, T).

Wir zeigen, daß der in Beispiel 2.2. (v) definierte limesforme Operator $L_{\mathfrak{X}}$ eine Topoformität auf der Menge X ist.

Zu (Lf_1): trivial.

Zu (Lf_2): Wir zeigen, daß für $A_1 \subset A_2$ die folgende Inklusion gilt: $\mathfrak{X}(A_2) \subset \mathfrak{X}(A_1)$. $F \in \mathfrak{X}(A_2)$ impliziert die Existenz einer offenen Menge $O \in \mathfrak{X}$ mit $F \supset O \supset A_2$, mithin gilt $F \supset O \supset A_1$, also ist $F \in \mathfrak{X}(A_1)$.

Zu (Lf_3): trivial.

Zu (Lf_4): trivial.

Zu (N): Sei $\mathfrak{R} \in L_{\mathfrak{X}}(\emptyset)$, so gilt $\mathfrak{X}(\emptyset) \ll \mathfrak{R}$. Da $\emptyset \in \mathfrak{X}(\emptyset)$, folgt $\emptyset \in \mathfrak{R}$.

Zu (R): Sei $\mathfrak{R}\in L_{\mathfrak{X}}(A)$, so folgt $\mathfrak{X}(A)\ll\mathfrak{R}$. Es gilt $\mathfrak{X}(A)\in L_{\mathfrak{X}}(A)$, mithin zeigen wir, daß $\mathfrak{X}(A)$ ein Raster ist. $\mathfrak{X}(A)\neq\emptyset$, weil $X\in\mathfrak{X}(A)$ gilt. Seien F_1, $F_2\in\mathfrak{X}(A)$, so existieren offene Mengen O_1, $O_1\in\mathfrak{X}$ mit $A\subset O_1\subset F_1$ und $A\subset O_2\subset F_2$. Mit (Top_2) gilt $O_1\cap O_2\in\mathfrak{X}$, so daß $F_1\cap F_2\in\mathfrak{X}(A)$ resultiert. Nach Konstruktion erfüllt $L_{\mathfrak{X}}$ die Axiome (K) bzw. (T).

Zu (V): Sei $\mathfrak{R}\in L_{\mathfrak{X}}(\bigcup\mathfrak{U})$, so folgt $\mathfrak{X}(\bigcup\mathfrak{U})\ll\mathfrak{R}$. Für $A\in\mathfrak{U}$ gilt $\mathfrak{X}(A)\in L_{\mathfrak{X}}(A)$, mithin zeigen wir, daß folgende Inklusion gilt: $V\{\mathfrak{X}(A)\mid A\in\mathfrak{U}\}\ll\mathfrak{X}(\bigcup\mathfrak{U})$. Sei $F\in V\{\mathfrak{X}(A)\mid A\in\mathfrak{U}\}$, so existiert für jedes $A\in\mathfrak{U}$ eine Menge $F^A\in\mathfrak{X}(A)$ mit $F = \bigcup\{F^A\mid A\in\mathfrak{U}\}$. Wähle für jedes $A\in\mathfrak{U}$ eine offene Menge $O^A\in\mathfrak{X}$ mit $F^A\supset O^A\supset A$, so folgt: $F\supset\bigcup\{O^A\mid A\in\mathfrak{U}\}\supset\bigcup\mathfrak{U}$. Mit ($\mathrm{Top}_3$) gilt dann $\bigcup\{O^A\mid A\in\mathfrak{U}\}\in\mathfrak{X}$, so daß die Aussage $F\in\mathfrak{X}(\bigcup\mathfrak{U})$ resultiert. Trivialerweise ist 1_X: $(X, T)\to(X, T_t)$ eine limesforme Abbildung. Sei nun (Y, S) ein topoformer Raum f: $(X, T)\to(Y, S)$ eine limesforme Abbildung, wir zeigen: f: $(X, T_t)\to(Y, S)$ ist eine limesforme Abbildung. Für $\mathfrak{R}\in T_t(A)$ folgt $\mathfrak{Y}_T(A)\ll\mathfrak{R}$. Da S die Axiome (K) und (T) erfüllt, zeigen wir: $\mathfrak{Y}_S(f[A])\ll f\mathfrak{R}$.

Sei dazu $F\in\mathfrak{Y}_{\mathfrak{S}}(f[A])$, so folgt die Existenz einer offenen Menge $O\in\mathfrak{Y}_S$ mit $f[A]\subset O\subset F$. Da f nach Voraussetzung eine limesforme Abbildung ist, folgt die Stetigkeit derselben, so daß $f^{-1}[O]\in\mathfrak{Y}_T$ resultiert. Wir haben $A\subset f^{-1}[f[A]]\subset f^{-1}[O]\subset f^{-1}[F]$, woraus $f^{-1}[F]\in\mathfrak{Y}_T(A)$ folgt. $f^{-1}[F]\supset N\in\mathfrak{R}$, so daß schließlich $F\in$ stack $f\mathfrak{R}$ resultiert, was zu beweisen war.

Zu (b): Unter Beachtung von Bemerkung 7.4., Lemma 7.5. und Definition 7.6. ist für einen topoformen Raum (X, T) das Mengensystem $\mathfrak{Y}_T$ eine Topologie auf X, während umgekehrt für eine Topologie $\mathfrak{X}$ auf X der limesforme Operator $L_{\mathfrak{X}}$ in Beispiel 2.2. (v) eine Topoformität auf dieser Menge ist.

Wir zeigen:

(i) $\mathfrak{Y}_{L_{\mathfrak{X}}} = \mathfrak{X}$;

(ii) $L_{\mathfrak{Y}_T} = T$.

Zu (i): „$\supset$“: Für $O\in\mathfrak{X}$ sei $\mathfrak{R}\in L_{\mathfrak{X}}(O)$, so gilt $\mathfrak{X}(O)\ll\mathfrak{R}$. Da $O\in\mathfrak{X}(O)$ ist, resultiert $O\in$ stack $\mathfrak{R}$, was die Inklusion beweist.

„$\subset$“: $O\in\mathfrak{Y}_{L_{\mathfrak{X}}}$ impliziert die Aussage

$$O\in\bigcap\{\text{stack }\mathfrak{R}\mid\mathfrak{R}\in L_{\mathfrak{X}}(O)\}$$

Da $\mathfrak{X}(O)\in L_{\mathfrak{X}}(O)$ gilt insbesondere $O\in\mathfrak{X}(O)$. Wähle $O'\in\mathfrak{X}$ mit $O\subset O'\subset O$, woraus sich $O\in\mathfrak{X}$ ergibt.

Zu (ii): „$\supset$“: Für $A\in\mathfrak{P}X$ sei $\mathfrak{R}\in T(A)$, wir zeigen: $\mathfrak{Y}_T(A)\ll\mathfrak{R}$. Sei $F\in\mathfrak{Y}_T(A)$, so folgt die Existenz einer offenen Menge $O\in\mathfrak{Y}_T$ mit $A\subset O\subset F$. Wir haben $O\in$ stack $VT(O) = \bigcap\{\text{stack }\mathfrak{M}\mid\mathfrak{M}\in T(O)\}$. Da nach $(Lf_2)T(A)\subset T(O)$ gilt, folgt $\mathfrak{R}\in T(O)$, mithin ist $O\in$ stack $\mathfrak{R}$ und schließlich $F\in$ stack $\mathfrak{R}$, was die Behauptung beweist.

„$\subset$": $\mathfrak{R}\in L_{\mathfrak{Y}_T}(A)$ impliziert $\mathfrak{Y}_T(A)\ll\mathfrak{R}$. Da T das Axiom (T) erfüllt, folgt $VT(A)\ll\mathfrak{Y}_T(A)$. $VT(A)\in T(A)$, weil T(K) genügt, woraus mit $(Lf_1)\mathfrak{R}\in T(A)$ resultiert.

Schließlich zeigen wir, daß für topoforme Räume (X, L) und (Y, T) die folgenden Aussagen äquivalent sind:

(iii) $f\colon (X, L)\to(Y, T)$ ist limesforme Abbildung;
(iv) $f\colon (X, \mathfrak{X}_L)\to(Y, \mathfrak{X}_T)$ ist stetig.

(iii) impliziert (iv): Für $O\in\mathfrak{X}_T$ sei $\mathfrak{R}\in L(f^{-1}[O])$, so gilt mit Voraussetzung $f\mathfrak{R}\in L(O)$. Wir haben $O\in$ stack $f\mathfrak{R}$, was die Existenz einer Menge $N\in\mathfrak{R}$ zur Folge hat mit $O\supset f[N]$, mithin gilt $f^{-1}[O]\supset f^{-1}[f[N]]\supset N$. Somit resultiert $f^{-1}[O]\in$ stack $\mathfrak{R}$, was zu zeigen war.

(iv) impliziert (iii): Für $A\in\mathfrak{P}Y$ sei $\mathfrak{R}\in L(f^{-1}[A])$, so folgt $\mathfrak{X}_L(f^{-1}[A])\ll\mathfrak{R}$. Da T die Axiome (T) und (K) erfüllt, genügt zu zeigen: $\mathfrak{X}_T(A)\ll f\mathfrak{R}$. $F\in\mathfrak{X}_T(A)$ impliziert die Existenz einer offenen Menge $O\in\mathfrak{X}_T$ mit $A\subset O\subset F$. Da f nach Voraussetzung stetig ist, gilt $f^{-1}[O]\in\mathfrak{X}_L$. Wir haben $f^{-1}[F]\supset f^{-1}[O]\supset f^{-1}[A]$, woraus $f^{-1}[F]\in\mathfrak{X}_L(f^{-1}[A])$ resultiert. $f^{-1}[F]\supset N\in\mathfrak{R}$, mithin gilt: $F\supset\supset f[f^{-1}[F]]\supset f[N]\in f\mathfrak{R}$, was die Behauptung beweist.

7.8. Satz: *Jeder topoforme Raum (X, L) ist ein sternformer Raum.*

Beweis: Für $A\in\mathfrak{P}X$ sei $\mathfrak{R}\in L(A)$, so folgt wegen (K), (T) und $(Lf_1)\mathfrak{X}_L(A)\in\in L(A)$. Sei $F^*\in\mathfrak{X}_L(A)$, so folgt die Existenz einer offenen Menge $O\in\mathfrak{X}_L$ mit $F^*\supset O\supset A$. Da $O\in$ stack $VL(O)$, folgt mit $(Lf_2)O\in$ stack $\mathfrak{R}$ und $F^*\in$ stack $VL(O)$, woraus unter nochmaliger Anwendung des Axioms (K) die Bedingung (S) resultiert (vergl. auch Definition 4.18.).

8. Uniforme Strukturen

8.1. Bemerkung: Da uniforme Räume bis auf Isomorphie spezielle merotope Räume sind, lassen sich die Vorgenannten unter Beachtung des Satzes 3.2. als geeignete limesforme Räume auffassen. Ausgehend von einer uniformen Konvergenzstruktur (die wiederum eine Verallgemeinerung einer Uniformität darstellt) zeigen wir die Existenz einer „unterliegenden" Sternformität auf einer vorgegebenen Menge. Hierbei wollen wir unter einer uniformen Konvergenzstruktur auf einer Menge X eine Menge J von Präuniformitäten auf X verstehen, die folgenden Axiomen genügt:

(UL_a) Die diskrete Präuniformität $\mathfrak{P}(X\times X)$ ist ein Element von J,

(UL_b) $\mathfrak{U}\cap\mathfrak{V}\in J$ genau dann, wenn $\mathfrak{U}, \mathfrak{V}\in J$,

(UL_1) $\mathfrak{D}_X\in J$, wobei $\mathfrak{D}_X=\{U\in\mathfrak{P}(X\times X)\mid\Delta_X\subset U\}$,

(UL_2) $\mathfrak{U}\in J$ impliziert $\mathfrak{U}^{-1}\in J$,

(UL_3) $\mathfrak{U}\in J$ und $\mathfrak{V}\in J$ implizieren $\mathfrak{U}\circ\mathfrak{V}\in J$.

Hierbei heißt $\mathfrak{U}\in\mathfrak{P}(X\times X)$ Präuniformität auf X; wenn folgende Axiome gelten:

(U_a) $X\times X\in\mathfrak{U}$,
(U_b) $U\cap V\in\mathfrak{U}$ genau dann, wenn $U, V\in\mathfrak{U}$.

8.2. Lemma: Für einen uniformen Konvergenzraum (X, J) ist der durch $L_J(A) = \{\mathfrak{N}\in\mathfrak{P}^2X \mid \exists\mathfrak{U}\in J\ \mathfrak{U}(A)\ll\mathfrak{N}\}$ mit $U(A) = \{U(A) \mid U\in\mathfrak{U}\}$ definierte limesforme Operator eine Sternformität auf X.

Beweis:
Zu (Lf_1)—(Lf_3): trivial.
Zu (Lf_4): Sei $x\in X$, so wähle mit (UL_1) $\mathfrak{D}_X\in J$, wir zeigen: $\mathfrak{D}_X(\{x\})\ll\{\{x\}\}$. Für $F\in\mathfrak{D}_X(\{x\})$ gibt es eine Relation $U\in\mathfrak{D}_X$ mit $F = U(\{x\})$, mithin gilt $\Delta_X\subset U$, was die Inklusion $\{x\}\subset U(\{x\})$ nach sich zieht.

Zu (S): Für $A\in\mathfrak{P}X$ sei $\mathfrak{N}\in L_J(A)$, so folgt die Existenz einer Präuniformität $\mathfrak{U}\in J$ mit $\mathfrak{U}(A)\ll\mathfrak{N}$. Mit ($UL_3$) folgt $\mathfrak{U}\circ\mathfrak{U}\in J$. Setze $\mathfrak{N}^* = (\mathfrak{U}\circ\mathfrak{U})(A)$, so gilt $\mathfrak{N}^*\in L_J(A)$. Für $F^*\in\mathfrak{N}^*$ gibt es eine Relation $R\in\mathfrak{U}\circ\mathfrak{U}$ mit $F^* = R(A)$. Es folgt $R\supset U^2$ für ein $U\in\mathfrak{U}$ unter Beachtung des Axioms (U_b) für eine Präuniformität. Setze $C = U(A)$ und $\mathfrak{R} = U(C)$, so gelten die folgenden Aussagen: $\mathfrak{R}\in L_J(C)$, $C\in\operatorname{stack}\mathfrak{N}$ und $U^2(A) = U(U(A)) = U(C)$, woraus $F^*\in\operatorname{stack}\mathfrak{R}$ resultiert. Damit ist aber L_J eine Sternformität auf X.

8.3. Bemerkung: Man kann nun die interessante Frage stellen, ob jede Sternformität uniformisierbar ist, d.h. ob zu einer vorgegebenen Sternformität T auf einer Menge X eine uniforme Konvergenzstruktur J (oder allgemeiner eine quasiuniforme Konvergenzstruktur — das ist eine Menge von Präuniformitäten, die alle Axiome bis auf (UL_2) erfüllt —) existiert derart, daß $L_J = T$ ist.

Diese Bemerkung soll jedoch nicht Gegenstand dieses Papiers sein und an anderer Stelle wieder erörtert werden.

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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF THE SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH DISTRIBUTED DELAY

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1. Introduction

The oscillatory and asymptotic properties of the solutions of functional differential equations of retarding type have been an object of investigation of many works. An extensive bibliography on this subject is given in [1]. We shall note that analogous results obtained for equations of neutral type are quite few. Sufficient conditions for oscillation of the solutions of equations of neutral type in the case of one discrete delay are obtained in the works [2]—[6].

In the work [7] it is proved that for the oscillation of all solutions of the equation

$$x'(t)+px'(t-\tau)+qx(t-\sigma) = 0,\ t\geq t_0,\ \tau,\sigma>0,$$

it is necessary and sufficient that the respective characteristic equation has no real roots. In the present paper analogous results are obtained for a considerably wider class of equations of the form

$$\frac{d}{dt}\left(x(t)+\delta_1\int_0^{\tau_1} x(t-s)dr_1(s)\right)+\delta_2\int_0^{\tau_2} x(t-s)dr_2(s) = 0, \tag{1}$$

where $\delta_1,\delta_2 = \pm 1$.

We shall say that conditions (A) are fulfilled if the following conditions hold:

A1. The functions r_i: $[0,\tau_i]\to[0,\infty), i = 1, 2$ are non-decreasing.

A2. $\tau_i>0$, $r_i(0)>0$, $i = 1, 2$, $r_1(0^+) = 0$ and $r_2(\tau_2)>0$.

Our main conjecture is that if conditions (A) hold, for the oscillation of all non-zero solutions of equation (1) it is necessary and sufficient that the respective charasteristic equation

$$Q(z) = z(1+\delta_1\int_0^{\tau_1} e^{-zs}dr_1(s))+\delta_2\int_0^{\tau_2} e^{-zs}dr_2(s) = 0, \tag{2}$$

has no real roots.

The necessity of this condition is obvious since if $z^*\in\mathbf{R}$ is a root of equation (2), then the function $x(t) = e^{z^*t}$ is a non-oscillatory solution of equation (1). Hence, in particular, it follows that our main conjecture holds in the following three cases:

(i) $\delta_2 = -1$,
(ii) $\delta_1 = \delta_2 = +1$, $\tau_1>\tau_2$, $r_1(\tau_1)>r_1(\tau_2)$,
(iii) $\delta_1 = \delta_2 = +1$, $\tau_1 = \tau_2$, $r_1(\tau_1)>r_1(\tau_1^-)$.

since in each one of cases (i) — (iii) equation (2) has at least one real root.

In view of the above arguments we shall assume without a special stipulation that $\delta_2 = +1$.

2. Preliminary notes and auxiliary assertions

Definition 1. We shall say that the function f: $[t_f,\infty)\to\mathbf{R}$ ultimately possesses prorerty P if there exists an interval $[t_f^*,\infty)\subseteq[t_f,\infty)$ in which f possesses property P.

Definition 2. We shall say that the function f: $[t_f,\infty)\to\mathbf{R}$ is oscillatory if there exists a strictly increasing sequence $\{t_i\}_{i=1}^{\infty}\subset[t_f,\infty)$, $\lim_{i\to\infty} t_i = \infty$ such that $f(t_i).f(t_{i+1})<0$.

Definition 3. We shall say that the function f: $[t_f,\infty)\to\mathbf{R}$ is monotone if it is non-decreasing or non-increasing.

Definition 4. We shall say that the function f:$[t_f, \infty)\to\mathbf{R}$ is vanishing if ultimately $f(t)\equiv 0$.

Definition 5. We shall call the continuous function x: $[t_x, \infty)\to\mathbf{R}$ a solution of equation (1) and we shall write $x\in(1)$ if the function

$$x(t)+\delta_1\int_0^{\tau_1} x(t-s)dr_1(s)$$

is continuously differentiable for $t\geq t_x+\tau_1$ and equation (1) is satisfied by x for $t\geq t_x+\tau$, $\tau = \max(\tau_1, \tau_2)$.

Lemma 1. Let conditions (A) hold, the function r:$[0, \sigma]\to\mathbf{R}$ be of bounded variations and x: $[t_x, \infty)\to\mathbf{R}$ be a solution of equation (1).

Then the function y: $[t_y,\infty)\to\mathbf{R}$, $y(t) = \int_0^{\sigma} x(t-s)dr(s)$, $t_y = t_x+\sigma$ is also a solution of equation (1).

PROOF. Since $y(t)$ is a continuous function, then if we introduce the notation

$$y_1(t) = x(t)+\delta_1\int_0^{\tau_1} x(t-s)dr_1(s),\ t\geq t_x+\tau_1$$

and change the order of integration, we obtain for $t\geq t_y+\tau_1$ the equalities

$$y(t)+\delta_1\int_0^{\tau_1} y(t-s)dr_1(s) = \int_0^{\sigma} x(t-s)dr(s)+\int_0^{\tau_1}\left(\int_0^{\sigma} x(t-s-h)dr(h)\right)dr_1(s) =$$
$$= \int_0^{\sigma} y_1(t-h)dr(h).$$

The above equality implies that the function $y(t)+\delta\int_0^{\tau_1} y(t-s)dr_1(s)$ is continuously differentiable with respect to t with a derivative

$$\int_0^{\sigma} y_1(t-h)dr(h) = -\int_0^{\sigma}\left(\int_0^{\tau_2} x(t-h-s)dr_2(s)\right)dr(h) =$$
$$= -\int_0^{\tau_2}\left(\int_0^{\sigma} x(t-s-h)dr(h)\right)dr_2(s) = -\int_0^{\tau_2} y(t-s)dr_2(s).$$

This completes the proof of lemma 1.

LEMMA 2. Let conditions (A) hold. Then if the function $x\in(1)$ is bounded, then $\liminf_{t\to\infty}|x(t)| = 0$.

PROOF. Let $\liminf_{t\to\infty}|x(t)| = c>0$. Then, integrating the two sides of equation (1) from $t_0\geq t_x+\tau$ to $t>t_0$, we obtain the equality

$$x(t)+\delta_1\int_0^{\tau_1} x(t-s)dr_1(s)-\left|x(t_0)+\delta_1\int_0^{\tau_1} x(t_0-s)dr_1(s)\right| =$$
$$= -\int_t^{t}\left(\int_0^{\tau_2} x(t'-s)dr_2(s)\right)dt'.$$

For $t\to\infty$ the left-hand side of the equality is bounded and the module of its right-hand side is not smaller than the expression $c_2\, r_2(\tau_2)t+o(t)$ which is impossible.

Lemma 2 is proved.

COROLLARY 1. If conditions (A) hold and the function $x\in(1)$ is monotone, then $x(\infty)\in\{0,-\infty,\ \infty\}$.

REMARK 1. Lemma 2 and corollary 1 are immediately generalized for the considerably wider class of equations of the form

$$\frac{d}{dt}\left(x(t)+\int_0^{\tau_1(t)} x(t-s)d_s r_1(t,\ s)\right)+\int_0^{\tau_2(t)} x(t-s)d_s r_2(t,\ s) = 0,$$

where $\tau_i(t)>0$, $\lim_{t\to\infty}(t-\tau_i(t))=\infty$, $i=1, 2$, the function $r_1(t, s)$ is of uniformly bounded variation with respect to s, the function $r_2(t, s)$ is monotone with respect to s and $\inf_t |r_2(t, \tau_2(t))-r_2(t, 0)|>0$.

Let $x\in(1)$ and introduce the notations

$$(3)\qquad y_0(t)=x(t),\ y_i(t)=y_{i-1}(t)+\delta_1\int_0^{\tau_1} y_{i-1}(t-s)dr_1(s),\ i=1, 2, \ldots .$$

By induction on i and by lemma 1 we deduce that each function $y_i(t)$, $i=1, 2, \ldots$ is defined in the interval $[t_x+i\tau_1, \infty)$, continuously differentiable in it and is a solution of equation (1). Moreover, if $x(t)$ is a non-oscillatory function, then from equation (1) by induction on i we obtain that the functions $y_i(t)$, $i=1, 2, \ldots$ are ultimately monotone and by corollary 1

$$y_i(\infty)\in\{0, -\infty, \infty\}.$$

Lemma 3. Let conditions (A) hold and one of the following two conditions be fulfilled:

(iv) $\delta_1=+1$,

(v) $\delta_1=-1$, $r_1(\tau_1)<1$.

Then if the function $x\in(1)$ is non-negative and non-vanishing (positive), then all functions $y_i(t)$, $i=1, 2, \ldots$ are ultimately positive and ultimately non-increasing (decreasing).

Proof. It suffices to consider only the case $i=1$ and after that the proof is completed by induction on i.

If condition (iv) holds, then all assertions of lemma 3 follow immediately from equation (1). Provided that condition (v) holds, it suffices to show that $y_1(\infty)=0$ and then the assertions needed follow immediately from equation (1). In fact, if we suppose that $y_1(\infty)=-\infty$, then in view of $y_1\in(1)$ from equation (1) we deduce that $y_2'(\infty)=\infty$, hence ultimately the following inequality holds

$$y_1(t)-\int_0^{\tau_1} y_1(t-s)dr_1(s)=y_2(t)>0.$$

Since the function $y_1(t)$ is ultimately non-increasing, then ultimately the following inequality holds

$$y_1(t)-\int_0^{\tau_1} y_1(t-s)dr_1(s)\le y_1(t)(1-r_1(\tau_1))<0,$$

which is impossible.

This completes the proof of lemma 3.

LEMMA 4. Let conditions (A) hold and $\delta_1 = -1$ and $r_1(\tau_1) \geq 1$.

Then if the function $x \in (1)$ is non-negative and non-vanishing, then

$$y_i(\infty) = (-1)^i \infty, \; i = 1, 2, \ldots$$

PROOF. It is immediately seen that it suffices to prove that $y_1(\infty) = -\infty$.

If we assume that $y_1(\infty) = 0$, then $y_2(\infty) = 0$ as well and since $x(t)$ is ultimately non-negative and $y_1(t)$ and $y_2(t) \in (1)$, then we deduce that the functions $y_1(t)$ and $y_2(t)$ are ultimately positive and non-increasing which implies that ultimately the following inequalities hold

$$y_1(t) >= \int_0^{\tau_1} y_1(t-s) dr_1(s) \geq \int_0^{\tau_1} y_1(t) dr_1(s) = r_1(\tau_1) y_1(t),$$

which is impossible.

Lemma 4 is proved.

COROLLARY 2. Let the conditions of lemma 3 (lemma 4) hold.

Then if equation (1) has a non-negative non-vanishing solution, then it has a positive decreasing (increasing) solution.

3. Main results

THEOREM 1. *Let conditions* (A) *hold and* $\delta_1 = -1$ *and* $r_1(\tau_1) < 1$.

Then in order all non-vanishing solutions of equation (1) *to be oscillatory it is necessary and sufficient that equation* (2) *has no real roots.*

PROOF. The necessity of the condition equation (2) to have no real roots was considered in section 1.

Sufficiency. Suppose that equation (2) has no real roots and equation (1) has a non-negative non-vanishing solution $x(t)$.

Then by corollary 2 and lemma 3 the functions $y_i(t)$, $i = 0, 1, \ldots$ are ultimately positive and ultimately decreasing. On the other hand, from the fact, that the quasi-polynomial $Q(z)$ has no real roots it follows that at least one of the functions $r_i(t)$, $i = 1, 2$ is not constant for $t > 0$, hence $Q(-\infty) = \infty$, whence we deduce that the following inequality holds

$$q := \min_{z \in [0, \infty]} Q(-z) > 0.$$

Introduce the sets

$$Z_1 = \{z \in [0, \infty) \mid y_i'(t) + z y_i(t) \leq 0 \text{ ultimately}\}, \; i = 1, 2, \ldots \tag{4}$$

Since $0 \in Z_i$, $i = 1, 2, \ldots$, then the set $Z_1 \neq \emptyset$ for any $i = 1, 2, \ldots$

On the other hand, if $z \in Z_i$, then we can conclude that $y_i(t) = O(e^{-zt})$ for $t \to \infty$. In view of the theorem of D. HENRY [8] which states that a non-vanishing solution of equation (1) cannot tend to zero for $t \to \infty$ faster than any exponent, we conclude that each set Z_i is bounded.

Choose $z^*\in Z_1$ such that $z^*>\sup Z_1-q$ and introduce the notation $\varphi(t)=e^{z^*t}y_1(t)$. From relation (4) it follows that ultimately $\varphi'(t)\leq 0$. From this inequality, the definition of the constant q and the fact that $y_1(t)$ is a solution of equation (1) it follows that ultimately the following estimate holds

$$y_1'(t)+(z^*+q)y_1(t)=\int_0^{\tau_1}(e^{-z^*(t-s)}\varphi(t-s))'dr_1(s)-\int_0^{\tau_1}e^{-z^*(t-s)}\varphi(t-s)dr_2(s)+$$

$$+(z^*+q)e^{-z^*t}\varphi(t)=-z^*\int_0^{\tau_1}e^{-z^*(t-s)}\varphi(t-s)dr_1(s)+\int_0^{\tau_1}e^{-z^*(t-s)}\varphi'(t-s)dr_1(s)-$$

$$-\int_0^{\tau_2}e^{-z^*(t-s)}\varphi(t-s))'dr_2(s)+(z^*+q)e^{-z^*t}\varphi(t)\leq$$

$$\leq e^{-z^*t}\varphi(t)[z^*-z^*\int_0^{\tau_1}e^{z^*s}dr_1(s)-\int_0^{\tau_2}e^{z^*s}dr_2(s)+q]=e^{-z^*t}\varphi(t)[q-Q(-z^*)]<0.$$

Hence $z^*+q\in Z_1$ which contradicts the choice of z^*.

This completes the proof of theorem 1.

THEOREM 2. *Let the following conditions be fulfilled:*

1. *Conditions* (A) *hold.*
2. $\delta_1=1$, $\tau_2>\tau_1$ *and* $r_2(\tau_2)>r_2(\tau_1^+)$.
3. *The equation*

$$\tilde{Q}(z)=z(1+\int_0^{\tau_1}e^{-zs}dr_1(s))+\int_{\tau_1}^{\tau_2}e^{-zs}dr_2(s)=0$$

has no real roots.

Then each non-vanishing solution of equation (1) *oscillates.*

PROOF. Suppose that equation (1) has a non-vanishing solution $x:[t_x,\infty)\to\mathbf{R}$ which is ultimately with constant signs. Without loss of generality, in view of corollary 2, we can assume that the function $x(t)$ is ultimately positive and decreasing.

Then from equality (3) and equation (1) we deduce that the functions $y_i(t)$, $i=1,2,\ldots$ are positive, decreasing and convex for $t\geq t_i=t_x+i\tau_2$.

If we choose the number $\tau_3\in(\tau_1,\tau_2)$ so that the inequality $b:=r_2(\tau_2)-r_2(\tau_3)>0$ holds, then equation (1) for $t\geq t_i+\tau_2$ implies the inequality

$$y_i'(t-\tau_1)(1+r_1(\tau_1))+by_i(t-\tau_3)\leq 0.$$

Integrating last inequality from $t+\tau_1\geq t_i+\tau_2$ to $t+2\tau_1$ we obtain that for $t\geq t_{i+1}-\tau_1$ the following inequalities hold

$$y_i(t)\big(1+r_1(\tau_1)\big)\geq y_i(t+\tau_1)\big(1+r_1(\tau_1)\big)+b\int_{t+\tau_1}^{t+2\tau_1}y_i(s-\tau_3)ds\geq b\tau_1y_i\big(t-(\tau_3-\tau_1)\big).$$

From last inequalities, shoosing $c_i > 0$ small enough, we deduce for $t \geq t_i$ the inequality

$$y_i(t) \geq c_i \left(\frac{b\tau_1}{1+r_1(\tau_1)} \right)^{\frac{t}{\tau_3-\tau_1}},$$

which implies that all sets Z_i satisfy the estimate

$$\sup_z Z_i \leq z_0 := \frac{1}{\tau_3-\tau_1} \ln \frac{1+r_1(\tau_1)}{b\tau_1}, \; i = 1, 2, \ldots$$

($Z_i \neq \emptyset$ since $0 \in Z_i$, $i = 1, 2, \ldots$).

In view of $\tilde{Q}(-\infty) = \infty$, setting

$$h := \left(1+\int_0^{\tau_1} e^{z_0 s}\, dr_1(s)\right)^{-1} \min_{z\in[0,\infty)} \tilde{Q}(-z)$$

we conclude that $h > 0$.

Let $z^* \in Z_i$ for some $i \geq 1$ and set $\phi_i(t) = y_i(t)e^{z^* t}$. Then the function $\phi'(t) \leq 0$ ultimately and the following inequalities hold

$$y'_{i+1}(t)+(z^*+h)y_{i+1}(t) = -\int_0^{\tau_2} e^{-z^*(t-s)}\phi_i(t-s)dr_2(s)+(z^*+h)e^{-z^* t}\phi_i(t)+$$

$$+(z^*+h)\int_0^{\tau_2} e^{-z^*(t-s)}\phi(t-s)dr_1(s) \leq$$

$$\leq e^{-z^* t}\phi_1(t-\tau_1)\left\{-\int_{\tau_1}^{\tau_2} e^{-z^* s}dr_2(s)+z^*+z^*\int_0^{\tau_1} e^{-z^* s}dr_1(s)+h\left|1+\int_0^{\tau_1} e^{-z^* s}dr_1(s)\right|\right\} =$$

$$= e^{-z^* t}\phi_1 t-\tau_1)\left\{\min_{z\in[0,\infty)} \tilde{Q}(-z)-\tilde{Q}(-z^*)\right\} \leq 0.$$

Hence $z^*+h \in Z_{i+1}$ whence it follows that for each $i \geq 1$ the inequality $\sup_z Z_{i+1} \geq \sup_z Z_i + h$ holds which is impossible since $\sup_i \sup_z Z_i \leq z_0$.

This completes the proof of theorem 2.

THEOREM 3. *Let conditions* (A) *hold,* $\delta_1 = -1$, $\tau_2 < \tau_1$, $r_1(\tau_2) = 0$ *and* $r_1(\tau_1) > 1$.

Then in order all non-vanishing solutions of equation (1) *to be non-oscillatory it is necessary and sufficient that equation* (2) *has no real roots.*

PROOF. Suppose that the quasi-polynomial $Q(z)$ has no real roots and equation (1) has a positive strictly increasing solution $x(t)$.

Introduce the functions

$$\bar{y}_i(t) = (-1)^i y_i(t), \; i = 0, 1, \ldots \; t \geq t_x + i\tau_1.$$

It is immediately verified that they satisfy the relations

$$\bar{y}_i(t) = \int_0^{\tau_1} \bar{y}_{i-1}(t-s)dr_1(s) - \bar{y}_{i-1}(t),$$

$$\bar{y}_i'(t) = \int_0^{\tau_2} \bar{y}_{i-1}(t-s)dr_2(s), \qquad i \geq 1.$$

which imply that all functions $\bar{y}_i(t)$, $i = 1, 2, \ldots$ are ultimately positive, ultimately increasing and ultimately convex.

Introduce the sets

$\bar{Z}_i = \{z \in [0, \infty) \mid \bar{y}_i'(t) - z\bar{y}_i(t) \geq 0$ ultimately$\}$, $i = 1, 2, \ldots$

It is immediately seen that $0 \in \bar{Z}_i$, hence $\bar{Z}_i \neq \emptyset$ for $i = 1, 2, \ldots$. On the other hand, from equation (1) it follows that ultimately the following inequalities hold

$$0 > \bar{y}_i'(t) - \int_{\tau_2}^{\tau_1} \bar{y}_i'(t-s)dr_1(s) > \bar{y}_i'(t) - r_1(\tau_1)\bar{y}_i'(t-\tau_2). \tag{5}$$

From inequalities (5) we deduce $\bar{y}_i' = O([r_1(\tau_1)]^{\frac{t}{\tau_2}})$ for $t \to \infty$, whence we conclude that $\bar{y}_i(t) = O([r_1(\tau_1)]^{\frac{t}{\tau_2}})$ for $t \to \infty$, hence

$$\sup_z \bar{Z}_i \leq z_0 := \frac{1}{\tau_2} \ln r_1(\tau_1), \; i = 1, 2, \ldots.$$

In view of $Q(\infty) = \infty$, setting

$$h := \frac{1}{r_1(\tau_1)} \min_{z \in [0, \infty]} Q(z)$$

we conclude that $h > 0$.

Let $z^* \in \bar{Z}_i$ for some $i \geq 1$ and set $\phi_i(t) = \bar{y}_i(t)e^{-z^*t}$. Then the function $\phi_i'(t) \geq 0$ ultimately and the following inequalities hold

$$\bar{y}_{i+1}'(t) - (z^*+h)\bar{y}_{i+1}(t) = \int_0^{\tau_2} (e^{z^*(t-s)}\phi_i(t-s))dr_2(s) - (z^*+h)[\int_0^{\tau_1} (e^{z^*(t-s)}\phi_i(t-s)$$

$$dr_1(s) - e^{z^*t}\phi_i(t)] \geq e^{z^*t}\phi_i(t-\tau_2)\int_0^{\tau_2} (e^{-z^*s}dr_2(s) - (z^*+h)e^{z^*t}\phi_i(t-\tau_2)$$

$$[\int_0^{\tau_1} e^{-z^*s}dr_1(s) - 1] \geq e^{z^*t}\phi_i(t-\tau_2)[\int_0^{\tau_2} (e^{-z^*s}dr_2(s) + z^* - z^*\int_{\tau_2}^{\tau_1} e^{-z^*s}dr_1(s) -$$

$$-hr_1(\tau_1)] = e^{z^*t}\phi_i(t-\tau_2)[Q(z^*) - hr_1(\tau_1)] \geq 0.$$

Hence $z^*+h \in \bar{Z}_{i+1}$, whence it follows that for each $i \geq 1$ the inequality $\sup_z \bar{Z}_{i+1} \geq \sup_z \bar{z}_i + h$ holds which is impossible since $\sup_i \sup_z \bar{Z}_i \leq z_0$.

Theorem 3 is proved.

THEOREM 4. *Let conditions* (A) *hold,* $\delta_1 = -1$, $\tau_2 < \tau_1$, $r_1(\tau_2) = 0$ *and* $r_1(\tau_1) = 1$.

Then each non-vanishing solution of equation (1) *oscillates.*

PROOF. As in the proof of theorem 3 we conclude that ultimately inequalities (5) hold which contradict the ultimate convexity of the functions $\bar{y}_i(t)$.

This completes the proof of theorem 4.

COROLLARY 3. Let conditions (A) hold, $\delta_1 = -1$, $\tau_2 < \tau_1$ and $r_1(\tau_2) = 0$.

Then the main conjecture is valid.

The proof of corollary 3 follows from theorems 1, 3 and 4.

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SUR L'UTILISATION DES DÉTER-CUBES DANS LES SYSTÈMES LINÉAIRES DE QUADRIQUES

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Introduction

Le problème de déterminer tous les systèmes linéaires possibles de quadriques à Jacobienne identiquement nulle $L_{d/m}$, où d est la dimension du système et m est la caractéristique de la Jacobienne, a été affronté pour la prémière fois par BONFERRONI (1914) (voir: [2]), qui a déterminé tous les systèmes de ce type dans S_3.

Successivement TERRACINI (1919) (voir: [3]) a mis en évidence l'importance de ce problème en démontrant que: « *Les systèmes apolaires des systèmes linéaires de quadriques à Jacobienne indentiquement nulle individualisent dans S_r des variétés, dont les espaces tangents recouvrent une variété de dimension inférieure à l'ordinaire*».

MURACCHINI (1953) (voir: [4]) détermina tous ceux de S_4 et généralisa le problème. Les variétés bases des systèmes déterminés par cet auteur sont constituées seulement par des espaces linéaires.

DEGOLI (1982) (voir: [7]) a mis en évidence que à partir de S_5 il existe des systèmes linéaires de quadriques à Jacobienne identiquement nulle qui ont pour bases des S_i—V_{1+i}^{r-i} et des variétés de Segre du même type.

Récemment il a démontré (voir: [12]) que les variétés bases de ces systèmes sont seulement: où un espace linéaire double, où une variété réductible composée par deux sous-variétés rationelles, ou bien une variété rationnelle irréductible.

Naturellement les quadriques de ces systèmes linéaires $L_{d/m}(m \leq d)$ satisfont à $d-m+1$ liens fonctionnellement indépendants.

Nous avons observé que jusqu'ici les quadriques des systèmes $L_{d/m}$, individualisés par BONFERRONI, TERRACINI, MURACCHINI et DEGOLI satisfont à des liens exprimables par des déterminants du IIIe, IVe, V^e etc. ordre.

Par exemple les quadriques suivantes:

$$A=x_0x_2-x_1^2 \qquad D=x_1x_4-x_2x_3$$
$$B=x_0x_4-x_1x_3 \qquad E=x_1x_5-x_2x_4$$
$$C=x_1x_5-x_1x_4 \qquad F=x_3x_5-x_4^2$$

qui déterminent un système $L_{5/5}$ de S_5 et ont pour variété base la surface rationnelle normale de S_5, satisfont au lien:

$$AF-BE+CD = 0$$

parce que le determinant:

$$\begin{vmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \\ x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{vmatrix}$$

est identiquement nul pour avoir deux couples de lignes égales et A, B, C, D, E, F apparaissent comme mineurs dans le déterminant signalé.

Donc certains déterminants, lorsqu'ils sont identiquement nuls, conduisent à des liens parmi des quadriques linéairement indépendantes.

Nous nous sommes demandé s'il pouvait exister d'autres objets ayant la même propriété et nous avons inventé: *les déter-cubes.*

Ceux-ci pourrons dans l'avenir servir pour d'autres buts; pour le moment ils permettent d'individualiser de nombreux systèmes lináaires de quadriques $L_{d/m}$ et ce fait nous semble justifier leur introduction en géometrie à cause du théorème de TERRACINI.

Le but de cet article est d'exposer les principales propriétés des déter-cubes et de montrer quelques-unes de leurs applications aux systèmes linéaires de quadriques à Jacobienne identiquement nulle.

1. DÉFINITION.

Nous nommerons déter-cubes du second ordre de l'espace Euclidien réel R^1, ou R^2, ou R^3, ou R^4 . . . ou R^n des ensembles de nombres complexes (éléments du déter-cube) disposés de manière à occuper respectivement un des sommets d'un segment, d'un carré, d'un cube, d'un hypercube de R^4 etc. d'un hypercube de R^n Euclidien.

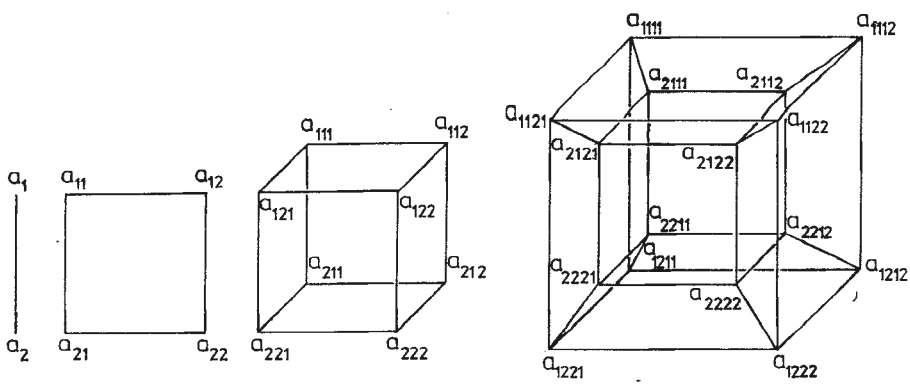

Fig. 1.

On convient d'indiquer les élements du déter-cube par des indices 1 et 2 en nombre égal aux dimensions de l'espace ambiant. Ces déter-cubes seront indiqués respectivement par: $D_{2/1}$, $D_{2/2}$, $D_{2/3}$, $D_{2/4}$, . . . , $D_{2/n}$.

Un déter-cube $D_{2/n}$ sera répresenté pour $n \geq 2$ par une matrice carrée $[D_{2/n}]$ d'ordre 2^{n-1}, dont 2^n éléments ne sont pas obligatoirement nuls.

Nous pouvons écrire:

$$[D_{2/1}] = (a_1, a_2),$$

$$[D_{2/2}] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$[D_{2/3}] = \begin{vmatrix} a_{111} & a_{112} & 0 & 0 \\ a_{121} & a_{122} & 0 & 0 \\ 0 & 0 & a_{211} & a_{212} \\ 0 & 0 & a_{221} & a_{222} \end{vmatrix},$$

$$[D_{2/4}] = \begin{vmatrix} a_{1111} & a_{1112} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{1121} & a_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1211} & a_{1212} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1221} & a_{1222} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{2111} & a_{2112} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{2121} & a_{2122} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{2211} & a_{2212} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{2221} & a_{2222} \end{vmatrix}.$$

Observons tout de suite que: *le nombre des éléments d'un déter-cube $D_{2/n}$ est 2^n.*

Nous appelerons *soutien d'un déter-cube* l'hypercube, dont les sommets sont les éléments du déter-cube.

Les matrices $[D_{2/n}]$ ont un centre de symétrie et 2 éléments symétriques par rapport à ce centre représentent 2 sommets opposés de l'hypercube soutien du déter-cube.

Est évident le:

LEMME 1. *Dans un déter-cube du second ordre $D_{2/n}$, si l'on fixe un indice, en variant tous les autres les manières possibles, on obtient un déter-cube $D_{2/n-1}$.*

Par exemple: Fixons en $D_{2/4}$ le troisième indice, on obtient (a_{1111}, a_{1112}, a_{1211}, a_{1212}, a_{2111}, a_{2112}, a_{2211}, a_{2212}) et après son élimination:

$$\begin{vmatrix} a_{111} & a_{112} & 0 & 0 \\ a_{121} & a_{122} & 0 & 0 \\ 0 & 0 & a_{211} & a_{212} \\ 0 & 0 & a_{221} & a_{222} \end{vmatrix} = [D_{2/3}].$$

LEMME 2. *On peut décomposer un déter-cube du second ordre de R^n de n manières en deux déter-cubes de R^{n-1}.*

En effet il suffit de fixer un indice quelconque dans $D_{2/n}$ en appliquant le numéro 1 et on obtient, par le Lemme précédent un $D_{2/n-1}$. Après on fixe le même indice en appliquant le numéro 2 et on obtient un second déter-cube de R^{n-1}: $D'_{2/n-1}$ *·$D_{2/n}$ est évidemment composé par* $D_{2/n-1}$ et $D'_{2/n-1}$.

LEMME 3. *Il est toujours possible d'assigner aux éléments d'un déter-cube du second ordre le signe + ou le signe — de manière que sur la même arête il n'y a pas deux signes égaux.*

Il suffit en effet d'assigner le signe + (ou —) à tous les éléments dont la somme des indices est pair et le signe — (ou +) à tous ceux dont la somme des indices est impaire.

Quand les signes sont mis de cette manière on dit qu'ils sont *assignés à échiquier.*

2. DÉFINITION. *Nous appelerons développement d'un déter-cube du second ordre $D_{2/n}$ la somme algébrique des produits de chacun élément par l'élément qui occupe le sommet opposé du soutien avec son signe ou avec signe opposé suivant que dans le produit considéré la somme des indices de l'élément, dont le premier indice est 1, est paire ou impaire.*

Notons que si le premier élément a h indices égaux à 1 et k indices égaux à 2, l'élément qui occupe le sommet opposé a les indices intervertis: donc h indices égaux à 2 et k indices égaux à 1.

A cause de cela si n est paire la somme des indices est paire ou impaire en même temps dans l'un et l'autre élément, tandis que si n est impair, la somme a parité opposée dans les deux éléments considérés.

Le résultat de l'opération précédente sera nommé *valeur du déter-cube.*

Notons que à cause de cette définition du déter-cube, $D_{2/2}$ résulte un banal déterminant du second ordre.

Notons que les éléments, dont le primier indice est 1, qui déterminent le signe du produit, s'ils sont sur la même arête du soutien du déter-cube $D_{2/n-1}$, qui les contient, déterminent toujours un signe différent. En effet les signes assignés aux éléments de $D_{2/n-1}$ sont à échiquier.

Exemples:

$$\|D_{2/3}\| = -a_{111}a_{222}+a_{112}a_{221}-a_{122}a_{211}+a_{121}a_{212},$$

$$\|D_{2/4}\| = a_{1111}a_{2222}-a_{1112}a_{2221}+a_{1122}a_{2211}-a_{1121}a_{2212}-a_{1211}a_{2122}+a_{1212}a_{2121}+$$
$$+a_{1222}a_{2111}+a_{1221}a_{2112}.$$

Nous appellerons *correspondants* deux éléments qui appartiennent au même produit, c'est à dire: deux éléments qui occupent deux sommets opposés du déter-cube.

Nous avons indiqué par $\|D_{2/n}\|$ la valeur du déter-cube $D_{2/n}$.

Le déterminant de la matrice $[D_{2/n}]$, qui n'est pas en général égal à la valeur du déter-cube sera noté $|D_{2/n}|$, tandis qu'on notera Vabs $(D_{2/n})$ la valeur absolue du déter-cube $D_{2/n}$.

En résumant: à tout sommet P d'un hypercube de R^n (espace Euclidien réel à n dimensions) ayant 2^n sommets on associe $(k_1, k_2, \ldots, k_n)$ où $k_i = 1$ ou 2.

Si à P est associé $(k_1, k_2, \ldots, k_n)$ au sommet opposé P' est associé $(k_1', k_2', \ldots, k_n')$ avec $k_i' = 1$ ou 2 et avec, pour $i = 1, 2, \ldots, n$, $k_i + k_i' = 3$. Il y a 2^{n-1} couples (P, P').

On obtient le déter-cube $D_{2/n}$ en associant à $(k_1, k_2, \ldots, k_n)$ le complexe noté $(k_1', k_2', \ldots, k_n')$ et on pose:

$$\|D_{2/n}\| = \sum_{(1, k_2, \ldots, k_n)} (-1)^{1+\sum_{i=2}^{n} k_i} a_{1k_2 k_3 \ldots k_n} a_{2k_2' k_3' \ldots k_n'}.$$

3. Définition. *Nous appelerons face d'indice $s^{\text{ième}}$ du déter-cube $D_{2/n}$, le déter-cube $D_{2/n-1}$ qu'on obtient en fixant l'indice $s^{\text{ième}}$ et en variant tous les autres de toutes les manières possibles.*

Deux faces seront dites parallèles si l'une et l'autre ont l'indice fixe $s^{\text{ième}}$, qui pour la première est 1 et pour la seconde est 2.

Elles seront indiquées par

$$D_{2/n-1}^{s/1} \quad \text{et} \quad D_{2/n-1}^{s/2}$$

Naturellement le nombre des faces d'un déter-cube $D_{2/n}$ est $2n$.

Comme on a déjà vu: un déter-cube $D_{2/n}$ a 2^n éléments, un déter-cube $D_{2/n-1}$ a 2^{n-1} éléments. Si, fixant un indice dans $D_{2/n}$, on obtient un $D_{2/n-1}$, le complémentaire de ce $D_{2/n-1}$ par rapport à $D_{2/n}$ est un autre $D_{2/n-1}$, disons $D_{2/n-1}'$ et on a:

$$D_{2/n-1} \cup D_{2/n-1}' = D_{2/n}$$

Une face d'un $D_{2/n}$ est un $D_{2/n-1}$ dont le soutien est une face de l'hypercube soutien de $D_{2/n}$ et deux faces du déter-cube $D_{2/n}$ sont parallèles si leurs soutiens sont des faces parallèles du soutien de $D_{2/n}$.

Théorème 1. *Quand on permute deux faces parallèles dans le déter-cube $D_{2/n}$ on obtient un autre déter-cubo $D_{2/n}^*$ et on a Vabs $(D_{2/n}^*)$ = Vabs $(D_{2/n})$, mais $\|D_{2/n}^*\| = -\|D_{2/n}\|$ sauf dans le cas où n est impair et où l'on permute deux faces qui ont des premiers indices différents, cas où l'on a*

$$\|D_{2/n}^*\| = \|D_{2/n}\|.$$

Démonstration. Si nous permutons la face $D_{2/n-1}^{s/2}$ avec la face $D_{2/n-1}^{s/2}$. l'élément: $a_{k_1 k_2 \ldots k_{s-1} 1 k_{s+1} \ldots k_n}$ s'intervertit avec l'élément

$$a_{k_1 k_2 \ldots k_{s-1} 2 k_{s+1} \ldots k_n}.$$

Ces deux éléments occupent la même arête du soutien du déter-cube, parcequ'ils différent par un seul indice. Pour cela ils déterminent nécessairement un signe différent. En effet on peut constater trois cas:

a) $k_1 = 1$.

En ce cas les deux éléments déterminent les signes des produits et puisqu'ils appartiennent à la même face $D'_{2/n-1}$ et à la même arête nécessairement ils déterminent des signes opposés.

b) $k_1 = 2$.

Tous les deux éléments appartiennent à la face $D''_{2/n-1}$ parallèle à $D'_{2/n}$ et leurs signes dépendent des deux éléments correspondants de $D'_{2/n-1}$. Mais ces derniers occupent la même arête parce-que les indices $k_i (i \neq s)$ sont égaux. Donc aussi ces éléments ont des signes différent.

c) $k_1 = s$ $(s = 1, 2)$.

Si dans un élément le premier indice est 1, dans l'autre il est 2 et viceversa. Indiquons par: $a_{1\,k_2\,k_3\,\ldots\,k_n}$ et $a_{2\,k_2\,k_3\,\ldots\,k_n}$ les deux éléments.

Le premier élément détermine le signe du produit avec son correspondant, tandis que le second élément participe du signe individualisé par son correspondant: $a_{1\,k'_2\,k'_3\,\ldots\,k'_n}$.

Supposons que n soit pair et $1+k_2+k_3+\ldots+k_n$ = numéro pair, il s'ensuit que: $2+k_2+k_3+\ldots+k_n$ = numéro impair.

Donc: $1+k'_2+k'_3+\ldots+k'_n$ = numéro impair. A ' cause de cela les deux éléments déterminent deux signes contraires.

Si: $1+k_2+k_3+\ldots+k_n$ est impair, il s'ensuit: $2+k_2+k_3+\ldots+k_n$ = = numéro pair et pour cela: $1+k'_2+k'_3+\ldots+k'_n$ = numéro pair. Aussi dans ce cas les signes sont opposés.

Supposons au contraire que n soit impair. Si: $1+k_2+k_3+\ldots+k_n$ = numéro impair, il s'ensuit: $2+k_2+k_3+\ldots+k_n$ = numéro pair et pour cela: $1+k'_2+k'_3+\ldots+k'_n$ = numéro impair.

Si: $1+k_2+k_3+\ldots+k_n$ est pair, il s'ensuit: $2+k_2+k_3+\ldots+k_n$ = numéro impair et pour cela: $1+k'_2+k'_3+\ldots+k'_n$ = numéro pair.

Comme on voit dans ce dernier cas les deux éléments déterminent le même signe, donc la valeur du déter-cube ne change pas.

4. Définition. *Nous nommerons déter-cubes du troisième ordre les ensembles de nombres complexes qu'on obtient en divisant un segment en deux segments égaux au moyen d'un point, un carré en 2^2 carrés égaux avec des segments, un cube en 2^3 cubes égaux avec des carrés, un hypercube de R^4 en 2^4 hypercubes égaux avec des cubes, etc., un hypercube de R^n en 2^n hypercubes égaux avec des hypercubes de S_{n-1} et en posant dans les sommets des hypercubes obtenus par division un nombre complexe (élément du déter-cube).*

Le déter-cube de R^n du troisième ordre sera indique avec $D_{3/n}$. Les matrices représentantes $[D_{3/n}]$ pourrons être individualisées par des éléments à trois indices (1, 2, 3) de manière analogue à la précédente.

Exemples:

$$[D_{3/1}] = (a_1,\ a_2,\ a_3),$$

$$[D_{3/2}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$[D_{3/3}] = \begin{vmatrix} a_{111} & a_{112} & a_{113} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{121} & a_{122} & a_{123} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{131} & a_{132} & a_{133} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{211} & a_{212} & a_{213} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{221} & a_{222} & a_{223} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{231} & a_{232} & a_{233} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{311} & a_{312} & a_{313} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{321} & a_{322} & a_{323} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{331} & a_{332} & a_{333} \end{vmatrix}.$$

Nous appelerons *face* de $D_{2/3}$ un quelconque déter-cube du second ordre qu'on obtient en fixant un indice et en variant tous les autres de toutes les manières possibles.

Nous appelerons complément algébrique d'un élément ω d'un déter-cube $D_{3/n}$, le déter-cube du second ordre, qu'on obtient en éliminant dans $D_{3/n}$ toutes les faces qui ont en commun l'élément ω.

Par exemple dans $D_{3/3}$ le complément algébrique de a_{123} est:

$$[D_{2/3}] = \begin{vmatrix} a_{211} & a_{212} & 0 & 0 \\ a_{231} & a_{232} & 0 & 0 \\ 0 & 0 & a_{311} & a_{312} \\ 0 & 0 & a_{331} & a_{332} \end{vmatrix}$$

qui est évidemment la matrice d'un déter-cube du second ordre.

Nous appelerons développement d'un déter-cube $D_{3/n}$ la somme algébrique des produits des éléments, dont le premier indice est 1, *par la valeur de leurs compléments algébriques avec le même signe ou avec le signe opposé selon que la somme des indices des éléments considérés est paire ou impaire.*

5. En itérant le procédé déjà employé on peut introduire les déter-cubes du IVième, du Vième, du VIième etc. ordre. Il suffit d'augmenter dans chacun élé-

ment le nombre d'indices de 3 à 4, de 4 à 5 etc. Le déter-cube du $r^{\text{ième}}$ ordre sera indique par $D_{r/n}$.

Par exemple on peut écrire les matrices des déter-cubes du quatrième ordre par:

$$[D_{4/1}] = (a_1,\ a_2,\ a_3,\ a_4),$$

$$[D_{4/2}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

$$[D_{4/3}] = \begin{vmatrix}
a_{111} & a_{112} & a_{113} & a_{114} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{121} & a_{122} & a_{123} & a_{124} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{131} & a_{132} & a_{133} & a_{134} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{141} & a_{142} & a_{143} & a_{144} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{211} & a_{212} & a_{213} & a_{214} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{221} & a_{222} & a_{223} & a_{224} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{231} & a_{232} & a_{233} & a_{234} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{241} & a_{242} & a_{243} & a_{244} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{311} & a_{312} & a_{313} & a_{314} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{321} & a_{322} & a_{323} & a_{324} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{331} & a_{332} & a_{333} & a_{334} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{341} & a_{342} & a_{343} & a_{344} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{411} & a_{412} & a_{413} & a_{414} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{421} & a_{422} & a_{423} & a_{424} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{431} & a_{432} & a_{433} & a_{434} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{441} & a_{442} & a_{443} & a_{444}
\end{vmatrix}.$$

Nous appelerons complément algébrique d'un élément ω d'un déter-cube du $r^{\text{ième}}$ ordre $D_{r/n}$ le déter-cube $D_{r-1/n}$ qu'on obtient en éliminant tous les déter-cubes $D_{r/n-1}$, qui ont en commun l'élémént ω consideré.

Nous appelerons développement du déter-cube $D_{r/n}$ la somme algébrique des produits des éléments dont le premier indice est 1, par la valeur de leurs compléments algébriques avec le même signe ou le signe contraire suivant que la somme des indices des éléments considérés est pair ou impaire.

Nous appelerons face d'un déter-cube $D_{r/n}$ le déter-cube $D_{r/n-1}$ qu'on obtient en fixant un indice quelconque et en variant tous les autres de toutes les manières possibles.

Deux faces sont dites *parallèles* quand elles ont tous les indices des éléments égaux excepté un: $1^{\prime}r^{\text{ième}}$.

Par la définition précédente il s'ensuit que le développement d'un déter-cube $D_{r/n}$ est donné par la somme algébrique des produits formé de manière que deux facteurs n'appartiennent jamais à la même face.

Le nombre des éléments d'un déter-cube $D_{r/n}$ est évidemment r^n.

Nous écrivons:

$$ND_{r/n} = r^n.$$

Il s'ensuit que le nombre des produits ($NPD_{r/n}$) de son développement est donné par:

$$\begin{aligned} NPD_{r/n} &= ND_{r/n-1}\, NPD_{r-1/n-1} = \\ &= ND_{r/n-1}\, ND_{r-1/n-1}\, ND_{r-2/n-1} \ldots ND_{2/n-1}\, ND_{1/n-1} = \\ &= r^{n-1}\,(r-1)^{n-1}\,(r-2)^{n-1} \ldots 2^{n-1}\, 1^{n-1} = \\ &= (r!)^{n-1}. \end{aligned}$$

Dans $D_{r/n}$ il est possible d'individualiser des déter-cubes $D_{p/n}$ avec $p<n$ qui seront dit: *mineurs des $D_{r/n}$ d'ordre p.*

Chaque mineur $D_{p/n}$, individualise un autre mineur $D_{n-p/n}$, qu'on obtient en supprimant en $D_{r/n}$ les façes, qui passent éléments de $D_{p/n}$. On dira que $D_{n-p/n}$ est le *complément algébrique de $D_{p/n}$.*

Considérons maintenant p faces parallèles quelconques de $D_{r/n}$, des quelles il est possible extraire $\binom{r}{p}^{n-1}$ mineurs $D_{p/n}$, chacun des quels fournit $(p!)^{n-1}$ facteurs. Le complèment algébrique $D_{r-p/n}$ fournit des facteurs pour le produits en nombre de:

$$[(r-p)!]^{n-1}.$$

A cause de cela le total des produits obtenus en développant le déter-cube $D_{r/n}$ selon les mineurs $D_{p/n}$ est:

$$(p!)^{n-1}\,\binom{r}{p}^{n-1}\,[(r-p)!]^{n-1} = (r!)^{n-1},$$

c'est à dire autant de produits qu'il y a de termes dans le développement ordinaire.

Donc, puisque chaque produit est formé par les éléments qui appartiennent à un seule face, les produits considerés sont les mêmes.

Par ce fait il est possible de développer le déter-cube selon les mineurs extraits de plusieurs faces parallèles pourvu qu'on établisse les signes suivant la règle précedente.

THÉORÈME 2. *Quand on permute deux faces parallèles dans le déter-cube $D_{r/n}$, le déter-cube obtenu a même valeur absolue, mais la valeur change de signe, faite exception pour le cas que n soit impair quand on permute les faces qui ont divers le premier indice. Dans ce dernier cas la valeur du déter-cube reste inal terée aussi en signe.*

DÉMONSTRATION. Considéros toutes les faces de $D_{r/n}$ qui ont le même indice $s^{\text{ième}}$.

Les éléments correspondants dans une éventuelle permutation sont respectivement dans chaque face:

$$a_{k_1k_2\ldots k_{s-1}\,1\,k_{s+1}\ldots k_n},\ a_{k_1\,k_2\ldots k_{s-1}\,2\,k_{s-1}\ldots k_n},\ a_{k_1k_2\ldots k_{s-1}nk_{s+1}\ldots k_n}.$$

Supposons qu'on échange deux faces, dont l'indice k_s dans l'une est h et dans l'autre est $h+m$ $(1<m\leq r-1)$.

Dans ce cas nous pouvons développer le déter-cube selon les $\binom{r}{2}^{n-1}$ mineurs du second ordre, qu'on peut extraire des deux faces parallèles.

Si les faces sont permutées, tous les mineurs restent inaltérés en valeur, mais ils changent de signe par le théorème 1, sauf l'exception bien connue contemplée dans l'énoncé. Donc le théorème est démontré.

On vérifie facilement les propriétés suivantes:

I. *Lorsque tous les éléments d'une face sont nuls, la valeur du déter-cube est égal à zéro.*

II. *Pour multiplier la valeur d'un déter-cube par k, il suffit multiplier par k les éléments d'une face.*

III. *Si les éléments de deux faces parallèles ne différent que par un facteur constant, le déter-cube a valeur nulle, excepté le cas que n soit impair et que les éléments des deux faces aient divers le premier indice.*

Le déter-cube $D_{r/n}$ jouit d'autres nombreuses propriétés analogues à celles valables pour les déterminants.

5. APPLICATIONS. Les déter-cubes peuvent individualiser plusieurs systèmes linéaires de quadriques à Jacobienne identiquement nulle.

Dans l'espace linéaire complexe S_r de coordonnées projectives homogènes $x_i (i = 0, 1, \ldots, r)$ choisissons $d+1$ quadriquas linéairement indépendantes:

$$f_0 = 0,\quad f_1 = 0,\quad f_2 = 0, \ldots, f_d = 0$$

avec:

$$f_q = \sum_{i,\,k=0}^{r} a_q^{ik} x_i x_k \qquad (a_q^{ik} = a_q^{ki}).$$

Le système linéaire L_d de dimension d, qui en résulte, est exprimé par l'équation:

$$\sum_{q=0}^{d} \rho_q f_q = 0.$$

Supposons que la matrice Jacobienne à $r+1$ lignes et $d+1$ colonnes:

$$J = \left\| \frac{\partial f_q}{\partial x_i} \right\| \quad \begin{pmatrix} q = 0, 1, \ldots, d \\ i = 0, 1, \ldots, r \end{pmatrix}$$

soit de caractéristique $m \leq d$.

Nous avons indiqué un tel système par $L_{d/m}$ $(m \leq d)$. Cela signifie que seulement m quadriques du système sont outre que linéairement indépendantes aussi fonctionnellement indépendantes. Les autres sont des fonctions quelconques des précédentes.

Parce que ce fait arrive, il faut que parmi les $d+1$ quadriques on établisse $d+m+1$ *liens fonctionnellement indépendantes.*

Ces liens peuvent être donnés par des déter-cubes. Nous en donnons trois exemples significatifs.

a) Considérons, par exemple, le système linéaire de quadriques, qu'on n'a jamais étudié précédemment:

$$\rho_0 (x_0x_5 - x_4x_1) + \rho_1 (x_0x_6 - x_4x_2) + \rho_2 (x_0x_7 - x_4x_3) + \rho_3 (x_1x_6 - x_5x_2) +$$
$$\rho_4 (x_1x_7 - x_5x_3) + \rho_5 (x_2x_7 - x_3x_6) + \rho_6 (x_4x_9 - x_8x_5) + \rho_7 (x_4x_{10} - x_8x_6) +$$
$$\rho_8 (x_4x_{11} - x_8x_7) + \rho_9 (x_5x_{10} - x_9x_6) + \rho_{10} (x_5x_{11} - x_9x_7) + \rho_{11} (x_6x_{11} - x_{10}x_7) = 0$$

le quel prouve ce que nous avons affirmé.

Il s'agit d'un $L_{11/9}$ de S_{11}.

En effet désignons en ordre par:

$$A, B, C, D, E, F, G, H, I, L, M, N$$

les douze quadriques, qui composent le système; on voit tout de suite qu'elles vérifient les liens suivants fonctionnellement indépendants:

$$AF - BE + CD = 0$$
$$GN - HM + IL = 0$$
$$AN - BM + CL + DI - EH + FG = 0$$

parce que les trois déterminants:

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \\ x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \end{vmatrix} \qquad \begin{vmatrix} x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \end{vmatrix} \qquad \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \end{vmatrix}$$

qui possèdent les quadriques dans leurs mineurs du second ordre, sont identiquement nuls pour avoir qui une, qui deux couples, de lignes égales.

Par conséquent les quadriques fonctionnellement indépendantes sont seulement: 12—3 = 9 et ce nombre est aussi la caracteristique de la Jacobienne du système.

A cause d'un théorème bien connu (voir: [6]) il faut que par un point générique de S_{11} passent ∞^2 cordes de la variété base du système constituant un S_3.

En effet la variété base est constituée par un S_7 d'équations:

$$x_4 = 0, \quad x_5 = 0, \quad x_6 = 0, \quad x_7 = 0$$

el par une variété V_5 dont les équations parametriques sont:

$$\begin{array}{llll} x_0 = 1 & x_1 = \mu & x_2 = \nu & x_3 = \vartheta \\ x_4 = \sigma & x_5 = \mu\sigma & x_6 = \nu\sigma & x_7 = \vartheta\sigma \\ x_8 = \tau & x_9 = \mu\tau & x_{10} = \nu\tau & x_{11} = \vartheta\tau \end{array}$$

En coordonnées cartesiennes on a:

$$\frac{x_0}{x_4} = \frac{x_1}{x_5} = \frac{x_2}{x_6} = \frac{x_3}{x_7}$$

et

$$\frac{x_4}{x_8} = \frac{x_5}{x_9} = \frac{x_6}{x_{10}} = \frac{x_7}{x_{11}}$$

ce qui prouve que V_5 est l'intersection de deux variétés particulières de Segre. En choisissant un point générique P de S_{11} et en projectant S_7 on détermine un S_8, qui entrecoupe V_5 selon un plan, ce qui demontre que les cordes qui passent par P constituent un S_3.

b) De manière analogue appliquons le déter-cube du IIIe ordre à la recherche d'un système linéaire de quadriques à Jacobienne identiquement nulle. Pour faciliter l'écriture nous employons les variables $a, b, c, \ldots$ au lien de $x_0, x_1, x_2, \ldots$.

Considérons le système linéaire de quadriques:

$$\rho_0 A+\rho_1 B+\rho_2 C+\rho_3 D+\rho_4 E+\rho_5 F+\rho_6 G+\rho_7 H+\rho_8 I+$$
$$+\rho_9 A'+\rho_{10} B'+\rho_{11} C'+\rho_{12} D'+\rho_{13} E'+\rho_{14} F'+\rho_{15} G'+\rho_{16} H'+\rho_{17} I' = 0$$

dans lequel:

$$\begin{array}{lll} & A = av & A' = eu-iq+hr-ft \\ & B = bv & B' = du-ip+gr-fs \\ & C = cv & C' = dt-hp+gq-es \\ & D = dv & D' = bu-im+hn-ct \\ (1) & E = ev & E' = au-il+gn-cs \\ & F = fv & F' = at-lh+gm-bs \\ & G = gv & G' = br-fm+en-cq \\ & H = hv & H' = ar-fl+dn-cp \\ & I = iv & I' = aq-el+dm-bp \end{array}$$

Ces 18 quadriques linéairement indépendantes satisfont au lien fonctionnel:

$$AA'+BB'+CC'-DD'-EE'-FF'+GG'+HH'+II' = 0$$

En effet le lien est déterminé par le développement du déter-cube suivant:

$$[D_{3/3}] = \begin{vmatrix} av & a & l & 0 & 0 & 0 & 0 & 0 & 0 \\ bv & b & m & 0 & 0 & 0 & 0 & 0 & 0 \\ cv & c & n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & dv & d & p & 0 & 0 & 0 \\ 0 & 0 & 0 & ev & e & q & 0 & 0 & 0 \\ 0 & 0 & 0 & fv & f & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & gv & g & s \\ 0 & 0 & 0 & 0 & 0 & 0 & hv & h & t \\ 0 & 0 & 0 & 0 & 0 & 0 & iv & i & u \end{vmatrix} = 0$$

en partant de la face verticale à gauche.

Puisqu'il existe seulement ce lien entre les 18 quadriques, il s'agit d'un système $L_{17/17}$ de S_{18}.

En effet en remplaçant dans cinq quadriques de (1) $q, r, t, u,$ tirés de quatre des autres quadriques, on obtient le système algébrique homogène:

$$\begin{aligned} & m\,(di+fg)+n\,(-dh-eg)+p\,(ch-bi)+s\,(ce-bf) &= 0 \\ l\,(-di-gf)+ \quad & +n\,(2dg) \quad +p\,(ai-cg)+s\,(af-cd) &= 0 \\ l\,(\,dh+ge)+m\,(-2dg)+ \quad & +p\,(bg-ah)+s\,(bd-ae) &= 0 \\ l\,(\,bi-ch)+m\,(cg-ai)+n\,(ah-bg) & &= 0 \\ l\,(\,bf-ce)+m\,(cd-af)+n\,(ae-bd) & &= 0 \end{aligned}$$

dont le déterminant principal est symétrique gauche d'ordre impair et pour cela nul.

Donc la prémière équation est combinaison linéaire des autres. En résolvant le système des équations restantes on obtient:

$$l = a\frac{s}{g},\ m = b\frac{s}{g},\ n = c\frac{s}{g},\ p = d\frac{s}{g}$$

et pour cela:

$$q = e\frac{s}{g},\quad r = f\frac{s}{g},\quad t = h\frac{s}{g},\ u = i\frac{s}{g}.$$

Il en résulte qu'une variété commune à routes les quadriques a les équations paramétriques:

$$a = 1,\ b = \beta,\ c = \gamma,\ d = \delta,\ e = \epsilon,\ f = \phi,\ g = \psi,\ h = \rho,\ i = \sigma,$$
$$l = \alpha,\ m = \alpha\beta,\ n = \alpha\gamma,\ p = \alpha\delta,\ q = \alpha\epsilon,\ r = \alpha\phi,\ s = \alpha\psi,\ t = \alpha\rho,$$
$$u = \alpha\sigma,\ v = 0.$$

Il s'agit d'une V_9 de S_{17}, située sur l'hyperplan $v = 0$. En éliminant les paramètres on obtient les équations cartesiennes:

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{d}{p} = \frac{e}{q} = \frac{f}{r} = \frac{g}{s} = \frac{h}{t} = \frac{i}{u}. \tag{2}$$

Démontrons qu'il s'agit d'une variété particulière de Segre.

Considérons une droite et un S_7 de S_9 non incidents, dont les équations peuvent être écrites réspectivement par:

$$x_0 = 0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0,\ x_6 = 0,\ x_7 = 0,\ x_8 = 0,$$
$$x_9 = 0.$$

Les quadriques qu'on obtient en accouplant un point de S_1 avec un point de S_7 sont:

$$\begin{aligned} y_{08} &= x_0x_8,\ y_{18} = x_1x_8,\ \ldots,\ y_{78} = x_7x_8, \\ y_{09} &= x_0x_9,\ y_{19} = x_1x_9,\ \ldots,\ y_{79} = x_7x_9. \end{aligned} \tag{3}$$

Ces quadriques individualisent un système à Jacobienne identiquement nulle parce-que elles ont comme variété base un S_1 et un S_7 de S_9 (voir: [7]). En éliminant x_i ($i = 0, 1, \ldots, 9$) on obtient:

$$\frac{y_{08}}{y_{09}} = \frac{y_{18}}{y_{19}} = \ldots = \frac{y_{78}}{y_{79}}$$

qui donnent exactement (2).

Mais outre à la variété V_9 de Segre de S_{17} toutes les quadriques du système donné ont pour base un S_9 d'équations:

$$a = 0,\ b = 0,\ c = 0,\ d = 0,\ e = 0,\ f = 0,\ g = 0,\ h = 0,\ i = 0.$$

En projetant S_9 par un point générique P de S_{18}, on obtient un S_{10}, qui entrecoupe V_9 selon une droite externe à la variété commune entre V_9 et S_9 et cela confirme que par P ils passent ∞^1 cordes de la variété base du système formé par V_9 et S_9.

En effet ces cordes appartiennent toutes à un plan. A cause de cela la Jacobienne a caracteristique $18-1 = 17$ comme il fallait prouver.

c) Considérons le système linéaire de 18 quadriques de S_{15}:

$$\tau_0A+\tau_1B-\tau_2C+\tau_3D+\tau_4E+\tau_5F+\tau_6A_1+\tau_7B_1+\tau_8C_1+\tau_9D_1+\tau_{10}E_1+\tau_{11}F_1+$$
$$+\tau_{12}A_2+\tau_{13}B_2+\tau_{14}C_2+\tau_{15}D_2+\tau_{16}E_2+\tau_{17}F_2 = 0$$

dont les quadriques exprimées par les variables: $a, b, c, d, e, f, g, h, a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1$, sont:

$$\begin{array}{lll}
A = ad_1-bc_1+da_1-cb_1 & A_1 = eh-fg & A_2 = e_1h_1-f_1g_1 \\
B = eh_1-fg_1+he_1-gf_1 & B_1 = ad-bc & B_2 = a_1d_1-b_1c_1 \\
C = af_1-be_1+fa_1-cb_1 & C_1 = ch-dg & C_2 = c_1h_1-d_1g_1 \\
D = ch_1-dg_1+hc_1-gd_1 & D_1 = af-be & D_2 = a_1f_1-b_1e_1 \\
E = ah_1-bg_1+ha_1-gb_1 & E_1 = cf-de & E_2 = c_1f_1-d_1e_1 \\
F = cf_1-de_1+fc_1-ed_1 & F_1 = ah-bg & F_2 = a_1h_1-b_1g_1.
\end{array}$$

Ces 18 quadriques de S_{15} satisfont identiquement aux cinq liens suivants fonctionnellement indépendants et compatibles entre eux:

I. $AB-CD+EF+A_1B_2+B_1A_2-C_1D_2-D_1C_2+E_1F_2+F_1E_2 = 0$
II. $AA_1+BB_1-CC_1-DD_1+EE_1+FF_1 = 0$
III. $AA_2+BB_2-CC_2-DD_2+EE_2+FF_2 = 0$
IV. $A_1B_1-C_1D_1+E_1F_1 = 0$
V. $A_2B_2-C_2D_2+E_2F_2 = 0$

Le premier lien dépend du développement du déter-cube du IV ordre, dont la matrice est:

$$[D_{4/3}] = \begin{vmatrix}
a & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & f & e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g & h & g & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & b_1 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & d_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e_1 & f_1 & e_1 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_1 & h_1 & g_1 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & a & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & f & e & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & h & g & h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & a_1 & b_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & d_1 & c_1 & d_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 & f_1 & e_1 & f_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & h_1 & g_1 & h_1
\end{vmatrix} = 0$$

qui est identiquement nul parce qu'il a 2 couples de faces verticales égales.

On peut développer ce déter-cube selon les 36 mineurs du second ordre extraits de deux faces parallèles verticales laterales à gauche (ou à droite).

On verifie que les signes doivent être pris dans la manière suivante:

$$\begin{array}{cccccc} + & - & + & + & - & + \\ + & - & + & + & - & + \\ + & - & + & + & - & + \\ + & - & + & + & - & + \\ + & - & + & + & - & + \\ + & - & + & + & - & + \end{array}$$

Le deuxième et le troisième lien sont exprimés par deux déter-cubes, dont les matrices respectives sont:

$$[D'_{4/3}] = \begin{vmatrix} a & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e & f & e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g & h & g & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & b_1 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & d_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1 & f_1 & e_1 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & h_1 & g_1 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & f & e & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & h & g & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & f & e & f \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & h & g & h \end{vmatrix} = 0,$$

$$[D''_{4/3}] = \begin{vmatrix} a_1 & b_1 & a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & d_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_1 & f_1 & e_1 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_1 & h_1 & g_1 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & f & e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & h & g & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & a_1 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & d_1 & c_1 & d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 & f_1 & e_1 & f_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & h_1 & g_1 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & a_1 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & d_1 & c_1 & d_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1 & f_1 & e_1 & f_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & h_1 & g_1 & h_1 \end{vmatrix} = 0.$$

Ils sont identiquement nuls pour les mêmes motifs que ceux enoncés précédemment en leurs mineurs du second ordre obtenus en développant tous les deux selon deux faces verticales laterales sont les mêmes quadrique du système.

Les liens IV et V sont exprimés par des banaux déterminants du quatrième ordre:

$$[D_{4/2}] = \begin{vmatrix} a & c & e & g \\ b & d & f & h \\ a & c & e & g \\ b & d & f & h \end{vmatrix} = 0, \qquad [D'_{4/2}] = \begin{vmatrix} a_1 & c_1 & e_1 & g_1 \\ b_1 & d_1 & f_1 & h_1 \\ a_1 & c_1 & e_1 & g_1 \\ b_1 & d_1 & f_1 & h_1 \end{vmatrix} = 0.$$

Puisque les quadriques de S_{15} satisfont à 5 liens indépendantes, la caracteristique de leur système est 18—5 = 13. Il s'agit donc d'un $L_{17/13}$ de S_{15}.

A' cause d'un théorème connu (voir: [7]) par un point générique de S_{15} il devra passer ∞^2 cordes de la variété base du système constituant un S_3.

On verifie immédiatement que la variété base du système est la V_8^8 de Segre de S_{15} d'équations cartesiennes:

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \frac{a_1}{b_1} = \frac{c_1}{d_1} = \frac{e_1}{f_1} = \frac{g_1}{h_1}.$$

Il suffit en effet d'écrire ses équations paramétriques:

$$a = 1, \quad c = \mu, \quad e = \nu, \quad g = \sigma, \quad a_1 = \vartheta, \quad c_1 = \phi, \quad e_1 = \psi, \quad g_1 = \rho,$$
$$b = \xi, \quad d = \mu\xi, \quad f = \nu\xi, \quad h = \sigma\xi, \; b_1 = \vartheta\xi, \; d_1 = \phi\xi, \; f_1 = \psi\xi, \; h_1 = \rho\xi$$

et de verifier qu'elles annullent toutes les quadriques du système.

Mais les cordes de cette variété sont en nombre tel que par un point de S_{15} en passent ∞^2 constituant un S_3. On le démontre facilement en choisissant un point quelconque de S_{15}: $P(a, b, c, d, e, f, g, h, a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1)$ et en imposant que deux points de la variété de Segre: $Q(1, \xi, \mu, \mu\xi, \ldots)$, $R(1, \xi_1, \mu_1, \mu_1\xi_1, \ldots)$ soient alignés avec P.

Parce-qu'il arrive ce fait il faut que tous les mineurs du troisième ordre extraits de la matrice:

$$\left\| \begin{matrix} a & b & c & d & e & f & g & h & a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 \\ 1 & \xi & \mu & \mu\xi & \nu & \nu\xi & \sigma & \sigma\xi & \vartheta & \vartheta\xi & \phi & \phi\xi & \psi & \psi\xi & \rho & \rho\xi \\ 1 & \xi_1 & \mu_1 & \nu_1\xi_1 & \nu_1 & \nu_1\xi_1 & \sigma_1 & \sigma_1\xi_1 & \vartheta_1 & \vartheta_1\xi_1 & \phi_1 & \phi_1\xi_1 & \psi_1 & \psi_1\xi_1 & \rho_1 & \rho_1\xi_1 \end{matrix} \right\|$$

soient tous nuls, et pour cela il est suffisant que soient nuls le premier et tous les autres 13, qu'on obtient en y remplacant la dernière colonne par toutes les autres.

On obtient ainsi un système algébrique de 14 équations en 16 inconnues. En éliminant les racines qui correspondent aux droites tangentes à la variété ($\xi = \xi_1$, $\mu = \mu_1$, etc.) tel système résulte symétriqu e II degré et il fournit toujours une solution dépendant linéairement de deux paramètres et sa symétrique, celle qui correspond à la permutation de R avec Q et cela prouve l'assertion.

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A NEW CHARACTERIZATION OF THE RIESZ CLASS A_p

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Introduction. The so-called Riesz class A_p has been introduced by Riesz in [2] in the following way: A real function u, defined on a not necessarily bounded interval (a, b), belongs to the class $A_p = A_p(a, b)$ with $1<p<\infty$ if and only if u is absolutely continuous in the interval (a, b) and its derivative belongs to the space $L_p = L_p(a, b)$. In the same paper, the following characterization of the class A_p was also proved: A real function u defined on the interval (a, b) belongs to the class A_p if and only if there exist a constant $K>0$ such that, for each system $\{(a_i, b_i)\}_{i=1}^n$ of pair-wise disjoint intervals of finite measure, contained in (a, b), we have

$$\sum_{i=1}^{n} \frac{|u(b_i)-u(a_i)|^p}{|b_i-a_i|^{p-1}} \leq K. \tag{1}$$

The sum (1) is called a Riesz sum.

If the interval $I = (a, b)$ is bounded then the Riesz class is identical with the Sobolev space $W^{1,p}(I)$, equipped with the usual norm

$$\|u\|_{W^{1,p}(I)} = \|u\|_{L_p} + \|u'\|_{L_p}.$$

In the present paper we shall give a new characterization of the class A_p, in term of maximal functions.

1. In this section we recall some known results on the so-called Hardy-Littlewood maximal functions and establish a characterization of the class A_p via such a maximal function.

Suppose that a real function u, defined on an interval (a, b), is locally integrable. Then the Hardy-Littlewood maximal function of u is given by

$$(\Lambda u)(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |u(\lambda)|\, d\lambda \quad (x\in(a, b)) \quad [x-h, x+h]\subset(a, b).$$

and this function satisfies the Hardy-Littlewood inequality, that is, if $1<p<\infty$, then there exists a constant $K>0$ such that, for all $u\in L_p$,

$$\|\Lambda(u)\|_{L_p}\leq K\|u\|_{L_p}$$

and for a.e. $x\in(a, b)$ we have

$$(\Lambda(u))(x)\geq |u(x)|$$

The following proposition in an easy consequence of the Hardy-Littlewood inequality (see [3]).

PROPOSITION. Let u be a real function, defined on the interval (a, b) and suppose that $1<p<\infty$. Then the function u belongs to the class A_p if and only if u is absolutely continuous in (a, b) and $\Lambda(u')\in L_p$. Moreover, there exists a constant $K>0$ such that, for all $u\in A_p$,

$$\|u'\|_{L_p}\leq\|\Lambda(u')\|_{L_p}\leq K\|u'\|_{L_p}.$$

COMMENT. In the above statement, the condition of absolute continuity of u is important. Indeed, the so-called Cantor function does not belong to the class $A_p(0, 1)$ although the Hardy-Littlewood maximal function of the derivative of the Cantor function belongs to the space $L_p(0, 1)$.

2. In this section the so-called symmetrical maximal function Λ^*u is introduced and, in terms of the latter, a characterization of the Riesz class A_p is given. Namely, it is shown that $u\in A_p$ if and only if $\Lambda^*u\in L_p$.

DEFINITION. Let u be a real function defined on an interval (a, b). The symmetrical maximal function Λ^*u of u is defined as follows

$$(\Lambda^*u)(x) = \sup_{\substack{h>0\\ [x-h,\, x+h]\subset(a,\, b)}} \frac{|u(x+h)-u(x-h)|}{2h}\quad(x\in(a, b)).$$

THEOREM. *Let u be a real function defined on the interval (a, b). Then $u\in A_p$ if and only if $\Lambda^*u\in L_p$.*

PROOF. Suppose that $u\in A_p$. Then u is absolutely continuous in (a, b) which implies

$$\frac{u(x+h)-u(x-h)}{2h} = \frac{1}{2h}\int_{x-h}^{x+h} u'(\lambda)d\lambda\quad(x\in(a, b)),$$

provided that $h>0$ and $[x-h, x+h]\subset(a, b)$. Hence, for all $x\in[a, b]$, we have

$$(\Lambda^*u)(x)\quad\sup_{\substack{h>0\\ [x-h,\, x+h]\subset(a,\, b)}}\frac{1}{2h}\int_{x-h}^{x+h}|u'(\lambda)|d\lambda = (\Lambda u')(x).$$

Let A be the set defined as follows

$$A = \left\{(x, h)\colon h\in\left(0, \frac{b-a}{2}\right), x\in(a+h, b-h)\right\}.$$

Then

$$A = \{(x, h) : (x-h, x+h) \subset (a, b)\}.$$

Let F be a real function defined on $B = (a, b) \times \left(0, \frac{b-a}{2}\right)$ by

$$F(x, h) = \begin{cases} \dfrac{|u(x+h)-u(x-h)|}{2h} & \text{if } (x, y) \in A \\ 0 & \text{if } (x, h) \in B-A. \end{cases}$$

Since u is measurable then $F(x, h)$ is measurable and we have that

$$\sup_{h>0} F(x, h) = \sup_{h>0} \frac{|u(x+h)-u(x-h)|}{2h} = \Lambda^*(u)(x)$$

is also measurable.

Moreover, by proposition, we have $\Lambda^* u \in L_p$ and there is a constant $k>0$ such that

$$\|u'\|_{L_p} \leq \|\Lambda^* u\|_{L_p} \leq \|\Lambda u'\|_{L_p} \leq K\|u'\|_{L_p}.$$

Assume now that $\Lambda^* u \in L_p$, and consider a system $\{(a_i, b_i)\}_{i=1}^n$ of pair-wise disjoint intervals of finite measure, contained in (a, b). For all $i = 1, \ldots, n$, define the sets E_i^1, E_i^2, D_i^1, D_i^2, and F_i in the following way:

$$E_i^1 = \left\{x \in (a_i, b_i) : \left| \frac{u(x)-u(a_i)}{x-a_i} \right| \geq \left| \frac{u(b_i)-u(a_i)}{b_i-a_i} \right| \right\},$$

$$E_i^2 = (a_i, b_i) - E_i^1 = \left\{x \in (a_i, b_i) : \left| \frac{u(x)-u(b_i)}{b_i-x} \right| > \left| \frac{u(b_i)-u(a_i)}{b_i-a_i} \right| \right\},$$

$$D_i^1 = a_i + \frac{1}{2}(E_i^1 - a_i), \quad D_i^2 = b_i + \frac{1}{2}(E_i^2 - b_i)$$

and

$$F_i = D_i^1 \cup D_i^2.$$

The above sets have the following properties

(a) $E_i^1 \cap E_i^2 = \emptyset$, $D_i^1 \cap D_i^2 = \emptyset$

(b) measure (F_i) = measure D_i^1 + measure $D_i^2 = \frac{1}{2}(b_i - a_i)$.

For all $x \in F_i$, we have

$$\frac{|u(b_i)-u(a_i)|}{b_i-a_i} \leq (\Lambda^* u)(x).$$

Indeed, let $x \in F_i$. Then $x \in D_i^1$ or $x \in D_i^2$. Suppose that $x \in D_i^1$. Then there exists an $x_i^1 \in E_i^1$ such that

$$x = a_i + \frac{1}{2}(x_i^1 - a_i).$$

Taking $h_i^1 = \frac{1}{2}(x_i^1 - a_i)$, we obtain

$$\frac{|u(x+h_i^1) - u(x-h_i^1)|}{2h_i^1} = \frac{|u(x_i^1) - u(a_i)|}{|x_i^1 - a_i|}.$$

Since $x_i^1 \in E_i^1$, the definition of $\Lambda^* u$ implies that for all $x \in D_i^1$,

$$(\Lambda^* u)(x) \geq \frac{|u(b_i) - u(a_i)|}{b_i - a_i}.$$

This inequality is similarly obtained for $x \in D_i^2$. Indeed, let $x \in D_i^2$, then there exists an $x_2^i \in E_i^2$ such that

$$x = b_i + \frac{1}{2}(x_i^2 - b_i).$$

Taking $h_i^2 = \frac{1}{2}(x_i^2 - b_i)$, we obtain

$$\frac{u(x+h_i^2) - u(x-h_i^2)}{2h_i^2} = \frac{u(x_i^2) - u(b_i)}{x_i^2 - b_i}.$$

Since $x_i^2 \in E_i^2$, the difinition of $\Lambda^* u$ implies that for all $x \in D_i^2$

$$(\Lambda^* u)(x) \geq \left| \frac{u(b_i) - u(a_i)}{b_i - a_i} \right|.$$

From the above inequality, we obtain

$$\frac{|u(b_i) - u(a_i)|^p}{|b_i - a_i|^{p-1}} \leq (\inf_{x \in F_i} (\Lambda^* u)^p(x))(b_i - a_i).$$

Hence

$$\sum_{i=1}^{n} \frac{|u(b_i) - u(a_i)|^p}{|b_i - a_i|^{p-1}} \leq 2 \sum_{i=1}^{n} \inf (\Lambda^* u)^p(x) \frac{(b_i - a_i)}{2} \leq 2 \int_{F_1 \cup F_2 \ldots \cup F_n} (\Lambda^* u)^p(x) dx \leq$$

$$\leq \int_a^b (\Lambda^* u)^p(x) dx < \infty.$$

In other words, $u \in A_p$. Moreover, by the Hardy-Littlewood maximal theorem and from the definition of $\Lambda^* u$ we get the existence of a constant $K>0$ such that

$$\|u'\|_{L_p} \leq \|\Lambda^* u'\|_{L_p} \leq \|\Lambda u'\|_{L_p} \leq K\|u'\|_{L_p}$$

COMMENT. Using a maximal function $\Lambda^* u$ defined by

$$(\Lambda^* u)(x) = \sup_{h \neq 0} \left| \frac{u(x+h)-u(x)}{h} \right|$$

a similar characterization of the class A_p can be obtained. The proof is similar to that of theorem.

If we assume that $u \in L_p(I)$ than the statement of the theorem is an easy consequence of a well-known characterization of the Sobolev spaces $W^{1,p}(I)$ where I is on arbitrary open interval in R (see e.g. (1), proposition VIII.3.).

Thus the main result of the present paper is that in the one-dimensional case the measurabily condition may be omitted.

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SOME PROPERTIES OF π-t GROUPS

By

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This paper is a generalization of the results of the author's earlier paper: "On π-t groups"[1]. We extend the theorems concerning π-solvable π-t groups to the case of arbitrary π-t groups.

We recall the following definitions and theorems.

DEFINITION. A t-group is a group G whose subnormal subgroups are all normal in G.

DEFINITION. We say that a subgroup H of a group G is pronormal in G if and only if for all $g \in G$, H and H^g are conjugate in $\langle H, H^g \rangle$.

GASHÜTZ [2] showed the following

THEOREM A. *The subgroups of solvable t-groups are again t-groups.*

In 1969 PENG [3] proved

THEOREM B. *G is a solvable t-group if and only if all subgroups of prime power order of G are pronormal is G.*

In 1977 ASAAD [4] SHOWED

THEOREM C. *If each subgroup of prime power order of G' is pronormal in G and G' is a π-Hall subgroup of G then G is a solvable t-group.*

The converse of this theorem is not true.

THEOREM D. (PENG: *personal communication.*) *Let N be a normal π-Hall subgroup of G. If G/N is a solvable t-group, and each subgroup of prime power order of N is pronormal in G, then G is a solvable t-group.*

In [1] we proved the following.

THEOREM 1 of [1]. *Let G be a finite group of odd order. G is a solvable t-group if and only if each subgroup of prime power order of G' is pronormal in G, and G' is a π-Hall subgroup in G.*

For the proof of this theorem we showed the following

LEMMA 1 of [1]. Let G be a finite group, $G' < G$. $P \in \mathrm{Syl}_p(G)$, $p \in \pi(G')$, $p \neq 2$. Suppose, that each p- subgroup of $N_G(P)$ is normal in $N_G(P)$. Then either $P \leq G'$ or $C_G(P) = N_G(P)$ or both.

In [1] we introduced the concept of a π-t group.

DEFINITION. Let G be a finite group, $\pi \subset \pi(G)$. G is called a π-t group if and only if its subnormal π-subgroups are all normal in G.

In [1] we generalized the result of PENG.

THEOREM 2 of [1]. *Let G be a finite group, and $\pi \subset \pi(G)$.* Let N be a π-solvable *normal subgroup of G such that $(|G : N|, p) = 1$ for all $p \in \pi \cap \pi(N)$. If G/N is a π-t group, and each p-subgroup of N where $p \in \pi$, is pronormal in G, then G is a π-t group.*

For the proof we showed the following.

LEMMA 2 of [1]. Let U be a finite group, $\pi \subset \pi(U)$. Let L be a solvable π-subgroup of U and $L \lhd\lhd U$. Suppose that each p-subgroup of L is pronormal in U. Then $L \lhd U$.

In [1] we also proved the following

THEOREM 3 of [1]. *Let G be a π-solvable finite group of odd order, and $\pi \subset \pi(G)$. G is a π-t group such that each subgroup of G is again a π-t group if and only if each p-subgroup of G', where $p \in \pi$, is pronormal in G, and $(|G : G'|, p) = 1$ for all $p \in \pi \cap \pi(G')$.*

Now we generalize Theorem 2 of [1]

THEOREM 1. *Let G be a finite group, and $\pi \subset \pi(G)$. Let N be a normal subgroup of G such that $(|G : N|, p) = 1$ for all $p \in \pi \cap \pi(N)$. If G/N is a π-t group and each p-subgroup of N, where $p \in \pi$, is pronormal in G then G is a π-t group.*

PROOF. Let L be a π-subgroup of G with $L \lhd\lhd G$. Clearly $L \cap N \lhd\lhd N$. The assumptions of our theorem imply that each p-subgroup of $L \cap N$ is pronormal in $L \cap N$. Using the Theorem of Peng (Theorem B) we get $L \cap N$ is a solvable t-group. Applying Lemma 2 of [1] to N and $L \cap N$, we obtain $L \cap N \lhd N$. We have $L \cap N \lhd N \lhd G$. Applying again Lemma 2 of [1] to G and $L \cap N$, $L \cap N \lhd G$ follows. By the conditions of our theorem $(|L \cap N|, |L : L \cap N|) = 1$ holds. The Schur-Zassenhaus theorem implies, that there exists a subgroup K in L such that $L = (L \cap N)\, K$ and $(L \cap N) \cap K = 1$. Obviously $L \cap N \lhd NK$, $L \lhd\lhd NK$ and $(|K|, |N|) = 1$. On the basis of Sylow's theorems it is easy to see that $L \lhd NK$. As $L \lhd\lhd G$, therefore $LN/N \lhd\lhd G/N$. G/N is a π-t group by assumption, hence clearly $LN/N \lhd G/N$, so $LN \lhd G$ holds. Obviously $NK = NL$. Thus $L \lhd NK \lhd G$. K is a Hall subgroup in NK, $L \cap N \lhd G$. Using again Sylow's theorems, it is easy to see that $L \lhd G$.

COROLLARY. Let G be a finite group, and $\pi \subset \pi(G)$. If each p-subgroup of G, where $p \in \pi$ is pronormal in G, then G is a π-t group.

Now we extend Theorem 3 of [1].

THEOREM II. *Let G be a finite group of odd order, $\pi \subset \pi(G)$. G is a π-t group such that each subgroup of G is again a π-t group if and only if each p-subgroup of G' where $p \in \pi$, is pronormal in G, and $(|G:G'|, p)=1$ for all $p \in \pi \cap \pi(G')$.*

PROOF. First suppose G' satisfies the above conditions. In case $G' = 1$ the statement is trivial.

1. $G' = G$.

Then using Corollary of Theorem I we obtain that G is a π-t group.

Let L be an arbitrary subgroup of G. Clearly each p- subgroup of L, where $p \in \pi$, is pronormal in L, hence by the Corollary of Theorem I it follows that L is a π-t group.

2. $G' < G$.

Then using Theorem I with $N = G'$, we get that G is a π-t group.

Let L be an arbitrary subgroup of G. We have $L/L \cap G' \cong LG'/G'$. Hence $L/L \cap G'$ is abelian.

a) $L \cap G' = 1$, then L is abelian, i.e. L is a π-t group.

b) $L \cap G' = L$, then clearly each p-subgroup of L — where $p \in \pi$ — is pronormal in L, and using the Corollary of Theorem I, we get that L is π-t group.

c) $L \neq L \cap G' \neq 1$, then clearly $(|L:L \cap G'|, p) = 1$ for all $p \in \pi$, and each p-subgroup of $L \cap G'$ — where $p \in \pi$ — is pronormal in L. Applying Theorem I to L and $L \cap G'$ we obtain that L is a π-t group.

Conversely, assume that G is a π-t subgroup such that each subgroup is a π-t group again.

a) Set $S \leq G'$, $|S| = p^k$, $p \in \pi$. We show that S is pronormal in G. Let $g \in G$ be arbitrary. Consider $T = \langle S, S^g \rangle$. If $S \in \mathrm{Syl}_p(T)$ then $S^g = S^u$ for some $u \in T$. Thus we can assume, $S < P^* \in \mathrm{Syl}_p(T)$. Clearly there exists a $t \in T$ such that $S^{gt} < P^*$, $P^* \leq P \in \mathrm{Syl}_p(G)$. We apply now the Theorem of Alperin [5] for G with S and S^{gt}. Thus there exist elements g_i and Sylow p-subgroups Q_i of G, $1 \leq i \leq n$, furthermore $y \in N_G(P)$, which satisfy the following conditions.

(i) $gt = g_1 g_2 \cdot \cdot \cdot g_n y$

(ii) $P \cap Q_i$ is a tame intersection, $1 \leq i \leq n$

(iii) g_i is a p-element of $N_G(P \cap Q_i)$, $1 \leq i \leq n$

(iv) $S \leq P \cap Q_1$, while $S^{g_1 \cdots g_i} \leq P \cap Q_{i+1}$ where $1 \leq i \leq n-1$.

Clearly $S \lhd\lhd P \cap Q_1 \lhd N_G(P \cap Q_1)$. By assumptions $N_G(P \cap Q_1)$ is a π-t group, hence $S \lhd N_G(P \cap Q_1)$. As $g_1 \in N_G(P \cap Q_1)$, $S = S^{g_1}$ follows. Similarly we obtain $S^{g_1 \cdots g_n} = S$. *Thus* $S^{gt} = S^y$, *where* $y \in N_G(P)$. Clearly $S \lhd\lhd P \lhd N_G(P)$, since $N_G(P)$ is a π-t group, $S \lhd N_G(P)$ follows. So $S^{qt} = S$, i.e. $S^g = S^{t^{-1}}$ and S is pronormal in G.

Thus to complete the proof, it remains to show that $(|G:G'|, p) = 1$ for all $p \in \pi \cap \pi(G')$. If $G = G'$, *the statement is clear. So we can assume, that* $G' < G$. Set $r \in \pi(G') \cap \pi$, $R \in \mathrm{Syl}_r(G)$. Suppose that $R \not\leq G'$. Applying Lemma 1 of [1], $N_G(R) = C_G(R)$ follows. As R is a t-group of odd order, it is well known that R is abelian. Using $N_G(R) = C_G(R)$, G has a normal r-complement by the Theorem of Burnside. Thus $G = RK$, $R \cap K = 1$, $K \lhd G$. Obviously $G/K \cong R$. As R is Abelian, $G' \leq K$, whence $G' \cap R = 1$ follows, a contradiction.

This completes the proof.

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ON RIESZ BASES FROM VECTOR EXPONENTIALS I.

By

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To Béla Szőkefalvi-Nagy for his 75th birthday

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Introduction

The properties of the system $\{e^{i\lambda x} : \lambda \in \sigma\}$ where σ is a discrete subset of C were investigated by many mathematicians and have long history. Recently in [1], [2] were obtained the necessary and sufficient conditions of the Riesz basis property in $L^2(0, a)$ of such system for $0 < a < \infty$. The main idea in these papers is to consider the set $\{e^{i\lambda x}\} \subset L^2(0, a)$ as the projection of the same set defined on $(0,\infty)$. The necessary and sufficient condition of the Riesz basis property of $\{e^{i\lambda x}\}$ in its closed linear hull in $L^2(0,\infty)$ is known: this is the Carleson condition for the set σ (see [3]). The Riesz basis property of $\{e^{i\lambda x}\}$ in $L^2(0, a)$ is equivalent to the fact that the natural projection maps its closed linear hull in $L^2(0,\infty)$ isomorphically to the whole $L^2(0, a)$. This isomorphism is equivalent to the Muckenhoupt condition (see [1], [4], [5]). This condition gives a bound for the growth of a function of exponentional type a with nullset σ. This is called the generating function of the pair $(\{e^{i\lambda x}\}, a)$. A good summary of this theory is given in [4].

In this paper we shall investigate the set $\Phi = \{e^{i\lambda x}e_\lambda : \lambda \in \sigma\}$ where the vectors e_λ belong to some Hilbert space E, so we can assume that $\Phi \subset L^2(0, \infty; E)$ or $\Phi \subset L^2(0, a; E)$.

The properties of Φ in $L^2(0, \infty; E)$ were first investigated in [6] and for the space $L^2(0, a; E)$ the first study of the basis properties of Φ is given in [7]. In what follows we continue these considerations. The tools applied for the

scalar case (dim $E = 1$) will be used. In order to formulate our results, some definitions are needed. Denote $H^2_{\pm}(E)$ the Hardy space of E-valued analytic functions of the upper half-plane $\mathbf{C}_+$(resp. the lower half-plane $\mathbf{C}_-$) with the usual norm, see [11]. Denote $\ll . \gg$ resp. $<,>$ the norm and the scalar product in E.The $L(E)$-valued operator function S is called inner if it is analytic in $\mathbf{C}_+$, $S(\lambda)$ is a contraction for $\lambda \in \mathbf{C}_+$, and $S(x)$ is unitary for a.e. $x \in \mathbf{R}$. If S is inner, let $K_s := H^2_+(E) \ominus SH^2_+(E)$ and $P_s : H^2_+(E) \to K_s$ be the orthonormal projection onto K_s. It is known [14] that in case dim $E < \infty$ any inner operator function S has a factorization $S = \Pi\Theta' = \Theta\Pi'$ where Π, Π' are the Blaschke — Potapov products and Θ, Θ' are inner functions satisfying det $\Theta(\lambda) \neq 0$, det $\Theta'(\lambda) \neq 0$ for $\lambda \in \mathbf{C}_+$. An inner function which is an isomorphism of E at every $\lambda \in \mathbf{C}_+$ is called singular inner. This name comes from the scalar case where these functions are represented by Poisson type integral of a measure singular to the Lebesgue measure, see [8], [9]. It is known that the Fourier transform maps $H^2_+(E)$ onto $L^2(0, \infty; E)$, $H^2_-(E)$ onto $L^2(-\infty, 0; E)$ and K_a onto $L^2(0, a; E)$, where $\Theta_a(z) = e^{iza} I_E$ and I_E is the identity of E. In order to handle the problem by the theory of Hardy spaces we change the above notation and denote by Φ the Fourier transform of the original set Φ, namely

$$\Phi := \{x_\lambda e_\lambda : \lambda \in \sigma\}, \; x_\lambda(z) := \sqrt{\frac{Im\lambda}{\pi}} \cdot \frac{1}{z-\bar{\lambda}}$$

We allow that the multiplicity of a value λ in σ is greater than 1, assume that the vectors e_λ corresponding to the same value λ are orthonormal and denote by E_λ the closed linear hull of these vectors e_λ. If the set σ fulfils the Blaschke condition

$$\text{(B)} \qquad \sum_{\lambda \in \sigma} \frac{Im\lambda}{1+|\lambda|^2} < \infty,$$

then $\{x_\lambda\}$ is minimal in H^2_+ [3], so Φ is minimal in $H^2_+(E)$ hence there exists a system $\Psi = \{\psi_\lambda\}$ biorthogonal to Φ. Denote Π the Blaschke — Potapov product for which the nullset of det Π is σ and

(1) Ker $\Pi^*(\lambda) = E_\lambda$, $\lambda \in \sigma$.

It is well-known [3] that

(2) $\vee\Phi = \vee\Psi = K_\Pi$,

where $\vee\Phi$ denotes the closed linear hull of elements $\phi_\lambda \in \Phi$ in $H^2_+(E)$.

In this paper we shall investigate the properties of Φ in $H^2_+(E)$ and of $P_\Theta\Phi$ in K_Θ. Here the inner operator function Θ is not necessarily indentical with $\Theta_a$but we assume that Θ can be represented in the form

(3) $\Theta(z) = e^{iz\alpha}\Theta_0(z)$, $\alpha > 0$,

where Θ_0 is a bounded in $\mathbf{C}_+$(consequently inner) entire operator function.

DEFINITION 1. The operator function $F: C_+ \to L(E)$ is called strong H^2_+— function if $F(z)e/(z+\lambda) \in H^2_+(E)$, $e \in E$, $\lambda \in C_+$. A strong H^2_+ — function is called outer if

$$\bigvee_{e \in E,\, n \in N} \left(\frac{z-i}{z+i}\right)^n \frac{F(z)e}{z+i} = H^2_+(E).$$

The definition of strong, resp. strong outer H^2_- — operator functions is similar, only i is replaced by $-i$ and H^2_+ by H^2_-. The strong outer functions will be denoted by $F_e^{\pm}$, $\hat{F}_e^{\pm}$ etc.

DEFINITION 2. As above, let Π be defined by (1). An entire function F: $\mathbf{C} \to L(E)$ is called a generating function for $P_\Theta \Phi$ if it has the factorizations $F = \Pi F_e^+$ in $\mathbf{C}_+$, $F = \Theta F_e^-$ in $\mathbf{C}_-$. We shall prove the following main statements.

THEOREM A. *Let F be a generating function of $P_\Theta \Phi$. Then*

1. *The system $P_\Theta \Phi$ is minimal in K_Θ,*

2. *P_Θ is an isomorphism between K_Π and K_Θ if and only if the Hilbert operator*

$$(Hu)(x) := \frac{1}{2\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{u(t)}{x-t} dt$$

is bounded in the space $L^2_{F^*F}$ of vectorfunctions u with finite norm

$$\|u\| := \{\int_{\mathbf{R}} \langle F^*(x)F(x),\, u(x)\rangle dx\}^{1/2} < \infty,$$

3. *$P_\Theta \Phi$ is Riesz basis in K_Θ if and only if Φ is an L-basis in $H^2_+(E)$ (i.e. Riesz basis in its closed linear hull $\vee\Phi$) and the Hilbert operator is bounded in $L^2_{F^*F}$.*

REMARK. Some points of this theorem were proved in [5].

THEOREM B. *Let $P_\Theta \Phi$ be a complete and minimal system in K_Θ and $\Psi_\Theta = \{\Psi_{\Theta,\lambda}\} \subset K_\Theta$ be biorthogonal to $P_\Theta \Phi$. Suppose that Ψ_Θ is also complete in K_Θ. Then there exists a generating function F of $P_\Theta \Phi$ such that for some $\{\Psi_{\Theta,\lambda_r}\}_{r=1}^N \subset \Psi_\Theta$ we have*

$$F(z)e = \sum_{r=1}^{N} \langle e,\, \xi_r\rangle (z-\lambda_r) \Psi_{\Theta,\lambda_r}(z) \tag{4}$$

where $\{\xi_r\}_{r=1}^N$ is an orthonormal basis in E with real coordinates and $e \in E$.

THEOREM C. *Let* $\Phi = \{\phi_n\}_1^\infty$, $\phi_n = x_{\lambda_n} e_n$, $\lambda_n \neq \lambda_k$ *for* $n \neq k$ *and let* $P_\Theta \Phi$ *be a basis in* K_Θ *and let* $\hat{\Phi} = \{\hat{\phi}_n\}$, $\hat{\phi}_n = x_{\lambda_n} \hat{e}_n$ *be such a system for which* $\ll e_n - \hat{e}_n \gg \to 0$ *as* $n \to \infty$. *Then we have*

a) *The system* $\Phi_Q = \{\phi_n\}_1^Q \cup \{\hat{\phi}_n\}_{Q+1}^\infty$ *is basis in* K_Θ(*i.e.* $P_\Theta \Phi_Q$ *is basis in* K_Θ) *if* Q *is large enough.*

b) *For any* ϵ, $0 < \epsilon < \alpha$ (see (3)) *the system* $P_{\Theta_{-\epsilon}} \hat{\Phi}$ *is complete in* $K_{\Theta_{-\epsilon}}$, *where* $\Theta_{\pm\epsilon}(z = \Theta(z) e^{\pm iz\epsilon}$. *If there exists a subsystem of* $P_{\Theta_{-\epsilon}} \Phi$ *which is basis in* $K_{\Theta_{-\epsilon}}$ *then there exists a subsystem of* $P_{\Theta_\epsilon} \hat{\Phi}$ *which is basis in* $K_{\Theta_{-\epsilon}}$.

c) *For every* $\epsilon > 0$ *the system* $P_{\Theta_\epsilon} \hat{\Phi}$ *is an L-basis in* K_{Θ_ϵ}.

The proofs

DEFINITION 3. The angle between the subspaces X and Y of a Hilbert space H is $\phi(X, Y) \in [0, \pi/2]$ satisfying

$$\cos \phi(X, Y) = \sup_{x \in X, y \in Y} \frac{(x, y)}{\|x\| \cdot \|y\|}.$$

We shall use the

LEMMA 1 ([3], p. 258.).

1. The projection P_X: $Y \to X$ is an isomorphism onto $P_X Y$ if $\phi(X^\perp, Y) > 0$, $X^\perp := (X \vee Y) \ominus X$.

2. P_X: $Y \to X$ is an isomorphism onto X if and only if

$$\phi(X^\perp, Y) > 0, \ \phi(X, Y^\perp) > 0.$$

REMARK. We assume everywhere $\dim E < \infty$. In this case the L-basis property of Φ implies the so-called Carleson-Newman condition $\sigma(\Phi) \in (\mathrm{CN})$ (see [3] p. 212.). So well assume everywhwere in this paper that $\sigma \in (\mathrm{CN})$.

LEMMA 2.

a) The system $P_\Theta \Phi$ is an L-basis in K_Θ if and only if Φ is an L-basis in $H_+^2(E)$ and $\phi(K_\Theta^\perp, K_\Pi) > 0$, where $K_\Theta^\perp := (K_\Pi \vee K_\Theta) \ominus K_\Theta$.

b) The system $P_\Theta \Phi$ is basis in K_Θ if and only if Φ is an L-basis in $H_+^2(E)$ and $\varphi(K_\Theta^\perp, K_\Pi) > 0$, $\varphi(K_\Pi^\perp, K_\Theta) > 0$.

c) $P_\Theta \Phi$ is complete in K_Θ if and only if $K_\Theta \cap K_\Pi^\perp = \{0\}$.

d) If $P_\Theta \Phi$ is minimal in K_Θ then $K_\Theta^\perp \cap K_\Pi = \{0\}$.

PROOF. a) If $P_\Theta \Phi$ is an L—basis then it is uniformly minimal and hence Φ is also uniformly minimal, see [2] p.228. So from $\sigma(\Phi) \in (\mathrm{CN})$ it follows ([3] p. 212.) that Φ is an L—basis. Let $\varphi = \sum_{\lambda \in \sigma} c_\lambda \varphi_\lambda$, $(c_\lambda) \in l_2$. then

$$\|\varphi\|^2 \asymp \sum_{\lambda \in \sigma} |c_\lambda|^2 \asymp \Big\|\sum_{\lambda \in \sigma} c_\lambda P_\Theta \varphi_\lambda\Big\|^2 = \|P_\sigma \varphi\|^2.$$

We mean by $f \asymp g$ that $0 < c < f/g < C < \infty$ with some absolute constants c, C.

This means that P_θ maps isomorphically $\vee\Phi = K_\pi$, so by Lemma 1, $\varphi(K^\perp_\theta, K_\pi) > 0$. Conversely if P_θ is isomorphic and Φ is an L—basis then $P_\theta\Phi$ is obviously L—basis, too.

b) If $P_\theta\Phi$ is basis then the isomorphism P_θ maps basis into basis, so its range is the whole K_θ and Lemma 1, 2. applies. The converse is trivial.

c) From the trivial identity $(P_\theta\varphi, \psi) = (\varphi, P_\pi\psi)$, $\varphi\in K_\pi$ $\psi\in K_\theta$ it follows that $(P_\theta{:}K_\pi\to K_\theta)^* = P_\pi{:}K_\theta\to K_\pi$. Hence $P_\theta\Phi$ is complete $\Leftrightarrow \overline{P_\theta K_\pi} = K_\theta \Leftrightarrow P_\pi{:}K_\theta\to K_\pi$ is injective $\Leftrightarrow K^\perp_\pi \cap K_\theta = \{0\}$.

d) If $P_\theta\Phi$ is minimal then the system $\{\Phi_\lambda\} = \{x_\lambda E_\lambda\}$ is also minimal, and its biorthogonal system $\{\Psi_\lambda\}$ is also complete, see (2). Consequently

$$\bigcap_{\lambda\in\sigma} \vee(\Phi_{\lambda'}{:}\lambda'\neq\lambda,\ \lambda'\in\sigma) = \{0\}, \tag{5}$$

because any function from this intersection is orthogonal to all Ψ_λ. Suppose indirectly $K^\perp_\theta \cap K_\pi \neq \{0\}$ i. e. that there exists $\varphi\in K_\pi$, $\varphi \neq 0$ with $P_\theta\phi = 0$. By (5) the exists $\lambda\in\sigma$ and $e\in E$, $c \neq 0$ with $\varphi - cx_\lambda e\in\vee(\Phi_{\lambda'}{:}\lambda \neq \lambda')$, then

$$P_\theta x_\lambda \cdot e\in\vee(P_\theta\Phi_{\lambda'}{:}\lambda\neq\lambda')$$

which contradicts the minimality of $P_\theta\Phi$.

Remark. Denote $\Psi_\theta = \{\psi_{\theta,\lambda}\}$ the biorthogonal system of $P_\theta\Phi$. It is known [13] that

$$P_\pi\psi_{\theta,\lambda} = \psi_\lambda. \tag{6}$$

Indeed, $P_\theta\Phi_\lambda \perp \psi_{\theta,\lambda}$ implies $\Phi_\lambda \perp P_\pi\psi_{\theta,\lambda}$ hence $P_\pi\psi_{\theta,\lambda} \subset \psi_\lambda$ follows. The other implication follows from the equality of their dimensions, $\dim \psi_\lambda = \dim \Phi_\lambda = \dim P_\theta\Phi_\lambda = \dim \psi_{\theta,\lambda} = \dim P_\pi\psi_{\theta,\lambda}$. Here we used the injectivity of P_θ and P_π proved above. We state the following

Problem 1. If for some Φ, $P_\theta\Phi$ is complete and minimal in K_θ, is the biorthogonal system Ψ_θ complete? If the answer is yes we can eliminate from Theorem B the condition that Ψ_θ be complete.

Lemma 3 ([5]). The equalities

$$\begin{aligned} \varphi(K_\pi, K^\perp_\theta) &= \varphi(\Pi H^2_+, \Theta H^2_-), \\ \varphi(K^\perp_\pi, K_\theta) &= \varphi(\Pi H^2_-, \Theta H^2_+) \end{aligned} \tag{7}$$

hold.

Proof of Theorem A.

1. We shall prove that the system

$$\psi_{\theta,\lambda}(z) = \frac{F(z)}{z-\lambda} E^\lambda,\ E^\lambda := \operatorname{Ker} F(\lambda),\ \lambda\in\sigma \tag{8}$$

is biorthogonal to $P_\theta\Phi$. Indeed, using the factorizations of the generating

function F we get that

$$\frac{F(z)}{z-\lambda} E^{\lambda} \subset H_+^2 \cap \Theta H_-^2 = K_\Theta.$$

Let further $e_\lambda \in E_\lambda$, $e^\mu \in E^\mu$, then for $\lambda \neq \mu$ we have

$$\left\langle x_\lambda e_\lambda, \frac{F(z)}{z-\lambda} e^\mu \right\rangle = 2\pi i \left\langle e_\lambda, \frac{F(z)}{z-\lambda} e^\mu \right\rangle =$$

$$= 2\pi i \left\langle F^*(\lambda) e_\lambda, \frac{e^\mu}{\lambda-\mu} \right\rangle = 0$$

and for $\lambda = \mu$

$$\left\langle x_\lambda e_\lambda, \frac{F(z)}{z-\lambda} e^\lambda \right\rangle = 2\pi i \langle e_\lambda, F'(\lambda) e^\lambda \rangle.$$

We have only to show that for every $e_\lambda \in E_\lambda$ there exists $e^\lambda \in E^\lambda$ satisfying $\langle e_\lambda, F'(\lambda) e_\lambda \rangle \neq 0$. Suppose indirectly that for some e_λ, $e_\lambda \perp F'(\lambda) E_\lambda$ i. e.

$$F'(\lambda) E^\lambda \cap (E \ominus E_\lambda) = F'(\lambda) F^\lambda \cap F(\lambda) E \neq \{0\}. \tag{9}$$

Then for some $e^\lambda \in E^\lambda$ and $g \in E$: $0 = F'(\lambda) e^\lambda + F(\lambda) g$. But this is impossible for Blaschke — Potapov products and hence it is also contradictory for F as well. So (9) holds indeed and 1. is proved.

2. By Lemma 1, P_Θ: $K_\Pi \to K_\Theta$ is an (onto) isomorphism if and only if $\phi(K_\Pi^\perp, K_\Theta) > 0$ and $\phi(K_\Theta^\perp, K_\Pi) > 0$, and here it is enough to consider $0 < \phi(K_\Pi, K_\Theta^\perp) = \phi(\Pi H_+^2, \Theta H_-^2)$ since in this case $L^2 = \Pi H_+^2 \dotplus \Theta H_-^2$ because the density of this direct sum follows from the part 1., already proved. Hence

$$\phi(K_\Pi, K_\Theta^\perp) = \phi(\Pi H_+^2, \Theta H_-^2) = \phi(\Pi H_-^2, \Theta H_+^2) = \phi(K_\Pi^\perp, K_\Theta) > 0$$

(see [10] p. 410.). Consider the sets

$$D_+ := \left\{ \frac{e}{z+i} \left(\frac{z-i}{z+i} \right)^n : n \geq 0,\ e \in E \right\},$$

$$D_- := \left\{ \frac{e}{z-i} \left(\frac{z+i}{z-i} \right)^n : n \geq 0,\ e \in E \right\}.$$

We know that

$$\overline{FD_+} = \Pi H_+^2, \ \overline{FD_-} = \Theta H_-^2. \tag{10}$$

Consider the operator FP_F^{-1}: $FD_+ \dot{+} FD_- \to L^2$, where $P_$ is the orthoprojection $P_$: $L^2(\mathbf{R}, E) \to H^2_-(E)$. It is easy to see that $FP_F^{-1}|_{FD_+} = 0$, $FP_F^{-1}|_{FD_-} = I|_{FD_-}$. Using (10) we see that $\phi(K_\Pi, K_\theta^\perp) > 0$ if and only if FP_F^{-1} can be continuously extended to $H^2_+ \dot{+} H^2_- = L^2$. This latter is equivalent to the continuity of $P_: D_+ \dot{+} D_- \to L^2_{F^*F}$. Since the Hilbert operator satisfies

$$P_ = I + iH \tag{11}$$

hence 2. will be proved if we show that $D_+ \dot{+} D_-$ is dense in $L^2_{F^*F}$, or in other words

$$FD_+ \dot{+} FD_- \text{ is dense in } L^2(\mathbf{R}, E). \tag{12}$$

But this follows from (10) and we are ready.

3. is a trivial consequence of Lemma 2. Theorem A is proved.

Remark. In the scalar case ($E = \mathbf{C}$) the necessary and sufficient condition of the boundedness of H in a weighted space L^2_w is known, this is the Helson — Szegő condition, or, what is the same, the Muckenhoupt condition (see [2]). If $\dim E > 1$ then such an equivalent of the boundedness of H is not known yet; some results of this type can be found in [5].

Proof of Theorem B.

Observe first that for all $z_0 \in \mathbf{C} \setminus \sigma$ we have

$$\bigvee_{\lambda \in \sigma} \Psi_{\theta, \lambda}(z_0) = E. \tag{13}$$

Indeed, in the opposite case the function $P_\theta \dfrac{e}{z - \bar{z}_0}$, $e \in E \ominus \bigvee_{\lambda \in \sigma} \Psi_{\theta, \lambda}(z_0)$ would be orthogonal to Ψ_θ but this contradicts the completeness of Ψ_θ. By (13) there exists a basis $\{\Psi_{\theta, \lambda_r}(z_0)\}_{r=1}^N$ in E. Define F by (4), then $\det F$ will not be identically zero. Since Θ is entire, the elements of K_θ are entire functions [11], therefore F is entire as well. Now verify that the zero set of $\det F$ is σ and

$$E_\lambda = \operatorname{Ker} F^*(\lambda). \tag{14}$$

If $\phi \in K_\theta$, $g \in E$ then $\langle x_\lambda g, \phi \rangle = 2\pi i \langle g, \phi(\lambda) \rangle$, consequently $0 = \langle x_\mu e_\mu, \Psi_{\theta, \lambda} \rangle = 2\pi i \langle e_\mu, \Psi_{\theta, \lambda}(\mu) \rangle (\lambda \neq \mu, \lambda, \mu \in \sigma)$. Using the equality

$$F^*(z)e = \Sigma_{r=1}^N \langle e, \Psi_{\theta, \lambda_r}(z) \rangle (\overline{z - \lambda_r}) \xi_r$$

we get that $E_\lambda \subset \operatorname{Ker} F^*(\lambda)$. To see the converse, define

$$\chi_\lambda(z) := \frac{F(z)}{z - \lambda} E^\lambda,$$

$E^\lambda = \operatorname{Ker} F(\lambda)$ and

$$\chi(z) = \frac{F(z)}{z-\lambda} e^\lambda,$$

$e^\lambda \in E^\lambda$. Since

$$\langle x_\mu e_\mu, \chi_\lambda \rangle = 2\pi i \left\langle e_\mu, \frac{F(\mu)}{\mu-\lambda} e^\lambda \right\rangle = 0 (\lambda \neq \mu,\ \lambda,\ \mu \in \sigma),$$

hence $\chi_\lambda \subset \Psi_{\theta,\lambda}$. Now we have $\dim E_\lambda \leq \dim \operatorname{Ker} F^*(\lambda) = \dim \operatorname{Ker} F(\lambda) = \dim \chi_\lambda \leq \dim \Psi_{\theta,\lambda} = \dim E$ which proves (14). The step $\dim \operatorname{Ker} F(\lambda) = \dim \chi_\lambda$ follows from the fact that

$$\operatorname{Ker} F(\lambda) \cap \operatorname{Ker} F'(\lambda) = \{0\}. \tag{15}$$

But F is a strong H^2_+—function so it has a factorization $F = F_1 \Pi_1$ and if e is a vector from the left of (15) then $0 = F'(\lambda)e = F_1'(\lambda)\Pi_1(\lambda)e + F_1(\lambda)\Pi_1'(\lambda)e = F_1(\lambda)\Pi_1'(\lambda)e$ and $F_1(\lambda)\Pi_1'(\lambda)$ is isomorphic on $\operatorname{Ker} F(\lambda)$. So (15) hence (14) is verified. Now suppose that $g \in \operatorname{Ker} F^*(\mu)$, $\mu \notin \sigma$. Then $x_\mu g \perp \frac{F(z)}{z-\lambda} E^\lambda = \chi_\lambda$, $\lambda \in \sigma$ which means by the completeness of the biorthogonal system $\chi_\lambda = \Psi_{\theta,\lambda}$ that $P_\theta x_\mu g = 0$. We know [3] that $P_\theta x_\mu g = (I - \theta(z)\theta^*(\mu)) \frac{g}{z-\bar{\mu}}$. By (3)

$$\left| \theta(z)\theta^*(\mu) \frac{g}{z-\bar{\mu}} \right| \leq c e^{-\alpha y},$$

$z = iy$, $y \to \infty$ and $g/(z-\bar{\mu})$ does not decrease exponentially along the half-axis unless $g = 0$. So the nullset of $\det F$ is σ indeed. Since F is strong H^2_+—function and $\theta^{-1} F$ is strong H^2_- — function by (4), we have $F = \Pi \Theta_1 F_e^+$ and $\theta^{-1}(z)F(z) = \overline{\Theta_2(\bar{z})F_e^-(\bar{z})}$. We have to show that Θ_1, Θ_2 are constant unitary operators. From the decomposition $\ln|\det F| = \ln|\det \pi| + \ln|\det \Theta_1| + \ln|\det F_e^-|$ we get on the one hand that $\ln|\det \Theta_1|$ is continuous on the closed half-plane $\mathbf{C}_+$ at every point of $\mathbf{R}$ because the other three members have this property. Consequently from the scalar theory we get $\Theta_1(z) = e^{i\gamma z}$, $z \in \mathbf{C}_+$ for some $\gamma \geq 0$. On the other hand $\psi_{\theta,\lambda_r}(iy)$ does not increase exponentially as $y \to \infty$ (it tends to zero and hence $\det F$ does not increase exponentially. The same is known for the inverse of a Blaschke product [18], [19] and for the inverse of an outer function. So the inverse of $\det \Theta_1$ does not increase exponentially. This means that $\gamma = 0$, $\det \Theta_1 = \text{const}$ and then $\Theta_1 = \text{const}$ [11]. In this case Θ_1 can be joined into F_e^+ and we get $F = \Pi F_e^+$. On the lower half-plane we can similarly argue, hence Theorem B is proved. □

PROBLEM 2. Is the generating function of a fixed system $P_\theta \Phi$ unique up to a multiplicative constant N by N matrix?

LEMMA 4. Let $\Phi := \{x_\lambda e_\lambda\}$, $\hat{\Phi} := \{x_\lambda \hat{e}_\lambda\}$, $\lambda \in \sigma$. If Φ is an L-basis in H^2_+ (E) (resp. $P_\theta \Phi$ is a Riesz basis in K_θ) then $\hat{\Phi}$ is an L-basis (resp. $P_\theta \hat{\Phi}$ is a Riesz

basis) whenever $\delta(\Phi, \hat{\Phi}) := \sup_{\lambda\in\sigma} \ll e_\lambda - \hat{e}_\lambda \gg$ is small enough.

PROOF. Suppose that Φ is L-basis. The operator R: $x_\lambda e_\lambda \to x_\lambda \hat{e}_\lambda$ can be linearly extended to the linear hull of Φ. Since Φ is an L-basis, its spectrum σ is the union of N Carleson-sets [3] $\sigma = \bigcup_{j=1}^{N} \sigma_j, \sigma_j \in (C)$. Hence

$$\sum_{\lambda\in\sigma_j} |c_\lambda|^2 \asymp \left\| \sum_{\lambda\in\sigma_j} c_\lambda x_\lambda \right\|^2_{H^2_+(E)} \quad (j = 1, \ldots, N),$$

$$\left\| \sum_{\lambda\in\sigma_j} c_\lambda x_\lambda e_\lambda \right\|^2_{H^2_+(E)} \asymp \sum_{\lambda\in\sigma} |c_\lambda|^2.$$

Let $\sigma_o \subset \sigma$ be a finite set, $\sigma_{oj} := \sigma_o \cap \sigma_j$ and $f := \sum_{\lambda\in\sigma_0} c_\lambda x_\lambda e_\lambda$. Fix an orthonormal basis $\{\xi_r\}_{r=1}^N$ in E. Then

$$\left\| (I-R)f \right\|^2_{H^2_+} = \sum_{r=1}^{N} \left\| \langle \xi_r, (I-R)f \rangle \right\|^2_{H^2_+} = \sum_{r=1}^{N} \left\| \sum_{\lambda\in\sigma_o} c_\lambda x_\lambda \langle \xi_r, e_\lambda - \hat{e}_\lambda \rangle \right\|^2_{H^2_+} \leq$$

$$\leq \sum_{r=1}^{N} N \sum_{j=1}^{N} \left\| \sum_{\lambda\in\sigma_{oj}} c_\lambda x_\lambda \langle \xi_r, e_\lambda - \hat{e}_\lambda \rangle \right\|^2_{H^2_+} \leq,$$

$$\leq C \sum_{r=1}^{N} \sum_{j=1}^{N} \sum_{\lambda\in\sigma_{0j}} c_\lambda \langle \xi_r, e_\lambda - \hat{e}_\lambda \rangle \Big\|^2_{H^2_+} \leq C\delta(\Phi, \hat{\Phi}) \sum_{\lambda\in\sigma_0} |c_\lambda|^2 =$$

$$= C\delta(\Phi, \hat{\Phi}) \left\| \sum_{\lambda\in\sigma_0} c_\lambda x_\lambda e_\lambda \right\|^2_{H^2_+(E)}.$$

If $\delta(\Phi, \hat{\Phi})$ is small enough then $\|I-R\| < 1$ which ensures that R is isomorphic, so $R\Phi = \hat{\Phi}$ is also L-basis. Now suppose that $P_\Theta \Phi$ is Reisz basis in K_Θ. Then Φ is Reisz basis in K_Π and P_Θ: $K_\Pi \to K_\Theta$ is isomorphic. If $\delta(\Phi, \hat{\Phi})$ is small enough, then $P_\Theta R$: $K_\Pi \to K_\Theta$ is also isomorphic, since $P_\Theta R f = P_\Theta f + P_\Theta (R-I) f$. So the operator $R_\Theta := P_\Theta R [P_\Theta|_{K_\Pi}]^{-1}$: $K_\Theta \to K_\Theta$ is isomorphic and maps $P_\Theta \Phi$ to $P_\Theta \hat{\Phi}$, hence $P_\Theta \hat{\Phi}$ is Riesz basis indeed.

PROOF OF THEOREM C.

a) follows directly from the above Lemma 4.

b) Let $f \in K_{\Theta_{-\epsilon}}$ and $f \perp P_{\Theta_{-\epsilon}}\hat{\Phi}$; we have to show that $f = 0$. From $K_{\Theta_{-\epsilon}} \subset K_\Theta$ it follows that $0 = \langle f, P_{\Theta_{-\epsilon}}\hat{\phi}_n\rangle = \langle f, \hat{\phi}_n\rangle = \langle f, P_\Theta\hat{\phi}_n\rangle$ so f is orthogonal to $P_\Theta\Phi'_Q$, $\Phi'_Q := \{\hat{\phi}_n\}_{Q+1}^\infty$. Consequently f belongs to the linear hull of the first Q elements of the system $\{\psi_j^Q\}_{j=1}^\infty$ biorthogonal to the Riesz basis $P_\Theta\Phi_Q$. Let F_Q be the generating function of $P_\Theta\Phi_Q$, then $f = F_Q\tilde{f}$,

$$\tilde{f} = \sum_{n=1}^{Q} c_n \frac{g_n}{z-\lambda_n},$$

$g_n \in \operatorname{Ker} F_Q(\lambda_n)$ by (8). Since $F_Q = \Theta F^-_{Q,\epsilon}$ hence $e^{iz\epsilon} F^-_{Q,\epsilon}\tilde{f} = e^{iz\epsilon}\Theta^{-1}f \in H^2_-(E)$. We know that the inverse of an outer function can not increase exponentially along the half-lines parallel to the imaginary axis. More precisely for $x \in \mathbf{R}$ and $\delta > 0$ there exists $C_{\delta,x} < \infty$ with

$$\|F^-_{Q,\epsilon}{}^{-1}(x-iy)\| \le C_{\delta,x}\, e^{\delta y} (y > 0).$$

Consequently $e^{(\epsilon-\delta)y} \ll \tilde{f}(x-iy) \gg \le C_{\delta,x} \ll e^{i(x-iy)\epsilon} F^-_{Q,\epsilon}\tilde{f}(x-iy) \gg$. If $\tilde{f} \neq 0$ then $\ll \tilde{f}(x-iy) \gg$ decreases only polynomially for $y \to \infty$ and this contradicts the relation $e^{iz\epsilon}F^-_{Q,\epsilon}\tilde{f} \in H^2$. So we have $\tilde{f} = 0$ and then $f = 0$, $P_{\Theta_{-\epsilon}}\hat{\Phi}$ is indeed complete in $K_{\Theta_{-\epsilon}}$. To prove the remaining part of b), let $\Phi_1 \subset \Phi$ be a system such that $P_{\Theta_{-\epsilon}}\Phi_1$ is a Riesz basis in $K_{\Theta_{-\epsilon}}$ and let $\hat{\Phi}_1 \subset \hat{\Phi}$ be the system corresponding to Φ_1. In a) is proved that leaving from $P_{\Theta_{-\epsilon}}\hat{\Phi}_1$ finitely many elements we get an L-basis in $K_{\Theta_{-\epsilon}}$. Since $P_{\Theta_{-\epsilon}}\hat{\Phi}$ is complete in $K_{\Theta_{-\epsilon}}$, the L-basis can be completed by elements from $P_{\Theta_{-\epsilon}}\hat{\Phi}$ to a Riesz basis.

c) $P_\Theta\Phi_Q$ is Riesz basis in K_Θ hence L-basis in K_{Θ_ϵ}. We can suppose that $\phi_n \neq c\hat{\phi}_n$ $(n = 1, \ldots, Q)$. First we extend $P_\Theta\Phi_Q$ to a system minimal in K_{Θ_ϵ}. To this we define the function $f_\epsilon(z) := e^{i\epsilon z/2}\sin\dfrac{\epsilon(z-i\beta)}{2}$ where $\beta > 0$ is chosen such that the nullset of f_ϵ be disjoint from σ, the spectrum of Φ. Let further $\Phi_\epsilon^{(0)} := \{x_\lambda E : f_\epsilon(\lambda) = 0\}$, $\Phi_\epsilon := \Phi_Q \cup \Phi_\epsilon^{(0)}$. Since f_ϵ has the factorization $f_\epsilon = B f_e^+ = e^{i\epsilon z} f_e^-$ where B is the Blaschke product corresponding to the zero set of f_ϵ, we see that $F_\epsilon := F_Q f_\epsilon$ is the generating function of $P_{\Theta_\epsilon}\Phi_\epsilon$. By Theorem A 1. $P_{\Theta_\epsilon}\Phi_\epsilon$ is minimal. We show next that $P_{\Theta_\epsilon}\Phi''_Q$ is linearly independent, where $\Phi''_Q := \{\hat{\phi}_n\}_1^Q$. Indeed, let

$$\sum_{n=1}^{Q} c_n P_{\Theta_{-\epsilon}} \frac{\hat{e}_n}{z-\bar{\lambda}_n} = 0 \quad \text{then} \quad \sum_{n=1}^{Q} c_n \frac{\hat{e}_n}{z-\hat{\lambda}_n} \in \Theta_\epsilon H^2_+$$

hence

$$e^{-i\epsilon z}\sum_{n=1}^{Q} c_n \frac{\hat{e}_n}{z-\bar{\lambda}_n} \in H_+^2.$$

This is a contradiction unless $c_n = 0$, $n = 1, \ldots, Q$, because the left hand side has an exponential growth along the imaginary axis. Hence $P_{\Theta_\epsilon}\hat{\Phi}$ will be L-basis in K_{Θ_ϵ} if we verify that $\vee(P_{\Theta_\epsilon}\Phi_Q') \cap \vee(P_{\Theta_\epsilon}\Phi_Q'') = \{0\}$, $\Phi_Q' := \{\hat{\phi}_n\}_{n=Q+1}^{\infty}$. Suppose that

$$f = \sum_{n=1}^{Q} c_n P_{\Theta_\epsilon}\hat{\phi}_n = \sum_{n=Q+1}^{\infty} c_n P_{\Theta_\epsilon}\hat{\phi}_n,$$

we have to prove $f = 0$. As we have seen in proving Theorem A 1. the sets

$$\Psi_m := \frac{F_\epsilon(z)}{z-\mu_m}E, \ \mu_m \in \sigma(f_\epsilon)$$

are the members of the system biorthogonal to $P_{\Theta_\epsilon}\Phi_\epsilon$ corresponding to $\Phi_\epsilon^{(0)}$. The function $f \in \vee\Phi_Q'$ is orthogonal to all Ψ_m, i.e.

$$0 = \sum_{n=1}^{Q} c_n \left\langle P_{\Theta_\epsilon}\hat{\phi}_n, \frac{F_\epsilon(z)}{z-\mu_m}\right\rangle = \sum c_n \left\langle \hat{\phi}_n, \frac{F_\epsilon(z)\xi_j}{\lambda_n-\mu_m}\right\rangle = \sum c_n \left\langle F_\epsilon^*(\lambda_n)\hat{e}_n, \frac{\xi_j}{\lambda_n-\mu_m}\right\rangle$$

for all j and m where $\{\xi_j\}_1^N$ is a fixed orthonormal basis of E. Hence

$$\sum_{n=1}^{Q} c_n \frac{F_\epsilon^*(\lambda_n)\hat{e}_n}{\lambda_n-\mu_m} = 0, \ \mu_m \in \delta(f_\epsilon).$$

Recall the following lemma, proved by Benzinger [12]: If $\lambda_1, \ldots, \lambda_Q$, $\mu_1, \ldots, \mu_Q$ are arbitrary different complex numbers then

$$\det\left(\frac{1}{\lambda_n-\mu_m}\right)_{n,m=1}^{Q} \neq 0.$$

Using this we get that $c_n F_\epsilon^*(\lambda_n)\hat{e}_n = 0$, $n = 1, \ldots, Q$. From $\operatorname{Ker} F_\epsilon^*(\lambda_n) = \langle e_n \rangle \ni \hat{e}_n$ we finally obtain $c_n = 0$, $n = 1, \ldots, Q$. Theorem C is proved.

Problem 3. Can we assert analogous statements without assuming that $\lambda_n \neq \lambda_k$ if $n \neq k$?

Problem 4. Is it true in c) that $P_{\Theta_\epsilon}\hat{\Phi} \cup \{P_{\Theta_\epsilon}\phi_0^{(0)}\}$, $\phi_0^{(0)} = e_0(z-\bar{\lambda}_0)^{-2}$ is L-basis in K_{Θ_ϵ}? A weaker result will be proved in the following paper of the authours.

In Theorem C, b) the question arises: can we extract from $P_{\Theta_{-\epsilon}}\hat{\Phi}$ a Riesz basis in $K_{\Theta_{-\epsilon}}$? A positive answer can be given if the spectrum σ has a special structure. First we consider the case dim E = 1. It is more convenient here return to exponentials from their Fourier transforms.

LEMMA 5. Suppose $\delta_n \to 0 (n \to \pm\infty)$ and $n+\delta_n \neq m+\delta_m (n \neq m)$. Then

a) The system $\tilde{\Phi} = \{e^{i(n+\delta_n)x}\}_{-\infty}^{\infty}$ is Riesz basis in $L^2(0, 2\pi)$.

b) For $a \in (0, 1)$ we have a decomposition $\Phi = \bigcup_{j=0}^{M} \Phi_j$ where Φ_0 is Riesz basis, $\Phi_j (j = 1, \ldots, M)$ are L-bases in $L^2(0, 2\pi a)$ and $M < \infty$.

For the proof we need some notions.

DEFINITION 4 (Levin, see e.g. [2]).
The entire function f of exponential type is called to be sine type if

1. The roots $\{\lambda_n\}$ of f ly in some horizontal strip $\{z \in C: |\operatorname{Im} z| \leq h\}$, $h < \infty$,

2. For some fixed y_0, $|f(x+iy_0)| \asymp 1$, $x \in \mathbf{R}$.

DEFINITION 5 ([16]). Let $\Lambda = \{\lambda_n\}$ be a set on $\mathbf{C}$ with $|\operatorname{Im} \lambda_n| \leq h < \infty$. If there exists an increasing sequence $\{\alpha_j\} j \in \mathbf{Z}$ of real numbers and $\Lambda_j := \{\lambda_n \in \Lambda: \alpha_j \leq \operatorname{Re} \lambda_n < \alpha_{j+1}\}$, $l_j := \alpha_{j+1} - \alpha_j \leq C$, then the partition $\Lambda = \bigcup_j \Lambda_j$ is called to be a J-partition of Λ.

Proposition 6 ([16]). Let $\Lambda = \{\lambda_r\}$ be a nullset of a sine type function f and suppose that the width of the indicator diagram of f is b. Let $\{q_r\} \subset \mathbf{C}$ be a bounded sequence such that for some J-partition of Λ

$$|\sum_{\lambda_r \in \Lambda_j} \operatorname{Re} q_r| \leq q l_j, \quad j \in \mathbf{Z}, \quad q < 1/4. \tag{16}$$

If $\{\lambda_r + q_r\}$ is separable then $\{e^{i(\lambda_r+q_r)x}\}$ is Riesz basis in $L^2(0, b)$.

PROOF OF LEMMA 5.

a) follows directly from Proposition 6 if we take $f(z) = \sin \pi z$ and choose $|\alpha_0|$, α_1 sufficiently large, $\alpha_0 < 0 < \alpha_1$.

b) We construct Φ_0 starting from $\{e^{irx/a}\}_{-\infty}^{\infty}$ given by the zeros of $\sin a\pi z$; it is a basis in $L^2(0, 2\pi a)$. Let $n_r \in \mathbf{Z}$ be given such that $|r/a - n_r| = \min \{|r/a - n|: n \in \mathbf{Z}\}$. If for some a $\{r/a\} = 1/2$ occurs then we define $n_r = r/a \pm \pm 1/2$ alternatively in the successive occurrences. Let $q_r^{(0)} = n_r - r/a$. If a is rational then $\Sigma q_r^{(0)} = 0$ for a double period. If a is irrational then $\{r/a\}$ has uniform distribution mod 1. Using e.g. the Weyl integral criterium [17] we get that for any $q > 0$ (16) can be obtained whenever l_j is large enough. Let now $q_r = n_r - r/a + \delta_r$, this is the perturbation of the original exponents r/a to obtain $n_r + \delta_r$. Since $\delta_r \to 0$, the same J-partition applies as for $q_r^{(0)}$, only some blocks Λ_j are to be unified around 0. Proposition 6 shows that the system

$\Phi_0 = \{e^{i(n_r+\delta_r)x}\}$ is Riesz basis in $L^2(0, 2\pi a)$. Now let $M>1/a$, $M\in\mathbf{N}$. Of course, any of the systems $\{e^{i(p+mM)x} : m\in\mathbf{Z}, p=1, \ldots, M\}$ is basis in $L^2\left(0, \frac{2\pi}{M}\right)$ and as in a) we can show that the system $\{e^{i(p+mM+\delta_{p+mM})x}\}_{m\in\mathbf{Z}}$, $p= 1, 2, \ldots, M$ have the same property, and hence there are L-basis in $L^2(0, 2\pi a)$. Since these systems cover the whole $\hat{\Phi}$, the proof is ready. □

We get then the corresponding result for vector—exponentials.

THEOREM D. *Let* $\lambda_n^{(j)} = a_j\, n+b_j$, $a_j>0$, $b_j\in\mathbf{C}$ *and* $\{\delta_n^{(j)}\}\subset\mathbf{C}$ *be sequences tending to* 0 *as* $n\to\pm\infty$ *and suppose* $\lambda_n^{(j)}+\delta_n^{(j)}\neq\lambda_m^{(j)}+\lambda_m^{(j)}$, $n\neq m$, $j = 1, \ldots, N$. *Then we have*

a)
$$\Phi := \bigcup_{j=1}^{N} \Phi_j,\ \Phi_j := \{x_{\lambda_n^j+\delta_n^j}\xi_j\}_{n\in\mathbf{Z}}$$

is a basis in K_Θ, $\Theta := diag\ [e^{iz2\pi/a}]$. *Here* $\{\xi_j\}$ *is an orthonormal basis in* E.

b) If $\langle\langle e_n^j-\xi_j\rangle\rangle\to 0$ $(n\to\pm\infty)$ *then for any* $0<\epsilon<\min 2\pi/a_j$ *there exists a basis* $P_{\Theta_j}\hat{\Phi}_0$,

$$\hat{\Phi}_0\subset\hat{\Phi} := \{x_{\lambda_n^{(j)}+\delta_n^{(j)}}\, e_n^{(j)} : j = 1, \ldots, N,\ n\in\mathbf{Z}\}$$

such that $\hat{\Phi}\backslash\hat{\Phi}_0$ *is infinite.*

PROOF.

a) Let $\Phi_j := \mathrm{diag}\ [1, \ldots, 1, \exp iz2\pi/a_j, 1, \ldots, 1]$, then $\Theta H_+^2 = \bigcap_{j=1}^{N} \Phi_j H_+^2$ hence $K_\Theta = \vee\{K_{\Theta_j} : j = 1, \ldots, N\}$. *Let* $\vartheta_j := \exp\ [iz\ 2\pi/a_j]$, then $K_{\Theta_j} = (0, \ldots, 0, K_{\vartheta_j}, 0, \ldots, 0)$, $P_{\Theta_j} = \mathrm{diag}\ [0, \ldots, 0, P_{\vartheta_j}, 0, \ldots, 0]$, $K_\Theta = (K_{\vartheta_1}, \ldots, K_{\vartheta_N})$, $P_\Theta = \mathrm{diag}\ [P_{\vartheta_1}, \ldots, P_{\vartheta_N}]$ which shows that $P_\Theta\Phi$ is Riesz basis in K_Θ if and only if $K_{\Theta_j}\Phi_j$ are Riesz bases in K_{Θ_j}. Now Proposition 6 implies that $K_{\Theta_j}\Phi_j$ are bases hence a) is proved.

b) Using again that $P_{(\Theta_j)_{-\epsilon}}\Phi_j = K_{\Theta_{-\epsilon}}\Phi_j$ we can apply the previous Lemma in each coordinata to show that there is a system $\Phi_0\subset\Phi$ such that $P_{\Theta_{-\epsilon}}\Phi_0$ is Riesz basis in $K_{\Theta_{-\epsilon}}$. It can be seen from the proof that $\Phi\backslash\Phi_0$ is infinite. Now Theorem C, b) guarantees a subsystem $\hat{\Phi}_0\subset\hat{\Phi}$ such that $P_{\Theta_{-\epsilon}}\hat{\Phi}_0$ is also a Riesz basis and, as we have seen while proving it, $\hat{\Phi}_0$ differs from the perturbed system of Φ_0 in finitely many elements. Consequently $\hat{\Phi}\backslash\hat{\Phi}_0$ is also infinite. The Theorem is proved. □

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ON RIESZ BASES FROM VECTOR EXPONENTIALS II.

By

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In this paper we continue the investigations of [1] on vector exponentials. In [1] is given the notion of the generating function for vector—valued case, the equivalency of the basis property and of the boundedness of the Hilbert operator in the appropriately weighted L^2 — space further some stability properties. This paper has less theoretical character. First we obtain an efficient basis test of vector exponentials on $(0, \infty)$ then we present an example how the vector exponentials arise while investigating the control problem of a string controlled at the endpoints.

We recall some notions and known theorems. First let $\lambda \in \mathbf{C}_+ := \{z \in \mathbf{C}: \operatorname{Im} z > 0\}$,

$$\chi_\lambda(z) := \sqrt{\frac{\operatorname{Im} \lambda}{2}} \frac{1}{z-\bar{\lambda}}, \quad z \in \mathbf{C}_+;$$

this is the Fourier transform of $\sqrt{2 \operatorname{Im} \lambda}\, e^{-i\bar{\lambda}x} \in L^2(0, \infty)$. As the Paley – Wiener theorem states, the Fourier transform of $L^2(0, \infty)$ is the Hardy space, H^2_+. It is known [2] that

1. The set $\{x_\lambda : \lambda \in \sigma\}$, $\sigma \subset \mathbf{C}_+$ is minimal if and only if the Blanschke condition

$$(B) \qquad \sum_{\lambda \in \sigma} \frac{\operatorname{Im} \lambda}{1+|\lambda|^2} < \infty$$

holds.

2. The set $\{x_\lambda\}$ is uniformly minimal if and only if the Carleson condition

$$(C) \qquad \lim_{\lambda\in\sigma} \prod_{\mu\in\sigma,\ \mu\neq\lambda} \left| \frac{\lambda-\mu}{\lambda-\bar{\mu}} \right| > 0$$

holds.

3. If $\{x_\lambda\}$ is uniformly minimal then it is L—basis (Riesz basis in its closed linear hull).

4. If $\sigma\subset B_+ := \{z\in\mathbf{C}: 0<\alpha\leq \operatorname{Im} z\leq\beta<\infty\}$, then (C) is equivalent to the gapness condition

$$(G) \qquad \inf_{\lambda,\ \mu\in\sigma,\ \lambda\neq\mu} \left| \frac{\lambda-\mu}{\lambda-\bar{\mu}} \right| > 0$$

and also to the separability condition

$$(S) \qquad \inf_{\lambda,\ \mu\in\sigma,\ \lambda\neq\mu} |\lambda-\mu| > 0.$$

Consider a finite dimensional Hilbert space E, $\dim E<\infty$. Given a set $\sigma\subset\mathbf{C}_+$ define for $\lambda\in\sigma$ a subspace $E_\lambda\subset E$ and consider the system $\Phi = \{x_\lambda E_\lambda : \lambda\in\sigma\}$. In [2] p. 211—212 is proved that

5. If Φ is uniformly minimal in $H^2_+(E)$ then Φ is L—basis if and only if σ satisfies the Carleson—Newman condition (CN) (which means that σ is the finite union of Carleson sets).

6. If Φ is uniformly minimal then σ is the union of $\dim E$ many gap sets.

7. If $\sigma\subset B_+$ then every uniformly minimal set $\Phi = \{x_\lambda E_\lambda\}$ is L—basis and σ is the union of N Carleson sets.

Introduce the hyperbolic distance of λ, $\mu\in\mathbf{C}_+$, denoted by $\rho(\lambda, \mu)$ by the equality $\operatorname{th}\frac{1}{2}\rho(\lambda, \mu) = \left|\frac{\lambda-\mu}{\lambda-\bar{\mu}}\right|$ and denote $D_\lambda(r) := \{\mu\in\mathbf{C}_+ : \rho(\lambda, \mu)<r\}$. We shall prove first the estimate

$$(1) \qquad \|x_\lambda - x_\mu\| \leq \rho(\lambda, \mu);\ \lambda, \mu\in\mathbf{C}_+.$$

Indeed, by

$$\langle x_\lambda, x_\mu\rangle = 2i\,\frac{\sqrt{\operatorname{Im}\lambda\operatorname{Im}\mu}}{\mu-\bar{\lambda}}$$

we get

we get

$$\|x_\lambda - x_\mu\|^2 = 2(1 - Re\langle x_\lambda, x_\mu\rangle) = 2\left(1 - 2\,\mathrm{Re}\, i\frac{\sqrt{\mathrm{Im}\,\lambda\,\mathrm{Im}\,\mu}}{\mu-\bar{\lambda}}\right) =$$

$$= 2\left(1 + 2\frac{\sqrt{\mathrm{Im}\,\lambda\,\mathrm{Im}\,\mu}\,\mathrm{Im}\,(\bar{\mu}-\lambda)}{|\mu-\bar{\lambda}|^2}\right) = 2\left(1 - 2\frac{\sqrt{\mathrm{Im}\,\lambda\,\mathrm{Im}\,\mu}\,(\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)}{|\mu-\bar{\lambda}|^2}\right) =$$

$$= \frac{2}{|\mu-\bar{\lambda}|^2}\left[(Re\mu - Re\lambda)^2 + (\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)^2 - 2\sqrt{\mathrm{Im}\,\mu\,\mathrm{Im}\lambda}\,(\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)\right] =$$

$$= \frac{2}{|\mu-\bar{\lambda}|^2}\left[(Re\mu - Re\lambda)^2 + (\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)(\sqrt{\mathrm{Im}\,\mu} - \sqrt{\mathrm{Im}\,\lambda})^2\right].$$

Now from

$$(\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)(\sqrt{\mathrm{Im}\,\mu} - \sqrt{\mathrm{Im}\,\lambda})^2 = \frac{(\mathrm{Im}\,\mu + \mathrm{Im}\,\lambda)(\mathrm{Im}\mu - \mathrm{Im}\,\lambda)^2}{(\sqrt{\mathrm{Im}\,\mu} + \sqrt{\mathrm{Im}\,\lambda})^2}$$

it follows that

$$\|x_\mu - x_\lambda\|^2 \le \frac{2}{|\mu-\bar{\lambda}|^2}\left[(Re\mu - Re\lambda)^2 + (\mathrm{Im}\,\mu - \mathrm{Im}\,\lambda)^2\right] = 2\left|\frac{\mu-\lambda}{\mu-\bar{\lambda}}\right|^2$$

and

$$\|x_\mu - x_\lambda\|^2 \ge \frac{2}{|\mu-\bar{\lambda}|^2}\left[(Re\mu - Re\lambda)^2 + \frac{1}{2}(\mathrm{Im}\,\mu - \mathrm{Im}\,\lambda)^2\right] \ge \left|\frac{\mu-\lambda}{\mu-\bar{\lambda}}\right|^2$$

i.e.

$$(1') \qquad \left|\frac{\mu-\lambda}{\mu-\bar{\lambda}}\right| \le \|x_\lambda - x_\lambda\| \le 2\left|\frac{\mu-\lambda}{\mu-\bar{\lambda}}\right|.$$

Now since $2\,\mathrm{arth}\,x = \ln\frac{1+x}{1-x} \ge 2x,\ 0 \le x \le 1$ we get

$$\|x_\mu - x_\lambda\| \le 2\,\mathrm{arth}\left|\frac{\mu-\lambda}{\mu-\bar{\lambda}}\right| = \rho\,(\lambda, \mu)$$

and (1) is proved.

Remark that $\left|\dfrac{\mu-\lambda}{\mu-\bar{\lambda}}\right| \leq \delta < 1$ implies $\rho(\lambda, \mu) \leq c(\delta)\|x_\lambda - x_\mu\|$. Now let σ be the union of N Carleson sets. It may happan that $\sigma \notin (C)$, hence $\{x_\lambda\}$ is not an L-basis. However, if we join the functions x_λ with "very close" values λ then the generated subspaces give an L-basis:

LEMMA 1. ([2] p. 296). Let σ be the union of N Carleson sets σ_j, $j = 1, \ldots, N$ and let $G_m(r)$ be the topological components of $\bigcup_{\lambda\in\sigma} D_\lambda(r)$. Define further $\Lambda_m := \{\lambda\in\sigma : \lambda\in G_m(r)\}$,

$$L_m(r) := \bigvee_{\lambda\in\Lambda_m(r)} x_\lambda.$$

Then the set $\{L_m(r)\}$ is an L-basis and $\dim L_m(r) \leq N$ if

$$r < r_0 := \frac{1}{2N} \min_j \inf_{\lambda,\, \mu\in\sigma_j} \rho(\lambda, \mu).$$

Consider the operators

$$\Gamma_m = \Gamma_m(r) : \bigoplus_{\lambda\in\Lambda_m(r)} E_\lambda \to \bigoplus_{\lambda\in\Lambda_m(r)} E_\lambda$$

$$(e_\lambda)_{\lambda\in\Lambda_m} \to \Big(\delta_\lambda\big(\sum_{\mu\in\Lambda_m} e_\mu\big)\Big)_{\lambda\in\Lambda_m},$$

where $\delta_\lambda : E \to E$ are the orthoprojections onto E_λ.

THEOREM 1. *Let* $\sigma\in$ (CN). *Then* $\Phi = \{x_\lambda E_\lambda\}$ *is L-basis if and only if there exists* $r > 0$ *satisfying*

$$\sup_m \|\Gamma_m^{-1}(r)\| < \infty. \tag{2}$$

Here we allow that E be an infinite dimensional Hilbert space.

PROOF. Analyze the condition (2). If it holds then $\sum_{\lambda\in\Lambda_m} \|e_\lambda\|^2 = 1$ implies that $\sum_{\lambda\in\Lambda_m} e_\lambda$ is not too close to zero; this is equivalent to the condition that the angle of the subspace E_λ, $\lambda\in\Lambda_m$ and of the closed linear hull of the other subspaces $\vee(E_\mu : \mu \neq \lambda,\ \mu\in\Lambda_m)$ has a lower bound uniformly also in m. Conversely if this is satisfied then

$$\sum_{\lambda\in\Lambda_m} \|\delta_\lambda(\sum_{\lambda\in\Lambda_m} e_\mu)\|^2$$

can not be small (see the Bari's theorem in [2]) and this implies (2). So we obtained that (2) is equivalent to the uniform minimality of the system $\{E_\lambda:\lambda\in\Lambda_m\}$ in E uniformly also in m. Consequently if (2) holds for an r then it holds a fortiori for any $r'<r$. Hence we can suppose that Lemma 1 can be applied. By Lemma 1 the sytem $\{\bigvee_{\lambda\in\Lambda_m} x_\lambda E\}$ and so $\{\bigvee_{\lambda\in\Lambda_m} x_\lambda E_\lambda\}$ *is* L-basis in $H^2_+(E)$. Here we have to drop the parentheses i.e. to show the basis property of $\{x_\lambda E_\lambda;\ \lambda\in\sigma\}$. This can be done if and only if the angle between any $x_\lambda E_\lambda$ and $\vee(x_\mu E_\mu:\mu\neq\lambda,\ \mu\in\Lambda_m)$ has a lower bound independent of m. Since

$$x_\lambda e_\lambda-\sum_{\mu\neq\lambda} x_\mu e_\mu = x_\lambda(e_\lambda-\sum_{\mu\neq\lambda} e_\mu)+\sum_{\mu\neq\lambda} e_\mu(x_\lambda-x_\mu),$$

we get by (1) that (2) implies the L-basis property of Φ. Conversely if (2) do not hold, then for every $r>0$, $\epsilon>0$ there exist m, $\lambda\in\Lambda_m$ (r) and $\|e_\lambda\|=1$, $\|e_\mu\|=1$, $\lambda\neq\mu\in\Lambda_m(r)$ with

$$\|e_\lambda-\sum_{\mu\neq\lambda} e_\mu\|<\epsilon.$$

So we have

$$\|x_\lambda e_\lambda-\sum_{\mu\neq\lambda} x_\mu e_\mu\|\leq C_N\{\|e_\lambda-\sum_{\mu\neq\lambda} e_\mu\|+\rho(\lambda,\mu)\}\leq C_N\{\epsilon+2Nr\}$$

and hence is not an L-basis. Theorem 1 is proved.

COROLLARY. If $\dim E<\infty$, then Φ is L-basis in $H^2_+(E)$ if and only if $\sigma\in$ (CN) and there exists $r>0$ satisfying (2).

REMARK. Obviously, if $K\subset\mathbf{C}_+$ is a given compact set then $\rho(\lambda,\mu)\asymp|\lambda-\mu)|$; $\lambda,\mu\in K$ and these inequalities remain valid if we translate K parallel to the real axis. Consequently in case $\sigma\subset B_+$ the statements of Lemma 1 and Theorem 1 remain valid if we take not hyperbolic but euclidean disks around the points $\lambda\in\sigma$.

At last we give two systems whose basis property can be verified at once by Theorem 1. We use euclidean metric in defining $G_m(r)$. The projections δ_λ are one — dimensional hence it is more convenient to consider a system of vector functions instead of subspaces.

EXAMPLE 1. Let

$$\Phi:=\left\{\frac{1}{z-2n+i}\begin{pmatrix}1\\0\end{pmatrix}\right\}\cup\left\{\frac{1}{z-2n-1+i}\begin{pmatrix}0\\1\end{pmatrix}\right\}\cup$$

$$\cup \left\{ \frac{1}{z-n-\gamma_n+i} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\},$$

$n \in \mathbf{Z}$, $|\gamma_n| < 1/4$. Here $\sigma = \{n+i\} \cup \{n+\gamma_n+i\}$; if $r < 1/2$ then $|\Lambda_m(r)| \leq 2$ and obviously the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ has a positive lower bound.

EXAMPLE 2.

$$\Phi: = \left\{ \frac{1}{z-n+i} \begin{pmatrix} \sin \alpha_n \\ \cos \alpha_n \end{pmatrix} \right\} \cup \left\{ \frac{1}{z-n-\gamma_n+i} \begin{pmatrix} \sin (\alpha_n+\beta) \\ \cos (\alpha_n+\beta) \end{pmatrix} \right\}$$

where $\alpha_n \in \mathbf{R}$, $\beta \in (0, \pi)$ and $|\gamma_n| < 1/4$. Here $|\Lambda_m(r)| \leq 2$ if $r < 1/2$ and the corresponding vectors

$$\begin{pmatrix} \sin \alpha_n \\ \cos \alpha_n \end{pmatrix}, \begin{pmatrix} \sin (\alpha_n+\beta) \\ \cos (\alpha_n+\beta) \end{pmatrix}$$

form a constant angle β, $0 < \beta < \pi$, so the L-basis property follows.

In the second part of this paper we show the connection between control theory and vector exponentials. Consider the non — homogeneous string

$$\rho(x) y_{tt}(x, t) = y_{xx}(x, t), \; 0 < x < X, \; 0 < t < T. \tag{3}$$

Assume $\rho \in C^1[0, X]$ and $\rho(x) > 0$, $x \in [0, X]$. The system is controlled at the boundary points:

$$y_x(0, t) = u_1(t), \; y_x(X, t) = u_2(t); \; u_1 \; u_2 \in L^2(0,T) \tag{4}$$

and assume that also the initial conditions

$$y(x, 0) = y_0(x), \; y_t(x, 0) = y_1(x); \; y_0 \in W_2^1(0, X), \; y_1 \in L^2(0, X) \tag{5}$$

hold. We are looking for the generalized solutions of (3) — (5) i.e. the functions $y(x, t) \in L^2([0, X] \times [0, T])$ such that for every $z(x, t) \in C^2([0, X] \times [0, T])$, $z(.,T) = z_t(., T) = 0$, $z_x(0, t) = z_x(X, t) = 0$ the equation

$$(6)\qquad \int_0^T\int_0^X y(\rho\, z_{tt}-z_{xx})dx\, dt = \int_0^X \rho(x)[y_1(x)z(x,0)-y_0(x)z_t(x,0)]dx+$$

$$+\int_0^X [u_1(t)z(0,t)-u_2(t)z(X,t)]dt$$

holds.

Definition. Denote $G(T)$ the set of all initial conditions $(y_0, y_1)\in W_2^1(0,X)\oplus L^2(0,X) =: Y$ for which there exists controls u_1, u_2 such that for the generalized solution $y(x, t)$ of (3)—(5) $y(.,T) = y_t(.,T) = 0$ holds. The string (3)—(5) is called controllable in time T if $G(T) = Y$.

An analogous problem is to describe the set of reachable movement states in time T if the string is relaxed at $t = 0$. Both questions were investigated by many authors, in several generalized form, see [3]—[15].

Consider the boundary value problem

$$v''(x,\lambda)+\lambda\,\rho(x)v(x,\lambda) = 0,\ v'(0,\lambda) = v'(X,\lambda) = 0.$$

It is well known that the eigenfunctions $v_n(x)$, $n = 0, 1, \ldots$ form an orthonormal basis in $L^2(0, X)$, all of the eigenvalues are simple, $\lambda_0 = 0$, $\lambda_n>0$ $(n>0)$. Further we have the following asymptotical formulas [16]:

$$(7)\qquad \lambda_n = \left(\frac{\pi n}{T_0}\right)^2+O(1),$$

$$v_n(O) = d_0+o(1),\ v_n(X) = d_1(-1)^n+o(1),$$

where

$$d_0:=\sqrt{2/T_0}\,\rho^{-\frac{1}{4}}(0),\ d_1:=\sqrt{2/T_0}\,\rho^{-\frac{1}{4}}(X),\ T_0:=\int_0^X\sqrt{\rho}.$$

T_0 is the so — called optical length of the string.

We shall looking for the solution of (3) —(5) in the form

$$(8)\qquad y(x,t) = \sum_{n=0}^{\infty} c_n(t)\, v_n(x)$$

and rewrite the initial conditions in the form

$$y_0(x) = \sum_{n=0}^{\infty} c_n^{(0)} v_n(x),\ y_1(x) = \sum_{n=0}^{\infty} c_n^{(1)} v_n(x). \tag{9}$$

Observe that

$$\|(y_0, y_1\|_Y^2 \asymp |c_0^{(0)}|^2 + \sum_{n=1}^{\infty} n^2 |c_n^{(0)}|^2 + \sum_{n=1}^{\infty} |c_n^{(1)}|^2. \tag{10}$$

Indeed, the $L_\rho^2(0, X)$ — norm is equivalent to the L^2 $(0, X)$ — norm further $\{v_n' / \sqrt{\lambda_n}\}$ is an orthonormal sequence, we get for

$$y = \sum_{n=0}^{\infty} c_n v_n$$

that

$$\|y\|_{L^2(0, X)}^2 \asymp \sum_{n=0}^{R} |c_n|^2$$

further

$$\|y'\|_{L^2(0, X)}^2 \asymp \sum_{n=1}^{\infty} \lambda_n |c_n|^2.$$

Hence (10) follows, taking into account (7).

Now define $z(x, t) = b(t)\ v_n(x)$, where $b \in C^2[0, T]$, $b(T) = b'(T) = 0$. From (6) we get for all such b:

$$\int_0^T (c_n b'' + \lambda_n c_n b) dt = c_n^{(1)} b(0) - c_n^{(0)} b'(0) + \int_0^T (u_1 v_n(0) - u_2 v_n(X))\ b\ dt$$

and hence

$$c_n''(t) + \lambda_n c_n(t) = u_n(t),\ c_n(0) = c_n^{(0)},\ c_n'(0) = c_n^{(1)} \tag{11}$$

holds, where $u_n(t) := \langle u(t), e_n \rangle$,

$$u := \{u_1, u_2\} \in H(T) := L^2(0, T; \mathbf{C}^2),\ e_n := \begin{pmatrix} v_n(0) \\ -v_n(X) \end{pmatrix}.$$

Introduce the notations: $\omega_n := \sqrt{\lambda_n}$, $\omega_{-n} = -\omega_n (n = 0, 1, \ldots)$, $b_n(t) := i\omega_n c_{|n|}(t) + c'_{|n|}(t)$, $b_n^{(0)} := i\omega_n c_{|n|}^{(0)} + c_{|n|}^{(1)}$, $n \in \mathbf{Z}$. Then we have

$$c_n(t) = (b_n(t) - b_{-n}(t))/2i\omega_n, \; n = 1, 2, \ldots, \tag{12}$$

$$c_0(t) = c_0^{(0)} + \int_0^t b_0, \; c_n'(t) = (b_n(t) + b_{-n}(t))/2.$$

The coefficients b_n satisfy the equations $b_n'(t) = i\omega_n b_n(t) + u_{|n|}(t)$, $b_n(0) = b_n^{(0)}$ hence

$$b_n(t) = b_n^{(0)} e^{i\omega_n t} + \int_0^T u_{|n|}(\tau) e^{i\omega_n (t-\tau)} d\tau, \; n \in \mathbf{Z} \setminus \{0\}, \tag{13}$$

$$b_0(t) = b_0^{(0)} + \int_0^t u_0(\tau) d\tau.$$

We shall prove the

LEMMA 2. For any boundary and initial conditions the problem (3)—(5) has a unique solution $y(x, t)$ and the phase trajectory $t \to \{y(.,t), y_t(.,t)\}$ is continuous in the space $Y = W_2^1(0, X) \oplus L^2(0, X)$

PROOF. From (13) we get that

$$|b_n(t)| \le b_n^{(0)} + |(u_1, v_n(0) e^{i\omega_n \tau})_{L^2(0,t)}| + |(u_2, v_n(X) e^{i\omega_n \tau})_{L^2(0,t)}|.$$

We know that $v_n(0)$, $v_n(X)$ are bounded in n further using (7) and ([1], Lemma 5) that the system $\{e^{i\omega_n \tau}\}$ is the union of finitely many L-bases, consequently

$$\sum_{n=-\infty}^{\infty} |b_n(t)|^2 \le c\{\sum |b_n^{(0)}|^2 + \|u_1\|^2_{L^2(0,t)} + \|u_2\|^2_{L^2(0,t)}\}$$

which means by (10) that

$$\sum_{n=-\infty}^{\infty} |b_n(t)|^2 \le c\{\|\{y_0, y_1\}\|_Y^2 + \|u_1\|^2_{L^2(0,t)} + \|u_2\|^2_{L^2(0,t)}\}$$

and hence

$$|c_0(t)|^2 + \sum_{n=1}^{\infty} \omega_n^2 |c_n(t)|^2 + \sum_{n=0}^{\infty} |c_n'(t)|^2 \le c\{\|\{y_0, y_1\}\|_Y^2 + \|u_1\|^2_{L^2(0,t)} + \|u_2\|^2_{L^2(0,t)}\}.$$

We have seen above that if $y(x, t)$ is a generalized solution then expanding it by $\{v_n\}$ the coefficient functions $c_n(t)$ must satisfy (12), (13) hence the unicity follows. If y is written by these functions $c_n(t)$ then (6) is satisfied for the linear combinations of the functions of the from $z(x, t) = v_n(x)b(t)$. To prove that (6) holds in general we can use a limit argument, see in [11], [12]; we omit the details. The proof of (10) also gives

$$\|\{y(.,t), y_t(.,t)\}\|_Y^2 \asymp |c_0(t)|^2+\sum_{n=1}^{\infty} \lambda_n |c_n(t)|^2+$$

$$+\sum_{n=0}^{\infty} |c_n'(t)|^2 \asymp \sum_{n=-\infty}^{\infty} |b_n(t)|^2+|c_0(t)|^2$$

hence $\{y(.,t), y_t(.,t)\}\in Y$. The continuity of the trajectory can be proved similarly. Lemma 2 is proved. □

Introduce the system $\tilde{\Phi}: = \{\tilde{\phi}_0^{(0)}\}\cup\{\tilde{\phi}_n\}_{n\in Z}$, $\tilde{\phi}_0^{(0)}(t): = t\, e_0$, $\tilde{\phi}_n^{(0)} = e_n \exp(i\omega_n t)$ and define the operators $J_{\tilde{\Phi}}:H(T)\to l_2$,

$$u\to\{\int_0^T \langle u, \tilde{\phi}_0^{(0)}\rangle\}\cup\{\int_0^T \langle u, \tilde{\phi}_n\rangle\}_{n\in Z}.$$

Clearly $J_{\tilde{\Phi}}$ maps into l_2, (see [17] p. 162). Let $A:Y\to l_2$, $\{y_0, y_1\}\to\{c_0^{(0)}, b_n^{(0)}; n\in Z\}$. By (10) the mapping A is an isomorphism of Y onto the whole l_2. If the string can be relaxed from the initial state $\{y_0, y_1\}$ then

$$0 = c_0(T) = c_0^{(0)}+\int_0^T\int_0^t u_0(\tau)d\tau = c_0^{(0)}+\int_0^T u_0(\tau)(T-\tau)d\tau,$$

$$0 = b_0(T) = b_0^{(0)}+\int_0^T u_0(\tau)d\tau,$$

$$0 = b_n(T) = b_n^{(0)}e^{i\omega_n T} + \int_0^T u_{|n|}(\tau)e^{i\omega_n(T-\tau)}d\tau.$$

Taking into account that $u_n(t) = \langle u(t), e_n\rangle$ we see that for the isomorphism $A':\{y_0, y_1\}\to\{c_0^{(0)}-b_0^{(0)}, -b_n^{(0)}:n\in Z\}$ we have

$$A'(G(T)) = \operatorname{Im} J_{\tilde{\Phi}}.$$

We need the following result on abstract moment problem:

LEMMA 3 ([10], [18]). Let $\Phi = \{\phi_n\}$ be an arbitrary almost normed system in a Hilbert space H and denote $J_\Phi: H \to l_2$, $f \to \{(f, \phi_n)_H\}$, $D_\Phi: = \{f \in H: J_\Phi f \in l_2\}$.

a) If $\operatorname{Im} J_\Phi = l_2$ then Φ is uniformly minimal. If Φ is an L-basis then $\operatorname{Im} J_\Phi = l_2$.

b) If $\overline{\operatorname{Im} J_\Phi} = l_2$ then Φ is ω-linearly independent (i.e. if $\sum c_n \Phi_n$ converges to zero and $\{c_n\} \in l_2$, then $c_n = 0$ $n = 1, 2, \ldots$). If Φ is minimal, then $\overline{\operatorname{Im} J_\Phi} = l_2$.

c) Suppose that there exists $\Phi_0 \subset \Phi$ such that Φ_0 is an L-basis and $\Phi \backslash \Phi_0 \subset \vee \Phi_0$ (i.e. belongs to the closed linear hull of Φ_0). Then $\operatorname{Im} J_\Phi$ is closed in l_2 and $\operatorname{codim} \operatorname{Im} J_\Phi = \operatorname{card} (\Phi \backslash \Phi_0)$.

Now we can prove the following description of the set $G(T)$ of initial states relaxable in time T.

THEOREM 2. *For $T > T_0$ we have $G(T) = Y$. If $T \leq T_0$, then $G(T)$ is a subspace in Y, $G(T)$ Y, the codimension of which is finite if $T = T_0$ and infinite if $T < T_0$.*

PROOF. By (14) we see that we have to consider $\operatorname{Im} J_{\tilde{\Phi}}$ in l_2. Since the multiplication by e^{-t} is an isomorphism of $\mathbf{H}(T)$ we can write in $\tilde{\Phi}$ $\omega_n + i$ instead of ω_n. Taking the Fourier transform we get the system $\Phi = \{\phi_0^{(0)}\} \cup \Phi_0 = \{\phi_n\}_{-\infty}^{\infty}$, $\phi^{(0)} = e_0(z+i)^{-2}$, $\phi_n = \phi_n(z+\omega_n+i)^{-1}$. The Fourier transform of $H(T)$ is $K_T = H_+^2(\mathbf{C}^2) \ominus \exp(izT) H_+^2(\mathbf{C}^2)$ (see [1]). We know that

$$e_{2n} \to e' = \begin{pmatrix} d_0 \\ -d_1 \end{pmatrix}, \quad e_{2n+1} \to e'' = \begin{pmatrix} d_0 \\ d_1 \end{pmatrix}.$$

From $d_0, d_1 \neq 0$ it follows that e' and e'' span the space $\mathbf{C}^2$. Denote

$$\Phi^{(0)} := \left\{ e'\left(z + \frac{2n}{T_0} + i\right)^{-1} \right\}_{-\infty}^{\infty} \cup \left\{ e''\left(z + \frac{2n+1}{T_0} + i\right)^{-1} \right\}_{-\infty}^{\infty},$$

$$\Phi^{(1)} = \{e'(z+\omega_{2n}+i)^{-1}\}_{-\infty}^{\infty} \cup \{e''(z+\omega_{2n+1}+i)^{-1}\}.$$

Obviously $P_{T_0} \Phi^{(0)}$ is Riesz basis in K_{T_0} and using the last theorem in [1] we see that $P_{T_0} \Phi^{(1)}$ is also Riesz basis in K_{T_0}, further in case $T < T_0$ the system $P_T \Phi_0$ contains a Riesz basis in K_T and we have to remove infinitely many elements to obtain it. Let $T = T_0$, then Theorem C in [1] shows that $P_{T_0} \Phi_0$ becomes Riesz basis in K_{T_0} if we substitute finitely many elements by their "non-perturbed form" from $P_{T_0} \Phi^{(1)}$. Finally suppose that $T > T_0$. Then by Theorem C in [1] $P_T \Phi_0$ is an L-basis in K_T. Suppose indirectly that $P_T \phi_0^{(0)} \in V$ $P_T \Phi_0$. Taking the Fourier transform we get that $e_0 t = \Sigma\, c_n e_n e^{i\omega_n t}$. By repeated integration in t we get $e_0 t$, $e_0 t^2$, $e_0 t^3$, $\ldots \in V_{L^2(0,T)}(e_n e^{i\omega_n t})$ and hence

$e_0L^2(0, T) \subset V(e_n e^{i\omega_n t})$. Take some $\omega \neq \omega_n$, then $e_0 e^{i\omega t} = \Sigma a_k e_{n_k} e^{i\omega_{n_k} t}$, $a_k \neq 0$. Denote $H_1 := V_{L^2(0, T)}(e_{n_k} e^{i\omega_{n_k} t})$, then by the above arguments: $e_0 L^2(0, T)$, $e_{n_k} L^2(0, T) \subset H_1$. On the other hand $H_1 \neq L^2(0, T, \mathbf{C}^2)$ because $\{e_n e^{i\omega_n t}\}$ *is* L-basis also in $L^2(0, T', \mathbf{C}^2)$, $T_0 < T' < T$. Consequently e_0 and e_{n_k} are parallel. Since here infinitely many k occur we must have e.g. $e_0 = e_{n_k} = e'$ (or e''). In this case the maximal possible system contains all $e^{i\omega_{2n} t}$ and finitely many $e^{i\omega_{2n+1} t}$; but this system can not be complete in $L^2(0,T)$. The contradiction proves that $P_T \Phi$ is L-basis in K_T consequently by the Bari theorem [2] its moment space is l_2. The statement of Theorem 2 in case $T \leq T_0$ follows from Lemma 3. c). Theorem 2 is proved.

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PROJECTIVE METRICS AND HYPERBOLIC VOLUME

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The Bolyai-Lobachevskian n-space, briefly hyperbolic n-space, is a Riemannian n-manifold of constant negative curvature. Therefore, the fundamental tensor enables us the computation of arc length and any dimensional areal measure. In spite of this fact many mathematicians, from GAUSS and LOBACHEVSKI via SCHLÄFLI up to now, have been trying to elaborate a volume calculus, as elementary as possible, for hyperbolic (spherical) simplices and, in such a way, for polyhedra. For the history of the topic and for other references we refer the monograph [3] of J. BÖHM and E. HERTEL. In [17] you find a survey on related topics as well.

The most relevant recent result in this field may be the paper [8] of U. HAAGERUP and H. MUNKHOLM proving the regular ideal simplex (with the vertices on the sphere at infinity) to be of maximal volume among all simplices. This result has already got an impressive application in the very elegant proof, due to GROMOV, of MOSTOW's rigidity theorem for oriented closed hyperbolic 3-manifolds. The theorem states that for $n \geqq 3$ two oriented, closed, hyperbolic n-manifolds which have isomorphic fundamental groups are automatically isometric.

Searching for such a compact hyperbolic space form, the metric data of the fundamental domain are of great importance. In many cases this domain can be constructed by means of special simplices or so-called truncated simplices. Thus the computation of metric data, e.g. the volume of the space form, can be reduced. For further information about these investigations we refer to [1], [9], [12], [13], [14], [18].

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The classical formula of SCHLÄFLI expresses the differential of the volume of an n-simplex in a non-Euclidean n-space of constant curvature $K \neq 0$ by the differentials of the dihedral angles of the n-simplex ($n \geqq 2$) (see [2], [3], [5], [10], [16]). However, Schläfli's differential form has some disadvantages in computing the volume of a given simplex or polyhedron.

The purpose of this paper is to give an outline of the computations in hyperbolic n-space on the base of projective metrics. The classical machinery is reviewed e.g. in [15]. However, it seems to be a new idea that I extend the exterior differential calculus (see e.g. in [7], [11]) to projective-metric n-spaces. Thus we get a unified method for concrete numerical computations by computer and for theoretical considerations as well. We shall discuss some illustrative examples and derive the Schläfli's differential form. This derivation seems to be simpler, than that of chapter 5 in [3]. I think, these initial results can be further developed in the future (see also [4]).

I should like to thank Professor JOHANNES BÖHM for the very stimulating conversations and for his friendly help.

1. The real projective n-space $\mathcal{P}^n(\mathbf{R})$

1.1. The classical *projective space* $\mathcal{P}^n(\mathbf{R})$ can be interpreted as subspace structure of the $(n+1)$—dimensional linear space $\mathbf{V}^{n+1}(\mathbf{R})$ and of its dual $\mathcal{V}_{n+1}(\mathbf{R})$ over the field $\mathbf{R}$ of real numbers. Elements of

$$\mathbf{V}^{n+1}(\mathbf{R}) = \{\mathbf{0}, \mathbf{a}, \mathbf{b}, \ldots, \mathbf{x}, \mathbf{y}, \ldots\}$$

are called *vectors*. A 1-dimensional subspace $(\mathbf{a})$ spanned by $\mathbf{a}$ ($\neq \mathbf{0}$) represents a *point of* $\mathcal{P}^n(\mathbf{R}) =: \mathcal{P}$. We introduce *r-planes* as $(r+1)$-subspaces of $\mathbf{V}^{n+1}(\mathbf{R}) =: \mathbf{V}$, in case $r = n-1$ the r-planes are called *hyperplanes*. The incidence structure of $\mathcal{P}$ is clearly defined by the inclusion relations. Hyperplanes of $\mathcal{P}$ can be characterized by elements, usually called *forms*, of the dual space

$$\mathcal{V}_{n+1}(\mathbf{R}) = \{o, \mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{x}, \mathfrak{y} \ldots\}.$$

A form $\mathfrak{u}(\neq o)$, more exactly the 1-subspace of $\mathcal{V}_{n+1}(\mathbf{R}) =: \mathcal{V}$ spanned by $\mathfrak{u}$, determines a hyperplane $(\mathfrak{u})$ by the equation

(1) $\mathbf{x}\,\mathfrak{u} = 0(\in \mathbf{R}) \Leftrightarrow (\mathbf{x})\ \mathrm{I}\ (\mathfrak{u})$ (incidence relation).

An r-plane of $\mathcal{P}$ can be described by $r+1$ linearly independent vectors $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r$ as well as by $n-r$ independent forms $\mathfrak{u}^{r+1}, \ldots, \mathfrak{u}^n$, i.e.,

(2) $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r) \equiv (\mathfrak{u}^{r+1}, \ldots, \mathfrak{u})^n \Leftrightarrow \mathbf{a}_i\,\mathfrak{u}^s = 0$

for all $i = 0, 1, \ldots, r$; $s = r+1, \ldots, n$.

If $\{\mathbf{e}_i\} = \{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis in $\mathbf{V}$ and $\{e^j\} = \{e^0, e^1, \ldots, e^n\}$ is its dual basis in $\mathcal{V}$, i.e. $\mathbf{e}_i e^j = \delta_i^j$ (the Kronecker symbol), then the form $\mathfrak{u} = e^j u_j$ takes the value $\mathbf{x}\mathfrak{u} = x^i u_i$ $(\in \mathbf{R})$ on the vector $\mathbf{x} = x^i \mathbf{e}_i$. We apply the summation convention for the same upper and lower indices.

To characterize the r-planes of $\mathcal{P}$ more concisely, we shall introduce either $(r+1)$-*vectors* of the space $\wedge^{r+1}\mathbf{V}$ or $(n-r)$-*forms* of $\wedge_{n-r}\mathcal{V}$ (in analogy to introducing Plücker's line coordinates in the projective 3-space $\mathcal{P}^3$).

1.2. The spaces $\wedge^{r+1}\mathbf{V}$ $(r+1 = 0, 1, \ldots, n+1)$ are defined inductively. We begin with $\wedge^0\mathbf{V} := \mathbf{R}$, $\wedge^1\mathbf{V} := \mathbf{V}$, $\wedge^2\mathbf{V}$ consists of all formal linear combinations of 2-vectors $\mathbf{a}_0\wedge\mathbf{a}_1$. For such exterior products we postulate the following identities.

$$(3)\qquad \begin{aligned} &\text{(i)}\quad (a\mathbf{x}+b\mathbf{y})\wedge\mathbf{a}_1 = a(\mathbf{x}\wedge\mathbf{a}_1)+b(\mathbf{y}\wedge\mathbf{a}_1) \text{ (distributivity)}\\ &\text{(ii)}\quad (\mathbf{a}_0\wedge\mathbf{a}_1) = -(\mathbf{a}_1\wedge\mathbf{a}_0) \text{ (anti-commutativity);}\end{aligned}$$

for all $a, b\in\mathbf{R}$; $\mathbf{x}, \mathbf{y}, \mathbf{a}_0, \mathbf{a}_1\in\mathbf{V}$.

Suppose $\{\mathbf{e}_i\}$ form a basis of $\mathbf{V}$ and $\mathbf{x} = x^i\mathbf{e}_i$, $\mathbf{y} = y^j\mathbf{e}_j$, then $\mathbf{x}\wedge\mathbf{y} = (x^i\mathbf{e}_i)\wedge(y^j\mathbf{e}_j) = x^iy^j(\mathbf{e}_i\wedge\mathbf{e}_j)$. We rearrange this as follows. From (ii) we deduce $\mathbf{e}_i\wedge\mathbf{e}_j = 0$ for $i = j$, and $\mathbf{e}_j\wedge\mathbf{e}_i) = -(\mathbf{e}_i\wedge\mathbf{e}_j)$ for $i<j$. Hence

$$(4)\qquad \mathbf{x}\wedge\mathbf{y} = \sum_{i<j} (x^iy^j-x^jy^i)(\mathbf{e}_i\wedge\mathbf{e}_j).$$

The 2-vectors $\{\mathbf{e}_i\wedge\mathbf{e}_j : 0\leqq i<j\leqq n\}$ form a basis of $\wedge^2\mathbf{V}$, hence $\dim \wedge^2\mathbf{V} = \binom{n+1}{2}$.

In general, we form $\wedge^{r+1}\mathbf{V}$ $(1\leqq r\leqq n)$ by the same idea. It consists of all formal linear combinations of $(r+1)$-vectors $\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r$. For such exterior products we postulate

$$(5)\qquad \begin{aligned} &\text{(i)}\quad (a\mathbf{x}+b\mathbf{y})\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r = a(\mathbf{x}\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)+b(\mathbf{y}\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r;\\ &\text{(ii)}\quad \mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r \text{ changes sign if any two } \mathbf{a}_i \text{ are interchanged;}\end{aligned}$$

for every $a, b\in\mathbf{R}$; $\mathbf{x}, \mathbf{y}, \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r\in\mathbf{V}$. Exactly as in the case $r+1 = 2$, we can show that if $\{\mathbf{e}_j\}$ form a basis of $\mathbf{V}$ and $\mathbf{a}_i = a_i^j\mathbf{e}_j (i = 0, 1, \ldots, r)$, then

$$(6)\qquad \mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r =$$

$$= \sum_{0\leqq i_0<i_1<\ldots<i_r\leqq n} \begin{vmatrix} a_0^{i_0} & a_0^{i_1} & \ldots & a_0^{i_r}\\ a_1^{i_0} & a_1^{i_1} & \ldots & a_1^{i_r}\\ \vdots & & & \\ a_r^{i_0} & a_r^{i_1} & \ldots & a_r^{i_r}\end{vmatrix} (\mathbf{e}_{i_0}\wedge\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_r}),$$

i.e. the coefficients are $(r+1)$-minor determinants from the matrix (a_i^j).

The $(r+1)$-vectors $\{\mathbf{e}_{i_0}\wedge\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_r}: 0\leqq i_0<i_1<\ldots<i_r\leqq n\}$ form a basis of $\wedge^{r+1}\mathbf{V}$, hence $\dim\wedge^{r+1}\mathbf{V}=\binom{n+1}{r+1}$.

We now observe that our spaces $\wedge^0\mathbf{V}:=\mathbf{R}$, $\wedge^1\mathbf{V}:=\mathbf{V}, \ldots, \wedge^{n+1}\mathbf{V}$ have a built-in multiplication process called *exterior multiplication* and denoted by $\wedge$ for obvious reasons. We multiply an $(r+1)$-vector $\alpha=(\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)$ by an s-vector $\beta=(\mathbf{b}_{r+1}\wedge\ldots\wedge\mathbf{b}_{r+s})$ to obtain an $(r+1+s)$-vector $\alpha\wedge\beta$ by the following definition

$$(7)\qquad (\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)\wedge(\mathbf{b}_{r+1}\wedge\ldots\wedge\mathbf{b}_{r+s}) := \\ := (\ldots((\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)\wedge\mathbf{b}_{r+1})\wedge\ldots)\wedge\mathbf{b}_{r+s}$$

then we extend this multiplication linearly to all $(r+1)$-vectors and s-vectors. The basic properties of this exterior product are

$$(8)\qquad \begin{aligned} &\text{(i)}\ \ (a\alpha+b\beta)\wedge\gamma=a(\alpha\wedge\gamma)+b(\beta\wedge\gamma),\ \text{linearity}\ (a,\ b\in\mathbf{R});\\ &\text{(ii)}\ \ \alpha\wedge(\beta\wedge\gamma)=(\alpha\wedge\beta)\wedge\gamma,\ \text{the associative law;}\\ &\text{(iii)}\ \ \beta\wedge\alpha=(-1)^{(r+1)s}\alpha\wedge\beta. \end{aligned}$$

Of course, the product in (7) equals to zero if $r+1+s>n+1$.

Analogously, we define the spaces $\wedge_{n-r}\mathcal{V}$ of $(n-r)$-forms and the exterior forms $(0\leqq n-r\leqq n+1)$. *We establish that* $\wedge_p\mathcal{V}$ *can be identified with the dual space of* $\wedge^p\mathbf{V}$. Especially, the p-form $u^1\wedge\ldots\wedge u^p$ takes the value

$$(9)\qquad (\mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_p)(u^1\wedge\ldots\wedge u^p) := \begin{vmatrix} \mathbf{x}_1u^1 & \ldots & \mathbf{x}_1u^p \\ \vdots & & \vdots \\ \mathbf{x}_pu^1 & \ldots & \mathbf{x}_pu^p \end{vmatrix} =: \det(\mathbf{x}_iu^j)$$

on the p-vector $\mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_p$.

1.3. Now, we fix a basis $\{\mathbf{e}_i\}=\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and its dual $\{e^j\}=\{e^0, e^1, \ldots, e^n\}$. A *natural* (bijective) *polarity map*

$$(10)\qquad (\,^\circ): \wedge^{r+1}\mathbf{V}\to\wedge_{n-r}\mathcal{V},\quad (\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_r)\mapsto(u^{r+1}\wedge\ldots\wedge u^n)$$

will be defined inductively. (Its inverse will be denoted by $(\,_\circ)$.)

The space $\wedge^{n+1}\mathbf{V}$ of $(n+1)$-vectors is 1-dimensional, and

$$(11)\qquad (\mathbf{a}_0\wedge\mathbf{a}_1\wedge\ldots\wedge\mathbf{a}_n)(e^0\wedge e^1\wedge\ldots\wedge e^n) = \begin{vmatrix} a_0^0 & a_0^1 & \ldots & a_0^n \\ a_1^0 & a_1^1 & \ldots & a_1^n \\ \vdots & & & \\ a_n^0 & a_n^1 & \ldots & a_n^n \end{vmatrix} =: \det(a_i^j)$$

holds if $\mathbf{a}_i = a_i^j \mathbf{e}_j$, i.e. $\mathbf{a}_i e^j = a_i^j$. Let $\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{n-1}$ be an n-vector. For every $\mathbf{x} \in \wedge^1 \mathbf{V} := \mathbf{V}$

$$\mathbf{x} \to (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{n-1} \wedge \mathbf{x})(e^0 \wedge e^1 \wedge \ldots \wedge e^n) =: \mathbf{x} u^n \tag{12}$$

defines a 1-form

$$u^n := (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_{n-1})^\circ = e^0 (-1)^n \begin{vmatrix} a_0^1 & \ldots & a_0^n \\ \vdots & & \vdots \\ a_{n-1}^1 & \ldots & a_{n-1}^n \end{vmatrix} +$$

$$+ e^1 (-1)^{n-1} \begin{vmatrix} a_0^0 & a_0^2 & \ldots & a_0^n \\ \vdots & & & \vdots \\ a_{n-1}^0 & a_{n-1}^2 & \ldots & a_{n-1}^n \end{vmatrix} + \ldots + e^n \begin{vmatrix} a_0^0 & \ldots & a_0^{n-1} \\ \vdots & & \vdots \\ a_{n-1}^0 & \ldots & a_{n-1}^{n-1} \end{vmatrix}.$$

The equality $\mathbf{x} u^n = 0$ holds, iff $\mathbf{x}$ is a linear combination of $\mathbf{a}_0, \ldots, \ldots, \mathbf{a}_{n-1}$.

Let $\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r$ be an $(r+1)$-vector. For every $\mathbf{x}_{r+1} \wedge \ldots \wedge \mathbf{x}_n \in \wedge^{n-r} \mathbf{V}$

$$\mathbf{x}_{r+1} \wedge \ldots \wedge \mathbf{x}_n \mapsto (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{x}_{r+1} \wedge \ldots \wedge \mathbf{x}_n) \cdot \tag{13}$$
$$\cdot (e^0 \wedge e^1 \wedge \ldots \wedge e^n) =: (\mathbf{x}_{r+1} \wedge \ldots \wedge \mathbf{x}_n) u \in \mathbf{R}$$

defines an $(n-r)$-form $u := (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r)^\circ = u^{r+1} \wedge \ldots \wedge u^n$ where the 1-forms u^s $(r+1 \leqq s \leqq n)$ can be chosen by

$$\mathbf{x} u^s := (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{a}_{r+1} \wedge \ldots \wedge \overset{s}{\overset{\vee}{\mathbf{x}}} \wedge \ldots \wedge \mathbf{a}_n)(e^0 \wedge e^1 \wedge \ldots \wedge e^n) \tag{14}$$

with any vectors $\mathbf{a}_{r+1}, \ldots, \mathbf{a}_n$ for which

$$(\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r \wedge \mathbf{a}_{r+1} \wedge \ldots \wedge \mathbf{a}_n)(e^0 \wedge e^1 \wedge \ldots \wedge e^n) = 1 \ (\in \mathbf{R}). \tag{15}$$

Thus we arrived at the natural polarity map (10). We remark only that another basis $\{\mathbf{e}_{i'}\} = \{\mathbf{e}_{0'}, \mathbf{e}_{1'}, \ldots, \mathbf{e}_{n'}\}$ and its dual $\{e^{j'}\} = \{e^{0'}, \ldots, e^{n'}\}$ yield the same polarity map iff the transition equations $\mathbf{e}_{i'} = \mathbf{e}_{i'}^i \mathbf{e}_i$ are of unit determinant, i.e.,

$$\det (e_{i'}^i) := \begin{vmatrix} e_{0'}^0 & e_{0'}^1 & \ldots & e_{0'}^n \\ e_{1'}^0 & e_{1'}^1 & \ldots & e_{1'}^n \\ \vdots & & & \\ e_{n'}^0 & e_{n'}^1 & \ldots & e_{n'}^n \end{vmatrix} = 1$$

(Einstein-Schouten convention).

Thus the construction of polarity map (10) only depends on giving an $(n+1)$-form called the volume-form of $\mathbf{V}$. Hence the polarity map $(\circ)$ is determined up to a constant factor from $\mathbf{R}$.

As a corollary of (13—15) we mention a lemma from the matrix theory [7] formulating in that form we need later on:

LEMMA 1. *Let (b^{ij}) be a regular $(n+1)\times(n+1)$ matrix and denote by (a_{ij}) its inverse, i.e. $b^{ij}a_{jk} = \delta^i_k$.*

Then for any $(r+1)$-minor determinant of (a_{ij}) and the complementary $(n-r)$-minor of (b^{ij}) holds the following equality

$$\begin{vmatrix} a_{i_0 j_0} & \dots & a_{i_0 j_r} \\ \vdots & & \vdots \\ a_{i_r j_0} & \dots & a_{i_r j_r} \end{vmatrix} = \det(a_{ij}) \begin{vmatrix} b^{i_{r+1} j_{r+1}} & \dots & b^{i_{r+1} j_n} \\ \vdots & & \vdots \\ b^{i_n j_{r+1}} & \dots & b^{i_n j_n} \end{vmatrix} \cdot \sigma.$$

Here $\sigma = \operatorname{sign}(i_0, \dots, i_r, i_{r+1}, \dots, i_n) \cdot \operatorname{sign}(j_0, \dots, j_r, j_{r+1}, \dots, j_n)$ denotes the sign product of the corresponding permutations of the elements $0, 1, \dots, n$.

1.4. Now, we represent an r-plane of $\mathcal{P}$ either by an $(r+1)$-vector $\mathbf{a}_0\wedge\wedge\mathbf{a}_1\wedge\dots\wedge\mathbf{a}_r$ or by an $(n-r)$-form $u^{r+1}\wedge\dots\wedge u^n$, in accordance with (2), up to a constant factor from $\mathbf{R}$. Indeed, if $\mathbf{a}_{0'}, \mathbf{a}_{1'}, \dots, \mathbf{a}_{r'}$ also span the subspace $(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_r)$, and $\mathbf{a}_{i'} = a^i_{i'}\mathbf{a}_i$ (Einstein-Schouten convention!), then

$$\mathbf{a}_{0'}\wedge\mathbf{a}_{1'}\wedge\dots\wedge\mathbf{a}_{r'} = \begin{vmatrix} a^0_{0'} & \dots & a^r_{0'} \\ \vdots & & \vdots \\ a^0_{r'} & \dots & a^r_{r'} \end{vmatrix} \mathbf{a}_0\wedge\mathbf{a}_1\wedge\dots\wedge\mathbf{a}_r (\neq 0). \tag{16}$$

Analogous statement holds for $(n-r)$-forms.

Note that not every $(r+1)$-vector of $\wedge^{r+1}\mathbf{V}$ describes an r-plane of $\mathcal{P}$, only the decomposable $(r+1)$-vectors do it. We mention a criterion from [7].

LEMMA 2. *Let α be an $(r+1)$-vector from $\wedge^{r+1}\mathbf{V}$ and $\mathbf{V}_\alpha$ the subspace of $\mathbf{V}$ consisting of all vectors $\mathbf{x}$ satisfying the equation*

$$\alpha\wedge\mathbf{x} = 0. \tag{17}$$

Then $\dim(\mathbf{V}_\alpha) = r+1$ *holds, iff there are vectors $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_r$ in $\mathbf{V}$ so that $\alpha = \mathbf{a}_0\wedge\mathbf{a}_1\wedge\dots\wedge\mathbf{a}_r$, i.e. α is decomposable.*

2. Projectiv-metric n-spaces and non-Euclidean geometries

2.1. We introduce a *projective metric* in $\mathcal{P} := \mathcal{P}^n(\mathbf{R})$ by giving a *polarity* as a linear map on $\mathcal{V} := \mathcal{V}_{n+1}(\mathbf{R})$ into $\mathbf{V} := \mathbf{V}^{n+1}(\mathbf{R})$

$$(1) \qquad (_*): \mathcal{V} \to \mathbf{V},\ \alpha \mapsto \alpha_* = :\mathbf{a};$$

or, equivalently, by defining a *symmetric bilinear form*

$$(2) \qquad \langle ; \rangle: \mathcal{V} \times \mathcal{V} \to \mathbf{R},\ \langle u; v \rangle \mapsto u_* v$$

in the usual way (cf. [15], here we give a modified interpretation). In projective geometric formulation, for every hyperplane (α) in $\mathcal{P}$ we associate a point $(\mathbf{a})$ in $\mathcal{P}$ as the *pole* for the *polar* hyperplane. If $\beta_* = \mathbf{b}$ defines another polar (β) and pole $\mathbf{b}$), then the equality

$$(3) \qquad \alpha_* \beta = \mathbf{a}\beta = \langle \alpha; \beta \rangle = \langle \beta; \alpha \rangle = \mathbf{b}\alpha = \beta_* \alpha = 0$$

means equivalently that the hyperplanes (α) and (β) are *orthogonal*, the pole $(\mathbf{a})$ of (α) lies in the hyperplane (β) and vice versa, the poles $(\mathbf{a})$ and $(\mathbf{b})$ are *conjugate.*

In the following we restrict ourselves to regular polarities. So we exclude the singular metrics, for instance, the Euclidean and pseudo-Euclidean geometries, when the image space of $(_*)$ is a hyperplane (the poles lie in the ideal hyperplane at infinity). In these regular cases

$$(4) \qquad (^*): \mathbf{V} \to \mathcal{V},\ \mathbf{x} \to \mathbf{x}^* = :\mathbf{x}$$

denotes the inverse polarity and

$$(5) \qquad \langle ; \rangle: \mathbf{V} \times \mathbf{V} \to \mathbf{R},\ \langle \mathbf{x}; \mathbf{y} \rangle := \mathbf{x}\mathbf{y}^* = \langle \mathbf{x}^*; \mathbf{y}^* \rangle$$

defines the induced bilinear form.

According to the polarity $(_*)$ *and its inverse* $(^*)$ *we can change our viewpoint, a form and its polar vector can be identified. Geometrically, a polar hyperplane and its pole can be arbitrarily changed in formulating facts for the projective metric space* $\mathcal{P}$.

Now we fix a basis $\{\mathbf{e}_i\}$ in $\mathbf{V}$ and its dual basis $\{e^j\}$ in $\mathcal{V}$. If

$$(6) \qquad \mathbf{e}^j := e^j_* = g^{ji}\mathbf{e}_i$$

defines the symmetric matrix (g^{ji}) of the polarity $(_*)$, then the inverse matrix $(g_{ij}) = (g_{ji})$ belongs to the inverse $(^*)$ by

(7) $$e_i := \mathbf{e}_i^* = e^j g_{ji}, \text{ where } g^{ji} g_{ik} = \delta_k^j.$$

A form $u = e^j u_j$, representing the plane (u), has its pole

(8) $$u_* = (e^j u_j)_* = u_j e^j_* = u_j g^{ji} \mathbf{e}_i = :u^i \mathbf{e}_i$$

in coordinates. The vector $\mathbf{x} = x^i \mathbf{e}_i$, representing the point $(\mathbf{x})$, has its polar hyperplane in the form

(9) $$\mathbf{x}^* = (x^i \mathbf{e}_i)^* = \mathbf{e}_i^* x^i = e^j g_{ji} x^i =: e^j x_j.$$

It is clear that the matrices $(c \cdot g^{ji})$ and $\left(g_{ji} \cdot \frac{1}{c}\right)$, respectively, define the same projective polarity in $\mathcal{P}$ as before $(0 \neq c \in \mathbf{R})$.

2.2. Any (regular) *linear transformation* α and its dual (adjoint) α^*, i.e.,

(10) $$\alpha : \mathbf{V} \to \mathbf{V},\ \mathbf{x} \mapsto \mathbf{x}\alpha \text{ and } \alpha^* : \mathcal{V} \to \mathcal{V},\ v \mapsto \alpha^* v$$

defined by $\mathbf{x}(\alpha^* v) := (\mathbf{x}\alpha)v$ for every $\mathbf{x} \in \mathbf{V}$ and $v \in \mathcal{V}$, determine the *projectivity of* $\mathcal{P}$ for the points and the hyperplanes, respectively. It is called a *metric projectivity* iff it preserves the polarity, i.e., the bilinear form is invariant up to a multiplication by a non-zero constant.

We mention only that the reflection formulas for vectors and forms are

(11) $$\mathbf{x} \mapsto \mathbf{x} - \frac{2}{(u_* u)} (\mathbf{x}u) u_*;$$

$$v \mapsto v - u(v_* u) \frac{2}{(u_* u)},$$

where the form u and its polar vector $\mathbf{u} := u_*$ characterize the reflection hyperplane and its pole, respectively. The group of metric projectivities of $\mathcal{P}$ can be generated by hyperplane reflections.

2.3. In the sense of section **1.3.** we extend projective metric, i.e. the polarity, to r-planes. We apply the construction of $(r+1)$-vectors in $\wedge^{r+1}\mathbf{V}$ and their polar $(n\text{-}r)$-forms in $\wedge_{n-r}\mathcal{V}$ by formulas **1** (13—15). The polarity $(_*)$ can be extended to a linear map on $\wedge_{n-r}\mathcal{V}$ into $\wedge^{n-r}\mathbf{V}$ by the following definition

(12) $$(_*) : (u^{r+1} \wedge u^{r+2} \wedge \ldots \wedge u^n)_* = (u_*^{r+1} \wedge u_*^{r+2} \wedge \ldots \wedge u_*^n).$$

If the polarity $(_*)$ is regular then we can also introduce the *Hodge star operator* associating any $(r+1)$-vector with an $(n\text{-}r)$-vector:

(13) $$* = (^\circ)(_*) : (\mathbf{a}_0 \wedge a_1 \wedge \ldots \wedge \mathbf{a}_r) \overset{\circ}{\mapsto} (u^{r+1} \wedge \ldots \wedge u^n) \overset{*}{\mapsto} (u_*^{r+1} \wedge \ldots \wedge u_*^n).$$

According to 1(9) the bilinear form on $\wedge_p \mathcal{V}$ can be defined by

$$(14)\qquad \langle u^1\wedge\ldots\wedge u^p; v^1\wedge\ldots\wedge v^p\rangle = (u^1_*\wedge\ldots\wedge u^p_*)(v^1\wedge\ldots\wedge v^p) =$$

$$= \begin{vmatrix} u^1_* v^1 & \ldots & u^1_* v^p \\ \vdots & & \vdots \\ u^p_* v^1 & \ldots & u^p_* v^p \end{vmatrix} = \begin{vmatrix} \langle u^1; v^1\rangle & \ldots & \langle u^1; v^p\rangle \\ \vdots & & \vdots \\ \langle u^p_*; v^1\rangle & \ldots & \langle u^p; v^p\rangle \end{vmatrix} =: \det\langle u^i; v^j\rangle.$$

With respect to (6) we can express this with the oriented bases $\{\mathbf{e}_i\}$, $\{e^j\}$. For any indices $0 \le i_1 < \ldots < i_p \le n$

$$(15)\qquad (e^{i_1}\wedge\ldots\wedge e^{i_p})_* = (g^{i_1 j_1}\,\mathbf{e}_{j_1}\wedge\ldots\wedge g^{i_p j_p}\,\mathbf{e}_{j_p}) =$$

$$= \sum_{0\le j_1<\ldots<j_p\le n} \begin{vmatrix} g^{i_1 j_1} & \ldots & g^{i_1 j_p} \\ \vdots & & \vdots \\ g^{i_p j_1} & \ldots & g^{i_p j_p} \end{vmatrix} \cdot \mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_p}$$

and

$$(16)\qquad \langle e^{i_1}\wedge\ldots\wedge e^{i_p}; e^{j_1}\wedge\ldots\wedge e^{j_p}\rangle = \begin{vmatrix} g^{i_1 j_1} & \ldots & g^{i_1 j_p} \\ \vdots & & \vdots \\ g^{i_p j_1} & \ldots & g^{i_p j_p} \end{vmatrix}.$$

Particularly, for $(e^0\wedge e^1\wedge\ldots\wedge e^n)$ and $(\mathbf{e}_0\wedge\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_n)$ the equalities

$$(17)\qquad \begin{aligned} &\langle e^0\wedge\ldots\wedge e^n; e^0\wedge\ldots\wedge e^n\rangle = \det(g^{ij}); \\ &\langle \mathbf{e}_0\wedge\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_n; \mathbf{e}_0\wedge\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_n\rangle = \det(g_{ij}) = \frac{1}{\det(g^{ij})} \end{aligned}$$

hold. If we fix the volume form to take value 1 ($\epsilon\mathbf{R}$) on the $(n+1)$-vectors of oriented orthonormed bases in $\mathbf{V}$, then we find the formulas in (17) expressing the volume quadrats, up to a sign ϵ, of the parallelotops spanned by $\{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^n\}$ and $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbf{V}$, respectively. The sign ϵ will depend on the signature of the polarity as we can easily see. Thus we can define a volume form in the metric vector space $\mathbf{V}$

$$(18)\qquad \omega = (e^0\wedge e^1\wedge\ldots\wedge e^n)\sqrt{\epsilon\det(g_{ij})} \quad \text{with } \langle\omega;\omega\rangle = 1.$$

If $\mathbf{a}_0, \mathrm{a}_1, \ldots, \mathbf{a}_n$ span a positively oriented $(n+1)$-dimensional parallelotop in $\mathbf{V}$, then its volume can be expressed in both terms

$$(19) \qquad (\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n)\omega = \sqrt{\epsilon \det \langle \mathbf{a}_i; \mathbf{a}_j \rangle},$$

where the sign ϵ depends on the signature of the polarity as before, this ϵ makes the root term to be positive.

The *Gramian of the vectors* $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ is defined by

$$(20) \qquad \begin{aligned} \mathrm{Gr}\,(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n) &:= \det \langle \mathbf{a}_i; \mathbf{a}_j \rangle = \\ &= \langle \mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n; \mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n \rangle. \end{aligned}$$

Analogously we define the $(r+1)$-dimensional volume of the $(r+1)$-parallelotop spanned by $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r$ in $\mathbf{V}$:

$$(21) \quad \begin{aligned} \mathrm{Vol}\,(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r) &= \sqrt{\epsilon\, \mathrm{Gr}\,(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r)} = \sqrt{\epsilon \det \langle \mathbf{a}_i, \mathbf{a}_j \rangle} = \\ &= \sqrt{\epsilon\, \langle \mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r; \mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r \rangle}. \end{aligned}$$

The sign ϵ depends on the signature of the subspace $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_r)$ again.

2.4. Although the machinery, introduced here, is applicable more generally [15], we restrict ourselves to the *elliptic and hyperbolic projective metric.* In the first case the bilinear form $\langle ; \rangle$ is of index 0, i.e., it has the signature $(+, +, \ldots, +)$ (or equivalently $(-, -, \ldots, -)$). In this elliptic projective space $\mathcal{P}_0^n := \mathcal{S}$ *self-orthogonal* hyperplanes and *self-conjugate* points do not exist, because no pole coincides with its polar. In strict connection with the spherical geometry, the elliptic geometry is the simpler case. Sometimes later we only refer to it [3].

We are mainly interested in the hyperbolic projective space $\mathcal{P}_1^n := \mathcal{H}$, when the bilinear form $\langle ; \rangle$ is of index 1, i.e., it has the signature $(-;+,\ldots,+)$ (or equivalently $(+; -, \ldots, -)$).

Fixing an orthonormed basis $\{\mathbf{e}_i\}$ and its dual $\{e^j\}$, we uniformly handle the two cases by defining the bilinear forms

$$(22) \qquad \begin{aligned} \langle \mathbf{x};\mathbf{y} \rangle &= \langle x^i \mathbf{e}_i; y^j \mathbf{e}_j \rangle := \epsilon x^0 y^0 + x^1 y^1 + \ldots + x^n y^n \text{ in } \mathbf{V}, \\ \langle u;v \rangle &= \langle e^i u_i; e^j v_j \rangle := \epsilon u_0 v_0 + u_1 v_1 + \ldots + u_n v_n \text{ in } \mathcal{V}. \end{aligned}$$

Then $\epsilon = 1$ is taken in the *elliptic case,* $\epsilon = -1$ holds in the *hyperbolic case,* coherently with the earlier formulas (18), (19), (21).

In the hyperbolic case the self-conjugate points $(\mathbf{a})$ of $\mathcal{H}$, i.e., ideal point at infinity, are represented in $\mathbf{V}$ by the *"light cone"* (in the therminology of the *space-time geometry;* see *Fig. 1* where $n = 2$)

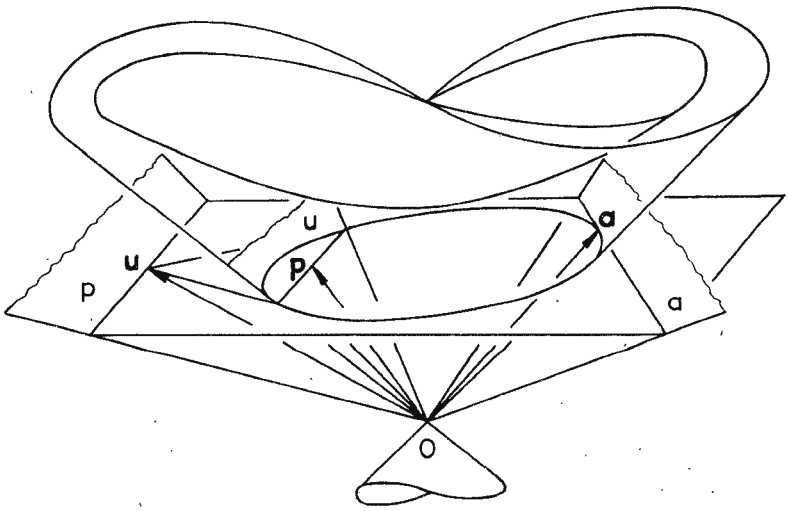


Fig. 1.

(23) $$\langle \mathbf{a};\mathbf{a}\rangle = 0. \quad \langle \mathfrak{a};\mathfrak{a}\rangle = 0$$

describes in $\mathcal{V}$ the self-orthogonal hyperplanes $(\mathfrak{a})$ touching the light cone. The equations in (23) define what we shall call the *absolut of* $\mathcal{H}^n$.

The *proper points* of $\mathcal{H}$ are described by *"time-like"* vectors in **V**, satisfying the inequality

(24) $$\langle \mathbf{p};\mathbf{p}\rangle < 0. \quad \langle \mathfrak{p};\mathfrak{p}\rangle < 0$$

characterizes the polar hyperplanes of the proper points. In such a hyperplane $(\mathfrak{p})$ lie only improper points $(\mathbf{u})$ whose vectors $\mathbf{u}$ are *"space-like"*:

(25) $$\langle \mathbf{u};\mathbf{u}\rangle > 0. \quad \langle \mathfrak{u};\mathfrak{u}\rangle > 0$$

describes the proper hyperplanes $(\mathfrak{u})$ of $\mathcal{H}$ which intersect the light cone. In $(\mathfrak{u})$ lie proper points (with time-like vectors), self-conjugate points (with light vectors) and improper points (with space-like vectors) as well.

Introducing the n-dimensional hyperplane in **V** by the equation

(26) $$\mathbf{x}_{\mathfrak{e}^0} = \langle \mathbf{x};\mathbf{e}^0\rangle = x^0 = 1$$

(where the induced bilinear form is of index 0 because of $\langle \mathbf{e}^0,\mathbf{e}^0\rangle = -1$), we get the classical *Cayley-Klein model* in the Euclidean n-space (Fig. 1.).

Both models (completed by ideal points obviously) have the characteristic property that *each 1-dimensional subspace of* V *meets them in one point. This way we can introduce in* V *any "transversal" hypersurface to model the projectiv-metric n-space or at least a part of it. The bilinear form* $\langle ; \rangle$ *of* V *induces an infinitesimale metric on the hypersurface and this can be applied to our computations.*

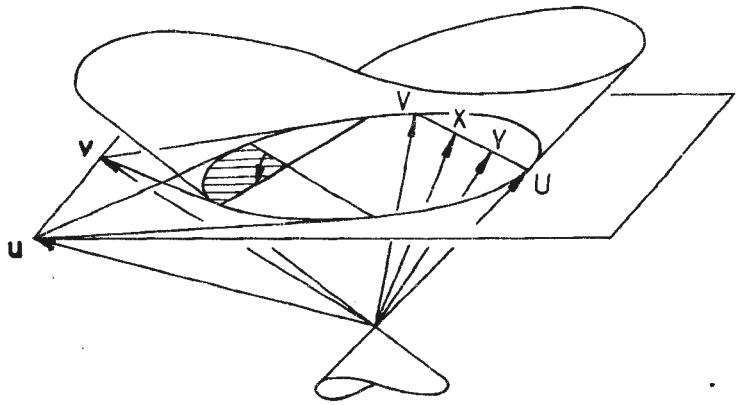


Fig. 2.

3. Distance, angular measure and volume in projectiv-metric n-spaces

3.1. Let $X(x)$, $Y(y)$ be two proper points in the hyperbolic n-space $\mathcal{H}$ with $\langle x;y \rangle < 0$ (*Fig. 2.*).

The last inequality guarantees that x and y is directed into the same component of the light cone. This is fulfilled, if the e_0-components are positive in the orthonormed basis of 2 (22), i.e., the vectors x, y point into the *positive half cone*

$$\mathbf{C}^+ = \{\mathbf{z} = z^0\mathbf{e}_0 + z^1\mathbf{e}_1 + \ldots + z^n\mathbf{e}_n : \langle \mathbf{z};\mathbf{z} \rangle < 0,\ 0 < z^0\} \subset \mathbf{V}. \tag{1}$$

Then the *distance* $s(\mathbf{x},\mathbf{y})$ can be defined equivalently by the following formulas

$$s(\mathbf{x}, \mathbf{y}) = \frac{k}{2} \ln \frac{-\langle \mathbf{x};\mathbf{y} \rangle + \sqrt{\langle \mathbf{x};\mathbf{y} \rangle^2 - \langle \mathbf{x};\mathbf{x} \rangle \langle \mathbf{y};\mathbf{y} \rangle}}{-\langle \mathbf{x};\mathbf{y} \rangle - \sqrt{\langle \mathbf{x};\mathbf{y} \rangle^2 - \langle \mathbf{x};\mathbf{x} \rangle \langle \mathbf{y};\mathbf{y} \rangle}}; \tag{2}$$

$$\operatorname{ch}\left(\frac{1}{k} s(\mathbf{x},\mathbf{y})\right) = \frac{-\langle \mathbf{x};\mathbf{y} \rangle}{\sqrt{\langle \mathbf{x};\mathbf{x} \rangle \langle \mathbf{y};\mathbf{y} \rangle}}, \tag{3}$$

$$\operatorname{sh}\left(\frac{1}{k} s(\mathbf{x},\mathbf{y})\right) = \sqrt{\frac{\langle \mathbf{x};\mathbf{y} \rangle^2 - \langle \mathbf{x};\mathbf{x} \rangle \langle \mathbf{y};\mathbf{y} \rangle}{\langle \mathbf{x};\mathbf{x} \rangle \langle \mathbf{y};\mathbf{y} \rangle}}.$$

Formula (2) namely expresses the same as

$$(2') \qquad s(\mathbf{x}, \mathbf{y}) = \frac{k}{2} \ln (XYUV).$$

Here in the *cross-ratio* those points $U(\mathbf{u})$, $V(\mathbf{v})$ occur in which the line XY intersects the absolut of $\mathcal{H}$. We know that if Z is an arbitrary point on the line XY, then the equality

$$(XZUV)(ZYUV) = (XYUV)$$

holds and this guarantees that we have in (2) an *additive distance function*, indeed.

The constant $k = \sqrt{\frac{-1}{K}}$ is the natural length unit in $\mathcal{H}^n$, K will be the *constant negative sectional curvature.* Analogous formulas hold in the elliptic space $\mathcal{S}$. *The following formulas*

$$(2^*) \qquad s(\mathbf{x}, \mathbf{y}) = \frac{r}{2i} \ln \frac{\epsilon\langle \mathbf{x}; \mathbf{y}\rangle + \sqrt{\langle \mathbf{x}; \mathbf{y}\rangle^2 - \langle \mathbf{x}; \mathbf{x}\rangle\langle \mathbf{y}; \mathbf{y}\rangle}}{\epsilon\langle \mathbf{x}; \mathbf{y}\rangle - \sqrt{\langle \mathbf{x}; \mathbf{y}\rangle^2 - \langle \mathbf{x}; \mathbf{x}\rangle\langle \mathbf{y}; \mathbf{y}\rangle}}$$

$$(3^*) \qquad \cos\left(\frac{1}{r} s(\mathbf{x}; \mathbf{y})\right) = \frac{\epsilon\langle \mathbf{x}; \mathbf{y}\rangle}{\sqrt{\langle \mathbf{x}; \mathbf{x}\rangle\langle \mathbf{y}; \mathbf{y}\rangle}},$$

$$\sin\left(\frac{1}{r} s(\mathbf{x}; \mathbf{y})\right) = \frac{1}{i} \sqrt{\frac{\langle \mathbf{x}; \mathbf{y}\rangle^2 - \langle \mathbf{x}; \mathbf{x}\rangle\langle \mathbf{y}; \mathbf{y}\rangle}{\langle \mathbf{x}; \mathbf{x}\rangle\langle \mathbf{y}; \mathbf{y}\rangle}}.$$

provide both cases: $\epsilon = 1$ stands for the elliptic space, then $r = \sqrt{\frac{1}{K}}$ is the natural length unit, K is the constant positive curvature; $\epsilon = -1$ holds for hyperbolic space, then $r = ki$. This is in the sense of 2 (22). We applied the identities

$$(4) \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix});$$

$$(5) \qquad \cos \frac{x}{i} = \operatorname{ch} x, \ \sin \frac{x}{i} = \frac{1}{i} \operatorname{sh} x, \ (i \cdot i = -1).$$

3.2. Let (u) and (v) be two intersecting proper hyperplanes in $\mathcal{H}$ with poles $(\mathbf{u})$ and $(\mathbf{v})$ respectively. Fixing the positive half cone $\mathbf{C}^+$ in $\mathbf{V}$, e.g. by (1), then

$$(6) \qquad (u, v)\measuredangle = \{(\mathbf{x}) : \mathbf{x} \in \mathbf{C}^+, \ \mathbf{x}_u = \langle \mathbf{x}; \mathbf{u}\rangle > 0, \ \mathbf{x}_v = \langle \mathbf{x}; \mathbf{v}\rangle > 0\}$$

defines one of the four *proper angular domains* of $\mathcal{H}$ determined by the hyperplanes (Fig. 2). The other three ones will be $(-u, v)\sphericalangle$, $(-u, -v)\sphericalangle$ and $(u, -v)\sphericalangle$. The measure α of $(u, v)\sphericalangle$ let be defined by

$$\cos(\alpha(u, v)) := \frac{-\langle u, v\rangle}{\sqrt{\langle u;u\rangle\langle v;v\rangle}} = \frac{-\langle \mathbf{u};\mathbf{v}\rangle}{\sqrt{\langle \mathbf{u};\mathbf{u}\rangle\langle \mathbf{v};\mathbf{v}\rangle}} =: \tag{7}$$

$$=: \operatorname{ch}\left(\frac{1}{k}s(\mathbf{u}, \mathbf{v})\right), \; 0 \leqq \alpha \leqq \pi.$$

At the same time we have defined the *imaginary distance* of the poles (**u**) and (**v**) by means of function ch extended onto the complex field **C** (5). If $\cos\left(\frac{1}{r}\cdot s(\mathbf{u}, \mathbf{v})\right)$ stands on the right side, then we get the elliptic case.

The formula (7) extends to parallel hyperplanes and to *not intersecting hyperplanes.* Moreover, *improper hyperplanes* may occur, too. All these generalizetions can be very useful, e.g., for deriving the *generalized trigonometry* uniformly in the hyperbolic and elliptic planes [15].

We mention only a simple example in $\mathcal{H}^2$.

Let $\mathfrak{b}^0$, $\mathfrak{b}^1$, $\mathfrak{b}^2$ be the forms representing the sides of a hyperbolic triangle with angles α^{01}, α^{12}, α^{20} *(Fig. 3)*. By (7) the symmetric matrix

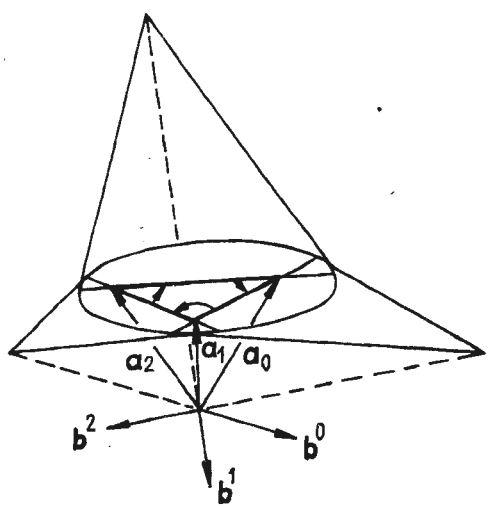


Fig. 3.

$$(\langle \mathfrak{b}^i;\mathfrak{b}^j\rangle) := \begin{pmatrix} 1 & -\cos\alpha^{01} & -\cos\alpha^{02} \\ -\cos\alpha^{10} & 1 & -\cos\alpha^{12} \\ -\cos\alpha^{20} & -\cos\alpha^{21} & 1 \end{pmatrix} =:$$

$$=:(\langle \mathbf{b}^i;\, \mathbf{b}^j\rangle) = (b^{ij})$$

induces a bilinear from in $\mathcal{V}_3$ or in $\mathbf{V}^3$ which os of index 1, iff

$\det(b^{ij}) := 1-\cos^2\alpha^{01}-\cos^2\alpha^{12}-\cos^2\alpha^{20}-2\cos\alpha^{01}\cos\alpha^{12}\cos\alpha^{20}<0.$

Let the vectors $\mathbf{a}_i = a_{ik}\mathbf{b}^k$ represent the vertices of the triangle. Since $(\mathbf{a}_i)$ I $(\mathfrak{b}^j)$ if $i\neq j$, and $(\mathbf{a}_i)$ lies in the positive half plane of $(\mathfrak{b}^i)$, we can assume that the equalities

$$\delta_i^j = \mathbf{a}_i\mathfrak{b}^j = \langle \mathbf{a}_i;\mathbf{b}^j\rangle = \langle a_{ik}\mathbf{b}^k;\, \mathbf{b}^j\rangle = a_{ik}b^{kj} \text{ and}$$

$$\langle \mathbf{a}_i;\, \mathbf{a}_k\rangle = \langle a_{ir}\mathbf{b}^r;\, a_{ks}\mathbf{b}^s\rangle = a_{ir}b^{rs}a_{sk} = \delta_i^s a_{sk} = a_{ik} \text{ hold.}$$

Inverting the matrix (b^{ij}) we get (a_{ik}) and so we can express any side length, by the angles. For instance by (3)

$$\operatorname{ch}\frac{1}{k}d_{12} = \frac{-a_{12}}{\sqrt{a_{11}a_{22}}} = \frac{\cos\alpha^{12}+\cos\alpha^{01}\cos\alpha^{20}}{\sin\alpha^{20}\sin\alpha^{01}}$$

holds, which formula is well known from the hyperbolic trigonometry.

3.3. The machinery, developed in Sections **1** and **2**, can be applied in the infinitesimal calculus as we indicated at the end of Section **2**. First, we give the so-called hyperbolic line element which can easily be derived from the second formula of (3)

$$\text{(8)} \qquad ds = \frac{k}{-\langle \mathbf{x};\mathbf{x}\rangle}\sqrt{\langle \mathbf{x};d\mathbf{x}\rangle^2-\langle \mathbf{x};\mathbf{x}\rangle\langle d\mathbf{x};d\mathbf{x}\rangle}.$$

Here $(\mathbf{x}$ and $(\mathbf{x}+d\mathbf{x})$ represent "infinitesimally near" points, as in the classical surface theory. More precisely, a smooth curve

$$[t_0,t_1] \ni t \mapsto \mathbf{x}(t) \in \mathbf{C}^+$$

has a hyperbolic arc length (see also the formula *2* (20))

$$\text{(9)} \qquad s(t_0,t_1) = \int_{t_0}^{t_1} \frac{k}{-\langle \mathbf{x}(t);\mathbf{x}(t)\rangle}\left\{-\mathrm{Gr}\left(\mathbf{x},\frac{d\mathbf{x}}{dt}\right)\right\}^{1/2} dt.$$

One can choose the curve $\mathbf{x}(t)$ with some projective freedom

$$\text{(10)} \qquad \mathbf{x}(t) \Leftrightarrow c(t)\cdot\mathbf{x}(t), \quad 0<c(t)\in C^1$$

($c(t)$ is an arbitrary positive smooth function).

We analogously define an r-dimensional infinitesimal volume element as an exterior r-form defined on the r-product of the tangent space to each point of an r-surface selected almost arbitrarily.

$$\text{(11)}\quad d\omega^r = k^r(-\langle \mathbf{x};\mathbf{x}\rangle)^{-(r+1)/2}\left\{-\mathrm{Gr}\left(\mathbf{x},\frac{\partial \mathbf{x}}{\partial u^1},\frac{\partial \mathbf{x}}{\partial u^r}\right)\right\}^{1/2} du^1\wedge\ldots\wedge du^r.$$

Here a smooth r-surface

$$\text{(12)}\qquad \mathfrak{U}\ni(u^1,\ldots,u^r)\mapsto \mathbf{x}(u^1,\ldots,u^r)\in\mathbf{C}^+$$

is defined with some projective freedom

$$\text{(13)}\qquad \mathbf{x}(u^i)\Leftrightarrow c(u^i)\mathbf{x}(u^i),\qquad 0<c(u^i)\in C^1(r)$$

($c(u^i)$ is an arbitrary positive smooth function). This property, namely the projectiv-metric invariance shows the definition to be adequate.

Consider an illustrative example. We introduce the polar coordinates in $\mathfrak{K}$. In the bases $\{\mathbf{e}_i\}$, $\{e^j\}$ of **2** (22) let $\mathbf{x} = x^i\mathbf{e}_i$ be defined by

$$\text{(14)}\quad \begin{array}{lll} x^0 & = k\,\mathrm{ch}\,\varphi^1 & 0\leqq\varphi^1<\infty;\\ x^1 & = k\,\mathrm{sh}\,\varphi^1\cos\varphi^2 & 0\leqq\varphi^j<\pi\ (j=2,\ldots,n-1)\\ \vdots & & \vdots\\ x^{n-1} & = k\,\mathrm{sh}\,\varphi^1\sin\varphi^2\sin\varphi^3\ldots\cos\varphi^n & 0\leqq\varphi^n<2\pi\\ x^n & = k\,\mathrm{sh}\,\varphi^1\sin\varphi^2\sin\varphi^3\ldots\sin\varphi^n. & \end{array}$$

Since $\langle \mathbf{x}(\varphi^i);\mathbf{x}(\varphi^i)\rangle = -k^2$ for every (φ^i), hence $\left\langle \mathbf{x};\dfrac{\partial \mathbf{x}}{\partial\varphi^i}\right\rangle = 0$.

Moreover, the matrix $(p_{ij}) = \left(\left\langle\dfrac{\partial \mathbf{x}}{\partial\varphi^i};\dfrac{\partial \mathbf{x}}{\partial\varphi^j}\right\rangle\right)$ can be computed:

$$\text{(15)}\qquad \left(\left\langle\frac{\partial \mathbf{x}}{\partial\varphi^i};\frac{\partial \mathbf{x}}{\partial\varphi^j}\right\rangle\right) =$$

$$= \begin{bmatrix} k^2 & 0 & 0 & 0\\ 0 & k^2\mathrm{sh}^2\varphi^1 & 0 & 0\\ 0 & 0 & k^2\mathrm{sh}^2\varphi^1\sin^2\varphi^2 & 0\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & k^2\mathrm{sh}^2\varphi^1\sin^2\varphi^2\ldots\sin^2\varphi^{n-1} \end{bmatrix}.$$

Then we have the line element by (8)

(16) $$ds = \sqrt{p_{ij}\, d\varphi^i d\varphi^j} =$$

$$= k\sqrt{d\varphi^1 d\varphi^1 + \mathrm{sh}^2\varphi^1 d\varphi^2 d\varphi^2 + \ldots + \mathrm{sh}^2\varphi^1 \sin^2\varphi^2 \ldots \sin^2\varphi^{n-1} d\varphi^n d\varphi^n},$$

and the n-dimensional volume element by (11)

(17) $$d\omega^n = \sqrt{\det(p_{ij})}\, d\varphi^1 \wedge \ldots \wedge d\varphi^n =$$
$$= k^n sh^{n-1}\varphi^1 \sin^{n-2}\varphi^2 \ldots \sin\varphi^{n-1}\, d\varphi^1 \wedge \ldots \wedge d\varphi^n.$$

This is in accordance with the Riemannian geometry. Taking $n = 3$ and the parameter domain

$$\mathfrak{U}:\ 0 \leqq \varphi^1 \leqq \frac{R}{k},\ 0 \leqq \varphi^2 < \pi,\ 0 \leqq \varphi^3 < 2\pi,$$

we compute by (17) the volume of the ball of radius R in $\mathfrak{H}^3$

(18) $$\omega(B(R)) = k^3 \int_{\varphi^3=0}^{2\pi} \left(\int_{\varphi^2=0}^{\pi} \left(\int_{\varphi^1=0}^{\frac{R}{k}} \mathrm{sh}^2\varphi^1 \sin\varphi^2 d\varphi^1 \right) \wedge d\varphi^2 \right) \wedge d\varphi^3 =$$

$$= 2\pi k^3 \left(\mathrm{ch}\frac{R}{k}\, \mathrm{sh}\frac{R}{k} - \frac{R}{k} \right).$$

Let $n = 2$ and take a hyperbolic circle of radius R

$$x^0(t) = k\,\mathrm{ch}\frac{R}{k},\ x^1(t) = k\,\mathrm{sh}\frac{R}{k}\cos t,\ x^2(t) = k\,\mathrm{sh}\frac{R}{k}\sin t,$$
$$0 \leqq \varphi^2 = t < 2\pi,$$

The perimeter of the circle can obviously be computed by (9) or (16)

(19) $$p(C(R)) = k\int_0^{2\pi} \mathrm{sh}\frac{R}{k}\, dt = 2\pi k\, \mathrm{sh}\frac{R}{k}.$$

3.4. We interpret the formula (11) in the following heuristic way (cf. [1], Chapter V, Exercises). We consider in $\mathbf{C}^+ \subset \mathbf{V}$ an infinitesimal parallelotop spanned by $r+1$ vectors

(20) $$\mathbf{x},\ \frac{\partial \mathbf{x}}{\partial u^1} du^1,\ \frac{\partial \mathbf{x}}{\partial u^2} du^2,\ \ldots,\ \frac{\partial \mathbf{x}}{\partial u^r} du^r$$

associated to each point $\mathbf{x}(u^i)$ of the r-surface (12). We take the volume of this parallelotop by 2 (21) with a norm factor $k^r(-\langle \mathbf{x};\mathbf{x}\rangle)^{-(r+1)/2}$ to get a projectiv-metric invariant volume element (11) as it was expressed at (13).

The volume of the infinitesimal parallelotop (20) may be rewritten as follows. We take the $(r+1)$-vector

$$\mathbf{x}\wedge\frac{\partial \mathbf{x}}{\partial u^1}du^1\wedge\frac{\partial \mathbf{x}}{\partial u^2}du^2\wedge\ldots\wedge\frac{\partial \mathbf{x}}{\partial u^r}du^r, \tag{21}$$

expressed by means of $\mathbf{e}_0\wedge\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_r$, where $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_r\}$ is a fixed basis, assumed to exist in the $(r+1)$-subspace at (20), with volume form

$$\omega = (e^0\wedge e^1\wedge\ldots\wedge e^r\sqrt{-\det\langle \mathbf{e}_i;\mathbf{e}_j\rangle}. \tag{22}$$

All these considerations are in the sense of 2 (18—21). Then, carrying out the infinitesimal computations, we get

$$\left\{-\mathrm{Gr}\left(\mathbf{x},\frac{\partial \mathbf{x}}{\partial u^1},\ldots,\frac{\partial \mathbf{x}}{\partial u^r}\right)\right\}^{1/2} du^1\wedge\ldots\wedge du^r = \tag{23}$$

$$= \left(\mathbf{x}\wedge\frac{\partial \mathbf{x}}{\partial u^1}du^1\wedge\ldots\wedge\frac{\partial \mathbf{x}}{\partial u^r}du^r\right)\omega =$$

$$= \left(x^{i_0}\,\mathbf{e}_{i_0}\wedge\frac{\partial x^{i_1}}{\partial u^1}du^1\mathbf{e}_{i_1}\wedge\ldots\wedge\frac{\partial \mathbf{x}^{i_r}}{\partial u^r}du^r\mathbf{e}_{i_r}\right)\omega =$$

$$= \left(x^0\,\mathbf{e}_0\wedge\frac{\partial \mathbf{x}^1}{\partial u^{j_1}}du^{j_1}\mathbf{e}_1\wedge\ldots\wedge\frac{\partial \mathbf{x}^r}{\partial u^{j_r}}du^{j_r}\mathbf{e}_r+\right.$$

$$+\frac{\partial x^0}{\partial u^{j_0}}du^{j_0}\,\mathbf{e}_0\wedge\mathbf{x}^1\mathbf{e}_1\wedge\ldots\wedge\frac{\partial \mathbf{x}^r}{\partial u^{j_r}}du^{j_r}\,\mathbf{e}_r+\ldots$$

$$\left.\ldots+\frac{\partial \mathbf{x}^0}{\partial u^{j_0}}du^{j_0}\,\mathbf{e}_1\wedge\frac{\partial x^1}{\partial u^{j_1}}du^{j_1}\mathbf{e}_1\wedge\ldots\wedge x^r\mathbf{e}_r\right)\omega =$$

$$= (x^0dx^1\wedge\ldots\wedge dx^r+(-1)^1x^1dx^0\wedge dx^2\wedge\ldots\wedge dx^r+\ldots+$$

$$+(-1)^rx^rdx^0\wedge dx^1\wedge\ldots\wedge dx^{r-1})\sqrt{-\det\langle \mathbf{e}_i;\mathbf{e}_j\rangle}.$$

We applied the usual formal conventions, for instance

$$(24)\qquad dx^i := \frac{\partial x^i}{\partial u^r} du^r,\ \operatorname{sign}(i_0, \ldots, i_r) = \operatorname{sign}(i_0, \ldots, i_r)^{-1} = \operatorname{sign}(j_0, \ldots, j_r).$$

We summarize the considerations in the following

PROPOSITION 3. *Let* $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_r\}$ *be a basis, with* $\langle \mathbf{e}_i;\mathbf{e}_j \rangle =: g_{ij}$, *in an* $(r+1)$*-subspace of* $\mathbf{V}$ *where an r-dimensional subspace of* $\mathfrak{K}$ *is modelled. Then the r-form*

$$(25)\qquad d\omega^r = k^r(-\det(g_{ij}))^{1/2} \cdot (-x^i g_{ij} x^j)^{-(r+1)/2} \left\{ \sum_{k=0}^{r} (-1)^k x^k dx^0 \wedge \overset{k}{\vee} \wedge dx^r \right\}$$

is a projectiv-metrically invarian r-dimensional volume element equivalent to (11) *with the assumptions mentioned above. The vector* $\mathbf{x} = x^i \mathbf{e}_i$ *runs over an arbitrary smooth r-surface* (12). (In (25) $\overset{k}{\vee}$ denotes that the differential dx^k is missing.)

Now we turn to our main subject.

4. Simplices with proper, ideal and improper faces and vertices

4.1. An *n-dimensional simplex* S in $\mathfrak{K}$ (or in $\mathfrak{S}$) will be represented by its side face hyperplanes $(\mathfrak{b}^0), (\mathfrak{b}^1), \ldots, (\mathfrak{b}^n)$ or its vertices $(\mathbf{a}_0), (\mathbf{a}_1), \ldots, (\mathbf{a}_n)$. We assume that

$$(1)\qquad \mathbf{a}_i \mathfrak{b}^j = \delta_i^j,$$

i.e., the face $(\mathfrak{b}^j)$ contains all vertices exept $(\mathbf{a}_j)$, and $(\mathbf{a}_j)$ lies in the positive half space of $(\mathfrak{b}^j)$ (see also **3.2.** and Fig. 3). Let the simplex S have the dihedral angles of measure

$$(2)\qquad \alpha(\mathfrak{b}^r, \mathfrak{b}^s) = \alpha^{rs} = \alpha^{sr} = \alpha^{(rs)}$$

Considering the bilinear form in $\mathfrak{V}$ or in $\mathbf{V}$, we normalize the forms $\mathfrak{b}^j$, in the sense of **3**(7), to get the Schläfli matrix

$$(3)\qquad (\langle \mathfrak{b}^i;\mathfrak{b}^j \rangle) = (\langle \mathbf{b}^i;\mathbf{b}^j \rangle) := \begin{bmatrix} 1 & -\cos\alpha^{01} & \ldots & -\cos\alpha^{0n} \\ -\cos\alpha^{10} & 1 & \ldots & -\cos\alpha^{1n} \\ \vdots & \vdots & & \vdots \\ -\cos\alpha^{n0} & -\cos\alpha^{n1} & \ldots & 1 \end{bmatrix} =: (b^{ij})$$

$$\mathfrak{b}^i,\ \mathfrak{b}^j \in \mathcal{V},\ \mathfrak{b}^i_* =: \mathbf{b}^i,\ \mathfrak{b}^j_* =: \mathbf{b}^j \in \mathbf{V}.$$

The dihedral angles (2) *characterize a proper hyperbolic simplex* (Fig. 3) *iff*

$$B = \det(b^{ij}) < 0 \tag{4}$$

but all other principal minor determinants are positive (see also [18]). For the elliptic case $B > 0$ stands instead of (4). ($B = 0$ determines the Euclidean simplices).

Then *the Schläfli matrix* (3) *fulfils the general criterion: it defines a hyperbolic projective metric iff the bilinear form*

$$\langle u;v\rangle = \langle \mathfrak{b}^i u_i; \mathfrak{b}^j v_j\rangle = u_i b^{ij} v_j = \langle \mathbf{u};\mathbf{v}\rangle \tag{5}$$

is of index 1. In the elliptic case the bilinear form (5) *is (positive) definite.*

The vertices $(\mathbf{a}_i)$ of the simplex S are characterized by the equations

$$\mathbf{a}_i := a_{ir}\mathbf{b}^r,\ \delta^j_i = \mathbf{a}_i \mathfrak{b}^j = \langle \mathbf{a}_i;\mathbf{b}^j\rangle = \langle a_{ir}\mathbf{b}^r;\mathbf{b}^j\rangle = a_{ir}b^{rj}, \tag{6}$$

$$\langle \mathbf{a}_i;\mathbf{a}_j\rangle = \langle a_{ir}\mathbf{b}^r;\mathbf{a}_{js}\mathbf{b}^s\rangle = a_{ir}b^{rs}a_{sj} = \delta^s_i a_{sj} = a_{ij},$$

$$A := \det(a_{ij}) = 1/\det(b^{ij}) = 1/B.$$

The requirements at (4) involve, by Lemma 1, that

$$a_{ij} = \langle \mathbf{a}_i;\mathbf{a}_j\rangle < 0 \quad (i, j = 0, 1, \ldots, n). \tag{7}$$

Therefore, each $\mathbf{a}_i$ is time-like vector, i.e., every simplex

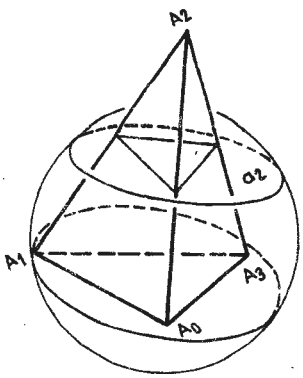


Fig. 4.

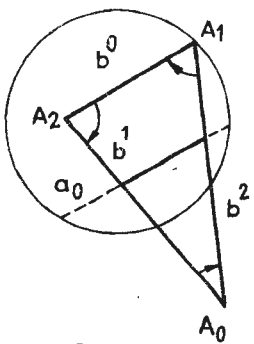


Fig. 5.

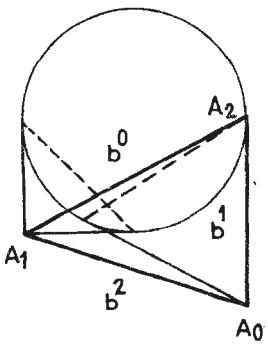


Fig. 6.

vertex is proper point of $\mathfrak{K}$. *The principal minor determinants of* (a_{ij}) *are called also Gramians of the corresponding vertices*

$$(8) \qquad \mathrm{Gr}\,(\mathbf{a}_{i_0}, \mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}) = \begin{vmatrix} a_{i_0 i_0} & a_{i_0 i_1} & \ldots & a_{i_0 i_r} \\ a_{i_1 i_0} & a_{i_1 i_1} & \ldots & a_{i_1 i_r} \\ \vdots & & & \vdots \\ a_{i_r i_0} & a_{i_r i_1} & \ldots & a_{i_r i_r} \end{vmatrix} < 0.$$

Although the bilinear form (5) is of index 1, the simplex S may have proper vertex, say $(\mathbf{a}_0)$, indeal vertex, say $(\mathbf{a}_1)$, and improper vertex, say $(\mathbf{a}_2)$, if $a_{00} < 0$, $a_{11} = 0$, $a_{22} > 0$, respectively (in *Fig.4* $n = 3$). Having an improper vertex $(\mathbf{a}_2)$, we can truncate the simplex by the polar hyperplane (α_2) which is prependicular to the side faces incident to $(\mathbf{a}_2)$, since $\mathbf{a}_2 \mathfrak{b}^j = \langle \alpha_2; \mathfrak{b}^j \rangle = 0$ if $j \neq 2$. All these combinations can be discussed in the theory. Moreover, simplices with parallel faces or not intersecting faces can occur by 3 (7), e.g. in *Fig. 5* ($n = 2$),

$$(b^{ij}) = \begin{bmatrix} 1 & -\cos\alpha^{01} & -1 \\ -\cos\alpha^{10} & 1 & -\mathrm{ch}\,(a^{12}/k) \\ -1 & -\mathrm{ch}\,(a^{21}/k) & 1 \end{bmatrix}.$$

Here the faces $(\mathfrak{b}^0)$ and $(\mathfrak{b}^2)$ are parallel (with angle 0); $(\mathfrak{b}^1)$ and $(\mathfrak{b}^2)$ do not intersect in proper point (its distance is even a^{12}).

The simplex in *Fig. 6* has ideal and improper faces as well. Its Schläfli matrix is of form

$$(b^{ij}) = \begin{bmatrix} 1 & 0 & b^{02} \\ 0 & 0 & b^{12} \\ b^{20} & b^{21} & -1 \end{bmatrix}; \; b^{12} \neq 0.$$

4.2. As an illustration, we compute for a simplex S the radii of the inscribed and circumscribed balls, respectively (see application in [4]).

First, we write down the distance of the point $(\mathbf{x})$ and the hyperplane (u) by means of **2** (11) and **3**(3)

$$\mathrm{sh}\,\frac{d}{k} = \frac{\mathbf{x}u}{\sqrt{-\langle \mathbf{x};\mathbf{x}\rangle \langle u;u\rangle}}; \; \mathbf{x}u > 0, \; \langle u;u\rangle > 0, \; \langle \mathbf{x};\mathbf{x}\rangle < 0$$

Let r denote the radius, $(\mathbf{x})$ be the centre of the inscribed ball. The form $\mathfrak{b}^i$ shall represent the proper faces of S with the normalizing assumptions $\langle \mathfrak{b}^i; \mathfrak{b}^i \rangle = \langle \mathbf{b}^i; \mathbf{b}^i \rangle = 1$.

Then

$$\operatorname{sh}\frac{r}{k} = \frac{\langle \mathbf{x};\mathbf{b}^j\rangle}{\sqrt{-\langle \mathbf{x};\mathbf{x}\rangle}} = \frac{x^j}{\sqrt{-x^r a_{rs} x^s}} \quad \text{for each } j = 0, 1, \ldots, n$$

holds in the sense of 2 (8)—(9), i.e., $x^0 = x^1 = \ldots = x^n$ and we get

$$\operatorname{sh}\frac{r}{k} = \frac{1}{\sqrt{\left(-\sum_{r,s=0}^{n} a_{rs}\right)}}. \tag{9}$$

We have got also the *criterion of existence: the "vertex matrix"* (a_{ij}), *i.e. the inverse of* (b^{ij}), *must have a negative sum of its entries.* Then the root term in (9) will be positive.

Second, let R denote the radius, $(\mathbf{y})$ be the centre of the circumscribed ball. The proper vertices of S are represented by the vectors $\mathbf{a}_i$ with the given matrix of negative entries

$$(\langle \mathbf{a}_i;\mathbf{a}_j\rangle) = (a_{ij}),\ a_{ij}<0 \ \ (i, j = 0, 1, \ldots, n).$$

Then, with $(a_{ij})^{-1} = (b^{rs})$ by 2 (8)—(9), we get

$$\operatorname{ch}\frac{R}{k} = \frac{-\langle \mathbf{y};\mathbf{a}_j\rangle}{\sqrt{\langle \mathbf{y};\mathbf{y}\rangle\langle \mathbf{a}_j;\mathbf{a}_j\rangle}} = \frac{-y_j}{\sqrt{(y_r b^{rs} y_s)(a_{jj})}};\ j = 0, 1, \ldots, n.$$

Hence $y_j = c\sqrt{-a_{jj}}$ and we have the formula

$$\operatorname{ch}\frac{R}{k} = \frac{1}{\sqrt{-\left(\sum_{r,s=0}^{n} \sqrt{-a_{rr}}\, b^{rs}\sqrt{-a_{ss}}\right)}}. \tag{10}$$

The *criterion of existence* is again: *the root term in* (10) *must be positive.* Normalizing the vertex matrix (a_{ij}), with $a_{ii} = -1$ for each i, its inverse, the face matrix (b^{ij}) must have negative sum of entries.

5. Simplex volume and Schläfli's differential form

Now, we apply the theory developed. If (b^{ij}) is the face matrix, (a_{ij}) is the vertex matrix of the n-simplex S, by **4** (1)—(7) then we can describe the points

of the vertex vectors $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$. Then we can compute the volume $\omega(S)$ of the simplex S by 3(25). The implicit dependence of $\omega(S)$ from the face matrix (b^{ij}) and so from the angles of S enables us to form the Schläfli differential of $\omega(S)$ by the angles.

5.1. The simplest way, to give a parametrized n-surface in V for the simplex S, is

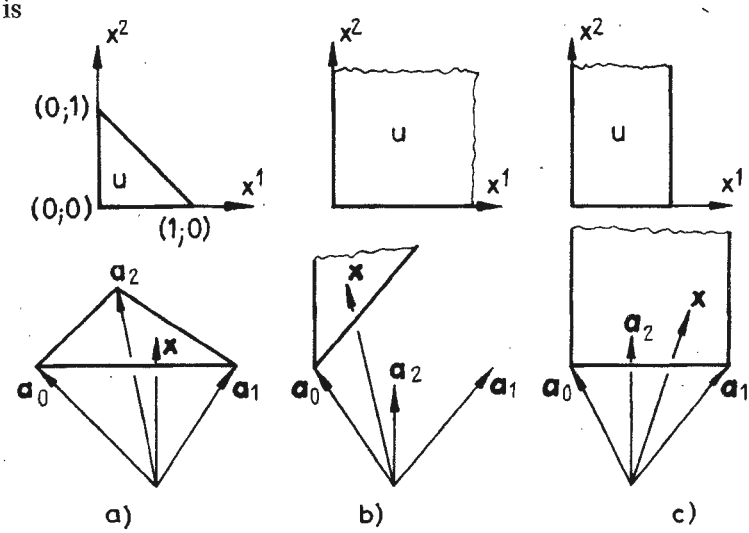


Fig. 7.

(1) $$\mathbf{x} = x^0\mathbf{a}_0 + x^1\mathbf{a}_1 + \ldots + x^n\mathbf{a}_n =: x^i\mathbf{a}_i$$

(2) $$x^0 + x^1 + \ldots + x^n = 1, \quad x^0, x^1, \ldots, x^n \geqq 0$$

(Fig, 7.a).

This seems to be most convenient for numerical computation by a computer. Then we apply formula 3 (25) with taking

$$r=n;\ (g_{ij}) = (a_{ij}) = (b^{ij})^{-1};\quad x^0 = 1 - x^1 - \ldots - x^n,\ dx^0 = -dx^1 - \ldots - dx^n,$$

$$\sum_{k=0}^{n} (-1)^k x^k\, x^0 \wedge \ldots \overset{k}{\vee} \ldots \wedge dx^n = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n;$$

(3) $$\omega(S) =$$

$$= \int_{\substack{x^1, \ldots, x^n \geqq 0 \\ x^1 + x^2 + \ldots + x^n \leqq 1}} \cdots \int \frac{k^n[-\det(a_{ij})]^{1/2} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n}{\left\{ -\left[a_{00} - 2\sum_{r=1}^{n} x^r(a_{00} - a_{r0}) + \sum_{i,j=1}^{n} x^i x^j (a_{00} - a_{i0} - a_{0j} + a_{ij}) \right] \right\}^{\frac{n+1}{2}}}$$

The parameter domain $\mathfrak{U}$ is the fixed normal simplex in the n-dimensional numerical space $\mathbf{R}^n$. Different simplices occur only by varying matrices (a_{ij}).

Up to now we assumed that the hyperbolic simplex S has only proper vertices, i.e. $a_{ii}<0$ for each $i = 0, 1, \ldots, n$. If, say, $(\mathbf{a}_0)$ is ideal vertex, then (3) is a convergent improper integral. A simplex with only ideal vertices has also finite volume [3], [5], [8], [16],[17].

If, say, $(\mathbf{a}_j)$ is improper vertex, $a_{jj}>0$, then we can truncate the simplex S with the polar hyperplane (α_j) to get a body of finite volume. The additional inequality

$$\mathbf{x}\alpha_j = \langle \mathbf{x};\mathbf{a}_j\rangle = \langle x^r\mathbf{a}_r;\mathbf{a}_j\rangle = x^r a_{rj}<0; \text{ i.e.,} \tag{4}$$
$$a_{0j}+x^1(a_{1j}-a_{0j})+\ldots+x^n(a_{nj}-a_{0j})<0$$

represents a hyperplane in $\mathbf{R}^n$ truncating the parametrizing normal simplex. Other truncations can be made in similar way.

Another parametrization *(Fig. 7.b)* to (1) is

$$x^0 = 1,\ 0 \leqq x^1<\infty, \ldots, 0 \leqq x^n<\infty; \tag{5}$$
$$dx^0 = 0,\ \sum_{k=0}^{n} (-1)^k x^k dx^0\wedge \overset{k}{\vee} .\wedge dx^n = dx^1\wedge dx^2\wedge \ldots \wedge dx^n$$

which leads to the improper integral

$$\omega(S) = \int_{\substack{0\leqq x^1<\infty \\ \vdots \\ 0\leqq x^n<\infty}} \cdots \int \frac{k^n[-\det(a_{ij})]^{1/2} dx^1\wedge dx^2\wedge \ldots \wedge dx^n}{\left\{-\left[a_{00}+2\sum_{r=1}^{n} x^r a_{r0}+\sum_{i,j=1}^{n} x^i x^j a_{ij}\right]\right\}^{\frac{n+1}{2}}} \tag{6}$$

This has some benefits in theoretical investigations ((see at (20)).

The parametrization *(Fig. 7. c)*

$$0\leqq x^n<\infty,\ x^0+x^1+\ldots+x^{n-1} = 1,\ x^0, x^1, \ldots, x^{n-1}\geqq 0 \tag{7}$$
$$x^0 = 1-x^1-\ldots-x^{n-1},\ \sum_{k=0}^{r} (-1)^k x^k dx^0\wedge \overset{k}{\vee} .\wedge dx^n = dx^1\wedge dx^2\wedge \ldots \wedge dx^n$$

which leads to the improper integral

$$(8)\qquad \omega(S) = \int\cdots\int_{\substack{0\leq x^n<\infty\\ x^1,\ldots,x^{n-1}\geq 0\\ x^1+\ldots+x^{n-1}\leq 1}} \frac{k^n[-\det(a_{ij})]^{1/2}\, dx^1\wedge dx^2\wedge\ldots\wedge dx^n}{\left\{-\left[Q(n-1)+2x^n a_{n0}+2\sum_{s=1}^{n-1} x^n x^s(a_{ns}-a_{n0})+x^n x^n a_{nn}\right]\right\}^{\frac{n+1}{2}}}$$

where $Q(n-1)$ is a quadratic form of $n-1$ variables, such as in denominator of (3). This can be useful in the reduction theory, e.g., for regular simplex $a_{ns}-a_{n0} = 0$ holds, for ideal vertex (a_n) $a_{nn} = 0$ stands.

5.2. To derive the Schläfli's differential form uniformly for the hyperbolic and elliptic case, we write down the volume of the simplex S by **3** (25) in the form

$$(9)\qquad \omega(S) = \int_{\mathfrak{U}} \frac{[\epsilon\det(a_{ij})]^{1/2}\left(\sum_{k=0}^{n}(-1)^k x^k dx^0\wedge\ldots\overset{k}{\vee}\ldots\wedge dx^n\right)}{\{\epsilon(x^i a_{ij}x^j)\}^{(n+1)/2}},$$

where $\epsilon = 1$ stands for the elliptic, $\epsilon = -1$ for the hyperbolic case, the constant factor k^n(or, r^n) is omitted ($K = \pm 1$).

We refer to Section **4**, where the face matrix (b^{ij}) and its inverse (a_{ij}) are introduced. The fixed n-surface $\mathfrak{U}$ is assumed as in **5.1** or in **3.3—4.**

Let $\alpha^{(rs)}$ the angle of the faces $(\mathfrak{b}^r)$ and $(\mathfrak{b}^s)$ as in **4** (2). We have to take by the chain rule

$$(10)\qquad \frac{\partial\omega}{\partial\alpha^{(rs)}} = \frac{\partial\omega}{\partial a_{ij}}\cdot\frac{\partial a_{ij}}{\partial b^{rs}}\cdot\frac{\partial b^{rs}}{\partial\alpha^{(rs)}} + \frac{\partial\omega}{\partial a_{ij}}\cdot\frac{\partial a_{ij}}{\partial b^{sr}}\cdot\frac{\partial b^{sr}}{\partial\alpha^{(rs)}}.$$

$$(11)\qquad \frac{\partial b^{rs}}{\partial\alpha^{(rs)}} = \sin\alpha^{(rs)} = (b^{rr}b^{ss}-b^{rs}b^{sr})^{1/2}$$

holds by **4** (3). Since $a_{ij}b^{jk}\,\delta^{ik}$, hence by derivation

$$0 = \frac{\partial a_{ij}}{\partial b^{rs}}b^{jk}+a_{ij}\frac{\partial b^{jk}}{\partial b^{rs}} = \frac{\partial a_{ij}}{\partial b^{rs}}b^{jk}+a_{ij}\delta^j_r\delta^k_s \quad\text{and}\quad \frac{\partial a_{ij}}{\partial b^{rs}}b^{jk} = -a_{ir}\delta^k_s.$$

Multiplying by a_{kl} and summing up for index k in both terms:

$$\frac{\partial a_{ij}}{\partial b^{rs}}b^{jk}a_{kl} = -a_{ir}\delta^k_s a_{kl} \Rightarrow \frac{\partial a_{ij}}{\partial b^{rs}}\,\delta^j_l = -a_{ir}a_{sl}$$

and we get (with $l = j$)

$$(12)\qquad \frac{\partial a_{ij}}{\partial b^{rs}} = -a_{ir}a_{sj}.$$

Using the derivation rule of determinant det $(a_{ij}) =: A$, we have

$$\frac{\partial}{\partial a_{ij}}[\epsilon A]^{1/2} = \frac{1}{2}[\epsilon A]^{-1/2}[\epsilon A]b^{ji} = \frac{1}{2}[\epsilon A]^{1/2} \cdot b^{ji}$$

and so

$$\text{(13)} \quad \frac{\partial}{\partial a_{ij}}\{[\epsilon A]^{1/2}\} \cdot \frac{\partial a_{ij}}{\partial b^{rs}} = \frac{1}{2}[\epsilon A]^{1/2} b^{ji}(-a_{ir}a_{sj}) = -\frac{1}{2}[\epsilon A]^{1/2} a_{sr}.$$

Furthermore:

$$\text{(14)} \quad \frac{\partial}{\partial a^{ij}}\{\epsilon(x^i a_{ij} x^j)\}^{-(n+1)/2} = -\frac{n+1}{2}\{\epsilon(x^i a_{ij} x^j)\}^{-(n+3)/2} x^i x^j \cdot \epsilon.$$

Now, we have by (10) and by the symmetry $b^{rs} = b^{sr}$

$$\text{(15)} \quad \frac{1}{2}\frac{\partial \omega}{\partial \alpha^{(rs)}} = \int_{\mathfrak{U}} \frac{-1}{2}[\epsilon A(b^{rr}b^{ss} - b^{rs}b^{sr})]^{1/2} \cdot$$

$$\cdot\{[\epsilon(x^i a_{ij} x^j)]^{-(n+1)/2} a_{sr} - \epsilon(n+1)[\epsilon(x^i a_{ij} x^j)]^{-(n+3)/2}(x^i a_{ir})(x^j a_{sj})\} \cdot$$

$$\left(\sum_{k=0}^{n} (-1)^k x^k dx^0 \wedge \overset{k}{\vee} \wedge dx^n\right).$$

We prepare the following steps by taking

$$\text{(16)} \quad \frac{\partial}{\partial x^r}[\epsilon \cdot (x^i a_{ij} x^j)]^{-(n+1)/2} = -\epsilon(n+1)[\epsilon(x^i a_{ij} x^j)]^{-(n+3)/2}(a_{ru} x^u);$$

$$\text{(17)} \quad \frac{\partial}{\partial x^r}\left[\{\epsilon(x^i a_{ij} x^j)\}^{-(n+1)/2}(x^u a_{su})\right] =$$

$$= \left(\frac{\partial}{\partial x^r}\{\epsilon(x^i a_{ij} x^j)\}^{-(n+1)/2}\right)(x^u a_{su}) + \{\epsilon(x^i a_{ij} x^j)\}^{-(n+1)/2} a_{sr} =$$

$$= \frac{\partial}{\partial x^r}\left[\frac{\partial}{\partial x^s}\left(-\frac{\epsilon}{n-1}\{\epsilon(x^i a_{ij} x^j)\}^{-(n-1)/2}\right)\right].$$

As a result we can read from (15—17)

$$\frac{\partial\omega}{\partial\alpha^{(rs)}} = \tag{18}$$

$$= \frac{\epsilon}{n-1}\int_{\mathfrak{U}} [\epsilon A(b^{rr}b^{ss}-b^{rs}b^{sr})]^{1/2}\left[\frac{\partial}{\partial x^r}\left(\frac{\partial}{\partial x^s}\{\epsilon(x^i a_{ij}x^j)\}^{-(n-1)/2}\right)\right]\cdot$$

$$\cdot\left(\sum_{k=0}^{n}(-1)^k x^k dx^0\wedge\overset{k}{\vee}\wedge dx^n\right).$$

First, we observe by Lemma 1 and by **4** (8), that

$$A(b^{rr}b^{ss}-b^{rs}b^{sr}) = \mathrm{Gr}(\mathbf{a}_0,\mathbf{a}_1,\overset{r}{\vee},\ \overset{s}{\vee},\ \mathbf{a}_n) =: A^{(rs)} \tag{19}$$

is just the Gramian of the vertices in the intersections of hyperplanes $(\mathfrak{b}^r)$ and $(\mathfrak{b}^s)$ where only $(\mathbf{a}_r)$ and $(\mathbf{a}_s)$ are missing.

Second, instead of using a general *Stokes formula,* we simply apply Newton-Leibniz formula in (18) two times with respect to the parametrization specialized in (5). Without loss of generality we may assume that $r = 1, s = 2$. Let $Q(x^1,x^2,\ldots,x^n)$ denote the quadratic form in denominator of (6):

$$\frac{\partial\omega}{\partial\alpha^{(12)}} = \tag{20}$$

$$= \frac{\epsilon}{n-1}\int_{\substack{0\leq x^1<\infty\\ \vdots\\ 0\leq x^n<\infty}}\cdots\int[\epsilon A^{(12)}]^{1/2}\left[\frac{\partial}{\partial x^1}\left(\frac{\partial}{\partial x^2}\{\epsilon Q(x^1,x^2,\ldots,x^n)\}^{-(n-1)/2}\right)\right]\cdot$$

$$\cdot dx^1\wedge dx^2\wedge\ldots\wedge dx^n=$$

$$= \frac{\epsilon}{n-1}\int_{x^1\to\infty}\cdots\int[\epsilon A^{(12)}]^{1/2}\left(\frac{\partial}{\partial x^2}\{\epsilon Q(x^1,x^2,\ldots,x^n)\}^{-(n-1)/2}\right)\cdot$$

$$\cdot dx^2\wedge dx^3\wedge\ldots\wedge dx^n-$$

$$-\frac{\epsilon}{n-1}\int_{x^1=0}\cdots\int[\epsilon A^{(12)}]^{1/2}\left(\frac{\partial}{\partial x^2}\{\epsilon Q(0,x^2,\ldots,x^n)\}^{-(n-1)/2}\right)\Bigg]\cdot$$

$$\cdot dx^2\wedge dx^3\wedge\ldots\wedge dx^n=$$

$$= 0 - \frac{\epsilon}{n-1} \int_{\substack{x^1=0 \\ x^2=\infty}} \cdots \int [\epsilon A^{(12)}]^{1/2} \{\epsilon Q(0, x^2, \ldots, x^n)\}^{-(n-1)/2} \cdot dx^3 \wedge \ldots \wedge dx^n +$$

$$+ \frac{\epsilon}{n-1} \int_{\substack{x^1=0 \\ x^2=0}} \cdots \int [\epsilon A^{(12)}]^{1/2} \{\epsilon \cdot Q(0, 0, \ldots, x^n)\}^{-(n-1)/2} \cdot dx^3 \wedge \ldots \wedge dx^n =$$

$$= \frac{\epsilon}{n-1} \int_{\substack{x^1=x^2=0 \\ 0 \leq x^3, \ldots, x^n < \infty}} \cdots \int \frac{[\epsilon A^{(12)}]^{1/2}\, dx^3 \wedge \ldots \wedge dx^n}{\left\{\epsilon \left[a_{00} + 2\sum_{r=3}^{n} x^r a_{r0} + \sum_{i,j=3}^{n} x^i x^j a_{ij}\right]\right\}^{\frac{n-1}{2}}}.$$

If $n = 2$, then we have at the last integration

$$\frac{\epsilon}{n-1} [\epsilon A^{(12)}]^{1/2} \{\epsilon a_{00}\}^{-1/2} = \frac{\epsilon}{n-1} \cdot 1.$$

If $n \geqq 3$ then the last integral is just the $(n-2)$-dimensional volume of the subsimplex $S^{(12)}$ where the vertices (a_1) and (a_2) are missing. We summarize the result in

PROPOSITION 4. (Schläfli's formula.) *Let S be a simplex in a non-Euclidean n-space of constant sectional curvature K. Let $\alpha^{(rs)}$ denote the measure of dihedral angle of the r-th and s-th face intersecting each other in the $(n-2)$-dimensional subsimplex $S^{(rs)}$ $(0 \leqq r < s \leqq n)$. Then Schläfli's differential form reads*

$$d\omega^n(S) = \frac{1}{K(n-1)} \sum_{\substack{r,s=0 \\ r<s}}^{n} \omega^{n-2}[S^{(rs)}] d\alpha^{(rs)}, \quad n \geqq 2 \tag{21}$$

(with $\omega^{n-2}(S^{(rs)}) = 1$, if $n = 2$).

The method, applied here, seems to provide other possibilities of generalizations, e.g. for truncated simplices of different types.

6. Computational results in $\mathcal{H}^3$

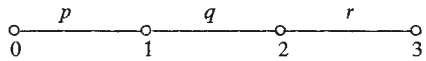
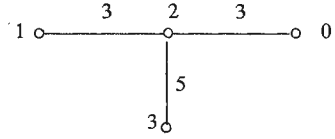
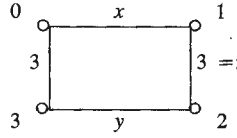
6.1. There are a lot of important simplices in $\mathcal{H}^3$ which have *natural dihedral angles* of the form $\alpha^{(rs)} = \frac{\pi}{m(rs)}$, where $2 \leqq m(rs) \in \mathbf{N}$ (natural numbers) [1], [4], [18]. Moreover, these kinds of *truncated simplices* do exist [9], [12], [13], [14]. Ideal vertices are also permitted.

The reflections in the side face hyperplanes of such a simplex (by formulas **2** (11)) generate a discrete transformation group, called *Coxeter group*, whose fundamental domain is the simplex itself. The volumes of these simplices seem to be important invariants for these groups (see introduction). There are 9 such compact simplices in $\mathcal{H}^3$ which were first enumerated by F. LANNER (1950) (see e.g. in [18]). Their *Coxeter diagrams* (graphs) and brief notations are indicated in the following table. The *nodes* (vertices) of such a diagram represent the planes of the corresponding simplex. A *branch* (edge) between the nodes r and s is indicated by $m(rs) \in \mathbf{N}$ if the dihedral angle $\alpha^{(rs)}$ of the simplex is even $\frac{\pi}{m(rs)}$. There is no branch if the dihedral angle is $\frac{\pi}{2}$.

We indicated the distances d_{ij} between the simplex vertices i and j (a face i does not contain the vertex i) and the volume of the simplex up to accuracy of 3 decimals. These latter computations were carried out by L. BREZOVICH with *Commodore* 64 on the base of formula **5** (3).

ADDED IN PROOF. E. B. VINBERG informed me about his computations concerning the volumes of Lanner's orthoschemes on the base of Lobachevski function. He also told that R. MEYERHOFF computed all the volumes of Lanner simplices (these estimates are published in *Duke Math. J.*, **57** (1988), 185—203). I thank him for this information. In a forthcoming paper with B. CSIKÓS we have computed all the finite volumes of simplices with natural dihedral angles. I inserted the corresponding results in the table.

TABLE

Simplex S	d_{01}	d_{02}	d_{03}	d_{12}	d_{13}	d_{23}	$\omega(S)$
Orthoschemes		 =: (p, q, r)					$k = 1$ E. B. Vinberg
(3, 5, 3)	0,8683	0,9727	1,3826	0,3942	0,9727	0,8683	0,039 (0,0391)
(4, 3, 5)	0,8085	1,0613	1,2265	0,6269	0,8425	0,5306	0,036 (0,0359)
(5, 3, 5)	0,9964	1,4391	1,9028	0,9136	1,4391	0,9964	0,093 (0,0933)
		 = L					$k = 1$ B. Csikós E. Molnár
L	1,6169	1,0613	1,2265	1,0613	1,2265	0,5306	0,072 (0,0718)
		 =: (x, y)					$k = 1$
(4, 3)	0,7691	1,1284	1,0148	1,0148	1,1284	1,1284	0,086 (0,0858)
(4, 4)	1,3843	1,7000	1,5668	1,5668	1,7000	1,3843	0,222 (0,2222)
(5, 3)	0,9291	1,7611	1,6267	1,6267	1,7611	2,0444	0,205 (0,2053)
(5, 4)	1,4871	2,2731	2,1328	2,1328	2,2731	2,2240	0,359 (0,3586)
(5, 5)	2,3016	2,8264	2,6838	2,6838	2,8264	2,3016	0,502 (0,5021)

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ON EIGENFUNCTION EXPANSIONS

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This paper is devoted to the study of two expansion problems. In the first one we prove exact order for the strong summability of expansions by a complete orthonormal system consisting of eigenfunctions of a Schrödinger operator. As G. ALEXITS and D. KRÁLIK [7] proved, for the strong trigonometric means the classical estimates of Bernstein concerning Lipschitz classes remain valid.

Recently I. Joó proved the strong summability for the expansions with respect to the eigenfunctions of a linear differential operator ([8]). In the next theorem we prove a lower estimate for the Lip 1 class, proved by I. Joó and N. BRUJ for the trigonometric case ([14]). In the second part of this paper we prove an equiconvergence theorem between the Laguerre and trigonometric expansions. For the Hermite and trigonometric expansions the analoguous result was proved by I. Joó in [8].

Consider the Schrödinger operator

$$Lu := -u'' + qu$$

in a finite open interval G and with $q \in L^1(G)$ and suppose that $(u_n)_1^\infty$ is a complete orthonormal system in $L^2(G)$ of eigenfunctions of L:

$$Lu_n = \lambda_n u_n$$

with positive eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$. It is known ([1], [2]) that in this case there exists a constant $M > 0$ such that

$$0 < c_1 \leq \sum_{|\sqrt{\lambda_k} - \mu| \leq M} 1 \leq c_2 < \infty \quad (\mu > 0) \tag{1}$$

with c_1, c_2 independent of μ. Denote

$$S_k^L(f, x) := \sum_{j=1}^{k} \langle f, u_j \rangle u_j(x)$$

the k-th partial sum of the expansion of $f \in L^2(G)$ by the system (u_n). Consider further for $0 \leq \alpha \leq 1$ the function spaces ([3]) $B_\alpha(G)$ consisting of all functions f defined on $\bar{G} = [a, b]$ and representable in the form

$$f(x) = \int_a^x (x-t)^\alpha d\rho_1(t) + \int_x^b (t-x)^\alpha d\rho_2(t)$$

where $\rho_1, \rho_2 \in BV[a, b]$. It is known that in this case $f \in BV[a, b]$. I. Joó and A. B. Nersesjan proved the following

Theorem A [3]. *Let* $0 \leq \alpha \leq 1$, $f \in B_\alpha(G)$; *then*

$$\langle df, e^{-i\lambda x} \rangle = \int_G e^{i\lambda x} df(x) = O(\lambda^{-\alpha}) \quad (0 < \lambda). \tag{2}$$

Remark that $B_0 = BV \supset W_1^1$, $B_1 = \{f : f' \in BV\} \supset W_1^2$ and in general

$$B_\alpha \supset W_1^{1+\alpha} \quad (\alpha \geq 0)$$

where $W_1^{1+\alpha}$ denotes the Soboleff space of fractional order $1+\alpha$. We shall prove the following

Theorem 1. *Let* $0 < \alpha < 1$ *and* $[a, b] \subset G$. *Suppose that* $q \in L^1(G) \cap L^p_{loc}$ for some $1 < p < \infty$. *Then for any* $f \in B_\alpha(G)$ *such that* f *is absolutely continuous in* $[a, b]$ *and* $f' \in$ Lip $(\alpha, [a, b])$ *we have*

$$\frac{1}{n} \sum_{k=1}^{n} |S_k^L(df, x) - f'(x)| \leq cn^{-\alpha} \quad (x \in [a', b']) \tag{3}$$

$$S_k^L(df, x) := \sum_{j=1}^{k} \langle df, u_k \rangle u_k(x)$$

for any subsegment $[a', b'] \subset (a, b)$, *and the constant* $c = c(a, b, a', b', \alpha, q, f)$ *is independent of* x *and* n.

Proof. Denote for $\mu > 0$

$$S_\mu^{T,R}(df, x) := \frac{1}{\Pi} \int_{x-R}^{x+R} \frac{\sin(\mu+1/2)(y-x)}{y-x} f'(y)dy$$

and for $g \in L^1(0, 2\Pi)$ let

$$S_\mu^T(g, x) := \frac{1}{\Pi} \int_0^{2\Pi} \frac{\sin(\mu+1/2)(y-x)}{2\sin\frac{y-x}{2}} g(y)dy$$

denote the usual μ-th partial sum of the trigonometric series of g. Let $x \in [a', b']$, $0 < R_0 < \frac{1}{3}\min(a'-a, b-b')$ and $\frac{R_0}{2} \le R \le R_0$. We can suppose that $0 < a' < b' < 2\Pi$.

Define

$$g(y) := \begin{cases} f'(y) & \text{if } a' \le y \le b' \\ 0 & \text{if } y \in [0, 2\Pi) \setminus [a', b'] \\ g(y \pm 2\Pi) = g(y) \end{cases}$$

then obviously

$$S_\mu^{T,R}(df, x) = S_\mu^{T,R}(g, x).$$

Now define the average operation

$$D_{R_0}[F] := \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} F(R)dR$$

and consider the decomposition

$$(4) \qquad S_k^L(df, x) - f'(x) = \left\{S_k^L(df, x) - D_{R_0}\left[S_{\sqrt{\lambda_k}}^{T,R}(f', x)\right]\right\} +$$

$$+ \left\{D_{R_0}\left[S_{\sqrt{\lambda_k}}^{T,R}(f', x)\right] - S_{\sqrt{\lambda_k}}^{T,R_0}(f', x)\right\} + \left\{S_{\sqrt{\lambda_k}}^{T,R_0}(f', x) - S_{\sqrt{\lambda_k}}^{T}(g, x)\right\} +$$

$$+ \left\{S_{\sqrt{\lambda_k}}^{T}(g, x) - g(x)\right\} =: I_1 + I_2 + I_3 + I_4.$$

We shall show that

$$(5) \qquad |I_j| = O(\lambda_k^{-\frac{\alpha}{2}}) \qquad j = 1, 2, 3.$$

By (1) this will imply

$$(5') \qquad |I_j| = O(k^{-\alpha}) \qquad j = 1, 2, 3.$$

Estimate first I_1. Define

$$\delta(\mu, \sqrt{\lambda_k}) := \begin{cases} 1 & \text{if} \quad \mu > \sqrt{\lambda_k} \\ \frac{1}{2} & \text{if} \quad \mu = \sqrt{\lambda_k} \\ 0 & \text{if} \quad \mu < \sqrt{\lambda_k}. \end{cases}$$

Then

$$S_n^L(df, x) = \sum_{k=1}^{\infty} \langle df, u_k \rangle u_k(x) \delta(\mu, \sqrt{\lambda_k}) \quad \text{if} \quad \sqrt{\lambda_n} < \mu < \sqrt{\lambda_{n+1}}. \tag{6}$$

Define further for fixed μ, $R>0$ and x the function

$$w(x+t) := \begin{cases} \sin \dfrac{(\mu+1/2)t}{\Pi t} & \text{if} \quad |t| < R \\ 0 & \text{otherwise.} \end{cases}$$

Using the Titchmarsh—formula

$$u_k(x+t) + u_k(x-t) = 2u_k(x) \cos \sqrt{\lambda_k}\, t + \tag{7}$$

$$+ \int_{x-t}^{x+t} \frac{\sin \sqrt{\lambda_k}\,(t - |x-\xi|)}{\sqrt{\lambda_k}} q(\xi) u_k(\xi) d\xi$$

we get

$$\langle u_n, w \rangle = \int_0^R \frac{\sin(\mu+1/2)t}{\Pi t} [u_k(x+t) + u_k(x-t)] dt =$$

$$= 2u_k(x) \int_0^R \frac{\sin(\mu+1/2)t}{\Pi t} \cos \sqrt{\lambda_k}\, t\, dt +$$

$$+ \int_0^R \frac{\sin(\mu+1/2)t}{\Pi t} \int_{x-t}^{x+t} \frac{\sin \sqrt{\lambda_k}\,(t - |x-\xi|)}{\sqrt{\lambda_k}} q(\xi) u_k(\xi) d\xi\, dt.$$

Since $D_{R_0} w \in L^2(G)$, the series

$$D_{R_0} w = \Sigma D_{R_0} \langle u_n, w \rangle u_n$$

corverges in L^2-norm. We have for $\sqrt{\lambda_n} < \mu < \sqrt{\lambda_{n+1}}$

$$(8)\qquad S_n^L(df, x) - D_{R_0}\left[S_\mu^{T,R}(df, x)\right] = \langle df(t), \sum_{k=1}^{\infty} u_k(t)\{c_{k,1}(x)+c_{k,2}(x)\}\rangle$$

with

$$c_{k,1}(x) := u_k(x)\left[\delta(\mu, \sqrt{\lambda_k}) - D_{R_0}\left[\frac{2}{\Pi}\int_0^R \frac{\sin(\mu+1/2)t}{t}\cos\sqrt{\lambda_k}\, t\, dt\right]\right],$$

$$c_{k,2}(x) := D_{R_0}\left[\int_0^R \frac{\sin(\mu+1/2)t}{\Pi t}\int_{x-t}^{x+t}\frac{\sin\sqrt{\lambda_k}(t-|x-\xi|)}{\sqrt{\lambda_k}}\, q(\xi)u_k(\xi)d\xi dt\right].$$

The following estimates are known ([5], [4]):

$$(9)\qquad \|u_k\|_{L^\infty(G)} \le c\|u_k\|_{L^2(G)} = c,$$

$$(10)\qquad |c_{k,1}(x)| \le c(R_0)\frac{|u_k(x)|}{1+|\mu-\sqrt{\lambda_k}|^2},$$

$$(11)\qquad \left|\int_a^R \frac{\sin\mu t}{\Pi t}\,\frac{\sin\sqrt{\lambda_k}(t-a)}{\sqrt{\lambda_k}}\, dt\right| \le$$

$$\le c(R_0, \epsilon)a^{-\epsilon}\min\{\mu^{-1}(1+\sqrt{\lambda_k})^{-\epsilon}+\mu^{-2}\sqrt{\lambda_k},\ (1+\sqrt{\lambda_k})^{-2}\mu\}\ (\mu>1,\ 0<a<R_0)$$

for any fixed $\epsilon>0$. Combining (1), (9) and (10) we get that

$$\sum_{k=1}^{\infty}|u_k(t)c_{k,1}(x)| \le c(R_0)\sum_{j=0}^{\infty}\frac{1}{1+|\mu-j|^2}\sum_{j^2\le\lambda_k<(j+1)^2}|u_k(t)u_k(x)| \le$$

$$\le c(R_0)\sum_{j=0}^{\infty}\frac{1}{1+|\mu-j|^2} \le c(R_0).$$

Change the order of integrations in $c_{k,2}(x)$:

$$\int_0^R \frac{\sin(\mu+1/2)t}{\Pi t}\int_{x-t}^{x+t}\frac{\sin\sqrt{\lambda_k}(t-|x-\xi|)}{\sqrt{\lambda_k}}\, q(\xi)u_k(\xi)d\xi\, dt =$$

$$= \int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin(\mu+1/2)t}{\Pi t}\,\frac{\sin\sqrt{\lambda_k}(t-|x-\xi|)}{\sqrt{\lambda_k}}\, dt\, d\xi.$$

If $\epsilon<1-\frac{1}{p}$ then $q(\xi)|x-\xi|^{-\epsilon}\in L^1(x-R, x+R)$ and by (11) we obtain

$$\sum_{k=1}^{\infty} |u_k(t)c_{k,2}(x)| \le c(R_0, \epsilon, q)\sum_{k=1}^{\infty} |u_k(t)|\,\|u_k\|_{L^\infty(x-R, x+R)}\cdot$$

$$\cdot \min\{\mu^{-1}(1+\sqrt{\lambda_k})^{-\epsilon}+\mu^{-2}\sqrt{\lambda_k}\,,\ (1+\sqrt{\lambda_k})^{-2}\mu\} \le$$

$$\le c(R_0, \epsilon; q)\sum_{j=0}^{\infty} \min\{\mu^{-1}(1+j)^{-\epsilon}+\mu^{-2}j,\ (1+j)^{-2}\mu\} \le$$

$$\le c(R_0, \epsilon, q)\left\{\sum_{j=1}^{[\mu]} (\mu^{-1}j^{-\epsilon}+\mu^{-2}j)+\sum_{j=[\mu]+1}^{\infty} j^{-2}\mu\right\} \le c(R_0, \epsilon, q).$$

So we showed that the series

$$\sum_{k=1}^{\infty} u_k(t)\{c_{k,1}(x)+c_{k,2}(x)\}$$

converges uniformly for x, $t\in G$ and then the scalar product in (8) can be taken term by term:

$$|S_n^L(df, x)-D_{R_0}[S_\mu^{T,R}(df, x)]| \le \sum_{k=1}^{\infty} |\langle df, u_k\rangle|\,|c_{k,1}(x)+c_{k,2}(x)|.$$

It follows from NEUMARK [9], Theorem 1, p. 42 that $u_k(t)$ has a representation

$$u_k(t) = a_k(e^{i\sqrt{\lambda_k}t}+O(k^{-1}))+b_k(e^{-i\sqrt{\lambda_k}t}+O(k^{-1})).$$

Now (9) yields that

$$a_k^2+b_k^2 \le \text{const}$$

and then (1) and (2) imply

$$|\langle df, u_k\rangle| \le ck^{-\alpha}.$$

Consequently

$$|S_n^L(df, x)-D_{R_0}[S_\mu^{T,R}(df, x)]| \le$$

$$\le c(R_0, \epsilon, q)\left\{\sum_{j=1}^{\infty} j^{-\alpha}\frac{1}{1+|\mu-j|^2}+\sum_{j=1}^{[\mu]} j^{-\alpha}(\mu^{-1}(1+j)^{-\epsilon}+\mu^{-2}j)+\right.$$

$$+ \sum_{j=[\mu]+1}^{\infty} j^{-\alpha-2}\mu \Bigg\} = :c(R_0, \epsilon, q)\{\Sigma_1+\Sigma_2+\Sigma_3\}.$$

Obviously $\Sigma_3 \le c\mu^{-\alpha}$ and if $\alpha+\epsilon<1$ then $\Sigma_2 \le c\mu^{-\alpha}$.

The same statement for Σ_1 can be proved by a decomposition:

$$\sum_{j=1}^{\left[\frac{\mu}{2}\right]} j^{-\alpha} \frac{1}{1+|\mu-j|^2} \le c\mu^{-2}\sum_{j=1}^{\left[\frac{\mu}{2}\right]} j^{-\alpha} \le c\mu^{-1-\alpha},$$

$$\sum_{j=\left[\frac{\mu}{2}\right]+1}^{\infty} j^{-\alpha} \frac{1}{1+|\mu-j|^2} \le c\mu^{-\alpha} \sum_{j=1}^{\infty} \frac{1}{1+|\mu-j|^2} \le c\mu^{-\alpha}.$$

The desired estimate (5) is proved for I_1. In the next step estimate I_2. Obviously

$$D_{R_0}\left[S^{T,\epsilon}_{\sqrt{\lambda_k}}(f', x)\right] = \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} \int_{-R}^{R} \frac{\sin(\sqrt{\lambda_k}+1/2)t}{\Pi t} f'(x+t)dt\, dR =$$

$$= \int_{-R_0}^{R_0} \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} \chi_{[-R,R]}(t)dR \frac{\sin(\sqrt{\lambda_k}+1/2)t}{\Pi t} f'(x+t)dt =$$

$$= \int_{-R_0}^{R_0} \frac{\sin(\sqrt{\lambda_k}+1/2)t}{\Pi t} f'(x+t) \min\left\{1, \frac{2}{R_0}(R_0-|t|)\right\} dt.$$

We can suppose that

$$[a, b] \subset (0, 2\Pi)$$

since we can shift arbitrarily the arguments of the functions. Define the function

$$h(t) := \begin{cases} \dfrac{f'(x+t)+f'(x-t)}{\Pi t}\left(2-\dfrac{2}{R_0}t\right) & \text{if } \dfrac{R_0}{2} \le t \le R_0 \\ 0 & \text{if } t \in [0, 2\Pi)\setminus\left[\dfrac{R_0}{2}, R_0\right] \\ h(t \pm 2\Pi) = h(t). \end{cases}$$

Then using the estimate of ALIMOV and Joó [6], stating that for $f \in L^1(0, 2\Pi)$

$$\left| \int_0^{2\Pi} f(x) e^{i\mu x} dx \right| \leq c \int_0^{2\Pi - \frac{\Pi}{\mu}} \left| f(x) - f\left(x + \frac{\Pi}{\mu}\right) \right| dx + \frac{c}{\mu} \int_0^{2\Pi} |f| \tag{16}$$

we get

$$|D_{R_0}[S^{T,R_0}_{\sqrt{\lambda_k}}] - S^{T,R_0}_{\sqrt{\lambda_k}}| = \left| \frac{1}{\Pi} \int_0^{2\Pi} h(t) \sin\left(\sqrt{\lambda_k} + \frac{1}{2}\right) t \, dt \right| \leq$$

$$\leq c\omega_1\left(h, \frac{\Pi}{\sqrt{\lambda_k} + \frac{1}{2}}\right) + \frac{c}{\mu} \|h\|_1$$

where

$$\omega_1(h, \delta) := \sup_{t \leq \delta} \int_0^{2\Pi - t} |h(x+t) - h(x)| dx$$

is the usual L^1—modulus of continuity.

It follows easily from the equality

$$F(y+\delta)G(y+\delta) - F(y)G(y) =$$
$$= (F(y+\delta) - F(y))G(y+\delta) + F(y)(G(y+\delta) - G(y))$$

and from

$$f' \in \mathrm{Lip}\,(\alpha, [a, b])$$

that

$$\omega_1(h, \delta) \leq c(R_0)\delta^\alpha + c(R_0, \|f\|_{L^\infty(a, b)})\delta \leq c(R_0, f, a, b)\delta^\alpha$$

hence I_2 also satisfies (5). The estimate of I_3 is similar:

$$|S^{T,R_0}_{\sqrt{\lambda_k}}(f', x) - S^T_{\sqrt{\lambda_k}}(g, x)| = \left| \frac{1}{\Pi} \int_0^{2\Pi} \sin\left(\sqrt{\lambda_k} + \frac{1}{2}\right)(y-x) g(y) \cdot \right.$$

$$\left. \cdot \left\{ \frac{\chi_{(x-R_0, x+R_0)}(y)}{y-x} - \frac{1}{2 \sin \frac{x-y}{2}} \right\} dy \right| =$$

$$= \left| \int_0^{2\Pi} \sin\left(\sqrt{\lambda_k}+\frac{1}{2}\right) t\{h_1(t)-h_2(t)\}\, dt \right| \leq$$

$$\leq c\omega_1\left(h_1, \frac{\Pi}{\sqrt{\lambda_k}+\frac{1}{2}}\right) + c\omega_1\left(h_2, \frac{\Pi}{\sqrt{\lambda_k}+\frac{1}{2}}\right) +$$

$$+\frac{c}{\sqrt{\lambda_k}}\,(\|h_1\|_1+\|h_2\|_1) = O(k^{-\alpha})$$

where

$$h_1(t) := \begin{cases} \dfrac{1}{\Pi}\, g(x+t)\left\{\dfrac{\chi_{(x-R_0,\, x+R_0)}(x+t)}{t} - \dfrac{1}{2\sin\frac{t}{2}}\right\} & \text{if } 0 \leq t \leq 2\Pi - x \\ 0 & \text{if } 2\Pi - x < t < 2\Pi \\ h_1(t \pm 2\Pi) = h_2(t), \end{cases}$$

$$h_2(t) := \begin{cases} \dfrac{1}{\Pi}\, g(x-t)\left\{\dfrac{\chi_{(x-R_0,\, x+R_0)}(x-t)}{t} - \dfrac{1}{2\sin\frac{t}{2}}\right\} & \text{if } 0 \leq t \leq x \\ 0 & \text{if } x < t < 2\Pi \\ h_2(t \pm 2\Pi) = h_2(t). \end{cases}$$

So (5) is proved and it follows from

$$\frac{1}{n}\sum_{k=1}^{n} k^{-\alpha} \leq \frac{c}{n} n^{1-\alpha} = \frac{c}{n^{\alpha}}$$

that (3) is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} |S^T_{\sqrt{\lambda_k}}(g, x) - g(x)| \leq \frac{c}{n^{\alpha}} \quad (x \in [a', b']). \tag{17}$$

Show first that if $\mu_n \to +\infty$, $\sup_n |\mu_n - \nu_n| < \infty$ then for the above defined function $g(x)$ we have

$$\mu_n^{\alpha} |S^T_{\mu_n}(g, x) - S^T_{\nu_n}(g, x)| \leq c. \tag{18}$$

Indeed, let

$$h_3(t):=\begin{cases}\dfrac{1}{\Pi}f'(x+t)\dfrac{\sin\frac{\mu_n-\nu_n}{2}t}{2\sin\frac{t}{2}} & \text{if } 0\le t\le 2\Pi-x\\ 0 & \text{if } 2\Pi-x<t<2\Pi\\ h_3(t\pm 2\Pi)=h_3(t), & \end{cases}$$

$$h_4(t):=\begin{cases}\dfrac{1}{\Pi}f'(x-t)\dfrac{\sin\frac{\mu_n-\nu_n}{2}t}{2\sin\frac{t}{2}} & \text{if } 0\le t\le x\\ 0 & \text{if } x<t<2\Pi\\ h_4(t\pm 2\Pi)=h_4(t); & \end{cases}$$

then

$$|S^T_{\mu_n}(g,x)-S^T_{\nu_n}(g,x)|=\left|\int_0^{2\Pi}\{h_3(t)+h_4(t)\}\cos\frac{\mu_n+\nu_n+1}{2}t\,dt\right|\le$$

$$\le c\left\{\omega_1\left(h_3,\frac{2\Pi}{\mu_n+\nu_n+1}\right)+\omega_1\left(h_4,\frac{2\Pi}{\mu_n+\nu_n+1}\right)+\right.$$

$$\left.+\frac{1}{\mu_n}(\|h_3\|_1+\|h_4\|_1)\right\}\le\frac{c}{\mu_n^{\alpha}}$$

i.e. (18) is proved. Using (1) we can shift the square roots $\sqrt{\lambda_k}$ such that the new sum

$$\frac{1}{n}\sum_{m<\sqrt{\lambda_n}}c_m|S^T_m(g,x)-g(x)|$$

satisfies

$$\text{(19)}\quad \frac{1}{n}\left|\sum_{k=1}^{n}|S^T_{\sqrt{\lambda_k}}(g,x)-g(x)|-\sum_{m<\sqrt{\lambda_n}}c_m|S^T_m(g,x)-g(x)|\right|\le\frac{c}{n^{\alpha}},$$

$$\text{(20)}\qquad 0<\inf c_m\le\sup c_m<\infty.$$

The known result of ALEXITS and KRÁLIK [7] states that

$$\frac{1}{\sqrt{\lambda_n}} \sum_{m<\sqrt{\lambda_n}} |S_m^T(g, x)-g(x)| \leq \frac{c}{\sqrt{\lambda_n^{\alpha}}} \tag{21}$$

and this combined with (19), (20) gives (17) and hence (3).

The Theorem 1. is proved.

THEOREM 2. *Let* $[a, b] \subset G := (0, 2\Pi)$, $f \in BV[0, 2\Pi] \cap \mathrm{Lip}(1, [a, b])$ *and let* q *satisfy the same assumptions as in Theorem 1. Then*

$$\frac{1}{n} \sum_{k=1}^{n} |S_k^L(f, x)-f(x)| \leq c \frac{\ln n}{n} \quad (x \in [a', b']) \tag{22}$$

for any subsegment $[a', b'] \subset (a, b)$. *The estimate is exact in the sense that there exists* $f \in \mathrm{Lip}(1, [0, 2\Pi])$ *for which the converse of the inequality* (22) *holds.*

Remark that a function $f \in L^1(0, 2\Pi)$ is called of power type if its Fourier coefficients of negative indices vanish.

PROOF. The inequality (22) can be proved as (3). The above proof gives (5′) with $\alpha = 1$. So (22) can be reduced to the analogous assertion for the trigonometric expansion which is known ([16]).

Now prove the exactness of (22). I. JOÓ and I. BRÚJ ([14]) proved that there exists a power type function

$$f \in \mathrm{Lip}(1, [0, 2\Pi])$$

such that $f(0) = f(2\Pi) = 0$ and

$$\frac{1}{n} \sum_{k=1}^{n} |S_k^T(f, \Pi)-f(\Pi)| \geq c \frac{\ln n}{n}. \tag{23}$$

Define a 2Π-periodec function $f^* \in \mathrm{Lip}\,1$ with compact support in $(0, 2\Pi)$, $\operatorname{supp} f^* \subset [\delta, 2\Pi-\delta]$ and such that

$$f^* = f \quad \text{in} \quad [2\delta, 2\Pi-2\delta].$$

Then for $x \in [3\delta, 2\Pi-3\delta]$

$$S_k^T(f, x)-S_k^T(f^*, x) = \frac{1}{\Pi} \int_0^{2\Pi} \frac{\sin\left(k+\frac{1}{2}\right)(t-x)}{2\sin\frac{t-x}{2}} (f(t)-f^*(t))dt =$$

$$= \frac{1}{2\Pi} \int_0^{2\Pi} \frac{\cos(k+\frac{1}{2})(t-x)}{k+\frac{1}{2}} \frac{d}{dt}\left(\frac{f(t)-f^*(t)}{\sin\frac{t-x}{2}}\right) dt = o(k^{-1})$$

uniformly in $x\in[3\delta, 2\Pi-3\delta]$. Indeed, if there exists $x_k\in[3\delta, 2\Pi-3\delta]$ with

$$\left|\int_0^{2\Pi} \cos\left(k+\frac{1}{2}\right)(t-x_k)\frac{d}{dt}\left(\frac{f(t)-f^*(t)}{\sin\frac{t-x_k}{2}}\right)dt\right| \geq c_0>0$$

then supposing $x_k\to x_0$ we get

$$c_0\leq\left|\int_{x_k-x_0}^{x_k-x_0+2\Pi} \cos\left(k+\frac{1}{2}\right)(t-x_0)\cdot\frac{d}{dt}\left(\frac{f(t+x_k-x_0)-f^*(t+x_k-x_0)}{\sin\frac{t-x_k}{2}}\right)dt\right|\leq$$

$$\leq 2\left|\int_{x_k-x_0}^{x_k-x_0+2\Pi} \cos\left(k+\frac{1}{2}\right)(t-x_0)\frac{d}{dt}\left(\frac{f(t)-f^*(t)}{\sin\frac{t-x_0}{2}}\right)dt\right|$$

since

$$\frac{d}{dt}\left(\frac{f(t+x_k-x_0)-f^*(t+x_k-x_0)}{\sin\frac{t-x_0}{2}}\right)\xrightarrow{L^1}\frac{d}{dt}\left(\frac{f(t)-f^*(t)}{\sin\frac{t-x_k}{2}}\right);$$

but this contradicts the Riemann-Lebesgue lemma. Consequently

$$\frac{1}{n}\sum_{k=1}^{n}|S_k^T(f, x)-S_k^T(f^*, x)| = o\left(\frac{\ln n}{n}\right) \quad (x\in[3\delta, 2\Pi-3\delta]). \tag{24}$$

Since f^* has compact support, we have

$$|S_{\mu_n}^T(f^*, x)-S_{\nu_n}^T(f^*, x)| = o\left(\frac{1}{\mu_n}\right)$$

uniformly in $x\in[3\delta, 2\Pi-3\delta]$ where

$$\mu_n\to\infty \text{ and } \nu_n-\mu_n = O(1).$$

Using this and (1) we can define the coefficients

$$0<c\leq c_m\leq C<\infty$$

such that

$$\sum_{k=1}^{\infty} |S^T_{\sqrt{\lambda_k}}(f^*, x)-f^*(x)| - \sum_{m<\sqrt{\lambda_n}} c_m |S^T_m(f^*, x)-f^*(x)| = o(\ln n) \tag{25}$$

uniformly on $[3\delta, 2\Pi-3\delta]$. Since f^* is absolutely continuous with compact support we can apply Theorem 2. of [15] which asserts that

$$|S^L_k(f^*, x)-S^T_{\sqrt{\lambda_k}}(f^*, x)| = o(k^{-1}) \tag{26}$$

locally uniformly in $(0, 2\Pi)$. Finally we shall prove.

$$|S^L_k(f, x)-S^L_k(f^*, x)| = o(k^{-1}) \tag{27}$$

uniformly for $x\in[3\delta, 2\Pi-3\delta]$. To this we shall use the asymmetric Titchmarsh-formula

$$u_k(t) = u_k(0)\cos\sqrt{\lambda_k}\,t+\frac{u_k'(0)}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\,t+$$

$$+\int_0^t q(\xi)u_k(\xi)\frac{\sin\sqrt{\lambda_k}\,(t-\xi)}{\sqrt{\lambda_k}}\,d\xi$$

which is easy to obtain if we substitute $q(\xi)u_k(\xi) = \lambda_k u_k(\xi)+u_k''(\xi)$ and integrate twice by parts. It follows from $q\in L^1(0, 2\Pi)$ that ([5])

$$\|u_k'\|_\infty\leq c\sqrt{\lambda_k}\;\|u_k\|_\infty\leq c\sqrt{\lambda_k}.$$

Since $f(0) = f(2\Pi) = 0$, an integration by parts shows that

$$\int_0^{2\Pi}[f(t)-f^*(t)]\cos\sqrt{\lambda_k}\,t\,dt,\ \int_0^{2\Pi}[f(t)-f^*(t)]\sin\sqrt{\lambda_k}\,t\,dt = o(k^{-1})$$

by the Riemann-Lebesgue lemma. On the other hand we change the order of integrations to obtain

$$\int_0^{2\Pi}(f(t)-f^*(t))\int_0^t q(\xi)u_k(\xi)\frac{\sin\sqrt{\lambda_k}\,(t-\xi)}{\sqrt{\lambda_k}}\,d\xi\,dt =$$

$$= \int_0^{2\Pi} q(\xi)u_k(\xi) \int_\xi^{2\Pi} (f(t)-f^*(t)) \frac{\sin\sqrt{\lambda_k}\,(t-\xi)}{\sqrt{\lambda_k}}\, dt\, d\xi = O(k^{-2}) = o(k^{-1})$$

So we have proved that

$$\langle f-f^*, u_k\rangle = o(k^{-1}).$$

Now since $f = f^*$ on $[2\delta, 2\Pi-2\delta]$, we have for $R_0<\delta$

$$D_{R_0}\,[S^{T,R}_{\sqrt{\lambda_k}}\,(f-f^*, x)] = 0 \qquad (x\in[3\delta, 2\Pi-3\delta]).$$

On the other hand, as we showed in proving Theorem 1, for $\sqrt{\lambda_k}<\mu<\sqrt{\lambda_{k+1}}$ we have

$$|S^L_k(f-f^*, x)-D_{R_0}\,[S^{T,R}_\mu\,(f-f^*, x)]| \le$$

$$\le \sum_{j=1}^{\infty} |\langle f-f^*, u_j\rangle|\,|c_{j,1}(x)+c_{j,2}(x)|$$

and

$$|c_{j,1}(x)| \le c\,(R_0)\,\frac{|u_j(x)|}{1+|\mu-\sqrt{\lambda_j}|^2}$$

$$|c_{j,2}(x)| \le c(R_0, \xi, q)\|u_j\|_\infty\cdot$$

$$\cdot\min\{\mu^{-1}(1+\sqrt{\lambda_j})^{-c}+\mu^{-2}\sqrt{\lambda_j},\ (1+\sqrt{\lambda_j})^{-2}\mu\} \quad (\mu>1).$$

Now

$$\sum_{j=1}^{\infty} |\langle f-f^*, u_j\rangle|\,|c_{j,1}(x)| \le$$

$$\le c(R_0)\sum_{j=1}^{\infty} o(j^{-1})\,\frac{1}{1+|\mu-j|^2} = c(R_0)\left(\sum_{j=1}^{\left[\frac{\mu}{2}\right]}+\sum_{j=\left[\frac{\mu}{2}\right]+1}^{\infty}\right) =$$

$$= O\left(\frac{\ln\mu}{\mu^2}\right)+o(\mu^{-1}) = o(\mu^{-1}).$$

We intend to prove that for any fixed $\delta > 0$

$$|S_k^L(f-f^*, x) - D_{R_0}[S_\mu^{T,R}(f-f^*, x)]| \le \frac{\delta}{\mu} \qquad (u \ge \mu(\delta)).$$

We proved above that

$$\sum_{j=1}^{\infty} |\langle f-f^*, u_j\rangle| \, |c_{j,1}(x)| \le \frac{\delta}{4\mu} \qquad (\mu \ge \mu(\delta)).$$

On the other hand for large $j(\delta)$ we have

$$\sum_{j=j(\delta)}^{\infty} |\langle f-f^*, u_j\rangle| \, |c_{j,2}(x)| \le$$

$$\le c \sum_{j=j(\delta)}^{[\mu]} o(j^{-1})(\mu^{-1} j^{-\epsilon} + \mu^{-2} j) + c \sum_{j=[\mu]+1}^{\infty} o(j^{-1}) j^{-2} \mu \le$$

$$\le \frac{\delta}{4\mu} \qquad (\mu \ge \mu(\delta),\ u \ge u(\delta)).$$

So it remains to prove that for any fixed $D = D(\delta)$

$$|c_{j,2}(x)| \le \frac{D\delta}{\mu} \qquad (1 \le j < j(\delta),\ \mu \ge \mu(\delta)).$$

To see this, write

$$|c_{j,2}(x)| =$$

$$= \left| D_{R_0} \left[\int_{x-R}^{x+R} q(\xi) u_j(\xi) \int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t - |x-\xi|)}{\sqrt{\lambda_j}} \frac{\sin \mu t}{\Pi t} \, dt \, d\xi \right] \right| =$$

$$= \left| \int_{x-R_0}^{x+R_0} q(\xi) u_j(\xi) \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} \chi_{(x-R,\, x+R)}(\xi) \cdot \right.$$

$$\left. \cdot \int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t - |x-\xi|)}{\sqrt{\lambda_j}} \frac{\sin \mu t}{\Pi t} \, dt \, dR \, d\xi \right|.$$

An integration by parts gives

$$\int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t-|x-\xi|)}{\sqrt{\lambda_j}} \frac{\sin\mu t}{\Pi t}\,dt =$$

$$= -\frac{\cos\mu R}{\Pi\mu R} \frac{\sin\sqrt{\lambda_j}\,(R-|x-\xi|)}{\sqrt{\lambda_j}} + \int_{|x-\xi|}^{R} \cos\sqrt{\lambda_j}\,(t-|x-\xi|) \frac{\cos\mu t}{\Pi\mu t}\,dt -$$

$$- \int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t-|x-\xi|)\cos\mu t}{\Pi\mu t^2}\,dt.$$

Consequently, taking the absolute value in the integrals, we get

$$\left| \int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t-|x-\xi|)}{\sqrt{\lambda_j}} \frac{\sin\mu t}{\Pi t}\,dt \right| \leq \frac{c_0}{\mu\,|x-\xi|^{\epsilon}}$$

and then for small $\eta = \eta\,(\epsilon, q)$

$$\left| \int_{|x-\xi|<\eta} q(\xi)u_j(\xi) \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} \chi_{(x-R,\,x+R)}(\xi) \cdot \right.$$

$$\left. \cdot \int_{|x-\xi|}^{R} \frac{\sin\sqrt{\lambda_j}\,(t-|x-\xi|)}{\sqrt{\lambda_j}} \frac{\sin\mu t}{\Pi t}\,dt\,dR\,d\xi \right| \leq \frac{D\delta}{2\mu}.$$

It remains to show the same estimate for the expression where the outer integral is over $\eta \leq |x-\xi| \leq R_0$. But it is easy: since the sets

$$\left\{ \frac{\sin\sqrt{\lambda_j}\,(R-a)}{\sqrt{\lambda_j}\,R} \chi_{\left(\frac{R_0}{2},\,R_0\right)}(R):\ \eta \leq a \leq R_0,\ 1 \leq j < j(\delta) \right\},$$

$$\left\{ \frac{\cos\sqrt{\lambda_j}\,(t-a)}{\Pi t} \chi_{(a,\,R)}(t),\ \frac{\sin\sqrt{\lambda_j}\,(t-a)}{\Pi t^2} \chi_{(a,\,R)}(t);\right.$$

$$\left. \eta \leq a \leq R,\ \frac{R_0}{2} \leq R \leq R_0,\ 1 \leq j < j(\delta) \right\}$$

are compact in $L^1(0, R_0)$, their Fourier coefficients with respect to $\cos \mu R$ resp. $\cos \mu t$ tend uniformly to zero when $\mu \to +\infty$. So for $\mu \geq \mu(\delta)$ we have indeed $|c_{j,2}(x)| \leq \frac{D\delta}{\mu}$ if $1 \leq j < j(\delta)$ and hence (27) is proved. Taking into account the estimates (23) — (27) we get (22) for the same function f as in (23).

Theorem 2 is proved.

In fact we have showed the following

COROLLARY. If $q \in L^p_{\mathrm{loc}}(G)$ for some $1 < p$, if (1) holds and if $f \in L^1(G)$ has coefficients

$$\langle f, u_j \rangle = o(j^{-1})$$

then

$$S_n^L(f, x) - D_{R_0}[S_\mu^{T,R}(f, x)] = o(n^{-1})$$

where

$$\sqrt{\lambda_n} < \mu < \sqrt{\lambda_{n+1}}.$$

REMARK. The Jacobi polynomials are defined for $\alpha, \beta > -1$ by

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \delta_{n,k}.$$

The functions

$$u_n(\Theta) := \left(\sin\frac{\Theta}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos\frac{\Theta}{2}\right)^{\alpha+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos\Theta)$$

satisfy the equation

$$-u_n'' + qu_n = \lambda_n u_n \qquad (\Theta \in (0, \Pi))$$

where

$$\lambda_n = \left(n + \frac{\alpha+\beta+1}{2}\right)^2$$

$$q(\Theta) = \frac{\alpha^2 - \frac{1}{4}}{4\sin^2\frac{\Theta}{2}} + \frac{\beta^2 - \frac{1}{4}}{4\cos^2\frac{\Theta}{2}}$$

(see[10]). Here (1) is satisfied and $q \in L^1_{\mathrm{loc}}(0, \Pi)$ since q is continuous. Hence the proof of Theorems 1 and 2 can be repeated to obtain these results for Jacobi expansions, too.

Now consider summability problems for Laguerre expansions. If $\alpha > -1$, the normed Laguerre polynomials $l_k^{(\alpha)}(x)$ are defined by

$$\int_0^\infty l_n^{(\alpha)}(x)\, l_k^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx = \delta_{n,k}.$$

Let

$$v_n(x) := \sqrt{2}\, x^{\alpha+\frac{1}{2}} e^{-\frac{x^2}{2}} l_n^{(\alpha)}(x^2) \qquad (x>0)$$

then ([10])

$$-v_n'' + q v_n = \lambda_n v_n$$

with

$$\lambda_n = 4_n$$

$$q(x) = x^2 - 2\alpha - 2 - \frac{\frac{1}{4} - \alpha^2}{x^2}.$$

We see that in contrast to the case of finite interval, the eigenvalues are linearly distributed so (1) no longer holds. In what follows we shall suppose that

$$\alpha \geq -\frac{1}{2}$$

since in this case

$$|v_n(x)| \leq \text{const}$$

independently of n and x. But we shall prove more

LEMMA 1. For $\alpha \geq -\frac{1}{2}$ the estimate

$$\sum_{j^2 \leq n < (j+1)^2} v_n^2(x) \leq c\sqrt{j} \qquad (j = 1, 2, \ldots; x>0) \tag{28}$$

holds.

PROOF. Use the Mehler-type formula ([10])

$$\sum_{n=0}^\infty l_n^{(\alpha)}(x)\, l_n^{(\alpha)}(y) z^n = \frac{\exp\left\{-(x+y)\frac{z}{1-z}\right\}}{1-z} \cdot \frac{J_\alpha\left(2i\frac{\sqrt{xyz}}{1-z}\right)}{(i\sqrt{xyz})^\alpha} \quad (|z|<1).$$

Put $x = y$ and use the Cauchy integral formula to obtain

$$|l_n^{(\alpha)}(x)|^2 = \frac{1}{2\Pi}\int_0^{2\Pi} \frac{\exp\left\{-\dfrac{2x\,re^{i\phi}}{1-re^{i\phi}}\right\}}{1-re^{i\phi}} \cdot$$

$$\cdot \frac{J_\alpha\left(2ix\,\dfrac{r^{1/2}\,e^{i\phi/2}}{1-re^{i\phi}}\right)}{(ixr^{1/2}\,e^{i\phi/2})^\alpha} \cdot \frac{1}{r^n\,e^{in\phi}}\,d\phi$$

for any $0<r<1$. Since ([11])

$$(29) \qquad |J_\alpha(z)| \le \frac{c(\alpha)}{\sqrt{|z|}}\,e^{|\mathrm{Im}\,z|} \qquad \left(z\neq 0,\ |\arg z|<\Pi,\ \alpha\ge -\frac{1}{2}\right)$$

and

$$\mathrm{Re}\,\frac{re^{i\phi}}{1-re^{i\phi}} = \mathrm{Re}\,\frac{1}{1-re^{i\phi}} - 1 = \frac{1-r\cos\phi}{(1-r\cos\phi)^2+(r\sin\phi)^2} - 1 =$$

$$= \frac{1-r\cos\phi}{2(1-r\cos\phi)-(1-r)^2} - 1 \ge -\frac{1}{2}\,;$$

$$\mathrm{Im}\,\frac{ie^{i\phi/2}}{1-re^{i\phi}} = \mathrm{Im}\,\frac{i}{e^{-i\phi/2}-e^{i\phi/2}+e^{i\phi/2}(1-r)} =$$

$$= \mathrm{Im}\,\frac{1}{-2\sin\dfrac{\phi}{2}+(1-r)\sin\dfrac{\phi}{2}-i(1-r)\cos\dfrac{\phi}{2}} =$$

$$= \frac{(1-r)\cos\dfrac{\phi}{2}}{(1+r)^2\sin^2\dfrac{\phi}{2}+(1-r)^2\cos^2\dfrac{\phi}{2}}\,,$$

we see that for $\frac{1}{2} \leq r < 1$

$$\left| \frac{e^{-2x \frac{re^{i\phi}}{1-re^{i\phi}}}}{1-re^{i\phi}} \cdot \frac{J_\alpha\left(2ix \frac{r^{1/2} e^{i\phi/2}}{1-re^{i\phi}}\right)}{(ixr^{1/2} e^{i\phi/2})^\alpha} \cdot \frac{1}{r^n e^{in\phi}} \right| \leq$$

$$\leq \frac{c(\alpha)}{\sqrt{|1-e^{i\phi}|}} x^{-\alpha-1/2} \exp\left\{ 2xr^{1/2} \frac{(1-r)\left|\cos\frac{\phi}{2}\right|}{(1+r)^2 \sin^2\frac{\phi}{2} + (1-r)^2 \cos^2\frac{\phi}{2}} - \right.$$

$$\left. -2x \frac{1-r\cos\phi}{2(1-r\cos\phi)-(1-r^2)} \right\} =$$

$$= \frac{c(\alpha)}{\sqrt{|1-e^{i\phi}|}} x^{-\alpha-1/2} \exp\left\{ 2x \frac{r^{1/2}(1-r)\left|\cos\frac{\phi}{2}\right| - (1-r\cos\phi)}{(1+r)^2 \sin^2\frac{\phi}{2} + (1-r)^2 \cos^2\frac{\phi}{2}} \right\}.$$

Since $r^{1/2}(1-r)\left|\cos\frac{\phi}{2}\right| \leq 1-r \leq 1-r+r(1-\cos\phi)$ hence we see from the Lebesgue dominated convergence theorem that the limit $r \to 1-0$ can be taken under the integral and

$$|l_n^{(\alpha)}(x)|^2 = \frac{1}{2\Pi} \int_0^{2\Pi} \frac{e^{-ix \operatorname{ctg}\frac{\phi}{2}}}{1-e^{i\phi}} e^{x-in\phi} \frac{J_\alpha\left(2ix \frac{e^{i\phi/2}}{1-e^{i\phi}}\right)}{(ixe^{i\phi/2})^\alpha} d\phi.$$

Since

$$\sum_{j^2 \leq n < (j+1)^2} e^{-in\phi} = e^{-ij^2\phi} \frac{1-e^{-i(2j+1)\phi}}{1-e^{-i\phi}}$$

hence we get

$$x^{\alpha}e^{-x}\sum_{j^2\le n<(j+1)^2}|l_n^{(\alpha)}(x)|^2 = \tag{30}$$

$$= \frac{i}{4\Pi}\int_0^{2\Pi}\frac{e^{-ix\operatorname{ctg}\frac{\phi}{2}}}{\sin\frac{\phi}{2}}\,\frac{J_\alpha\left(-\dfrac{x}{\sin\frac{\phi}{2}}\right)}{(ie^{i\phi/2})^{\alpha}}\,e^{-i(j^2+j+1/2)\phi}\,\frac{\sin\left(j+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}}\,d\phi.$$

This integral will be estimated separately for

$$\int_0^{2\Pi} = \int_0^{j^{-1}} + \int_{j^{-1}}^{2\Pi-j^{-1}} + \int_{2\Pi-j^{-1}}^{2\Pi}$$

by (29). Namely,

$$\left|\int_0^{j^{-1}}\right| \le c(\alpha)\int_0^{j^{-1}}\sqrt{\frac{\phi}{x}}\,\frac{j}{\phi}\,d\phi \le \frac{c(\alpha)}{\sqrt{x}}\sqrt{j}$$

$$\left(\text{we used the estimate}\left|\frac{\sin\left(j+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}}\right| \le cj\right).$$

Analogously

$$\left|\int_{2\Pi-j-1}^{2\Pi}\right| \le \frac{c(\alpha)}{\sqrt{x}}\sqrt{j}.$$

Finally we have

$$\left|\int_{j^{-1}}^{\Pi}\right| \le c(\alpha)\int_{j^{-1}}^{\Pi}\sqrt{\frac{\phi}{x}}\,\phi^{-2}\,d\phi \le \frac{c(\alpha)}{\sqrt{x}}\sqrt{j}$$

and analogously

$$\left| \int_{\Pi}^{2\Pi - j^{-1}} \right| \leq c(\alpha) \sqrt{\frac{j}{x}}.$$

So

$$x^{\alpha+1/2} e^{-x} \sum_{j^2 \leq n < (j+1)^2} |l_n^{(\alpha)}(x)|^2 \leq c(\alpha) \sqrt{j}.$$

The Lemma 1 is proved.

We shall need another estimate, too:

LEMMA 2. Define for $\mu \geq 1$ and $\lambda_k \geq 1$, a, $r > 0$ the number

$$\gamma_n := \int_a^R \sin \sqrt{\lambda_n}\,(t-a) \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}}\, dt.$$

Fix an ϵ, $0 < \epsilon < 1$. Then

a) $$|\gamma_n| \leq \frac{c}{\mu} \begin{cases} \dfrac{\sqrt{\lambda_n}}{\sqrt{\mu}} a^{-r} + \mu^{r-1/2} \dfrac{\sqrt{\lambda_n^{\epsilon}}}{a^{1-\epsilon}} & \text{if } r < 1 \\ \mu^{r-1/2+\epsilon} a^{\epsilon-1} \left(1 + \dfrac{\sqrt{\lambda_n}}{\mu}\right) & \text{if } r \geq 1, \end{cases}$$

b) $$|\gamma_n| \leq \frac{c}{\sqrt{\lambda_n}} \begin{cases} \mu^{r-1/2+\epsilon} a^{\epsilon-1} + \mu^{1/2} a^{-r} & \text{if } r < 1 \\ \mu^{r-1/2+\epsilon} a^{\epsilon-1} & \text{if } r \geq 1. \end{cases}$$

Here $c = c(r, \epsilon, R_0)$ does not depend on a, $\sqrt{\lambda_n}$, μ $\left(\text{and } \dfrac{R_0}{2} \leq R \leq R_0\right)$.

PROOF. We shall use (29), the rules ([11])

(31) $$\int z^{-\alpha} J_{\alpha+1}(\mu z)\, dz = -\mu^{-1} z^{-\alpha} J_{\alpha}(\mu z),$$

$$\int z^{\alpha+1} J_{\alpha}(\mu z)\, dz = \mu^{-1} z^{\alpha+1} J_{\alpha+1}(\mu z)$$

and the estimate

$$|J_\alpha(x)| \leq c(\alpha)\, x^\alpha \qquad (x>0)$$

which gives with (29) that

$$(32) \qquad |J_\alpha(x)| \leq c(\alpha)\, x^\gamma \qquad \left(x>0,\ -\frac{1}{2} \leq \gamma \leq \alpha\right).$$

a) Integration by parts gives

$$\gamma_n = \int_a^R \frac{\sin\sqrt{\lambda_n}\,(t-a)}{t}\,\frac{J_{r+1/2}(\mu t)}{t^{r+1/2}}\,dt =$$

$$= -\mu^{-1}\left[\frac{J_{r-1/2}(\mu t)}{t^{r-1/2}}\,\frac{\sin\sqrt{\lambda_n}\,(t-a)}{t}\right]_{t=a}^{R} -$$

$$-\frac{r+\frac{1}{2}}{\mu}\int_a^R \frac{J_{r-1/2}(\mu t)\sin\sqrt{\lambda_n}\,(t-a)}{t^{r+3/2}}\,dt+$$

$$+\frac{\sqrt{\lambda_n}}{\mu}\int_a^R \frac{J_{r-1/2}(\mu t)\cos\sqrt{\lambda_n}\,(t-a)}{t^{r+1/2}}\,dt =: I_1+I_2+I_3.$$

By (29) we see that

$$|I_1| \leq c(r, R_0)\,\mu^{-3/2}.$$

Take the estimate

$$|J_{r-1/2}(\mu t)| \leq c(r)(\mu t)^{r-1/2}$$

and

$$|\sin\sqrt{\lambda_n}\,(t-a)| \leq |\sin\sqrt{\lambda_n}\,(t-a)|^\epsilon \leq (\sqrt{\lambda_n}t)^\epsilon$$

to obtain

$$|I_2| \leq c(r, R_0, \epsilon)\, \mu^{r-3/2} \sqrt{\lambda_n^{\epsilon}}\, a^{\epsilon-1}\,.$$

In I_3 estimate

$$J_{r-1/2}(\mu t) \text{ by } c(r)(\mu t)^{-1/2} \quad \text{if} \quad r<1 \text{ and by}$$

$$c(r)(\mu t)^{r-3/2+\epsilon} \text{ if } r \geq 1;$$

then

$$|I_3| \leq c(r, R_0, \epsilon)\, \frac{\sqrt{\lambda_n}}{\mu} \begin{cases} \mu^{-1/2}\, a^{-r} & \text{if } r<1 \\ \mu^{r-3/2+\epsilon}\, a^{\epsilon-1} & \text{if } r \geq 1. \end{cases}$$

b)
$$\gamma_n = -\left[\frac{\cos\sqrt{\lambda_n}\,(t-a)}{\sqrt{\lambda_n}}\, \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}} \right]_{t=a}^{R} -$$

$$-\mu \int_a^R \frac{\cos\sqrt{\lambda_n}\,(t-a)}{\sqrt{\lambda_n}}\, \frac{J_{r+3/2}(\mu t)}{t^{r+1/2}}\, dt = :I_4 + I_5.$$

By

$$|J_{r+1/2}(\mu t)| \leq c(r)(\mu t)^{r-1/2+\epsilon}$$

we get

$$|I_4| \leq c(r, R_0)\, \frac{\mu^{r-1/2+\epsilon}}{\sqrt{\lambda_n}}\, a^{\epsilon-1}$$

Estimate

$$J_{r+3/2}(\mu t) \text{ by } c(r)(\mu t)^{-1/2} \text{ for } r<1 \text{ and by}$$

$$c(r)(\mu t)^{r-3/2+\epsilon} \text{ for } r \geq 1,$$

then we obtain

$$|I_5| \leq c(r, R_0, \epsilon)\, \frac{\mu}{\sqrt{\lambda_n}} \begin{cases} \mu^{-1/2}\, a^{-r} & \text{if } r<1 \\ \mu^{r-3/2+\epsilon}\, a^{\epsilon-1} & \text{if } r \geq 1. \end{cases}$$

The Lemma 2 is proved.

Introduce the Riesz means of order r of the Fourier-Laguerre expansions of a function $f \in L^1_{x^{-1/4}}(0, \infty)$ as

$$(33) \quad R^r_\mu(f, x) := \sum_{\sqrt{\lambda_k} < \mu} \langle f, l^{(\alpha)}_k t^{\alpha/2} e^{-t/2} \rangle \, l^{(\alpha)}_k(x) \, x^{\alpha/2} e^{-x/2} \left(1 - \frac{\lambda_k}{\mu^2}\right)^r$$

and the corresponding means of the ordinary Fourier series of f by the formula

$$(34) \qquad S^{r,R}_\mu(f, x) := \frac{1}{\sqrt{2}\, x^{1/4}} \int_{\sqrt{x}-R}^{\sqrt{x}+R} \sqrt{y}\, f(y^2)\, W^r_R(y - \sqrt{x})\, dy$$

where

$$W^r_R(t) := \begin{cases} a(r)\mu^{1/2-r} \dfrac{J_{r+1/2}(\mu|t|)}{|t|^{r+1/2}} & \text{if } |t| \le R \\ 0 & \text{if } |t| > R \end{cases}$$

and

$$a(r) := 2^r \frac{\Gamma(r+1)}{\sqrt{2\Pi}} .$$

THEOREM 3. *Let* $\alpha \ge -\frac{1}{2}$, $r \ge \frac{1}{4}$ *and* $K \subset (0, \infty)$ *be a compact set,* $R_0 < 2\operatorname{dist}(0, K)$ *and* $\frac{R_0}{2} \le R \le R_0$. *Then for every* $f \in L^1_{x^{-1/4}}(0, \infty)$ *we have*

$$(35) \qquad R^r_\mu(f, x) - S^{r,R}_\mu(f, x) = o(1) \qquad (\mu \to \infty)$$

uniformly on K.

PROOF. We know that the above introduced Laguerre functions $v_n(x)$ form a complete orthonormal system of $L^2(0, \infty)$ further $v_n \in L^\infty(0, \infty)$ for all n. Define

$$f^*(t) := \sqrt{2}\, t^{1/2} f(t^2)$$

then $f \in L^1_{x^{-1/4}}(0, \infty)$ if and only if $f^* \in L^1(0, \infty)$ and

$$\|f^*\|_{L^1(0,\infty)} = \frac{1}{\sqrt{2}} \|f\|_{L^1_{x^{-1/4}}(0,\infty)},$$

further

$$\int_0^\infty f(x)\, x^{\alpha/2}\, e^{-x/2}\, l_n^{(\alpha)}(x)\, dx = \int_0^\infty 2t^{1+\alpha}\, e^{-t^2/2} f(t^2)\, l_n^{(\alpha)}(t^2)\, dt =$$

$$= \int_0^\infty f^*(t)\, v_n(t)\, dt$$

and then the Riesz means of f^* with respect to the system (v_n) sytisfy

$$R_\mu^r(f^*, x) := \sum_{\sqrt{\lambda_k}<\mu} \langle f^*, v_k\rangle\, v_k(x) \left(1-\frac{\lambda_k}{\mu^2}\right)^r = \tag{36}$$

$$= \sum_{\sqrt{\lambda_k}<\mu} \langle f, l_k^{(\alpha)}\, t^{\alpha/2}\, e^{-t/2}\rangle \sqrt{2}\, x^{\alpha+1/2}\, e^{-x^2/2}\, l_n^{(\alpha)}(x^2) \left(1-\frac{\lambda_k}{\mu^2}\right)^r =$$

$$= \sqrt{2x}\, R_\mu^r(f, x^2)$$

and

$$S_\mu^{r,R}(f^*, x) := \int_{x-R}^{x+R} f^*(y)\, W_R^r(y-x)\, dy = \tag{37}$$

$$= \sqrt{2} \int_{x-R}^{x+R} \sqrt{y}\, f(y^2) W_R^r(y-x)\, dy = \sqrt{2x}\, S_\mu^{r,R}(f, x^2).$$

Hence (35) is equivalent to

$$R_\mu^r(f^*, x) - S_\mu^{r,R}(f^*, x) = o(1) \qquad (\mu\to\infty) \tag{35'}$$

uniformly for $x\in K$. We shall prove (35').

Consider the Fourier-Laguerre coefficients (for $x\in K$)

$$w_n := \int_0^\infty W_R^r(t-x) v_n(t)\, dt = \int_0^R W_R^r(t)\, [v_n(x+t)+v_n(x-t)]dt =$$

$$= 2v_n(x) \int_0^R W_R^r(t) \cos\sqrt{\lambda_n}\, t\, dt +$$

$$+\int_0^R W_R^r(t)\int_{x-t}^{x+t} q(\xi)v_n(\xi)\,\frac{\sin\sqrt{\lambda_n}\,(t-|x-\xi|)}{\sqrt{\lambda_n}}\,d\xi\,dt$$

(we used the Titchmarsh formula (7)). Now the difference of the kernels in (35') is

$$(38)\quad W_R^r(t-x)-\sum_{\sqrt{\lambda_k}<\mu} v_k(x)\,v_k(t)\left(1-\frac{\lambda_k}{\mu^2}\right)^r = W_R(t-x)-D_{R_0}[W_R^r(t-x)]+$$

$$+\sum_{k=0}^{\infty} v_k(t)\left\{D_{R_0}[w_k]-\delta(\mu,\sqrt{\lambda_k})\left(1-\frac{\lambda_k}{\mu^2}\right)^r v_k(x)\right\}.$$

We shall use the following estimates proved in [12]:

$$(39)\qquad |W_R^r(t-x)-D_{R_0}[W_R^r(t-x)]|\le\frac{c(r,R_0)}{\mu^r}\quad(t>0,\ x\in K)$$

$$(40)\qquad \left|D_{R_0}\left[2\int_0^R W_R^r(t)\cos\sqrt{\lambda_k}\,t\,dt\right]-\delta(\mu,\sqrt{\lambda_k})\left(1-\frac{\lambda_k}{\mu^2}\right)^r\right|\le$$

$$\le\frac{c(R_0)}{\mu^r}\,\frac{1}{1+|\mu-\sqrt{\lambda_k}|^2}.$$

Remark that (39) can be directly verified; (40) is a special case of Lemma 1 of [1] having a very tedious proof, but (40) can be shortly proved by repeating the method of the proof of Lemma 6 in [13]. The estimate (40) and

$$(41)\qquad \sum_{j^2\le\lambda_k<(j+1)^2} v_k^2(x)\le c(K)\quad\text{for}\quad x\in K$$

([10]) implies

$$\sum_{k=0}^{\infty}\left|v_k(t)\,v_k(x)\left\{D_{R_0}\left[2\int_0^R W_R^r(t)\cos\sqrt{\lambda_k}\,dt\right]-\delta(\mu,\sqrt{\lambda_k})\left(1-\frac{\lambda_k}{\mu^2}\right)^r\right\}\right|\le$$

$$\leq \sum_{k=0}^{\infty} \frac{c(R_0)}{\mu^r} \frac{|v_k(t)\, v_k(x)|}{1+|\mu-\sqrt{\lambda_k}|^2} =$$

$$= \sum_{j=0}^{\infty} \frac{c(R_0)}{\mu^r} \frac{1}{1+|\mu-j|^2} \sum_{j^2 \leq \lambda_k < (j+1)^2} |v_k(t)\, v_k(x)| \leq$$

$$\leq \frac{c(R_0)}{\mu^r} \sum_{k=0}^{\infty} \frac{1}{1+|\mu-j|^2} \left(\sum_{j^2 \leq \lambda_k < (j+1)^2} |v_k(t)|^2 \right)^{1/2} \left(\sum_{j^2 \leq \lambda_k < (j+1)^2} |v_k(x)|^2 \right)^{1/2} \leq$$

$$\leq \frac{c(R_0)}{\mu^r} \sum_{k=1}^{\infty} \frac{j^{1/4}}{1+|\mu-j|^2}.$$

Now

$$\sum_{j=1}^{\left[\frac{\mu}{2}\right]} \frac{j^{1/4}}{1+|\mu-j|^2} \leq \frac{c}{\mu^2} \sum_{j=1}^{\left[\frac{\mu}{2}\right]} j^{1/4} \leq \frac{c}{\mu^{3/4}},$$

$$\sum_{j=\left[\frac{\mu}{2}\right]+1}^{3\left[\frac{\mu}{2}\right]} \frac{j^{1/4}}{1+|\mu-j|^2} \leq c\mu^{1/4} \sum_{k=0}^{\infty} \frac{1}{1+|\mu-j|^2} \leq c\mu^{1/4}$$

$$\sum_{j=3\left[\frac{\mu}{2}\right]+1}^{\infty} \frac{j^{1/4}}{1+|\mu-j|^2} \leq c \sum_{j=3\left[\frac{\mu}{2}\right]+1}^{\infty} j^{-7/4} \leq \frac{c}{\mu^{3/4}}.$$

Consequently

(42)

$$\sum_{k=0}^{\infty} \left| v_k(t)\, v_k(x) \left\{ D_{R_0} \left[2 \int_0^R W_R^r(t) \cos \sqrt{\lambda_k}\, t\, dt \right] - \delta(\mu, \sqrt{\lambda_k}) \left(1 - \frac{\lambda_k}{\mu^2} \right)^r \right\} \right| \leq$$

$$\leq \frac{c(R_0)}{\mu^{r-1/4}} \leq c(R_0).$$

Form Lemma 2 it follows that

$$\sum_{\sqrt{\lambda_k}<\mu} \left| v_k(t)\, D_{R_0} \left[\int_0^R W_R^r(t) \int_{x-t}^{x+t} q(\xi)\, v_k(\xi)\, \frac{\sin\sqrt{\lambda_k}\,(t-|x-\xi|)}{\sqrt{\lambda_k}}\, d\xi\, dt \right] \right| =$$

$$= \sum_{\sqrt{\lambda_k}<\mu} \left| v_k(t)\, D_{R_0} \left[\int_{x-R}^{x+R} q(\xi)\, v_k(\xi) \int_{|x-\xi|}^{R} W_R^r(t) \frac{\sin\sqrt{\lambda_k}\,(t-|x-\xi|)}{\sqrt{\lambda_k}}\, dt\, d\xi \right] \right| =$$

$$\leq c(r, R_0, \epsilon) \sum_{\sqrt{\lambda_k}<\mu} |v_k(t)|\, \|v_k\|_{L^\infty(x-R_0,\, x+R_0)} \frac{\mu^{1/2-r}}{\sqrt{\lambda_k}} \frac{1}{\mu} \cdot$$

$$\cdot \begin{cases} \sqrt{\dfrac{\lambda_k}{\mu}} + \mu^{r-1/2} \sqrt{\lambda_k^\epsilon} & \text{if } r<1 \\ \mu^{r-1/2+\epsilon} & \text{if } r\geq 1. \end{cases}$$

In case $r<1$, using the Cauchy-Schwarz inequality we can further estimate by

$$c(r, R_0, \epsilon) \sum_{j=1}^{[\mu]} j^{1/4} \left(\mu^{-r-1} + \frac{j^{\epsilon-1}}{\mu} \right) \leq$$

$$\leq c(r, R_0, \epsilon)\, (\mu^{1/4-r} + \mu^{\epsilon-3/4}) \leq c(r, R_0, \epsilon)$$

if $\epsilon < \frac{3}{4}$. Take the case $r\geq 1$, then

$$c(r, R_0, \epsilon) \sum_{j=1}^{[\mu]} \mu^{\epsilon-1} j^{-3/4} \leq c(r, R_0, \epsilon)\, \mu^{\epsilon-3/4} \leq c(r, R_0, \epsilon).$$

Analogously

$$\sum_{\sqrt{\lambda_k}<\mu} \left| v_k(t)\, D_{R_0} \left[\int_{x-R}^{x+R} q(\xi)\, v_k(\xi) \int_{|x-\xi|}^{R} W_R^r(t) \frac{\sin\sqrt{\lambda_k}\,(t-|x-\xi|)}{\sqrt{\lambda_k}}\, dt\, d\xi \right] \right| \leq$$

$$\leq c(r, R_0, \epsilon) \sum_{j=\left[\frac{\mu}{2}\right]}^{\infty} j^{1/4} \frac{\mu^{1/2-r}}{j} \begin{cases} \dfrac{\sqrt{\lambda}}{j} & \text{if } r<1 \text{ and } \epsilon<1-r \\ \dfrac{\mu^{r-1/2+\epsilon}}{j} & \text{if } r\geq 1 \end{cases}$$

$$\leq c(r, R_0, \epsilon) \begin{Bmatrix} \mu^{1-r} \dfrac{1}{\mu^{3/4}} \\ \mu^{\epsilon} \dfrac{1}{\mu^{3/4}} \end{Bmatrix} \leq c(r, R_0, \epsilon).$$

The above calculations proved that the difference of the kernels (38) is uniformly bounded (independently of μ). The operator sequence

$$T_\mu : L^1 (0, \infty) \to C(K),$$

$$T_\mu f^* := R_\mu^r (f^*, x) - S_\mu^{r, R}(f^*, x)$$

has common bound,

$$\|T_\mu\| \leq c(r, R_0, \epsilon) \qquad (\mu>0).$$

The estimate (35') then follows from the Banach-Steinhaus theorem since for the finite sums

$$\sum_{k=0}^{N} c_k v_k$$

which run over a dense subset of $L^1(0, \infty)$, the Riesz means R_μ^r converge since the partial sums are constant for $\mu > \sqrt{\lambda_n}$ and the means $S_\mu^{r, R}$ converge locally uniformly. Indeed, we know ([11], p. 59) that

$$\int_0^\infty \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}} dt = \frac{2^{-r-1/2} \Gamma\left(\frac{1}{2}\right)}{\mu^{1/2-r} \Gamma(1+r)}$$

which implies

$$\int_0^\infty W_R^r(t)\,dt = 2^r \frac{\Gamma(r+1)}{\sqrt{2\Pi}} \mu^{1/2-r} \int_0^\infty \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}}\,dt = \frac{1}{2}$$

hence

$$|S_\mu^{r,R}(f,x)-f(x)| \le$$

$$\le \left| 2^r \frac{\Gamma(r+1)}{\sqrt{2\Pi}} \mu^{1/2-r} \int_0^R [f(x+t)+f(x-t)-2f(x)] \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}}\,dt \right| +$$

$$+ \left| 2f(x)\, 2^r \frac{\Gamma(r+1)}{\sqrt{2\Pi}} \mu^{1/2-r} \int_R^\infty \frac{J_{r+1/2}(\mu t)}{t^{r+1/2}}\,dt \right| = :I_1+I_2.$$

Now

$$|I_2| \le c(r, R_0, f, K)\, \mu^{-r} \int_R^\infty t^{-r-1}\,dt \le$$

$$\le c(r, R_0, f, K)\, \mu^{-r} \qquad (x \in K).$$

Since

$$|f(x+t)+f(x-t)-2f(x)| \le c(R_0, f, K)\, t^2 \qquad (x \in K)$$

for $f \in C^2(\mathbf{R})$ hence in case $r<2$

$$|I_1| \le c\, \mu^{1/2-r} \int_0^R \mu^{-1/2} t^{1-r}\,dt \le c\, \mu^{-r}$$

and in case $r \ge 2$

$$|I_1| \le c\, \mu^{1/2-r} \int_0^R t^2 \frac{(\mu t)^{r-2}}{t^{r+1/2}}\,dt \le c\, \mu^{-3/2}$$

holds with $c = c(r, R_0, f, K)$. This proves the locally uniform convergence of the means $S_\mu^{r,R}$, so the Banach-Steinhaus theorem applies.

The Theorem 3 is proved.

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EXISTENCE AND LIPSCHITZIAN DEPENDENCE THEOREMS IN COMPLETE SPACES

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1. INTRODUCTION

We shall give two very elementary conditions each of which implies the existence of a zero of a non-negative real valued function defined on a complete metric space. The validity of these existence theorems is equivalent to the completeness of the metric space. In section 3 we derive from our existence theorems a number of fixed point theorems, Lyusternik-type theorems, open mapping theorems, implicit- and inverse multifunction theorems. Most of these latter theorems are known, some of them are slight generalizations of known theorems, but the unified and elementary approach given here seems to be novel.

Some words about terminology and notations. Let (M, d) be a metric space, $u \in M$ and $r \geq 0$. The closed resp. (in the case $r > 0$.) the open ball about u with radius r will be denoted by $B(u,r)$ resp. $\dot{B}(u,r)$. The distance of an element $x \in M$ from a nonempty subset H of M will be denoted by $d(x,H)$. Analogously, if $\emptyset \neq H \subset M$ then

$$B(H,r) := \{x \in M \mid d(x,H) \leq r\}$$

and

$$\dot{B}(H,r) := \{x \in M \mid d(x,H) < r\}.$$

Let (M,d), (N,e) be metric spaces, $f: N \to 2^M$ a multifunction and $(\bar{y},x) \in$ graph f. f is said to be *pseudo-Lipschitzian* at $(\bar{y},x)$ with Lipschitz constant $L > 0$ (see [10]) if there exist a neighbourhood W of $\bar{y}$ and a neighbourhood U of x such that if $y \in W$, $z \in W$ and $u \in f(z) \cap U$ then there exists a $v \in f(y)$ with $d(u,v) \leq Le(z,y)$. Let M and N be topological spaces, $f: N \to 2^M$ a multifunction and $(y,x) \in$ graph f. f is said to be *lower semicontinuous* at (y,x) if for each neighborhood V of x the set $\{z \in Y \mid f(z) \cap V \neq \emptyset\}$ is a neighborhood of y. If a multifunction is pseudo-Lipschitzian at a point then it is lower semicontinuous at the same point (this follows immediately from the definitions).

Finally, some special notations that will be used troughout the paper. If (M, d) is a metric space and $g:M\to\mathbf{R}_+$ then one can introduce an order relation $\leq_g$ on the set M in the following way: $v\leq_g u$ if and only if $d(v,u)+g(v)\leq g(u)$ ($d(u,u) = 0$ implies reflexivity, $d(u,v)\geq 0$ implies antisymmetry and the triangle inequality implies transitivity). If $u\in M$ then the set $\{x\in M \mid x\leq_g u\}$ will be denoted by $]-\infty,u]_g$ and the set $\{x\in M\setminus\{u\} \mid x\leq_g u\}$ by $]-\infty,u[_g$. Note that $]-\infty,u]_g\subset B(u,g(u))$.

2. TWO EXISTENCE THEOREMS AND COMPLETENESS

THEOREM 2.1. *Let (M,d) be a complete metric space, $u\in M$, $0\leq p<1$, and $g:M\to\mathbf{R}_+$ a function. Suppose that*

(2.1) $$\forall v\in]-\infty,u]_g\,\exists\, w\in B(v,(1-p)g(v)) \text{ such that } g(w)\leq pg(v),$$

if (x_n) is a convergent sequence in $]-\infty,u]_g$ such that

(2.2) $$\lim(g(x_n)) = 0 \text{ then } g(\lim(x_n)) = 0.$$

Then there exists a $z\in]-\infty, u]_g$ with $g(z) = 0$.

PROOF. The proof is essentially the same as that of the Banach contraction principle. We construct recursively a Cauchy sequence (x_n) in $]-\infty,u]_g$ such that $\lim(g(x_n)) = 0$. Starting with $x_0 := u$ we select an $x_1\in B(u,(1-p)g(u))$ according to (2.1):

$$g(x_1)\leq pg(x_0).$$

Suppose that $n\in\mathbf{N}$ and the elements $x_1,\ldots,x_n$ have been chosen in such a way that, for $i = 1,\ldots, n$,

$$d(x_{i-1},x_i)\leq(1-p)g(x_{i-1}) \text{ and}$$

(2.3) $$g(x_i)\leq pg(x_{i-1})\leq p^i g(u).$$

Then

(2.4) $$d(u,x_n)\leq\sum_{i=1}^{n}d(x_{i-1},x_i)\leq(1-p)\sum_{i=1}^{n} g(x_{i=1})\leq$$

$$\leq(1-p)\sum_{i=0}^{n-1}p^i g(u) = (1-p^n)g(u)\leq g(u)-g(x_n),$$

that is $x_n\in]-\infty,u]_g$. According to (2.1), an element x_{n+1} can be chosen in such a way that (2.3) holds for $i = 1,\ldots,n, n+1$. Thus we have constructed a sequence (x_n) in $]-\infty,u]_g$ such that (2.3) holds for all $i\in\mathbf{N}$. This implies that

$\lim (g(x_n)) = 0$ and that the series $\Sigma d(x_{i-1},x_i)$ converges, therefore (x_n) is a Cauchy sequence. (M,d) is complete, thus $z:=\lim(x_n)$ exists and from (2.3) we get $g(z) = 0$. Finally, by (2.4) we have $d(u,x_n) \leq g(u)$, $B(u,g(u))$ is closed, so $d(u,z) \leq g(u)$, that is $z \in]-\infty,u]_g$. □

REMARK 2.2. If the function $g:M \to \mathbf{R}_+$ is lower semicontinuous then it satisfies (2.2), but the condition of lower semicontinuity instead of (2.2) does not always work in the applications (see remark 3.9).

For our second existence theorem we shall need the following Theorem 2.3, which is not only a special case of Ekeland's variational principle but is actually equivalent to it. For a proof see for example Chapter 5 of [1] (Corollary 3.2), [2] (Theorem 2.1), [3] (Theorem 2.3) or [5] (Theorem 1.1).

THEOREM 2.3. *If (M,d) is a complete metric space, $u \in M$ and $g:M \to \mathbf{R}_+$ a lower semicontinuous function, then there exists a $z \in]-\infty,u]_g$ such that $]-\infty,z[_g = \emptyset$.*

THEOREM 2.4. *Let (M,d) be a complete metric space, $u \in M$ and $g:M \to \mathbf{R}_+$ a lower semicontionuous function. Suppose that*

$$]-\infty,v[_g \neq \emptyset \text{ whenever } v \leq_g u \text{ and } g(v) \neq 0. \tag{2.5}$$

Then there exists a $z \in]-\infty,u]_g$ with $g(z) = 0$.

PROOF. Indeed, if $g(v) \neq 0$ held for all $v \in]-\infty,u]_g$ then the interval $]-\infty,v[_g$ would be nonempty for all such v—contrary to Theorem 2.3. □

REMARK 2.5. (2.1) clearly implies (2.5), therefore Theorem 2.4 seems to be more general than Theorem 2.1 for lower semicontinuous g. Nevertheless Theorem 2.4 itself implies that if for a lower semicontinuous $g:M \to \mathbf{R}_+$ (2.5) holds then (2.1) holds as well with $p = 0$, that is, for every $v \in]-\infty,u]_g$ there exists a $w \in B(v,g(v))$ with $g(w) = 0$. Namely, Theorem 2.4 can be applied for the point v instead of u (note that $]-\infty,v]_g \subset B(v,g(v))$).

REMARK 2.6. The condition of lower semicontinuity in Theorem 2.4 cannot be relaxed to condition (2.2). This is shown by the following trivial counterexample: $M: = \mathbf{R}$, $d(x,y) := |x-y|$, $u := 0$

$$g(x):=\begin{cases} \frac{1}{2}+|x|, & \text{if } 0<|x|\leq 1, \\ 1, & \text{if } x = 0. \end{cases}$$

To $v = 0$ one can choose any $w \in [-1/4,1/4]\setminus\{0\}$, if $0<|v|\leq 1$ then w can be chosen between 0 and v arbitrarily.

REMARK 2.7. Theorem 2.1 characterizes the complete spaces. Moreover, one can prove the following assertion: If $p \in]0,1[$ and the metric space (M,d) has the property that for all $u \in M$ and every lower semicontinuous $g:M \to \mathbf{R}_+$

satisfying (2.1) has a zero, then (M,d) is complete. Indeed, if (x_n) is a Cauchy sequence then the sequence of numbers $d(x,x_n)$ is convergent for all $x\in X$. Take a $u\in X$ and if $u\neq\lim(x_n)$ then for all $x\in M$, $g(x) := 2\lim(d(x,x_n))/(1-p)$. g is easily seen to be lower semicontinuous. If $v\in]-\infty,u]_g$ and $g(v)\neq 0$ (in the case $g(v)=0$ $w:=v$) then for $n>n_1$ we have $d(v,x_n)\leq 2\lim(d(v,x_n))=(1-p)\cdot g(v)$; for $n>n_2$ and $m>n_2$ it holds $d(x_1,x_m)\leq(1-p)pg(v)/2$, hence for $n>n_2$ we have $g(x_n)\leq pg(v)$, therefore one can take $w:=x_n$ with any $n>\max\{n_1,n_2\}$. The zero of g is of course the limit of (x_n). Theorem 2.4 characterizes the complete spaces, too. This follows already from the assertion proved just now, because (2.1) implies (2.5). Similar characterizations of completeness can be found for example in [7], [11] and [12].

3. CONSEQUENCES OF THE EXISTENCE THEOREMS

Let us start with the trivial remark that the Banach contraction principle is a corollary of Theorem 2.1. Indeed, if (M,d) is a complete metric space and $f:M\to M$ is Lipschitzian with Lipschitz constant $p\in[0,1[$ then for any $u\in X$ one can define $g(x) := d(x,f(x))/(1-p)$ and $w := f(v)$.

Now we show that the set-valued contraction principle (see [8], Theorem 5), is also a special case of Theorem 2.1.

THEOREM 3.1. *Let (M,d) be a complete metric space, let the values of the multifunction $F:M\to 2^M$ be nonempty closed subsets of M, and suppose that*

$$d(w,F(y))\leq Ld(x,y) \tag{3.1}$$

holds for some $L\in]0,1[$ and for all $y\in M$, $x\in M$, $w\in F(x)$. Then there exists a $z\in M$ with $z\in F(z)$.

PROOF. For any $u\in M$ and $p\in]L,1[$, Theorem 2.1 can be applied to the function g defined by $g(x) := pd(x,F(x))/L(1-p)$. Indeed, to each $v\in X$ one can find a $w\in F(v)$ such that $d(v,w)\leq pd(v,F(v))/L = (1-p)g(v)$ (because $p/L>1$), and applying (3.1) with $y := w$ and $x := v$ we obtain

$$d(w,F(w))\leq Ld(v,w)\leq pd(v,F(v)),$$

which proves (2.1). Suppose that $\lim(x_n) = y$, $\lim(g(x_n)) = 0$, and that $w_n\in F(x_n)$ satisfies $d(x_n,w_n)\leq d(x_n,f(x_n))+1/n$. Applying (3.1) with $x := x_n$ and $w := w_n$ we have, for all positive integers n,

$$\begin{aligned}
d(y,F(y)) &\leq d(y,w_n)+d(w_n,F(y))\leq\\
&\leq d(y,x_n)+d(x_n,w_n)+Ld(x_n,y)\leq\\
&\leq (1+L)d(x_n,y)+d(x_n,\ F(x_n))+\frac{1}{n}\leq\\
&\leq (1+L)d(x_n,y)+\frac{L(1-p)}{p}g(x_n)+\frac{1}{n},
\end{aligned}$$

hence $g(y) = 0$. Of course, any zero of the function g is a fixed point of the multifunction F. □

The next corollary to Theorem 2.1 will be the so-called generalized Lyusternik theorem (see § 0.2 of [6]). In the proof of it we shall use the following trivial and well-known corollary to the Banach open mapping theorem.

LEMMA 3.2 Let X and Y be Banach spaces and $P:X\to Y$ a continuous linear surjection. Then there exists a positive number r such that $B(0_Y,r)\subset \subset PB(0_X, 1)$, consequently for each

$$k>c(P) := \frac{1}{\sup\{r>0 \mid B(0_y,r)\subset PB(0_X,1)\}}$$

and each $y\in Y$ there exists an $x\in X$ such that $Ax = y$ and $\|x\|\leq k\|y\|$.

THEOREM 3.3. *Let X and Y be Banach spaces, $P:X\to Y$ a continuous linear surjection, $x_0\in X$, X, $R>0$, $0<b<\min\{1/2c(P), 1/2\}$, and $f:B(x_0, 2R)\to Y$ a mapping such that for each $x,z\in B(x_0,2R)$,*

$$\|f(x)-f(z)-P(x-z)\|\leq b\|x-z\|. \tag{3.2}$$

Then there exist a neighborhood $U\subset B(x_0,R)$ of x_0 and a function $q:U\to \to B(0_x, R)$ such that, for $t\in U$, $f(t+q(t)) = f(x_0)$ and $\|q(t)\|\leq \|f(t)-f(x_0)\|/b$.

PROOF. As (3.2) implies the continuity of f, there exists a neighborhood $U\subset B(x_0,R)$ of x_0 such that

$$\|f(t)-f(x_0)\|\leq bR \text{ for all } t\in U.$$

Fix an arbitrary $t\in U$. We claim that $u: = 0_X$, $M := B(0,r)$ with $r := \|f(t)-f(x_0)\|/b$, $d(x,y) := \|x-y\|$, $p := 1-b$ and the function $g:M\to \mathbf{R}_+$ defined by $g(x) := \|f(t+x)-f(x_0)\|/b$ satisfy the conditions of Theorem 2.1. (2.3) holds by the continuity of f. As $1/2b>c(P)$, by lemma 3.2 we have for each $v\in M$ a $w\in X$ such that

$$P(w-v) = -2b(f(t+v)-f(x_0)) \tag{3.3}$$

and

$$\|w-v\|\leq \|f(t+v)-f(x_0)\| = bg(v). \tag{3.4}$$

Observe that if $v\leq_g 0$ that is $\|v\|\leq g(0)-g(v) = r-g(v)$ then $\|w\|\leq \|w-v\|+\|v\| = r-pg(v)\leq r$ that is $w\in M$. Defining the function $h:B(x_0,R)xB(x_0,R)\to Y$ by

$$h(x,z) := \begin{cases} 0, \text{ if } x = z \\ [f(x)-f(z)-P(x-z)]/\|x-z\|, \\ \text{if } x \neq z \end{cases} \tag{3.5}$$

and using (3.3) we obtain

$$f(t+w)-f(x_0) = f(t+v)-f(x_0)+P(w-v)+h(t+w,t+v)\|w-v\| =$$
$$= (1-2b)[f(t+v)-f(x_0)]+h(t+w,t+v)\|w-v\|,$$

hence (3.4) and (3.2) yield

$$g(w) = \|f(t+w)-f(x_0)\|/b \leq (1-2b)g(v)+bg(v) = pg(v).$$

Applying Theorem 2.1 we get a zero $q(t)\in]-\infty,u]_g \subset B(u,g(u))$ of the function g.□

REMARK 3.4. It is obvious that if f has a strict Fréchet derivative P at the point x_0 (this means $P:X\to Y$ is a continuous, linear mapping and for each positive ϵ there exists a positive δ such that if $\|x-x_0\|<\delta$ and $\|z-z_0\|<\delta$ then $\|h(x,z)\|<\epsilon$ where $h(x,z)$ is definied by (3.5)) then (3.2) holds for some $R>0$. It is also well known (being a corollary to the Lagrange inequality) that if f has a Fréchet derivative $Df(x)$ in each point of a neigborhood of x_0 and the mapping $x\to Df(x)$ is continuous at x_0, then $Df(x_0)$ is a strict Fréchet derivative.

A natural extension of Theorem 3.3 is the following.

THEOREM 3.5. *Let T be a nonempty set, X and Y Banach spaces, $P:X\to y$ a continuous linear surjection, $x_0\in X$, $R>0$, $0<b<\min\{1/2c(P),\ 1/2\}$ and $f:T\times B(x_0,R)\to Y$ such that for all $x,z\in B(x_0,R)$ and $t\in T$*

$$\|f(t,x)-f(t,z)-P(x-z)\| \leq b\|x-z\|. \tag{3.6}$$

Then

A) for each $t\in V := \{t\in T \mid \|f(t,x_0)\|<bR\}$ there exists a zero of the function $f(t,\cdot)$ in the ball $B(x_0,\|f(t,x_0)\|/b)$;

B) if, in addition, T is a topological space, f is continuous, $\|u-x_0\|<R$ and $f(t_0,u) = 0$, then the multifunction $F:V\to 2^{B(x_0,R)}$ defined by

$$F(t) := \{x\in B(x_0,R) \mid f(t,x) = 0\}$$

is lower semicontinuous at (t_0,u);

C) if, in addition, (T, d) is a metric space and f is locally Lipschitzian with respect to its first variable at (t_0, u) [this means it satisfies

$$\|f(t_1, x)-f(t_2, x)\| \leq Ld(t_1, t_2) \tag{3.7}$$

for some $L>0$, $r>0$ and for all $t_1, t_2\in B(t_0, r)$, $x\in B(u, r)$] then F is pseudo-Lipschitzian at (t_0, u).

PROOF. *A*) In the same way as in the previous proof, we can apply Theorem 2.1 with $u := x_0$, $r := \|f(t, u)\|/b$, $M := B(u, r)$, $d(x, y) := \|x-y\|$ and $g(x) := \|f(t, x)\|/b$.

B) For each neighborhood U of u there exists a positive number s such that $B(u, s)\subset U\cap B(x_0, R)$. f is continuous, therefore $W := \{t\in T \mid \|f(t, u)\|<bs\}$ is a neighborhood of t_0. Now we can apply part *A*) with (x_0, R) replaced by (u, s) and this yields the desired result: $F(t)\cap U\neq\emptyset$ for all $t\in W$.

C) Let L and r be chosen according to (3.7). We may and shall suppose that $r<R-d(x_0, u)$. Let W be the closed ball about t_0 with radius min $\{r, br/4L\}$ and $U := B(u, r/2)$. If $t_1, t_2 \in W$ and $x_1 \in F(t_1) \cap U$ then $f(t_1, x_1) = 0$ and $d(t_1, t_2) \leq br/2L$. Part *A*) can be applied to $(x_1, R/2)$ instead of (x_0, R): there exists an $x_2 \in X$ with

$$\|x_1-x_2\| \leq \|f(t_2, x_1)\|/b = \|f(t_2, x_1)-f(t_1, x_1)\|/b \leq Ld(t_2, t_1)/b \leq r/2$$

such that $f(t_2, x_2) = 0$. Thus x_2 is in $F(t_2)$ and F is pseudo-Lipschitzian at (t_0, u) with Lipschitz constant L/b.

REMARK 3.6. If S is a topological space, X and Y Banach spaces, $G \subset X$ an open set, $t_0 \in S$, $x_0 \in G$ and $f: S \times G \to Y$ a function such that

a) for each $t \in S$ the function $f(t, \cdot)$ has a Fréchet derivative at every point $x \in G$ denoted by $D_2 f(t, x)$ and

b) the function $D_2 f$ is continuous at (t_0, x_0), then for each $b>0$ there exist a neighborhood T of t_0 and an $R>0$ such that (3.6) holds with $P := D_2 f(t_0, x_0)$. This is an immediate corollary to the Lagrange inequality:

$$\|f(t, x)-f(t, z)-P(x-z)\| \leq \sup_{s \in]0,1[} \|D_2 f(t, sx+(1-s)z)-D_2 f(t_0, x_0)\| \cdot \|x-z\|.$$

Theorem 3.8 is the multivalued version of Dmitruck's theorem (that of Theorem 1.5 in [4]), which is a common generalization of several Lyusternik type theorems. In this theorem the notion of p-chain will be used in the following "$\leq$-sense":

DEFINITION 3.7. Let Y be a metric space, $H \subset Y$, $K \subset Y$ and $p>0$. The set H is called a p-chain for K if, for each $y \in K$, $H \cap B(y, p) \neq \emptyset$.

THEOREM 3.8. *Let (X, d) be a complete metric sapce, (Y, e) a metric sapce, $x_0 \in X$, $R>0$ and $0<b<a$. Suppose that $f: B(x_0, R) \to 2^Y$ is a multifunction with closed graph such that $f(B(v, s))$ is a bs-chain for the set $B(f(v), as)$ whenever*

$$d(x_0, v)+\frac{as}{a-b} \leq R. \tag{3.8}$$

Then

A) *$B(f(u), (a-b)r) \subset f(B(u, r))$ whenever*

$$d(x_0, u)+r \leq R, \tag{3.9}$$

consequently $\dot{B}(f(u), (a-b)r) \subset f(\dot{B}(u, r))$ for these pairs (u, r);

B) *the inverse F of the restriction of f to the open ball $\dot{B}(x_0, R)$ is pseudo-Lipschitzian at each point of its graph (with Lipschitz-constant $1/(a-b)$).*

PROOF. *A*) Suppose that (u, r) satisfies (3.9) and $y \in B(f(u), (a-b)r)$. We must prove that $e(y, f(z)) = 0$ for some $z \in B(u, r) =: M$. We shall apply Theorem 2.1 to the function $g: B(u, r) \to \mathbf{R}_+$ defined by $g(x) := e(y, f(x))/(a-b)$ and to the number $p := b/a$. (The restriction of d to $M \times M$ will be denoted again by d). If $d(u, v)+g(v) \leq g(u)$ then using (3.9) and the triangle inequality we get

$$d(x_0, v)+g(v) \leq R,$$

thus (3.8) holds with $s := e(y, f(v))/a = (1-p)g(v)$. As $f(B((v, (1-p)g(v)))$ is a $pe(y, f(v))$-chain for the set $B(f(v), e(y, f((v)))$ and y is an element of this latter set, there exist a $w \in B(v, (1-p)g(v))$ and a $\bar{y} \in f(w)$ such that $e(y, \bar{y}) \leq \leq pe(y, f(v))$, therefore

$$g(w) \leq e(y, f(w))/(a-b) \leq e(x, \bar{y})/(a-b) \leq pe(y, f(v))/(a-b) = pg(v).$$

Hence (2.1) holds. Now suppose that $z = \lim (x_n)$ and $0 = \lim (g(x_n)) = = \lim (e(y, f(x_n))/(a-b)) = 0$. Then there exists a sequence (y_n) such that for each $n \in \mathbf{N}$ we have $y_n \in f(x_n)$ and $e(y, y_n) < e(y, f(x_n)) + 1/n$, so that $y = \lim (y_n)$. The graph of f is closed, therefore $y \in f(z)$, $e(y, f(z)) = 0$, $g(z) = 0$. Hence (2.2) holds and Theorem 2.1 yields the desired result: $y \in f(B(u, r))$.

B) If $u \in \dot{B}(x_0, R)$ and $z \in f(u)$, that is, $(z, u) \in$ graph F, then there exists a positive number r such that $d(x_0, u) + r < R$. Using the notations

$$W := \dot{B}(z, (a-b)r/3) \text{ and } U := \dot{B}(u, r/3),$$

we have for each $z_1, z_2 \in W$ and $x_1 \in U \cap F(z_1)$,

$$e(z_1, z_2) < \frac{2}{3} r(a-b) < (a-b)(r - d(u, x_1)),$$

hence if $z_1 \neq z_2$ then the pair $(x_1, e(z_1, z_2)/(a-b)) =: (u, r)$ satisfies (3.9). By *A*)

$$z_2 \in B(f(x_1), e(z_1, z_2)) \subset f(B(x_1, e(z_1, z_2)/(a-b))),$$

that is, $z_2 \in f(x_2)$ for some $x_2 \in B(x_1, e(z_1, z_2)/(a-b))$, thus

$$x_1 \in B(F(z_2), e(z_1, z_2)/(a-b)). \square$$

REMARK 3.9. The function g in the proof of Theorem 3.8 need not be lower semicontinuous even f is single-valued (this fact justifies the choice of condition (2.2) instead of the lower semicontinuity condition). For instance, if $X = \mathbf{R}$, $d(x, y) := |x-y|$, Y is the space of all bounded $\mathbf{R} \to \mathbf{R}$ functions with the metric induced by the norm $\|h\| := \sup |h(t)|$, $y: R \to \{0\}$, and $f: X \to Y$ is defined by

$$f(x)(t) := \begin{cases} 2, & \text{if } x = 0 \text{ and } t \in \mathbf{R}, \\ 1, & \text{if } x \neq 0 \text{ and } t = x, \\ 0, & \text{if } x \neq 0 \text{ and } t \neq x, \end{cases}$$

then the graph of f is closed (since $\|f(x_1) - f(x_2)\| \geq 1$ for all $x_1 \neq x_2$, the graph has no accumulation points) but g is not lower semicontinuous at the point 0:

$$g(x) = \|f(x) - y\| = \|f(x)\| = \begin{cases} 2, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0. \end{cases}$$

Now we prove that a sharpening of a theorem of J. P. PENOT (see [9], Theorem 2.3) is a special case of the previous theorem. Penot's theorem can be reformulated as follows.

THEOREM 3.10'. Let (X, d) be a complete metric space, Y a normed linear space, $R>0$, $0<b<a$, $x_0 \in X$, $h:B(x_0, R) \to Y$ a Lipschitzian mapping with Lipschitz constant b and $P:X \to Y$ a continuous mapping satisfying $B(P(v), as) \subset P(B(v, s))$ for all $v \in X$ and $s>0$. Then for the mapping $f := P+h$ we have $B(f(x_0), R(a-b)) \subset f(B(x_0,R))$.

The sharpened version reads:

THEOREM 3.10. *Let (X, d) be a complete metric space, Y a normed linear space, $R>0$, $0<b<a$, $x_0 \in X$, $h:B(x_0, R) \to Y$ a Lipschitzian mapping with Lipschitz constant b and $P:B(x_0, R) \to Y$ a mapping with closed graph satisfying*

$$B(P(v),as) \subset P(B(v, s))$$

whenever

$$d(x_0, v)+\frac{as}{a-b} \leq R.$$

Then

A) the mapping $f:P+h$ satisfies $B(f(u), (a-b)r) \subset f(B(u, r))$ whenever

$$d(x_0, u)<d(x_0, u)+r \leq R, \tag{3.10}$$

consequently in this case it satisfies $\dot{B}(f(u), (a-b)r) \subset f(\dot{B}(u, r))$ as well,

B) the inverse of the restriction of f to the open ball $\dot{B}(x_0, R)$ is pseudo-Lipschitzian at each point of its graph (with Lipschitz constant $1/(a-b)$).

PROOF. If we prove that $f(B(v, s))$ is a bs-chain for the set $B(f(v), as)$ whenever the pair (v, s) satisfies (3.8), then everything will follow from Theorem 3.8. Let y be in $B(f(v),as)$. Then $y-h(v) \in B(P(v),as) \subset P(B(v, s))$, that is, $y-h(v) = \|P(w)$ with some $w \in B(v,s)$. Therefore

$$\|y-f(w)\| = \|P(w)+h(v)-P(w)-h(w)\| = \|h(v)-h(w)\| \leq bd(v, w) \leq bs.$$

□

REMARK 3.11 In the previous theorem Y need not be a normed linear space. The same proof works for example if Y is a commutative topological group whose topology is induced by a translation invariant metric.

Now we derive from Theorem 2.4 an abstract "implicit multifunction Theorem" which is a generalization of parts A) and C) of Theorem 3.5.

THEOREM 3.12. *Let (X, d) be a complete metric space, (Y, e) a metric space, $\bar{x} \in X$, R a positive number, for each $y \in Y$ $g_y:B(\bar{x}, R) \to \mathbf{R}_+$ a lower semicontinuous function and $M_y := \{v \in B(\bar{x}, R) \mid d(\bar{x}, v)<d(\bar{x}, v)+g_y(v)<R$.*

Suppose that

$$g_y(\bar{x})<R \quad \text{for all} \quad y \in Y, \tag{3.11}$$

(3.12) $$\forall y \in Y,\ \forall v \in M_y\ ]-\infty,\ v[_y \neq 0,$$
$$\forall (x, \bar{y}) \in \dot{B}(\bar{x}, R) x Y\ \exists r > 0 \exists L > 0$$

such that

(3.13) $$|g_y(u) - g_z(u)| \leq Le(y, z)$$

for all

$$u \in B(x, r),\ y \in B(\bar{y}, r) \quad \textit{and} \quad z \in B(\bar{y}, r).$$

Then

$$F(y) := \{x \in \dot{B}(\bar{x}, R) \mid g_y(x) = 0\} \neq \emptyset \quad \textit{for all} \quad y \in Y,$$

and the multifunction $F: Y \to 2^{\dot{B}(\bar{x}, R)}$ *is pseudo-Lipschitzian in each point of its graph.*

Proof. Theorem 2.4 can be applied with $u := \bar{x}$, $M := B(u, g_y(u))$ and $g(x) := g_y(x)$, therefore $F(y) \neq \emptyset$.

Let $(\bar{y}, x) \in$ graph F, that is, $\bar{y} \in Y$, $x \in \dot{B}(\bar{x}, R)$ and $g_{\bar{y}}(x) = 0$. Fix the positive numbers L and r according to (3.13), we may and shall assume that $L > 1/2$ and $r < (R - d(\bar{x}, x))/2$. Then $W := B(\bar{y}, r/2L) \subset B(\bar{y}, r)$ and for every $u \in U := B(x, r)$ we have $B(u, r) \subset \dot{B}(\bar{x}, R)$ If $z \in W$, $y \in W$ and $u \in F(z) \cap U$ then (3.13) implies

$$g_y(u) = |g_y(u) - g_z(u)| \leq Le(y, z) \leq r,$$

hence $g_y(u) + d(u, \bar{x}) \leq g_y(u) + d(\bar{x}, x) + d(x, u) \leq 2r + d(\bar{x}, x) < R$, therefore $M(g, u) \subset M_y$ where $g := g_{y|B(u, r)}$. Theorem 2.4 applies with $M := B(u, r)$ (the elements w_y *are contained in* $B(u, r)$) and yields an $x_1 \in B(u, r) \subset \dot{B}(\bar{x}, R)$ such that $g_y(x_1) = 0$ and

$$d(x_1, u) \leq g_y(u) \leq Le(y, z). \qquad \square$$

Remark 3.13. One could formulate the generalization of part *B*) of theorem 3.5, too. Similarly one could derive another abstract implicit multifunction theorem from Theorem 2.1.

Next we shall give a short proof for the Aubin-Ekeland inverse multifunction theorem (Chapter 7 of [1], Theorem 5.4) based on Theorem 3.12. First we recall the definition of contingent vectors and that of the contingent derivative of a multifunction.

Definition 3.14. If X is a closed subset of a normed linear space E and $v \in X$ then a vector $y \in E$ is called a contingent vector of X at v iff there exist a sequence (s_n) of positive numbers converging to $0 \in \mathbf{R}$ and a sequence (h_n) in E converging to $0 \in E$ such that

$$v + s_n y + s_n h_n \in X \quad \text{for all} \quad n \in \mathbf{N}.$$

Definition 3.15 If X_1 and X_2 are normed linear spaces, $P: X_1 \to 2^{X_2}$ is a multifunction with closed graph and $v \in$ graph P, then the contingent derivative of P at v is the multifunction $DP(v): X_1 \to 2^{X_2}$ the graph of which is the set (cone) of all contingent vectors of graph P at v.

THEOREM 3.16 *Let X_1 and X_2 be Banach spaces, $P:X_1 \to 2^{X_2}$ a multifunction with closed graph, $\bar{x} \in X :=$ graph P, R, c and q three positive numbers with $q<1$. Suppose that for all $v \in \dot{B}(\bar{x}, R)$ (open ball in the subspace X) and for all $z \in X_2$ there exist $x \in X_1$ and $\bar{z} \in X_2$ such that*

$$z-\bar{z} \in DP(v)x, \quad \|x\| \le c\|z\| \quad \text{and} \quad \|\bar{z}\| \le q\|z\|. \tag{3.14}$$

Then there exists a neighborhood Y of x_2 such that for each $v_2 \in Y$ one can find a $v_1 \in X_1$ such that $(v_1, v_2) \in \dot{B}(\bar{x}, R)$, moreover, the inverse multifunction P^{-1} is pseudo-Lipschitzian at (v_2, v_1) whenever $v_2 \in Y$ and $(v_1, v_2) \in \dot{B}(\bar{x}, R)$.

PROOF. Let Y be the open ball with center $\bar{x}_2$ and radius $R(1-q)/2(c+2)$ as a subspace of X_2. The subspace X of $X_1 \times X_2$ (with the metric $d(w, v) := \|w_1-v_1\|+\|w_2-v_2\|$) is a complete metric space. A ball about a point of X always means a ball in this subspace. For each $y \in Y$, $g_y:B(\bar{x}, R) \to \mathbf{R}_+$ is defined by

$$g_y(x) := \frac{2(c+2)}{1-q}\|x_2-y\|,$$

which is obviously continuous. (3.11) holds by the choice of Y and (3.13) holds because of the triangle inequality ($L = 2(c+2)/(1-q)$). We must check (3.12). Let $v \in \dot{B}(\bar{x}, R)$, and $y \in Y$ be such that $z := y-v_2 \neq 0$, then there exist $x \in X_1$ and $\bar{z} \in X_2$ such that (3.14) holds. As $(x, z-\bar{z})$ is a contingent vector of X at v, there exist a number $s \in]0,1[$ and a vector $h \in X_1 \times X_2$ such that

$$\|h_1\| \le \frac{1-q}{2} \cdot \|z\|, \quad \|h_2\| \le \frac{1-q}{2}\|z\|, \tag{3.15}$$

and the vector $w \in X_1 \times X_2$ with components defined by

$$w_1 := v_1+sx+sh_1, \quad w_2 := v_2+s(z-\bar{z})+sh_2 \tag{3.16}$$

belongs to X. Here $w_2 \neq v_2$ and therefore $w \neq v$. Indeed, $w_2 = v_2$ would imply $\|z-\bar{z}\| = \|h_2\|$ but by (3.15) and (3.14) we have

$$\|h_2\| \le \frac{1-q}{2}\|z\| < (1-q)\|z\| \le \|z\|-\|\bar{z}\| \le \|z-\bar{z}\|.$$

Since

$$w_2-y = (s-1)z-s\bar{z}+sh_2,$$

$$\|w_2-y\| \le (1-s)\|z\|+sq\|z\|+s\frac{1-q}{2}\|z\|,$$

we get

$$\|v_2-y\|-\|w_2-y\| \ge s\|z\|\left(1-q-\frac{1-q}{2}\right) = s\frac{1-q}{2}\|z\|. \tag{3.17}$$

Finally, from (3.16), (3.14), (3.15) and (3.17) we obtain

$$\|w_1-v_1\| \le sc\|z\|+s\frac{1-q}{2}\|z\|,$$

$$\|w_2-v_2\| \leq s\|z\|+sq\|z\|+s\frac{1-q}{2}\|z\|,$$

$$d(w, v) \leq s(c+2)\|z\| \leq g_y(v)-g_y(w).$$

$v \in M_y$ implies $w \in B(\bar{x}, R)$, thus (3.12) holds, too. Now everything follows from Theorem 3.12 because the distance of any two points in $\dot{B}(\bar{x}, R)$ is not lessthen the distance of the first components.

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ON THE EQUICONVERGENCE OF THE RIESZ MEANS FOR THE FUNCTIONS FROM THE CLASS L_2^s

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The theory of non-selfadjoint differential operators has great importance in several applications. In the last years a new and fruitful method was developed by V. A. IL'IN for the investigation of non-selfadjoint differential operators. By the aid of this method in this paper we shall prove an equiconvergence of the Riesz means for functions from the class L_2^s.

Let $G = (a, b)$ be a finite interval on the real line and consider the Schrödinger operators:

$$Lu: = -u'' + q(x)\, u(x),\ \hat{L}u: = -u'' + \hat{q}(x)\, u(x),$$

where $q(x)$, $\hat{q}(x)$, be an arbitrary non-negative function from the class $L_2(G)$. Denote by $\{u_k\}_{k=1}^{\infty}$ and $\{\hat{u}_k\}_{k=1}^{\infty}$ the complete (in $L_2(G)$) orthonormal systems of the eigenfunctions of the operators L and $\hat{L}$, respectively, with positive eigenvalues $\{\lambda_k\}_{L=1}^{\infty}$ and $\{\hat{\lambda}\}_{k=1}^{\infty}$. We may suppose $\lambda_k \geq 1$, $\hat{\lambda}_k \geq 1$ $(k = 1, 2, \ldots)$. For the sake of brevity we use the notation $\mu_k := \sqrt{\lambda_k}$. For any $f \in L_1(G)$, $\mu > 0$, $s \in \left[0, \frac{1}{2}\right)$ consi der the sth Riesz means of the spectral expansions of f:

$$\sigma_\mu^s(f, x): = \sum_{\mu_k < \mu} (f, u_k)\, u_k(x) \left(1 - \frac{\mu_k^2}{\mu^2}\right)^s,$$

$$\hat{\sigma}_\mu^s(f, x): = \sum_{\hat{\mu}_k < \mu} (f, \hat{u}_k)\, \hat{u}_k(x) \left(1 - \frac{\hat{\mu}_k^2}{\mu^2}\right)^s.$$

The aim of the present paper is to prove the following

THEOREM. *Given any compact subset $K \subset G$, for any $f \in \mathring{L}_2^s(G)$, $0 \le s < \frac{1}{2}$, $x \in K$, $\mu \ge 1$ then*

$$|\sigma_\mu^s(f, x) - \hat{\sigma}_\mu^s(f, x)| \le C \|f\|_{L_2^s} \mu^{-s},$$

where

$$\mathring{L}_2^s(G) := (L_2, \mathring{W}_2^{1/2})_{2s,2}.^{(*)}$$

The constant C does not depend on f. □

We recall some well-known results which are necessary for our proof.

(1) $$|(f, u_k)| \le C_1 \|f\|_{L_2^s} \cdot \mu_k^{-s} \quad (f \in L_2^s,\ k = 1, 2, \ldots)$$

(cf. [1], (2. 13)).

(2) $$\sum_{k=1}^{\infty} \frac{1}{1+(\mu-\mu_k)^2} \le C_2 < \infty$$

(cf. [2], Lemma 2).

(3) $$|u_k(x)| \le C_3 \quad (a \le x \le b,\ k = 1, 2, \ldots)$$

(cf. [4], (8)).

(4) $$\left| \int_0^R g_0(u_k, \mu_k, x, t)\, t^{-s-1/2} J_{s+1/2}(\mu t)\, dt \right| \le C_4 \min\left\{ \frac{1}{\mu^{3/2}}, \frac{\mu^{1/2}}{\mu_k^2} \right\}$$

where

$$g_0(u_k, \mu_k, x, t) := \int_{x-t}^{x+t} \frac{\sin \mu_k (t - |x-\xi|)}{\mu_k} q(\xi)\, u_k(\xi)\, d\xi,$$

$J_{s+1/2}(\mu t)$ is Bessel function; and

(*) *The notations were used in* H. TRIEBEL 's book (cf. [7], 4.3.2/2]).

$$|\mathfrak{D}_{R_0} K_{\mu_k}^{\mu}(R)| \leq \frac{C(K, s)}{1+(\mu-\mu_k)^2}, \tag{5}$$

where

$$K_{\mu_k}^{\mu}(R) := \mu^{1/2} \int_R^{\infty} t^{-s-1/2} J_{s+1/2}(\mu t) \cos \mu_k t \, dt,$$

$$D_{R_0} g := \frac{2}{R_0} \int_{\frac{R_0}{2}}^{R_0} g(R)\, dR \quad \left(0 < R_0 < \frac{1}{2} \operatorname{dist}(K, \partial G)\right)$$

(cf. [5]).

Now we return to the proof of our theorem.

Given any compact $K \subset G$ denote R an arbitrary number from the interval $(0, \operatorname{dist}(K, \partial G))$. Fixing $x \in K$ arbitrarily and define $W_R^s : G \to \mathbf{R}$ with

$$W_R^s(x+t) := \begin{cases} a(s)\mu^{1/2-s}|t|^{-s-1/2} J_{s+1/2}(\mu|t|) & \text{if } |t| \leq R, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where

$$a(s) := 2^s (2\pi)^{-1/2} \Gamma(s+1).$$

Proof. Computing the Fourier coefficients of the function $W_R^s(x+t)$ with respect to the system $\{u_k\}$ we get

$$W_k^s = (u_k, W_R^s) = \int_{x-R}^{x+R} W_R^s(|x-y|)\, u_k(y)\, dy = \int_0^R W_R^s(t)\, [u_k(x-t)+u_k(x+t)]\, dt.$$

Using the Titchmarsh formula [8]:

$$u_k(x+t)+u_k(x-t) = 2\, u_k(x) \cos \mu_k t + \int_{x-t}^{x+t} q(\xi)\, u_k(\xi) \frac{\sin \mu_k (t-|x-\xi|)}{\mu_k}\, d\xi$$

and applying the integral transformation $\int_0^R = \int_0^\infty - \int_R^\infty$ we have

$$(7) \qquad W_k^s = 2u_k(x) \int_0^R W_R^s(t) \cos \mu_k t \, dt + \int_0^R W_R^s(t) \, g_0(u_k, \mu_k, x, t) \, dt =$$

$$= u_k(x) \, 2 \, a(s) \mu^{-s} \left\{ \mu^{1/2} \int_0^\infty t^{-s-1/2} J_{s+1/2}(\mu t) \cos \mu_k t \, dt - K_{\mu_k}^{\mu}(R) \right\} +$$

$$+ a(s) \, \mu^{-s} \left\{ \mu^{1/2} \int_0^R t^{-s-1/2} J_{s+1/2}(\mu t) \, g_0(u_k, \mu_k, x, t) \, dt \right\}.$$

It is well-known (cf. [9], p. 107, (34)) that

$$(8) \qquad 2 \, a(s) \, \mu^{1/2} \int_0^\infty t^{-s-1/2} J_{s+1/2}(\mu t) \cos \mu_k t \, dt = \delta_{\mu_k}^{\mu} \left(1 - \frac{\mu_k^2}{\mu^2} \right)^s,$$

where

$$\delta_{\mu_k}^{\mu} := \begin{cases} 1 & \text{if} \quad \mu_k < \mu, \\ 0 & \text{if} \quad \mu_k > \mu. \end{cases}$$

Replacing (8) into (7) we obtain

$$W_k^s = \delta_{\mu_k}^{\mu} u_k(x) \left(1 - \frac{\mu_k^2}{\mu^2} \right)^s - 2 \, a(s) \, \mu^{-s} \, u_k(x) \, K_{\mu_k}^{\mu}(R) +$$

$$+ a(s) \, \mu^{-s} \left\{ \mu^{1/2} \int_0^R t^{-s-1/2} J_{s+1/2}(\mu t) \, g_0(u_k, \mu_k, x, t) \, dt \right\}.$$

According to the fact that for any fixed $x \in K$ and $\mu > 0$ the function $W_R^s(|x-y|)$ as function of y belongs to $L_2(G)$, we have the following equality in $L_2(G)$-convergence in y:

$$W_R^s(|x-y|) - \sum_{\mu_k < \mu} u_k(x) \, u_k(y) \left(1 - \frac{\mu_k^2}{\mu^2} \right)^s = -2a(s)\mu^{-s} \sum_{k=1}^{\infty} u_k(x) \, u_k(y) \, K_{\mu_k}^{\mu}(R) +$$

$$+a(s)\mu^{-s}\sum_{k=1}^{\infty}\left\{\mu^{1/2}\int_0^R t^{-s-1/2}J_{s+1/2}(\mu t)\,g_0(u_k,\mu_k,x,t)\,dt\right\}u_k(y).$$

Apply the operation $\mathfrak{D}_{R_0}$ term by term on both sides of the last equality we get

$$\mathfrak{D}_{R_0}W_R^s(|x-y|)-\sum_{\mu_k<\mu}u_k(x)\,u_k(y)\left(1-\frac{\mu_k^2}{\mu^2}\right)^s= \tag{9}$$

$$=-2a(s)\,\mu^{-s}\sum_{k=1}^{\infty}u_k(x)\,u_k(y)\,\mathfrak{D}_{R_0}K_{\mu_k}^{\mu}(R)+$$

$$+a(s)\,\mu^{-s}\sum_{k=1}^{\infty}\mathfrak{D}_{R_0}\left\{\mu^{1/2}\int_0^R t^{-s-1/2}J_{s+1/2}(\mu t)\,g_0(u_k,\mu_k,x,t)\,dt\right\}u_k(y).$$

It is easy to prove that after multiplication of both sides of (9) by any $f\in L_2^s$, one can integrate the resulting equality term by term over G in y. Introducing the notations:

$$P_1:=-2a(s)\,\mu^{-s}\sum_{k=1}^{\infty}u_k(x)\,f_k\,\mathfrak{D}_{R_0}K_{\mu_k}^{\mu}(R),$$

$$P_2:=a(s)\,\mu^{-s}\sum_{k=1}^{\infty}\mathfrak{D}_{R_0}\left\{\mu^{1/2}\int_0^R t^{-s-1/2}J_{s+1/2}(\mu t)\,g_0(u_k,\mu_k,x,t)\,dt\right\}f_k,$$

where $f_k:=(f,u_k)$ we get

$$\int_a^b\mathfrak{D}_{R_0}W_R^s(|x-y|)\,f(y)\,dy-\sigma_\mu^s(f,x)=P_1+P_2. \tag{10}$$

Using (1)—(5) we obtain

$$|P_i|\le C(K,q,s)\,\|f\|_{L_2^s}\cdot\mu^{-s}\quad(\mu\ge1,\ i=1,2).$$

Consequently

$$\left|\int_a^b\mathfrak{D}_{R_0}W_R^s(|x-y|)\,f(y)\,dy-\sigma_\mu^s(f,x)\right|\le C\,\|f\|_{L_2^s}\cdot\mu^{-s},$$

similarly

$$\left| \int_a^b \mathfrak{D}_{R_0} W_R^s (|x-y|) f(y)\, dy - \hat{\sigma}_\mu^s (f, x) \right| \leq C \|f\|_{L_2^s} \cdot \mu^{-s}.$$

At last we get the desired estimate by the triangle inequality.

Theorem is proved. □

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POLYGONAL CYCLES WITH A LARGE NUMBER OF COMPONENTS

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Introduction

In this note plane means the Euclidean plane, the Hyperbolic plane, or the surface of the 3-dimensional sphere. Let $P_0, P_1, \ldots, P_n$ be points on the plane, not necessarily distinct. In case of the surface of the sphere we assume that the points P_i, P_{i+1} are not antipodal for $i = 0, 1, \ldots, n$ and by the term "segment" we will mean the shortest arc connecting two non-antipodal points. The union of the segments $P_i P_{i+1}$ $(i = 0, 1, \ldots, n-1)$ is a cycle of n segments. The points $P_0, \ldots, P_n$ are the vertices, P_0 and P_n are the endpoints of the cycle. If $P_0 = P_n$ then the cycle is said to be closed, otherwise it is said to be open.

The problem of determining the maximum number of connected components of the complement of an open (closed) cycle of n segments was posed by L. Fejes Tóth .

The regular star polygon S_{2k+1} is defined by taking $2k+1$ points equally distributed on a circle and connecting any two of them if they are separated by $k-1$ points. Since it is easy to visualize S_{2k+1} and since in some cases the extremal configuration is derived from S_{2k+1}, we formulate our theorems using an abbreviation for the number of components into which S_{2k+1} divides the plane. Let $N(2k+1) = 2k^2-k+1$.

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THEOREM. *The maximum number of components into which a cycle of n sides can divide the plane is*

a) $N(2k+1)$ *if the cycle is open and* $n = 2k+1$,
b) $N(2k+1)-2k+1$ *if the cycle is open and* $n = 2k$,
c) $N(2k+1)$ *if the cycle is closed and* $n = 2k+1$,
d) $N(2k+1)-2k$ *if the cycle is closed and* $n = 2k$

where $N(2k+1) = 2k^2-k+1$ *is the number of components into which the regular star* $(2k+1)$*-gon divides the plane.*

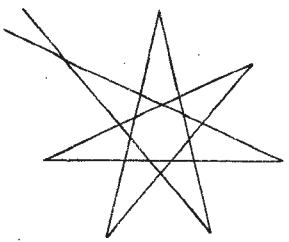

Fig. 1/a

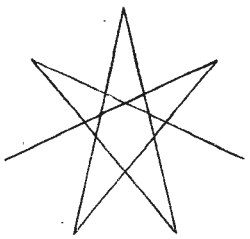

Fig. 1/b

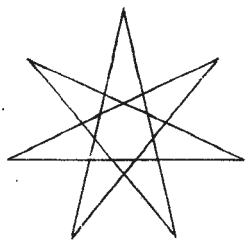

Fig. 1/c

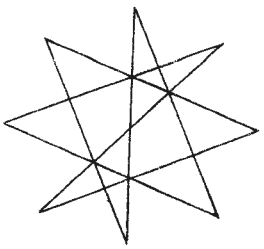

Fig. 1/d

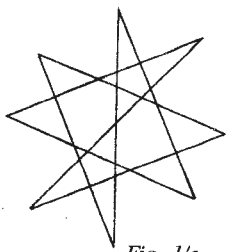

Fig. 1/e

Extreme cycles. We describe for each of the above four cases one cycle determining the maximal number of components. It is left to the reader to verify — after reading the proof of the Theorem — that, combinatorically, these are the only extreme cycles.

Case a) Extend two consecutive sides of S_{2k+1} through the common vertex by a small segment (*Fig. 1/a*).

Case b) Remove one side of S_{2k+1} (*Fig. 1/b*).

Case c) Consider S_{2k+1}(*Fig. 1/c*).

Case d) Label the vertices of a regular $2k$-gon clockwise by $1, 2, \ldots, 2k$. Let the segments $(1, k+1)$, $(2k, k)$ and $(i, i+k+1)$, $(i+1, i+k)$ $i = 2, \ldots, k-1$ form the cycle (*Fig. 1/d*). Move the vertices in a random direction by small distances so that no three sides have a common point (*Fig. 1/e*). This modified cycle determines the maximum number of components in case d).

PROOF OF THEOREM. The Cayley-Klein model for representing the cycles of the hyperbolic plane reduces the general case to the Euclidean and spherical cases. Our forthcoming argument works in both cases.

We start with two technical lemmas which will simplify the enumeration of the components.

LEMMA 1. The assumption that in a cycle of n sides determining the maximum number of components no two consecutive sides have common inner points does not reduce the generality of the conclusion of the Theorem.

LEMMA 2. We may suppose that in a cycle of n sides determining the maximum number of components no three sidelines have a point in common.

The idea at proving these lemmas is that we start with an arbitrary cycle which may not satisfy the above conditions and then replace its vertices, always one at a time, by small distances so that the number of components does not decrease while the number of "bad" pairs of sides (or the number of "bad" intersection points) will be zero after finitely many steps.

PROOF OF LEMMA 1. To the contrary, let us suppose that for the cycle $P = P_0 \ldots P_n$ of n sides there exists an i such that either P_{i+1} belongs to the interior of P_{i-1} P_i or $P_{i+1} = P_{i-1}$. Let e denote the sideline of P_{i-1} P_i. For the sake of brevity we will list the cases we have to consider and only the corresponding motions for reducing the number of bad pairs of sides. It is easy and is left to the reader to see that the number of components does not decrease if the motions are small enough.

1. $P_{i+1} = P_n$.
   Move P_{i+1} off the line $P_{i+1}P_i$.
2. $P_{i+2} \notin e$.
   Move P_{i+1} toward P_{i+2}.
3. $P_{i+2} \in P_{i-1}\, P_{i+1}$.
   Replace the subcycle $P_i\, P_{i+1}\, P_{i+2}$ by one segment $P_i\, P_{i+2}$.
4. $P_{i-1} \in P_{i+1}\, P_{i+2}$ and either P_{i-1} is an endpoint or $P_{i-2} \notin e$.
   Proceed as in 1. and 2.
5. $P_{i-1} \in P_{i+1}\, P_{i+2} \cap P_{i-2}\, P_{i+2}$.
   Replace the subcycle $P_{i-2}\, P_{i-1}\, P_i$ by the segment $P_{i-2}P_i$.
6. $P_{i-1} \in P_{i+1}\, P_{i+2}$, $P_{i-2} \in P_{i-1}\, P_{i+2}$.
   Move P_{i-1} off the line e.
7. $P_{i-1} \in P_{i+1}\, P_{i+2}$, $P_{i+2} \in P_{i-2}\, P_{i-1}$.
   Move P_{i+1} off the line e.
8. $P_{i+1} \in P_{i-1}\, P_{i+2}$.
   Replace the subcycle $P_i\, P_{i+1} P_{i+2}$ by the segment $P_i\, P_{i+2}$.

PROOF OF LEMMA 2. In view of Lemma 1 we may suppose that no two consecutive sides of the cycle $P = P_0 \ldots P_n$ have common interior points. P must have three consecutive noncolinear vertices P_{i-1}, P_i, P_{i+1}, since otherwise all vertices are on a line and the complement consists of one component only. Using the condition of Lemma 1 it is easy to see that we can move P_i either toward P_{i+1} or away from P_{i+1} (depending on the cycle) on the line off P_i P_{i+1} so that the cycles corresponding to certain initial interval divide the plane into at least as many components as P. Stop this motion at a point P_i^* such that

i) The cycle $P_0 \ldots P_{i-1} P_i^* P_{i+1} \ldots P_n$ divides the plane into at least as many components as P.

ii) The line $P_{i-1}P_i^*$ intersects all sidelines of the new cycle and in case P is an open cycle the line should also intersect $P_n P_0$.

iii) The line $P_{i-1} P_i^*$ punctured at P_{i-1} does not contain the intersection points of other sidelines of the new cycle.

In view of ii) we can iterate the above procedure for the vertex P_{i-1}, then the vertices $P_{i-2}, \ldots, P_0$. *Treating* P_0P_n as a side we continue by replacing P_n, $P_{n-1}, \ldots$ and stop after replacing P_{i+1}. In view of iii) the new cycle $P_0^* P_1^* \ldots P_n^*$ almost has the desired property, except that on each sideline $P_{i-1}^*P_i^*$ there is one point (P_{i-1}) where more than two sidelines may meet. If in the original cycle P_i was the intersection of two sidelines only, then this cannot happen. Therefore by repeating the whole procedure for $P_0^* P_1^* \ldots P_n^*$ we get a cycle satisfying the desired condition.

PROOF OF STATEMENTS a), b) OF THEOREM. Suppose that the open cycle $P = P_0 \ldots P_n$ of n sides satisfies the condition of Lemma 2 and no two sidelines of P meet in a point belonging to the segment $P_0 P_n$. Let v_0 be the number of those sides of P which have a common interior point with the interior of $P_0 P_n$ and let v be the number of points belonging to the interiors of two sides of the cycle. If the cycle P divides the plane into f components, then the closed cycle $P_0 P_1 \ldots P_n P_0$ of $n+1$ sides should determine $f+v_0+1$ components and then by Euler theorem

$$(f+v_0+1)+(n+1+v+v_0) = (2v+2v_0+n+1)+2$$
$$f = v+1.$$

Therefore it is enough to maximize v. It is obvious that the sides $P_0 P_1, P_{n-1} P_n$ can be intersected by the interior of at most $n-2$ sides and any other side can be intersected by the interior of at most $n-3$ sides. Thus

$$f = v+1 \leq \frac{1}{2}(2(n-2)+(n-2)(n-3))+1 = \frac{n^2-3n+4}{2}$$

i.e. if $n = 2k+1$, then $f \leq 2k^2-k+1$, and if $n = 2k$, then $f \leq 2k^2-3k+2$, which is what we wanted to prove.

PROOF OF STATEMENTS c), d) OF THEOREM. Suppose that the closed cycle $P = P_1 \ldots P_n$ of n sides has the condition of Lemma 2. Let v, f denote the same numbers as that in the previous argument. By the Euler theorem

$$f+(n+v) = (n+2v)+2$$
$$f = v+2.$$

It is enough to maximize v again. It is obvious that each of the sides can be intersected by the interior of at most $n-3$ sides. But if $n = 2k$ we can state more:

LEMMA 3. Any closed cycle $P = P_1 \ldots P_{2k}$ of $2k$ sides has at most two sides whose interior is intersected by $2k-3$ sides.

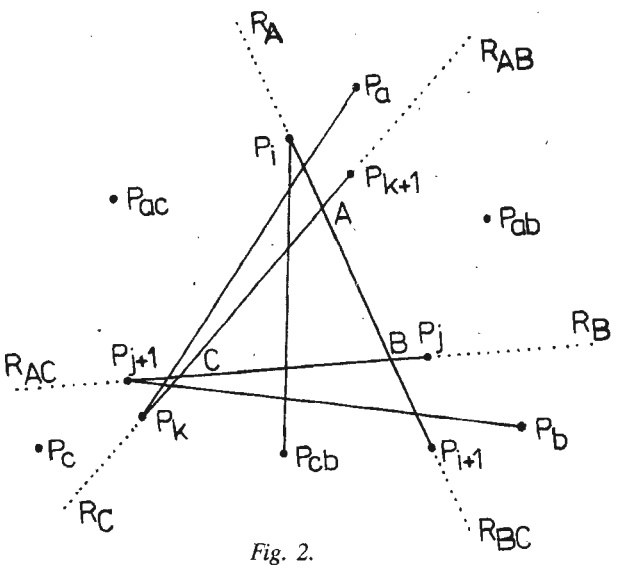

Fig. 2.

PROOF OF LEMMA 3. Suppose to the contrary that there are three sides $P_i P_{i+1}$, $P_j P_{j+1}$, $P_k P_{k+1}$ of P whose interior is intersected by $n-3$ sides.

First consider the case where no two of these sides are adjacent in the cycle. The intersections $A = P_i P_{i+1} \cap P_k P_{k+1}$, $B = P_i P_{i+1} \cap P_j P_{j+1}$, $C = P_j P_{j+1} \cap P_k P_{k+1}$ exist and are different from the six endpoints (Figure 2). Since we use the indices for identifying the vertices only, we do not have to bother with the order of the vertices P_i, P_{i+1} (P_k, P_{k+1}; P_j, P_{j+1} resp.). The side lines $P_i P_{i+1}$, $P_j P_{j+1}$, $P_k P_{k+1}$ divide the plane into seven regions denoted by R_A, R_{AB}, R_B, R_{BC}, R_C, R_{AC}, R_{ABC} where the letters in the indices show which of the vertices A, B, C belong to the boundary of the particular region. Since the triangle R_{ABC} cannot contain any of the vertices of P and each of the sidelines $P_i P_{i+1}$, $P_j P_{j+1}$, $P_k P_{k+1}$ splits the vertices into two classes of $k-1$ elements

each, an easy enumeration shows that if the interior of R_A (R_B resp.) contains x (y resp.) vertices then the regions R_{AB}, R_{BC}, R_C, R_{AC} contain respectively $(k-3-x-y)$, x, $(k-3-x-y)$, y vertices. If $x \geq 1$, then it is easy to see that P has an open subcycle of $2x-1$ sides with vertices belonging alternatively to the interiors of R_A, and R_{BC}. Denote the endpoints by P_a and P_{bc} so that $P_a \in R_A$ and $P_{bc} \in R_{BC}$. In the same way, if $y \geq 1$ and $(k-3-x-y) \geq 1$, we can speak about two other subcycles. One starts at $P_C \in$ int R_C and ends at $P_{ab} \in$ int R_{AB} and has $2(k-3-x-y)$ sides. The other starts at $P_b \in$ int R_B and ends at $P_{ac} \in R_{AC}$ and has $2y-1$ sides. All three of the above subcycles exist, since otherwise there would be less than 6 "free" ends to join to the vertices P_i, P_{i+1}, P_j, P_{j+1}, P_k, P_{k+1}. One of the segments P_aP_k, P_aP_{i+1} must be a side of P. At this point, without loss of generality, we may suppose that P_aP_k is a side of P. One of the vertices P_i, P_{k+1} is connected to P_{bc} through a side of P. P_{k+1} fails, because then P would have a proper closed subcycle. Then P_i cannot be connected to P_b by a side, which means that $P_{j+1}P_b$ is a side too. Notice that both P_i and P_{j+1} are connected to two vertices and neither of them is P_{ab}, which is a contradiction.

Essentially, the same argument leads to a contradiction if among the sides P_iP_{i+1}, P_jP_{j+1}, P_kP_{k+1} there are two consecutive sides.

In view of Lemma 3 we have that if $n = 2k$, then

$$f = v+2 \leq \frac{1}{2}(2(2k-3)+(2k-2)(2k-4))+2 = 2k^2-4k+3,$$

which is what we wanted to prove.

Finally if $n = 2k+1$, then a very obvious estimation leads to the result

$$f = v+2 \leq \frac{1}{2}((2k+1)(2k-2))+2 = 2k^2-k+1.$$

ON THE SPECTRA OF SINGULARITIES OF FOURIER SERIES EQUATIONS

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1. Introduction

We consider linear Fourier series operators Q of the form

$$(1.1)\qquad (Q\psi)(x) = (2\pi)^{-n}\Sigma_{l\in Z^n}\left(\int_W Q(x,y,l)\psi(y)e^{i(x-y,l)}dy\right),\ x\in R^n.$$

Here W is the cube $\{x\in R^n \mid -\pi<x_j<\pi;\ j=1,\ldots,n\}$. The functions $Q(\cdot,\ y,\ l)$; $y\in W$, $l\in Z^n$ and $Q(x,\ \cdot,\ l)$; $x\in W$, $l\in Z^n$ are assumed to belong to the space C_π^∞ of smooth periodic functions. Furthermore, we suppose that

$$(1.2)\qquad \sup_{x,\ y\in W} |D_x^\alpha D_y^\beta Q(x,\ y,\ l)| \le C_{\alpha,\ \beta}(1+|l|^2)^{\mu_\alpha+\delta|\beta|)/2},\ \text{for } l\in Z^n,$$

where $C_{\alpha,\ \beta}>0$, $\mu_\alpha\in\mathbf{R}$ and $\delta<1$ are constants. This assumption guarantees that Q maps C_π^∞ continuously into C_π^∞ (Theorem 3.1).

Supposing that $Q(\cdot,\ \cdot,\ l)$ is of the form $Q(x,\ y,\ l) = L(x,\ l)$ where $L(\cdot,\ l)\in C_\pi^\infty$; $l\in Z^n$ and where

$$(1.3)\qquad \sup_{x\in W} |D_x^\alpha L(x,\ l)| \le C_\alpha(1+|l|^2)^{\mu_\alpha/2},\ l\in Z^n,$$

the operator Q gets the form

$$(1.4)\qquad (Q\psi)(x) = (2\pi)^{-n}\Sigma_{l\in Z^n} L(x,\ l)\psi_l e^{i(l,\ x)},\ x\in R^n.$$

In this case one often denotes $Q = L(x,\ D)$. Especially, one sees that every linear partial differential operator $\Sigma_{|\sigma|\le r}\ a_\sigma(x)D^\sigma$ with C_π^∞ — coefficients a_σ can be expressed in the form (1.4) and hence in the form (1.1). For the properties of periodic partial differential operators we refer to [1], [2], [3] and [5]. In [8] boundedness and compactness criteria are obtained for the Fourier series operators (1.4). We also refer to [7] for the related topic.

In this paper we establish the existence of the continuous formal transpose $Q' : C_\pi^\infty \to C_\pi^\infty$ of Q in the case when $\mu_\alpha = \mu + \delta|\alpha|$, $\alpha \in N_0^n$ (Theorem 3.2). Furthermore, we show that the operator (1.1) can always be expressed in the form (1.4) (Theorem 3.7). Suppose that $L(\cdot, \cdot)$ obeys (1.3) with $\mu_\alpha = \mu + \delta|\alpha|$. Denote by $L^\# : D_\pi' \to D_\pi'$ the continuous extension of $L'(x, D)$. In the fourth chapter we prove the inclusion

$$\Sigma_\pi(L^\# u) \subset \Sigma_\pi(u), \text{ for } u \in D_\pi'. \tag{1.5}$$

(Theorem 4.1). Here $(Z^n - \{0\}) - \Sigma_\pi(u)$ is the conic subset of $Z^n - \{0\}$, which descripes the directions, where the Fourier coefficients u_l of u are rapidly decreasing. The set $\Sigma_\pi(u)$ is analogical notion with the set $\Sigma(u)$, $u \in E'$, in the study of singularitis of pseudodifferential equations (cf. [4], pp. 252—256 and [6], pp. 126—129).

2. General notations

We denote by W the cube of R^n which is defined by $W = \{x \in R^n \mid -\pi < x_j < \pi, \forall j = 1, \ldots, n\}$. C_π^∞ is subspace of $C^\infty(R^n)$ such that the functions $x_j \to \psi(x_1, \ldots x_j, \ldots x_n)$ are 2π-periodic for each $j = 1, \ldots, n$. The space C_π^∞ is equipped with topology defined by the semi-norms $\psi \to \sup_{x \in W} |(D^\sigma \psi)(x)|$, $\sigma \in N_0^n$. Then C_π^∞ is a Frechet space. The dual of C_π^∞ is denoted by D_π'. D_π' is called the space of periodic distributions and D_π' is equipped with the weak dual topology. We recall that a distribution $T \in D_\pi'$ lies in C_π^∞ if and only if

$$\sup_{l \in Z^n} (1 + |l|^2)^{N/2} |T_l| \leq C_N < \infty \text{ for each } N \in \mathrm{N}, \tag{2.1}$$

where

$$T_l = T(e^{-i(l, \cdot)})$$

The definition of the topology implies that for each $T \in D_\pi'$ there exist constants $C > 0$ and $M \in R$ such that

$$|T_l| \leq C(1 + |l|^2)^{M/2} \text{ for all } l \in Z^n. \tag{2.2}$$

Suppose that $\psi \in C_\pi^\infty$ and $T \in D_\pi'$. Then by (2.1) and (2.2) the series $\Sigma_{l \in Z^n} T_l \psi_{-l}$ is absolutely convergent and one has

$$T(\psi) = (2\pi)^{-n} \sum_{l \in Z^n} T_l \psi_{-l}, \tag{2.3}$$

where

$$\psi_l = \psi(e^{-i(l,\,\cdot)} := \int_W \psi(x)e^{-i(l,\,x)}dx. \tag{2.4}$$

This follows from the fact that $\psi_n := (2\pi)^{-n}\Sigma_{|l|\leq n}\psi_l e^{i(l,\,\cdot)}$ is converging to ψ in C_π^∞

The product $\varphi T \in D_\pi'$ of $\varphi \in C_\pi^\infty$ and $T \in D_\pi'$ is defined by $(\varphi T)(\psi) = T(\varphi\psi)$ for $\psi \in C_\pi^\infty$.

Then one sees that

$$\begin{aligned}(\varphi T)_l &= (\varphi T)(e^{-i(l,\,\cdot)}) = T(\varphi e^{-i(l,\,\cdot)}) = \\ &= (2\pi)^{-n}\Sigma_{z\in Z^n}T_z(\varphi e^{-i(l,\,\cdot)})_{-z} = (2\pi)^{-n}\Sigma_{z\in Z^n}T_z\varphi_{l-z}\end{aligned} \tag{2.5}$$

and especially for φ, $\psi \in C_\pi^\infty$ one has

$$(\varphi\psi)_l = (2\pi)^{-n}\Sigma_{z\in Z^n}\,\varphi_z\psi_{l-z}. \tag{2.6}$$

For the theory of periodic distribution we refer to [1], pp.

Let K_π be the totality of weight functions $Z^n \to \mathbf{R}$ such that with $c>0$, $C>0$, $m\in\mathbf{N}$ and $M\in\mathbf{N}$ one has

$$ck_{-m}(l) \leq k(l) \leq Ck_M(l) \quad \text{for all} \quad l\in Z^n, \tag{2.7}$$

where $k_s(l) = (1+|l|^2)^{s/2}$, $s\in\mathbf{R}$. For the definition and for the basic properties of spaces $B_{p,k}^\pi$, $B_{\infty,k}^\pi$; $p\in[1,\infty[$, $k\in K_\pi$ we refer to [8]. We recall that

$$\mathbf{C}_\pi^\infty \subset \mathbf{B}_{p,k}^\pi \subset D_\pi',\ p\in[1,\infty[,\ k\in K_\pi$$

and

$$C_\pi^\infty \subset B_{\infty,k}^\pi \subset D_\pi',\ k\in K_\pi.$$

Furthermore, the imbeddings $C_\pi^\infty \to B_{p,k}^\pi \to D_\pi'$ and $C_\pi^\infty \to B_{\infty,k}^\pi \to D_\pi'$ are continuous. In addition, one has

$$C_\pi^\infty = \bigcap_{k\in K_\pi} B_{p,k}^\pi = \bigcap_{k\in K_\pi} B_{\infty,k}^\pi \tag{2.8}$$

and

$$D_\pi' = \bigcup_{k\in K_\pi} B_{p,k}^\pi = \bigcup_{k\in K_\pi} B_{\infty,k}^\pi. \tag{2.9}$$

For $p\in[1,\infty]$ and $k\in K_\pi$ we define $p'\in[1,\infty]$ and $k^\vee\in K_\pi$ by $1/p+1/p' = 1$ and $k^\vee(l) = k(-l)$.

3. The class of Fourier series operators

3.1. Let $Q(\cdot, \cdot, \cdot); R^n \times R^n \times Z^n \to \mathbf{C}$ be a function such that the mappings

$$x \to Q(x, y, l),\ y \in R^n,\ l \in Z^n$$

and

$$y \to Q(x, y, l),\ x \in R^n,\ l \in Z^n$$

belong to C_π^∞. We shall consider linear operators Q in C_π^∞ which are of the form

$$(3.1) \qquad (Q\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} \Big(\int_W Q(x, y, l)\psi(y) e^{i(x-y, l)} dy \Big),\ x \in R^n$$

Using the notations given above $(Q\psi)(x)$ has also the expression

$$(3.2) \qquad (Q\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} (Q(x, \cdot, l)\psi)_l e^{i(x, l)}.$$

Suppose that $L(\cdot, \cdot): R^n \times Z^n \to \mathbf{C}$ in a function such that the mapping $x \to L(x, l)$, $l \in Z^n$ belongs to C_π^∞. Then one sees that the operator $L(x, D)$ defined by

$$(3.3) \qquad (L(x, D)\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} L(x, l)\psi_l\, e^{i(l, x)},\ x \in R^n$$

is obtained from (3.2) by choosing $Q(x, y, l) = L(x, l)$.

At the first instance we show a condition under which Q maps C_π^∞ continuously into itself.

THEOREM 3.1. *Suppose that there exists a constant* $\delta \in [0, 1[$ *such that the following condition holds: For each* $(\alpha, \beta) \in N_0^n \times N_0^n$ *there exist constants* $C_{\alpha, \beta} > 0$ *and* $\mu_\alpha \in \mathbf{R}$ *such that*

$$(3.4) \qquad \lim_{x, y \in W} |D_y^\alpha D_y^\beta Q(\alpha, y, l)| \le C_{\alpha, \beta}\, k_{\mu_\alpha + \delta|\beta|}(l),\ \textit{for } l \in Z^n.$$

Then Q maps C_π^∞ continuously into itself.

PROOF. A. At first we show that $(Q\psi)(x) \in \mathbf{C}$ for each $\psi \in C_\pi^\infty$ and $x \in R^n$. For all $l \in Z^n$ and for every $\beta \in N_0^n$ we obtain

$$(3.5) \qquad |l^\beta (Q(x, \cdot, l)\psi)_l| = |(D_y^\beta Q(x, \cdot, l)\psi)_l| \le$$

$$\le \Sigma_{t \le \beta} \binom{\beta}{t} |(D_y^{\beta - t} Q(x, \cdot, l) D^t \psi)_l| \le$$

$$\le \Sigma_{t \le \beta} \binom{\beta}{t} \sup_{x, y \in W} |D_y^{\beta - t} Q(x, y, l)|\ \|D^t \psi\|_{L_1(W)} \le$$

$$\leq \Sigma_{t\leq\beta} \binom{\beta}{t} C_{0,\beta-t} \|D^t \psi\|_{L_1(W)} k_{\mu_0+\delta|\beta-t|}(l) \leq$$

$$\leq C_{\beta,\psi} k_{\mu_0+\delta|\beta|}(l),$$

with suitable constant $C_{\beta,\psi}>0$. Here we applied the relation

$$l^\beta \psi_l = (D^\beta \psi)_l, \tag{3.6}$$

which holds for smooth W-periodic functions. Due to (3.5) for each $m\in\mathbf{N}$ we find a constant $C_{m,\psi}>0$ such that

$$|(Q(x,\cdot,l)_l \leq C_{m,\psi} k_{\mu_0+(\delta-1)m}(l), \text{ for all } l\in Z^n. \tag{3.7}$$

Choose $m\in\mathbf{N}$ such that $\mu_0+(\delta-1)m\leq-(n+1)$. Then one sees that the series

$$(2\pi)^{-n} \Sigma_{l\in Z^n} (Q(x,\cdot,l)\psi)_l e^{i(x,l)}$$

is absolutely convergent, which shows that $(Q\psi)(x)\in\mathbf{C}$.

B. To verify the smoothness of $Q\psi$ and the continuity of Q we use the expression 3.1 for Q. Let α be in N_0^n. There we have

$$D_x^\alpha (Q(x,y,l)\psi(y)e^{i(x-y,l)}) =$$

$$= \Sigma_{\gamma\leq\alpha} \binom{\alpha}{\gamma} (D_x^{\alpha-\gamma} Q)(x,y,l)\psi(y)l^\gamma e^{i(x-y,l)} =$$

$$= \Sigma_{\gamma\leq\alpha} \binom{\alpha}{\gamma} Q_{\alpha,\gamma}(x,y,l)\psi(y)e^{i(x-y,l)},$$

where we denoted $Q_{\alpha,\gamma}(x,y,l) = (D_x^{\alpha-\gamma} Q)(x,y,l)l^\gamma$. For esch α and $\gamma\in N_0^n$ we get

$$\sup_{x,y\in W} |D_x^\omega D_y^\beta Q_{\alpha,\gamma}(x,y,l)| = \sup_{x,y\in W} |D_x^{\alpha-\gamma+\omega} D_y^\beta Q(x,y,l)l^\gamma| \leq$$

$$\leq C_{\alpha-\gamma+\omega,\beta} k_{\mu_{\alpha-\gamma+\omega}+|\gamma|+\delta|\beta|}(l) \quad \text{for } l\in Z^n,$$

and then the functions $Q_{\alpha,\gamma}(\cdot,\cdot,\cdot)$ satisfy the estimate like (3.4). As in the Part A one sees that for each $m\in\mathbf{N}$ there exists a constant $C_{m,\psi}>0$ such that

$$(3.9)\qquad \left|\int_W Q_{\alpha,\gamma}(x, y, l)\psi(y)e^{i(x-y,\, l)}dy\right| = \left|(Q_{\alpha,\gamma}(x, \cdot, l)\psi)_l\, e^{i(x,\, l)}\right| \leq$$

$$\leq C_{m,\psi}\, k_{\mu_{\alpha-\gamma}+(\delta-1)m}(l) \quad \text{for } l\in Z^n$$

and then by (3.8) for each $m\in\mathbf{N}$ there exists a constant $C_{m,\psi}>0$ such that

$$(3.10)\qquad \left|\int_W D_x^\alpha\left(Q(x, y, l)\psi(y)e^{i(x-y,\, l)}\right)dy\right| \leq$$

$$\leq C_{m,\psi}\, k_{N_\alpha+(\delta-1)m}(l) \quad \text{for } l\in Z^n,$$

where $N_\alpha := \max_{\gamma\leq\alpha}\{\mu_{\alpha-\gamma}\}$. Since $Q(\cdot, \cdot, l)\psi e^{i((\cdot)-(\cdot),\, l)} \in C^\infty(R^n\times R^n)$ for each $l\in Z^n$ and since W is bounded, the equality

$$D_x^\alpha\left(\int_W Q(x, y, l)\psi(y)e^{i(x-y,\, l)}\, dy\right) = \int_W D_x^\alpha\left(Q(x, y, l)\psi(y)e^{i(x-y,\, l)}\right) dy$$

holds. Choosing $m\in\mathbf{N}$ such that $N_\alpha+(\delta-1)m\leq-(n+1)$ one sees that the series

$$(2\pi)^{-n}\,\Sigma_{l\in Z^n} D_x^\alpha\left(\int_W Q(x, y, l)\psi(y)e^{i(x-y,\, l)}\, dy\right)$$

is absolutely and (in x) uniformly convergent for each $\alpha\in N_0^n$. Hence $D_x^\alpha(Q\psi)(x)$ exists for each $x\in R^n$ and one has

$$(3.11)\qquad D_x^\alpha(Q\psi)(x) = (2\pi)^{-n}\,\Sigma_{l\in Z^n}\int_W D_x^\alpha\left(Q(x, y, l)\psi(y)e^{i(x-y,\, l)}\, dy\right) dy.$$

Hence $Q\psi$ lies in C_π^∞ and Q defines a linear operator $C_\pi^\infty\to C_\pi^\infty$.

C. We establish the continuity of Q. Let $\alpha\in N_0^n$. Then by (3.11) and by (3.8) one sees that

$$(3.12)\qquad |D_x^\alpha(Q\psi)(x)| \leq (2\pi)^{-n}\,\Sigma_{l\in Z^n}\,\Sigma_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\,|(Q_{\alpha,\gamma}(x, \cdot, l)\psi)_l|.$$

Furthermore, one has (cf. (3.5))

$$(3.13)\qquad \left|\left(l^\beta(Q_{\alpha,\gamma}(x, \cdot, l)\psi)_l\right)\right| \leq$$

$$\leq \Sigma_{t\leq\beta}\binom{\beta}{t} C_{\alpha-\gamma,\beta-t}\,\|D^t\psi\|_{L_1(W)}\, k_{\mu_{\alpha-\gamma}+|\gamma|+\delta|\beta-t|}(l), \text{ for } l\in Z^n.$$

Thus for each $m\in\mathbf{N}$ we find a constant $C_{\alpha,\gamma,m}>0$ such that

$$|(Q_{\alpha,\gamma}(x, \cdot, l)\psi)_l| \leq C_{\alpha,\gamma,m}\,\Sigma_{|t|\leq m}\,\|D^t\psi\|_{L_1(W)}\, k_{\mu_{\alpha-\gamma}+|\gamma|+(\delta-1)m}(l).$$

Choose $m \in \mathbf{N}$ such that $\max_{\gamma \leq \alpha} \{\mu_{\alpha-\gamma} + |\gamma|\} + |\alpha| + (\delta - 1)\, m \leq -(n+1)$. Then by (3.12) we obtain

$$|D_x^\alpha (Q\psi)(x)| \leq (2\pi)^{-n}\, C_\alpha\, \Sigma_{|t| \leq m} \|D^t \psi\|_{L_1(W)}$$

$$\leq (2\pi)^{-n}\, C_{\alpha\mu}(W)\, \Sigma_{|t| \leq m} \sup_{x \in W} |(D^t \psi)(x)|,$$

where C_α is a suitable constant and where $\mu(W)$ denotes the Lebesgue measure of W. This completes the proof.

3.2. Let Q be a linear operator from C_π^∞ into itself. We say that Q has a formal transpose (denoted by Q') if it is possible to find another linear operator $Q' : C_\pi^\infty \to C_\pi^\infty$ such that

$$(3.14) \qquad (Q\psi)(\phi) := \int_W (Q\psi)(x)\phi(x)dx = \phi(Q'\phi) \text{ for } \phi,\ \phi \in C_\pi^\infty.$$

In the sequal we establish a sufficient condition for the existence of Q' when Q is of the form (3.1).

THEOREM 3.2. *Suppose that there exist constants $\mu \in \mathbf{R}$ and $\delta \in [0, 1[$ such that the following condition holds: For each $(\alpha, \beta) \in N_0^n$ there exists a constant $C_{\alpha,\beta} > 0$ such that*

$$(3.15) \qquad \sup_{x, y \in W} |D_x^\alpha D_y^\beta Q(x, y, l)| \leq C_{\alpha,\beta}\, k_{\mu+\delta|\alpha+\beta|}(l), \quad \textit{for } l \in Z^n.$$

Then the operator Q define by (3.1) *maps C_π^∞ into itself and the continuous formal transpose $Q' : C_\pi^\infty \to C_\pi^\infty$ of Q exists.*

PROOF. A. In virtue of Theorem 3.1 Q maps C_π^∞ into C_π^∞. Similarly, Theorem 3.1 implies that the operator Q' defined by

$$(3.16) \qquad (Q'\phi)(y) = (2\pi)^{-n}\, \Sigma_{l \in Z^n} \Big(\int_W Q(x, y, -l)\phi(x) e^{i(y-x, l)}\, dy \Big)$$

maps continuously C_π^∞ into C_π^∞, since by (3.15) the mapping $Q(y, x, l) = Q(x, y, l)$ satisfies the estimate (3.4).

B. We have so show the validity of (3.14). Define a linear operator Q_j: $C_\pi^\infty \to C_\pi^\infty$ by

$$(3.17) \qquad (Q_j\psi)(x) = (2\pi)^{-n}\, \Sigma_{|l| \leq j} \Big(\int_W Q(x, y, l)\psi(y) e^{i(x-y, l)}\, dy \Big).$$

We prove that $(Q_j\psi)(x)$ is in R^n uniformly converging to $(Q\psi)(x)$. In the proof of Theorem 2.1, Part A we verified that for each $m \in \mathbf{N}$ there exists a constant $C_{m,\psi} > 0$ such that

$$(3.18)\qquad \left| \int_W Q(x, y, l)\psi(y)e^{i(x-y, l)}\, dy) \right| = \left| (Q(x, \cdot, l)\psi)_l \right| \leq$$

$$\leq C_{m,\psi}\, k_{\mu+(\delta-1)m}\,(l) \text{ for } l\in Z^n$$

Choosing $m\in\mathbf{N}$ such that $\mu+(\delta-1)m \leq -(n+1)$ we get

$$|(Q_j\,\psi)(x)-(Q\psi)(x)| = (2\pi)^{-n}\left|\Sigma_{|l|>j}\left(\int_W Q(x, y, l)\psi(y)e^{i(x-y, l)}\, dy\right)\right| \leq$$

$$\leq (2\pi)^{-n}\, C_{m,\psi}\, \Sigma_{|l|>j}\, k_{-(n+1)}\,(l),$$

where the right hand side is tending to zero with $j\to\infty$. This shows that $Q_j\,\psi$ uniformly converges in R^n to $Q\psi$. Similarly, one sees that $Q_j'\phi$ is uniformly converges in R^n to $Q'\phi$, where Q_j' is defined by

$$(3.19)\qquad (Q_j'\phi)(y) = (2\pi)^{-n}\, \Sigma_{|l|\leq j}\left(\int_W Q(x, y, -l)\phi(x)e^{i(y-x, l)}\, dx\right).$$

C. For each $j\in\mathbf{N}$ one has

$$(Q_j\,\psi)(\phi) = (2\pi)^{-n}\int_W \left(\Sigma_{|l|\leq j}\left(\int_W Q\,(x, y, l)\psi(y)e^{i(x-y, l)}\, dy\right)\right)\phi(x)dx =$$

$$= (2\pi)^{-n}\, \Sigma_{|l|\leq j}\int_W \left(\int_W Q(x, y, l)\psi(y)e^{i(x-y, l)}\,\phi(x)dy\right)dx =$$

$$(3.20)\qquad = (2\pi)^{-n}\, \Sigma_{|l|\leq j}\int_W \left(\int_W Q(x, y, l)\phi(x)e^{i(x-y, l)}\, dx\right)\psi\,(y)dy =$$

$$\int_W \psi(y)(2\pi)^{-n}\, \Sigma_{|l|\leq j}\left(\int_W Q(x, y, -l)\phi(x)e^{i(y-x, l)}\, dx\right)dy = \psi(Q_j'\phi).$$

Since $Q_j\,\psi\to Q\psi$ and $Q_j'\phi\to Q'\phi$ uniformly, we obtain from (3.20) the validity of relation (3.14). This finishes the proof.

REMARK 3.3. Suppose that the assumption of Theorem 2.2 is valid. Then by (3.16) the formal transpose Q' of Q has the expression

$$(3.21)\qquad (Q'\phi)(y) = (2\pi)^{-n}\, \Sigma_{l\in Z^n}\left(\int_W Q(x, y, -l)\phi(x)e^{i(y-x, l)}\, dx\right) =$$

$$= (2\pi)^{-n}\, \Sigma_{l\in Z^n}\,(Q(\cdot, y, -l)\phi)_l\, e^{i(y, l)}.$$

Especially one sees that the 'symbol' $Q'\,(\cdot, \cdot, \cdot)$ also satisfies the condition (3.15).

In the case when $Q(x, y, l) = L(x, l)$(cf. Section 2.) one gets

COROLLARY 3.4. Suppose that for each $\alpha \in N_0^n$ there exist constants $C_\alpha > 0$ and $\mu_\alpha \in R$ such that

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu_\alpha}(l) \text{ for } l \in Z^n. \tag{3.22}$$

Then the operator $L(x, D)$ defined by (3.3) maps C_π^∞ continuously into C_π^∞.

COROLLARY 3.5. Suppose that there exist constants $\mu \in \mathbf{R}$ and $\delta \in [0, 1[$ such that the following condition holds: For each $\alpha \in N_0^n$ there exists $C_\alpha > 0$ such that

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu+\delta|\alpha|}(l) \text{ for } l \in Z^n. \tag{3.23}$$

Then the operator $L(x, D)$ defined by (3.3) maps C_π^∞ into C_π^∞ and the continuous formal transpose $L'(x, D): C_\pi^\infty \to C_\pi^\infty$ of $L(x, D)$ exists. Furthermore $L'(x, D)$ has the expression

$$(L'(x, D)\phi)(y) = (2\pi)^{-n} \Sigma_{l \in Z^n} \left(\int_W L(x, -l)\phi(x) e^{i(y-x, l)} dx \right). \tag{3.24}$$

REMARK 3.6. From [8] we know that $L'(x, D)$ has the expression

$$(L'(x, D)\phi)(y) = (2\pi)^{-n} \Sigma_{l \in Z^n} L'(y, l)\phi l e^{i(y, l)},$$

where $L'(\cdot, \cdot)$ is a mapping such that $L'(\cdot, l) \in C_\pi^\infty$, $l \in Z^n$ and that

$$|D_x^\alpha L'(x, l)| \leq C_\alpha k_{\mu_\alpha}(l),$$

(when $L(\cdot, \cdot)$ satisfies (3.23)). Furthermore, the symbols $L(\cdot, \cdot)$ and $L'(\cdot, \cdot)$ are in the following relation

$$(L'(\cdot, z))_l = (L(\cdot, -(z+1))_l \text{ for } l, z \in Z^n.$$

3.3 In this section we establish a sufficient criteria, which guarantees that the operator (3.1) can be expressed in the form (3.3). In fact we have

THEOREM 3.7. *Suppose that there exists a constant* $\delta \in [0, 1[$ *such that the estimate* (3.4) *holds for* $Q(\cdot, \cdot, \cdot)$. *Then there exists* $L(\cdot, \cdot): R^n \times Z^n \to \mathbf{C}$ *such that* $L(\cdot, l) \in C_\pi^\infty$ *for* $l \in Z^n$, *that*

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu_\alpha}(l) \text{ for } l \in Z^n. \tag{3.25}$$

and that

$$Q = L(x, D). \tag{3.26}$$

PROOF. A. We have by (2.3)

$$(Q\psi)(x) = (2\pi)^{-n} \Sigma_{l\in Z^n} (Q(x, \cdot, l)\, \psi)_l\, e^{i(l, x)} =$$

$$= (2\pi)^{-2n} \Sigma_{l\in Z^n} \left(\Sigma_{z\in Z^n} (Q(x, \cdot, l))_{l-z}\, \psi_z\right) e^{i(l, z)} = \tag{3.27}$$

$$= (2\pi)^{-2n} \Sigma_{l\in Z^n} \left(\Sigma_{z\in Z^n} (Q(x, \cdot, l))_{l-z}\, e^{i(l-z, x)}\, e^{i(z, x)}\, \psi_z\right).$$

For each $\beta\in N_0^n$ one has

$$|(l-z)^\beta (Q(x, \cdot, l))_{l-z}| = |(D_y^\beta Q(x, \cdot, l))_{l-z}| \leq$$

$$\leq C_{0,\beta}\, k_{\mu_0+\delta|\beta|}(l) \text{ for } z, l\in Z^n, x\in W \tag{3.28}$$

and then for each $m\in N$ there exists a constant $C_m>0$ such that

$$|(Q(x, \cdot, l))_{l-z}| \leq C_m\, k_{\mu_0+\delta m}(l)\, k_{-m}(l-z) \leq$$

$$\leq C'_m\, k_m(z)\, k_{\mu_0+(\delta-1)m}(l) \text{ for } z, l\in Z^n, x\in W. \tag{3.29}$$

Choosing $m\in N$ such that $\mu_0+(\delta-1)m\leq -(n+1)$ one sees that the series

$$\Sigma_{l, z\in Z^n} (Q(x, \cdot, l))_{l-z}\, e^{i(l-z, x)}\, e^{i(z, x)}\psi_z$$

is absolutely and uniformly convergent in R^n. Hence we can change the order summation in (3.27), which gives

$$(Q\psi)(x) = (2\pi)^{-2n} \Sigma_{z\in Z^n} \left(\Sigma_{l\in Z^n} (Q(x, \cdot, l))_{l-z}\, e^{i(l-z, x)}\right)\psi_z\, e^{i(z, x)}. \tag{3.30}$$

In virtue of (3.29) the function $L(\cdot, \cdot)$: $R^n\times Z^n\to C$ defined by

$$L(x, z) = (2\pi)^{-n} \Sigma_{l\in Z^n} (Q(x, \cdot, l))_{l-z}\, e^{i(l-z, x)} \tag{2.31}$$

is well-defined and $L(\cdot, z)$ is a continuous periodic function.

B. Our task is to show that $L(\cdot, z)\in C_\pi^\infty$ and that the estimate (3.25) holds. Let α be in N_0^n. Since $(Q(x, \cdot, l))_{l-z} = \int_W Q(x, y, l)e^{i(z-l, y)}\, dy$ one sees that the function $f_{l,z}: x\to (Q(x, \cdot, l))_{l-z}$ lies in C_π^∞ and that

$$D_x^\alpha f_{l,z})(x) = \int_W (D_x^\alpha Q)(x, y, l)\, e^{i(z-l, y)}\, dy = ((D_x^\alpha Q)(x, \cdot, l))_{l-z}. \tag{3.32}$$

Hence we obtain

$$L(x, z) = (2\pi)^{-n} \Sigma_{l\in Z^n} f_{l,z}(x) e^{i(l-z, x)}, \tag{3.33}$$

where

$$|D_x^\alpha (f_{l,z}\, e^{i(l-z,\,\cdot)})(x)| =$$

$$= |\Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} (D_x^{\alpha-\gamma} f_{l,z})(x)(l-z)^\gamma e^{i(l-z, x)}| \le \tag{3.34}$$

$$\le \Sigma_{\gamma\in\alpha} \binom{\alpha}{\gamma} |((D_x^{\alpha-\gamma}Q)(x, \cdot, l))_{l-z} (l-z)^\gamma|.$$

For each $\beta\in N_0^n$ one has

$$|(l-z)^{\beta+\gamma} ((D_x^{\alpha-\gamma}Q)(x, \cdot, l))_{l-z}| = |((D_x^{\alpha-\gamma} D_y^{\beta+\gamma}Q)(x, \cdot, l))_{l-z}| \le$$

$$\le C_{\alpha-\gamma, \beta+\gamma}\, \mu(W)\, k_{\mu_{\alpha-\gamma}+\delta|\beta+\gamma|}(l)$$

and then for each $m\in\mathbf{N}$, α, $\gamma\in N_0^n$ there exists a constant $C_{\alpha,\gamma,m}>0$ such that

$$|(l-z)^\gamma ((D_x^{\alpha-\gamma}Q)(x, \cdot, l))_{l-z}| \le$$

$$\le C_{\alpha,\gamma,m}\, k_{\mu_{\alpha-\gamma}+\delta|\gamma|+\delta m}(l)\, k_{-m}(l-z) \le \tag{3.35}$$

$$\le C'_{\alpha,\gamma,m}\, k_m(z)\, k_{\mu_{\alpha-\gamma}+\delta|\gamma|+(\delta-1)m}(l), \text{ for } z, l\in Z^n.$$

Choose $m\in\mathbf{N}$ such that $\max_{\gamma\le\alpha} \{\mu_{\alpha-\gamma}\}+\delta|\alpha|+(\delta-1)m\le-(n+1)$. Then one sees by (3.34) that the series

$$\Sigma_{l\in Z^n} D_x^\alpha (f_{l,z}\, e^{i(l-z,\,\cdot)})(x)$$

is absolutely and uniformly convergent in R^n for each $\alpha\in N_0^n$ and $z\in Z^n$. Hence $D_x^\alpha L(x, z)$ exists and

$$D_x^\alpha L(x, z) = (2\pi)^{-n} \Sigma_{l\in Z^n} D_x^\alpha (f_{l,z}\, e^{i(l-z,\,\cdot)})(x) \tag{3.36}$$

The estimate (3.25) follows (3.36) and (3.35), which completes the proof.

COROLLARY 3.8. Suppose that there exist constants $\mu\in\mathbf{R}$ and $\delta\in[0, 1[$ such that the condition (3.15) holds. Then there exists $L(\cdot, \cdot):R^n\times Z^n\to\mathbf{C}$ such that $L(\cdot, z)\in C_\pi^\infty$ for $l\in Z^n$, that with $\epsilon>0$

$$\sup_{x\in W} |D_x^\alpha L(x, z)| \le C_\alpha\, k_{[(\mu+n+\epsilon)/(1-\delta)+(\delta/1-\delta)|\alpha|]}(z), \text{ for } z\in Z^n \tag{3.37}$$

[1] $[s]$ denotes the smallest integer $n\in N$ which satisfies $s\le n$.

and that

$$Q = L(x, D). \tag{3.38}$$

PROOF. We have only to verify the estimate (3.37). In virtue of (3.36) and (3.34) one finds

$$\begin{aligned} |D_x^\alpha L(x, z)| &\le (2\pi)^{-n} \Sigma_{l\in Z^n} \Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} |((D_x^{\alpha-\gamma}Q)(x, \cdot, l))_{l-z} (l-z)^\gamma| = \\ &= (2\pi)^{-n} \Sigma_{l\in Z^n} \Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} |((D_x^{\alpha-\gamma} D_y^\gamma Q)(x, \cdot, l))_{l-z}| \end{aligned} \tag{3.39}$$

and for each $\beta\in N_0^n$ we have

$$\begin{aligned} |(l-z)^\beta((D_x^{\alpha-\gamma} D_y^\gamma Q)(x, \cdot, l))_{l-z}| &= |((D_x^{\alpha-\gamma} D_y^{\gamma+\beta} Q)(x, \cdot, l))_{l-z}| \le \\ &\le C_{\alpha-\gamma, \beta+\gamma}\, \mu(W)\, k_{\mu+\delta|\alpha+\beta|}(l), \text{ for } l\in Z^n, \end{aligned} \tag{3.40}$$

and then for each $m\in\mathbf{N}$

$$\begin{aligned} |((D_x^{\alpha-\gamma} D_y^\gamma Q)(x, \cdot, l))_{l-z}| &\le C_{\alpha, \gamma, m}\, k_{\mu+\delta|\alpha|+\delta m}(l)\, k_{-m}(l-z) \le \\ &\le C'_{\alpha, \gamma, m}\, k_{\mu+\delta|\alpha|+(\delta-1)m}(l)\, k_m(z) \end{aligned}$$

Choosing $m = [(\mu+\delta|\alpha|+n+\epsilon)/(1-\delta)]$, one gets the estimate (3.37).

REMARK 3.9. In the case when $\delta<\frac{1}{2}$ the symbol $L(\cdot, \cdot)$ of Corollary 3.8 obeys the estimate (3.23) (since $[s]\le s+1$).

4. Spectra of singularities

4.1. Suppose that the operator $L(x, D)$ defined by (3.3) maps continuously C_π^∞ into C_π^∞ and that the continuous formal transpose $L'(x, D): C_\pi^\infty \to C_\pi^\infty$ exists. Then we can define the continuous operators $L'^{\#}$ and $L^{\#}: D'_\pi \to D'_\pi$ by the requirements

$$(L'^{\#} u)(\psi) = u(L'(x, D)\psi), \ \psi\in C_\pi^\infty \tag{4.1}$$

and

$$(L^{\#} u)(\psi) = u(L(x, D)\psi), \ \psi \in C_{\pi}^{\infty}. \tag{4.2}$$

One sees that $L'^{\#}$ is the extension of $L(x, D)$ and that $L^{\#}$ is the extension of $L(x, D)$.

In the sequel we shall consider the spectra of singularities of the equation

$$L^{\#} u = f; \ u, f \in D_{\pi}'. \tag{4.3}$$

Let l be in $Z^n \setminus \{0\}$. We say that a set $U \subset R^n \setminus \{0\}$ in a conic neighbourhood of l if one has: $l \in U$; U is open in R^n; $c\, z \in U$ for each $c > 0$ and $z \in U$. For $u \in D_{\pi}'$ we define a subset $\Sigma_{\pi}(u) \subset Z^n \setminus \{0\}$ as follows: $l \in (Z^n \setminus \{0\}) \setminus \Sigma_{\pi}(u)$ if and only if one can find a conic neighbourhood U_l of l such that for each $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$|u_z| \leq C_m\, k_{-m}(z), \text{ for } z \in U_l \cap Z^n. \tag{4.4}$$

One sees that $\Sigma_{\pi}(u) = \phi$ if and only if u belongs to C_{π}^{∞} (cf. (2.1) and note that $S(0, 1) := \{x \in R^n \mid \|x\| = 1\}$ is compact).

We obtain

THEOREM 4.1. *Suppose that there exist constant $\mu \in \mathbf{R}$ and $\delta \in [0, 1[$ such that the following condition holds: For each $\alpha \in N_0^n$ there exists $C_{\alpha} > 0$ such that*

$$\sup_{x \in W} |D_x^{\alpha} L(x, l)| \leq C_{\alpha}\, k_{\mu + \delta|\alpha|}(l), \textit{ for } l \in Z^n. \tag{3.23}$$

Then the inclusion

$$\Sigma_{\pi}(L^{\#} u) \subset \Sigma_{\pi}(u), \textit{ for } u \in D_{\pi}' \tag{4.5}$$

holds.

PROOF. A. For each $l \in Z^n$ we obtain

$$\begin{aligned} (L^{\#} u)_l &= (L^{\#} u)(e^{-i(l, \cdot)}) = u(L(x, D)(e^{-i(l, \cdot)})) = \\ &= u(L(\cdot, -l)e^{-i(l, \cdot)}) = (uL(\cdot, -l))_l, \end{aligned} \tag{4.6}$$

since

$$\begin{aligned} (L(x, D)(e^{-i(l, \cdot)}))(x) &= (2\pi)^{-n} \Sigma_{z \in Z^n} L(x, z)(e^{-i(l, \cdot)})_z\, e^{(z, x)} = \\ &= L(x, -l)e^{-i(l, x)}. \end{aligned}$$

Hence in virtue of (2.5) one has

$$(L^{\#} u)_l = (2\pi)^{-n} \Sigma_{z \in Z^n} u_{l-z}\, (L(\cdot, -l))_z. \tag{4.7}$$

Let α belong to N_0^n. Then we get by (3.23)

$$(4.8)\quad |z^\alpha (L(\cdot, -l))_z| = |(D_x^\alpha L(\cdot, -l))_z| \le C_\alpha \mu(W) k_{\mu+\delta|\alpha|}(l) \text{ for } z, l \in Z^n,$$

which implies that for each $m \in \mathbf{N}$ there exists a constant $C'_m > 0$ such that

$$k_m(z) |(L(\cdot, -l))_z| \le C'_m k_{\mu+\delta(n+1+m)}(l) k_{-(n+1)}(z)$$

(apply (4.8) with $|\alpha| \le n+1+m$). Hence we find that

$$\begin{aligned} &\Sigma_{z\in Z^n} k_m(z) |(L(\cdot, -l))_z| \le \\ (4.9)\qquad &\le C'_m (\Sigma_{z\in Z^n} k_{-(n+1)}(z)) k_{\mu+\delta(n+1+m)}(l) = \\ &= C''_m k_{\mu+\delta(n+1+m)}(l) \text{ for } l \in Z^n. \end{aligned}$$

B. Suppose that $l_0 \in (Z^n \setminus \{0\}) \setminus \Sigma_\pi(u)$. Let U_{l_0} be a conic neighbourhood of l_0 such that

$$(4.10)\qquad |u_l| \le C_m k_{-m}(l), \text{ for } l \in U_{l_0} \cap Z^n$$

Let ξ'_0 be in $S(0, 1) \cap U_{l_0}$ such that $\xi'_0 = l_0 / |l_0|$ where $S(0, 1) = \{\xi \in \mathbf{R}^n | \xi| = 1\}$. Chosse $0 < \epsilon < 1$ so that the ball $B(\xi'_0, 2\epsilon) \subset U_{l_0}$. Define the cone U'_{l_0} by $U'_{l_0} = \{\eta \in \mathbf{R}^n | \eta = c\xi \ ; \ c > 0$ and $\xi \in B(\xi'_0, \epsilon) \cap S(0, 1)\}$. Then for each $\eta = c\xi \in U'_{l_0}$ the set $V_\eta := \{\omega \in \mathbf{R}^n \| \omega - \eta| < \epsilon |\eta|\}$ is a subset of U_{l_0} : For $\omega \in V_\eta$ one has

$$|\omega/|\eta| - \xi'_0| \le |\omega/|\eta| - \eta/|\eta|| + |\xi'_0 - \eta/|\eta|| < \epsilon + |\xi'_0 - \xi| < 2\epsilon$$

and so $\omega/|\eta| \in B(\xi'_0, 2\epsilon) \subset U_{l_0}$. Hence $\omega \in U_{l_0}$.

C. In virtue of (4.7) we see by the Taylor formula that

$$|l^\alpha (L^\# u)_l| = (2\pi)^{-n} \Sigma_{z\in Z^n} \left(\Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} (l-z)^{\alpha-\gamma} u_{l-z} z^\gamma (L(\cdot, -l))_z\right) \le$$

$$\begin{aligned} &\le (2\pi)^{-n} \Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} \Sigma_{|z|\ge\epsilon|l|} |(l-z)^{\alpha-\gamma} u_{l-z} z^\gamma (L(\cdot, -l))_z| + \\ (4.12)\qquad &+ (2\pi)^{-n} \Sigma_{\gamma\le\alpha} \binom{\alpha}{\gamma} \Sigma_{|z|<\epsilon|l|} |(l-z)^{\alpha-\gamma} u_{l-z} z^\gamma (L(\cdot, -l))_z| = \\ &=: S_{1,l} + S_{2,l} \quad \text{for each } l \in Z^n. \end{aligned}$$

Consider the first sum $S_{1,l}$: In view of (2.2) there exists $M \in \mathbf{N}$ such that

$$|u_l| \leq C k_M(l) \text{ for all } l \in Z^n. \tag{4.13}$$

When $|z| \geq \epsilon |l|$ one sees that $|z-l| \leq (1+\epsilon^{-1})|z|$ and then we obtain by (4.13)

$$\begin{aligned} |(l-z)^{\alpha-\gamma} u_{l-z} z^\gamma| &\leq C k_{|\alpha-\gamma|}(l-z) k_M(l-z) k_{|\gamma|}(z) \\ &\leq C(1+\epsilon^{-1})^{|\alpha-\gamma|+M} k_{|\alpha|+M}(z) \text{ for } |z| \geq \epsilon |l|. \end{aligned} \tag{4.14}$$

Thus in virtue of (4.9) we find that for all $l \in Z^n$

$$S_{1,l} \leq (2\pi)^{-n} \Sigma_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C(1+\epsilon^{-1})^{|\alpha-\gamma|+M} \Sigma_{|z| \geq \epsilon |l|} k_{|\alpha|+M}(z) |(L(\cdot, -l))_z| \leq$$

$$\leq (2\pi)^{-n} \Sigma_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C(1+\epsilon^{-1})^{|\alpha-\gamma|+M} C''_{|\alpha|+M} k_{\mu+\delta(n+1+M)+\delta|\alpha|}(l) = \tag{4.15}$$

$$= C''_{\alpha,u} k_{\mu+\delta(n+1+M)+\delta|\alpha|}(l).$$

Consider the last sum $S_{2,l}$: Since $|z| < \epsilon |l|$, one sees that $l-z \in V_l \cap Z^n$ and then $l-z \in U_{l_0} \cap Z^n$ when $l \in U'_{l_0} \cap Z^n$. Thus by (4.10) we get

$$|(l-z)^{\alpha-\gamma} u_{l-z} z^\gamma (L(\cdot, -l))_z| \leq C_{|\alpha-\gamma|} |z^\gamma (L(\cdot, -l))_z| \tag{4.16}$$

and then by (4.9)

$$S_{2,l} \leq (2\pi)^{-n} \Sigma_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \Sigma_{|z| < \epsilon |l|} C_{|\alpha-\gamma|} k_{|\gamma|}(z) |(L(\cdot, -l))_z| \leq$$

$$\leq (2\pi)^{-n} \Sigma_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{|\alpha-\gamma|} C''_{|\gamma|} k_{\mu+\delta(n+1)+\delta|\gamma|}(l) \leq \tag{4.17}$$

$$\leq C''_{\alpha,u} k_{\mu+\delta(n+1)+\delta|\alpha|}(l) \text{ for } l \in U'_{l_0} \cap Z^n.$$

The inequalities (4.15), (4.17) and (4.12) imply that for each m and $m' \in \mathbf{N}$ there exists a constant $C_{m,m',u} > 0$ such that for all $l \in U'_{l_0} \cap Z^n$.

$$|k_m(l)(L^\# u)_l| \leq C_{m,m',u} k_{\mu+\delta(n+1+M+m)+(\delta-1)m'}(l)$$

(apply the inequalities in question with $|\alpha| \leq m+m'$). Choosing m' large enough one sees that

$$|k_m(l)(L^{\#} u)_l| \leq C_m \text{ for all } l \in U'_{l_0} \cap Z^n.$$

This shows that $l_0 \in (Z^n \setminus \{0\}) \setminus \Sigma_\pi(L^{\#} u)$, which completes the proof.

From Corollaries 3.4—3.5 and from Theorem 4.1 we get

COROLLARY 4.2. Suppose that $L(\cdot) : Z^n \to \mathbf{C}$ satisfies

$$|L(l)| \leq C k_\mu(l) \text{ for all } l \in Z^n. \tag{4.18}$$

Then the operator $L(D)$ defined by

$$(L(D)\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} L(l)\psi_l e^{i(l, z)} \tag{4.19}$$

maps C_π^∞ continuously into C_π^∞. The continuous formal transpose $L'(D)$: $C_\pi^\infty \to C_\pi^\infty$ of $L(D)$ exists and

$$(L'(D)\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} L(-l)\psi_l e^{i(l, z)}. \tag{4.20}$$

Furthermore, the continuous extension $L^{\#}$: $D'_\pi \to D'_\pi$ of $L'(D)$ satisfies

$$\Sigma_\pi (L^{\#} u) \subset \Sigma_\pi (u), \text{ for } u \in D'_\pi. \tag{4.21}$$

REMARK 4.3. In the case when (4.18) holds one sees that

$$(L^{\#} u)_l = u\,(L(D)(e^{-i(l, \cdot)})) = L(l)u_l \tag{4.22}$$

Hence in this case the validity of (4.21) is obvious.

REMARK 4.4. Suppose that $L(\cdot, \cdot): R^n \times Z^n \to \mathbf{C}$ satisfies

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu+|\alpha|}(l), \; l \in Z^n. \tag{4.23}$$

Then $L(x, D)$ has not necessarily the continuous formal transpose $L'(x, D): C_\pi^\infty \to C_\pi^\infty$: Choose $L(x, l) = e^{-i(x, l)}$. Then one sees that

$$(L(x, D)\psi)(x) = (2\pi)^{-n} \Sigma_{l \in Z^n} \psi_l = \psi(0).$$

Suppose that $L'(x, D)$ exists, that is,

$$(L(x, D)\psi)(\phi) = \psi(L'(x, D)\phi), \text{ for } \psi, \phi \in C_\pi^\infty.$$

Then we find that (by choosing $\psi = e^{i(l, x)}$, $\phi = e^{i(z, x)}$)

$$e^{i(l,\,0)}(\phi) = \int_W e^{i(l,\,x)}\, L'(x, z) e^{i(z,\,x)}\, dx$$

and then

$$(L'(\cdot\,, z)e^{i(z,\,\cdot)})_l = \int_W e^{i(z,\,x)} dx \qquad \text{for } l,\, z \in Z^n$$

which implies easily a contradiction.

Here we used the fact that $(L'(x, D)\phi)(x) = (2\pi)^{-n}\, \Sigma_{z\in Z^n}\, L'(x, z)\, \phi_z\, e^{i(z,\,x)}$ where $L'(\cdot\,, z) \in C_\pi^\infty$ such that $|D_x^\alpha L'(x, z)| \leq C_\alpha\, k_{\mu_\alpha}(z)$(cf. [7]).

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ON ORDERING THE FREE PRODUCT OF ORDERED SEMIGROUPS

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1. Introduction

It was proved by A. A. VINOGRADOV that the free product of two linearly ordered groups is always linearly orderable. His construction has been generalized and simplified by R. JOHNSON [2], where it is shown that the free product of two cancellative semigroups without idempotens is also orderable. The question of finding a necessary and sufficient condition for the orderability of the free product of two arbitrary semigroups was posed by E. YU. GABOVICH [1]. He showed that if $C = A * B$ is orderable, then either A or B contains no idempotent elements.

Assume that $C = A * B$ is orderable. Then we show that, in fact, neither A nor B contains any idempotens. (*Corollary* 1.) It follows that the ordering of C can be extended to the monoid C^1 (*Proposition* 1.), which is the free product of A^1 and B^1 in the category of monoids. Thus our problem is reduced to monoids. Next we prove that both A^1 and B^1 must be one-sided cancellative from the same side. (*Proposition* 2 and 3.) This condition is the strongest one that can be imposed on A in order that $A * B$ be orderable for some B. Indeed, if A^1 is orderable and one-sided cancellative, then $A^1 * F$ is also orderable for the free semigroup F on one generator.

Next we generalize the result of R. JOHNSON mentioned above and solve Gabovich's question for the commutative case. Our results are mentioned in the Abstract (*Theorem* 4 and 5). Finally we provide a description of the least convex congruence of an ordered semigroup S. (*Theorem* 6).

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2. Necessary Conditions

LEMMA 1. If the free product $A * B$ is orderable, then for every, $a,b \in A$ either $ab \neq a$ or $ba \neq a$ holds.

PROOF. Assume by way of contradiction that $ab = ba = a$ holds for some $a,b \in A$. Let x be an arbitrary element of B. We may suppose that $bx < xb$. (If not, we can consider the reverse order of $A * B$.)

Thus $abx \leqq axb$, i.e., $ax < axb$ and so $x < xb$, $x^2 < xbx$ holds.

On the other hand $bxa \leqq xba$, therefore $bxa < xa$ and so $bx < x$, $xbx < x^2$ holds, contradicting $x^2 < xbx$.

COROLLARY 1. Let A and B be semigroups. If their free product $A * B$ is orderable, then neither A nor B contains idempotent elements.

PROOF. *Lemma* 1 says that $a^2 \neq a$ for every $a \in A$. A similar argument works for B.

LEMMA 2. If $A * B$ is orderable, then $A * B$ contains no idempotents.

PROOF. Let $l(u)$ denote the length of a word $u \in A * B$. (We can consider the elements of $A * B$ as sequences of letters taken alternately from A and B.)

If u^2 equals u, then obviously $l(u) = l(u^2) \geqq 2l(u) - 1$ holds, and so $1 \geqq l(u)$, i.e., u is an element of A or B. By the previous corollary however, $A \cup B$ contains no idempotents.

Let us adjoin a unit element 1 to the semigroup $A * B$ (1 stands for the empty word), and extend the order of $A * B$ to $(A * B) \cup \{1\}$. Next we will show that we can do this.

PROPOSITION 1. If $A * B$ is orderable then this order can be extended uniquely to $(A * B) \cup \{1\}$.

PROOF. By *Lemma* 2 u is different from u^2 for every $u \in A * B$ so the only way to extend this order is to define

$$1 < u \overset{\text{def}}{\leftrightarrow} u < u^2.$$

First we show that this extended binary relation is compatible with the multiplication, i.e., $u < u^2$ implies $v \leqq vu$ and $v \leqq uv$ for every $v \in A * B$.

Assume the contrary that, e.g., $u < u^2$ and $v > vu$ holds.

We can suppose that the last letter of u is an element of A and let x be an arbitrary element of B. Hence $ux \neq u^2x$ and so $ux < u^2x$, $x < ux$, $vx \leqq vux$ hold. On the other hand, from $v > vu$ it follows $vx \geqq vux$ and so $vx = vux$.

Obviously

$$l(vux) \geqq l(v) + l(x) = l(v) + 1,$$

because the last element of u is from A and $x \in B$.

Hence

$$l(v) + 1 \leqq l(vux) = l(vx) \leqq l(v) + 1$$

and so $l(vx) = l(v) + 1$. Thus the last letter of v cannot be an element of B, because we had simply to write x after v in the shortest representation of vx, we get $vu = v$, contradicting $vu < v$.

Similar reasoning yields the assertion in the other cases. It remains only to show, that the extended relation is transitive (the irreflexivity and the antisymmetry is trivial).

Thus we will prove that $x<y$ and $y<z$ implies $x<z$ for every $x,y,z\in(A*B)\cup\{1\}$. If $x,y,z\in A*B$, then we have nothing to prove because $<$ was an order on $A*B$.

Case 1. $z=1$. Suppose that $x<y$ and $y<1$, i.e., $y^2<y$ for $x,y\in A*B$. From $x<y$ it follows $xy\leqq y^2$ and so by $y^2<y$, $xy<y$ holds. We have proved that $x<x^2$ would imply $y<xy$ so $xy<y$ implies $x^2<x$, i.e. $x<1$.

Case 2. $y=1$. Suppose that $x<1$ and $1<z$ hold for $x,z\in A*B$. We have proved that $x<1$ i.e. $x^2<x$ implies $xz\leqq z$. Similarly from $1<z$ it follows $x\leqq xz$. Thus by the transitivity of the ordering of $A*B$ $x\leqq xz$ and $xz\leqq z$ implies $x\leqq z$. $x\neq z$ follows trivially from $x^2<x$ and $z<z^2$.

Case 3. $x=1$. This case is similar to *Case* 1. (It is the same if we consider the reverse order.)

Therefore if $A*B$ is orderable, then the monoids $A\cup\{1\}$ and $B\cup\{1\}$ and their free product $(A*B)\cup\{1\}$ are also orderable. (We consider 1 as "the common unit element of all monoids".) So the necessity of the one-side cancellativity of A and B follows from the next proposition.

PROPOSITION 2. If $S_1\neq\{1\}$ and $S_2\neq\{1\}$ are ordered monoids and S_1*S_2 is orderable, then S_1 must be one-sided cancellative.

PROOF. Assume by way of contradiction that there are elements a_1, a_2, b_1, b_2, c, $d\in S_1$ such that $a_1c=a_2c$, $a_1\neq a_2$, $db_1=db_2$ and $b_1\neq b_2$. Without loss of generality we can confine ourselves to the case $a_1<a_2$, $b_1<b_2$ and let x be an arbitrary element of S_2, different from 1.

Case 1. If $b_1xa_2<b_2xa_1$ then $db_1xa_2<db_2xa_1$, (the two sides are not equal because of $a_1\neq a_2$ and $x\neq 1$), and so (deleting $db_1x=db_2x$ on both sides) $a_2<a_1$ follows, which is a contradiction.

Case 2. If $b_1xa_2>b_2xa_1$ then similarly we have $b_1xa_2c>b_2xa_1c$ and hence $b_1>b_2$, a contradiction.

Now we will prove that S_1*S_2 must also be one-sided cancellative. This statement follows trivially from

PROPOSITION 3. If $S_1\neq\{1\}$and $S_2\neq\{1\}$ are ordered monoids and S_1*S_2 is orderable but S_1 and S_2 are not both left cancellative, then they are both right cancellative.

PROOF. Assume the contrary: for example that $ac=bc$, $a<b$ holds for certain $a,b,c\in S_1$ and there exist elements $x,y,z\in S_2$ such that $x<y$, $zx=zy$.

Case 1. $xb<ya$. Then $zxb<zya$ thus by $zx=zy$ we have $b<a$.

Case 2. $xb>ya$. Then $xbc>yac$ thus by $ac=bc$ we have $x>y$.

We obtained again a contradiction in both cases.

We shall call an ordered semigroup C *left-arranged* if C statisfies the following conditions

(i) $C \cup \{1\}$ is left-cancellative.

(ii) The order on C is extendable to the monoid $C \cup \{1\}$.

Analogously we shall also speak of *right-arranged* semigroups.

So far we have proved the following

THEOREM 1. *If the free product $A * B$ of the ordered semigroups A and B is orderable, then $A * B$ as well as A and B are arranged from the same side.*

Note, that from the one-sided cancellativity of a monoid it follows that the only idempotent element is 1. So if C is one-sided arranged then C contains no idempotents, and there is a unique way to extend the order to $C \cup \{1\}$. (Setting $1 < c$ iff $c < c^2$ for every $c \in C$.)

In Section 3 we shall see that there are no more "symmetrical" necessary conditions in the sense that there are no more conditions which are to be satisfied by the elements of A and by the elements of B as well.

At the end of this Section we prove a "non-symmetrical" necessary condition. Let us call an element a of a semigroup S a *local left unit* if there is an element b such that $ab = b$. Analogously we shall also speak of *local right units.*

THEOREM 2. *Let $S_1 * S_2$ be the free product of the monoids $S_1 \neq \{1\}$ and $S_2 \neq \{1\}$, and suppose that $S_1 * S_2$ is orderable. If $S_1 * S_2$ is left cancellative and S_1 contains a local left unit different from* 1, *then S_2 is cancellative.*

PROOF. We have to prove only that S_2 is right cancellative. Assume there were elements $x,y,z \in S_2$ such that $x < y$, $xz = yz$.

By the hypothesis there are elements $a,b \in S_1$ such that $a \neq 1$ and $ab = b$. We can suppose that $a < 1$ holds. (If not, we have to change the role of x and y in the following argument.)

Thus we have $ya < y$ and $yaz < yz = xz$. From $yaz < xz$ it follows $ya < x$, therefore $yab < xb$, i.e., $yb < xb$, $y < x$ a contradiction.

REMARK 1. If a semigroup has no idempotens and is left cancellative then it cannot contain local right units, because from $ba = b$ it follows $ba^2 = ba$ and so by left cancellativity, $a = a^2$.

So we have seen that if $A * B$ is orderable then at most one of the semigroups may contain local left or right units, and only one of the two types.

EXAMPLE 1. Let a and b be symbols and consider the semigroup

$$A := \langle a,b \mid ab = b \rangle$$

(A is given by the defining relation $ab = b$).

It is easy to prove that every element of $A \cup \{1\}$ can be written uniquely in the form $b^k \cdot a^l$, where k and l are nonnegative integers, and

$$(b^k a^l)\,(b^m a^n) = \begin{cases} b^{k+m} a^n & \text{if } m \neq 0 \\ b^k a^{l+n} & \text{if } m = 0 \end{cases}$$

$A \cup \{1\}$ is left-cancellative and the ordering

$$b^k a^l < b^m a^n \quad \text{iff} \quad \begin{cases} k < m \text{ or} \\ k = m \text{ and } l < n \end{cases}$$

is compatible with the multiplication.

So A is a left-arranged semigroup with a local left unit a. By *Theorem* 2, if for a semigroup B the free product $A * B$ is orderable then B is cancellative.

3. Sufficient conditions

First we prove a theorem which shows that if we are interested in the orderability of the free products, then we have to consider exactly the one-sided arranged semigroups.

THEOREM 3. *Let A be an arbitrary ordered semigroup. There is a semigroup B such that $A * B$ is orderable iff A is left- or right-arranged.*

PROOF. In Section 2 we have proved that A must be one-sided arranged. We have to find only a semigroup B such that $A * B$ is orderable provided that A is, say, left arranged.

Let B be the free semigroup generated by a single element x. We can uniquely write the elements of $(A * B) \cup \{1\}$ in the form $a_1 x a_2 x \ldots x a_n$ where $a_i \in A \cup \{1\}$ for $i = 1, 2, \ldots, n$. Let $r(u)$ be the number of occurrences of the letter x in u for every $u \in (A * B) \cup \{1\}$.

If $r(u) < r(v)$, then $r(zu) < r(zv)$ and $r(uz) < r(vz)$ hold because of $r(uv) = r(u) + r(v)$.

We can define an order on $(A * B) \cup \{1\}$ in the following way. Let

$$u < v \quad \text{iff} \quad \begin{cases} r(u) < r(v) \text{ or} \\ r(u) = r(v) \;\; u = a_1 x a_2 x \ldots x a_n, \; v = b_1 x b_2 x \ldots x b_n, \\ a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, \; a_i < b_i \text{ for an } i \in \{1, 2, \ldots, n\} \end{cases}$$

It is easy to check that the compatibility of this ordering follows from the properties of r and left cancellativity.

REMARK 2. Let B be Mal'cev's example for a cancellative semigroup which cannot be embedded into a group. In [3] it was proved that B is orderable. Similarly to the previous proof one can show that $A * B$ is orderable for every one-sided arranged A.

The next theorem generalizes the main result of [2]. We call a congruence relation convex if its equivalence classes are convex.

THEOREM 4. *Let A and B be left cancellative ordered semigroups. If there is a convex congruence relation ϕ of A and a convex congruence relation ψ of B such that A/ϕ and B/ψ are cancellative and have no idempotents, then $A * B$ is orderable by extending the order of A and B.*

PROOF. Assume that such convex congruence relations ϕ and ψ exist. The order on A and B induces a natural order on A/ϕ and B/ψ. We can uniquely extend this order to $(A/\phi)\cup\{1\}$ and $(B/\psi)\cup\{1\}$. This assertion follows from

LEMMA 3. If C is an ordered cancellative semigroup without idempotents, then we can uniquely extend its order to $C\cup\{1\}$.

PROOF. Since $c\neq c^2$ for every $c\in C$, the only way of extending the order to $C\cup\{1\}$ is $1<c$ iff $c<c^2$.

This ordering is compatible, because for example $c<c^2$, $b>bc$ would imply $bc>bc^2$ ($bc\neq bc^2$ by cancellativity) and so $c>c^2$ a contradiction. For proving the transitivity of the extended order we can repeat the argument in *Proposition* 1.

So — continuing the proof of the theorem — we can use [2] to extend the order of $(A/\phi)\cup\{1\}$ and $(B/\psi)\cup 1$ to $(A/\phi)*(B/\psi)\cup\{1\}$.

Let ν be the natural homomorphism of $A*B$ onto $(A/\phi)*(B/\psi)$.

If $\nu(u)\neq\nu(v)$ for $u,v\in A*B$, then $\nu(uz)\neq\nu(vz)$ and $\nu(zu)\neq\nu(zv)$ for every $z\in A*B$ by the cancellativity of $(A/\phi)*(B/\psi)$.

If $\nu(u) = \nu(v)$, then the shortest form of u and v must have the same number of letters and the first letter of u is from A iff the first letter of v is from A.

So we can define an order in the following way

$$u<v \text{ iff } \begin{cases} \nu(u)<\nu(v) \text{ in } (A/\phi)*(B/\psi) \\ \text{or} \\ \nu(u) = \nu(v) \text{ and the first letter that is different in } u \text{ and } v \text{ (from the} \\ \text{left) is less in } u \text{ than in } v \text{ (by the order of } A \text{ resp. } B) \end{cases}$$

It is easy to check that the compatibility of this order follows from the properties of ν and left cancellativity.

The restrictions of this order on A or on B are the original orders of A resp. B.

EXAMPLE 2. Let A be the semigroup generated by the symbols a and b subjected to the defining relations $ab = b^2$, $ba = a^2$. It is easy to check that the elements of A are $a, a^2, \ldots a^n, \ldots b, b^2, \ldots b^n, \ldots$ and $a^kb^l = b^{k+l}$, $b^ka^l = a^{k+l}$.

The order

$$u^k<v^l \text{ iff } \begin{cases} k<l \\ \text{or} \\ k = l,\ u = a,\ v = b \end{cases}$$

for $u, v\in\{a, b\}$ is compatible with the multiplication.

The following relation: $(a^k, b^k)\in\theta$ for every $k\in\mathbf{N}$ and $(c, c)\in\theta$ for every $c\in A$, is a convex congruence relation, and A/θ is isomorphic to the free semigroup generated by a single element. Thus by *Theorem* 4 $A*A$ is orderable.

4. The solution of Gabovich' problem for commutative semigroups

The results, which we have obtained so far, allow us to formulate a necessary and sufficient condition of the orderability of the free product $A*B$ of the ordered commutative semigroups A and B.

THEOREM 5. *The free product $A*B$ of the ordered commutative semigroups A and B is orderable if and only if A and B are cancellative, without idempotents.*

PROOF. If $A*B$ is orderable then by *Corollary* 1 and *Theorem* 1 A and B are one-sided cancellative, without idempotents. However the commutativity of A and B implies in this case two-sided cancellativity in A and B.

On the other hand if A and B are cancellative, without idempotents, then setting $\phi = \{(a, a)\,|\,a\in A\}$ $\psi = \{(b, b)\,|\,b\in B\}$ it is easy to check that the conditions of *Theorem* 4 are trivially fulfilled and so $A*B$ is orderable.

5. Further remarks on convex cancellative congruence relations

A congruence relation θ is called cancellative if $(ca, cb)\in\theta$ or $(ac, bc)\in\theta$ implies $(a, b)\in\theta$. (See [4].) Thus a congruence relation is cancellative iff the factor by this relation is a cancellative semigroup. Obviously the intersection of all cancellative congruence relations of a semigroup is cancellative and so it is the least cancellative congruence relation of the semigroup.

Similarly the intersection of all convex cancellative relations of an ordered semigroup is the least convex cancellative congruence relation. Analogously to the corresponding result of [4] we can prove the following.

THEOREM 6. *Let C be an arbitrary ordered semigroup. Then we can construct the least convex congruence relation θ_0 of C in the following way:*

1. Let θ_1 be $\{(c, c)\,|\,c\in C\}$.
2. If θ_n is defined, let $(a, b)\in\theta_{n+1}$ iff one of the following conditions is satisfied
   (1) $\exists c\in C$ such that $(a, c)\in\theta_n$ and $(c, b)\in\theta_n$,
   (2) $\exists a', b', c\in C$ such that $(a', b')\in\theta_n$
   and $a'c = a$, $b'c = b$ or $ca' = a$, $cb' = b$ hold,
   (3) $\exists c\in C$ such that $(ca, cb)\in\theta_n$ or $(ac, bc)\in\theta_n$,
   (4) $\exists c\in C$ such that $(a, c)\in\theta_n$ and $a\leqq b\leqq c$
   or $(b, c)\in\theta_n$ and $b\leqq a\leqq c$ hold.
3. $\theta_0 = \bigcup_{n=1}^{\infty}\theta_n$.

PROOF. 1. First we show that $\theta_{n+1} \supseteq \theta_n$ holds for every $n \geq 1$. This can be proved by induction on n. For $n = 1$ $(a, a) \in \theta_1$ implies by (1) $(a, a) \in \theta_2$, so $\theta_1 \supseteq \theta_2$. If the claim is proved for $\theta_{\nu-1}$ and so $\theta_\nu \supseteq \theta_1$ holds, then $(a, b) \in \theta_n$ $(b, b) \in \theta_n$ implies $(a, b) \in \theta_{n+1}$ by (1).

2. θ_n is symmetric for every n.

We prove this also by induction on n. θ_1 is obviously symmetric. Assume that θ_n is symmetric and consider $(a, b) \in \theta_{n+1}$.

If there is a $c \in C$ such that $(a, c) \in \theta_n$ and $(c, b) \in \theta_n$, then from $(c, a) \in \theta_n$ and $(b, c) \in \theta_n$ it follows $(b, a) \in \theta_{n+1}$ by (1).

If $(a, b) \in \theta_{n+1}$, because (2) or (3) holds, then $(b', a') \in \theta_n$ or $(cb, ca) \in \theta_n$ or $(bc, ac) \in \theta_n$ implies $(b, a) \in \theta_{n+1}$.

If $(a, b) \in \theta_{n+1}$ follows from (4), then $(b, a) \in \theta_{n+1}$ follows from the symmetry of this condition.

Thus $\bigcup_{n=1}^{\infty} \theta_n$ is also symmetric.

3. $\bigcup_{n=1}^{\infty} \theta_n$ is a congruence relation.

$\theta_1 \subseteq \bigcup_{n=1}^{\infty} \theta_n$ implies the reflexivity of this relation. In 2. we have proved that $\bigcup_{n=1}^{\infty} \theta_n$ is symmetric. If $(a, b) \in \bigcup_{n=1}^{\infty} \theta_n$, and $(b, c) \in \bigcup_{n=1}^{\infty} \theta_n$ then there are integers k, m such that $(a, b) \in \theta_k$ and $(b, c) \in \theta_m$. For $n = \max(k, m)$ by 1. it follows $(a, b) \in \theta_n$ and $(b, c) \in \theta_n$, so by (1) $(a, c) \in \theta_{n+1}$, i.e., $(a, c) \in \bigcup_{n=1}^{\infty} \theta_n$. Therefore $\bigcup_{n=1}^{\infty} \theta_n$ is transitive. So far we have proved that $\bigcup_{n=1}^{\infty} \theta_n$ is an equivalence relation.

$\bigcup_{n=1}^{\infty} \theta_n$ is a congruence relation because of (2), it is cancellative because of (3) and it is convex because of (4).

So $\theta_0 \subseteq \bigcup_{n=1}^{\infty} \theta_n$ holds by the minimality of θ_0.

4. On the other hand we can prove by induction on n that $\theta_n \subseteq \theta_0$ holds, and so $\bigcup_{n=1}^{\infty} \theta_n \subseteq \theta_0$ is also true.

The next proposition may be useful in applying Theorem 4.

PROPOSITION 4. Let C be an arbitrary ordered semigroup. There is a convex congruence relation ϕ such that C/ϕ is a cancellative semigroup without idempotents iff C/θ_0 contains no idempotents. Here θ_0 denotes the least convex cancellative congruence relation of C.

PROOF. If C/ϕ is cancellative, then ϕ is a convex cancellative congruence relation and so $\phi \supseteq \theta_0$ hold. Thus if C/θ_0 contains an idempotent element, i.e., $(a, a^2) \in \theta_0$ for certain $a \in C$ then (a, a^2) also belongs to ϕ, so C/ϕ contains an idempotent element too. So if C/ϕ is cancellative without idempotents, then C/θ_0 is also cancellative and without idempotents. The other direction of the implication is trivial.

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CONVERGENCE OF SINGULAR INTEGRALS

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1. Introduction

It was proved by LEBESGUE (see [4]) that Fejér means of integrable functions converge to the function a. e. Setting out from a condition connected with this the concept of the so-called Lebesgue point was introduced and on the one hand it was shown that in case of integrable function a. e. point is Lebesgue point, on the other hand Fejér means converge to the value of the functions in Lebesgue points. By the help of the property of humped majorant that can be integrated uniformly a necessary and sufficient condition was given by FADYEYEV for the convergence of singular integrals in Lebesgue points (see e. g. ALEXITS [1]). Using this theorem it can be shown that (C,α) means of the trigonometrical system converge a. e. The theorem, however, can not be applied for $(C,1)$ means of the Walsh system. Using the dyadic derivative of measure the analogue of the definition of Lebesgue point will be introduced in this paper and it will be called Walsh-Lebesgue point. Then a necessary and sufficient condition will be given for the convergence of a singular integral in Walsh-Lebesgue points. Prooving this we need a dual space of an interesting space of sequences and we shall characterize it. Applying the necessary and sufficient condition we get easily that $(C,1)$ means of the Walsh system converge a. e.

2. Walsh-Lebesgue Points

Every point $x\in[0,1]$ can be uniquely written in the following way:

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)} \ (x_k \in \{0,1\}).$$

If there are two forms the one must be chosen for which $\lim_{k\to\infty} x_k = 0$. Introduce an addition on interval [0,1):

$$x \dot{+} y := \sum_{k=0}^{\infty} |x_k - y_k| \cdot 2^{-(k+1)} \quad (x, y \in [0, 1))$$

where $x = \sum_{k=0}^{\infty} x_k \cdot 2^{-(k+1)}$, $y = \sum_{k=0}^{\infty} y_k \cdot 2^{-(k+1)}$.

The following notations will be used:

$$x \dot{+} I := \{x \dot{+} y : y \in I\} \quad (x \in [0,1),\ I \subset [0,1)),$$

$$I_n(x) := \{y \in [0,1) : y_i = x_i \ (i = 0,1,\ldots,n-1)\}$$

$$(x \in [0,1],\ n \in \mathbf{N} \setminus \{0\})\ I_0(x) := [0,1].$$

Moreover, let $L_p := L_p([0,1), B, \lambda)$ $(0 < p \leqq \infty)$ where B denotes the aggregate of Borel sets, λ denotes the Lebesgue-measure. How do we get to the concept of the Lebesgue point? It is known that for any integrable function f

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, d\lambda(t) = 0 \quad \text{for a. e. } x \in [0,1).$$

The following condition is stricter: let $c \in \mathbf{R}$ be a number for which

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f - c|\, d\lambda = 0.$$

Satisfying this condition points x are called Lebesgue points. It can be proved that a. e. point of a function $f \in L^1$ is a Lebesgue one. Substituting the so-called dyadic differential ratio for the differential ratio we get to the derivative of measure and to the concept of Walsh-Lebesgue point. We mean that the dyadic differential of a finite additive measure of bounded oscillation v is

$$d_n v(x) := \sum_{i=0}^{n} 2^{i-1} v(I_n(x) \dot{+} 2^{-i-1}).$$

It is known (see SCHIPP [5]) that for a. e. $x \in [0,1)$

$$\lim_{n\to\infty} d_n v(x) = v'(x)$$

where $v':[0,1)\to\mathbf{R}$ is the derivative of the measure v. Suppose that v is totally continuous relative to λ, i. e.

$$v(H) = \int_H f d\lambda \quad (f\in L_1, H\in B)$$

then

$$\lim_{n\to\infty} \sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} f\, d\lambda = v'(x) = f(x) \tag{1}$$

for a. e. $x\in[0,1)$.

Now we give the definition of Walsh-Lebesgue point: a point $x\in[0,1)$ is said to be a Walsh-Lebesgue point of $f\in L_1$ if and only if there exists $c\in\mathbf{R}$ such that

$$\lim_{n\to\infty} \sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} |f-c|\, d\lambda = 0.$$

THEOREM 1. *If $f\in L_1$ then a. e. point $x\in[0,1)$ is Walsh-Lebesgue point of f with the constant $c = f(x)$.*

PROOF. Take an order $(r_k, k\in\mathbf{N})$ of rational numbers. Denote by E_n the set of the points $x\in[0,1)$ for which

$$\lim_{n\to\infty} \sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} |f-r_n|\, d\lambda = |f(x)-r_n|$$

is not satisfied. By (1) we get $\lambda(E_n) = 0$, consequently, $E := \bigcup_{n=0}^{\infty} E_n$ is a set of measure zero. We show that all points of $[0,1)\setminus E$ are Walsh-Lebesgue points. Let $x\in[0,1)\setminus E$ be arbitrary and $\epsilon>0$ be arbitrary but fixed number. Let r_k be given as follows:

$$|f(x)-r_k| < \epsilon/2.$$

Then for every n that is large enough

$$\sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} |f(t)-f(x)|\, d\lambda(t) \leqq$$

$$\leqq \sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} |f(t)-r_n|\, d\lambda(t) +$$

$$+\sum_{i=0}^{n} 2^{i-1} \int_{I_n(x)+2^{-i-1}} |f(x)-r_n|\, d\lambda(t) \leqq \epsilon/2+\epsilon/2 = \epsilon.$$

Thus x is really a Walsh-Lebesgue point of f.

3. A Duality Theorem

In this section we shall give the dual space of a double-indexed space of sequences which has an important role while prooving Theorem 3.

Let $\emptyset \neq H_n \subset \mathbf{N}\times\mathbf{N}$ $(n\in\mathbf{N})$ have the following properties:

(i) $$\bigcup_{n=0}^{\infty} H_n = \mathbf{N}\times\mathbf{N}$$

(2) (ii) if $(i,j)\in H_n$ and $(i,j)\in H_k$ $(n\leqq k)$ then for all $n\leqq l\leqq k$ $(i,j)\in I_l$

(iii) $H_n \neq H_k$ $(k\neq n)$.

Denote by X the space of sequences as follows:

$$\alpha = (\alpha_{i,j}; i,j\in \mathbf{N}) \in X \quad (\alpha_{i,j}\in \mathbf{R})$$

if

$$\sup_{n\in\mathbf{N}} \sum_{(i,j)\in H_n} |\alpha_{i,j}| < +\infty$$

and let this space be endowed with the norm

$$\|\alpha\| := \sup_{n\in\mathbf{N}} \sum_{(i,j)\in H_n} |\alpha_{i,j}|.$$

Note that if in 2(i) we take finite union then everything below holds the same way and then in special case this space corresponds to the space l^1.

Now we are to give the norm of the linear function

(3) $$A(\alpha) := \sum_{i,j\in\mathbf{N}} \alpha_{i,j} a_{i,j} \quad (\alpha = (\alpha_{i,j}; i,j\in \mathbf{N})\in X)$$

where $a = (a_{i,j}; i,j\in\mathbf{N})$ is a sequence of real numbers for which

$$\sum_{i,j\in\mathbf{N}} |\alpha_{i,j} a_{i,j}| < \infty \text{ for every } \alpha\in X.$$

THEOREM 2. *If condition* (2) *is satisfied then for the above linear functional* A *one has*

$$\|A\| = \sup_{\substack{\|\alpha\|\leqq 1 \\ \alpha_{i,j}=0 \text{ or } 1 \\ (i,j\in\mathbf{N})}} \sum_{i,j\in\mathbf{N}} |\alpha_{i,j}|\cdot|a_{i,j}|.$$

PROOF. It is easy to see that to give the norm of A we can assume that $a_{i,j} \geqq 0$, $\alpha_{i,j} \geqq 0$ $(i,j \in \mathbf{N})$. Let A be restricted the following way:

$$A_n(\alpha) := \sum_{\substack{i,j \in \mathbf{N} \\ (i,j) \in \bigcup_{k=0}^{n} H_k}} \alpha_{i,j} a_{i,j}.$$

Prooving Theorem 2 it is enough to show that

$$\|A_n\| = \sup_{\substack{\|\alpha\| \leqq 1 \\ \alpha_{i,j} = 0 \text{ or } 1 \\ (i,j \in \mathbf{N})}} \sum_{\substack{i,j \\ (i,j) \in \bigcup_{k=0}^{n} H_k}} \alpha_{i,j} a_{i,j} \tag{4}$$

because if $n \leqq m$ then $\|A_n\| \leqq \|A_m\|$, moreover, $\|A_n\| \leqq \|A\|$. Since for any $\epsilon > 0$ there exists $\alpha = (\alpha_{i,j};\ i,j \in \mathbf{N})$, $\|\alpha\| \leqq 1$ such that $\sum_{i,j} \alpha_{i,j}\, a_{i,j} \geqq \|A\| - \epsilon$ and then for every n that is large enough

$$\|A_n\| \geqq \sum_{\substack{i,j \\ (i,j) \in \bigcup_{k=0}^{n} H_k}} \alpha_{i,j} a_{i,j} \geqq \sum_{i,j} \alpha_{i,j}\, a_{i,j} - \epsilon \geqq \|A\| - 2\epsilon.$$

$\|A_n\| \to \|A\|$ also holds if $n \to \infty$. Introduce a sequence of sets $(G_n^k,\ k \in \mathbf{N})$ $(n \in \mathbf{N})$ with the following properties:

(5)
(i) fixing n it is nondecreasing

(ii) for all $n, k \in \mathbf{N}$ $\quad G_n^k \subset \mathbf{N} \times \mathbf{N}$ is finite

(iii) $\bigcup_{k=0}^{\infty} G_n^k = \bigcup_{l=0}^{n} H_l$

(iv) for all $k, l \in \mathbf{N}$, $0 \leqq l \leqq n \quad G_n^k \cap H_l \neq \emptyset$.

Restrict further the functionals A_n as follows:

$$A_n^k(\alpha) := \sum_{\substack{i,j \\ (i,j) \in G_n^k}} \alpha_{i,j} a_{i,j}.$$

By the above idea

$$\|A_n^k\| \leqq \|A_n^m\| \quad \text{if} \quad k \leqq m,\ \|A_n^k\| \leqq \|A_n\| \ (k \in \mathbf{N})$$

and if $k \to \infty$ then $\|A_n^k\| \to \|A_n\|$. Consequently, prooving Theorem 2 namely to show (4) it is enough to get

$$\|A_n^k\| = \sup_{\substack{\|\alpha\| \leqq 1 \\ \alpha_{i,j} = 0 \text{ or } 1 \\ (i,j \in \mathbf{N})}} \sum_{\substack{i,j \\ (i,j) \in G_n^k}} \alpha_{i,j} a_{i,j}.$$

$$(\alpha = (\alpha_{i,j};\ i,j \in \mathbf{N}))$$

As the set G_n^k is finite to show this equality is a problem of linear programming. Number the pairs of numbers $(i,j) \in G_n^k$ to a sequence having one index; then from the sequence $(\alpha_{i,j};(i,j) \in G_n^k)$ we get $x = (x_l,\ 1 \leqq l \leqq |G_n^k|)$ and from the sequence $(a_{i,j};\ i,j \in G_n^k)$ we get $(a'_l;\ 1 \leqq l \leqq |G_n^k|)$ where $|G_n^k|$ is the number of the elements of the set $|G_n^k|$. Let **B** be a matrix with elements 0 or 1 and let the condition $\|(\alpha_{i,j};\ (i,j) \in G_n^k)\| \leqq 1$ be equivalent to $\mathbf{Bx} \leqq [1]$ where the above vector **x** is a column vector, [1] is the column vector having element 1 only. As by the condition (2) in the definition of the norm it is enough to take the supremum only over indexes being not larger than n because of (5) (iv) **B** has n rows and $|G_n^k|$ columns exactly.

The maximum of the objective function of this linear programming problem $\sum_{l=1}^{|G_n^k|} a'_l x_l$ has to be given. We can suppose that $|G_n^k| \geqq n$ (it holds for every k that is large enough) so let us complement the problem with variables $y_1, \ldots, y_n$, take an $(n \times n)$—form unit matrix after **B**, finally denote by **C** the matrix we got this way. **C** has n rows and $|G_n^k| + n$ columns. Then the problem can be written in the following form:

$$\mathbf{Cx}' = [1],$$

$$x_i \geqq 0 \qquad i = 1, \ldots, |G_n^k|,$$

$$y_i \geqq 0 \qquad i = 1, \ldots, n,$$

$$\max(\mathbf{c}^* \cdot \mathbf{x}')$$

where $\mathbf{x}' = (x_1, \ldots, x_{|G_n^k|}, y_1, \ldots, y_n)$ is a column vector,

$$\mathbf{c} = (a'_1, \ldots, a'_{|G_n^k|}, \underbrace{0, \ldots, 0}_{n \text{ pieces}})$$

is a column vector, [1] is the column vector containing n pieces numbers 1 and in case of any matrix **D** its transposed matrix is denoted by $\mathbf{D}^*$.

Obviously, there exists a primal allowable solution of our problem, however, it also has dual allowable solution because $\mathbf{y}^*\mathbf{C} \geqq \mathbf{c}^*$ has an allowable solution since every element of $\mathbf{C}$ is 0 or 1 and it follows from (5) (iii) that in every column of $\mathbf{C}$ there is a number one at least. Then by well-known linear programming theorems there exists an optimal solution of the primal problem and it also has an optimal basis-solution. Choose a basis and denote by $\mathbf{C}_s$ the submatrix of $\mathbf{C}$, by $\mathbf{x}_s$ the subvector of $\mathbf{x}'$ which are appropriate to it. Note that complementing $\mathbf{x}_s$ with zeros we get an optimal solution. Then $\mathbf{C}_s\mathbf{x}_s=[1]$.

From the well-known formula for inverse matrix we get

$$\mathbf{C}_s^{-1} = \begin{bmatrix} \frac{d_{11}}{d}, \ldots, & \frac{d_{n,1}}{d} \\ \vdots & \vdots \\ \frac{d_{1,n}}{d}, \ldots, & \frac{d_{n,n}}{d} \end{bmatrix}$$

where $d\ (\neq 0)$ is the determinant of $\mathbf{C}_s$, $d_{k,l}\ (1 \leqq k,\ l \leqq n)$ is the determinant of the apropriate submatrix $\mathbf{C}_{k,l}$. We prove that the determinant of those matrixes containing 0 and 1 only of which columns do not contain zero between two numbers 1 is 1 or —1 or 0. By (2) (ii) $\mathbf{C}$ has this property so $\mathbf{C}_s$ and $\mathbf{C}_{k,l}$ also have it $(1 \leqq k,\ 1 \leqq n)$. In order to prove the statement we use totally induction; if $n = 1$ or $n = 2$ it is evident. Let us consider an $(n \times n)$—form matrix. Take the columns of which first element is 1. If there is no this kind of column then the determinant is 0. If there is an only one expand the determinant by the first row using the inductive condition we get that the determinant is 0 or 1 or —1. If there exist k columns with this property then we take the one which has least number 1, this must be subtracted from the above $(k$—1) columns. Operating this way the determinant does not change, furthermore there is no zero between two numbers 1 and in the first row there is an only 1.

As $\mathbf{x}_s = \mathbf{C}_s^{-1}$ [1] and by the above idea the elements of $\mathbf{C}_s^{-1}$ are integers, so the elements of $\mathbf{x}_s$ are also integers. Consequently, by the conditions of our problem we get that the elements of $\mathbf{x}_s$ can be only 0 or 1. Theorem 2 is complete.

Since $X \supset l^1$ every $A \in X^*$ has (3) form. Then it is easy to verify by Theorem 2 that the dual of X is the space containing sequences $a = (a_{i,j}; i, j \in \mathbf{N})$ such that

$$\sup_{\substack{\|\alpha\| \leqq 1 \\ \alpha_{i,j} = 0 \text{ or } 1 \\ (i,j \in \mathbf{N})}} \sum_{i,j \in \mathbf{N}} |\alpha_{i,j}| a_{i,j}| < +\infty$$

with the norm

$$\|a\| =: \sup_{\substack{\|\alpha\| \leq 1 \\ \alpha_{i,j} = 0 \text{ or } 1 \\ i,j \in \mathbf{N})}} \sum_{i,j \in \mathbf{N}} |\alpha_{i,j}| \, |a_{i,j}|.$$

Introduce a partial ordering on $\mathbf{N} \times \mathbf{N}$: $(i_1, j_1) < (i_2, j_2)$ if there is no n for which $(i_1, j_1) \in H_n$ and $(i_2, j_2) \in H_n$ and if

$$\min \{n : (i_1, j_1) \in H_n\} < \min \{n : (i_2, j_2) \in H_n\}.$$

Using this Theorem 2 can be written as

$$\|A\| = \sup \{ \sum_{k \in \mathbf{N}} |a_{i_k, j_k}| : ((i_k, j_k); k \in \mathbf{N}) \text{ is nondecreasing sequence}\}.$$

Further on Theorem 2 will be applied only for a special space X_0:

$$H_n := \{(i, j) : i \leq n, j \geq n+1\}$$

and for $\alpha = (\alpha_{i,j}; i, j \in \mathbf{N}) \in X$ $\alpha \in X_0$ if $\alpha_{i,j} = 0$ when $j \leq i$. The above partial ordering means the following: $(i_1, j_1) < (i_2, j_2)$ if $i_2 \geq j_1$.

Troughout this article this partial ordering will be used.

4. Convergence of Singular Integrals

First we define the concept of a singular integral. Let $x \in [0,1)$ be fixed and $y \in [0,1)$ be chosen such that $x \notin I_k(y)$. Suppose that

$$\lim_{n \to \infty} \int_{I_k(x)} \phi_n(t, x) \, d\lambda(t) = 1 \quad \text{and}$$

(6)
$$\lim_{n \to \infty} \int_{I_k(y)} \phi_n(t, x) \, d\lambda(t) = 0$$

where

$$k \in \mathbf{N} \quad \text{and} \quad \phi_n(t, x) \in L^\infty \quad (n \in \mathbf{N}).$$

Assume that for all fixed k there exists $\phi(k) > 0$ such that

(7)
$$\operatorname*{ess\,sup}_{t \in [0,1) \setminus I_k(x)} |\phi_n(t, x)| \leq \phi(k) \quad \text{for all } n \in \mathbf{N}.$$

Note that $\phi(k)$ depends on x. If the kernel functions $\phi_n(t, x)$ satisfy (6) and (7) for fixed x then the sequence of integrals

$$U_n(f, x) := \int_0^1 f(t)\phi_n(t, x)\, d\lambda(t) \qquad (f \in L^1)$$

is called singular integral (with the singular point x).

A norm will be introduced on kernel functions $\phi(\cdot, x) \in L^\infty$. Decompose the interval $\left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right)$ $(i \in \mathbf{N})$ the following way:

$$\left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right) = \bigcup_{j=i+1}^{\infty} \left(\frac{1}{2^{i+1}} + \frac{1}{2^{j+1}}, \frac{1}{2^{i+1}} + \frac{1}{2^j}\right)$$

and let

$$I_{i,j}(x) := x + \left[\frac{1}{2^{i+1}} + \frac{1}{2^{j+1}}, \frac{1}{2^{i+1}} + \frac{1}{2^j}\right)$$

$$(i \in \mathbf{N}, j \geqq i+1, I_{i,j} := I_{i,j}(0)).$$

Furthermore, let

$$a_{i,j} := \frac{1}{2^{i-1}} \operatorname*{ess\,sup}_{I_{i,j}(x)} |\phi|. \tag{8}$$

The norm will be defined by the remark in the end of Chapter 3:

$$\|\phi\| := \sup \{ \sum_{k \in \mathbf{N}} a_{i_k j_k} : j_k \geqq i_k + 1,\ i_{k+1} \geqq j_k,\ k \in \mathbf{N} \}.$$

Note that this definition can be illustrated geometrically: if (i_k, j_k) has already been given then

$$I_{i_{k+1}, j_{k+1}}(x) \subset x + [0, 2^{-j_k}).$$

Now the main result can be interpreted.

THEOREM 3. *In a Walsh-Lebesgue point x of any function $f \in L^1$ $U_n(f, x) \to c$ $(n \to \infty)$ if and only if*

(i) *the sequence of integrals $U_n(\cdot, x)$ is a singular integral*

(ii) *there exists K such that for every $n \in \mathbf{N}$*

$$\|\phi_n(\cdot, x)\| \leq K$$

where c is the constant in the definition of Walsh-Lebesgue point of f.

PROOF. It is enough to verify the theorem in case $x = 0$, moreover, it is easy to show that by (6) we can assume that $c = f(0) = 0$. Let the space of this kind of function $f \in L^1$ be denoted by L_0 and endowed by the norm

$$\|f\| = \sup_{n \in \mathbf{N}} \sum_{i=0}^{n} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f| d\lambda$$

by the definition of Walsh-Lebesgue point. Show that L_0 is a Banach space with this norm. Let $(f_n, n \in \mathbf{N})$ be a Cauchy sequence in L_0, i. e. $\|f_n - f_m\| \to 0$ if $n, m \to \infty$. Then there exists a subsequence of it for which $\|f_{\nu_{n+1}} - f_{\nu_n}\| < 1/2^n$ $(n \in \mathbf{N})$. Consequently, there exists $f \in L_1$ to which in $n \to \infty$ case f_{ν_n} converges λ-a. e. and in L_1-norm, too. Obviously, $f(0) = 0$ can be supposed. We prove that the point 0 is a Walsh-Lebesgue point of f with the value $c = 0$. For any $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\|f - f_{\nu_N}\| \leq \sum_{k=N}^{\infty} \|f_{\nu_{k+1}} - f_{\nu_k}\| \leq \sum_{k=N}^{\infty} \frac{1}{2^k} < \epsilon/2.$$

Moreover, since the point 0 is a Walsh-Lebesgue point of f_{ν_n} with the constant $c = 0$ for $n \in \mathbf{N}$ that is large enough

$$\sum_{i=0}^{n} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f_{\nu_n}| d\lambda < \epsilon/2.$$

Consequently,

$$\sum_{i=0}^{n} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f| d\lambda \leq$$

$$\leq \sum_{i=0}^{n} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f - f_{\nu_n}| d\lambda +$$

$$+ \sum_{i=0}^{n} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f_{\nu_n}| d\lambda \leq \epsilon.$$

Thus 0 is a Walsh-Lebesgue point of f therefore $f \in L_0$, i. e. L_0 is really a Banach space.

Endowe L_0 with a norm equivalent to the above one:

$$\|f\|_* := \sup_{n\in\mathbf{N}\setminus\{0\}} \sum_{i=0}^{n-1} 2^{i-1} \int_{[0,2^{-n})+2^{-i-1}} |f|d\lambda \quad (f\in L_0).$$

It is clear that $\|f\|_* \leqq \|f\|$. On the other hand

$$2^n \int_{[0,2^{-n})} |f|d\lambda = 2^n \sum_{j=n}^{\infty} \int_{1/2^{j+1}}^{1/2^j} |f|d\lambda$$

and

$$2^j \int_{1/2^{j+1}}^{1/2^j} |f|d\lambda \leqq \|f\|_* \quad (n,j\in\mathbf{N}).$$

Thus

$$2^n \int_{[0,2^{-n})} |f|d\lambda \leqq 2^n \sum_{j=n}^{\infty} \frac{\|f\|_*}{2^j} = 2\|f\|_*.$$

Finally we get $\|f\| \leqq 3\|f\|_*$

Later on we use the following notations: $\phi_n(t):=\phi_n(t,0)$, $U_n(f):=U_n(f,0)$. In order to apply Banach-Steinhaus's theorem we give the norm of the linear functional

$$U(f) := \int_0^1 \phi f\, d\lambda \quad (\phi\in L^\infty, f\in L_0).$$

$$\|U\| := \sup_{\substack{f\in L_0\\ \|f\|_*\leqq 1}} |U(f)| = \sup_{\substack{f\in L_1\\ \|f\|_*\leqq 1}} |U(f)|$$

because if $f\in L_1$ and $\|f\|_* \leqq 1$ then for the functions

$$f_n := \begin{cases} 0 & \text{if } x\in[0,2^{-n}) \\ f & \text{else} \end{cases}$$

$f_n\in L_0$, $\|b_n\|_* \leqq 1$ and in $n\to\infty$ case $f_n\to f$ in L_1-norm.

It is easy to see that to give the norm of U we can suppose that ϕ and f are positive. Introduce the following notations:

$$f_{i,j} := \begin{cases} 2^{i-1} f & \text{if } x \in I_{i,j} \\ 0 & \text{else,} \end{cases}$$

$$\alpha_{i,j} := \int_{I_{i,j}} f_{i,j} \qquad (i \in \mathbf{N},\ j \geqq i+1).$$

Then

$$\|f\|_* = \sup_{n \in \mathbf{N} \setminus \{0\}} \sum_{i=0}^{n-1} 2^{i-1} \int_{2^{-i-1}}^{2^{-i-1}+2^{-n}} f d\lambda =$$

$$= \sup_{n \in \mathbf{N} \setminus \{0\}} \sum_{i=0}^{n-1} 2^{i-1} \sum_{j=n}^{\infty} \int_{I_{i,j}} \frac{f_{i,j}}{2^{i-1}} d\lambda = \sup_{k \in \mathbf{N}} \sum_{i=0}^{k} \sum_{j=k+1}^{\infty} \alpha_{i,j} = \|\alpha\|$$

where the norm of $\alpha = (\alpha_{i,j};\ i \in \mathbf{N}, j \geqq i+1) \in X_0$ was defined in Chapter 3. Then on the one hand

$$\|U\| = \sup_{\substack{\|f\|_* \leqq 1 \\ f \geqq 0}} \int_0^1 \phi f d\lambda \leqq$$

$$\leqq \sup_{\substack{\|f\|_* \leqq 1 \\ f \geqq 0}} \sum_{\substack{i \in \mathbf{N} \\ j \geqq i+1}} a_{i,j} \int_{I_{i,j}} f_{i,j} d\lambda = \sup_{\|\alpha\| \leqq 1} \sum_{\substack{i \in \mathbf{N} \\ j \geqq i+1}} a_{i,j} \alpha_{i,j}$$

where the definition of $a = (a_{i,j};\ i \in \mathbf{N},\ j \geqq i+1)$ can be found in (8). On the other hand for an abitrary $\epsilon > 0$ let

$$E_{i,j}(\epsilon) := \left\{ x \in I_{i,j} : \frac{\phi}{2^{i-1}} \geqq a_{i,j} - \frac{\epsilon}{2^j} \right\}$$

then it is clear that $\lambda(E_{i,j}(\epsilon)) > 0$. Let $f \in L_1$ be a positive function such that $\|f\|_* \leqq 1$, $f_{i,j}(x) = 0$ if $x \notin E_{i,j}(\epsilon)$. Then

$$\int_0^1 \phi f d\lambda = \sum_{\substack{i \in \mathbf{N} \\ j \geqq i+1}} \int_{I_{i,j}} \phi f d\lambda \geqq \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \left(a_{i,j} - \frac{\epsilon}{2^j} \right) \alpha_{i,j}.$$

Since

$$\sum_{j=i+1}^{\infty} \alpha_{i,j} \leqq 1 \qquad (i \in \mathrm{N})$$

thus

$$\sum_{j=i+1}^{\infty} \frac{\epsilon}{2^j} \alpha_{i,j} \leqq \sum_{j=i+1}^{\infty} \frac{\epsilon}{2^j} \leqq \frac{\epsilon}{2^i}.$$

Therefore

$$\int_0^1 \phi f \, d\lambda \geqq \sum_{\substack{i \in \mathrm{N} \\ j \geqq i+1}} \alpha_{i,j} a_{i,j} - 2\epsilon,$$

consequently, by Theorem 2 one has $\|U\| = \|\phi\|$ where the norm of a kernel function was introduced before expressing Theorem 3. Thus $\|U_n\| = \|\phi_n\|$ is verified.

Let M denote the subspace containing the functions $f \in L_1$ for which there exists a natural number $k = k(f)$ such that if $x \in [0, 2^{-k})$ then $f(x) = 0$. Obviously, $M \subset L_0$ and it is easy to see that M is dense everywhere in L_0.

Let us set about the direct proof of Theorem 3.

a) Suppose that for the functions $f \in L_1$ which has the point 0 as their Walsh-Lebesgue point with the constant c one has $U_n(f) \to c$ if $n \to \infty$. Then (6) obviously holds and if $f \in L_0$ then $U_n(f) \to 0$ $(n \to \infty)$. Applying Banach-Steinhaus's theorem we get that there exists $K > 0$ such that $\|U_n\| \leqq K$ $(n \in \mathrm{N})$. Furthermore, if $f \in L_1$ for which $f(x) = 0$ if $x \in [0, 2^{-k})$, then $U_n(f) \to 0$ $(n \to \infty)$. Using Banach-Steinhaus's theorem again we have that the L_∞-norms of the kernel functions ϕ_n considering on $[2^{-k}, 1)$ are uniformly bounded, namely (7) holds.

b) Conversely, assume that conditions (i) and (ii) are satisfied. Using (7) the second part of (6) and Banach-Steinhaus's theorem we get that $U_n(f) \to 0$ if $f \in M$ and $n \to \infty$. Applying (ii) and once more Banach-Steinhaus's theorem it comes that $U_n(f) \to 0$ in case $f \in L_0$, too $(n \to \infty)$. Thus by the first part of (6) the theorem holds. The proof of Theorem 3 is finished.

Now we are to show that the Fejér-kernel functions (K_n) of the Walsh system satisfy the conditions of the theorem. It is known that these kernel functions satisfy the following conditions (see [3]):

(i) $$(2^n + k) K_{2^n+k} = 2^n K_{2^n} + k D_{2^n} + W_{2^n} k K_k$$

where

$$D_{2^n}(x := \begin{cases} 2^n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases}$$

and W_n is the nth Walsh function ($n \in \mathbf{N}$, $0 \leqq k < 2^n$)

(ii) $$k\,K_k = \sum_{i=1}^{\nu} 2^{n_i}\, W_{k-k^{(i)}}\, K_{2^{n_i}} + \sum_{i=1}^{\nu} k^{(i)} D_{2^{n_i}}$$

where $k = \sum_{i=1}^{\nu} 2^{n_i}$, $n_1 > n_2 > \ldots > n_\nu \geqq 0$,

$$k^{(1)} := k - 2^{n_1} \text{ and } \quad k^{(i)} := k^{(i-1)} - 2^{n_i} \quad (i = 2, \ldots, \nu)$$

(iii) $$K_{2^n} = \frac{1}{2}\left(2^{-n} D_{2^n} + \sum_{i=0}^{n} 2^{i-n}\, \tau_{2^{-i-1}}\, D_{2^n}\right)$$

where $(\tau_h f)(x) := f(x+h) \quad (h<1,\ x \in [0,1))$.

Using these three connections by an easy computation we get (6) and (7) for K_n. It is also easy to show that $\|K_{2^n}\| \leqq 3$, $\|D_{2^n}\| \leqq 2$ where the norm is the one in Theorem 3. Then by the above second connection we get

$$\|K_k\| \leqq \frac{1}{k} \sum_{i=1}^{\nu} 2^{n_i} \|K_{2^{n_i}}\| + \frac{1}{k} \sum_{i=1}^{\nu} k^{(i)} \|D_{2^{n_i}}\| \leqq$$

$$\leqq 3 + \frac{1}{k} k \|D_{2^{n_i}}\| \leqq 5.$$

Thus the following corollary holds:

COROLLARY. *$(C, 1)$ means of the Walsh system converge to the value of the function a. e.*

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A REMARK ON A PAPER OF P. VOLKMANN

By

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In [1], P. VOLKMANN proved the following inner product space characterization.

Let $(X, \|\cdot\|)$be a normed linear space. Then the following two statements are equivalent.

(1) $(X, \|\cdot\|)$ is an inner product space.

(2) For every one-codimensional closed subspace $Y \subset X$ there exists a $y^* \in X \setminus Y$ such that

$$\|y+ty^*\|^2 = \|y\|^2 + \|ty^*\|^2$$

for all $y \in Y$, $t \in \mathbf{R}$(or, in a complex space $\mathbf{C}$).

In this paper we give a generalization of this statement for real normed linear spaces of dimension ≥ 3. We now introduce some notations and conventions. We denote

$$P_M(x) = \{y \in M;\ \|x-y\| = \inf_{m \in M} \|x-m\|\}$$

for any subset $M \subset Y \cdot P_M$ is called the metric projection of M.

We denote the linear hull of a set by $\alpha(\ \ )$.

We say that y^* appearing above is the axis corresponding to the subspace Y.

The result

THEOREM. *Let* $(X, \|\cdot\|)$ *be a real normed linear space with* $\dim X \geq 3$. *Then the following two statements are equivalent.*

(1) $(X, \|\cdot\|)$ *is an inner product space.*

(2) *There exists a function* $F: \mathbf{R}^* \times \mathbf{R}^* \to \mathbf{R}^*$ *such that for every one-codimensional closed subspace* $Y \subset X$ *there is a* $y^* \in X \setminus Y$ *satisfying*

$$\|y+ty^*\|^2 = F(\|y\|, \|ty^*\|)$$

for all $y \in Y$ *and* $t \in \mathbf{R}$.

PROOF. $1 \to 2$ is left to the reader. (In this case $y^* \perp Y$, $F(u, v) = \sqrt{u^2+v^2}$.)

$2 \to 1$.

First we show that $(X, \|\cdot\|)$ is strictly convex. We prove this by contradiction.

Using that $S(0,1)$ contains a line segment, and (2), there exists an $\epsilon > 0$ such that for all $x \in S(0,1)$, there exists a closed hyperplane H with the property that H is tangent to $S(0,1)$, and there exists $x^* \in H$ with $x \in (S(x^*, \epsilon) \cap H) \subset S(0,1)$. This mean that all facets contain a one-codimensional unit ball.

Now, let us fix a one-codimensional closed subspace $Y \subset X$. (2) implies a "rotational symmetry" using the corresponding vector as axis, i. e. if

$$y_1, y_2 \in Y, \ \|y_1\| = \|y_2\|$$

then

$$\|y_1 + ty^*\| = \|y_2 + ty^*\|$$

for all $t \in \mathbf{R}$.

Using this and the above-proved facet-property, one can easily show that can find $a_1, b_1, a_2, b_2, \ldots \in \mathbf{R}$ such that

$$\|y+ty^*\| = \max_{n \in \mathbf{N}} a_n \|y\| + b_n t \|y^*\| \tag{3}$$

for all $y \in Y$, $t \in \mathbf{R}$. Clearly, these $a_1, b_1, a_2, b_2, \ldots$ determine F, so, these parameters are the same for all one-codimensional closed subspaces $Y \subset X$ and corresponding axes y^*.

Let $Z \subset Y$ be closed and one-codimensional in Y, and let $y \in Y$, $\|y\| = 1$. Let us define a, b, H so, that

$$H = \alpha(Z, ay + by^*)$$

and

$$\left| \frac{a}{b} \right| \neq \left| \frac{a_i}{b_i} \right|$$

for any $i = 1, 2, 3, \ldots$. One can easily show that no translate of H meets $S(0,1)$ in a one-codimensional unit ball.

This is a contradiction, $(X, \|\cdot\|)$ is strictly convex. Now we show that F is strictly monotonic in both variables.

Let $0 \leq a_1 < a_2$, $b \geq 0$. Let Y be one-codimensional and closed and y^* the corresponding axis. Let $t \in \mathbf{R}$ be such that $\|ty^*\| = b$, $y_2 \in Y$ with $\|y_2\| = a_2$. We define $\lambda = \frac{1}{2}\left(\frac{a_1}{a_2} + 1\right)$. One can easily check that

$$\|\lambda y_2 + (1-\lambda)(-y_2)\| = a_1.$$

The strict convexity implies

$$\lambda\|y_2+ty^*\|+(1-\lambda)\|-y_2+ty^*\|>\|\lambda y_2+(1-\lambda)(-y_2)+ty^*\|.$$

So,

$$\lambda F(a_2, b)+(1-\lambda)F(a_2, b)>F(a_1, b).$$

On the other hand, let $0\leq b_1<b_2$, $a\geq 0$, Y be one-codimensional and closed, y^* the corresponding axis. Let $t_2\in\mathbf{R}$ be such that $\|t_2y^*\| = b_2$, $y\in Y$ with $\|y\| = a$. We define $\lambda = \frac{1}{2}\left(\frac{b_1}{b_2}+1\right)$. Again by the strict convexity,

$$\lambda\|y+t_2y^*\|+(1-\lambda)\|y-t_2y^*\|>\|\lambda t_2y^*+(1-\lambda)(-t_2y^*)+y\|.$$

So,

$$\lambda F(a, b_2)+(1-\lambda)F(a, b_2)>F(a, b_1).$$

Now we fix Y again with $\|y^*\| = 1$.

Let $Z\subset Y$ be closed and one-codimensional in Y. We define $\tilde{Z} = \alpha(Z, y^*)$. Clearly, $\tilde{Z}\subset X$ is one-condimensional and closed.

Let $\underline{z}^*\in X\backslash\tilde{Z}$, $\|\underline{z}^*\| = 1$ be the corresponding axis. We show that $\underline{z}^*\in Y$. Assume the contrary. Clearly, $\underline{z}^*\notin\alpha(y^*)$. So, there exists an $\tilde{y}\in Y$ such that $\|\tilde{y}\|=1$ and $a, b>0$ with

$$\underline{z}^* = ay^*+b\tilde{y}.$$

By the axis-property of $\underline{z}^*$, $\|y^*-\underline{z}^*\|=F(1, 1)$. By the axis-property of y^*, $\|y^*-\underline{z}^*\|=F(b, |1-a|)$. These together mean

$$F(1, 1) = F(b, |1-a|).$$

Clearly, we have one of the following cases.

Case 1. $a\leq 2$, $b<1$,

$$F(b, |1-a|)<F(1,1).$$

This is a contradiction.

Case 2. $a = 2$, $b<1$,

$$1 = \|\underline{z}^*\|=F(b, a)>F(1, 0) = 1.$$

This is a contradiction.

Case 3. $a>2$, $b\leq 1$.

$$1 = \|\underline{z}^*\|=\|ay^*+b\tilde{y}\|\geq a-b>1.$$

This is a contradiction.

Case 4. a arbitrary, $b>1$,

$$1 = \|\underline{z}^*\| = F(b, a) > F(1, 0) = 1.$$

This is a contradiction. Therefore $z^* \in Y$.

For every $x \in X$, we find $C_1(x)$, $C_2(x) \in \mathbf{R}$, $\underline{z} \in Z$ satisfying

$$x = C_1(x)y^* + C_2(x)\underline{z}^* + \underline{z}.$$

Clearly, $C_1(x)$, $C_2(x)$, $\underline{z}$ are uniquely determined by x. So, the mapping $\underline{x} \to \underline{z}$ is linear.

Let $\underline{z}' \in Z$ be arbitrary. We have

$$\|\underline{x} - \underline{z}'\| = F(\|C_2(\underline{x})\underline{z}^* + \underline{z}'\|, |C_1(\underline{x}|) =$$

$$F(F(\underline{z} - \underline{z}'\|, |C_2(\underline{x})|), |C_1(\underline{x}|) \geq F(F(0,1\ C_2(\underline{x})|), |C_1(\underline{x})|).$$

In the last inequality we have equality if and only if $\underline{z} = \underline{z}'$. This implies that the metric projection onto Z is a linear operator, where Z is an arbitrary two-condimensional closed subspace of X. The following inner product space characterization is known [2], p. 249: Let $k \geq 1$, and $\dim X \geq k+1$.

Then the following two statements are equivalent.

(1) Every k-condimensional closed subspace has linear metric projection.

(2) $(X\|\cdot\|)$ is an inner product space.

Our result follows.

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ON COMPARISON OF HOMOGENEOUS MEANS

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1. Introduction

Define for real a and b the mean $M_{a,b}$ as follows:

$$M_{a,b}(x_1, \ldots, x_n) = \begin{cases} \left(\sum_{i=1}^{n} x_i^a / \sum_{i=1}^{n} x_i^b \right)^{1/(a-b)}, & \text{if } a \neq b, \\ \exp\left(\sum_{i=0}^{n} x_i^a \ln x_i / \sum_{i=1}^{n} x_i^a \right), & \text{if } a = b \end{cases} \tag{1}$$

and where $x_1, \ldots, x_n > 0$.

Obviously $M_{1,0}$ is the arithmetic mean, $M_{0,0}$ is the geometric mean and, more generally, $M_{p,0}$ is the p-th power mean.

Concerning the comparison of means of type (1), Daróczy and Losonczi [6] found the following result:

Theorem A. *The inequality*

$$M_{a,b}(x_1, \ldots, x_n) \leqq M_{c,d}(x_1, \ldots, x_n) \tag{2}$$

holds for all natural number n and $x_1, \ldots, x_n > 0$ *if and only if*

$$\min(a, b) \leqq \min(c, d) \text{ and } \max(a, b) \leqq \max(c, d).$$

This theorem obviously generalizes the arithmetic mean — geometric mean inequality and the well-known comparison theorem of the power means (see [8], Th. 16, p. 26).

LOSONCZI ([10], Th. 7) also obtained necessary and sufficient conditions for (2) to hold when the range of the variables x_i is only a proper subinterval $[u, v]$ of $]0, \infty[$. This result is contained in the Theorem below.

The main observation of this paper is the following: Formula (1) defines a mean not only for real but for certain complex parameters a, b, namely, when a and b are conjugate complex numbers. To prove this, let $a = \alpha+\beta i$ and $b = \alpha-\beta i$ where $\alpha, \beta \in \mathbf{R}$ and $\beta \neq 0$. Then

$$M_{a,b}(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} x_j^{\alpha+\beta i} \Big/ \sum_{j=1}^{n} x_j^{\alpha-\beta i}\right)^{1/(2\beta i)} =$$

$$= \left(\frac{\sum_{j=1}^{n} x_j^{\alpha}(\cos(\beta \ln x_j)+i \sin(\beta \ln x_j))}{\sum_{j=1}^{n} x_j^{\alpha}(\cos(\beta \ln x_j)-i \sin(\beta \ln x_j))}\right)^{\frac{1}{2\beta i}} =$$

$$= \left(\exp\left\{2i \arg\left[\sum_{j=1}^{n} x_j^{\alpha}(\cos(\beta \ln x_j)+i \sin(\beta \ln x_j))\right]\right\}\right)^{\frac{1}{2\beta i}} =$$

$$= \exp\left\{\frac{1}{\beta} \arctan\left(\frac{\sum_{j=1}^{n} x_j^{\alpha} \sin(\beta \ln x_j)}{\sum_{j=1}^{n} x_j^{\alpha} \cos(\beta \ln x_j)}\right)\right\}. \tag{3}$$

Since the range of the arctan function is the interval $]-\pi/2, \pi/2[$, we can see that the value $M_{a,b}(x_1, \ldots, x_n) = M_{\alpha+\beta i, \alpha-\beta i}(x_1, \ldots, x_n)$ is always between $\exp(-\pi/2|\beta|)$ and $\exp(\pi/2|\beta|)$. Therefore the maximal interval, where formula (3) can define a mean, is $]\exp(-\pi/2|\beta|), \exp(\pi/2|\beta|)[$. On the other hand, it is easy to check that $M_{a,b}(x_1, \ldots, x_n)$ is a mean value of $x_1, \ldots, x_n$ (i.e. it is between $\min(x_1, \ldots, x_n)$ and $\max(x_1, \ldots, x_n)$), if these values belong to the above interval. (We note that the above defined means were introduced and examined in detail by ACZÉL and DARÓCZY in [1].)

Summarizing our observations, we can state that the function $M_{a,b}$ defined by formula (1) is a mean on the interval $I_{a,b}$ if $(a, b) \in D$, where

$$D = \{(a, b) \in \mathbf{C}^2 \mid a+b,\ ab \in \mathbf{R}\} = \{(\alpha+\beta, \alpha-\beta) \mid \alpha \in \mathbf{R},\ \beta \in \mathbf{R} \cup (i\mathbf{R})\}$$

and

$$I_{a,b} = \begin{cases}]0, \infty[\ , & \text{if } a-b \in \mathbf{R} \\ ]\exp(-\pi/|a-b|), \exp(\pi/|a-b|)[\ , & \text{if } a-b \notin \mathbf{R}. \end{cases}$$

2. Results

First we define the quasiarithmetic means with weight function. Let $I \subseteq \mathbf{R}$ be a fixed proper interval, furthermore let $f: I \to \mathbf{R}_+$ be a positive valued function and $\Phi: I \to \mathbf{R}$ be a strictly monotonic continuous function. Then the mean $M_{\Phi,f}$ defined by

$$\mathrm{M}_{\Phi,f}(x_1, \ldots, x_n) := \Phi^{-1}\Big(\sum_{j=1}^{n} f(x_i)\Phi(x_i) / \sum_{i=1}^{n} f(x_i)\Big),\ n \in \mathbf{N},\ x_1, \ldots, x_n \in I$$

is called a quasiarithmetic mean with weight function (see Aczél, Daróczy [1], Bajraktarević [2]).

On the comparison of these means, Daróczy and Losonczi proved the following result (see [6] and [9]):

Theorem B. *Let $f, g: I \to \mathbf{R}_+$ be positive valued functions, and $\Phi, \Psi: I \to \mathbf{R}$ be differentiable functions with nonvanishing derivative. Then*

$$\mathrm{M}_{\Phi,f}(x_1, \ldots, x_n) \leqq \mathrm{M}_{\Psi,g}(x_1, \ldots, x_n) \tag{4}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ if and only if

$$\frac{f(x)\big(\Phi(x)-\Phi(y)\big)}{f(y)\Phi'(y)} \leqq \frac{g(x)\big(\Psi(x)-\Psi(y)\big)}{g(y)\Psi'(y)} \tag{5}$$

is satisfied for all $x, y \in I$.

To apply this result we observe that $M_{a,b}$ is a quasiarithmetic mean with weight function

$$f(x) = f_{a,b}(x) = \begin{cases} x^b & \text{if } a, b \in \mathbf{R}, \\ x^{(a+b)/2} \cos\left(\dfrac{|a-b|}{2} \ln x\right), & \text{if } a = \bar{b} \notin \mathbf{R} \end{cases} \tag{6}$$

and with

$$\Phi(x)=\Phi_{a,b}(x)=\begin{cases} x^{a-b}, & \text{if } a, b\in\mathbf{R},\ a\neq b,\\ \ln x, & \text{if } a=b\in\mathbf{R},\\ \tan\left(\dfrac{|a-b|}{2}\ln x\right), & \text{if } a=\bar{b}\notin\mathbf{R}\end{cases}$$

for all $(a, b)\in D$.

A direct computation shows that

$$(7)\quad \frac{f_{a,b}(x)\big(\Phi_{a,b}(x)-\Phi_{a,b}(y)\big)}{f_{a,b}(y)\Phi'_{a,b}(y)} = y(x/y)^{(a+b)/2}\ln(x/y)S\left(\frac{a-b}{2}\ln(x/y)\right)$$

where S: $\mathbf{C}\to\mathbf{C}$ is defined by

$$S(z) = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n+1)!}\left(=\frac{\sinh z}{z}=\frac{\sin iz}{iz}\right).$$

Clearly, $S(z)$ is a real value if $z\in\mathbf{R}\cup(i\mathbf{R})$. Using the above notations, we can prove the main result of this paper.

THEOREM. *Let (a, b), $(c, d)\in D$ and $0<u<v<\infty$. Assume that $[u, v]\subseteq \subseteq I_{a,b}\cap I_{c,d}$. Then*

$$(8)\qquad M_{a,b}(x_1, \ldots, x_n)\leqq M_{c,d}(x_1, \ldots, x_n)$$

holds for all $n\in\mathbf{N}$ and $x_1, \ldots, x_n\in[u, v]$ if and only if

$$(9)\qquad \frac{\ln[S((a-b)t)/S((c-d)t)]}{t}\leqq c+d-a-b$$

is satified for $t=\ln\sqrt{u/v}$ and $t=\ln\sqrt{v/u}$.

PROOF. We can see that (8) is equivalent to the inequality (4) if we take $f=f_{a,b}$, $\Phi=\Phi_{a,b}$, $g=f_{c,d}$, $\Psi=\Phi_{c,d}$. Now applying Theorem B and (7), we find that (8) is valid if and only if

$$(10)\qquad y(x/y)^{(a+b)/2}\ln(x/y)S\left(\frac{a-b}{2}\ln(x/y)\right)\leqq$$

$$\leqq y(x/y)^{(c+d)/2}\ln(x/y)S\left(\frac{c-d}{2}\ln(x/y)\right)$$

is satisfied for all $x, y \in [u, v]$. Putting $t = \ln \sqrt{x/y}$, we can see that (10) holds for all values $x, y \in [u, v]$ exactly when (9) is valid for all $t \in J := [\ln\sqrt{u/v}, \ln \sqrt{v/u}] \setminus \{0\}$.

Denote by $h(t)$ the left hand side of (9), and let $h(0) := 0$. Then, by L'Hospital's rule, one can see that h is a continuous function at $t = 0$. We show that it is also monotic on J.

A simple calculation yields that

$$\left(\frac{\ln S(\sigma t)}{t}\right) = \begin{cases} \dfrac{1}{t^2}\displaystyle\int_0^{|\sigma t|} \left(\frac{1}{u} - \frac{u}{\sinh^2 u}\right) du, & \text{if } \sigma \in \mathbf{R},\ t \neq 0, \\ \dfrac{1}{t^2}\displaystyle\int_0^{|\sigma t|} \left(\frac{1}{u} - \frac{u}{\sin^2 u}\right) du, & \text{if } \sigma \in i\mathbf{R},\ 0 < |\sigma t| < \pi. \end{cases}$$

Write $\beta := a - b$, $\delta := c - d$ and suppose first that $\beta^2 > \delta^2$. Then, for $t \in J$,

$$h'(t) = \begin{cases} \dfrac{1}{t^2}\displaystyle\int_{|\delta t|}^{|\beta t|} \left(\frac{1}{u} - \frac{u}{\sinh^2 u}\right) du, & \text{if } \beta^2 > \delta^2 \geqq 0, \\ \dfrac{1}{t^2}\left[\displaystyle\int_0^{|\beta t|} \left(\frac{1}{u} - \frac{u}{\sinh^2 u}\right) du + \int_0^{|\delta t|} \left(\frac{u}{\sin^2 u} - \frac{1}{u}\right) du\right], & \text{if } \beta^2 > 0 > \delta^2, \\ \dfrac{1}{t^2}\displaystyle\int_{|\beta t|}^{|\delta t|} \left(\frac{u}{\sin^2 u} - \frac{1}{u}\right) du, & \text{if } 0 \geqq \beta^2 > \delta^2. \end{cases}$$

Since $\sin u \leqq u \leqq \sinh u$ for $u > 0$, we have that $h'(t) > 0$ for all $t \in J$. Therefore, using the continuity at $t = 0$, we get that h is strictly increasing on $J \cup \{0\}$.

In the case $\beta^2 = \delta^2$, h turns out to be a constant; in the case $\beta^2 < \delta^2$ a similar argument shows that it is strictly decreasing. Therefore h is monotonic in each case. Now if (9) is satisfied by $t = \ln \sqrt{u/v}$ and $t = \ln \sqrt{v/u}$, then the monotonicity of h yields that (9) is also true for all intermediate values $t \in J$.

The proof is complete.

REMARK. If a, b, c, d are real parameters, then the above Theorem reduces to a result of LOSONCZI [10]. There exist also integral analogues of inequality (8) (the case of real parameters is discussed in [10]).

The theory of the $M_{a,b}(a, b \in \mathbf{R})$ means has a rich literature. Hölder-Minkowski,— and Ingham-Jessen-type inequalities have been proved for them (see [3], [4], [5], [7], [10], [11], [12], [13], [14].) These means are the only homogeneous and continuous quasiarithmetic means with weight function (see [1]).

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NECESSARY AND SUFFICIENT CONDITION FOR THE MAXIMAL INEQUALITY OF CONCAVE YOUNG-FUNCTIONS

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1. Concave Young-functions

Let ϕ be a right-continuous and decreasing function defined on $(0,\infty)$, such that it is integrable on every finite interval $(0, x)$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = 0$. It can be easily checked that

$$\Phi(x) = \int_0^x \phi(t)dt,\ x>0,$$

is a nonnegative, increasing and concave function with $\Phi(0) = 0$. We shall further assume that $\Phi(\infty) = \infty$. Φ is called "concave Young-function". One can easily show that $\frac{\Phi(x)}{x}$ is a nonnegative and decreasing function on $(0, \infty)$, such that

$$0 \le \alpha := \lim_{x\to\infty} \frac{\Phi(x)}{x} < \infty.$$

Also, for each constant $c > 1$, the inequality $\Phi(cx) \le c\,\Phi(x)$ holds for all $x > 0$.

2. Maximal inequality

Definition. We say that for the concave Young-function Φ the maximal inequality is valid with some positive constant K_Φ (depending only on Φ) if for an arbitrary nonnegative submartingale $(X_n, \mathfrak{F}_n)$, $n\ge 1$, the inequality

$$E\Phi\,(X_n^*) \le K_\Phi\,(1+E\,X_n) \tag{1}$$

holds for all $n \geq 1$, with $X_n^* = \max_{1 \leq k \leq n} X_k$.

The aim of this paper is to characterize all the concave Young-functions for which (1) holds.

Let us point out that inequality (1) fails to hold in general (cf. [1], Example 3.1, page 13).

THEOREM 1. *Let Φ be any concave Young-function. In order that Φ satisfy the above maximal inequality, it is necessary and sufficient that*

$$\int_1^\infty \frac{\phi(t)}{t}\, dt = : A < \infty. \tag{2}$$

PROOF. Although the sufficiency is already known (cf. [1], Theorem 3.2), for the sake of completeness we here give a proof to it.

Consider the maximal inequality of Doob

$$xP(X_n^* \geq x) \leq E\, X_n\, \chi(X_n^* \geq x),\ x > 0$$

to be integrated on $[1, \infty)$ with respect to the measure generated by the nondecreasing function

$$\int_1^\infty \frac{\phi(t)}{t}\, dt.$$

By the Fubini theorem it follows that

$$E \int_1^{X_n^* \vee 1} \phi(x)dx \leq E\, X_n \int_1^{X_n^* \vee 1} \frac{\phi(t)}{t}\, dt \leq A\, E\, X_n,$$

since

$$\int_1^{X_n^* \vee 1} \frac{\phi(t)}{t}\, dt \leq A.$$

Consequently,

$$E\, \Phi X_n^* \leq \Phi(1) + A\, E\, X_n.$$

By choosing $K_\Phi = \max\,(A, \Phi(1))$ we obtain (1).

NECESSITY. Suppose that the maximal inequality holds for Φ with some constant $K_\Phi > 0$. Let (x_n), $n \geq 1$, be an arbitrary sequence of real numbers such that $x_1 = 1$, $x_n < x_{n+1} < 2x_n$, $\lim_{n\to\infty} x_n = \infty$ and

$$A := \int_1^\infty \frac{\phi(t)}{t}\, dt = \lim_{n\to\infty} \int_1^{x_n} \frac{\phi(t)}{t}\, dt. \tag{3}$$

Consider the probability space $(\mathbf{N}, \mathfrak{A}, P)$ with $\mathfrak{A}$ being the σ-field of Borel measurable subsets of $\mathbf{N}$. The probability measure is defined on $(\mathbf{N}, \mathfrak{A})$ as follows:

$$P(\{n\}) = \frac{1}{x_n} - \frac{1}{x_{n+1}}, \quad n = 1, 2, \ldots .$$

Define for all $n \geq 1$, the r. v.'s

$$X_n(\omega) := x_n\, \chi(\omega \geq n), \quad \omega \in \mathbf{N},$$

and let $\mathfrak{F}_n := \sigma(\{1\}, \ldots, \{n-1\}, \{n, n+1, \ldots\})$ be the minimal σ-field generated by the measurable partition noted in the brackets.

It can be easily shown that $(X_n, \mathfrak{F}_n)$, $n \geq 1$, is a nonnegative martingale, and

$$X_n^* = \begin{cases} x_\omega, & \omega < n \\ x_n & \omega \geq n. \end{cases}$$

In virtue of the maximal inequality we have

$$\sum_{k=1}^{n-1} \Phi(x_k) \left(\frac{1}{x_k} - \frac{1}{x_{k+1}} \right) + \frac{\Phi(x_n)}{x_n} \leq 2K_\Phi, \tag{4}$$

since here $EX_n = 1$.

The sum of the left hand side of (4) can be estimated as follows:

$$\sum_{k=1}^{n-1} \Phi(x_k) \left(\frac{1}{x_k} - \frac{1}{x_{k+1}} \right) = \sum_{k=1}^{n-1} \Phi(x_k) \int_{x_k}^{x_{k+1}} \frac{1}{t^2}\, dt \geq \frac{1}{2} \int_1^{x_n} \frac{\Phi(t)}{t^2}\, dt =$$

$$= \frac{1}{2} \left\{ \Phi(1) - \frac{\Phi(x_n)}{x_n} + \int_1^{x_n} \frac{\phi(t)}{t}\, dt \right\}$$

(in virtue of the integration by parts).

Consequently, (4) implies

$$\int_1^{x_n} \frac{\phi(t)}{t}\, dt + \frac{\Phi(x_n)}{x_n} \leq 4K_\Phi - \Phi(1).$$

Passing to the limit, we observe by (3) that

$$\int_1^{\infty} \frac{\phi(t)}{t}\, dt \leq 4K_\Phi - \Phi(1) - \alpha < \infty.$$

This ends the proof of our theorem.

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