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ZUR THEORIE LIMESFORMER RÄUME II*

von

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(Eingegangen am 15. September 1987)

9. Die Kategorien Limform und Ultraform

9.1 LEMMA: X sei eine Menge und (Y, L) ein limesformer (ultraformer) Raum. Für eine Abbildung $f : X \to Y$ setze für $A \in \in \mathfrak{P}X : L_{\{f^{-1}\}}(A) := \{\mathfrak{N} \in \mathfrak{P}^2 X \mid f\mathfrak{N} \in L(f[A])\}$, so ist $L_{\{f^{-1}\}}$ ein limesformer (ultraformer) Operator auf X mit $f : (X, L_{\{f^{-1}\}}) \to (Y, L)$ ist eine limesforme Abbildung.

BEWEIS: Aus Gründen der Trivialität zeigen wir nur das Axiom (U) (vergl. Hinweis 4.10. bzw. Definition 4.11.).

zu (U): Für $A \in \mathfrak{P}X$ sei $\mathfrak{N} \in L_{\{f^{-1}\}}(\mathrm{cl}_{L_{\{f^{-1}\}}}(A))$, so gilt nach Definition von $L_{\{f^{-1}\}}$ $f\mathfrak{N} \in L(f[\mathrm{cl}_{L_{\{f^{-1}\}}}(A)])$. Wir zeigen : $f[\mathrm{cl}_{L_{\{f^{-1}\}}}(A)] \subset \mathrm{cl}_L(f[A])$. $y \in f[\mathrm{cl}_{L_{\{f^{-1}\}}}(A)]$ impliziert die Existenz eines Elementes $x \in \mathrm{cl}_{L_{\{f^{-1}\}}}(A)$ mit $y = f(x)$.

Wähle $\mathfrak{M} \in L_{\{f^{-1}\}}(A)$ mit $x \in \cap\mathfrak{M}$, so folgt $f\mathfrak{M} \in L(f[A])$. Wir haben $f(x) \in f\left[\cap\mathfrak{M}\right] \subset \cap f\mathfrak{M}$, woraus $y \in \mathrm{cl}_L(f[A])$ resultiert. Da $L(\mathrm{Lf}_2)$ erfüllt, folgt

$$L(f[\mathrm{cl}_{L_{\{f^{-1}\}}}(A)]) \subset L(\mathrm{cl}_L(f[A])) \subset L(f[A]),$$

weil L darüberhinaus Axiom (U) erfüllt. Somit ist schließlich $\mathfrak{N} \in \in L_{\{f^{-1}\}}(A)$. Trivialerweise ist $f : (X, L_{\{f^{-1}\}}) \to (Y, L)$ eine limesforme Abbildung.

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* Den ersten Teil s. in dieselber Zeitschrift, Band **32** (1989), 37–60.

9.2. LEMMA: Für eine Menge X bezeichne Limform(X) bzw. Ultraform(X) die Menge aller Limesformitäten bzw. Ultraformitäten auf X. Für $\emptyset \neq \mathfrak{B} \subset$ Limform(X), Ultraform(X) gilt dann: $\cap\mathfrak{B} \in$ Limform(X), Ultraform(X).

BEWEIS: Aus Gründen der Trivialität zeigen wir nur das Axiom (U).

zu (U): Für $A \in \mathfrak{P}X$ sei $\mathfrak{N} \in \cap\mathfrak{B}(\mathrm{cl}_{\cap\mathfrak{B}}(A))$ und $L \in \mathfrak{B}$, so gilt $\mathfrak{N} \in L(\mathrm{cl}_{\cap\mathfrak{B}}(A))$. Wir zeigen: $\mathrm{cl}_{\cap\mathfrak{B}}(A) \subset \mathrm{cl}_L(A)$. $x \in \mathrm{cl}_{\cap\mathfrak{B}}(A)$ impliziert die Existenz eines Mengensystems $\mathfrak{R} \in \cap\mathfrak{B}(A)$ mit $x \in \cap\mathfrak{R}$. Es folgt $\mathfrak{R} \in L(A)$, mithin ist $x \in \mathrm{cl}_L(A)$. Unter Anwendung der Axiome (Lf$_2$) bzw. (U) folgt dann $L(\mathrm{cl}_{\cap\mathfrak{B}}(A)) \subset L(\mathrm{cl}_L(A)) \subset L(A)$, woraus sich die Aussage $\mathfrak{N} \in \cap\mathfrak{B}(A)$ ergibt.

9.3. SATZ: *Limform (Ultraform) ist eine topologische Kategorie.*

BEWEIS: $(Y_i, L_i)_{i\in I}$ sei eine Familie von limesformen (ultraformen) Räumen, X sei eine Menge und $f_i : X \to Y_i$ für jedes $i \in I$ eine Abbildung.

Setze $L^{in} := \cap\{L_{i_{\{f_i^{-1}\}}} \mid i \in I\}$, so ist mit den Lemmata 9.1. bzw. 9.2. L^{in} eine Limesformität (Ultraformität) auf X derart, daß für jedes $i \in I$ $f_i : (X, L^{in}) \to (Y_i, L_i)$ eine limesforme Abbildung ist. Sei nun (Y, L) ein limesformer (ultraformer) Raum und $g : Y \to X$ eine Abbildung mit $f_i \circ g : (Y, L) \to (Y_i, L_i)$ ist für jedes $i \in I$ eine limesforme Abbildung, wir zeigen: $g : (Y, L) \to (X, L^{in})$ ist eine limesforme Abbildung.

Für $A \in \mathfrak{P}Y$ sei $\mathfrak{N} \in L(A)$, so folgt für jedes $i \in I$ $(f_i \circ g)\mathfrak{N} \in$ $\in L_i((f_i \circ g)[A])$. $(f_i \circ g)[A] = f_i[g[A]]$, so daß die Aussage $(f_i \circ g)\mathfrak{N} \in$ $\in L_i(f_i[g[A]])$ resultiert. Wir haben $(f_i \circ g)\mathfrak{N} = f_i(g\mathfrak{N})$, woraus $g\mathfrak{N} \in$ $\in L^{in}(g[A])$ folgt.

9.4. BEMERKUNG: Limform(X) und Ultraform(X) sind Mengen und für Elemente L_1, $L_2 \in$ Limform(X), bzw. L_1, $L_2 \in$ Ultraform(X) mit $1_X : (X, L_1) \to (X, L_2) \to (X, L_1)$ sei jeweils eine limesforme Abbildung folgt dann $L_1 = L_2$.

9.5. BEISPIEL: Auf einer einelementigen Menge X definiere die beiden folgenden *verschiedenen* Limesformitäten, wobei mit "x" das einzige Element von X bezeichnet wird.

(i) $K(A) := \mathfrak{P}^2X \setminus \{\emptyset\}$;

(ii) $$\begin{cases} P(\{x\}) := \{\{\emptyset\}, \{\emptyset, \{x\}\}, \{\{x\}\}\}, \\ P(\emptyset) := \{\{\emptyset\}, \ldots, \{\emptyset, \{x\}\}\}. \end{cases}$$

9.6. SATZ: *Die Kategorie Nullform ist eine passend gefaserte (properly fibred) topologische Kategorie.*

BEWEIS: Nach Satz 9.3. ist *Limform* eine topologische Kategorie, in der die volle Unterkategorie *Nullform* mit Satz 5.3. bikoreflektiv liegt. Für einen nullformen Raum (X, L) und einen limesformen Raum (Y, T) sei $f : (X, L) \to (Y, T)$ eine homöoforme Abbildung (vergl. Definition 2.3), so ist trivialerweise (Y, T) ein nullformer Raum. Mithin ist *Nullform* in *Limform* isomorphie-abgeschlossen. Mit einem Satz aus der Kategorientheorie folgt dann, daß *Nullform* eine topologische Kategorie ist. Darüberhinaus gilt Bemerkung 9.4. auch für die *Nullform-Faser* Nullform(X). Wir zeigen noch, daß auf einer einelementigen Menge $X = \{x\}$ Nullform(X) aus genau einem Element besteht.

Für $A \in \mathfrak{P}X$ setze $\dot{L}(A) = \{\mathfrak{N} \in \mathfrak{P}^2X \mid A \in \operatorname{stack}\mathfrak{R}\}$, so gilt trivialerweise für $T \in$ Nullform(X) die Beziehung $\dot{L} \leq T$. Umgekehrt sei für $A \in \mathfrak{P}X$ $\mathfrak{N} \in T(A)$:

1. Fall: $A = \emptyset$, so folgt mit Axiom (N) $\emptyset \in \operatorname{stack}\mathfrak{N}$, und es gilt $\mathfrak{N} \in \dot{L}(A)$.
2. Fall: $A = \{x\}$; da nach (Lf_3) $\mathfrak{N} \neq \emptyset$, folgt $\emptyset \in \mathfrak{N}$ oder $\{x\} \in \mathfrak{N}$.

In beiden Fällen gilt $A \in \operatorname{stack}\mathfrak{N}$, und somit ist $\mathfrak{N} \in \dot{L}(A)$. Insgesamt haben wir $\dot{L} = T$.

9.7. LEMMA: (X, L_1), (Y, L_2) seien limesforme Räume. Für eine Abbildung $f : X \to Y$ sind dann die folgenden Aussagen äquivalent:

(i) $f : (X, L_1) \to (Y, L_2)$ ist limesforme Abbildung;

(ii) $A \in \mathfrak{P}Y$ und $\mathfrak{N} \in L_1(f^{-1}[A])$ implizieren $f\mathfrak{N} \in L_2(A)$;

(iii) $L_1 \leq L_{2\{f^{-1}\}}$.

BEWEIS: trivial, vergl. auch Lemma 9.1. .

9.8. SATZ: *Die Kategorie Rastform (Filform) ist eine bikoreflektive Unterkategorie der Kategorie Limform (Nullform).*

BEWEIS: Zunächst beachte man die Definition 5.7. bzw. 6.5. Für einen limesformen (nullformen) Raum (X, L) setze für $A \in \mathfrak{P}X$: $L_r(A) := \{\mathfrak{N} \in \mathfrak{P}^2X \mid \exists \text{ Raster } \mathfrak{R} \in L(A)\ \mathfrak{R} \ll \mathfrak{N}\}$, so ist (X, L_r) ein rasterformer (filterformer) Raum, und $1_X : (X, L_r) \to (X, L)$ ist die *Rastform* (*Filform*) Bikoreflektion von (X, L). Desweiteren beachte man, daß die Bilder von Rastern wieder Raster sind.

9.9. LEMMA:

(i) *Rastform* ist isomorphie-abgeschlossen in *Limform*;

(ii) *Filform* ist isomorphie-abgeschlossen in *Nullform*.

BEWEIS: Trivial.

9.10. BEMERKUNG: Da *Rastform* eine bikoreflektive isomorphie-abgeschlossene volle Unterkategorie der topologischen Kategorie *Limform* ist, ist *Rastform* eine topologische Kategorie. Analog ist *Filform* als eine bikoreflektive isomorphie-abgeschlossene volle Unterkategorie der passend gefaserten topologischen Kategorie *Nullform* selbst eine passend gefaserte topologische Kategorie. Schließlich weisen wir darauf hin, daß ein analoger Satz und ein analoges Lemma gelten, sofern von der Kategorie *Ultraform* ausgegangen wird (vergl. auch Satz 9.3.), d.h. *R-Ultraform* ist isomorphie-abgeschlossen in *Ultraform*, und *F-Ultraform* ist isomorphie-abgeschlossen in *N-Ultraform*. Die Objekte von *I-Ultraform* (I:=(R), (F), (N)) sind dabei gerade diejenigen ultraformen Räume, welche die entsprechend genannten zusätzlichen Axiome erfüllen. In diesem Zusammenhang sei insbesondere auf Kapitel 3. bzw. Kapitel 6. hingewiesen. *R-Ultraform* ist bikoreflektiv in *Ultraform*, und *F-Ultraform* ist bikoreflektiv in *N-Ultraform*. Bei der zuletzt genannten Kategorie beachte man, daß der in Definition 4.3. erklärte Hüllenoperator cl_L das folgende zusätzliche Axiom erfüllt:

$$(\mathrm{et}_4) \qquad \mathrm{cl}(\emptyset) = \emptyset.$$

Aufgrund der vorangehenden Bemerkungen zeigen wir: L_r erfüllt das Axiom (U).

zu (U): Für $A \in \mathfrak{P}X$ sei $\mathfrak{N} \in L_r(\mathrm{cl}_{L_r}(A))$, so existiert ein Raster $\mathfrak{R} \in L(\mathrm{cl}_{L_r}(A))$ mit $\mathfrak{R} \ll \mathfrak{N}$. Wir zeigen: $\mathrm{cl}_{L_r}(A) \subset \mathrm{cl}_L(A)$. $x \in \mathrm{cl}_{L_r}(A)$ impliziert die Existenz eines Mengensystems $\mathfrak{M} \in L_r(A)$ mit $x \in \bigcap\mathfrak{M}$. Wähle einen Raster $\mathfrak{F} \in L(A)$ mit $\mathfrak{F} \ll \mathfrak{M}$, mithin folgt $x \in \bigcap\mathfrak{F}$, so daß die Aussage $x \in \mathrm{cl}_L(A)$ resultiert. Insgesamt folgt nun: $\mathfrak{N} \in L_r(A)$, da L nach Voraussetzung eine Ultraformität ist.

9.11. KOROLLAR:

(i) Die Kategorie *R-Ultraform* ist eine topologische Kategorie;

(ii) Die Kategorie *F-Ultraform* ist eine passend gefaserte topologische Kategorie.

10. Die Kategorien Rastform und Filform

10.1. LEMMA: (X, L) sei ein rasterformer Raum, Y eine Menge und $f : X \to Y$ eine surjektive Abbildung. Für $A \in \mathfrak{P}Y$ setze:

$$L_{\{f\}}(A) = \{\mathfrak{N} \in \mathfrak{P}^2Y \mid \exists\, \mathfrak{M} \in L(f^{-1}[A])\ f\mathfrak{M} \ll \mathfrak{N}\},$$

so ist $L_{\{f\}}$ final bezüglich $((X, L), f, Y)$, d.h. f is eine Quotientenabbildung in *Rastform*.

BEWEIS: Zu (Lf_1) – (Lf_3): trivial.

Zu (Lf_4) : $y \in Y$ impliziert die Existenz eines Elements $x \in X$ mit $f(x) = y$, da f nach Voraussetzung surjektiv ist. Es folgt $\{\{x\}\} \in$ $\in L(\{x\}) \subset L(f^{-1}[\{y\}])$ mit $f\{\{x\}\} \ll \{\{y\}\}$, so daß $\{\{y\}\} \in L_{\{f\}}(\{y\})$ resultiert.

Zu (R): trivial unter Beachtung der Tatsache, daß das Bild eines Rasters wieder ein Raster ist. Evidenterweise ist $f : (X, L) \to (Y, L_{\{f\}})$ eine limesforme Abbildung.

Sei nun (Z,T) ein rasterformer Raum und $g : Y \to Z$ eine Abbildung derart, daß $g \circ f : (X, L) \to (Z,T)$ eine limesforme Abbildung ist, wir zeigen: $g : (Y, L_{\{f\}}) \to (Z,T)$ ist eine limesforme Abbildung. Für $A \in \mathfrak{P}Y$ sei $\mathfrak{N} \in L_{\{f\}}(A)$, so folgt die Existenz eines Mengensystems $\mathfrak{M} \in L(f^{-1}[A])$ mit $f\mathfrak{M} \ll \mathfrak{N}$.

Es folgt nach Voraussetzung $(g \circ f)\mathfrak{M} \in T((g \circ f)[f^{-1}[A]])$. Nun gilt $(g \circ f)[f^{-1}[A]] = g\Big[f[f^{-1}[A]]\Big] \subset g[A]$, woraus $(g \circ f)\mathfrak{M} \in T(g[A])$ folgt, d.h. $g(f\mathfrak{M}) \in T(g[A])$, und wir haben mit ($\mathrm{Lf}_1$) $g\mathfrak{N} \in T(g[A])$.

10.2. BEMERKUNG: Für manche Zwecke ist es günstig zu einer vorgegebenen Kategorie isomorphe Kategorien anzugeben, die es gestatten, diffizile Sachverhalte der Ersteren in Letztere zu übertragen, um hier "übersichtlichere" Lösungen zu finden. Wir tun das im folgenden Fall und erklären:

10.3. DEFINITION: Für eine Menge X bezeichne $R(X)$ die Menge aller Raster über X. Eine Abbildung $T : \mathfrak{P}X \to \mathfrak{P}(R(X))$ heißt *Rasterlimitierung* auf X, und das Paar (X,T) heißt *Rasterlimesraum*, wenn die folgenden Axiome gelten:

(R_1) $A \in \mathfrak{P}X$, $\mathfrak{N}_1 \in T(A)$ und $\mathfrak{N}_1 \ll \mathfrak{N}_2 \in R(X)$ implizieren $\mathfrak{N}_2 \in T(A)$,

(R_2) A_1, $A_2 \in \mathfrak{P}X$ und $A_1 \subset A_2$ implizieren $T(A_1) \subset T(A_2)$,

(R_3) $T(\emptyset) \neq \emptyset$,

(R_4) $x \in X$ impliziert $\{\{x\}\} \in T(\{x\})$.

Für Rasterlimesräume (X, T_1), (Y, T_2) heißt eine Abbildung $f : X \to Y$ *limestreu* genau dann, wenn für alle $A \in \mathfrak{P}X$ die folgende Implikation gilt:

$$\mathfrak{N} \in T_1(A) \text{ impliziert } f\mathfrak{N} \in T_2(f[A]).$$

R-Lim bezeichne die Kategorie der Rasterlimesräume mit den limestreuen Abbildungen.

10.4. SATZ: *Die Kategorien Rastform und R-Lim sind isomorph.*

Beweis: Für einen Rasterlimesraum (X, T) setze für $A \in \mathfrak{P}X$:

$$L_T(A) := \{\mathfrak{N} \in \mathfrak{P}^2 X \mid \exists \mathfrak{R} \in T(A)\ \mathfrak{R} \ll \mathfrak{N}\}.$$

Umgekehrt betrachte für eine Rasterformität L' auf X den folgenden Operator:

$$T'_{L'}(A) := \{\mathfrak{R}' \in R(X) \mid \mathfrak{R}' \in L'(A)\}.$$

10.5. Satz: *Die Produktabbildung zweier Quotientenabbildungen in R–Lim ist wieder eine Quotientenabbildung in R–Lim.*

Beweis: (X, L_X), $(X', L_{X'})$, (Y, L_Y), $(Y', L_{Y'})$ seien Rasterlimesräume und $f : (X, L_X) \to (X' L_{X'})$, $g : (Y, L_Y) \to (Y', L_{Y'})$ seien Quotientenabbildungen in R–Lim, wir zeigen:

$$f \times g : \left(X \times Y, L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}\right) \to \left(X' \times Y', L_{X'_{\{p_{X'}^{-1}\}}} \cap L_{Y'_{\{p_{Y'}^{-1}\}}}\right)$$

ist eine Quotientenabbildung in R–Lim, wobei mit p_X, $p_{X'}$, p_Y, $p_{Y'}$ die jeweiligen Projektionen bezeichnet werden. Trivialerweise ist $f \times g$ surjektiv. Es bleibt zu zeigen, daß $L_{X'_{\{p_{X'}^{-1}\}}} \cap L_{Y'_{\{p_{Y'}^{-1}\}}}$ final bezüglich $\left((X \times Y, L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}\right), f \times g, X' \times Y')$ ist.

Wir zeigen zunächst:

$$f \times g : \left(X \times Y, L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}\right) \to \left(X' \times Y', L_{X'_{\{p_{X'}^{-1}\}}} \cap L_{Y'_{\{p_{Y'}^{-1}\}}}\right)$$

ist eine limestreue Abbildung.

Für $A \in \mathfrak{P}(X \times Y)$ sei $\mathfrak{N} \in L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}(A)$, so folgt $p_X\mathfrak{N} \in L_X(p_X[A])$ und $p_Y\mathfrak{N} \in L_Y(p_Y[A])$. Da f, g nach Voraussetzung limestreu sind, gilt

$$f(p_X\mathfrak{N}) \in L_{X'}\Big(f[p_X[A]]\Big) \text{ und } g(p_Y\mathfrak{N}) \in L_{Y'}\Big(g[p_Y[A]]\Big).$$

Wir zeigen:

$$(1) \qquad p_{X'}(f \times g\mathfrak{N}) \in L_{X'}\Big(p_{X'}[f \times g[A]]\Big),$$

$$(2) \qquad p_{Y'}(f \times g\mathfrak{N}) \in L_{Y'}\Big(p_{Y'}[f \times g[A]]\Big).$$

Dazu genügt es, folgende Beziehungen nachzuweisen:

(i) $\qquad f[p_X[A]] = p_{X'}[f \times g[A]],$
(ii) $\qquad g[p_Y[A]] = p_{Y'}[f \times g[A]],$
(iii) $\qquad f(p_X\mathfrak{N}) \ll p_{X'}(f \times g\mathfrak{N}),$
(iv) $\qquad f(p_Y\mathfrak{N}) \ll p_{Y'}(f \times g\mathfrak{N}).$

Aus Analogiegründen zeigen wir, daß die Aussage in (i) und (ii) gelten.

Zu (i): $f(p_X(z_1, z_2)) = f(z_1) = p_{X'}(f \times g(z_1, z_2))$.

Zu (iii): Sei $F \in f(p_X\mathfrak{N})$, so folgt $F = f[G]$ für ein $G \in p_X\mathfrak{N}$. Wir haben $G = p_X[N]$ für $N \in \mathfrak{N}$, mithin gilt $F = f[p_X[N]]$. Andererseits ist $f \times g[N] \in f \times g\mathfrak{N}$, woraus $p_{X'}[f \times g[N]] \in p_{X'}(f \times g\mathfrak{N})$ resultiert. Mit (i) folgt nun die Behauptung. Insgesamt erhalten wir $L_{X'}\Big(f[p_X[A]]\Big) = L_{X'}\Big(p_{X'}[f \times g[A]]\Big)$, mithin folgt $f(p_X\mathfrak{N}) \in L_{X'}\Big(p_{X'}[f \times g[A]]\Big)$, und schließlich gilt

$$p_{X'}(f \times g\mathfrak{N}) \in L_{X'}\Big(p_{X'}[f \times g[A]]\Big).$$

Sei nun weiter (Z, L) ein Rasterlimesraum und $h : X' \times Y' \to Z$ eine Abbildung derart, daß

$$h \circ (f \times g) : \Big(X \times Y, L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}\Big) \to (Z, L)$$

limestreu ist, wir zeigen:

$$h : \Big(X' \times Y', L_{X_{\{f\}\{p_{X'}^{-1}\}}} \cap L_{Y_{\{g\}\{p_{Y'}^{-1}\}}}\Big) \to (Z, L)$$

ist limestreu. Da $L_{X'}$ final bezüglich $\Big((X, L_X), f, X'\Big)$ und $L_{Y'}$ final bezüglich $\Big((Y, L_{Y'}), g, Y'\Big)$ sind, gilt $L_{X'} \le L_{X_{\{f\}}}$ bzw. $L_{Y'} \le L_{Y\{g\}}$, woraus die Aussagen $L_{X'_{\{p_{X'}^{-1}\}}} \le L_{X_{\{f\}\{p_{X'}^{-1}\}}}$ und $L_{Y'_{\{p_{Y'}^{-1}\}}} \le L_{Y_{\{g\}\{p_{Y'}^{-1}\}}}$ resultieren. Für $A \in \mathfrak{P}Z$ zei

$$\mathfrak{N} \in L_{X_{\{f\}\{p_{X'}^{-1}\}}} \cap L_{Y_{\{g\}\{p_{Y'}^{-1}\}}}(h^{-1}[A]),$$

wir zeigen: $h\mathfrak{N} \in L(A)$ (vergl. auch Lemma 9.7.).

Nach Definition gilt

$$p_{X'}\mathfrak{N} \in L_{X_{\{f\}}}\Big(p_{X'}[h^{-1}[A]]\Big) \quad \text{und} \quad p_{Y'}\mathfrak{N} \in L_{Y_{\{g\}}}\Big(p_{Y'}(h^{-1}[A]]\Big).$$

Wähle $\qquad \mathfrak{M}_X \in L_X\Big(f^{-1}[p_{X'}[h^{-1}[A]]]\Big) \quad$ mit $\quad f\mathfrak{M}_x \ll p_{X'}\mathfrak{N}$

und wähle $\qquad \mathfrak{M}_Y \in L_Y\Big(g^{-1}[p_{Y'}[h^{-1}[A]]]\Big) \quad$ mit $\quad g\mathfrak{M}_Y \ll p_{Y'}\mathfrak{N}.$

Wir zeigen nun:

(i) $$f^{-1}\Big[p_{X'}[h^{-1}[A]]\Big] \subset p_X\Big[(h\circ(f\times g))^{-1}[A]\Big],$$

(ii) $$g^{-1}\Big[p_{Y'}[h^{-1}[A]]\Big] \subset p_Y\Big[(h\circ(f\times g))^{-1}[A]\Big].$$

Zu (i): Sei $z \in f^{-1}\Big[p_{X'}[h^{-1}[A]]\Big]$, so folgt $f(z) \in p_{X'}[h^{-1}[A]]$, d.h. $f(z) = p_{X'}(\overline{z})$ für $\overline{z} \in h^{-1}[A]$. Zeige: $z = p_X(z^*)$ mit $h(f \times g(z^*)) \in A$. Nun gilt $\overline{z} = (x', y')$ mit $x' \in X'$ und $y' \in Y'$ und $h(\overline{z}) = h(x', y') \in A$. Wir haben $f(z) = p_{X'}(\overline{z}) = p_{X'}((x', y')) = x'$ und $g(y) = y'$ für ein geeingnetes $y \in Y$, da g nach Voraussetzung surjektiv ist. Setze $z^* := (z, y)$, so folgt $p_X(z^*) = p_X(z, y) = z$ und $h(f \times g(z^*)) = h(f \times g((z, y))) = h((f(z), g(y))) = h((x', y')) = h(\overline{z}) \in A$, woraus die Behauptung folgt.

Zu (ii): analog; benutze dabei die vorausgesetzte Surjektivität der Abbildung f.

Mit (i) gilt nun $\mathfrak{M}_X \in L_X\Big(p_X[(h\circ(f\times g))^{-1}[A]]\Big)$. Mit (ii) gilt entsprechend $\mathfrak{M}_Y \in L_Y\Big(p_Y[(h\circ(f\times g))^{-1}[A]]\Big)$. Setze $\mathfrak{M} := p_X^{-1}\mathfrak{M}_X \wedge \wedge p_Y^{-1}\mathfrak{M}_Y$ und beachte die Tatsache, daß die Urbilder von Rastern wieder Raster sind und für Raster $\mathfrak{R}_1$, $\mathfrak{R}_2$ gilt, daß $\mathfrak{R}_1 \wedge \mathfrak{R}_2$ ein Raster ist. Wir haben nun

$$\mathfrak{M} \in L_{X_{\{p_X^{-1}\}}} \cap L_{Y_{\{p_Y^{-1}\}}}((h\circ(f\times g))^{-1}[A]);$$

denn es gelten die Aussagen:

(iii) $\mathfrak{M}_X \ll p_X\mathfrak{M}$ und

(iv) $\mathfrak{M}_Y \ll p_Y\mathfrak{M}$.

Zu (iii): Sei $M_X \in \mathfrak{M}_X$: da $\mathfrak{M}_Y \neq \emptyset$ wähle $M_Y \in \mathfrak{M}_Y$, mithin gilt $p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y] \in \mathfrak{M}$, woraus $p_X\Big[p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y]\Big] \in p_X\mathfrak{M}$ folgt. Wir haben weiter

$$M_X \supset p_X\Big[p_X^{-1}[M_X]\Big] \cap p_X\Big[p_Y^{-1}[M_Y]\Big] \supset p_X\Big[p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y]\Big],$$

was die Behauptung beweist.

Zu (iv): analog! Da $h\circ(f\times g)$ nach Voraussetzung limestreu ist, folgt $h\circ(f\times g)\mathfrak{M} \in L(A)$.

Es bleibt zu zeigen: $h\circ(f\times g)\mathfrak{M} \ll h\mathfrak{N}$. Sei dazu $F \in h\circ(f\times g)\mathfrak{M}$, so ist $F = h\circ(f\times g)[M]$ für ein $M \in \mathfrak{M}$. Es folgt $M = p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y]$ für $M_X \in \mathfrak{M}_X$ und $M_Y \in \mathfrak{M}_Y$. Damit hat F die folgende Darstellung:

$F = h \circ (f \times g)\left[p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y]\right]$, und es gilt $f[M_X] \supset p_{X'}[N_1]$ für $N_1 \in \mathfrak{N}$ bzw. $g[M_Y] \supset p_{Y'}[N_2]$ für $N_2 \in \mathfrak{N}$.

Wir zeigen:

(v) $$(f \times g)\left[p_X^{-1}[M_X] \cap p_Y^{-1}[M_Y]\right] \supset p_{X'}^{-1}\left[f[M_X]\right] \cap p_{Y'}^{-1}\left[g[M_Y]\right];$$

denn dann gilt

$$p_{X'}^{-1}\left[f[M_X]\right] \supset p_{X'}^{-1}\left[p_{X'}[N_1]\right] \supset N_1 \text{ und}$$
$$p_{Y'}^{-1}\left[g[M_Y]\right] \supset p_{Y'}^{-1}\left[p_{Y'}[N_2]\right] \supset N_2,$$

woraus die Aussage

$$p_{X'}^{-1}\left[f[M_X]\right] \cap p_{Y'}^{-1}\left[g[M_Y]\right] \supset N_1 \cap N_2 \supset N_3 \in \mathfrak{N}$$

folgt (weil $\mathfrak{N}$ ein Raster ist). Damit gilt $F \supset h[N_3]$, was die Behauptung beweist.

Zu (v): Sei $z \in p_{X'}^{-1}\left[f[M_X]\right] \cap p_{Y'}^{-1}\left[g[M_Y]\right]$, so ist $z = (x', y')$ mit $x' \in X'$, $y' \in Y'$, und es gilt: $p_{X'}(z) = p_{X'}((x', y')) = x' \in f[M_X]$ bzw. $p_{Y'}(z) = p_{Y'}((x', y')) = y' \in g[M_Y]$. Wir haben $x' = f(x)$ für $x \in M_X$ und $y' = g(y)$ für $y \in M_Y$, so daß $z = (f(x), g(y)) = f \times g((x, y))$ ist mit $p_X((x, y)) \in M_X$ und $p_Y((x, y)) \in M_Y$, was zu zeigen war.

10.6. DEFINITION: Für einen Rasterlimesraum (X, T) heißt T *Filterlimitierung* auf X, und das Paar (X, T) heißt *Filterlimesraum*, wenn zusätzlich das Axiom (R_5) gilt, wobei:

(R_5) $\quad \mathfrak{R} \in T(\emptyset)$ impliziert $\emptyset \in \mathfrak{R}$.

F-Lim bezeichne die Kategorie der Filterlimesräume mit den limestreuen Abbildungen.

10.7. SATZ: *Die Kategorien Filform und F-Lim sind isomorph.*

BEWEIS: Man vergleiche den Beweis zu Satz 10.4. .

10.8. BEMERKUNG: Mit Bemerkung 9.10. und vorangehendem Satz ist *F-Lim* eine passend gefaserte topologische Kategorie, welche unter Beachtung des Satzes 10.5. die Eigenschaft hat, daß die Produktabbildung zweier Quotientenabbildungen in *F-Lim* wieder eine Quotientenabbildung in *F-Lim* ist. Trivialerweise erfüllt $L_{\{f\}}$ das Axiom (R_5), sofern L diese Eigenschaft hat (vergl. auch Lemma 10.1.). Damit erfüllt *F-Lim* die Voraussetzungen einer prekonvenienten Kategorie im Sinne von L. D. Nel.

*F-Lim** ist dann eine kartesisch abgeschlossene passend gefaserte topologische Kategorie, in der *F-Lim* bireflektiv eingebettet ist. Unter

Beachtung der Bemerkung 6.6. und der Sätze 6.8., 7.7. und 10.7. enthält die kartesisch abgeschlossene Kategorie *F–Lim** als Unterkategorien insbesondere die Kategorie *Top* der topologischen Räume und stetigen Abbildungen, sowie die Kategorie *LE–PROX* der Leader-Proximitätsräume und p-stetigen Abbildungen.

11. Kompaktheitsbegriffe

11.1. DEFINITIONEN: Für einen limesformen Raum (X, L) sei $\mathfrak{F}$ ein Präfilter (vergl. Bemerkung 5.8.) über X, $x \in X$ und A, $U \in \mathfrak{P}X$, so heißt

(i) x *L–Adhärenzpunkt* von $\mathfrak{F}$ genau dann, wenn es ein Mengensystem $\mathfrak{N} \in L(\{x\})$ gibt mit $\operatorname{stack}\mathfrak{N} \subset \sec\mathfrak{F}$,

(ii) x *L–Limes* von $\mathfrak{F}$ genau dann, wenn es ein Mengensystem $\mathfrak{N} \in$ $\in L(\{x\})$ gibt mit $\mathfrak{N} \ll \mathfrak{F}$; wir sagen dann auch, daß $\mathfrak{F}$ gegen x *L–konvergiert* oder $\mathfrak{F}$ gegen x *L–konvergent* ist,

(iii) U *L–Umgebung* von A genau dann, wenn die Aussage $U \in$ $\in \operatorname{stack} VL(A)$ gilt. Für $x \in X$ heißt $\mathfrak{U}_L(x) := \{U \subset X \mid U$ ist L–Umgebung von $\{x\}$ $\}$ *L–Umgebungssystem* von x.

11.2. BEMERKUNG: Man beachte insbesondere, daß jeder L–Limes eines Präfilters einen L–Adhärenzpunkt dieses Mengenssystems darstellt. Für Ultrafilter sind dagegen beide Begriffe äquivalent.

11.3. HINWEIS: Für einen rasterformen Raum (X, L) sei für einen Präfilter $\mathfrak{F}$ ein Punkt $x \in X$ L–Adhärenzpunkt von $\mathfrak{F}$, so gibt es einen Präfilter $\mathfrak{F}' \in L(\{x\})$ mit $\operatorname{stack}\mathfrak{F}' \subset \sec\mathfrak{F}$.

11.4. LEMMA: Für einen rasterformen Raum (X, L) sei $\mathfrak{F}$ ein Präfilter über X. Für einen Punkt $x \in X$ sind dann die folgenden Aussagen äquivalent:

(i) x ist L–Adhärenzpunkt von $\mathfrak{F}$;

(ii) Es existiert ein Präfilter $\mathfrak{R}$ über X mit $\mathfrak{F} \ll \mathfrak{R}$ und $\mathfrak{R}$ ist L–konvergent gegen x.

BEWEIS: (ii) impliziert (i): Wähle nach Voraussetzung einen Präfilter $\mathfrak{R}$ über X, der gegen x L–konvergiert, so daß $\mathfrak{F} \ll \mathfrak{R}$ gilt. Wir haben dann $\operatorname{stack}\mathfrak{N} \subset \operatorname{stack}\mathfrak{R} \subset \sec\mathfrak{R} \subset \sec\mathfrak{F}$, wobei $\mathfrak{N}$ ein $\{x\}$–limitiertes Mengensystem ist.

(i) impliziert (ii): Nach Voraussetzung existiert ein $\{x\}$–limitiertes Mengensystem $\mathfrak{N}$ derart, daß $\operatorname{stack}\mathfrak{N} \subset \sec\mathfrak{F}$ gilt. Mit Bemerkung 11.3. wähle einen Präfilter $\mathfrak{R} \in L(\{x\})$ mit $\mathfrak{R} \ll \mathfrak{N}$, mithin folgt

$\mathfrak{R} \subset \sec \mathfrak{F}$. Setze $\mathfrak{F}' := \mathfrak{R} \wedge \mathfrak{F}$, so ist $\mathfrak{F}'$ ein Präfilter mit den gewünschten Eigenschaften.

11.5. LEMMA: Für einen kernformen Raum $(X.L)$ und einen Präfilter $\mathfrak{F}$ über X sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:

(i) $\mathfrak{F}$ ist gegen x L–konvergent;

(ii) $\mathfrak{U}_L(x) \subset \operatorname{stack} \mathfrak{F}$.

BEWEIS: (ii) impliziert (i): Da L das Axiom (K) erfüllt, folgt insbesondere $VL(\{x\}) \in L(\{x\})$, und es gilt $VL(\{x\}) \subset \mathfrak{U}_L(x)$, woraus unter Anwendung der Voraussetzung die Behauptung folgt.

(i) impliziert (ii): Für $U \in \mathfrak{U}_L(x)$ wähle ein $\{x\}$–limitiertes Mengensystem $\mathfrak{N}$ mit $\mathfrak{N} \ll \mathfrak{F}$, so folgt $U \in \operatorname{stack} \mathfrak{N} \subset \operatorname{stack} \mathfrak{F}$.

11.6. LEMMA: Für einen kernformen Raum (X, L) und einen Präfilter $\mathfrak{F}$ über X sind für einen Punkt $x \in X$ die folgenden Aussagen äquivalent:

(i) x ist L–Adhärenzpunkt von $\mathfrak{F}$;

(ii) $\mathfrak{U}_L(x) \subset \sec \mathfrak{F}$.

BEWEIS: analog!

11.7. DEFINITION: Für einen limesformen Raum (X, L) heißt $\mathfrak{C} \subset \subset \mathfrak{P}X$ *L–Überdeckung* von X, wenn für jedes $x \in X$ und jedes $\{x\}$–limitierte Mengensystem $\mathfrak{N}$ es eine Menge $C \in \mathfrak{C}$ so gibt, daß $C \in \in \operatorname{stack} \mathfrak{N}$ gilt.

11.8. BEMERKUNG: Für einen limesformen Raum (X, L) ist dann offensichtlich jede L–Überdeckung von X eine Überdeckung dieser Menge unter Anwendung des Axioms (Lf_4).

Unter Beachtung von Beispiel 2.2. (v) ist jede $\mathfrak{X}$–offene Überdeckung von X eine $L_{\mathfrak{X}}$–Überdeckung dieser Menge. Sei dazu $x \in X$ und $\mathfrak{N} \in L_{\mathfrak{X}}(\{x\})$, so folgt $\mathfrak{X}(\{x\}) \ll \mathfrak{N}$, und es existiert eine Menge $C \in \mathscr{C} \cap \mathfrak{X}$ mit $x \in C$. Wir haben dann $C \in \mathfrak{X}(\{x\})$, woraus $C \in \operatorname{stack} \mathfrak{N}$ resultiert.

11.9. DEFINITIONEN: (X, L) sei ein limesformer Raum. L heißt

(i) ***kompakt*** genau dann, wenn für jede L–Überdeckung $\mathfrak{C} \subset \mathfrak{P}X$ es ein endliches Mengensystem $\mathfrak{C}^* \subset \mathfrak{C}$ so gibt, daß $\mathfrak{C}^*$ L–Überdeckung von X ist;

(ii) ***epikompakt*** genau dann, wenn für jede L–Überdeckung $\mathfrak{C} \subset \mathfrak{P}X$ es ein endliches Mengensystem $\mathfrak{C}^* \subset \mathfrak{C}$ so gibt, daß $\bigcup \mathfrak{C}^* = X$ ist.

11.10. BEMERKUNG: Offensichtlich ist mit Bemerkung 11.8. jede kompakte Limesformität epikompakt. Im nun Folgenden werden wir zeigen, daß im Fall von Topoformitäten beide Begriffe äquivalent sind.

Schließlich weisen wir darauf hin, daß von kompakten (epikompakten) Räumen gesprochen wird, sofern die entsprechende Limesformität auf ihrer Trägermenge diese Eigenschaft hat.

11.11. SATZ: *Für einen limesformen Raum (X, L) sind die folgenden Aussagen äquivalent:*

(i) L is epikompakt;

(ii) Jeder Präfilter über X hat einen L–Adhärenzpunkt;

(iii) Jeder Ultrafilter ist L-konvergent.

BEWEIS: (i) impliziert (iii): Angenommen es gäbe einen nicht L–konvergenten Ultrafilter über X, d.h. es existiert $\mathfrak{U}$ Ultrafilter, so daß für alle $x \in X$ und für alle $\{x\}$–limitierten Mengensysteme $\mathfrak{N}$ die Aussage $\text{stack}\,\mathfrak{N} \not\subset \mathfrak{U}$ gilt.

Setze $\mathfrak{C} := \{X \setminus F \mid F \in \mathfrak{U}\}$. Für $x \in X$ und $\mathfrak{N} \in L(\{x\})$ wähle $F \in \text{stack}\,\mathfrak{N}$ mit $F \notin \mathfrak{U}$. Es folgt $X \setminus F \in \mathfrak{U}$, da $\mathfrak{U}$ Ultrafilter ist, und es existiert ein $N \in \mathfrak{N}$ mit $F \supset N$. Wir haben $F \in \mathfrak{C}$, woraus die Aussage $\mathfrak{C}$ ist L–Überdeckung von X resultiert. Sei nun $\mathfrak{C}^* \subset \mathfrak{C}$ endlich, so gibt es ein $n \in \mathbb{N}$ mit $\mathfrak{C}^* := \{C_1, \ldots, C_n\}$, mithin gilt $C_i = X \setminus F_i$ für $F_i \in \mathfrak{U}$ mit $i \in \{1, \ldots, n\}$. Wir haben $\bigcup \mathfrak{C}^* = \bigcup_{i=1}^{n} C_i = \bigcup_{i=1}^{n} X \setminus F_i = X \setminus \bigcap_{i=1}^{n} F_i$ mit $\bigcap_{i=1}^{n} F_i \neq \emptyset$. Wähle $x \in \bigcap_{i=1}^{n} F_i$, so gilt $x \notin \bigcup \mathfrak{C}^*$, woraus folgt, daß L nicht epikompakt ist.

(iii) impliziert (i): Angenommen (X, L) ist nicht epikompakt, so wähle eine L–Überdeckung $\mathfrak{C} \subset \mathfrak{P}X$, so daß für alle $\mathfrak{C}^* \subset \mathfrak{C}$ endlich die Aussage $\bigcup \mathfrak{C}^* \neq X$ gilt.

Setze $\mathfrak{R} := \{X \setminus \bigcup \mathfrak{C}^* \mid \mathfrak{C}^* \subset \mathfrak{C} \text{ endlich}\}$, mithin ist $\mathfrak{R}$ ein Präfilter über X. Wir zeigen das Axiom (R_2): Seien $F_1, F_2 \in \mathfrak{R}$, so gilt $F_1 = = X \setminus \bigcup \mathfrak{C}_1^*$ und $F_2 = X \setminus \bigcup \mathfrak{C}_2^*$ für $\mathfrak{C}_1^*, \mathfrak{C}_2^* \subset \mathfrak{C}$ endlich. Setze $\mathfrak{C}^* := \mathfrak{C}_1^* \cup \mathfrak{C}_2^*$, so ist $\mathfrak{C}^* \subset \mathfrak{C}$ endlich und nach Voraussetzung gilt $\bigcup \mathfrak{C}^* \neq X$. Setze $F := X \setminus \bigcup \mathfrak{C}^*$, so ist $F \in \mathfrak{R}$ mit $F_1 \cap F_2 = (X \setminus \bigcup \mathfrak{C}_1^*) \cap (X \setminus \bigcup \mathfrak{C}_2^*) = = X \setminus (\bigcup \mathfrak{C}_1^* \bigcup \bigcup \mathfrak{C}_2^*) \supset X \setminus \bigcup \mathfrak{C}^* = F$. Wähle einen Ultrafilter $\mathfrak{U}$ mit $\mathfrak{R} \subset \mathfrak{U}$. Für $x \in X$ und $\mathfrak{N} \in L(\{x\})$ gilbt es nach Voraussetzung Mengen $C \in \mathfrak{C}$ und $N \in \mathfrak{N}$ derart, daß die Inklusion $N \subset C$ gilt.

Setze $\mathfrak{C}^* := \{C\}$, so folgt $X \setminus C \in \mathfrak{U}$, woraus die Aussagen $C \notin \mathfrak{U}$ und $C \in \text{stack}\,\mathfrak{N}$ resultieren.

(ii) impliziert (iii): Sei $\mathfrak{U}$ ein Ultrafilter über X und $x \in X$ ein L–Adhärenzpunkt desselben, so daß ein $\{x\}$–limitiertes Mengensystem

$\mathfrak{N}$ existiert, mit stack $\mathfrak{N} \subset \sec \mathfrak{U}$. Wir haben $\sec \mathfrak{U} \subset \mathfrak{U}$, woraus mit $\mathfrak{N} \ll \mathfrak{U}$ die Behauptung folgt.

(iii) impliziert (ii): Umgekehrt sei $\mathfrak{F}$ ein Präfilter über X. Wähle einen feineren Ultrafilter $\mathfrak{U}$ und ein $\{x\}$-limitiertes Mengensystem $\mathfrak{N}$ mit $\mathfrak{N} \ll \mathfrak{U}$ für ein $x \in X$. Wir haben stack $\mathfrak{N} \subset \mathfrak{U} \subset \sec \mathfrak{U} \subset \sec \mathfrak{F}$, so daß x ein L-Adhärenzpunkt von $\mathfrak{F}$ ist.

11.12. SATZ: *Für einen kernformen Raum (X, L) sind die folgenden Aussagen äquivalent:*

(i) *L is epikompakt;*

(ii) *Für $x \in X$ sei U_x eine L-Umgebung von x, so existiert eine endliche Menge $E \subset X$ mit $X = \cup\{U_{x_i} \,|\, i \in E\}$;*

(iii) *Für jeden Ultrafilter $\mathfrak{F}$ gibt es ein Element $x \in X$ mit $\mathfrak{U}_L(x) \subset \mathfrak{F}$;*

(iv) *Für jeden Präfilter $\mathfrak{R}$ gibt es ein Element $x \in X$ mit $\mathfrak{U}_L(x) \subset$ $\subset \sec \mathfrak{R}$.*

BEWEIS: Unter Beachtung des vorangehenden Satzes und der Lemmata 11.5. bzw. 11.6. genügt es, die Äquivalenz der Aussagen (ii) und (iv) zu beweisen.

(iv) impliziert (ii): Sei U_x eine L-Umgebung von x für jedes $x \in X$. Angenommen, für jede endliche Teilmenge $E \subset X$ gilt $X \not\subset \cup\{U_x | x \in E\}$. Setze $\mathfrak{R} := \Big\{X \setminus \cup\{U_x \,|\, x \in E\} \,|\, E \subset X \text{ endlich}\Big\}$ so ist $\mathfrak{R}$ ein Präfilter über X. Sei $z \in X$ L-Adhärenzpunkt von $\mathfrak{R}$, so folgt $U_z \in \sec \mathfrak{R}$, d.h. $U_z \cap (X \setminus U_z) \neq \emptyset$, was zu einem Widerspruch führt!

(ii) impliziert (iv): Angenommen es gibt einen Präfilter $\mathfrak{R}$ über X, der keinen L-Adhärenzpunkt besitzt. Für $x \in X$ sei $F_x \in \mathfrak{U}_L(x)$ mit $X \setminus F_x \supset R_x \in \mathfrak{R}$, so folgt, daß $X \setminus R_x$ eine L-Umgebung von x ist. Nach Voraussetzung existiert eine endliche Menge E mit $X =$ $= \cup\{X \setminus R_{x_i} \,|\, i \in E\}$. Nun gilt $\cup\{X \setminus R_{x_i} \,|\, i \in E\} = X \setminus \cap\{R_{x_i} \,|\, i \in E\}$ mit $\cap\{R_{x_i} \,|\, i \in E\} \supset R \in \mathfrak{R}$, was zum Widerspruch führt!

11.13. DEFINITIONEN: Für einen prätopoformen Raum (X, L) heißt für $x \in X$ eine Teilmenge $N \subset X$ *L-Nachbarschaft* von x, wenn eine offene Menge $O \in \mathfrak{D}_L$ so existiert, daß $x \in O \subset N$ gilt (vergl. auch Lemma 7.5.). Setze für $A \subset X$,

$$\overline{A}^L := \cap\{B \subset X \,|\, X \setminus B \in \mathfrak{D}_L \wedge B \supset A\},$$

so heißt $\overline{A}^L$ die *L-Nachbarschaftshülle* von A. Setze für $x \in X$:

$$\mathfrak{N}^L(x) := \{N \subset X \,|\, N \text{ ist } L\text{-Nachbarschaft von } x\},$$

so heißt $\mathfrak{N}^L(x)$ *L-Nachbarschaftssystem* von x.

11.14. Bemerkung: Man beachte, daß für einen prätopoformen Raum (X, L) und für jedes $x \in X$ $\mathfrak{N}^L(x)$ ein Präfilter über X ist.

11.15. Lemma: Für einen prätopoformen Raum (X, L) sind für $A \in \mathfrak{P}X$ die folgenden Aussagen äquivalent:

(i) $x \in \overline{A}^L$;

(ii) $A \in \sec \mathfrak{N}^L(x)$.

Beweis: (i) impliziert (ii): Für $A \notin \sec \mathfrak{N}^L(x)$ existiert eine L–Nachbarschaft N von x mit $N \cap A = \emptyset$. Wähle eine offene Menge $O \in \mathfrak{O}_L$ mit $x \in O \subset N$. Es folgt $A \subset X \setminus O$, woraus $x \notin \overline{A}^L$ resultiert.

(ii) impliziert (i): $x \notin \overline{A}^L$ impliziert die Existenz einer Menge $B \subset X$ mit $B \supset A$, $X \setminus B \in \mathfrak{O}_L$ und $x \notin B$. Setze $N := X \setminus B$, so ist N eine L–Nachbarschaft von x mit $N \cap A = \emptyset$.

11.16. Lemma: Für einen prätopoformen Raum (X, L) sind für $A \subset X$ die folgenden Aussagen äquivalent:

(i) $A = \overline{A}^L$;

(ii) $X \setminus A \in \mathfrak{O}_L$.

Beweis: (i) impliziert (ii): Sei $\mathfrak{N} \in L(X \setminus A)$, wir zeigen: $X \setminus A \in$ $\in \operatorname{stack} \mathfrak{N}$. Mit Voraussetzung gilt nach (Lf_2) $\mathfrak{N} \in L(X \setminus \overline{A}^L)$, und wir haben $\mathfrak{O}_L(X \setminus \overline{A}^L) \ll \mathfrak{N}$. Es folgt $X \setminus \overline{A}^L \in \mathfrak{O}_L(X \setminus \overline{A}^L)$, denn

$$X \setminus \overline{A}^L = X \setminus \bigcap \{B \subset X \mid X \setminus B \in \mathfrak{O}_L \wedge B \supset A\} = \bigcup \{X \setminus B \mid B \supset A \wedge X \setminus B \in \mathfrak{O}_L\},$$

und mit Lemma 7.5. gilt dann $X \setminus \overline{A}^L \in \mathfrak{O}_L$. Sofort folgt nun $X \setminus A \in$ $\in \operatorname{stack} \mathfrak{N}$.

(ii) impliziert (i): Aufgrund der Definition von $\overline{A}^L$ genügt zu zeigen: $\overline{A}^L \subset A$. $x \notin A$ impliziert $x \in X \setminus A$. Nach Voraussetzung gilt $X \setminus A \in \mathfrak{O}_L$, woraus $x \notin \overline{A}^L$ resultiert.

11.17. Definition: Für einen prätopoformen Raum (X, L) und einer Teilmenge $A \subset X$ heißt A *L–abgeschlossen* in X genau dann, wenn $A = \overline{A}^L$ gilt.

11.18. Bemerkung: Unter Beachtung von Lemma 11.16. gilt nun, daß für jede Teilmenge $A \subset X$ die L–Nachbarschaftshülle von A selbst in X L–abgeschlossen ist.

11.19. Lemma: Für einen topoformen Raum (X, L) gilt für jedes $x \in X$: $\mathfrak{U}_L(x) = \mathfrak{N}^L(x)$.

BEWEIS: Zu "$\subset$": Sei $U \in \mathfrak{U}_L(x)$, so folgt $U \in \operatorname{stack} VL(\{x\})$. Da L insbesondere das Axiom (T) erfüllt, gilt $U \supset F \in \mathfrak{D}_L(\{x\})$. Wähle $O \in \mathfrak{D}_L$ mit $x \in O \subset F \subset U$, so daß $U \in \mathfrak{N}^L(x)$ resultiert.

Zu "$\supset$": Umgekehrt sei $N \in \mathfrak{N}^L(x)$, so gibt es eine offene Menge $O \in \mathfrak{D}_L$ mit $x \in O \subset N$. Für $\mathfrak{R} \in L(\{x\})$ folgt mit (Lf_2) die Aussage $\mathfrak{R} \in L(O)$. Da $O \in \mathfrak{D}_L$ gilt, folgt $O \in \operatorname{stack}\mathfrak{R}$, woraus schließlich $N \in \operatorname{stack}\mathfrak{R}$ resultiert, so daß insgesamt $N \in \operatorname{stack} VL(\{x\})$ gilt.

11.20. LEMMA: Für einen topoformen Raum (X, L) sind für einen Präfilter $\mathfrak{F}$ die folgenden Aussagen äquivalent:

(i) x ist L–Adhärenzpunkt von $\mathfrak{F}$;

(ii) $x \in \bigcap\limits_{F \in \mathfrak{F}} \overline{F}^L$.

BEWEIS: (ii) impliziert (i): Setze $\mathfrak{R} := \mathfrak{F} \wedge \mathfrak{N}^L(x)$, so ist $\mathfrak{R}$ ein Präfilter mit $\mathfrak{F} \ll \mathfrak{R}$ (vergl. auch Lemma 11.15.). Mit Lemma 11.19. folgt dann $\mathfrak{U}_L(x) \subset \operatorname{stack}\mathfrak{R}$, d.h. mit Lemma 11.5. ist $\mathfrak{R}$ L–konvergent gegen x, woraus mit Lemma 11.4. die Behauptung folgt.

(i) impliziert (ii): Sei $F \in \mathfrak{F}$, wir zeigen unter Anwendung von Lemma 11.15. $F \in \sec\mathfrak{N}^L(x)$. Mit Lemma 11.6. gilt nun nach Voraussetzung $\mathfrak{U}_L(x) \subset \sec\mathfrak{F}$, woraus mit Lemma 11.19. $\mathfrak{N}^L(x) \subset \sec\mathfrak{F}$ resultiert. Wir haben dann $\mathfrak{F} \subset \sec\mathfrak{N}^L(x)$, so daß die Behauptung bewiesen ist.

11.21. SATZ: Für einen topoformen Raum (X, L) sind die folgenden Aussagen äquivalent:

(i) L is kompakt;

(ii) In jeder Familie $(A_i)_{i \in I}$ L–abgeschlossener Teilmengen von X mit $\bigcap\limits_{i \in I} A_i = \emptyset$ existieren endlich viele $A_{i_1}, \ldots, A_{i_n}$ derart, daß $\bigcap\limits_{k=1}^{n} A_{i_k} = \emptyset$ ist.

BEWEIS: (ii) impliziert (i): Sei $\mathfrak{C} \subset \mathfrak{P}X$ eine L–Überdeckung von X. Für $x \in X$ wähle $C_x \in \mathfrak{C}$, $F_x \in \mathfrak{D}_L(\{x\})$ mit $C_x \supset F_x$ (unter Beachtung der Axiome (K) und (T) bzw. der Definition einer L–Überdeckung von X).

Wähle $O_x \in \mathfrak{D}_L$ mit $x \in O_x \subset F_x$ und setze: $\mathfrak{U} := \{X \setminus O_x \mid x \in X\}$. Für $A \in \mathfrak{U}$ gilt dann mit Lemma 11.16. bzw. Definition 11.17., daß A L–abgeschlossen in X ist. Wir haben $\bigcap\mathfrak{U} = \emptyset$, andernfalls gibt es ein $x \in X$ mit $x \in \bigcap\limits_{y \in X} X \setminus O_y$, d.h. $x \in X \setminus \bigcup\limits_{y \in X} O_y$, woraus sich ein Widerspruch ergibt. Nach Voraussetzung existieren $x_1, \ldots, x_n \in X$ mit $\bigcap\limits_{i=1}^{n} X \setminus O_{x_i} = \emptyset$. Setze $\mathfrak{C}^* := \{C_{x_i} \mid i \in \{1, \ldots, n\}\}$, so ist $\mathfrak{C}^* \subset \mathfrak{C}$

endlich, und es bleibt zu zeigen, daß $\mathfrak{C}^*$ eine L–Überdeckung von X ist. Für $x \in X$ sei $\mathfrak{N} \in L(\{x\})$, so gilt $\mathfrak{D}_L(\{x\}) \ll \mathfrak{N}$ und $x \notin \bigcap_{i=1}^{n} X \setminus O_{x_i}$. Mithin existiert ein $j \in \{1,\ldots,n\}$, so daß die Aussage $x \notin X \setminus O_{x_j}$ erfüllt ist. Wir haben $x \in O_{x_j} \subset F_{x_j}$ und $x \in O_x$. Da $O_{x_j} \cap O_x \in \mathfrak{D}_L$ (vergl.Lemma 7.5.) gilt, folgt $C_{x_j} \supset F_{x_j} \supset F_{x_j} \cap F_x \supset O_{x_j} \cap O_x$, woraus die Aussage $C_{x_j} \in \mathfrak{D}_L(\{x\})$ resultiert. Es gilt nun $C_{x_j} \in \mathfrak{C}^*$ mit $C_{x_j} \in \operatorname{stack}\mathfrak{N}$, mithin ergibt sich die Behauptung.

(i) impliziert (ii): Sei $(A_i)_{i\in I}$ eine Familie L–abgeschlossener Teilmengen von X mit $\bigcap_{i\in I} A_i = \emptyset$. Setze $\mathfrak{C} := \{X \setminus A_i \mid i \in I\}$, so gilt, daß $\mathfrak{C}$ eine L–Überdeckung von X ist. Für $x \in X$ sei dazu $\mathfrak{N} \in L(\{x\})$, so folgt $x \notin \bigcap_{i\in I} A_i$. Wähle $j \in I$ mit $x \in X \setminus A_j$, mithin ist $\mathfrak{N} \in L(X \setminus A_j)$ unter Beachtung des Axioms (Lf_2). Wir haben $\mathfrak{D}_L(X \setminus A_j) \ll \mathfrak{N}$ mit $X \setminus A_j \in \mathfrak{D}_L(X \setminus A_j)$ unter Berücksichtigung von Lemma 11.16. bzw. Definition 11.17., woraus $X \setminus A_j \in \operatorname{stack}\mathfrak{N}$ resultiert. Nach Voraussetzung existiert ein $\mathfrak{C}^* \subset \mathfrak{C}$ endlich mit $\mathfrak{C}^*$ ist eine L–Überdeckung von X. Sei also $\mathfrak{C}^* = \{X \setminus A_{i_k} \mid k \in \{1,\ldots,n\}\}$ für ein $n \in \mathbb{N}$, und angenommen es gibt ein $x \in X$ mit $x \in \bigcap_{k=1}^{n} A_{i_k}$. Da $\mathfrak{C}^*$ insbesondere eine Überdeckung von X ist, wähle $j \in \{1,\ldots,n\}$ mit $x \in X \setminus A_{i_j}$, woraus sich ein Widerspruch ergibt, mithin gilt $\bigcap_{k=1}^{n} A_{i_k} = \emptyset$.

11.22. SATZ: *Für einen topoformen Raum (X, L) sind die folgenden Aussagen äquivalent:*

(i) *L ist epikompakt;*

(ii) *L ist kompakt;*

(iii) *In jeder Familie $(A_i)_{i\in I}$ L–abgeschlossener Teilemengen von X mit $\cap A_i = \emptyset$ existieren endlich viele $A_{i_1}, \ldots, A_{i_n}$ derart, daß $\bigcap_{k=1}^{n} A_{i_k} = \emptyset$ ist.*

(iv) *Für jeden Präfilter $\mathfrak{F}$ über X gibt es ein $x \in X$ mit $x \in \bigcap_{F\in\mathfrak{F}} \overline{F}^L$;*

(v) *Für $x \in X$ sei N_x eine L–Nachbarschaft von x, so existiert eine endliche Menge E mit $X = \bigcup\{N_{x_i} \mid i \in E\}$;*

(vi) *Für jeden Ultrafilter $\mathfrak{U}$ gibt es ein $x \in X$ mit $\mathfrak{N}^L(x) \subset \mathfrak{U}$.*

BEWEIS: Aufgrund der vorangehenden Ergebnisse genügt es, die Implikation "(iv) folgt (iii)" nachzuweisen.

Ist (iii) nicht erfüllt, so existiert eine Familie $(A_i)_{i\in I}$ L–abgeschlossener Mengen mit $\bigcap_{i\in I} A_i = \emptyset$ und für alle $I' \subset I$ endlich gilt $\bigcap_{i'\in I'} A_{i'} \neq \emptyset$. Setze $\mathfrak{F} := \left\{ \bigcap_{i'\in I'} A_{i'} \mid I' \subset I \text{ endlich} \right\}$, so ist $\mathfrak{F}$ ein Präfilter über X mit $\bigcap_{F\in\mathfrak{F}} \overline{F}^L = \bigcap_{i\in I} \overline{A_i}^L = \bigcap_{i\in I} A_i = \emptyset$.

11.23. LEMMA: (X, L), (Y, T) seien limesforme Räume. Für eine surjektive limesforme Abbildung $f : X \to Y$ sei L epikompakt, so gilt T ist epikompakt.

BEWEIS: Sei $\mathfrak{C} \subset \mathfrak{P}Y$ eine T-Überdeckung von Y. Setze $\mathfrak{C}^* :=$ $:= \{f^{-1}[C] | C \in \mathfrak{C}\}$, wir zeigen: $\mathfrak{C}^*$ ist eine L–Überdeckung von X. Sei dazu $x \in X$ und $\mathfrak{N} \in L(\{x\})$, so gilt nach Voraussetzung $f\mathfrak{N} \in$ $\in L\Big(\{f(x)\}\Big)$. Da $\mathfrak{C}$ eine T–Überdeckung von Y ist, existiert eine Menge $C \in \mathfrak{C}$ mit $C \in \operatorname{stack} f\mathfrak{N}$, d.h. $C \supset f[F]$ für ein $F \in \mathfrak{N}$. Es folgt $f^{-1}[C] \supset f^{-1}[f[F]] \supset F$, so daß die Aussage $f^{-1}[C] \in \operatorname{stack}\mathfrak{N}$ resultiert. Da L epikompakt ist, existieren endlich viele $C_1, \ldots, C_n \in \mathfrak{C}$ mit $\bigcup_{i=1}^{n} f^{-1}[C_i] = X$. Wir haben $f[\bigcup_{i=1}^{n} f^{-1}[C]] = f[X] = Y$, weil f nach Voraussetzung surjektiv ist, mithin gilt $\bigcup_{i=1}^{n} C_i = \bigcup_{i=1}^{n} f[f^{-1}[C_i]] = Y$, woraus die Behauptung resultiert.

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TOTAL BOUNDEDNESS AND COMPACTNESS FOR FILTER PAIRS

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Quasi–uniform Completeness

Among the various ideas of completeness which have been given for quasi–uniform spaces, we look below at a traditional one and one introduced in December, 1987 by D. DOITCHINOV at the conference "Convergence Structures in Topology and Analysis" at Oberwolfach (see [Do1], [Do2]). Á. CSÁSZÁR there asked whether in this new theory, as in the established one, the topology associated with a quasi–uniform space would be compact iff the space were complete and precompact. This author feels that the newer completeness notion is quite natural, and indeed knew the answer to CSÁSZÁR's question; this answer is discussed below.

Henceforth $\mathcal{F}$, $\mathcal{G}$, with or without subscripts, will denote filters, and $\mathcal{F} \times \mathcal{G} = \{U : F \times G \subset U \subset X \times X$ for some $F \in \mathcal{F}$, $G \in \mathcal{G}\}$. If f is a binary relation, we use $f[A]$ to denote the image of A under $f(= \{b :$ for some $a \in A$, $(a,b) \in f\})$; f^{-1} to denote the inverse of $f(= \{(b,a) : (a,b) \in f\})$, and if $x \in X$, then $f(x) = f[\{x\}]$. For $x \in X$, x^* denotes the principal ultrafilter $\{A \subset X : x \in A\}$. The closure of a set F (in the topology on X arising from the quasi–uniformity) is denoted $\mathrm{Cl}(F)$.

1. DEFINITION: Given a quasi-uniform space $(X, \mathcal{U})$, a filter $\mathcal{F}$ on X is *SP-Cauchy* if for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$, $x \in X$ such that $F \subset U(x)$.

A pair of filters $(\mathcal{F}, \mathcal{G})$ is *bi-Cauchy* if $\mathcal{U} \subset \mathcal{F} \times \mathcal{G}$. A filter $\mathcal{G}$ is *D-Cauchy* if for some filter $\mathcal{F}$, $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy.

A point $x \in X$ is a *limit of* $\mathcal{F}$ if for each $U \in \mathcal{U}$, $U(x) \in \mathcal{F}$, and is an *adherent to* $\mathcal{F}$ if for each $F \in \mathcal{F}$, $x \in \mathrm{Cl}(F)$.

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DEÁK (in [De1], [De2]) uses the notion of bi-Cauchy filter pairs (there called Cauchy filter pairs) to study extensions of bitopological spaces, while other properties of a notion of CAUCHY and limit equivalent to his are shown in [LF] to be sufficient to describe CSÁSZÁR's original notion of CAUCHY, [CS] in key cases (also see proposition 11).

2. PROPOSITION: A point $x \in X$ is a limit of $\mathcal{F}$ iff $(x^*, \mathcal{F})$ is bi-Cauchy.

PROOF: If x is a limit of $\mathcal{F}$, $U \in \mathcal{U}$ then $U \supset \{x\} \times U(x) \in x^* \times \mathcal{F}$, so $(x^*, \mathcal{F})$ is bi-Cauchy. Conversely, if $(x^*, \mathcal{F})$ is bi-Cauchy, and $U \in \mathcal{U}$ then for some $F \in \mathcal{F}$, $G \in x^*$, $G \times F \subset U$; thus $U \supset \{x\} \times F$, so $U(x) \supset F \in \mathcal{F}$.

3. THEOREM: *If $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy then each adherent to $\mathcal{F}$ is a limit of $\mathcal{G}$.*

PROOF If x adheres to $\mathcal{F}$ and $U \in \mathcal{U}$ we must show $U(x) \in \mathcal{G}$. Let $V \circ V \subset U$, $V \in \mathcal{U}$. Then for some $F \in \mathcal{F}$, $G \in \mathcal{G}$, $F \times G \subset V$; also since $x \in \mathrm{Cl}(F)$, $V(x)$ meets F, so let $y \in V(x) \cap F$. If $z \in G$, then $(x, y), (y, z) \in V$, so $U(x) \supset G \in \mathcal{G}$.

4. DEFINITION: $(X, \mathcal{U})$ is:

Doitchinov complete if each D-Cauchy $\mathcal{F}$ has a limit,

Pervin complete if each SP-Cauchy $\mathcal{F}$ has a limit,

Cauchy bounded if each ultrafilter is D-Cauchy,

precompact if for each $U \in \mathcal{U}$ there is a finite $S \subset X$ such that $U[S] = X$ (here we follow the nomenclature of [*FL*]; such spaces are often called totally bounded).

Below, (a) is immediate from 2, while proofs of (b), (c), (d) are routine.

5. PROPOSITION:

(a) Each filter with a limit is D-Cauchy.

(b) Each limit of a filter is adherent to it; and each adherent to an ultrafilter is a limit of it.

(c) If $\mathcal{F} \subset \mathcal{G}$ then each limit of $\mathcal{F}$ is a limit of $\mathcal{G}$, and each adherent to $\mathcal{G}$ is adherent to $\mathcal{F}$.

(d) If $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{F}$ is D-Cauchy then so is $\mathcal{G}$.

6. THEOREM: *X is compact in the topology arising from $\mathcal{U}$ iff X is Doitchinov complete and Cauchy bounded.*

PROOF: Consider these equivalences:

X is compact in the topology arising from $\mathcal{U} \Longleftrightarrow$

each filter of closed sets has nonempty intersection $\Longleftrightarrow$

each ultrafilter has adherents $\Longleftrightarrow$

(by 5 (b)) each ultrafilter has limits $\Longleftrightarrow$

X is Doitchinov complete and Cauchy bounded ($\Rightarrow$ uses 5 (a)).

The following proposition and example compare the two types of completeness and boundedness.

7. PROPOSITION: Each D-Cauchy filter is SP-Cauchy. Thus each Pervin complete space is Doitchinov complete and each Cauchy bounded space is precompact.

PROOF: If $\mathcal{G}$ is D-Cauchy find $\mathcal{F}$ such that $(\mathcal{F},\mathcal{G})$ is bi-Cauchy. Next let $U \in \mathcal{U}$; by assumption we can find $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \times G \subset U$. Then F is non-empty, so let $x \in F$; $G \subset (F \times G)(x) \subset U(x)$, showing $\mathcal{G}$ to be SP-Cauchy.

The first assertion of the second sentence is immediate, and the second follows from the well-known (e.g., [FL], 3.14) and easily shown fact that X is precompact iff each ultrafilter is SP-Cauchy.

8. EXAMPLE: Let X be the set of positive integers, $\mathcal{W}$ the quasi-uniformity induced by the quasimetric $d(x,y) = 1$ if $x > y$, 0 if $x = y$, $1/x$ if $x < y$.

First note that if $(\mathcal{F},\mathcal{G})$ is a bi-Cauchy ultrafilter pair, then for some $x \in X$, $(\mathcal{F},\mathcal{G}) = (x^*, x^*)$: For such $(\mathcal{F},\mathcal{G})$, if $\mathcal{F}$ were nonprincipal, then each $F \in \mathcal{F}$ would contain arbitrarily large elements. If $G \in \mathcal{G}$ choose $y \in G$, then $x \in F$ such that $x > y$; then $(x,y) \in F \times G - N_{1/2}$, contradicting the assumption that $(\mathcal{F},\mathcal{G})$ is bi-Cauchy. Thus we may let $\mathcal{F} = x^*$; if $\mathcal{G} \neq x^*$ then for each $G \in \mathcal{G}$ there is a $y \in G - \{x\}$, and if $F \in \mathcal{F}$ $y \neq x$, then $d(x,y) \geq 1/x$, so $(x,y) \in F \times G - N_{1/2x}$, again contradicting the assumption that $(\mathcal{F},\mathcal{G})$ is bi-Cauchy.

Hence only principal ultrafilters are D-Cauchy, and by 2, x is a limit for x^*. This and 5 (d) show that X is Doitchinov complete but not Cauchy bounded, and it follows that X is not compact (a fact easy to verify in other ways).

X is precompact: For any positive r we can find $x > 1/r$; if $y > x$ then $d(x,y) \leq 1/x < r$. Thus X is the finite union $N_r(0) \cup \ldots \cup N_r(x)$.

Since it is not compact, X is not Pervin complete. Direct proof: Since X is precompact, each ultrafilter is SP–Cauchy. If $\mathcal{F}$ is a nonprincipal ultrafilter then by our first comment, $(x^*, \mathcal{F})$ is not bi-Cauchy, so by 2, $\mathcal{F}$ has no limit, contradicting Pervin completeness.

Products and Symmetry

For the product space $X = \prod_I X_i$ we denote by p_i the i'th projection map (if $x \in X$ then $p_i(x) = x(i)$). If f, g are relations, $f \times g = \{((a,b),(c,d)) : (a,c) \in f,\ (b,d) \in g\}$. Products of filters are defined as are products of topologies or quasi-uniformities: $F \in \prod_I \mathcal{F}_i$, iff for some finite subset I' of I, $\prod_I F_i \subset F$, where for each $i \in I$, $F_i \in \mathcal{F}_i$ and if $i \notin I'$ then $F_i = X_i$. Also if $\mathcal{F}$ is a filter on $\prod_I X_i$ then for each $i \in I$ let $\mathcal{F}(i) = \{F \subset X_i : p_i^{-1}[F] \in \mathcal{F}\}$.

9. LEMMA:

(a) $\mathcal{F}(i)$ is a filter, an ultrafilter if $\mathcal{F}$ is one.

(b) $\prod_I \mathcal{F}(i) \subset \mathcal{F}$.

(c) If $\mathcal{F} \subset \mathcal{F}'$, $\mathcal{G} \subset \mathcal{G}'$, and $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy, then so is $(\mathcal{F}', \mathcal{G}')$.

(d) If for each i, $(\mathcal{F}_i, \mathcal{G}_i)$ is bi-Cauchy then so is $(\prod_I \mathcal{F}_i, \prod_I \mathcal{G}_i)$.

(e) For filters $\mathcal{F}, \mathcal{G}$ on $\prod_I X_i$, $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy iff for each $i \in I$, $(\mathcal{F}(i), \mathcal{G}(i))$ is bi-Cauchy. Thus x is a limit of $\mathcal{G}$ iff for each i, $x(i)$ is a limit of $\mathcal{G}(i)$.

PROOF: All parts are straightforward (and some are shown in [Do2]); we show (e) since its proof is typical but most intricate. Conversely, if each $(\mathcal{F}(i), \mathcal{G}(i))$ is bi-Cauchy, then so is $(\mathcal{F}, \mathcal{G})$ by $(b), (c)$. If $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy, $i \in I$, $U_i \in \mathcal{U}_i$, then $U = (p_i \times p_i)^{-1}[U_i] \in \mathcal{U}$, so for some $F \in \mathcal{F}$, $G \in \mathcal{G}$, $F \times G \subset U$. Set $F_i = p_i[F]$, $G_i = p_i[G]$, then $p_i^{-1}[F_i] \supset F \in \mathcal{F}$, so $F_i \in \mathcal{F}(i)$, and similarly $G_i \in \mathcal{G}(i)$. Finally, $F_i \times G_i = (p_i \times p_i)[F \times G] \subset (p_i \times p_i)[U] = U_i$, so $(\mathcal{F}(i), \mathcal{G}(i))$ is bi-Cauchy. The comment on limits then follows from 2.

10. THEOREM:

(a) Products of Doitchinov complete spaces are Doitchinov complete.

(b) Products of Cauchy bounded spaces are Cauchy bounded.

(c) (Tychonoff) Products of compact spaces are compact.

PROOF:

(a) If $\mathcal{G}$ is D-Cauchy then by 9 (e) so is $\mathcal{G}(i)$ for each $i \in I$, thus by Doitchinov completeness of the factors, find for each i, a limit x_i for $\mathcal{G}(i)$. Define x by $x(i) = x_i$. Each $(x_i^*, \mathcal{G}(i))$ is bi-Cauchy, therefore so is their product, thus by 9 (c), so is $(x^*, \mathcal{G})$, so x is a limit for $\mathcal{G}$.

(b) Let $\mathcal{G}$ be an ultrafilter on $\prod_I X_i$; then for each $i \in I$, $\mathcal{G}(i)$ is one on X_i by 9 (a), and is D-Cauchy since each X_i is Cauchy bounded. Thus let $(\mathcal{F}_i, \mathcal{G}(i))$ be bi-Cauchy; by 9 (d), $(\prod_I \mathcal{F}_i, \prod_i \mathcal{G}(i))$, is bi-Cauchy, and by 9 (b), (c) so is $(\prod_I \mathcal{F}_i, \mathcal{G})$, showing $\mathcal{G}$ to be D-Cauchy, thus $\prod_I X_i$ to be Cauchy bounded.

(c) is immediate from (a), (b), and 6 (since any space is quasi-uniformizable, see e.g. [FL] 2.1).

We close with a result relating the above to the usual symmetric (uniform space) theory, and noting the transitivity and non-reflexivity of the bi-Cauchy relation. For a quasi-uniformity $\mathcal{U}$, let $\mathcal{U}^{-1}$ denote the quasi-uniformity $\{U^{-1} : U \in \mathcal{U}\}$, $\mathcal{U}^S$ denote the uniformity generated by $\{U \cap U^{-1} : U \in \mathcal{U}\}$.

11. PROPOSITION:

(a) If $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{H})$ are bi-Cauchy with respect to $\mathcal{U}$ then so is $(\mathcal{F}, \mathcal{H})$; also $(\mathcal{G}, \mathcal{F})$ is bi-Cauchy with respect to $\mathcal{U}^{-1}$. $(\mathcal{F}, \mathcal{F})$ is bi-Cauchy with respect to $\mathcal{U}$ iff $\mathcal{F}$ is D-Cauchy with respect to $\mathcal{U}^S$. Thus for uniform spaces $\mathcal{G}$ is Cauchy iff $(\mathcal{G}, \mathcal{G})$ is bi-Cauchy.

(b) A uniform space X is Cauchy bounded iff it is precompact.

PROOF:

(a) Assume $(\mathcal{F}, \mathcal{G})$, $(\mathcal{G}, \mathcal{H})$ are bi-Cauchy with respect to $\mathcal{U}$, $U \in \mathcal{U}$. Then for some $F \in \mathcal{F}$, $G', G'' \in \mathcal{G}$, $H \in \mathcal{H}$, and $V \in \mathcal{U}$, $F \times G'$, $G'' \times H \subset V$ and $V \circ V \subset U$. But then $G = G' \cap G'' \in \mathcal{U}$, and $F \times H \subset (F \times G) \circ (G \times H) \subset V \circ V \subset U$, so $(\mathcal{F}, \mathcal{H})$ is bi-Cauchy with respect to $\mathcal{U}$; also $G' \times F \subset U^{-1}$, so $(\mathcal{G}, \mathcal{F})$ is bi-Cauchy with respect to $\mathcal{U}^{-1}$. By this last comment, if $(\mathcal{F}, \mathcal{F})$ is bi-Cauchy with respect to $\mathcal{U}$, then it is bi-Cauchy with respect to $\mathcal{U}^{-1}$, thus clearly with respect to $\mathcal{U}^S$. Conversely, if $\mathcal{F}$ is D-Cauchy with respect to $\mathcal{U}^S$ then find $\mathcal{G}$ such that $(\mathcal{G}, \mathcal{F})$ is is bi-Cauchy with respect to $\mathcal{U}^S$; then $(\mathcal{F}, \mathcal{G})$ is bi-Cauchy with respect to $(\mathcal{U}^S)^{-1} = \mathcal{U}^S$, so by transitivity $(\mathcal{F}, \mathcal{F})$ is bi-Cauchy with respect to $\mathcal{U}^S$, thus with respect to the smaller $\mathcal{U}$. The last comment is apparent from the usual (uniform) definition of Cauchy.

(b) By 7 each Cauchy bounded space is precompact. The converse is well-known (see [K]) for uniform spaces.

12. EXAMPLE: The referee and H. KÜNZI have pointed out that (despite assertions in earlier versions of this paper), subspaces of Cauchy bounded spaces need not be Cauchy bounded. Below, we use the notations of example 8. Extend d to $X \cup \{\infty\}$, and consider its conjugate, d^{-1} defined by $d^{-1}(x,y) = d(y,x)$. For $r > 0$, $N_r^{-1}(\infty) = \{x : x \geq 1/r\}$, so its topology is compact; by 6 its quasi-uniformity is Cauchy bounded.

Now consider the restriction of d^{-1} to X; it gives rise to $\mathcal{W}^{-1}$. The only filter pairs bi-Cauchy with respect to $\mathcal{W}$ are those of the form

(x^*, x^*), so by 11 (a), only those are bi-Cauchy with respect to $\mathcal{W}^{-1}$; thus as in 8, only principal ultrafilters are D-Cauchy with respect to $\mathcal{W}^{-1}$, so it is not Cauchy bounded.

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THE VIBRATION OF A MEMBRANE IN DIFFERENT POINTS

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The properties of vibrating membranes are studied in many papers, see [1]–[10]. In this paper we shall deal with a problem formulated by V. KOMORNIK in [7] which reads as follows. Let $\Omega = (0,\pi)\times(0,\pi)$ and define $\mathcal{U}$ as the vector space of all functions

$$u(t,x,y) \in C^\infty(\mathbf{R}\times\Omega)\cap C\left(\mathbf{R}\times\overline{\Omega}\right)$$

satisfying

$$u_{tt} = \Delta u \quad \text{on} \quad \mathbf{R}\times\Omega, \tag{1}$$

$$u = 0 \quad \text{on} \quad \mathbf{R}\times\partial\Omega. \tag{2}$$

We consider these functions as the possible movements in time t of the rectangular membrane Ω, fixed on its boundary. Given N distinct points $P_1,\dots,P_N\in\Omega$ consider the mapping

$$A:\mathcal{U}\to C^\infty(\mathbf{R})^N$$

$$Au := (u(\cdot,P_1),\dots,u(\cdot,P_N)). \tag{3}$$

We shall prove the above mentioned conjecture of V. KOMORNIK, namely the following

THEOREM *For any distinct N-tuples $P_1,\dots,P_N\in\Omega$ the range set $A\mathcal{U}$ is dense in $C^\infty(\mathbf{R})^N$.*

Before proving it we give some remarks and Lemmas.

First, it is enough to show that $A\mathcal{U}$ is dense in $L^2(0,T)^N$ for arbitrary $T>0$; the density in $C^\infty(\mathbf{R})^N$ follows easily from it, see [9]. Secondly, we can restrict ourselves to the solutions u of (1), (2) representable by a finite sum

$$u(t,x,y) = \sum\left(c_{n,k}e^{i\sqrt{n^2+k^2}\,t} + d_{n,k}e^{-i\sqrt{n^2+k^2}\,t}\right)\sin nx\sin ky.$$

Let further $P_j =: (x_j, y_j)$ and define

$$e_{n,k} = \begin{pmatrix} \sin nx_1 \sin ky_1 \\ \vdots \\ \sin nx_N \sin ky_N \end{pmatrix}.$$

Then our Theorem reduces to the following statement: the system of vector exponentials

$$e(\Lambda) := \left\{ e_{n,k} e^{\pm i\sqrt{n^2+k^2}\,t} \ : \ n,k = 1,2,\dots \right\}$$

is complete in $L^2(0,T)^N$ for any $T > 0$.

Below we shall use some ideas from [14].

LEMMA 1. The vectors $e_{n,k}$, $1 \le n,k \le 2N$ span the space $\mathbf{C}^N$.

PROOF. Let $\gamma \in \mathbf{C}^N$ be orthogonal to $e_{n,k}$, i.e.

$$\gamma_1 \sin nx_1 \sin ky_1 + \dots + \gamma_N \sin nx_N \sin ky_N = 0 \qquad (1 \le n,\ k \le 2N).$$

We know that $\frac{\sin nx}{\sin x}$ is a polynomial of $\cos x$ of degree $n-1$. Consequently we get by induction that

$$\delta_1 \cos^{n-1} x_1 \cos^{k-1} y_1 + \dots + \delta_N \cos^{n-1} x_N \cos^{k-1} y_N = 0 \quad (1 \le n,k \le 2N)$$

with

$$\delta_1 := \gamma_1 \sin x_1 \sin y_1, \dots, \delta_N := \gamma_N \sin x_N \sin y_N.$$

That is, given any (algebraic) polynomial of degree at most $2N-1$ we have

$$\delta_1 P(\cos x_1, \cos y_1) + \dots + \delta_N P(\cos x_N, \cos y_N) = 0.$$

Applying this to the polynomial

$$P(x,y) := \prod_{j=2}^{N} [(x - \cos x_j)^2 + (y - \cos y_j)^2]$$

we get $\delta_1 = 0$ and hence $\gamma_1 = 0$. Analogously $\gamma_2 = \dots = \gamma_N = 0$. Lemma 1 is proved.

LEMMA 2. Let $T > 0$ be arbitrary and $\{e_1, \dots, e_N\} \subset \{e_{n,k} : 1 \le \le n,k \le 2N\}$ be a vector-N-tuple which spans $\mathbf{C}^N$. The set $e(\Lambda)$ contains for small $\varepsilon > 0$ a subsystem

$$(4) \qquad \Phi = \{e_n^j e^{\pm i\lambda_{n,j} t} \ : \ j = 1,\dots,N;\ n = 1,2,\dots\} \cup \cup \{e_0^j e^{i\lambda_{0,j} t} \ : \ j = 1,\dots,N\}$$

such that

(a) $$|e_n^j - e_j| < \varepsilon, j = 1, \dots, N; \; n = 0, 1, 2, \dots$$

(b) $$\lambda_{n,j} = 2\pi\frac{n}{T} + O\Big(\frac{1}{\sqrt{n}}\Big), j = 1, \dots, N; n = 1, 2, \dots .$$

PROOF. Let $\varepsilon > 0$ be so small that modifying the vectors $e_1, \dots, e_N$ arbitrarily at a distance $\le \varepsilon$ they remain linearly independent. We shall use the following form of the Kronecker theorem. Given $x_1, \dots, x_N$, consider the following property of the real numbers $\alpha_1, \dots, \alpha_N$:

(*) $n_1x_1 + \dots + n_Nx_N \equiv 0 \pmod{2\pi}$ implies
$n_1\alpha_1 + \dots + n_N\alpha_N \equiv 0 \pmod{2\pi}$ for any $n_1, \dots, n_N \in \mathbf{Z}$.

Then for any $\varepsilon > 0$ there exists $X(\varepsilon) > 0$ such that for any reals $\alpha_1, \dots, \alpha_N$ satisfying (*) there exists $n_0 \in \mathbf{Z}, |n_0| \le X(\varepsilon)$ such that

(5) $$|||n_0x_1 - \alpha_1||| < \frac{\varepsilon}{2N}, \dots, |||n_0x_N - \alpha_N||| < \frac{\varepsilon}{2N}$$

where $|||x|||$ denotes the distance of x and the set $\{2r\pi : r \in \mathbf{Z}\}$ (see [12], p.58). In fact for any $n' \in \mathbf{Z}$ there exists $n_0 \in \mathbf{Z}$ with $|n' - n_0| \le X(\varepsilon)$ and satisfying (5); this follows if we set $\alpha_1 + n'x_1, \dots, \alpha_N + n'x_N$ instead of $\alpha_1, \dots, \alpha_N$; this does not violate the property (*). We can also suppose that for any $k' \in \mathbf{Z}$ there exists $k_0 \in \mathbf{Z}$, $|k' - k_0| \le X(\varepsilon)$ such that

(5′) $$|||k_0y_1 - \beta_1||| < \frac{\varepsilon}{2N}, \dots, |||k_0y_N - \beta_N||| < \frac{\varepsilon}{2N}$$

where $\beta_1, \dots, \beta_N$ satisfy the property

(*′) $n_1y_1 + \dots + n_Ny_N \equiv 0 \pmod{2\pi}$ implies
$n_1\beta_1 + \dots + n_N\beta_N \equiv 0 \pmod{2\pi}$ for any $n_j \in \mathbf{Z}$.

The estimates (5), (5′) will be applied in the case $\alpha_1 = n''x_1, \dots, \alpha_N = n''x_N, \beta_1 = k''y_1, \dots, \beta_N = k''y_N$ where $1 \le n'', k'' \le 2N$ and $e_{n'',k''}$ belongs to the set $\{e_1, \dots, e_N\}$. Define inductively the exponentials $e_n^j e^{\pm i\lambda_{n,j}t}$ as follows. Suppose first that $0 \le 2\pi\frac{n}{T} \le CX(\varepsilon)$, $C = C(T, \varepsilon)$ and let $e_j = e_{n'',k''}$. Choose $0 \le n_0 \le 2X(\varepsilon)$ satisfying (5) and choose k_0 satisfying (5′) from $2X(\varepsilon) + 1$ consecutive integers which do not contain k_0-type values constructed in the earlier steps. So a) is ensured and we can suppose that any interval $[\frac{2n-1}{T}\pi, \frac{2n+1}{T}\pi]$ contains no more than one value $\sqrt{n_0^2 + k_0^2}$. Now if $2\pi\frac{n}{T} > CX(\varepsilon)$ then choose

$$2\pi\frac{n}{T} - 2X(\varepsilon) \le n_0 \le 2\pi\frac{n}{T}$$

satisfying (5). Let $k' \in \mathbb{N}$ be the number such that the distance $\left|2\pi\frac{n}{T}-\sqrt{n_0^2+k'^2}\right|$ is the smallest possible and choose $k' \le k_0 \le k'+2X(\varepsilon)$ satisfying (5′). If the value $\sqrt{n_0^2+k_0^2}$ is already constructed, choose $k'+2X(\varepsilon)+1 \le k_0 \le k'+4X(\varepsilon)+1$ satisfying (5′). Denote finally

$$\lambda_{n,j} := \sqrt{n_0^2+k_0^2}, \qquad e_n^j := \begin{pmatrix} \sin n_0x_1 \sin k_0y_1 \\ \vdots \\ \sin n_0x_N \sin k_0y_N \end{pmatrix}.$$

If $C(T,\varepsilon)$ is large enough then $|\lambda_{n,j} - 2\pi\frac{n}{T}| < \frac{\pi}{T}$ and then the $\lambda_{n,j}$ constructed in different steps are different. We sketch the proof. First

$$\sqrt{n_0^2+(k+1)^2} - \sqrt{n_0^2+k^2} = \frac{2k+1}{\sqrt{n_0^2+(k+1)^2}+\sqrt{n_0^2+k^2}}$$

has the order $k/(2\pi\frac{n}{T})$. If $C(T,\varepsilon)$ is large enough then k'/n and so k_0/n is small enough to ensure $|\lambda_{n,j} - 2\pi\frac{n}{T}| < \frac{\pi}{T}$. Since $\sqrt{n_0^2+k'^2} - n_0 \le$ $\le 2X(\varepsilon)+1$ hence k' has the order at most $\sqrt{n}$ and so (b) follows. Lemma 2 is proved.

LEMMA 3. *Let the system $e_ne^{i\lambda_nx}$, $n \in \mathbb{Z}$ be Riesz basis in $L^2(0,T)^N$, $e_n \in \mathbb{C}^N$. Then there exists an $\varepsilon > 0$ such that*

$$|\lambda_n - \lambda_n'| < \varepsilon \qquad\qquad n \in \mathbb{Z}$$

implies that $e_ne^{i\lambda_n'x}$ is also a Riesz basis in $L^2(0,T)^N$.

PROOF. Recall the following theorem of N. K. BARI. If $(\varphi_n) \subset H$ is a Riesz basis in the Hilbert space H then there exists $L > 0$ with the following property: Given any system $(\psi_n) \subset H$ such that for any finite sequence (c_n) of complex numbers the estimate

$$\left\|\sum c_n(\varphi_n - \psi_n)\right\|_H^2 \le L\sum |c_n|^2$$

holds, the system (ψ_n) must be Riesz basis, too.

The proof is simple: the transform

$$A : \sum c_n\varphi_n \to \sum c_n(\varphi_n - \psi_n)$$

extends continously onto H and for small L we have $||A|| < 1$ and then $I-A$ is an isomorphism of H and maps φ_n into ψ_n. This will be applied to our situation. Using the identity

$$e^{i\lambda_nx} - e^{i\lambda_n'x} = e^{i\lambda_nx}(1-e^{i\delta_nx}) = -\sum_{k=1}^{\infty} e^{i\lambda_nx}\frac{(i\delta_n)^k}{k!}x^k,$$

$$\delta_n := \lambda'_n - \lambda_n$$

we get

$$\begin{aligned}
&\Big\|\sum_n c_n e_n (e^{i\lambda_n x} - e^{i\lambda'_n x})\Big\|_{L^2(0,T)^N} = \\
=&\Big\|\sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_n c_n e_n e^{i\lambda_n x}(i\delta_n)^k\Big\|_{L^2(0,T)^N} \le \\
\le&\sum_{k=1}^{\infty}\frac{T^k}{k!}\Big\|\sum_n c_n e_n (i\delta_n)^k e^{i\lambda_n x}\Big\|_{L^2(0,T)^N} \le \\
\le& c\sum_{k=1}^{\infty}\frac{T^k}{k!}\Big(\sum_n |c_n(i\delta_n)^k|^2\Big)^{\frac{1}{2}} \le \\
\le& c\sum_{k=1}^{\infty}\frac{(T\varepsilon)^k}{k!}\Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}} = c(e^{T\varepsilon}-1)\Big(\sum_n |c_n|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since for ε small enough we have

$$c(e^{T\varepsilon} - 1) < L,$$

the proof is complete.

LEMMA 4. Let the system $\{e_n e^{i\lambda_n t} : n \in \mathbb{Z}\}$ be complete in $L^2(0,T;\mathbb{C}^N)$. Then eliminating the term $e_0 e^{i\lambda_0 t}$ the remaining system is complete in $L^2(0,T';\mathbb{C}^N)$ for any $T' < T$.

PROOF. Suppose indirectly that the remaining system is not complete over $(0,T')$ for some $T' < T$. Then a fortiori it is not complete over $(0,T)$, hence its closed linear hull over $(0,T)$ is one-codimensional and the same is then true over $(0,T')$. This means that

$$\begin{aligned}
L^2(0,T;\mathbb{C}^N) &= H_T \dot{+} \langle e_0 e^{i\lambda_0 t}\rangle, \\
L^2(0,T';\mathbb{C}^N) &= H_{T'} \dot{+} \langle e_0 e^{i\lambda_0 t}\rangle
\end{aligned} \tag{6}$$

where H_T (resp. $H_{T'}$) denotes the closed linear hull of the system $\{e_n e^{i\lambda_n t} : n \neq 0\}$ over $(0,T)$ (resp. over $(0,T')$).

Clearly $H_{T'} \supset H_t\Big|_{(0,T')}$ and since $H_T\Big|_{(0,T')}$ has a codimension not greater than 1, we obtain

$$H_{T'} = H_T\Big|_{(0,T')}. \tag{7}$$

Any function $f \in L^2(0,T;\mathbf{C}^N)$ can be uniquely written in the form

$$f = ce_0 e^{i\lambda_0 t} + f_0, \quad f_0 \in H_T. \tag{8}$$

Then

$$f\Big|_{(0,T')} = ce_0 e^{i\lambda_0 t} + f_0\Big|_{(0,T')}$$

and by (7) it is the unique decomposition of $f|_{(0,T')}$ over $(0,T')$. Again by (7) we see that continuing any $f \in H_{T'}$ anyway over (T',T) in L^2, the resulting function belongs to H_T. Let $h(t)$ be a nonzero L^2-function orthogonal to H_T over $(0,T)$; then we must have

$$h\Big|_{(T',T)} = 0. \tag{9}$$

Consequently

$$0 = \int_0^{T'} \langle h(t), e_n e^{i\lambda_n t}\rangle \mathrm{d}t = \int_\varepsilon^{T'+\varepsilon} \langle h(t-\varepsilon), e_n e^{i\lambda_n (t-\varepsilon)}\rangle \mathrm{d}t =$$

$$= \int_\varepsilon^{T'+\varepsilon} \langle h(t-\varepsilon), e_n e^{i\lambda_n t}\rangle \mathrm{d}t \qquad (n \neq 0).$$

So the function $h_\varepsilon := h(t-\varepsilon)$ is also orthogonal to H_T for any $0 < \varepsilon < T-T'$ and this contradicts the fact that H_T is one-codimensional. Lemma 4 is proved.

Now the Theorem easily follows from the above Lemmas. Indeed, Lemma 2 a) shows that the system

$$\Phi_0 := \{e_n^j e^{\pm i 2\pi \frac{n}{T} x} : n = 0,1,\dots; j = 1,\dots,N\}$$

forms a Riesz basis in $L^2(0,T)^N$. This is based on the following Lemma of BARI [13]: a minimal system $(\varphi_n) \subset H$ is Riesz basis if and only if there exist constants $0 < c \le C < \infty$ such that

$$c\|f\|^2 \le \sum |\langle f, \varphi_n\rangle|^2 \le C\|f\|^2, \qquad f \in H;$$

in symbols

$$\|f\|^2 \asymp \sum |\langle f, \varphi_n\rangle|^2, \qquad f \in H.$$

In our case the minimality of Φ_0 is trivial. Suppose indirectly that

$$e_{n_0}^{j_0} e^{\overset{+}{(-)} i 2\pi \frac{n_0}{T} x} \in V\Big(\Phi_0 \backslash \Big\{e_{n_0}^{j_0} e^{\overset{+}{(-)} i 2\pi \frac{n_0}{T} x}\Big\}\Big). \tag{10}$$

Let $f_{n_0}^{j_0} \in \mathbf{C}^N$ be a vector satisfying

$$\langle f_{n_0}^{j_0}, e_{n_0}^{j_0}\rangle = 1, \langle f_{n_0}^{j_0}, e_{n_0}^{j}\rangle = 0 \quad \text{for} \quad j \neq j_0.$$

Taking the scalar product of both sides in (10) by $f_{n_0}^{j_0}$, the resulting relation contradicts the minimality of the system

$$\{e^{\pm i2\pi\frac{n}{T}} : n = 0, 1, \dots\} \quad \text{in} \quad L^2(0,T).$$

So Φ_0 is indeed minimal. For $f \in L^2(0,T)^N$ we have

$$||f||^2_{L^2(0,T)^N} \asymp \sum_{j=1}^{N} ||\langle f, e_j\rangle_{\mathbf{C}^N}||^2_{L^2(0,T)} \asymp$$

$$\asymp \sum_{j=1}^{N}\sum_{n\in\mathbf{Z}} \left|\left\langle \langle f, e_j\rangle_{\mathbf{C}^N}, e^{i2\pi\frac{n}{T}x}\right\rangle_{L^2(0,T)}\right|^2 =$$

$$= \sum_{n\in\mathbf{Z}}\sum_{j=1}^{N} \left|\left\langle \langle f, e^{i2\pi\frac{n}{T}x}\rangle_{L^2(0,T)}, e_j\right\rangle\right|^2 \asymp \sum_{n\in\mathbf{Z}} ||\langle f, e^{i2\pi\frac{n}{T}x}\rangle_{L^2(0,T)}||^2_{\mathbf{C}^N} \asymp$$

$$\asymp \sum_{n\in\mathbf{Z}}\sum_{j=1}^{N} |\langle f, e_n^j e^{i2\pi\frac{n}{T}x}\rangle_{L^2(0,T)^N}|^2$$

which implies that Φ_0 is Riesz basis in $L^2(0,T)^N$. Denote

$$\widehat{\Phi}_0 := \left\{e_n^j e^{\pm i2\pi\frac{n}{T}x} : n = 0, 1, \dots, N_0; j = 1, \dots, N\right\} \bigcup$$
$$\bigcup \left\{e_n^j e^{\pm i\lambda_{n,j}x} : n \geq N_0 + 1; j = 1, \dots, N\right\}.$$

Lemma 2. b) and Lemma 3 show that for N_0 sufficiently large the system $\widehat{\Phi}_0$ is Riesz basis in $L^2(0,T)^N$. Finally by Lemma 4 the remaining finitely many elements of $\widehat{\Phi}_0$ can be shifted into the corresponding elements of Φ, and then Φ is complete over any $(0,T')$, $T' < T$. The Theorem is completely proved.

The author learned after writing this paper that this statement was independently proved by S. A. Avdonin in Leningrad and I. Joó in Columbus, Ohio with a different proof, see [11].

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ABOUT FACTORIZATION OF IDENTITY OPERATOR FOR A GIVEN UNBOUNDED OPERATOR

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Let A be a densely defined operator with domain $D(A)$ in a Hilbert space H and let A^* be its adjoint operator. In this paper we give necessary and sufficient conditions for the existence of $C \in B(H)$, where $B(H)$ stands for the C^*-algebra of all bounded linear operators on H, such that $CAx = Ix = x$ $(x \in D(A))$.

If we take the condition $R(C^*) \subset \overline{R(A)}$, then C is unique. Here $R(A)$, $(R(C^*))$ stands for the range of A (C^*), bar denotes closure.

In particular, when A is closed we give necessary and sufficient conditions for the existence of $B \in B(H)$ such that $ABx = Ix = x$ $(x \in H)$.

We follow the notation of [1]. Previous extension results on Hilbert space operators are studied in SEBESTYÉN [2]–[4].

We begin with a lemma.

LEMMA 1. Let A be densely defined operator in a Hilbert space H with $R(A^*) = H$. Then there exists a unique $B \in B(H)$ such that

$$Ix = A^*Bx \qquad (x \in H) \qquad \text{and} \qquad R(B) \subset \overline{R(A)}.$$

PROOF. First, let $w \in H$ be arbitrary and let us prove that

(1) there exists a unique $v \in \overline{R(A)}$ such that $A^*v = w$.

Since $R(A^*) = H$ there exists u in H such that $A^*u = w$. Now let $p : H \to \overline{R(A)}$ the projection on $\overline{R(A)}$. We want to prove that $A^*(pu) = w$.

Since $u - pu \in R(A)^{\perp} = \left(\overline{R(A)}\right)^{\perp}$ we have

$$\langle z, 0\rangle = 0 = \langle Az, u - pu\rangle \qquad (z \in D(A))$$

and then $u - pu \in D(A^*)$ and $A^*(u - pu) = 0$ or equivalently $w = A^*u = A^*pu$. Since $pu \in \overline{R(A)}$ and $A^*pu = w$, to prove the unicity of v in (1) take $v \in \overline{R(A)}$ such that $A^*v = w$ and let us prove that

$$v = pu. \tag{2}$$

Since $A^*(v - pu) = w - w = 0$

$$0 = \langle x, A^*(v - pu)\rangle = \langle Ax, v - pu\rangle \qquad (x \in D(A)) \tag{3}$$

Now because $v - pu \in \overline{R(A)}$, it follows from (3) that (2) is true. Now let $B : H \to \overline{R(A)}$ given by

$$Bw = v \qquad \text{if} \qquad A^*v = w \qquad \text{and} \qquad v \in \overline{R(A)} \tag{4}$$

According to (1), B is well defined and linear. Moreover we have

$$(A^*B)w = A^*(Bw) = A^*v = w \qquad (w \in H), \tag{a}$$

and according to the definition

$$R(B) \subset \overline{R(A)}. \tag{b}$$

Now we prove that B is bounded.

Using the closed graph theorem we have only to show that B is closed. Take $\{w_n\} \subset H$ such that

$$w_n \to w \qquad \text{and} \qquad Bw_n \to v \tag{5}$$

and let us prove that

$$Bw = v. \tag{6}$$

From (a) and (5) we have $A^*(Bw_n) = w_n \to w$ and $Bw_n \to v$. A^* is known closed, therefore

$$v \in D(A^*) \qquad \text{and} \qquad A^*v = w. \tag{7}$$

Moreover from (b) we have

$$v = \lim_n \ Bw_n \in \overline{R(A)}. \tag{8}$$

From (1), (7) and (8) we see that (6) follows. (We remind that $A^*(Bw) = w$ and $Bw \in \overline{R(A)}$, according to (a) and (b) respectively.) Therefore B is bounded.

Finally we prove the uniqueness of such a B in $B(H)$. Let B and B_1 in $B(H)$ such that

$$A^*B = I = A^*B_1 \tag{9}$$

and

$$R(B) \subset \overline{R(A)} \qquad \text{and} \qquad R(B_1) \subset \overline{R(A)}. \tag{10}$$

From (9) and (10), if $x \in H$, we have that $A^*(Bx - B_1x) = x - x = 0$ and $Bx - B_1x \in \overline{R(A)}$. Then from these two relations and from (1) we have that $Bx = B_1x$. Now since x is arbitrary in H we have indeed that $B = B_1$.

THEOREM. *Let A be a densely defined operator in H. Then the following properties are equivalent:*

(i) *There exist $C \in B(H) : x = CAx \qquad (x \in D(A))$.*
(ii) *There exist $M \geq 0 : \|x\| \leq M\|Ax\| \qquad (A \in D(A))$*
(iii) *For each $y \in H$ there exists $M_y \geq 0$ such that $|\langle x, y\rangle| \leq M_y\|Ax\|$ $(x \in D(A))$.*
(iv) *$R(A^*) = H$.*
(v) *There exists $B \in B(H) : I = A^*B$.*

PROOF. (i)$\Rightarrow$ (ii):

$$\|x\| = \|CAx\| \leq \|C\| \, \|Ax\| \quad (x \in D(A))$$

(ii)$\Rightarrow$ (iii):

$$|\langle x, y\rangle| \leq \|x\|\|y\| \leq M\|Ax\|\|y\| = (M\|y\|)\|Ax\| \qquad (x \in D(A))$$

(iii) $\Rightarrow$ (iv): Let $\varphi_y : R(A) \to K$ be given by $\varphi_y(Ax) = \langle x, y\rangle$. φ_y is well defined because if $Ax_1 = Ax_2$ then by (iii)

$$\begin{aligned}|\langle x_1, y\rangle - \langle x_2, y\rangle| = |\langle x_1 - x_2, y\rangle| \leq M_y\|A(x_1 - x_2)\| = \\ = M\|Ax_1 - Ax_2\| = 0.\end{aligned}$$

Moreover φ_y is linear and bounded since

$$\left|\varphi_y(Ax)\right| = |\langle x, y\rangle| \leq M_y\|Ax\| \qquad (x \in D(A)).$$

Then φ_y has a unique bounded and linear extension to $\overline{R(A)}$ (Hilbert space) denoted also by φ_y.

Now, by the Riesz representation theorem, there exists $z \in \overline{R(A)}$ such that $\varphi_y(w) = \langle w, z\rangle \quad (w \in \overline{R(A)})$. In particular

$$\langle x, y\rangle = \varphi_y(Ax) = \langle Ax, z\rangle \qquad (x \in D(A)),$$

and therefore $z \in D(A^*)$ and $A^*z = y$.

Now since $y \in H$ was arbitrary it follows that $R(A^*) = H$.

(iv) $\Rightarrow$ (v): by Lemma 1.

(v) $\Rightarrow$ (i):

$$\langle B^*Ax, y\rangle = \langle Ax, By\rangle = \langle x, A^*By\rangle = \langle x, y\rangle \quad (x \in D(A)) \text{ and } (y \in H).$$

Therefore $B^*Ax = x$ $(x \in D(A))$.

COROLLARY. Suppose that A is densely defined closed operator in H.

Then the following properties are equivalent:

(i′) There exists $C \in B(H) : x = CA^*x \qquad (x \in D(A^*))$.

(ii′) There exists $M \geq 0 : \|x\| \leq M\|A^*x\| \qquad (x \in D(A^*))$.

(iii′) For each $y \in H$, there exist $M_y \geq 0$ such that $|\langle x, y\rangle| \leq$ $\leq M_y\|A^*x\| \qquad (x \in D(A^*))$.

(iv′) $R(A) = H$.

(v′) There exist $B \in B(H) : x = ABx \qquad (x \in H)$.

PROOF. Since A is closed by [1], Theorem 13.12 we have that $D(A^*) = H$ and $A^{**} = A$.

Then by Theorem 2 (applied to A^*) corollary follows.

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ON THE REPRESENTATION OF −1 AS A SUM OF FOURTH AND SIXTH POWERS

By

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1. Introduction.

In order to solve Waring's problem, HILBERT [5] and HAUSDORFF [4] established the following identity:

$$(1) \qquad (x_1^2+\ldots+x_r^2)^m=\sum_k \varrho_k(\beta_{1k}x_1+\ldots+\beta_{rk}x_r)^{2m}$$

where the β_{jk} are rationals and the ϱ_k are positive rationals, the number of terms in the summation being $(2m+1)^r$.

If K (a field of characteristic 0) is not formally real, i.e. -1 is a sum of squares in K, then every element of K is a sum of squares in K and so by (1) every m^{th} power in K is a sum of $2m^{\text{th}}$ powers in K. JOLY used this idea [6] to prove that in a field K of characteristic 0, -1 *is a sum of squares in K iff for every natural number m, -1 is a sum of $2m^{\text{th}}$ powers in K*. Here the number of terms involved in expressing -1 as a sum of $2m^{\text{th}}$ powers in enormous — much more than asserted by JOLY, as he made an error in carrying the result of HAUSSDORF in that he presumed that the number of terms on the right hand side of (1) is $2m+1$, whereas the number is actually $(2m+1)^r$ as already said above.

JOLY's result can be derived in a much simpler manner, at the same time considerably improving the bound on the number of terms involved in the expression for -1 as a sum of $2m^{\text{th}}$ powers, as follows:

We can find $x_1,x_2,\ldots,x_r\in K$, not all 0, such that $x_1^2+\ldots+x_r^2=0$. It follows, by (1), that $\sum\gamma_k y_k^{2m}=0$, where $y_k=\beta_{1k}x_1+\ldots+\beta_{rk}x_r\in$ $\in K$ and not all y_k are 0, as follows from the special nature of the β_{ij} in [4], page 304. We may suppose that the γ_j are sufficiently large positive integers. It now readily follows that a non-trivial sum of $(2m+1)^r G(2m)$ many $2m^{\text{th}}$ powers is 0 and hence that -1 *is a sum of* $(2m+1)^r \cdot G(2m) - 1$ *many* $2m^{\text{th}}$ *powers.*

Note that for $m = 2$, this number equals $19 \cdot 5^r - 1 = 95 \cdot 5^s - 1$ where $s = r - 1$ is the stufe of K. Better bounds than this are of course available using analytic methods ([1], [8]), but these are applicable only for algebraic number fields where the stufe $s = 1$, 2 or 4. Some elegant identities of type (1) are also available for small values of m ([2], [3]) but they are not applicable if the stufe of K is greater than 4.

For a natural number n, the least number of terms in expressing -1 as a sum of n^{th} powers in a non-formally real field K is called the n^{th} power stufe of K and will be denoted by $s_n(K)$. Thus, for odd n, we have $s_n(K) = 1$.

In this note, for non-formally real fields K of stufe s, we write -1 as a sum of $2^s + 2s + 2$ fourth powers and a sum of $\binom{m}{k} 2^{k-1} + g(6)(m+1) - 1$ sixth powers, the $m = 3[s/3] + 2$ and $k = (m+4)/3$. The terms involved are linear polynomials in $x_1, \ldots, x_s$ with rational coefficients, where $-1 = x_1^2 + \ldots + x_s^2$. This gives us upper bounds for $s_4(K)$ and $s_6(K)$ which are much smaller than those got by JOLY.

2. Upper bounds for $s_4(K)$ and $s_6(K)$.

Let K be a non-formally real field of characteristic 0 and stufe s. We have the following

THEOREM 1. *Suppose*

$$-1 = x_1^2 + \ldots + x_s^2 \qquad (x_j \in K). \tag{2}$$

Then

(i) $$-1 = 1 + 2\sum_{j=1}^{s} x_j^4 + \sum_{\text{all } \pm} \left(\frac{1 \pm x_1 \pm \ldots \pm x_s}{2^{s/4}}\right)^4 \qquad \text{if } s \geq 4,$$

(ii) $$-1 = x_1^4 + x_2^4 + 2\Big[\left(\frac{x_1 + x_2 + 1}{2}\right)^4 + \left(\frac{x_1 + x_2 - 1}{2}\right)^4 + \left(\frac{x_1 - x_2 + 1}{2}\right)^4 + \left(\frac{x_1 - x_2 - 1}{2}\right)^4\Big] \qquad \text{if } s = 2,$$

(iii) $$-1 = (1 + x_1)^4 + 1 + 1 + 1 \qquad \text{if } s = 1.$$

PROOF. Squaring (2) we get

$$1 = \sum x_j^4 + 2\sum x_i^2 x_j^2. \tag{3}$$

Now consider the sum $S = \sum(1 \pm x_1 \pm x_2 \pm \ldots \pm x_s)^4$, where the sum is for all different possibilities of signs $\pm$, so that there are 2^s terms in the sum. Expanding and simplifying we see that

$$S = 2^s\Big(1 + \sum x_j^4 + 6\sum x_i^2 x_j^2 + 6\sum x_j^2\Big) =$$
$$= 2^s\Big(1 + \sum x_j^4 + 3\Big(1 - \sum x_j^4\Big) + 6(-1)\Big),$$

using (1), (3)

$$S = 2^s\Big(-2 - 2\sum x_j^4\Big).$$

Now s is a power of 2 ([7]) and so 2^s is a fourth power if $s \geq 4$. Absorbing it in S, we get (i). (ii) and (iii) follow by direct easy verifications. This completes the proof of the theorem.

As a corollary we note that in all cases we have

$$s_4(K) \leq 2^s + 2s + 2.$$

THEOREM 2.

$$s_6(K) \leq \begin{cases} \binom{m}{k} \cdot 2^{k-1} + (m+1)g(6) - 1 & \text{if } s \geq 4, \text{ where } m = 3[s/3]+2 \\ & \text{and } k = (m+4)/3, \\ 27 & \text{if } s = 2, \\ 1 & \text{if } s = 1. \end{cases}$$

PROOF. First let $s \geq 4$. By adding one 0^2 to the right hand side of (2), if necessary, we can write (2) as

$$-1 = \sum_1^m x_j^2 \tag{4}$$

where $m \geq 5$ and $m \equiv 2\ (3)$. The reason for doing this will become clear in the sequel. Squaring (4), we get

$$1 = \sum x_j^4 + 2\sum x_i^2 x_j^2. \tag{5}$$

Multiplying (4) and (5) gives

$$-1 = \sum x_j^6 + 3\sum x_i^4 x_j^2 + 6\sum x_i^2 x_j^2 x_\ell^2.$$

Further $(-1)(\sum x_j^4) = (\sum x_i^2)(\sum x_j^4) = \sum x_j^6 + \sum x_i^4 x_j^2$. This now gives us the three sums $\sum x_i^2 x_j^2$, $\sum x_i^4 x_j^2$, and $\sum x_i^2 x_j^2 x_\ell^2$ in terms of sums of 4^{th} and 6^{th} powers viz.

$$2\sum x_i^2 x_j^2 = 1 - \sum x_i^4, \tag{6}$$

$$\sum x_i^4 x_j^2 = -\sum x_i^4 - \sum x_i^6, \tag{7}$$

$$6\sum x_i^2 x_j^2 x_\ell^2 = -1 + 2\sum x_i^6 + 3\sum x_i^4. \tag{8}$$

Now consider the sum

$$S = \sum (x_{i_1} \pm x_{i_2} \pm \ldots x_{i_k})^6 = \frac{1}{2}\sum (\pm x_{i_1} \pm x_{i_2} \pm \ldots x_{i_k})^6,$$

where the summation runs over all $i_1, i_2, \ldots, i_k$, with $1 \le i_1 < \ldots < < i_k \le m$ and all possible $\pm$, where $k (\ge 3)$ will be chosen suitably later. Then

$$S = 2^{k-1}\left[\binom{m-1}{k-1}\sum x_i^6 + \binom{6}{2}\binom{m-2}{k-2}\sum x_i^4 x_j^2 + \right.$$
$$\left. + \binom{6}{2}\binom{4}{2}\binom{m-3}{k-3}\sum x_i^2 x_j^2 x_\ell^2\right] =$$
$$= 2^{k-1}\left[-15\binom{m-3}{k-3} + \left\{45\binom{m-3}{k-3} - 15\binom{m-2}{k-2}\right\}\sum x_i^4 + \right.$$
$$\left. + \left\{\binom{m-1}{k-1} - 15\binom{m-2}{k-2} + 30\binom{m-3}{k-3}\right\}\sum x_i^6\right],$$

(using (4)–(8)).

We shall try to make coefficient of $\sum x_i^4$ equal to 0 and the coefficient of $\sum x_i^6$ negative. This will then give a non-trivial solution of the equation $X_1^6 + \ldots + X_N^6 = 0$. Now the coefficient of $\sum x_i^4$ is $45\binom{m-3}{k-3} - 15\binom{m-2}{k-2}$ and this equals 0 if $\frac{m-2}{k-2} = 3$ ie. $m = 3k-4$. So we take $k = (m+4)/3$, which is an integer ≥ 3. Then for this choice of k, the coefficient of $\sum x_i^6$ becomes

$$2^{k-1}\binom{m-3}{k-3}\left[3\left(\frac{m-1}{k-1}\right) - 15\right] =$$
$$2^{k-1}\binom{m-3}{k-3}\cdot\frac{3(m-1-5k+5)}{k-1} = \frac{3\cdot 2^{k-1}}{k-1}\cdot\binom{m-3}{k-3}(-2k) < 0.$$

Now we have

$$S + 15\binom{m-3}{k-3}2^{k-1} + \frac{3k\cdot 2^k}{k-1}\binom{m-3}{k-3}\sum x_i^6 = 0.$$

Since S has $\binom{m}{k}\cdot 2^{k-1}$ terms, it follows that a non-trivial sum of $\binom{m}{k}\cdot 2^{k-1} + g(6)(1+m)$ sixth powers is 0. This deals with the case $s \ge 4$.

Now let $s = 2$. Then

$$(x_1 + x_2)^6 + (x_1 - x_2)^6 + 10 + 8(x_1^6 + x_2^6) = 10[1 + (x_1^2 + x_2^2)^3] = 0;$$

so $s_6(K) \leq 27$.

Finally, for $s = 1$, since $-1 = x_1^2$, we have $-1 = x_1^6$ and so $s_6(K) = 1$.

This completes the proof of theorem 2.

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A NOTE ON OPERATOR RANGES

By

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In the theory of operator ranges (see [1]) the following theorem is proved for the ranges of positive operators A and $A^{1/2}$: $R(A) = R(A^{1/2})$ if only if $R(A)$ is closed. In [1] the proof uses the following corollary due to Douglas [2] (see also [3]).

Let A and B be bounded linear operators in a Hilbert space H. Then there exists an invertible bounded linear operator C in H with $A = BC$ if and only if A and B have the same range and nullity.

In the present paper we give a direct elementary proof of the above theorem without using the quoted result by DOUGLAS.

Finally I wish to thank prof. Zoltán Sebestyén for calling my attention for this question.

THEOREM. *Let $A : H \to H$ be a positive operator. Then $R(A) = R(A^{1/2})$ if and only if $R(A)$ is closed* (0).

PROOF. Suppose that $R(A) = R(A^{1/2})$ and take $w \in \overline{R(A)}$. Since $w \in \overline{R(A)} = \overline{R(A^{1/2})}$ there exists a sequence $\{x_n\} \subset H$ such that

$$A^{1/2}x_n \to w. \tag{1}$$

The continuity of $A^{1/2}$ implies that

$$Ax_n = A^{1/2}(A^{1/2}x_n) \to A^{1/2}w. \tag{2}$$

On the other hand $R(A) = R(A^{1/2})$ so that there exists $z \in H$ such that

$$A^{1/2}w = Az. \tag{3}$$

Now we prove that

$$w = A^{1/2}z. \tag{4}$$

From (3) we have $A^{1/2}(w - A^{1/2}z) = 0$ and hence that

$$\left\langle A^{1/2}x, A^{1/2}z - w \right\rangle = \left\langle x, A^{1/2}(A^{1/2}z - w) \right\rangle = 0 \qquad (x \in H). \tag{5}$$

Now, if we take $\{y_n\} = \{z - x_n\}$, then from (1) and (5) we get

$$\|A^{1/2}z - w\|^2 = \lim_{n\to\infty} \left\langle A^{1/2}z - A^{1/2}x_n, A^{1/2}z - w \right\rangle =$$
$$= \lim_{n\to\infty} \left\langle A^{1/2}y_n, A^{1/2}z - w \right\rangle = \lim_{n\to\infty} \left\langle y_n, A^{1/2}(A^{1/2}z - w) \right\rangle = 0.$$

Thus $A^{1/2}z = w$ indeed. Hence it follows that $R(A^{1/2}) = R(A)$.

Suppose now that $R(A)$ is closed. Since $A = A^{1/2}A^{1/2}$, we have that

$$R(A) \subset R(A^{1/2}). \tag{6}$$

Now we prove the inclusion

$$R(A^{1/2}) \subset R(A). \tag{7}$$

To this end it is enough to prove that

$$A^{1/2} = PA^{1/2} \tag{8}$$

where $P : H \to H$ is the orthogonal projection onto $R(A)$. Since $P(H) = R(A)$, $A \geq 0$ (and self-adjoint), we have

$$A = PA \quad \text{and } A = A^* = (PA)^* = A^*P^* = AP. \tag{9}$$

Before proving (8), we shall show that

$$A = A^{1/2}PA^{1/2}. \tag{10}$$

If $x \in H$ is arbitrary, then from (9) we have that

$$\|Ax - A^{1/2}PA^{1/2}x\|^2 =$$
$$= \|Ax\|^2 - \left\langle Ax, A^{1/2}PA^{1/2}x \right\rangle - \left\langle A^{1/2}PA^{1/2}x, Ax \right\rangle +$$
$$+ \left\langle A^{1/2}PA^{1/2}x, A^{1/2}PA^{1/2}x \right\rangle =$$

$$= \|Ax\|^2 - \left\langle A^{1/2}x, APA^{1/2}x \right\rangle - \left\langle APA^{1/2}x, A^{1/2}x \right\rangle +$$
$$+ \left\langle APA^{1/2}x, PA^{1/2}x \right\rangle =$$
$$= \|Ax\|^2 - \left\langle A^{1/2}x, AA^{1/2}x \right\rangle - \left\langle AA^{1/2}x, A^{1/2}x \right\rangle + \left\langle AA^{1/2}x, PA^{1/2}x \right\rangle =$$
$$= \|Ax\|^2 - \|Ax\|^2 - \|Ax\|^2 + \left\langle A^{1/2}x, APA^{1/2}x \right\rangle =$$
$$= -\|Ax\|^2 + \left\langle A^{1/2}x, AA^{1/2}x \right\rangle = -\|Ax\|^2 + \|Ax\|^2 = 0.$$

which implies (10).

Now let us prove (8). If $x \in H$, then by (10) we have that

$$\left\|A^{1/2}x - PA^{1/2}x\right\|^2 = \left\|A^{1/2}x\right\|^2 - \left\langle A^{1/2}x, PA^{1/2}x \right\rangle -$$
$$- \left\langle PA^{1/2}x, A^{1/2}x \right\rangle + \left\langle PA^{1/2}x, PA^{1/2}x \right\rangle =$$
$$= \langle Ax, x\rangle - \left\langle A^{1/2}PA^{1/2}x, x \right\rangle - \left\langle A^{1/2}PA^{1/2}x, x \right\rangle + \left\langle A^{1/2}P^2A^{1/2}x, x \right\rangle =$$
$$= \langle Ax, x\rangle - \langle Ax, x\rangle - \langle Ax, x\rangle + \left\langle A^{1/2}PA^{1/2}x, x \right\rangle =$$
$$= -\langle Ax, x\rangle + \langle Ax, x\rangle = 0.$$

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ON DOUBLE-LATTICELIKE SPHEREPACKING

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This paper is connected with the paper [1] of M. HOLLAI. In that paper, the author declared two very interesting theorems. However the proof of Theorem 2 is incomplete. We prove the statement in that case, when the radii of the balls of the spherepacking are equal to each other, namely the density of the densest congruent double-latticelike spherepacking is equal to the density of the densest latticelike spherepacking.

Notations, Lemmas

Let L be a three-dimensional lattice in E^3. A compact convex polyhedron in E^3 with vertices in L is called an L-polyhedron if there is a solid ball containing the vertices on its boundary and containing no other points of L (empty ball). The system of L-polyhedra forms a convex face-to-face tiling of E^3, the L-partition of L. (see: [4] or [2]) Let R be the radius of the maximal support ball of the 3-dimensional lattice L, the length of the radius of the disjoint balls with the centre $\mathbf{z}$ is one, where $\mathbf{z} \in L \cup L + \mathbf{a}$. (The length of the minimal distance of $L \cup L + \mathbf{a}$ is equal to 2.)

LEMMA 1. R greater or equal to 2.

LEMMA 2. The maximal support-polyhedron includes the centre of its circumscribed-ball. (The support-polyhedron is maximal if the radius of its circumscribed-ball is equal to R.)

The proofs of these Lemmas can be found in [1]. By the help of the fundamental work [3], we prove an important property of the 3-dimensional latticelike L-partition.

STATEMENT. Let L be a 3-dimensional lattice and denote $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ the successive minima of L. Then the parallelepiped $P[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ (which is spanned by the system $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$) can be decomposed by L-polyhedra.

PROOF. We know that the system of successive minima of a 3-dimensional lattice is a basis of this lattice. (see [3] § 36). For this reason the symplex $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$, which is spanned by the successive minima $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a basic simplex in the lattice.

(1) First we assume, that:

$$(\mathbf{e}_1,\mathbf{e}_2)\sphericalangle < \pi/2 \qquad 1 \le i < j \le 3.$$

At this time every faces of the symplex $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ are acute triangle (see: [3] § 36) for this reason $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ is an L-symplex. (see: [3] § 45). Regard now the union of the simpleces $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]+\mathbf{z}\ \mathbf{z}\in L$, and $S'[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]+\mathbf{z}\ \mathbf{z}\in L$ where S' is the image of S after reflecting S to the centre of the parallelepiped P. Since the complement of this set is the following one:

$$U\{P\setminus(S\cup S')+\mathbf{z}|\ \mathbf{z}\in L\},$$

where for arbitrary $\mathbf{z}_1,\mathbf{z}_2\in L$ the dimension of the set $(P\setminus(S\cup S')+$ $+\mathbf{z}_1)\cap(P\setminus(S\cup S')+\mathbf{z}_2)$ less then two, for this reason $P\setminus(S\cup S')$ can be decomposed by L-polyhedra. (The L-partition is unique!) The P also can be decomposed by L-polyhedra.

(2) Secondly assume, that two vectors are orthogonal to each other. (f.e $\mathbf{e}_1$ and $\mathbf{e}_2$) Regard the circumscribed-ball of the symplex $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$. If it isn't a "support-ball", then in its inside there is a lattice point. Decrease now the angle of the surfaces $P[\mathbf{e}_1,\mathbf{e}_3]$, $P[\mathbf{e}_2,\mathbf{e}_3]$ in a sufficiently small amount. Then the angle of the new vectors $\mathbf{e}'_1,\mathbf{e}'_2$ is an acute one, and the ball still contains in its inside a lattice point. Since the new parallelepiped $P[\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3]$ also satisfies the so-called "diagonalical condition" (see [3] § 36), then $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$ is a system of the successive minima. But by virtue of the first case (1), $S[\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3]$ is an L-symplex, namely its circumscribed-ball is empty. This is contradiction, for this reason the circumscribed-ball of the symplex $S[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ is empty, too. This means, that the polyhedron $Q[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ having its vertices on the sphere of the circumscribed-ball of the symplex S is an L-polyhedron. Repeating the method of the first case (1), we get that P can be decomposed by L-polyhedra, too.

LEMMA 3. If a double-lattice like congruent spherepacking is optimal, then $R=2$.

PROOF. Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be the successive minima of the lattice $L[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ and regard that point $\mathbf{x}$ of the lattice $L+\mathbf{a}$ which is in the parallelepiped $P[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. Since this parallelepiped can be decomposed by L-polyhedra (see: Statement) so if $L \cup L + \mathbf{a}$ is optimal, then it can be assumed that $\mathbf{x}$ is the centre of the maximal support polyhedron (see: Lemma 1) and for this reason, every latticepoint of L is outside of the ball with the radius R and centre $\mathbf{x}$, or it is on this sphere. Assume that $R > 2$! In this case there exists such an ε, that for every vector $\boldsymbol{\varepsilon}$, for which $|\boldsymbol{\varepsilon}| < \varepsilon$, $B(\mathbf{x} + \boldsymbol{\varepsilon}, 1) \cap B(\mathbf{y}, 1) = 0$ for every $\mathbf{y} \in L$. ($B(\mathbf{x} + \boldsymbol{\varepsilon}, 1)$ is the closed ball with the centre $\mathbf{x} + \boldsymbol{\varepsilon}$ and with the radius 1.) It's clear, that if there exists such an affine map $A \in E^{3\times 3}$ for which the minimum of $A(L)$ is equal to two, the volume of $A(P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3))$ is less than the volume $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $B(A(\mathbf{x}), 1) \cap B(A(\mathbf{y}), 1) = \emptyset \ \forall \mathbf{y} \in L$, then the system $L \cup L + \mathbf{a}$ isn't optimal one. It can be seen, that the last condition holds if the norm $||A - I||$ is "sufficiently small" where I is the unit matrix, and the norm is an arbitrary norm in $E^{3\times 3}$ (for example the row norm). We distinguish two cases:

(i) $$|\mathbf{e}_3| > |\mathbf{e}_1|.$$

Let A be such an orthogonal affin mapping for which $[\mathbf{e}_1, \mathbf{e}_2]$ is the fix plane and for which $v(A(P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3))) < v(P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3))$. It's easy to see that the three conditions hold for this A, and $L \cup L + \mathbf{a}$ isn't an optimal system. This is a contradiction.

(ii) $$|\mathbf{e}_1| = |\mathbf{e}_3| \ (= |\mathbf{e}_2|).$$

At this time we distinguish two cases, too:

(a) The angle of the vectors $\mathbf{e}_i, \mathbf{e}_j$ is acute for every pair of indices.

Look for the minimal vectors of $L[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$! Delone proved, that the minimal vectors of this lattice can be found among the diagonals of the parallelepiped $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. (see: [3]) For this reason we have to examine only the following vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k$, $i = j = k$, $i, j, k \in \{1, 2, 3\}$. We prove that for example the vector $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ isn't a minimal one. Since $|\mathbf{e}_3| < |\mathbf{e}_3 - \mathbf{e}_1|$, $|\mathbf{e}_3| < |\mathbf{e}_3 - \mathbf{e}_2|$ and the triangle OBC is an acute one, the orthogonal projection of the point A to the plane $OBCD$ is the angular domain $PA'Q$ where $A'P$ is the normal bisector of OB, and $A'Q$ is the normal bisector of OC. (see: figure 1)

For this reason the point A is inside of that half-space which contains the point O and the boundary plane of which is the normal bisector plane of the diagonal OD. This means, that the vectors $\mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_2$, $\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_1$ are not minimal vectors. For this reason the minima of L can be found among the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3$.

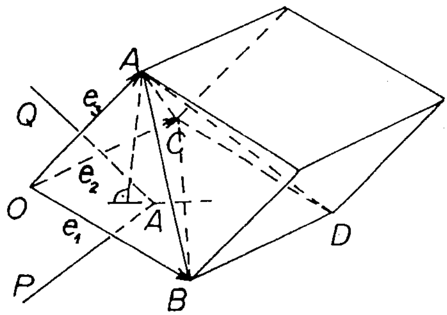

Fig. 1

Assume that one of these vectors isn't minimal. (for example $\mathbf{e}_1 - \mathbf{e}_3$) Regard the following mapping A: Decrease the angle of the surfaces OBC and OAC. It's easy to see, that this map A satisfies the earlier conditions and we have got a contradiction. This means, that it can be assumed that $|\mathbf{e}_i| = |\mathbf{e}_1 - \mathbf{e}_2| = |\mathbf{e}_1 - \mathbf{e}_3| = |\mathbf{e}_2 - \mathbf{e}_3|$ $i = 1,2,3$ but in this case the symplex $S(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is regular and it follows from the starting condition that $2 = |\mathbf{e}_i| > \max\{\min\{\mathbf{x} - \mathbf{y}, \mathbf{x}$ is a vertex of $P\} | \mathbf{y} \in P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\} \geq R > 2$. This is a contradiction too.

(b) If for example the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal to each other. Let A be the following map: Decrease the angle of the plane $[\mathbf{e}_2, \mathbf{e}_3]$ and the plane $[\mathbf{e}_1, \mathbf{e}_3]$. It's clear that this mapping A satisfies the earlier conditions namely the given double-latticelike system isn't optimal. In every case we get contradiction, so the indirect statement isn't true and we proved the Lemma 3.

LEMMA 4. Assume that $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3|$ and the system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a successive minimum system in the lattice L. Then the radius R is less then $|\mathbf{e}_1|$.

PROOF. Since the system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is one of the successive minima, the parallelepiped $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ can be decomposed by L-polyhedra. For this reason R is less or equal to the following number K_P:

$$K_P = \max\{\min\{|\mathbf{x} - \mathbf{y}| \quad \text{where } \mathbf{x} \text{ is a point of } L\} \mathbf{y} \in P\}.$$

We shall prove that the union of the open balls $G(\mathbf{x}, |\mathbf{e}_1|)$, where $\mathbf{x}$ is a lattice point in $P(\mathbf{e}_1, \mathbf{e}_2) \cup P(\mathbf{e}_1, \mathbf{e}_2) + \mathbf{e}_3$ cover the parallelepiped $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ namely $|\mathbf{e}_1| > K_P$. Regard those balls the centres of which are lattice points of the plane of the parallelogram $P(e_1, e_2)$. Since the distance of the bound of the union of these balls from the plane $[e_1, e_2]$ is greater or equal to the distance (d) of that sphere-point from the plane $[e_1, e_2]$, which is lying on the sphere $S(O, e_1)$ above the centre of the triangle $S(e_1, e_2)$, then the union set covers that stripe which is bounded by the planes $[e_1, e_2]$ and $[e_1, e_2]+de_3$. Let d^* be the following number:

$$d^* = \min\{d | \text{such } L[e_1, e_2] \text{ for which } |e_1| = |e_2| = \min L\}.$$

It's easy to see, that $d = \sqrt{|e_1|^2 - r^2}/|e_1|$ where r is the radius of the circumscribe ball of the triangle $S(e_1, e_2)$. It's minimal iff **r** is maximal iff e_1 is perpendicular to e_2. For this reason $d^* = \sqrt{2}/2 > 1/2$ and we proved that the union really cover the parallelepiped P. This means that $R < |K_P| < e_1|$ and we proved the Lemma 4.

The proof of the theorem

We'll prove the statement of Theorem 2 of the paper [1] in that case when the packing is congruent.

THEOREM. *The density of the densest double-latticelike spherepacking is equal to the density of the densest latticelike spherepacking.*

PROOF. Assume that the lattice $L[e_1, e_2, e_3]$ is spanned in every case by the system of the successive minima $\{e_1, e_2, e_3\}$. Then the point system can be written in the following form:

$$L[e_1, e_2, e_3] \cup L[e_1, e_2, e_3] + a.$$

We prove that if e_3 isn't perpendicular to the plane $[e_1, e_2]$ then the system $L \cup L + \mathbf{a}$ isn't an optimal one. From this statement already follows the statement of Theorem, because in the case of $e_3 \perp [e_1, e_2]$ where the vector e_3 is fix, the density is maximal if and only if the following number s is maximal:

$$s = \min\{|x - (e_3 - \frac{ae_3}{e_3^2} e_3)| \,\Big|\, x \text{ is a vertex of } P(e_1, e_2)\}.$$

A short calculation shows, that the quadratic form of the basic lattice L of the optimal systems are the following:

$$f_1(x_1, x_2, x_3) = 4(x_1^2 + x_2^2 + 2x_3^2)$$
$$f_2(x_1, x_2, x_3) = 4(x_1^2 + x_2^2 + x_1x_2 + (8/3)x_3^2).$$

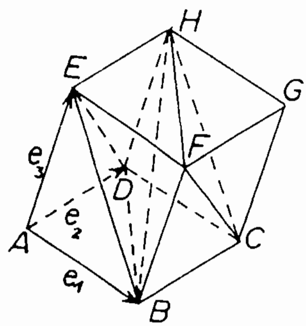

Fig. 2

Now we prove the above mentioned statement. We saw that one of the different types of support-balls are the followings: (see: figure 2, and the Statement)

(1) the circumscribed-ball of the tetrahedron $ABDE$ is the support ball with the greatest radius;

(2) the circumscribed-ball of the tetrahedron $EBFH$ is the maximal support ball;

(3) the circumscribed-ball of the tetrahedron $FBHC$ is the maximal support ball.

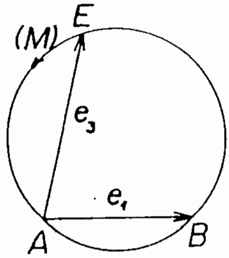

Fig. 3

Here we assumed that the shortest one among the diagonals of the octahedron $EBCHFD$ is BH and note that these balls may be equal to each other in some cases. In every case we shall give such a motion (M) of a vertex of the parallelepiped P, which is transforming the original system into a "better" system. This means that in the optimal case $\mathbf{e}_3$ is perpendicular to the plane $[\mathbf{e}_1, \mathbf{e}_2]$.

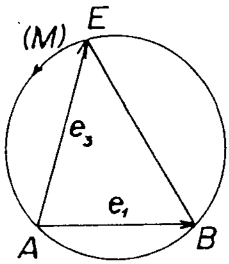

Fig. 4/a

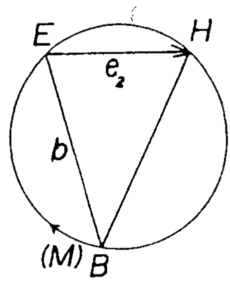

Fig. 4/b

(1) If $\mathbf{e}_3$ isn't perpendicular to the plane $[\mathbf{e}_1, \mathbf{e}_2]$ then for example $\mathbf{e}_3$ isn't perpendicular to $\mathbf{e}_i$ $(i = 1 \text{ or } 2)$. Using the Lemma 3 and Lemma 4 we get that $|\mathbf{e}_3| > |\mathbf{e}_1|$ and $|\mathbf{e}_3| \geq |\mathbf{e}_2|$ $|\mathbf{e}_3| \leq |\mathbf{e}_3 - \mathbf{e}_i|$ (using the definition of the successive minima), for this reason the motion which can be seen on the figure 3 is good. (The basic-volume of the system decreased and the minimal distance of the point-system didn't decrease so this motion gives a better system.)

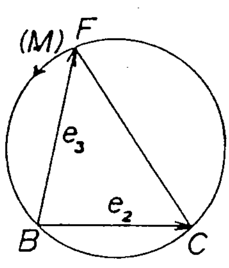

Fig. 5/a

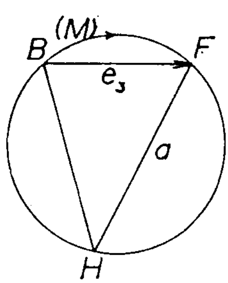

Fig. 5/b

(2) If the vector $\mathbf{e}_1$ isn't perpendicular to the vector $\mathbf{e}_3$, the good motion can be seen on the figure 4/a and if $\mathbf{e}_1$ is perpendicular to $\mathbf{e}_3$ then $|\mathbf{b}| \geq |\mathbf{e}_3| > |\mathbf{e}_1|$ and $\mathbf{b} = \mathbf{e}_3 - \mathbf{e}_1$ so $\mathbf{b}$ isn't perpendicular to $\mathbf{e}_2$ and the good motion can be seen on the figure 4/b.

(3) If the vector $\mathbf{e}_2$ isn't perpendicular to $\mathbf{e}_3$ then the good motion can be seen on the figure 5/a, and in the other case the good motion can be seen on the figure 5/b. (now $\mathbf{a} = \mathbf{e}_2 - \mathbf{e}_1$ isn't perpendicular to $\mathbf{e}_3$).

So we have verified the statement and the Theorem.

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SPIRAL CIRCLE PACKINGS ON A SPHERE

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1. Introduction

The Tammes problem is the following: how must n equal non-overlapping circles be packed on a sphere so that the angular diameter of the circles will be as great as possible? The solutions of this problem are proven only for some cases ($n = 3$ to 12 and $n = 24$), for other values of n there are only estimations given by dense arrangements. Many construction methods were applied, for example, axially symmetric packing (GOLDBERG, [3]), multibranched helical packing (SZÉKELY, [6]), multi-symmetric packing (ROBINSON, [5], TARNAI, [7], TARNAI and GÁSPÁR, [10]), construction of new packing by moving the graph of an existing packing (DANZER, [2], TARNAI and GÁSPÁR, [9]), by minimization of a repulsion-energy function (CLARE and KEPERT, [1]), on the basis of a similar principle by using a special iteration which works even in higher dimensions (LAZIĆ, ŠENK and ŠEŠKAR, [4]).

In this paper the algorithm to construct spiral arrangements is shown. In some cases (n = 28, 29, 53, 54, 55, 78, 79, 96, 198) these spiral arrangements give more dense packings than the previously best known ones. This method is suitable also for construction of quite dense arrangements in the case of large values of n, so it can give a lower estimation for the solution of the Tammes problem, or this spiral arrangement can be used as a starting-point for the improving method of moving the graph (TARNAI and GÁSPÁR, [9]).

2. Spiral arrangement

2.1. Prescribed diameter. The first task is to pack the maximum number of non-overlapping circles with prescribed diameter d on the unit sphere. The circles are placed one by one onto the sphere, attaching a serial number to each circle. The spiral arrangement will be achieved by positioning the first circle at the south pole, the second one touches the first one, and the positions of the following circles are determined by the following rules:

(i) circles must not intersect (overlap) each other

(ii) each circle has to touch the preceding one

(iii) each circle has to touch a circle placed before with the lowest possible serial number.

Let us assume, that j circles are already placed on the sphere. Let the kth circle be the circle with the lowest serial number touching the jth circle. The circles with serial numbers not smaller than k and not greater than j form a closed ring on the sphere. There are no circles to the north of this ring. The $(j+1)$th circle is placed on the north side of this ring.

The spherical coordinates $(\phi_{j+1}, \theta_{j+1})$ of the $(j+1)$st circle will be determined according to the following steps:

a) The $(j+1)$th circle is constructed as the common tangent circle of the jth and ith circle. (From the two possible alternatives we select the circle, which lies to the north of the geodetic line connecting the centres of the ith and jth circles. Should this line be identical with a great circle connecting the north and south pole, then we select the common tangent circle on the eastern side.) In order to comply with rule (iii) we first try the value $i = k$.

b) According to the notations of Fig. 1. the centre of the circle touching the ith and jth circle can be calculated as follows:

$$\begin{aligned}
\alpha &= \theta_i - \theta_j \\
c &= \operatorname{acos}(\cos\phi_j \cos\phi_i + \sin\phi_j \sin\phi_i \cos\alpha) \\
\beta &= \operatorname{atan}(\sin\alpha/(\sin\phi_j/\tan\phi_i - \cos\alpha\cos\phi_j)) \\
\delta &= \operatorname{asin}(\sin(c/2)/\sin d) \\
\gamma &= \operatorname{atan}(1/\cos d/\tan\delta) + \beta \\
\phi_{j+1} &= \operatorname{acos}(\cos\phi_j \cos d + \sin\phi_j \sin d \cos\gamma) \\
\theta_{j+1} &= \theta_j + \operatorname{atan}(\sin\gamma/(\tan d - \cos\gamma\cos d)).
\end{aligned}$$

c) We check the circles from the kth one to the jth one (which form a ring) whether they intersect the $(j+1)$st circle.

- If none of them intersects, than the coordinates ϕ_{j+1}, θ_{j+1} are correct and we can proceed by calculating the coordinates of the next circle.

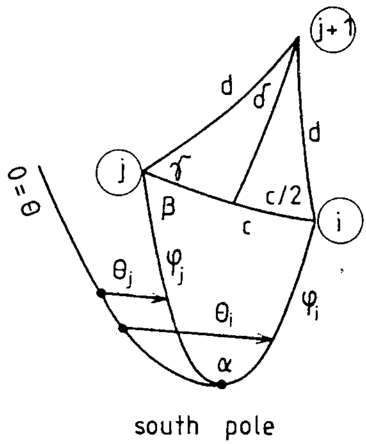

Fig. 1.

- If the lth circle and the $(j+1)$st circle intersect and $l < i$ then the packing is completed: in this system j circles of diameter d could be packed.
- If $l > i$ then we repeat the described procedure with $i = l$ from step b).

2.2. Prescribed number of circles. If we have packed n circles of diameter d, this does not mean that the packing of the same number of circles with greater diameter is not possible. The algorithm described in section 2.1 evaluates a step function $n(d)$ at arbitrary d.

If $n_1 \geq n$ circles of diameter d_1, and $n_2 < n$ circles of diameter d_2 can be packed, then the possible greatest diameter d of n circles packed in a spiral system can be determined numerically up to the computation accuracy by the interval-bisection process. In this limit case a new circle will touch three rather than two previous circles, and the graph of the packing will be rigid (TARNAI and GÁSPÁR, [9]).

3. Numerical results

Based on the above described algorithm a FORTRAN program was written running on an IBM–XT PC. By using this program we determined the possible greatest diameters from $n = 6$ to 100 and for numbers greater than 100 for the cases published in the paper of TARNAI [8].

Table 1.

Close packings of congruent circles on a sphere

n	Diameter (^{0})	Density	n	Diameter (^{0})	Density
28	39.14182205	0.80881	82	23.03645367	0.82569
29	38.40478112	0.80674	83	22.85331367	0.82257
53	28.63421300	0.82304	84	22.55208231	0.81075
54	28.47227658	0.82816	85	22.37301112	0.80746
55	27.95603488	0.81432	86	22.29319772	0.81116
61	26.61134482	0.81874	87	22.19002027	0.81304
62	26.43691398	0.82134	88	22.10755795	0.81630
63	26.13469033	0.81569	89	21.96701652	0.81515
64	25.93952876	0.81636	90	21.87783147	0.81765
65	25.61937674	0.80886	91	21.84907313	0.82457
66	25.46305514	0.81136	92	21.77852372	0.82827
67	25.12629608	0.80209	93	21.53588686	0.81878
68	25.07708133	0.81089	94	21.39654628	0.81694
69	24.95113696	0.81461	95	21.19571280	0.81024
70	24.89696276	0.82284	96	21.11601512	0.81264
71	24.71915675	0.82276	97	20.98828446	0.81123
73	24.23337447	0.81314	98	20.91392523	0.81382
74	24.13215720	0.81743	99	20.82574359	0.81522
75	23.93258807	0.81488	100	20.72996468	0.81592
76	23.73145671	0.81198	198	14.76641391	0.82082
77	23.56101859	0.81093	202	14.65440761	0.82476
78	23.46867252	0.81506	2610	4.07089108	0.82340
79	23.36084975	0.81796	2650	4.03686694	0.82210
81	23.06920105	0.81793			

In some cases ($n = 6, 7, 8, 9, 10, 12$) we get the provenly best possible result. In the case of $n = 10$ we arrived at the configuration given by DANZER [2], but we determined the diameter more precisely ($d = 66.14682198^0$).

In Table 1. we listed both the cases in which we achieved better results than the previously best known ones and the cases that have not been investigated yet.

For the cases n = 28, 29, 53, 54 and 55 the best results were published by SZÉKELY [6]. The packings constructed by the spiral method are more dense, but in the case of 28 circles the results of LAZIĆ et al. ($d_{28} = 39.35493^0$), in the case of 29 circles the results of TARNAI and GÁSPÁR [11] ($d_{29} = 38.67707^0$) and LAZIĆ and ŠENK ($d_{29} = 38.70737^0$) are even better. The stereographic projection of the graphs for the cases n = 53, 54 and 55 are illustrated in Figs. 2–4. (For the sake of simplicity, the edges of the graph are replaced usually by straight line segments, but to avoid intersections in the pictures two edges are drawn as a curve. The order of the packing is shown by heavy lines, and the last edges achieved by the iteration are drawn by broken lines.)

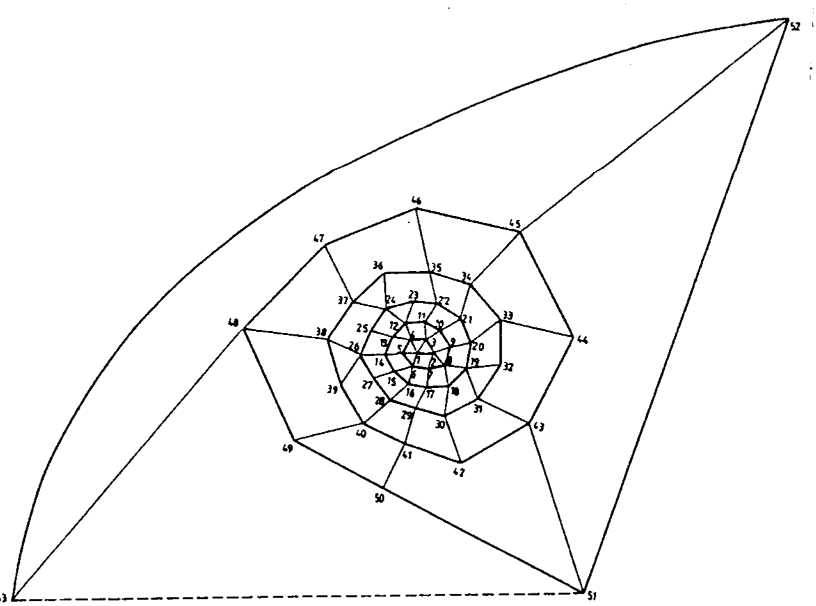

Fig. 2.

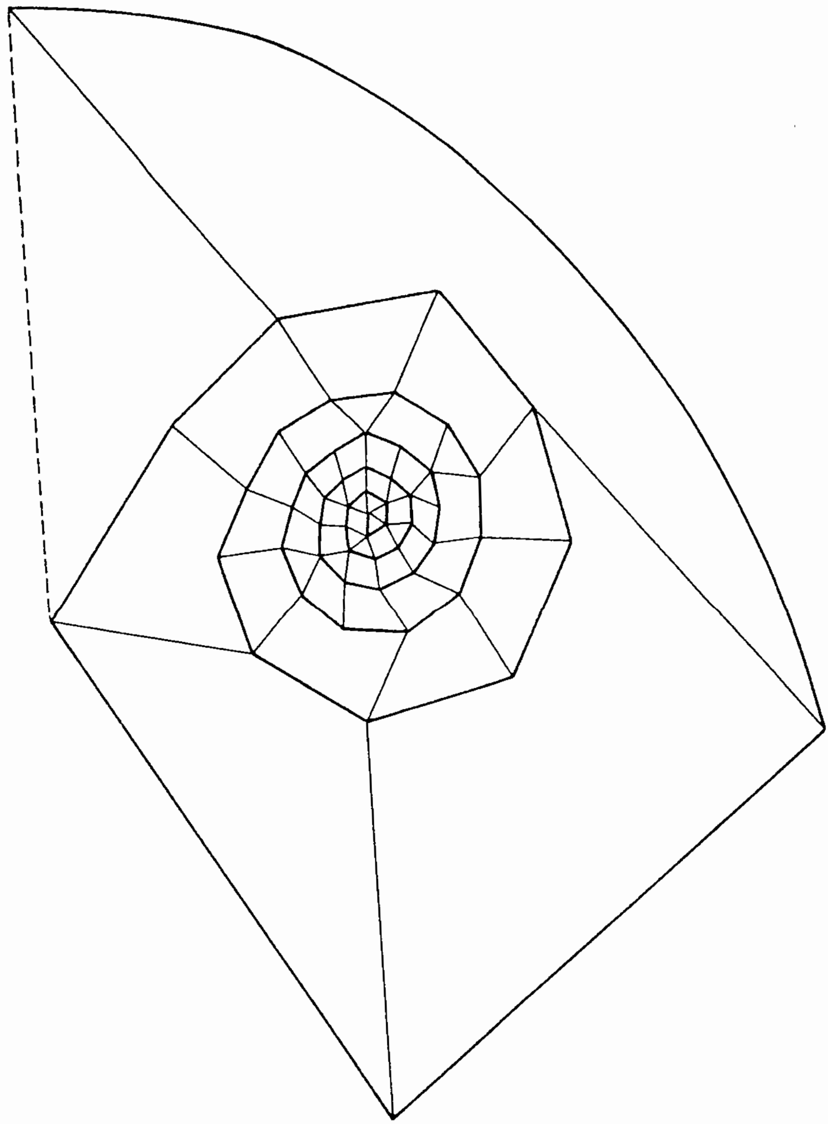

Fig. 3.

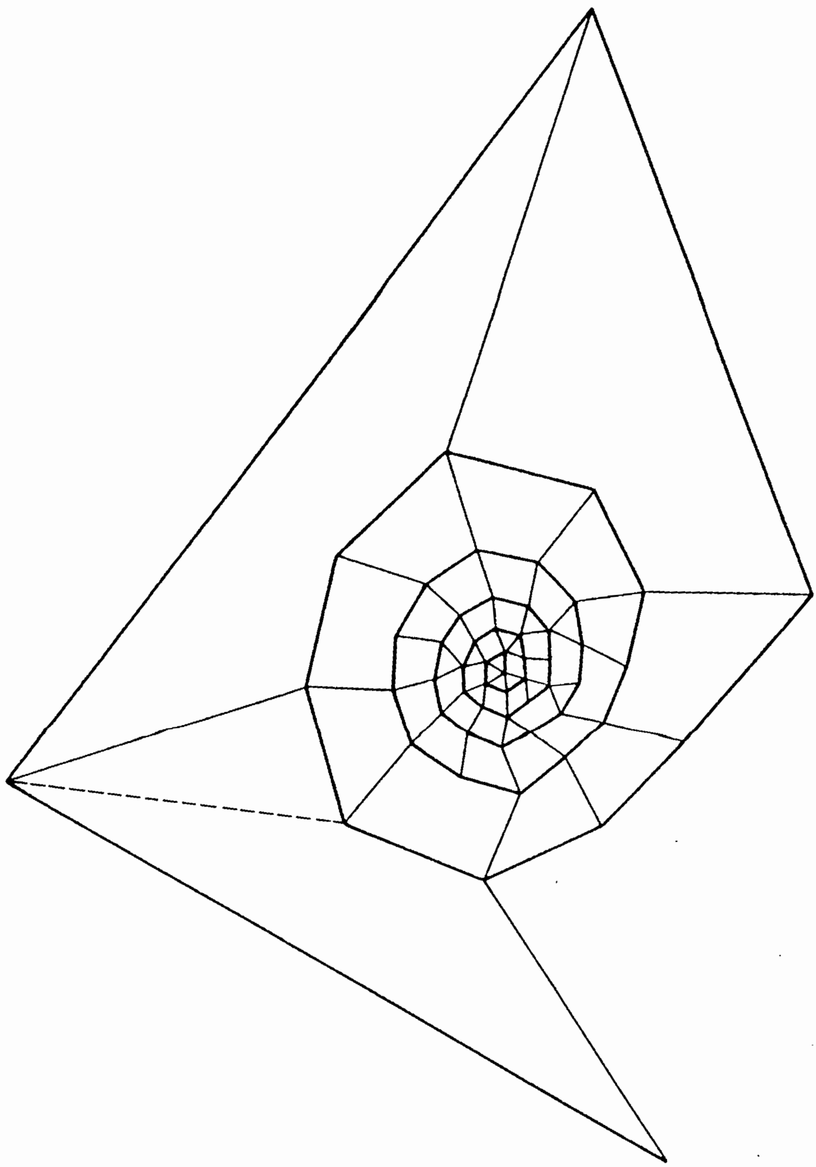

Fig. 4.

In the case $n < 100$ the topology of the graphs can be reconstructed by using Table 2. According to the algorithm the circles with adjacent serial numbers are tangent to each other, so these edges are not given in the table. The circles with $j > 3$ are generally touching one more circle with a lower serial number. The circles from 3 to 10 resulted in the same edges in each configuration: 1–3, 1–4, 1–5, 1–6, 2–7, 2–8, 2–9, 3–10. The second column of the table contains the neighbour (with the lowest serial number) of the circles with a serial number greater than 10. The last column specifies the last edge determined by the iteration. (Fig. 2 contains the serial numbers of the circles, so by comparing the figure and the data in the table the notation of the table may be easier to understand.)

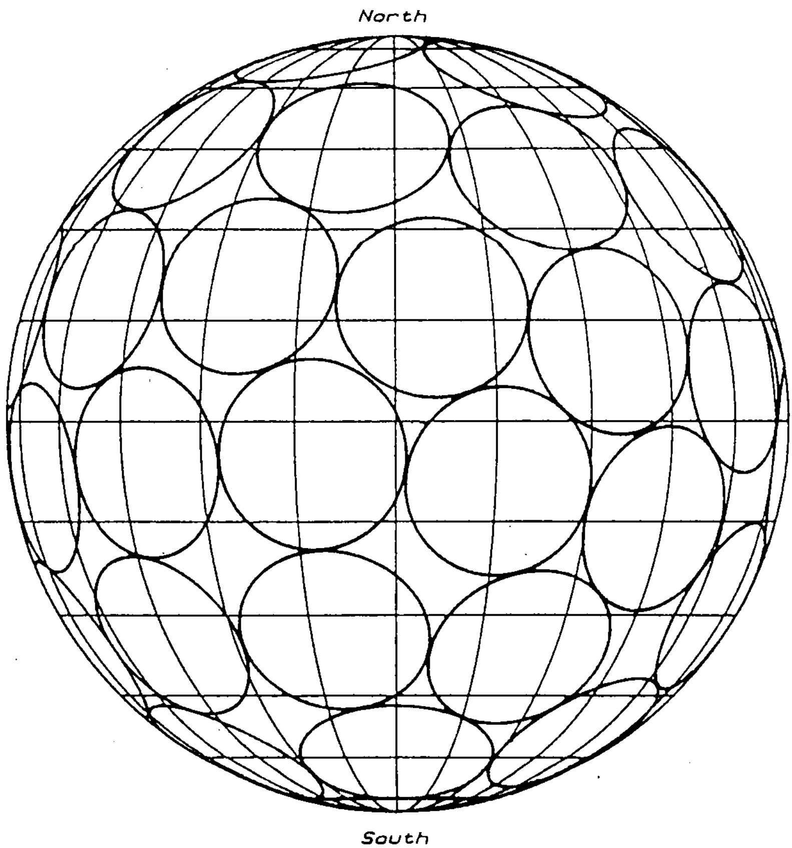

Fig. 5.

Using the results of spherical circle packings GÜNTHER KOLLER (Trollhätten, Sweden) prepared some beautiful drawings with the plotter of his PC. We demonstrate these plots for $n = 53$ in Fig. 5 and for $n = 2610$ in Fig. 6.

The research work was supported by OTKA Grant No. 744.

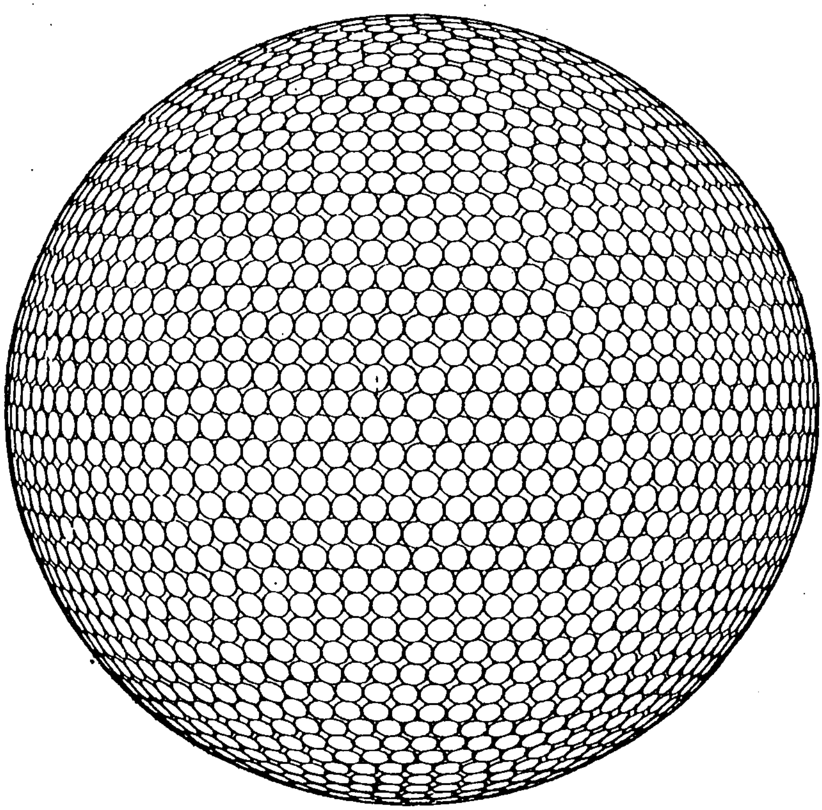

Fig. 6.

Table 2.
The graph of the packings

n	Neighbour of the jth circle $(j = 11, \ldots, n)$	Last edge
28	4,4,5,6,6,7,8,9,10,11,12,13,14,16,17,19,21,23	25–28
29	4,4,5,5,6,7,8,9,10,11,12,13,14,16,17,18,19,22,24	25–29
53	3,4,4,5,6,6,7,8,8,9,10,11,12,12,13,14,15,16,17,18,19,19,20,21,22 24,24,26,26,28,29,30,31,33,34,35,37,38,40,41,43,45,48	51–53
54	3,4,4,5,6,6,7,8,8,9,10,11,12,12,13,14,15,16,17,18,18,19,20,21,22 24,24,26,26,28,29,30,31,33,34,35,36,38,39,41,43,44,47,49	51–54
55	3,4,4,5,6,6,7,8,8,9,10,11,11,12,13,14,14,16,16,18,18,19,20,21,22 23,24,25,26,27,28,30,31,32,33,34,36,37,39,40,41,43,45,47,49	48–54
61	3,4,4,5,5,6,7,8,8,9,10,10,11,12,13,14,14,15,16,17,18,19,20,20,22 23,24,24,26,27,27,29,29,31,32,33,34,35,37,38,39,40,42,43,45,46 47,49,51,53,56	59–61
62	3,4,4,5,5,6,7,8,8,9,10,10,11,12,13,14,14,15,16,17,18,19,20,20,21 23,23,24,25,26,27,29,29,31,32,33,34,35,36,38,39,40,41,43,44,45 47,49,50,52,54,57	55–61
63	3,4,4,5,5,6,7,8,8,9,10,10,11,12,13,14,14,15,16,17,18,19,19,20,21 22,23,24,25,26,27,28,29,31,32,32,34,35,36,37,38,40,41,42,43,45 46,48,49,51,53,56,58	59–63
64	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,14,14,15,16,17,18,19,19,20,21 22,23,24,25,26,27,28,29,30,32,32,33,34,36,37,38,39,41,42,43,45 46,47,49,51,52,54,56,58	57–63
65	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,17,18,19,19,20,21 22,23,24,25,26,27,28,29,30,31,32,33,34,35,37,38,39,40,41,43,44 46,47,48,50,51,53,55,57,59	58–64
66	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,17,18,19,19,20,21 22,23,24,25,26,27,28,29,30,31,32,33,34,35,37,38,39,40,41,43,43 45,47,48,49,51,52,54,56,58,61	63–66
67	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,16,18,19,19,20,21 22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,40,41,42,43 44,46,47,48,50,51,53,54,56,59,61	62–67
68	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,16,18,19,19,20,21 22,23,24,25,26,27,28,29,29,31,32,33,34,35,36,37,38,40,41,42,43 44,46,47,48,50,51,52,54,56,58,60,63	64–68
69	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,16,18,19,19,20,21 22,23,24,25,26,27,27,29,29,30,32,33,34,35,36,37,38,39,41,42,43 44,45,47,48,49,51,52,54,55,57,59,61,64	65–69
70	3,4,4,5,5,6,7,8,8,9,9,10,11,12,13,13,14,15,16,16,18,19,19,20,21 22,23,24,25,26,27,27,29,29,30,32,33,34,35,36,37,38,39,41,41,43 44,45,47,48,49,51,52,53,55,56,58,60,62,65	68–70

71	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,16,16,17,18,19,20,21 22,23,24,24,26,27,27,28,29,30,32,33,33,34,36,37,38,39,40,41,42 44,45,46,47,48,50,51,53,54,56,57,59,60,63,66	67–71
73	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,16,16,17,18,19,20,21 22,23,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,40,41,42 43,44,45,47,48,49,50,52,53,55,56,57,59,61,63,65,67	70–73
74	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,16,16,17,18,19,20,21 22,23,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,41,42 43,44,45,47,48,49,50,51,53,54,56,57,58,61,62,64,67,68	70–74
75	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,16,16,17,18,19,20,21 21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,42 42,44,45,46,48,48,50,51,53,54,55,56,58,59,61,63,65,67,70	71–75
76	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,16,16,17,18,19,20,21 21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41 42,43,45,46,47,48,49,51,52,53,55,56,57,59,61,62,64,66,68,70	72–76
77	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,15,16,17,18,19,20,20 21,22,23,24,25,26,27,28,29,30,30,32,33,34,35,36,37,38,39,40,41 42,43,45,45,47,48,49,50,52,53,54,56,57,58,60,62,63,64,66,69,71	72–77
78	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,15,16,17,18,19,19,20 21,22,23,24,25,26,27,28,29,30,30,32,33,33,35,36,37,38,39,40,41 42,43,44,45,47,48,49,50,51,52,54,55,57,58,59,60,62,64,65,67,69 71	73–78
79	3,4,4,5,5,6,7,8,8,9,9,10,11,12,12,13,14,15,15,16,17,18,19,19,20 21,22,23,24,25,26,27,28,29,30,30,32,33,33,35,36,37,38,39,40,41 42,43,44,45,46,48,49,50,51,53,54,55,56,57,59,60,62,63,65,67,69 71,74	75–79
81	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,26,27,27,29,30,30,31,33,33,34,35,36,37,38,40,41 42,43,44,45,46,47,48,49,51,52,53,54,56,57,58,60,61,63,64,66,67 69,70,73,76	77–81
82	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,26,27,27,29,30,30,31,33,33,34,35,36,37,38,40,41 42,43,44,45,46,47,48,49,51,52,53,54,56,57,58,60,61,62,64,65,66 68,70,72,73,77	75–81
83	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,26,27,27,28,29,30,31,33,33,34,35,36,37,38,39,40 41,43,43,45,46,47,48,49,50,52,53,54,55,56,58,59,60,61,63,64,66 68,69,71,73,75,78	76–82
84	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,26,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40 41,42,43,44,46,46,48,49,50,51,52,54,55,56,57,59,60,61,62,64,65 67,68,70,72,74,76,78	80–84

85	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,26,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40 41,42,43,44,45,46,47,49,50,51,52,53,54,56,57,58,59,61,62,64,65 67,68,69,71,73,75,77,80	81–85
86	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40 41,42,43,44,45,46,47,49,50,51,52,53,54,56,57,58,59,61,62,63,65 66,67,69,70,72,73,75,78,80	82–86
87	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,25,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40 41,42,43,44,45,46,47,49,50,50,52,53,54,55,57,58,59,60,62,62,64 65,67,68,70,72,73,75,76,79,81	82–87
88	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40 41,42,43,44,45,46,47,48,49,50,52,53,54,55,56,58,59,60,61,62,64 65,66,68,69,71,72,74,75,78,80,83	84–88
89	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,24,24,25,26,27,28,29,30,31,32,33,34,34,36,36,38,39,40 41,42,43,44,45,46,47,48,49,50,51,52,53,55,56,57,58,59,61,62,63 65,66,67,68,70,72,73,75,76,79,81,84	85–89
90	3,4,4,5,5,6,6,7,8,9,9,10,11,11,12,13,14,14,15,16,17,18,19,19,20 21,22,23,23,24,25,26,27,28,29,30,30,32,33,34,34,36,36,37,38,40 41,42,43,44,45,46,47,48,49,50,51,52,53,55,56,57,58,59,61,62,63 64,66,67,68,69,71,73,74,76,77,79,81,85	86–90
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93	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,14,14,15,16,17,17,19,19,20 21,21,22,23,24,25,26,27,28,29,30,30,31,33,34,34,35,36,37,38,39 40,41,42,43,44,45,46,47,49,50,51,52,53,54,55,56,58,59,60,61,62 63,65,66,67,69,70,71,73,74,76,77,79,81,82,85,88	90–93
94	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,14,14,15,16,16,17,19,19,20 21,21,22,23,24,25,26,27,28,29,30,30,31,32,34,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,50,51,51,53,54,56,57,58,59,60,61,62 63,64,66,67,69,70,71,73,74,75,77,78,81,82,84,86,89	90–94
95	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,14,14,15,16,16,17,19,19,20 21,21,22,23,24,25,26,27,28,29,29,30,31,32,34,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,49,51,51,53,54,55,56,57,58,59,61,62 63,64,65,67,68,69,70,72,73,75,76,78,79,81,83,85,87,90	91–95

96	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,14,14,15,16,16,17,18,19,20 21,21,22,23,24,25,26,27,28,29,29,30,31,32,33,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,49,51,51,53,53,54,56,57,58,59,60,62 63,64,65,67,68,69,70,72,73,74,76,77,79,80,82,84,86,88,90	92–96
97	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,14,14,15,16,16,17,18,19,20 21,21,22,23,24,25,26,27,28,29,29,30,31,32,33,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,56,57,58,59,60,61 63,64,65,66,68,68,70,71,73,74,75,77,78,80,82,83,84,87,89,92	93–97
98	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,13,14,15,16,16,17,18,19,20 21,21,22,23,24,25,26,27,28,28,29,30,31,32,33,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,57,58,59,60,61 62,63,64,66,68,68,70,71,72,74,75,76,78,79,80,83,84,85,88,90,93	94–98
99	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,13,14,15,16,16,17,18,19,20 21,21,22,23,24,25,26,27,28,28,29,30,31,32,33,34,35,36,37,38,39 40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,57,58,59,60,61 62,63,64,66,67,68,69,70,72,73,75,76,77,79,80,82,83,85,86,88,90 94	95–99
100	3,4,4,5,5,6,6,7,8,8,9,10,11,11,12,13,13,14,15,16,16,17,18,19,20 21,21,22,23,24,25,26,27,28,28,29,30,31,32,33,34,35,36,37,38,39 40,41,42,42,44,45,46,47,48,49,50,51,52,53,54,55,56,57,57,59,61 62,63,64,65,67,68,69,70,71,73,74,75,77,78,79,81,83,84,86,88,90 92,95	96–100

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BIPYRAMIDAL NONCOMPACT HYPERBOLIC SPACE FORMS WITH FINITE VOLUME

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Following Poincaré's geometric method, we shall construct 8 non-isomorphic isometry groups G_i ($i = 1, 2, \dots, 8$), acting discontinuously and freely on the hyperbolic 3-space H^3. Each group is given by the same noncompact fundamental Dirichlet polyhedron $\mathcal{D}$ (Fig. 3.) and by pairing its faces via isometries in 12 different ways. The identifying isometries generate the corresponding group G_i. This way we can obtain all the hyperbolic space forms $\tilde{\mathcal{D}}_i = H^3/G_i$ whose fundamental polyhedron is the given bipyramid $\mathcal{D}$ if the six base edges of $\mathcal{D}$ are required to be G_i-equivalent. Since each group G_i (Fig. 5–12) contains orientation reversing isometries and all vertices of the polyhedron $\mathcal{D}$ are ideal points, each manifold $\tilde{\mathcal{D}}_i = H^3/G_i$ becomes a nonorientable noncompact hyperbolic space form with finite volume.

The constructions were motivated by the Euclidean compact nonorientable plane form $E^2/\mathbf{pg}$ of genus 2 (Klein bottle), which has two special fundamental hexagons (Fig. 1, 2). Describing the isometry group of H^3/G_i, we find some interesting phenomena.

We shall exploit the existence of a Coxeter supergroup C of the group G_i, generated by reflections (Fig. 4), for describing the metric data of the space form $\tilde{\mathcal{D}}_i = H^3/G_i$ ($i = 1, 2, \dots, 8$).

This paper is a continuation of [6], where more details of the method are described and not repeated here.

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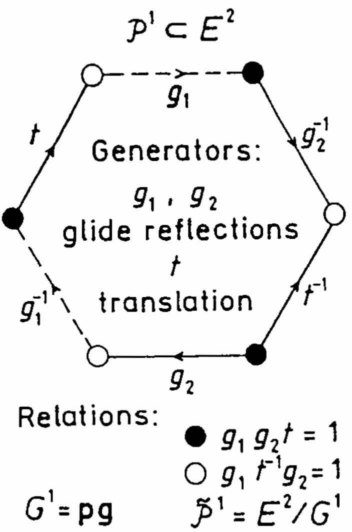


Fig.1.

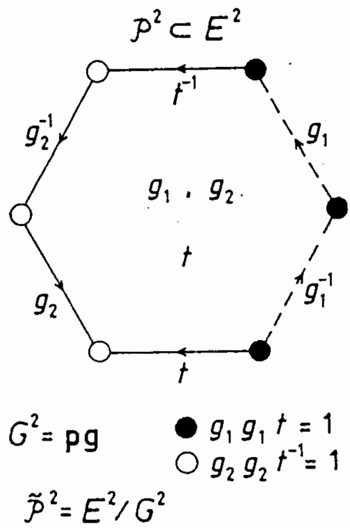


Fig. 2.

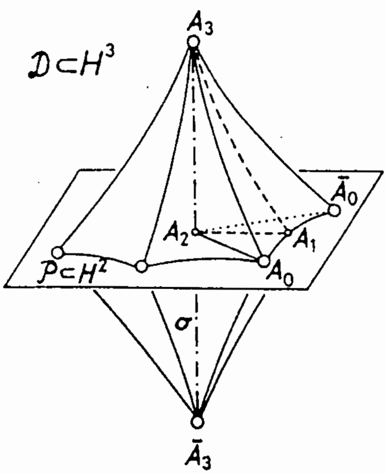


Fig.3.

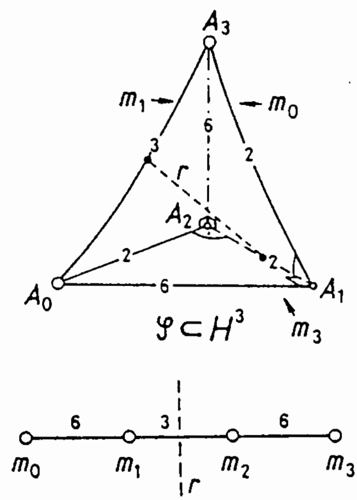


Fig.4.

1. The construction of the bipyramid $\mathcal{D}$, results

Take a triangle $A_0A_1A_2$ on a hyperbolic plane H^2 with a right angle at the vertex A_1, with an acute angle $A_1A_2A_0\sphericalangle = \frac{\pi}{6}$ and with the ideal vertex A_0 (Fig. 3). Thus the angle of parallelism $\prod(A_1A_2)$ equals $\frac{\pi}{6}$. Then, by reflections in the lines A_1A_2 and A_0A_2, we construct a regular hyperbolic hexagon $\mathcal{P}$, with all ideal vertices. On the plane containing the line A_1A_2 and the line σ perpendicular to the plane of the polygon $\mathcal{P}$ at the point A_2 we construct the rays A_1A_3 and $A_1\overline{A}_3$, such that both of them are parallel to the line σ, with the angle of parallelism $\prod(A_1A_2) = \frac{\pi}{6}$. We got the polyhedron $\mathcal{D}$, a regular hexagonal bipyramid with ideal vertices. All its faces form angles $\frac{\pi}{6}$ with the base, and all its intersecting faces, that are on the same side of the base, form angles $\frac{2\pi}{3}$. $\mathcal{D}$ is the required noncompact fundamental Dirichlet polyhedron.

Since all vertices of this polyhedron are ideal points, all its edges concurrent to such an ideal vertex belong to a parabolic bundle. A horosphere centred at this ideal vertex is orthogonal to this bundle and intersects the bipyramid $\mathcal{D}$ in a horosperical polygon. A horosphere is isometric to the Euclidean plane E^2, so these polygons are Euclidean. The angles of such a polygon are congruent to the dihedral angles of $\mathcal{D}$ where the edges of these dihedra are concurrent to the center of the horosphere, a so called parabolic vertex or end of $\mathcal{D}$.

In this sense, the apices of the bipyramid are related to Euclidean regular hexagons with angles $\frac{2\pi}{3}$, and the vertices of its base are related to Euclidean rhombs with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Some metric data of the polyhedron $\mathcal{D}$ will be described in Section 3 based on Fig. 3, 4.

In Section 2 we shall identify the faces of the bipyramid $\mathcal{D}$ in 12 different ways. So we obtain 8 non-isomorphic groups G_i. The result is symbolized in the Schlegel diagram of $\mathcal{D}_i$, and $\overline{\mathcal{D}}_i$ $(i = 1, 2, \dots, 8)$.

A face denoted, for example, by t_1^{-1} is mapped onto the face t_1 by the isometry t_1, moreover, $\tilde{\mathcal{D}}_i$ is mapped onto its neighbouring t_1-image $\tilde{\mathcal{D}}_i^{t_1}$. The inverse transformation t_1^{-1} maps the face t_1 onto the face t_1^{-1} and $\tilde{\mathcal{D}}_i$ onto $\tilde{\mathcal{D}}_i^{t_1^{-1}}$, the neighbour at the face t_1^{-1}.

Poincaré's polyhedron theorem (see [6], [3], [4] for more details) will guarantee the discontinuous and free action of each group G_i on H^3.

We summarize our results in two theorems.

THEOREM 1. *Each of the 12 pairings on the bipyramid $\mathscr{D}$, given in Section 2, generates the corresponding group G_i $(i = 1, 2, \dots, 8)$ which acts discontinuously on the hyperbolic space H^3. Each point of the identified $\tilde{\mathscr{D}}_i$ or $\bar{\mathscr{D}}_i$ has a trivial stabilizer subgroup in G_i. Every end of $\tilde{\mathscr{D}}_i$ has a stabilizer in G_i which is isomorphic to a Euclidean plane crystallographic group and acts discontinuously and freely on every horosphere centred in the end considered. Thus we have obtained hyperbolic space forms $\tilde{\mathscr{D}}_i = H^3/G_i$ of finite volume with some cusps.*

THEOREM 2. *For the given bipyramid $\mathscr{D}$ in H^3 there exist precisely 12 combinatorially different pairings and 8 non-isomorphic discontinuous freely acting isometry groups G_i whose fundamental domain is $\mathscr{D}$, equipped with corresponding identifications such that the six base edges of $\mathscr{D}$ are G_i-equivalent.*

These theorems will be proven at the end of Sections 2. Meanwhile we use the above notations for our space forms being constructed.

2. Identifications on $\mathscr{D}$, proofs

$i = 1$ $\qquad\qquad \tilde{\mathscr{D}}_1 = H^3/G_1$ $\qquad\qquad$ (Fig. 5).

The pairs of faces g_j^{-1} and g_j $(j = 1, 2, 3, 4)$ are identified by horospherical glide reflections g_j or their inverses g_j^{-1}, respectively.

The pairs of faces t_j^{-1} and t_j $(j = 1, 2)$ are identified by horospherical translations t_j.

The identifications of the faces of $\tilde{\mathscr{D}}_1$ are induced by those of the sides of the regular hexagon $\mathscr{P}^1$ in the Euclidean plane E^2 (Fig. 1), where g_j $(j = 1, 2)$ are glide reflections, and t is a translation in E^2. This observation motivated us to construct the group G_1 (Fig. 5, for a more general treatment of plane groups see [1]).

Now we list the edge equivalence classes of the polyhedron $\tilde{\mathscr{D}}_1$, induced by the above identifications, writing down for each edge class the defining relation which expresses that every point on these edges has trivial stabilizer. We establish incidentally the stabilizer subgroup for each end class (cusp).

The bipyramid $\tilde{\mathscr{D}}_1$ has 12 side edges, divided into 4 classes with 3 edges in each class:

(1) $\quad t_1 g_1^{-1} g_2^{-1} = 1$ at the edge class $\quad a$

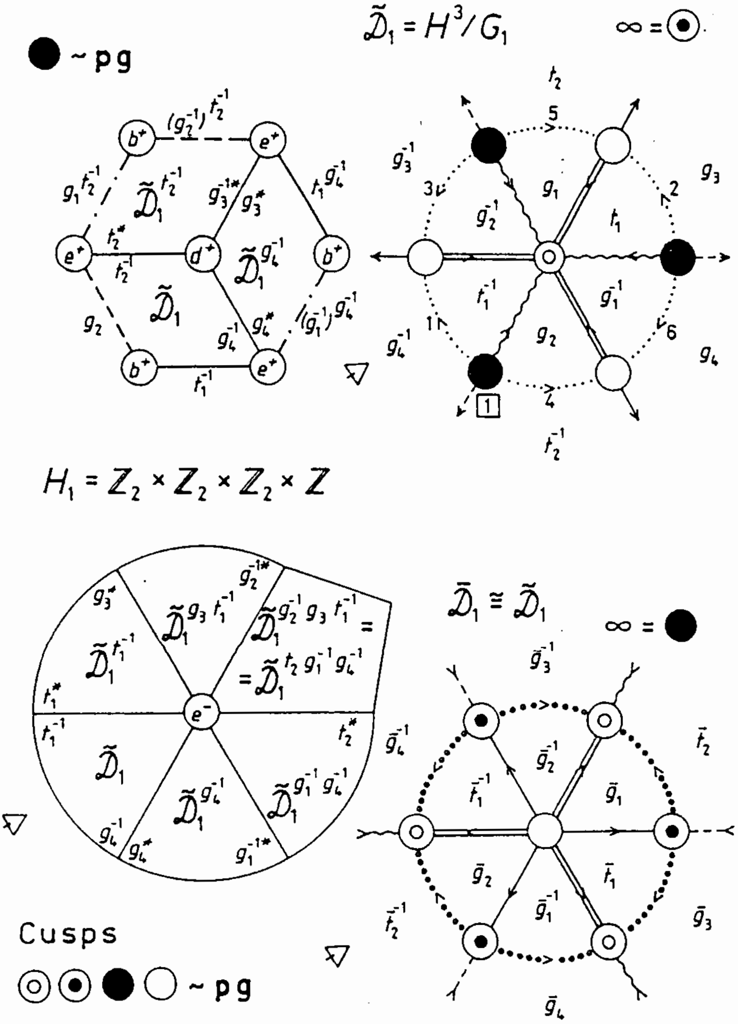

$\tilde{\mathcal{D}}_1 = H^3/G_1$
$\infty =$
~ pg
$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$
$\bar{\mathcal{D}}_1 \cong \tilde{\mathcal{D}}_1$
$\infty =$
Cusps
~ pg

Fig. 5.

(2) $t_1 g_1 g_2 = 1$ $\qquad b$ ∼∼∼∼>∼∼∼∼

(3) $t_2 g_3^{-1} g_4 = 1$ $\qquad c$ ———>———

(4) $t_2 g_3 g_4^{-1} = 1$ $\qquad d$ ---->---- (Fig. 5)

each of the related dihedral angle is $\frac{2\pi}{3}$, so $3 \cdot \frac{2\pi}{3} = 2\pi$ which guarantees the trivial stabilizer for points on these edges (see [6] for the general algorithm).

There are 6 base edges, belonging to the same class:

(5) $t_1 g_3^{-1} g_2 t_2 g_1^{-1} g_4^{-1} = 1$ $\qquad e$ ·····>······ (Fig. 5),

each of the related dihedral angles is $\frac{\pi}{3}$, so $6 \cdot \frac{\pi}{3} = 2\pi$, ensuring the trivial stabilizer for points on base edges.

We conclude that $\tilde{\mathcal{D}}_1$ is a fundamental polyhedron for the group G_1 with presentation

(6) $$G = (g_1, g_2, g_3, g_4, t_1, t_2 - t_1 g_1^{-1} g_2^{-1} = t_1 g_1 g_2 = t_2 g_3^{-1} g_4 = \\ = t_2 g_3 g_4^{-1} = t_1 g_3^{-1} g_2 t_2 g_1^{-1} g_4^{-1} = 1).$$

Forming the orbit O^{G_1}, where $O := A_2$ is the centre of the polyhedron $\tilde{\mathcal{D}}_1$ (Fig. 3), we obtain by a symmetry argument that $\tilde{\mathcal{D}}_1 = \tilde{\mathcal{D}}_{1O}$ is a Dirichlet polyhedron corresponding to O and its G_1-orbit.

The stabilizer for the end classes of A_3 and $\overline{A}_3$ will be the Euclidean plane crystallographic group $G^1 = \mathbf{pg}$ (Klein bottle group) corresponding to the fundamental hexagon $\tilde{\mathcal{P}}^1 = E^2/G^1$ and presentation given in Fig. 1. We have two end classes on the base of $\tilde{\mathcal{D}}_1$ corresponding to A_0 and $\overline{A}_0$ in Fig. 3, 4.

Each of them has a stabilizer group isomorphic to the Euclidean plane group **pg** again. For instance, at the end class ● we glue together 3 rhombs on a horosphere, centred in $\boxed{1}$, to get a fundamental hexagon for the stabilizer (Fig. 5). The sides of this hexagon are cut out by the faces

$$g_1^{t_2^{-1}}, (g_2^{-1})^{t_2^{-1}}, t_1^{g_4^{-1}}, (g_1^{-1})^{g_4^{-1}}, t_1^{-1}, g_2,$$

of

$$\tilde{\mathcal{D}}_1^{t_2^{-1}}, \tilde{\mathcal{D}}_1^{g_4^{-1}}, \tilde{\mathcal{D}}_1^{g_4 g_3^{-1} t_2^{-1}} = \tilde{\mathcal{D}}_1,$$

see relation (4). These 3 G_1-images of $\tilde{\mathcal{D}}_1$ are glued together along the edges d^+ -->--. The pairing

$$t_1^{-1} \to (t_1)^{g_4^{-1}}, g_2 \to (g_2^{-1})^{t_2^{-1}} = (g_2)^{g_2^{-1} t_2^{-1}},$$

$$(g_1^{-1})^{g_4^{-1}} \to (g_1)^{t_2^{-1}}$$

provides us the generators of the stabilizer: the horospherical glide reflections $t_1 g_4^{-1}$ and $g_2^{-1} t_2^{-1}$ and the horospherical translation $g_4 g_1 t_2^{-1}$, respectively. This means the stabilizer of the end class ● is isomorphic to **pg**, indeed (see also Fig. 1). Such an argument (also later on) needs an accurate analysis of the identification on $\tilde{\mathscr{D}}_i$ and of the tiling by $\tilde{\mathscr{D}}_i$ under G_i (see [2], [4] for a more systematic discussion).

By the general theory the first homology group of $\tilde{\mathscr{D}}_1 = H^3/G_1$ is $H_1 = G_1/[G_1, G_1]$, the abelianization of G_1. By the presentation (6) of G_1 we easily compute

$$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \tag{7}$$

where, in the direct product, $\mathbb{Z}_2$ is the cyclic group of order 2, $\mathbb{Z}$ is the infinite cyclic group.

Now, we geometrically illustrate relation (5) and, at the same time, describe another fundamental bipyramid $\overline{\mathscr{D}}_1$ for the group G_1. $\overline{\mathscr{D}}_1$ is obtained of $\tilde{\mathscr{D}}_1$ by cutting and gluing as follows. The bipyramid $\mathscr{D}$ in Fig. 3 can be cut into 6 congruent tetrahedra at the axis σ, one of them is $A_0\overline{A}_0A_3\overline{A}_3$. These 6 tetrahedra will be glued along the former faces of $\tilde{\mathscr{D}}_1$ at edge $A_0\overline{A}_0$ to form a new bipyramid $\overline{\mathscr{D}}_1$ congruent to $\mathscr{D}$ by angular conditions. In Fig. 5 we have chosen the tetrahedron at the 1st edge from the class $e = \cdot\cdot\!>\!\cdot\cdot$ where the faces t_1^{-1} and g_4^{-1} intersect. To the face t_1^{-1} we glue the image face, t_1^* of the corresponding tetrahedron from $\tilde{\mathscr{D}}_1^{t_1^{-1}}$. Then the t_1^{-1}-image of face g_3, denoted by $g_3^* := (g_3)^{t_1^{-1}}$, follows, furthermore the $g_3 t_1^{-1}$-image of the corresponding tetrahedron from $\tilde{\mathscr{D}}^{g_3 t_1^{-1}}$ comes and so on. The side edges of the new bipyramid $\overline{\mathscr{D}}_1$ are G_1-images of side edges of $\tilde{\mathscr{D}}_1$. G_1-images of $\sigma = A_3\overline{A}_3$ form the class $f =$⦿ ∘ ∘ > ∘ ∘ ◎ of base edges of $\overline{\mathscr{D}}_1$. We can also read off the new generators of G_1, corresponding to $\overline{\mathscr{D}}_1$:

$$\begin{aligned}
&\bar{t}_1 : \tilde{\mathscr{D}}_1^{t_1^{-1}} \to \tilde{\mathscr{D}}_1^{g_1^{-1} g_4^{-1}}, \text{ hence } \bar{t}_1 = t_1 g_1^{-1} g_4^{-1} = g_2 g_4^{-1} \text{ by (1);}\\
&\bar{g}_1 = g_4 t_2 g_1^{-1} g_4^{-1} = g_3 g_1^{-1} g_4^{-1} \quad \text{by (3);}\\
&\bar{g}_2 = t_1 g_3^{-1}; \qquad \bar{t}_2 = g_2^{-1} g_3 t_1^{-1};\\
&\bar{g}_3 = t_1 g_3^{-1} g_1^{-1} g_4^{-1}; \qquad \bar{g}_4 = t_1 g_4.
\end{aligned} \tag{8}$$

We see that $\tilde{\mathscr{D}}_1 \cong \overline{\mathscr{D}}_1$, i.e. there is an isometry φ_e mapping the axis σ onto the first e-edge and $\tilde{\mathscr{D}}_1$ onto $\overline{\mathscr{D}}_1$ so that φ_e preserves the G_1-equivalence. (We see, by comparing with Fig. 4, that φ_e is in strict connection with the half-turn r, the self-symmetry of the characteristic simplex $\mathcal{S}$ of bipyramid $\mathcal{D}$.) Thus φ_e is an isometry of the space form H^3/G_1 which induces an outer automorphism of G_1 defined by $t_j \to \bar{t}_j$ $(j = 1, 2)$ and $g_k \to \bar{g}_k$ $(k = 1, \ldots, 4)$ [8], [9], [10]. We can determine some isometries of H^3/G_1 induced by identification preserving self-symmetries of $\tilde{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_1$, furthermore by φ_e. We write $Sym\ H^3/G_1 \supset \langle \tilde{D}_1, \overline{D}_1, \varphi_e \rangle$ where $\overline{D}_1 = \varphi_e^{-1}\tilde{D}_1\varphi_e$ and $\tilde{D}_1$ is the dihedral motion group of order 4 generated by half-turns $\tilde{r_1}, \tilde{r_2}$:

$$\begin{aligned} \tilde{r_1} &: g_1 \leftrightarrow g_2, t_1^{-1} \leftrightarrow t_1, t_2^{-1} \leftrightarrow t_2, g_3^{-1} \leftrightarrow g_4; \\ \tilde{r_2} &: t_1 \leftrightarrow t_2, g_1 \leftrightarrow g_3, g_2^{-1} \leftrightarrow g_4. \end{aligned} \tag{9}$$

This is an interesting case where the essential assumption of [10]. Prop. 4 does not hold for $\tilde{\mathcal{D}}_1$. The above method yields also new isometries φ_a, φ_b, φ_c, φ_d, mapping the axis ⚬ of $\tilde{\mathcal{D}}_1$ onto edges from the classes a, b, c, d, respectively, such that these isometries preserve the G_1-equivalence. From Fig. 1 at ● we can deduce, e.g., that φ_d involves the automorphism

$$\begin{aligned} &t_1 \to \hat{t}_1 := t_2 g_1^{-1} g_4^{-1},\ t_2 \to \hat{t}_2 := t_2;\ g_1 \to \hat{g}_1 := g_4 t_1^{-1}, \\ &g_2 \to \hat{g}_2 := g_2^{-1} t_2^{-1},\ g_3 \to \hat{g}_3 := g_4^{-1},\ g_4 \to \hat{g}_4 := t_2 g_4^{-1}. \end{aligned}$$

$Sym\ H^3/G_1 = \langle \tilde{r_1}, \tilde{r_2}; \varphi_a, \varphi_b, \varphi_c, \varphi_d, \varphi_e \rangle$ is of order 24.

Such phenomena will also appear later on in cases i =1, 4, 5, 7. The first author intends to discuss more general criteria for determining isometries of space forms in a subsequent paper.

$i = 2$ $\qquad\qquad \tilde{\mathcal{D}}_2 = H^3/G_2 = \overline{\mathcal{D}}_2 \qquad\qquad$ (Fig. 6).

The pairs of faces g_j^{-1} and g_j $(j = 1, 2, 3, 4)$ are identified by horospherical glide reflections g_j. The faces t_j^{-1} and t_j $(j = 1, 2)$ are identified by horospherical translations t_j.

These identifications of the faces are induced by those of the sides of the regular fundamental hexagons $\mathcal{P}^1$ and $\mathcal{P}^2$ for the group **pg** in the Euclidean plane E^2, where g_j $(j = 1, 2)$ are glide reflections and t is a translation (Fig. 1, 2).

We list the edge equivalence classes of the polyhedron $\tilde{\mathcal{D}}_2$, induced by these identifications, writing down for each edge class the corresponding relation as before:

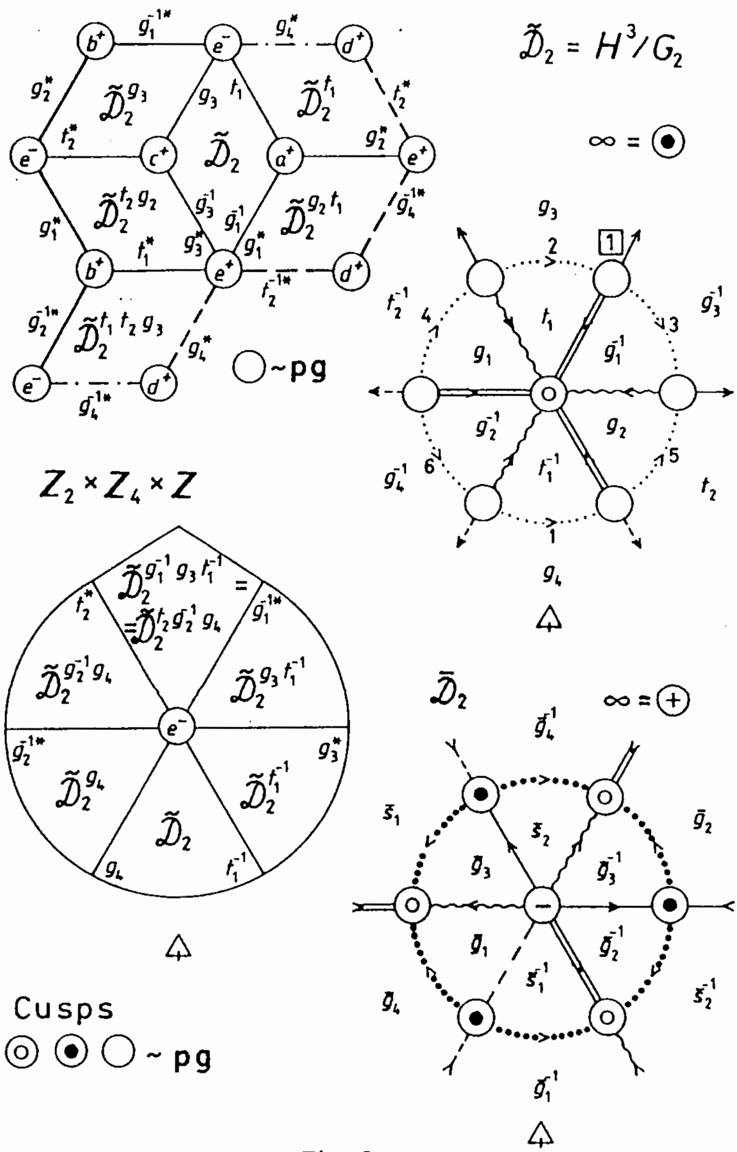

$\tilde{\mathcal{D}}_2 = H^3/G_2$
$\infty =$
~pg
$Z_2 \times Z_4 \times Z$
$\bar{\mathcal{D}}_2$
$\infty =$
Cusps
~ pg

Fig. 6.

$$
\begin{array}{lll}
(1) & t_1 g_1 g_2 = 1 \text{ at the edge class} & a \\
(2) & t_1 g_1^{-1} g_2^{-1} = 1 & b \\
(3) & g_3 g_3 t_2 = 1 & c \\
(4) & g_4 g_4 t_2^{-1} = 1 & d \\
(5) & t_1 g_3^{-1} g_1 t_2 g_2^{-1} g_4 = 1 & e \qquad \text{(Fig. 6).}
\end{array}
$$

We conclude that $\tilde{\mathcal{D}}_2$ is a fundamental polyhedron for the freely acting group G_2, with presentation

$$
\begin{aligned}
(6) \qquad G_2 = (g_1, g_2, g_3, g_4, t_1, t_2 - t_1 g_1 g_2 = t_1 g_1^{-1} g_2^{-1} = \\
= g_3 g_3 t_2 = g_4 g_4 t_2^{-1} = t_1 g_3^{-1} g_1 t_2 g_2^{-1} g_4 = 1).
\end{aligned}
$$

We obtain that $\tilde{\mathcal{D}}_2 = \tilde{\mathcal{D}}_{2O}$ is a Dirichlet polyhedron for $O := A_2$.

The stabilizer for the end classes of A_3 and $\overline{A}_3$ is **pg** again (see also Fig. 1, 2, 3). We have one end class (cusp) on the base of $\tilde{\mathcal{D}}_2$.

Its stabilizer is **pg**. Fig. 6 shows how to choose a fundamental domain for it on a horosphere centred in the end indicated by $\boxed{1}$. By gluing the g_2-images of $\tilde{\mathcal{D}}_2$ first around the edge $a^+ \Longrightarrow$, then around $c^+ \longrightarrow$, we obtain the six images of the end domain of $\tilde{\mathcal{D}}_2$ at O. The faces cut out of the horosphere centred at $\boxed{1}$ a fundamental polygon for **pg**, since $(g_2^{-1})^{t_1 t_2 g_3}$, $(g_1)^{t_2 g_3} \to (g_2)^{g_3}$, $(g_1^{-1})^{g_3}$ are mapped by

$$g_3^{-1} t_2^{-1} t_1^{-1} g_2 g_3 = g_3^{-1} t_2^{-1} g_1^{-1} g_3,$$

which is a horospherical glide reflection;

$$(t_2)^{t_1}, \quad (g_4^{-1})^{g_2 t_1} \to (t_2^{-1})^{g_2 t_1}, \quad (g_4)^{t_1 t_2 g_3}$$

are also mapped by a horospherical glide reflection

$$t_1^{-1} t_2^{-1} g_2 t_1 = t_1^{-1} g_2^{-1} g_4 t_1 t_2 g_3;$$

$(g_4)^{t_1} \to (g_4^{-1})^{t_1 t_2 g_3}$ is mapped by $t_1^{-1} g_4^{-1} t_1 t_2 g_3$ as a horospherical translation (c.f. Fig. 2). The first homology group of $\tilde{\mathcal{D}}_2 = H^3/G_2$ is

$$(7) \qquad H_1 = G_2/[G_2, G_2] = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}.$$

The method of cutting and gluing $\tilde{\mathcal{D}}_2$ into $\overline{\mathcal{D}}_2$, as before, provides us combinatorially different face identifications: There does not exist an identification preserving isometry which maps $\tilde{\mathscr{D}}_2$ onto $\overline{\mathscr{D}}_2$ (Fig. 6). So we have another presentation for

$$(8) \qquad G_2 = (\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4, \overline{s}_1, \overline{s}_2 - \overline{s}_1 \overline{g}_4^{-1} \overline{g}_2^{-1} = \overline{s}_2 \overline{g}_3 \overline{g}_1^{-1} = \\ = \overline{s}_2 \overline{g}_3^{-1} \overline{g}_2 = \overline{s}_1 \overline{g}_4 \overline{g}_1 = \overline{s}_1 \overline{g}_3^{-1} g_2^{-1} \overline{s}_2 \overline{g}_4 \overline{g}_1^{-1} = 1).$$

We could cut $\tilde{\mathscr{D}}_2$ into 3 pieces and glue at the edge class a (or equivalently at the class b) to get a new hexagonal bipyramid. But then t six base edges would not be G_2-equivalent although we required this property at the beginning. *Sym* H^3/G_2 consists of two elements, only.

It is generated by an identification preserving half-turn $\tilde{r}$ of $\tilde{\mathscr{D}}_2$ $\tilde{r}: t_1^{-1} \leftrightarrow t_1, t_2^{-1} \leftrightarrow t_2, g_1 \leftrightarrow g_2, g_3 \leftrightarrow g_4$, whose effect on H^3/G_2 is the same as that of

$$\overline{r}: \overline{s}_1 \leftrightarrow \overline{s}_2,\ \overline{g}_1 \leftrightarrow \overline{g}_2,\ \overline{g}_3 \leftrightarrow \overline{g}_4^{-1}, \text{a half turn of } \tilde{\mathscr{D}}_2.$$

$i = 3$ $\qquad$ $\tilde{\mathscr{D}}_3 = H^3/G_3 = \overline{\mathscr{D}}_3$ $\qquad$ (Fig. 7).

The faces g_j^{-1} and g_j (j = 1, 2, 3, 4) are identified by horospherical glide reflections g_j, the faces t_j^{-1} and t_j (j = 1, 2) are paired by horospherical translations t_j.

The identifications of the faces are induced by those of the sides of a regular hexagon $\mathscr{P}^2$ for the group **pg** in the Euclidean plane E^2 (Fig. 2), where g_j (j = 1, 2) are glide reflections and t is a translation.

We list the edge equivalence classes of the polyhedron $\tilde{\mathscr{D}}_3$, induced by the above identifications, writing down for each edge class the corresponding relation:

(1) $\quad g_1 g_1 t_1 = 1$ $\quad$ at edge class $\quad a$

(2) $\quad g_2 g_2 t_1^{-1} = 1$ $\quad b$

(3) $\quad g_3 g_3 t_2 = 1$ $\quad c$

(4) $\quad g_4 g_4 t_2^{-1} = 1$ $\quad d$

(5) $\quad t_1 g_3 g_1 t_2 g_2^{-1} g_4^{-1} = 1$ $\quad e$ $\quad$ (Fig. 7).

We conclude that $\tilde{\mathscr{D}}_3$ is a fundamental polyhedron for the freely acting group G_3 with a presentation above.

The stabilizer for the end class of A_3 resp. $\overline{A}_3$ is **pg** (Fig. 2). We have one end class on the base of $\tilde{\mathscr{D}}_3$. Its stabilizer is the Euclidean

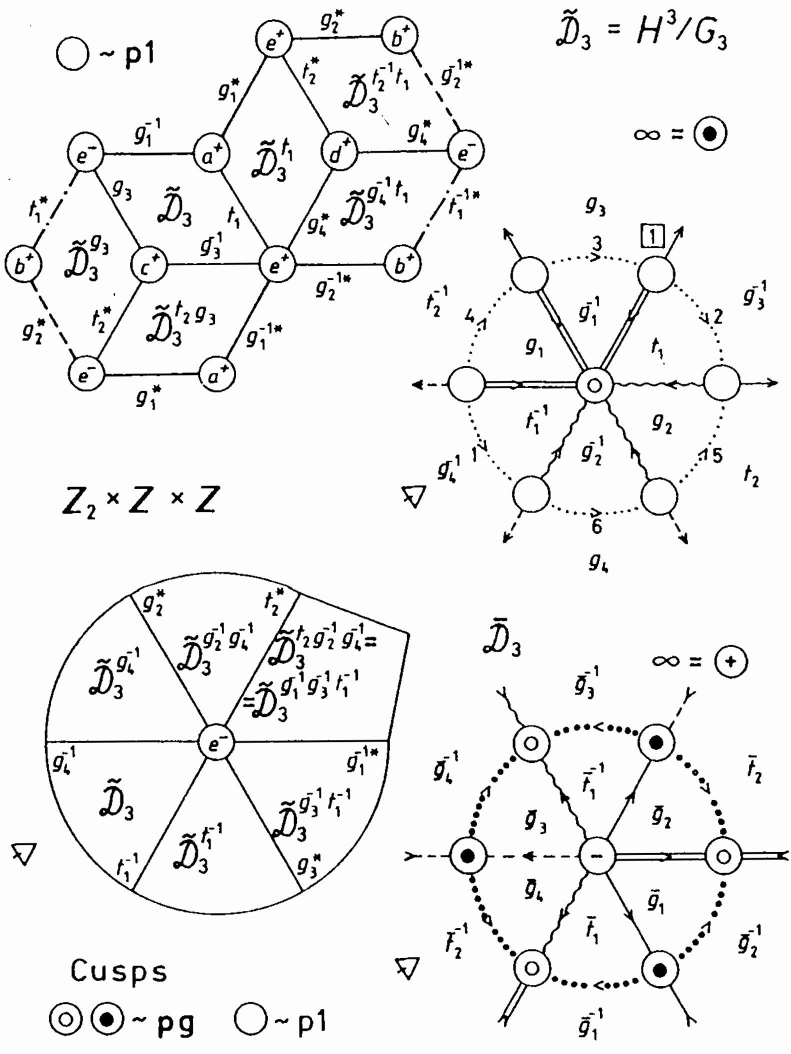


Fig. 7.

lattice translation group **pl**, $E^2/\mathbf{pl}$ is the flat torus. To show this surprising fact, we proceed by Fig. 7 as before. So we obtain the fundamental domain of the stabilizer on the horosphere centred at the end $\boxed{1}$ by gluing the sides. Now the sides

$$g_1^{-1}, (g_1)^{t_1}, (g_2)^{t_2^{-1}t_1} \to (g_1)^{t_2g_3}, (g_1^{-1})^{t_2g_3}, (g_2^{-1})^{g_4^{-1}t_1}$$

are mapped by $g_1t_2g_3 = t_1^{-1}g_1^{-1}t_2g_3 = t_1^{-1}t_2g_2^{-1}g_4^{-1}t_1$ which is a horospherical translation;

$$(t_1)^{g_3} \to (t_1^{-1})^{g_4^{-1}t_1} \quad \text{is mapped by} \quad g_3^{-1}t_1^{-1}g_4^{-1}t_1 \quad \text{and}$$
$$(g_2)^{g_3} \to (g_2^{-1})^{t_2^{-1}t_1} \quad \text{is mapped by} \quad g_3^{-1}g_2^{-1}t_2^{-1}t_1, \quad \text{both are}$$

horospherical translations. By the presentation of G_3 we have

$$H_1 = G_3/[G_3, G_3] = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}. \tag{6}$$

By cutting and gluing $\tilde{\mathcal{D}}_3$ into $\overline{\mathcal{D}}_3$ we have a combinatorially different face pairing and presentation for

$$\begin{aligned} G_3 = (\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4, \bar{t}_1, \bar{t}_2 - \bar{g}_4^{-1}\bar{g}_3\bar{t}_1 = \bar{g}_3\bar{g}_4^{-1}\bar{t}_2 = \bar{t}_2\bar{g}_2\bar{g}_1^{-1} = \\ = \bar{t}_1\bar{g}_1^{-1}\bar{g}_2 = \bar{t}_2\bar{g}_2^{-1}\bar{g}_1^{-1}\bar{t}_1^{-1}\bar{g}_3\bar{g}_4 = 1). \end{aligned} \tag{7}$$

Sym H^3/G_3 consist of 4 elements induced by identification preserving self-symmetries of $\tilde{\mathcal{D}}_3$ resp. $\overline{\mathcal{D}}_3$ analogously to earlier cases.

$i = 4$ $\qquad\qquad \tilde{\mathcal{D}}_4 = H^3/G_4 \qquad\qquad$ (Fig. 8)

The sides g_j^{-1} and g_j of $\tilde{\mathcal{D}}_4$ are paired by horospherical glide reflections g_1, g_2 and glide reflections g_3, g_4; the faces s_j^{-1} and s_j are mapped by screw motions s_j ($j = 1, 2$). The induced edge equivalence classes provide the corresponding relations as follows:

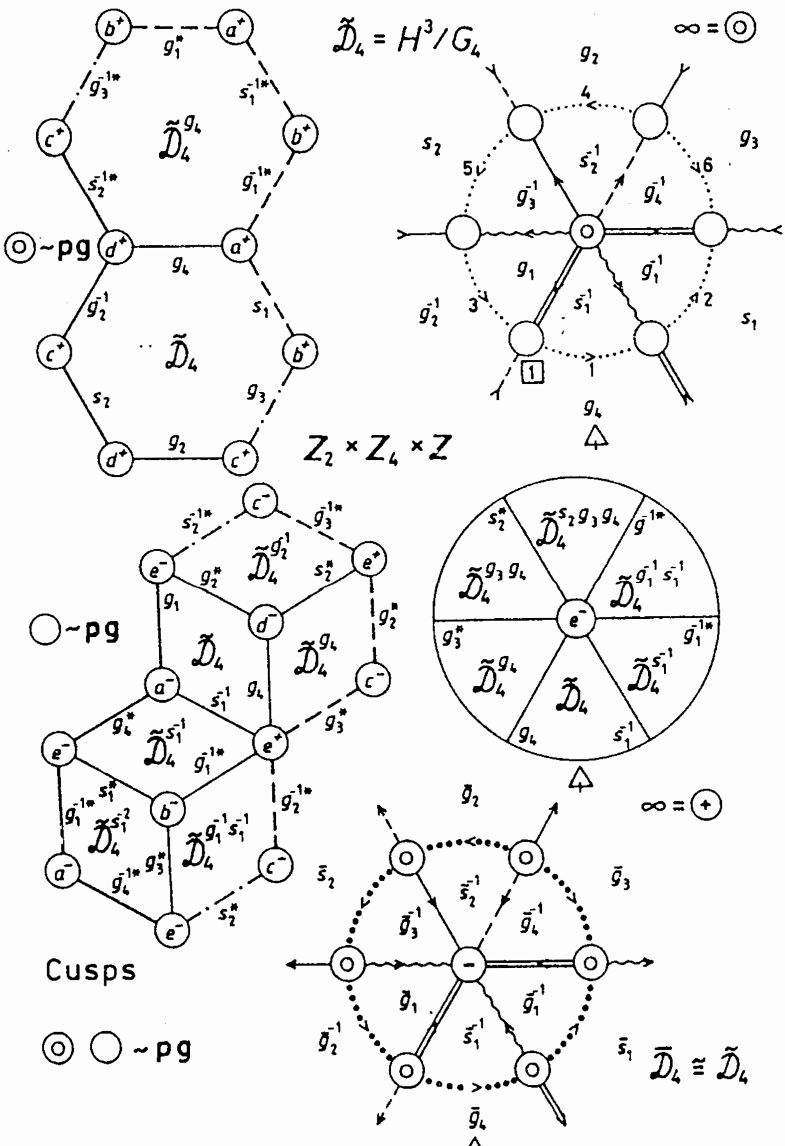
(1) $s_1g_4^{-1}g_1 = 1$ at edge class a

(2) $s_1^{-1}g_1g_3 = 1$ $\qquad b$

(3) $s_2g_2g_3^{-1} = 1$ $\qquad c$

(4) $s_2^{-1}g_4g_2 = 1$ $\qquad d$

(5) $s_1g_1g_2s_2g_3g_4 = 1$ $\qquad e$ $\qquad$ (Fig. 8).



$\tilde{\mathcal{D}}_4 = H^3/G_4$
$\infty = ◎$
◎~pg
$Z_2 \times Z_4 \times Z$
○~pg
$\infty = ⊕$
Cusps
◎ ○ ~pg
$\bar{\mathcal{D}}_4 \cong \tilde{\mathcal{D}}_4$

Fig. 8.

We conclude that $\tilde{\mathfrak{D}}_4$ is a fundamental polyhedron for the freely acting group G_4 with presentation above.

The fundamental domains for the stabilizer subgroups of the two end classes ◎ and ○ are described in Fig. 8 by our standard method.

For ◎ we glue two end domains of $\tilde{\mathfrak{D}}_4$ along the face g_4 of $\tilde{\mathfrak{D}}_4$. The other faces of $\tilde{\mathfrak{D}}_4$ and of $\tilde{\mathfrak{D}}_4^{g_4}$ running to the end ◎ determine the generators of the stabilizer as before. These are two horospherical glide reflections $g_2 = g_4^{-1}s_2$ and $g_4^{-1}g_1g_4 = s_1^{-1}g_4$, and the horospherical translation $g_3^{-1}g_4$.

So, it is the Euclidean group **pg** (Fig. 2) that corresponds to the cusp ◎. We similarly obtain the fundamental domain for the stabilizer of the other end class by gluing 6 images of end domains ○ from $\tilde{\mathfrak{D}}_4$. Thus, the group corresponds to the cusp ○ as well.

The homology group of H^3/G_4 is

(6) $H_1 = G_4/[G_4, G_4] = \mathbf{Z}_2 \times \mathbf{Z}_4 \times \mathbf{Z}$ by the presentation (1) – (5).

By cutting and gluing $\tilde{\mathfrak{D}}_4$ into $\overline{\mathfrak{D}}_4$ by the method developed at the group G_1, we obtain another fundamental bipyramid $\overline{\mathfrak{D}}_4$ endowed by the pairing

$$(7)\qquad \begin{aligned} &\bar{s}_1 := s_1^{-1},\ \bar{g}_1 := s_1g_4,\ \bar{g}_2 := g_4^{-1}s_2g_3g_4,\\ &\bar{s}_2 := g_4^{-1}g_3^{-1}s_2^{-1}g_3g_4,\ \bar{g}_3 := g_4^{-1}g_3^{-1}g_1^{-1}s_1^{-1},\ \bar{g}_4 := s_1g_1. \end{aligned}$$

Comparing $\tilde{\mathfrak{D}}_4$ and $\overline{\mathfrak{D}}_4$ geometrically, we see that the correspondence $s_j \to \bar{s}_j$, $g_j \to \bar{g}_j$ induce an automorphism of G_4 by an isometry φ of $\tilde{\mathfrak{D}}_4$ onto $\overline{\mathfrak{D}}_4$. The situation is similar to the case of G_1. The isometry φ preserves identifications and so does the motion group $\tilde{D}_4$ of the bipyramid $\tilde{\mathfrak{D}}_4$ of order two generated by the half-turn

$$\tilde{r} : s_1 \leftrightarrow s_2^{-1},\quad g_2^{-1} \leftrightarrow g_1,\quad g_3 \leftrightarrow g_4^{-1}.$$

Sym $H^3/G_4 = \langle \tilde{r}, \varphi \rangle$ is of order 4.

$i = 5$ $\qquad \tilde{\mathfrak{D}}_5 = H^3/G_5$ $\qquad$ (Fig. 9).

The pairs of faces g_j^{-1} and g_j (j = 1, 2, 3, 4) are identified by (non-horospherical) glide reflections g_j.

The faces t_j^{-1} and t_j ($i = 1, 2$) are paired by horospherical translations t_j.

We list the edge equivalence classes of $\tilde{\mathscr{D}}_5$, induced by these identifications, writing down for each class the defining relation:

(1)	$g_4 g_3 t_2 = 1$	at edge class	a
(2)	$g_3 g_4 t_1 = 1$		b
(3)	$g_1^{-1} g_2^{-1} t_1 = 1$		c
(4)	$g_2^{-1} g_1^{-1} t_2 = 1$		d
(5)	$g_1^{-1} g_3 t_1 g_2^{-1} g_4 t_2 = 1$		e (Fig. 9).

We conclude that $\tilde{\mathscr{D}}_5$ is a fundamental polyhedron for the freely acting group G_5 with the presentation above.

The stabilizer for both end classes ◎ and ○ is the torus group **pl** as indicated in Fig. 9. Gluing $\tilde{\mathscr{D}}_5$ and $\tilde{\mathscr{D}}_5^{g_4}$, we obtain three horospherical translations as generators for the stabilizer of cusp ◎:

$$t_1^{-1} = g_3 g_4 = g_4^{-1} t_2^{-1} g_4 : t_1,\ g_3^{-1}, (t_2)^{g_4} \to t_1^{-1}, (g_3)^{g_4}, (t_2^{-1})^{g_4};$$
$$g_1^{-1} g_4 : g_1 \to (g_1^{-1})^{g_4}; g_2 g_4; g_2^{-1} \to (g_2)^{g_4}.$$

Gluing six end domains of $\tilde{\mathscr{D}}_5$, we obtain three horospherical translations to generate the stabilizer of the cusp ○:

$$g_2^{-1} g_4 g_1 g_2 = g_2^{-1} g_4 t_2 = t_1^{-1} g_3^{-1} g_1 :$$
$$(g_1^{-1})^{g_4^{-1} g_2}, (t_2^{-1})^{g_4^{-1} g_2}, (g_3)^{t_1} \to (g_1)^{g_2}, t_2, (g_3^{-1})^{g_1};$$
$$g_2^{-1} t_2^{-1} t_1 g_1 : (t_1^{-1})^{t_2 g_2} \to (t_1)^{g_1} \quad \text{and}$$
$$g_2^{-1} t_2^{-1} g_4^{-1} t_1 : (g_4)^{t_2 g_2} \to (g_4^{-1})^{t_1}.$$

By the presentation of G_5 above we have

$$H_1 = G_5/[G_5, G_5] = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}. \tag{6}$$

Transforming $\tilde{\mathscr{D}}_5$ into the bipyramid $\overline{\mathscr{D}}_5$ by cutting and gluing in our standard way, we obtain $\overline{\mathscr{D}}_5$ from $\tilde{\mathscr{D}}_5$ by an identification preserving isometry φ (like r in Fig. 3, 4) inducing the automorphism of G_5 defined by

$$\begin{aligned} (7)\qquad & g_1 \to \bar{g}_1 = g_4 t_2,\ g_2 \to \bar{g}_2 := g_1^{-1} t_2 = g_2, \\ & g_3 \to \bar{g}_3 := t_2^{-1} g_4^{-1} g_3^{-1} g_1 = t_2^{-1} t_1 g_1,\ t_1 \to \bar{t}_1 := \\ & := g_1^{-1} g_3 t_2 = g_1^{-1} g_4^{-1},\ t_2 \to \bar{t}_2 := t_2^{-1} g_4^{-1} g_2. \end{aligned}$$

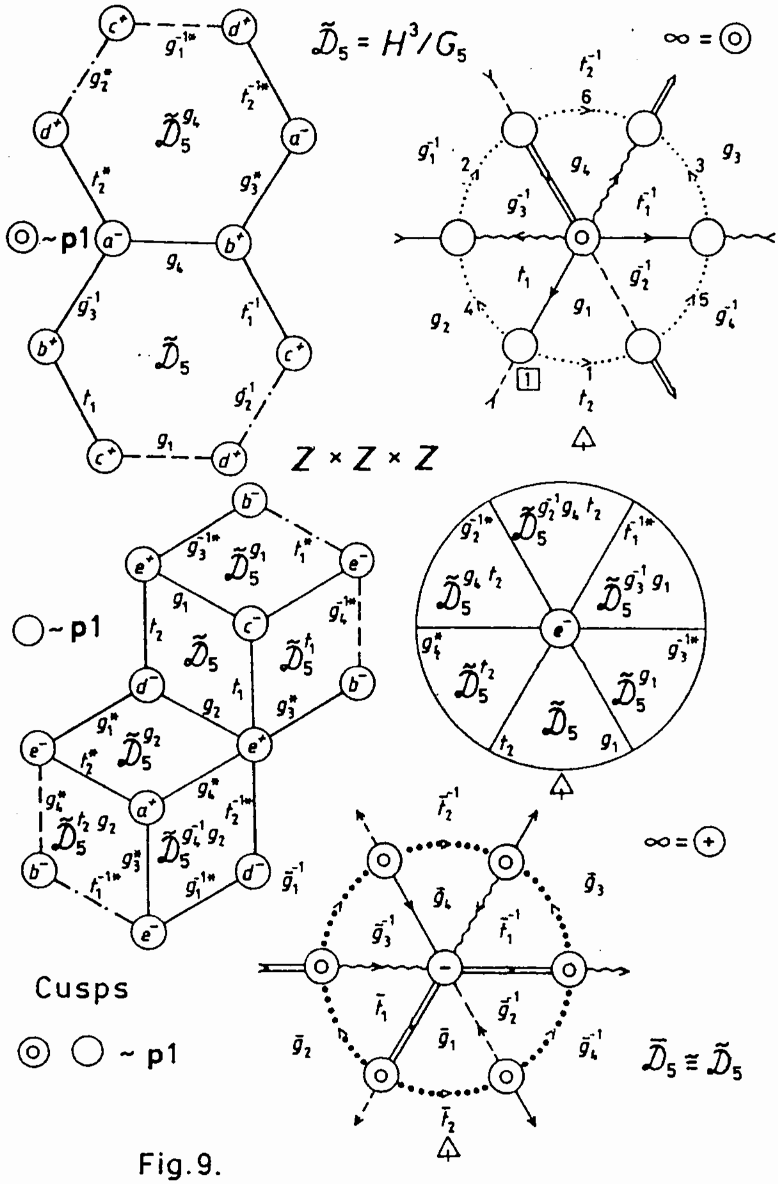

$\tilde{\mathcal{D}}_5 = H^3/G_5$
$\infty =$ ⊚
⊚ ~ p1
○ ~ p1
$Z \times Z \times Z$
$\infty =$ ⊕
$\bar{\mathcal{D}}_5 \cong \tilde{\mathcal{D}}_5$
Cusps
⊚ ○ ~ p1

Fig. 9.

The identification preserving isometries of $\tilde{\mathcal{D}}_5$ (resp. of $\overline{\mathcal{D}}_5$) from the dihedral motion group $\tilde{D}_5$ generated by two half-turns:

$$\tilde{r}_1 : g_1 \leftrightarrow g_4,\ g_2 \leftrightarrow g_3,\ t_1^{-1} \leftrightarrow t_1,\ t_2^{-1} \leftrightarrow t_2 \text{ and}$$
$$\tilde{r}_2 : t_1 \leftrightarrow t_2,\ g_1 \leftrightarrow g_2,\ g_3 \leftrightarrow g_4.$$

$Sym\ H^3/G_5 = \langle \varphi, \tilde{r}_1, \tilde{r}_2 \rangle$ is of order 8.

$i = 6$ $\qquad\qquad \tilde{\mathcal{D}}_6 = H^3/G_6 = \overline{\mathcal{D}}_6 \qquad\qquad$ (Fig. 10).

The faces s_j^{-1} and s_j $(j = 1, 2)$ are identified by screw motions s_j.

The faces g_j^{-1} and g_j $(j = 3, 4)$ are paired by (non-horospherical) glide reflections g_j.

The faces t_j^{-1} and t_j $(t = 1, 2)$ are identified by horospherical translations t_j.

We list the edge equivalence classes of the polyhedron $\tilde{\mathcal{D}}_6$, induced by these identifications, writing down for each class the corresponding relation:

(1) $\quad g_4 g_3 t_2 = 1 \quad$ at edge class $\quad a$

(2) $\quad g_3 g_4 t_1^{-1} = 1 \quad b$

(3) $\quad s_1 s_2 t_1^{-1} = 1 \quad c$

(4) $\quad s_2 s_1 t_2^{-1} = 1 \quad d$

(5) $\quad s_1 t_1 g_3^{-1} s_2 g_4 t_2 = 1 \quad e$

We conclude that $\tilde{\mathcal{D}}_6$ is a fundamental polyhedron for the freely acing group G_6 with the presentation above.

The stabilizer for both end classes ◎ and ○ is the Klein bottle group **pg** (Fig. 10). Gluing $\tilde{\mathcal{D}}_6$ and $\tilde{\mathcal{D}}_6^{g_4}$ we obtain the generators for the stabilizer of cusp ◎:

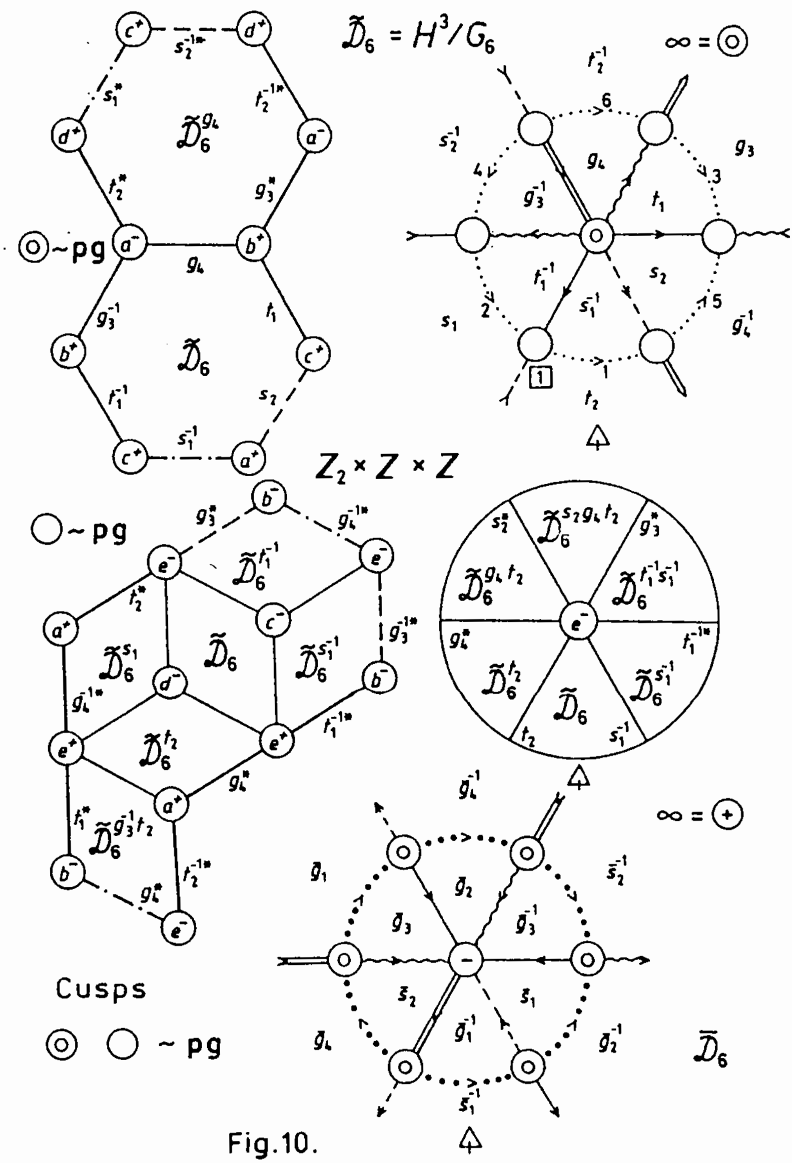
$$t_1 = g_3 g_4 = g_4^{-1} t_2^{-1} g_4 : t_1^{-1},\ g_3^{-1},\ (t_2)^{g_4} \to t_1,\ (g_3)^{g_4},\ (t_2^{-1})^{g_4}$$

is a horospherical translation,

$$s_1 g_4 : s_1^{-1} \to (s_1)^{g_4} \quad \text{and} \quad s_2^{-1} g_4 : s_2 \to (s_2^{-1})^{g_4}$$

are horospherical glide reflections.

Gluing six end domains of $\tilde{\mathcal{D}}_6$, we get the generators for the stabilizer of cusp ○: the horospherical glide reflections $t_2^{-1} g_3 t_2 s_1 =$



$\tilde{\mathcal{D}}_6 = H^3/G_6$
$\infty = ◎$
◎~pg
$Z_2 \times Z \times Z$
○~pg
Cusps
◎ ○ ~ pg
$\infty = ⊕$
$\bar{\mathcal{D}}_6$

Fig.10.

$= t_2^{-1} g_4^{-1} s_1 = s_1 t_1 g_3^{-1} t_2$ and $s_1 g_3 t_1^{-1}$, and the horospherical translation $t_2^{-1} g_3 g_4^{-1} t_1^{-1}$.

By the presentation of G_6 above we obtain

$$H_1 = G_6/[G_6, G_6] = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}. \tag{6}$$

Transforming $\tilde{\mathcal{D}}_6$ into $\overline{\mathcal{D}}_6$ by our standard method, see Fig. 10, we get another presentation

$$\begin{aligned} G_6 = (\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4, \overline{s}_1, \overline{s}_2 - \overline{g}_1 \overline{g}_4^{-1} \overline{s}_2 = \overline{g}_2 \overline{g}_3 \overline{s}_2^{-1} = \\ = \overline{g}_3 \overline{g}_2^{-1} \overline{s}_1 = \overline{g}_1 \overline{g}_4 \overline{s}_1 = \overline{s}_1 \overline{g}_2 \overline{g}_4 \overline{s}_2^{-1} \overline{g}_3 \overline{g}_1^{-1} = 1) \end{aligned} \tag{7}$$

Sym H^3/G_6 consists of 2 elements. It is generated an identification preserving half-turn $\tilde{r}$ of $\tilde{\mathcal{D}}_6$

$$\tilde{r} : t_1^{-1} \leftrightarrow t_2,\ s_1^{-1} \leftrightarrow s_1,\ s_2^{-1} \leftrightarrow s_2,\ g_3 \leftrightarrow g_4,$$

or, equivalently, that of $\overline{\mathcal{D}}_6$

$$\overline{r} : \overline{s}_1^{-1} \leftrightarrow \overline{s}_1,\ \overline{s}_2^{-1} \leftrightarrow \overline{s}_2,\ \overline{g}_1 \leftrightarrow \overline{g}_2,\ \overline{g}_3 \leftrightarrow \overline{g}_4.$$

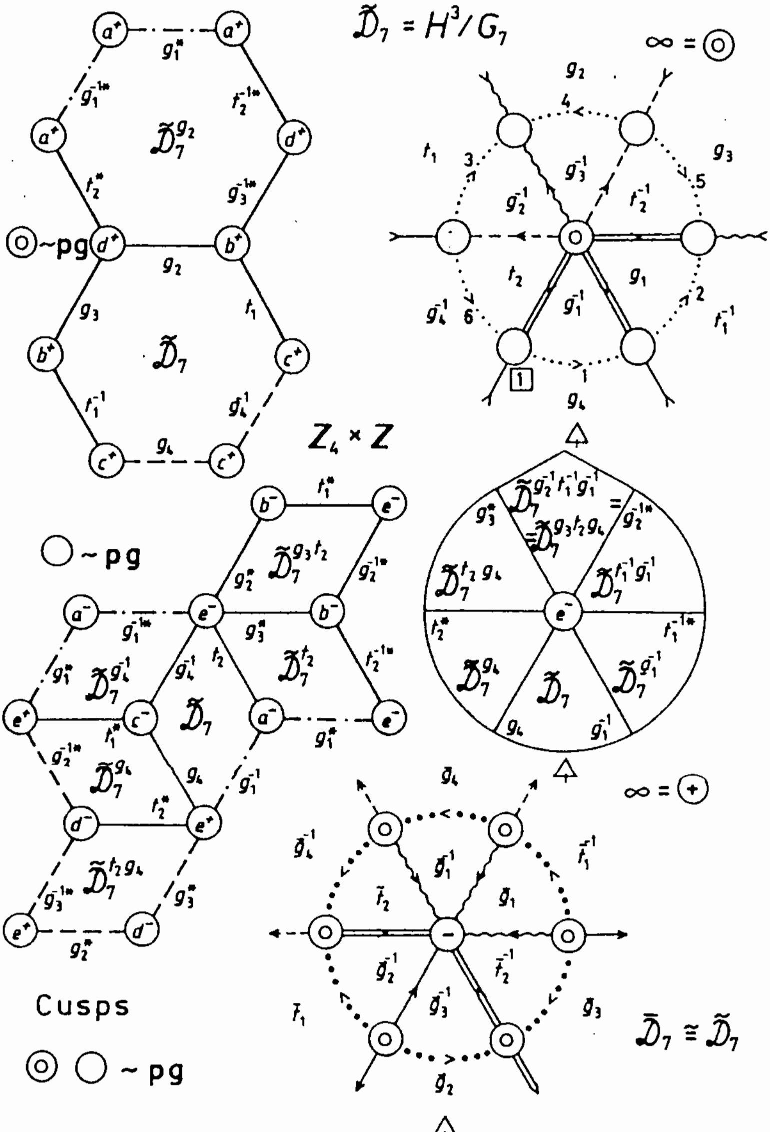
$i = 7$ $\tilde{\mathcal{D}}_7 = H^3/G_7$ (Fig. 11)

The faces g_j^{-1} and g_j of $\tilde{\mathcal{D}}_7$ are paired by horospherical glide reflections g_j (j = 1, 2, 3, 4). The faces t_j^{-1} and t_j are paired by horospherical translations t_j (j = 1, 2). The edge equivalence classes and the corresponding relations are

(1)	$g_1 g_1 t_2 = 1$	at edge class	a	
(2)	$t_1 g_2^{-1} g_3 = 1$		b	
(3)	$g_4 g_4 t_1 = 1$		c	
(4)	$t_2 g_2 g_3^{-1} = 1$		d	
(5)	$g_1 t_1 g_2 g_3 t_2 g_4 = 1$		e	(Fig. 11).

Again, $\tilde{\mathcal{D}}_7$ is a fundamental polyhedron for the freely acting group G_7 with the presentation above.

Fundamental domains for stabilizers of end classes ◎ resp. ○ are described by our standard method. Both stabilizers are isomorphic to **pg**.



$\tilde{\mathcal{D}}_7 = H^3/G_7$
$\infty =$
$Z_4 \times Z$
~pg
Cusps
~pg
$\bar{\mathcal{D}}_7 \cong \tilde{\mathcal{D}}_7$
$\infty =$

Fig.11.

As an illustration we compute the first homology group of H^3/G_7:

$$H_1 = G_7/[G_7, G_7] = \mathbb{Z}_4 \times \mathbb{Z}. \tag{6}$$

Expressing t_1 and t_2 from relations (1)-(5), we get

$$g_1^2 g_3 g_2^{-1} = 1;\ g_4^2 g_3^{-1} g_2 = 1;\ g_1 g_3^{-1} g_2 g_2 g_3 g_3 g_2^{-1} g_4 = 1. \tag{7}$$

Expressing $g_3 = g_2 g_4^2$ and $g_1 = g_4^{-1} g_2 (g_4^{-2} g_2^{-1})^2 g_2^{-2} (g_2 g_4^2)$ and putting into the first relation of (7), we obtain the 2-generator presentation

$$\begin{aligned} G_7 = (g_2, g_4 - 1 = \\ = g_4^{-1} g_2 g_4^{-2} g_2^{-1} g_4^{-2} g_2^{-2} g_4 g_2 g_4^{-2} g_2^{-1} g_4^{-2} g_2^{-2} g_4^2 g_2 g_4^2 g_2^{-1}) \end{aligned} \tag{8}$$

Factoring by the commutator subgroup of G_7, we have indeed

$$H_1 = (z = g_2 g_4;\ g_2 - z^4 = 1) = \mathbb{Z}_4 \times \mathbb{Z}. \tag{9}$$

By cutting and gluing $\tilde{\mathscr{D}}_7$ into $\overline{\mathscr{D}}_7$ by our standard method we get another pairing

$$\begin{aligned} &\bar{t}_1 := g_1(t_1 g_4) = g_1 g_4^{-1},\ \bar{t}_2 := (g_1 t_2) g_4 = g_1^{-1} g_4, \\ &\bar{g}_1 := g_1 t_1 (g_2 t_1^{-1}) g_1^{-1} = g_1 t_1 g_3 g_1^{-1},\ \bar{g}_2 := g_4^{-1},\ \bar{g}_3 := g_1^{-1}, \\ &\bar{g}_4 := g_4^{-1}(t_2^{-1} g_3) t_2 g_4 = g_4^{-1} g_2 t_2 g_4 \end{aligned} \tag{10}$$

Comparing $\tilde{\mathscr{D}}_7$ and $\overline{\mathscr{D}}_7$ geometrically, we see that the correspondences $t_j \to \bar{t}_j$, $g_j \to \bar{g}_j$ induce an automorphism of G_7 by an identification preserving isometry φ (as in case $i = 1$). Another identification preserving isometry of $\tilde{\mathscr{D}}_7$ is determined by the half-turn

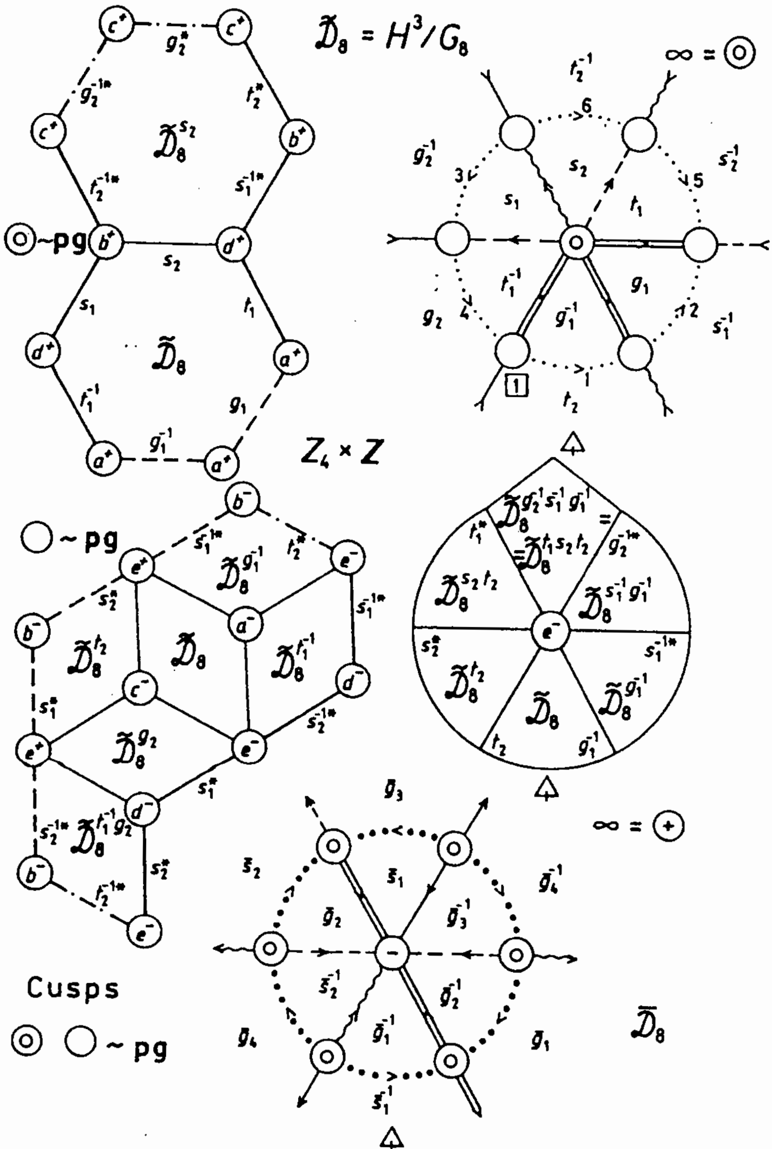
$$\tilde{r} : t_1^{-1} \leftrightarrow t_2,\ g_1^{-1} \leftrightarrow g_4,\ g_2^{-1} \leftrightarrow g_3.$$

Sym $H^3/G_7 = \langle \tilde{r}, \varphi \rangle$ is of order 4.

$i = 8$ $\qquad$ $\tilde{\mathscr{D}}_8 = H^3/G_8 = \overline{\mathscr{D}}_8$ $\qquad$ (Fig. 12)

The faces t_j^{-1} and t_j are paired by horospherical translations t_j ($j = 1, 2$), s_j^{-1} and s_j correspond by screw motions s_j ($j = 1, 2$), g_j^{-1} and g_j are by horospherical glide reflections g_j ($j = 1, 2$).

This pairing induces the edge equivalence classes and the corresponding relations as follows:



$\tilde{\mathcal{D}}_8 = H^3/G_8$
$\infty = \odot$
$Z_4 \times Z$
$\odot$~pg
$\bigcirc$~pg
$\infty = \oplus$
$\bar{\mathcal{D}}_8$
Cusps
$\odot$ $\bigcirc$ ~ pg

Fig.12.

(1)	$g_1 g_1 t_1^{-1} = 1$	at edge class	a
(2)	$t_2 s_1 s_2^{-1} = 1$		b
(3)	$g_2 g_2 t_2^{-1} = 1$		c
(4)	$t_1 s_2^{-1} s_1 = 1$		d
(5)	$g_1 s_1 g_2 t_1 s_2 t_2 = 1$		e (Fig. 12).

$\tilde{\mathcal{D}}_8$ is a fundamental polyhedron for the freely acting group G_8 with this presentation.

Fundamental domains for stabilizers of end classes ◎ and ○ are described in Fig. 12 by our standard method. Both stabilizers are isomorphic to **pg**, again.

The group G_8 also has a 3-generator presentation. Expressing t_1 and t_2 from relations (1)–(5), we get

$$g_1^2 s_2^{-1} s_1 = 1,\ g_2^2 s_1 s_2^{-1} = 1,\ g_1 s_1 g_2 g_1^2 s_2 g_2^2 = 1. \tag{6}$$

We can express, for example, $s_2 = g_2^2 s_1$ from the second equation, and put it into the others. Thus we obtain a desired 3-generator presentation:

$$G_8 = (g_1, g_2, s_1 - 1 = g_1^2 s_1^{-1} g_2^{-2} s_1 = g_1 s_1 g_2 g_1^2 g_2^2 s_1 g_2^2). \tag{7}$$

Turning to the abelianization of G_8, we have first

$$g_1^2 = g_2^2,\ g_2 s_1^2 g_1^7 = 1, \text{ then } (s_1 g_1^4)^4 = 1. \text{ Finally}$$

$$H_1 = (g_1,\ z = s_1 g_1^4 - z^4 = 1) = \mathbf{Z}_4 \times \mathbf{Z}. \tag{8}$$

Transforming $\tilde{\mathcal{D}}_8$ into $\overline{\mathcal{D}}_8$ by our standard method we get a combinatorially different fundamental polyhedron providing a new presentation

$$\begin{aligned} G_8 = (&\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4, \overline{s}_1, \overline{s}_2 - 1 = \overline{g}_1 \overline{s}_1 \overline{g}_2^{-1} = \overline{g}_1 \overline{g}_4 \overline{s}_2^{-1} = \overline{s}_1 \overline{g}_3 \overline{g}_4 = \\ &= \overline{g}_2 \overline{s}_2 \overline{g}_3^{-1} = \overline{g}_1 \overline{g}_2 \overline{s}_2^{-1} \overline{g}_4^{-1} \overline{g}_3 \overline{s}_1^{-1}), \text{ where} \end{aligned} \tag{9}$$

$$\begin{aligned} &\overline{g}_1 := g_1^{-1},\ \overline{g}_2 := g_1 s_2 t_2,\ \overline{g}_3 := g_1(s_1 t_1) s_2 t_2 = g_1 s_2^2 t_2, \\ &\overline{g}_4 := g_1 s_1 t_2,\ \overline{s}_1 := t_1 s_2 t_2,\ \overline{s}_2 := t_2^{-1} s_2 t_2. \end{aligned} \tag{10}$$

Sym $H^3/G_8 = \langle\tilde{r}\rangle = \langle\overline{r}\rangle$ consists of 2 elements. The identification preserving half-turn $\tilde{r}$ of $\tilde{\mathcal{D}}_8$ and the half turn $\overline{r}$ of $\overline{\mathcal{D}}_8$ yield the same involutive isometry of H^3/G_8.

$$\begin{aligned} &\tilde{r} : t_1^{-1} \leftrightarrow t_2,\ s_1^{-1} \leftrightarrow s_1,\ s_2^{-1} \leftrightarrow s_2,\ g_1^{-1} \leftrightarrow g_2; \\ &\overline{r} : \overline{s}_1^{-1} \leftrightarrow \overline{s}_1,\ s_2^{-1} \leftrightarrow \overline{s}_2,\ \overline{g}_1^{-1} \leftrightarrow \overline{g}_3,\ \overline{g}_2 \leftrightarrow \overline{g}_4. \end{aligned} \tag{11}$$

Now we turn to proving our theorems announced at the end of Sect. 1.

PROOF OF THEOREM 1. We summarize our facts.

We have seen that each pairing $\tilde{\mathcal{D}}_i$ or $\overline{\mathcal{D}}_i$, by isometries on the faces of the bipyramid $\mathcal{D}$, induces equivalence classes of edges such that Poincaré's angle conditions hold for each edge class. This guarantees the free action of G_i, generated by the pairing, at the points of edges of $\tilde{\mathcal{D}}_i$.

The orthogonal projections of the centre $O := A_2$ (Fig. 3, 4) on the faces and edges of $\mathcal{D}$ will be mapped by the pairing onto each other, respectively. Hence each element of G_i, preserving an end of $\tilde{\mathcal{D}}_i$, is parabolic. That means, only the fixed end is invariant at a non-trivial isometry from the stabilizer of that end, so each horosphere, centred at the end considered, is invariant at this stabilizer. Indeed, we have determined a fundamental end domain for each stabilizer and recognized the corresponding Euclidean plane group as one of **pl** and **pg**. So we have checked the so-called cusp condition which guarantees that G_i is discrete on H^3 and the hyperbolic metric of H^3/G_i is complete. (see [11] for locally hyperbolic manifolds of non-complete metric).

By the Poincaré theorem these facts are sufficient for $\tilde{\mathcal{D}}_i = H^3/G_i$ to be a hyperbolic space form.

PROOF OF THEOREM 2. The 8 groups G_i are not isomorphic and the spaces H^3/G_i are not isometric as we have seen their invariants: the minimal number of generators, the first homology groups, the cusps, the symmetry groups of H^3/G_i. (We remark that we do not use the rigidity theorem of MOSTOW and MARGULIS that the existence of an isomorphism between the fundamental groups G and G' of complete hyperbolic manifolds $\mathcal{M}$, respectively $\mathcal{M}'$ of finite volume is equivalent with the existence of on isometry between $\mathcal{M}$ and $\mathcal{M}'$ [8], [9].) We have pointed out which pairings of $\mathcal{D}$ lead to isomorphic groups. We have obtained 12 different pairings on $\mathcal{D}$. (Two pairings $\mathcal{I}_1$ and $\mathcal{I}_2$, as involutive maps of the faces of $\mathcal{D}$ onto themselves, are combinatorially eqivalent, if there is a self-isometry Ψ of $\mathcal{D}$ such that $\mathcal{I}_2 = \Psi^{-1}\mathcal{I}_1\Psi$.)

We have to show that no other pairings exist on $\mathcal{D}$, which satisfy the Poincaré's angle conditions and the six base edges of $\mathcal{D}$ are equivalent.

This is a question of systematic trying of all non-equivalent pairings which induce 6 base edges and $3 + 3 + 3 + 3$ side edges of $\mathcal{D}$ in the corresponding edge equivalence classes. For this bipyramid that has a relatively simple structure we have found 9 different pairings by hand. But a computer checking by ISTVÁN PROK (on the base of papers [2], [4]) discovered also the last 3 pairings in Fig. 11, 12. We thank him for his collegial help.

It is very interesting that all these space forms with the bipyramid $\mathcal{D}$ as a fundamental polyhedron are non-orientable. We work on the problem whether a space form among them can cover a smaller space form. We also work on the list of all space forms whose fundamental polyhedron is $\mathcal{D}$ without the condition on base edges.

3. Metric construction of the polyhedron $\mathcal{D}$ in the vector model of the hyperbolic space H^3

We are going to indicate the analitical treatment of our problem.

In order to embed H^3 into the real projective space $P^3(\mathbf{V}^4, \mathbf{V}^*)$ we introduce a hyperbolic projective metric by giving a bilinear form (see [5], [6] for more details):

$$\langle\ ;\ \rangle \quad : \quad \mathbf{V}_4^* \times \mathbf{V}_4^* \to \mathbf{R}, \ \langle b^i u_i; b^j v_j\rangle = u_i b^{ij} v_j,$$

by means of the Schläfli matrix

$$(1) \qquad (\langle b^i; b^j\rangle) = (b^{ij}) = \begin{pmatrix} 1 & -\cos\frac{\pi}{6} & 0 & 0 \\ -\cos\frac{\pi}{6} & 1 & -\cos\frac{\pi}{3} & 0 \\ 0 & -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{6} \\ 0 & 0 & -\cos\frac{\pi}{6} & 1 \end{pmatrix},$$

where the basis $\{b^j\}$ in the dual vector space $\mathbf{V}_4^*$ represents the planes m_j of P^3 in connection with the polyhedron $\mathcal{D}$ in $H^3 \subset P^3$. The planes $m_0 = A_1A_2A_3$ and $m_1 = A_0A_2A_3$ are the vertical symmetry planes of the polyhedron $\mathcal{D}$ (Fig. 3, 4). $m_3 = A_0A_1A_2$ is the base plane and $m_2 = A_0A_1A_3$ is a side plane of the bipyramid $\mathcal{D}$.

We simply compute

$$(2) \qquad B = \det(b^{ij}) = -\frac{3}{16} < 0.$$

We compute the inverse matrix (a_{ij}) of the matrix (b^{ij}). The equation $b^{ij}a_{jk} = \delta^i_k$ holds iff

$$(3) \qquad \begin{aligned} &a_{00} = 0,\ a_{01} = a_{10} = -\frac{2}{\sqrt{3}} < 0,\ a_{02} = a_{20} = -\frac{4}{\sqrt{3}} < 0, \\ &a_{03} = a_{30} = -2 < 0,\ a_{11} = -\frac{4}{3} < 0, \\ &a_{12} = a_{21} = -\frac{8}{3} < 0,\ a_{13} = a_{31} = -\frac{4}{\sqrt{3}} < 0, \\ &a_{22} = -\frac{4}{3} < 0,\ a_{23} = a_{32} = -\frac{2}{\sqrt{3}} < 0,\ a_{33} = 0. \end{aligned}$$

Let $\{\mathbf{a}_j\}$ be the basis in the vector space $\mathbf{V}^4$ dual to the given basis $\{\boldsymbol{b}^i\}$ in $\mathbf{V}_4^*$, defined by $\mathbf{a}_j\boldsymbol{b}^i = \delta_j^i$. Geometrically, the vectors $\mathbf{a}_j$ represent the vertices of the simplex $\mathcal{S} = A_0A_1A_2A_3$ in $H^3 \subset P^3$, whose side planes are described by the forms $\boldsymbol{b}^i$. The induced bilinear form

$$\langle\, ;\, \rangle \quad : \quad \mathbf{V}^4 \times \mathbf{V}^4 \to \mathbf{R},\ \langle x^i\mathbf{a}_i;\ y^j\mathbf{a}_j\rangle\ = x^i a_{ij} y^j$$

is defined by the matrix

$$(\langle\mathbf{a}_i;\ \mathbf{a}_j\rangle) = (a_{ij}) \tag{4}$$

with entries in (3).

We easily see that the bilinear form $\langle\, ;\, \rangle$ has a signature $(+,+,+,-)$; this means that the projective metric in $P^3(\mathbf{V}^4, \mathbf{V}_4^*)$ is hyperbolic, indeed [5].

In general, the proper points $X(\mathbf{x})$ of $H^3 \subset P^3$ are defined by "time-like" 1-subspaces of $\mathbf{V}^4$

$$\{(\mathbf{x}) \subset \mathbf{V}^4\ :\ \langle\mathbf{x};\mathbf{x}\rangle < 0\}. \tag{5}$$

$$\{(\mathbf{y}) \subset \mathbf{V}^4\ :\ \langle\mathbf{y};\mathbf{y}\rangle = 0\}$$

describes the ideal points of H^3 (the ends on the absolute). The proper planes $u(\boldsymbol{u})$ of H^3 are

$$\{(\boldsymbol{u}) \subset \mathbf{V}_4^*\ :\ \langle\boldsymbol{u};\boldsymbol{u}\rangle > 0\}. \tag{6}$$

As all elements of the main diagonal of the matrix (1) are positive, all planes m_i of the polyhedron $\mathcal{D}$, represented by the forms $\boldsymbol{b}^i$, are proper planes in the space $H^3 \subset P^3$.

From the relations (3) of this section, we see that the vertices $A_1(\mathbf{a}_1)$ and $A_2(\mathbf{a}_2)$ of the simplex $\mathcal{S}$ described by the vectors a_1 and $\mathbf{a}_2$ are proper points, while the vertices $A_0(\mathbf{a}_0)$ and $A_3(\mathbf{a}_3)$ of the simplex $\mathcal{S}$, described by the vectors $\mathbf{a}_0$ and $\mathbf{a}_3$, are ideal points (ends). This implies that $\mathcal{D}$ is a polyhedron with ideal vertices.

Applying formulas valid for $H^3 \subset P^3$ [5], other data of the simplex $\mathcal{S}$ can be computed from the matrices (a_{ij}) and (b^{ij}). Thus we can check that the Coxeter diagram (Fig. 4) correctly describes the dihedral angles of simplex $\mathcal{S}$, e.g., $\frac{\pi}{6}$ is the angle of planes $m_0 = A_1A_2A_3$ and $m_1 = A_0A_2A_3$, furthermore, m_0 is perpendicular to m_2 and m_3.

The Coxeter group C, generated by reflections in mirrors m_j ($j = 0$, 1, 2, 3) of the simplex $\mathcal{S}$, is a supergroup of index 24 for each group G_i, since $\mathcal{D}$ is the union of 24 congruent copies of simplex $\mathcal{S}$. Therefore, we could also express the generators of each group G_i by matrices with respect to the bases $\{\boldsymbol{b}^i\}$ or $\{\mathbf{a}_i\}$.

In general, isometries of H^3 can be described by linear transformations of $\mathbf{V}^4$, or $\mathbf{V}_4^*$ which preserve the bilinear form $\langle\ ;\ \rangle$. For instance, a reflection σ in the plane $u(\boldsymbol{u})$, represented by $\boldsymbol{u} \in \mathbf{V}_4^*$, can be given by

$$
\begin{aligned}
(7) \qquad & \sigma : \mathbf{V}^4 \to \mathbf{V}^4,\ \mathbf{x} \to \mathbf{x} - \frac{2\langle \mathbf{x}; \mathbf{u}\rangle}{\langle \mathbf{u}; \mathbf{u}\rangle}\mathbf{u} \quad \text{or} \\
& \sigma : \mathbf{V}_4^* \to \mathbf{V}_4^*,\ \boldsymbol{v} \to \boldsymbol{v} - \boldsymbol{u} \cdot \frac{2\langle \boldsymbol{v}; \boldsymbol{u}\rangle}{\langle \boldsymbol{u}; \boldsymbol{u}\rangle}.
\end{aligned}
$$

where $\mathbf{u}$ is the vector polar to $\boldsymbol{u}$ defined by $\mathbf{x}\boldsymbol{u} = \langle \mathbf{x}; \mathbf{u}\rangle$ for every $\mathbf{x} \in \mathbf{V}^4$.

The most important distance in the simplex $\mathcal{S}$ is $d(A_1A_2) := d_{12}$. Applying the distance formula of H^3, we obtain from (3) and (4)

$$
\begin{aligned}
(8) \qquad & \operatorname{ch}\frac{d_{12}}{k} = \frac{-\langle \mathbf{a}_1; \mathbf{a}_2\rangle}{\sqrt{\langle \mathbf{a}_1; \mathbf{a}_1\rangle\,\langle \mathbf{a}_2; \mathbf{a}_2\rangle}} = \frac{-a_{12}}{\sqrt{a_{11} \cdot a_{22}}} = 2; \\
& d_{12} = k \cdot 1,316958.
\end{aligned}
$$

Here $k = \sqrt{-1/K}$ is the metric constant of H^3, where $K < 0$ is the sectional curvature. The formula of Bolyai-Lobachevsky about the distance and angle of parallelism leads to the same result:

$$
(9) \qquad e^{d_{12}/k} = \operatorname{ctg}\frac{1}{2}\prod(d_{12}) = \operatorname{ctg}\frac{\pi}{12}.
$$

We also see that A_2 lies nearer the line A_0A_3 than d_{12}.

The radius ϱ of the inscribed ball of $\mathcal{D}$ can be computed. This is the distance of $A_2(\mathbf{a}_2)$ from the plane $m_2 = A_0A_1A_3$ represented by $\boldsymbol{b}^2$. We have

$$
\begin{aligned}
(10) \qquad & \operatorname{sh}\frac{\varrho}{k} = \frac{\mathbf{a}\boldsymbol{b}^2}{\sqrt{-\langle \mathbf{a}_2; \mathbf{a}_2\rangle\,\langle \boldsymbol{b}^2; \boldsymbol{b}^2\rangle}} = \frac{1}{\sqrt{-a_{22}}} = \frac{\sqrt{3}}{2}; \\
& \varrho = k \cdot 0,7833996.
\end{aligned}
$$

We conjecture that this ball has the largest radius among those contained in all space forms H^3/G_i. We refer to the reference list of [6] for more complete information.

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REGULARLY VARYING EXPECTED RESIDUAL LIFE AND DOMAINS OF ATTRACTION OF EXTREME VALUE DISTRIBUTIONS

By

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Let $X_1,\ X_2,\dots,\ X_n$ be independent random variables with common distribution function $F(x)$. Set $Z_n = \max(X_1, X_2, \dots, X_n)$. It is well known (see Chapter 2 in GALAMBOS (1987)) that there are only three types of nondegenerate distribution functions $H(x)$ such that, with suitable constants a_n and $b_n > 0$,

$$H_n(x) = P(Z_n \le a_n + b_n x) = F^n(a_n + b_n x) \to H(x) \tag{1}$$

weakly. The three types are

$$H_{1,\gamma}(x) = \begin{cases} \exp(-x^{-\gamma}) & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$H_{2,\gamma}(x) = \begin{cases} 1 & \text{if } x > 0 \\ \exp[-(-x)^{\gamma}] & \text{otherwise,} \end{cases} \tag{3}$$

and

$$H_{3,0}(x) = \exp(-e^{-x}),\ \text{all } x, \tag{4}$$

where $\gamma > 0$ is an arbitrary constant. When (1) holds we say that $F(x)$ is in the domain $D(H)$ of attraction of $H(x)$. For $H(x) = H_{3,0}(x)$, we write $D_3 = D(H)$. The following characterization of D_3 is due to L. DE HAAN (1970), who refined a classical result of GNEDENKO (1943). Let

$$\alpha = \alpha_F = \inf\{x:\ F(x) > 0\} \text{ and } \omega = \omega_F = \sup\{x:\ F(x) < 1\}.$$

For $\alpha < t < \omega$, define

$$R(t) = \frac{1}{1 - F(t)} \int_t^{\omega} (1 - F(y))dy, \tag{5}$$

which is possibly infinite. The function $R(t)$ is called the expected residual life (at t). Now, $F \in D_3$ if, and only if, $R(t)$ is finite and, for every real x,

$$\frac{1-F(t+xR(t))}{1-F(t)} \to e^{-x} \text{ as } t \nearrow \omega. \tag{6}$$

Our aim is to show that, in the special case when $R(t)$ is regularly varying, which covers several important distributions such as exponential, Weibull, normal and many others, the simple property $R(t)/t \to 0$ as $t \to +\infty$ if $\omega = +\infty$ (or $R(t)/(\omega - t) \to 0$ as $t \nearrow \omega$ if $\omega < +\infty$) is both necessary and sufficient for $F \in D_3$.

We say that $R(t)$ is regularly varying (at infinity) if, for fixed x,

$$R(tx)/R(t) \to u(x) \text{ as } t \to \infty, \tag{7}$$

where $0 < u(x) < +\infty$. This is equivalent to the existence of a positive function $L(t)$ and a number ϱ such that (see SENETA (1976), Chapter I)

$$R(t) = t^{\varrho} L(t) \tag{8}$$

and

$$L(tx)/L(t) \to 1 \text{ as } t \to +\infty. \tag{9}$$

The number ϱ is called the index of regular variation (of $R(t)$), and the function $L(t)$ is called a slowly varying function. We prove the following results.

THEOREM 1. *Let $\omega = +\infty$. Assume that $R(t)$ is finite for all t and regularly varying. Then $R(t)/t \to 0$ as $t \to +\infty$ is both necessary and sufficient for $F \in D_3$.*

REMARK. Note that the condition $R(t)/t \to 0$ as $t \to +\infty$ is automatically satisfied if the index ϱ of $R(t)$ is smaller than one, and it fails when it exceeds one. Hence, the characterizing condition of the theorem can be restated as "either $\varrho < 1$ or if $\varrho = 1$ then $R(t)/t \to 0$ as $t \to +\infty$". Since, with $\varrho = 1$, F can be in either $D(H_{1,\gamma})$ (e.g., $F(x) = 1 - 1/x^2$, $x \geq 1$, for which $R(t) = t$) or D_3 (e.g.,

$$F(x) = 1 - C[(\log x)/x] \exp[-(\log x)^2/2], \; x \geq 1,$$

where $C > 0$ is a constant, for which $R(t) = t/\log t$), the additional assumption on $R(t)$ is essential.

THEOREM 2. *Let $\omega < +\infty$. If $R(\omega - 1/t)$ is regularly varying at infinity, then $tR(\omega - 1/t) \to 0$ as $t \to +\infty$ is both necessary and sufficient for $F \in D_3$.*

PROOF OF THEOREM 1. It is well known (see DE HAAN (1970), Corollary 2.4.2 or GALAMBOS (1987), Lemma 2.7.2) that $F \in D_3$

implies that $R(t)/t \to 0$ as $t \to \infty$. Hence, only the sufficiency part of the theorem needs proof. Let $B > \alpha$ be a fixed number. Then, with a constant $C > 0$, we can write

$$1 - F(y) = [C/R(y)] \exp\left\{ - \int_B^y [1/R(u)] du \right\}, \quad y > \alpha. \tag{10}$$

By assumption, $R(t)$ satisfies (8) and (9) with some $\varrho \le 1$, and thus, for every x,

$$\frac{R(t + xR(t))}{R(t)} = \left[1 + x\frac{R(t)}{t}\right]^{\varrho} \frac{L(t + xR(t))}{L(t)} \to 1 \tag{11}$$

under our basic assumption $R(t)/t \to 0$ on account of the Uniform Convergence Theorem for slowly varying functions (see Theorem 1.1 in SENETA (1976)). Hence, by (10) and (11),

$$\frac{1 - F(t + xR(t))}{1 - F(t)} = C(x,t) \exp\left\{ - \int_t^{t+xR(t)} [1/R(u)] du \right\},$$

where $C(x,t) \to 1$ for every x as $t \to +\infty$. Upon substituting $u = zt$ in the last integral, the mean value theorem yields

$$\frac{1 - F(t + xR(t))}{1 - F(t)} = C(x,t) \exp[-xR(t)/R(st)],$$

where s is between 1 and $1+xR(t)/t$. Thus, just as at (11), the uniform convergence theorem for $L(t)$ yields (6), which completes the proof.

PROOF OF THEOREM 2. Since $\omega < +\infty$, $R(t)$ is finite for all $t < \omega$. Now, since

$$tR(\omega - 1/t) = R(y)/(\omega - y) \tag{12}$$

and $t \to +\infty$ is equivalent to $y \nearrow \omega$, we have from Corollary 2.4.2 of DE HAAN (1970) (or see Lemma 2.7.2 of GALAMBOS (1987)) that $tR(\omega - 1/t) \to 0$ as $t \to +\infty$ is necessary for $F \in D_3$. Next, note that

$$\lim_{y \nearrow \omega} \frac{1 - F(y + xR(y))}{1 - F(y)} = \lim_{t = +\infty} \frac{1 - F(\omega - 1/t + xR(\omega - 1/t))}{1 - F(\omega - 1/t)}, \tag{13}$$

and that by (10)

$$\frac{1 - F(\omega - 1/t + xR(\omega - 1/t))}{1 - F(\omega - 1/t)} = C^*(x,t) \exp\left\{ - \int_{a(t)}^{b(x,t)} [1/R(u)] du \right\},$$

where $a(t) = \omega - 1/t$, $b(x,t) = \omega - 1/t + xR(\omega - 1/t)$, and $C^*(x,t) \to 1$ for every x as $t \to +\infty$. Hence, by the successive substitutions $u = \omega - 1/v$ and $v = tz$, the mean value theorem yields for the right hand side

$$C^*(x,t)\exp[-xR(\omega - 1/t)/R(\omega - 1/st)],$$

where $1 < s < 1/[1 - xtR(\omega - 1/t)]$. The assumed regular variation of $R(\omega - 1/t)$ and the uniform convergence theorem for slowly varying functions imply the validity of (6) via (13). The proof is completed.

In applications, it is some time convenient to introduce the family of distributions

$$F_A(x) = 1 - \frac{1 - F(x)}{x^A}, \; x \geq 1.$$

Let $R_A(t)$ be the expected residual life function of $F_A(x)$. By l'Hospital's rule, $R_A(t)/R(t) \to 1$ as $t \to +\infty$ if $\omega_F = +\infty$ and if $R_A(t)/t \to 0$. Hence, by Theorem 1, if $R_A(t)$ is regularly varying for one $A \geq 0$ and if $F_A(x) \in D_3$ for this A then $F_A(x) \in D_3$ for all $A \geq 0$, in particular, $F_0(x) = F(x) \in D_3$. As an example, take $F(x) = 1 - \exp(-x^B)$, $0 < B < 1$, $x \geq 0$. Then, with $A = 1 - B$, $F_A(x) = 1 - x^{B-1}\exp(-x^B)$, $x \geq 1$, for which $R_A(t) = t^A$. Since $A < 1$, $F_A(x)$ as well as $F(x)$ belong to D_3. Next, take $F(x) = 1 - Ax^{A-1}\exp(-x^A)$, $x \geq 1$, $A \geq 1$. Then $R(t) = t^{1-A}$, so $F(x) \in D_3$ by Theorem 1. Note that $F_{A-1}(x)$ is a Weibull distribution, and by what was demonstrated at the beginning of the present paragraph, $F_{A-1}(x) \in D_3$ as well. Hence, all Weibull distributions belong to D_3. Finally, from the special Weibull distribution $F(x) = 1 - \exp(-x^2/2)$ we can conclude that the standard normal distribution $N(x) \in D_3$ by an appeal to the asymptotic formula $1 - N(x) \sim C[1 - F_1(x)]$ as $x \to +\infty$, where $C > 0$ is a constant (see GALAMBOS (1987), p. 69).

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ON HERMITE-CONJUGATE FUNCTION

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In [7] we have proved the following two theorems. Denote $\{h_n(z)\}$ the orthonormal Hermite polynomials, $\|h_n(z)e^{-z^2/2}\|_2 = 1$ and let $R(x,y,z) := \sum\limits_{n=0}^{\infty} e^{-\sqrt{2n}\,x} h_n(y)h_n(z)$. The Abel–Poisson means of a function f, $f(z)e^{-z^2/2} \in L^p(\mathbf{R})$ is defined by

$$f(x,y) := \int\limits_{-\infty}^{\infty} R(x,y,z)f(z)e^{-z^2}\,\mathrm{d}z \quad (x>0).$$

THEOREM A.

a) *If* $1 \le p < \infty$ $f(z)e^{-z^2/2} \in L^p(\mathbf{R})$ *and* $f(z) \sim \sum\limits_{n=0}^{\infty} a_n h_n(z)$ *is the Hermite–Fourier series of* f, *then* $f(x,y) = \sum\limits_{n=0}^{\infty} a_n e^{-\sqrt{2n}\,x} h_n(y)$.

b) *If* $1 < p \le \infty$ *then* $\|\sup\limits_{x>0} f(x,y)e^{-y^2/2}\|_p \le A_p\|fe^{-y^2/2}\|_p$, *where* A_p *depends only on* p *and for* $1 < p < \infty$ *we have:*

$$\lim_{x\to 0+} \left\| [f(x,y) - f(y)]e^{-y^2/2} \right\|_p = 0.$$

c) *If* $f(y)e^{-y^2/2} \in L^1(\mathbf{R})$ *and* $E_\alpha := \left(\sup\limits_{x>0} |f(x,y)|e^{-y^2/2} > \alpha\right)$ *then* $|E_\alpha| \le \frac{c}{\alpha}\|fe^{-y^2/2}\|_1$.

The conjugate Abel–poisson means of a function f, $f(z)e^{-z^2/2} \in \in L^p(\mathbf{R})$ is defined as

$$\tilde{f}(x,y) := \int_{-\infty}^{\infty} Q(x,y,z)f(z)e^{-z^2}\,\mathrm{d}z, \quad (x>0),$$

where $Q(x,y,z) := \sum_{n=1}^{\infty} e^{-\sqrt{2n}\,x} h_{n-1}(y)h_n(z)$.

THEOREM B.

a) *If* $f(y)e^{-y^2/2} \in L^1(\mathbf{R})$ *and* $E_\alpha := \left(\sup_{x>0} |\tilde{f}(x,y)|e^{-y^2/2} > \alpha\right)$ *then* $|E_\alpha| \le \frac{c}{\alpha}\|fe^{-y^2/2}\|_1$.

b) *If* $1<p<\infty$ *and* $f(y)e^{-y^2/2} \in L^p(\mathbf{R})$ *then*

$$\left\|\sup_{x>0} |\tilde{f}(x,y)|e^{-y^2/2}\right\|_p \le A_p \left\|fe^{-y^2/2}\right\|_p.$$

c) *If* $1<p<\infty$, $f(y)e^{-y^2/2} \in L^p(\mathbf{R})$ *then there exists a function* $\tilde{f}$, $\tilde{f}(y)e^{-y^2/2} \in L^p(\mathbf{R})$ *such that* $\tilde{f} \sim \sum_{n=1}^{\infty} a_n h_{n-1}$,

$$a_n = \int_{-\infty}^{\infty} f(x)h_n(x)e^{-x^2}\,\mathrm{d}x, \quad \lim_{x\to 0+} \left\|[\tilde{f}(x,y) - f(y)]e^{-y^2/2}\right\|_p = 0.$$

d) *If* $1 \le p < \infty$, $f(y)e^{-y^2/2} \in L^p(\mathbf{R})$ *then there exists* $\lim_{x\to 0+} \tilde{f}(x,y) = \tilde{f}(y)$ a.e. *(almost everywhere) (which is considered for* $p=1$ *as the definition of the function* $\tilde{f}$*).*

Remark that analogous theorems for the function class $\{f : f(y)e^{-y^2} \in L^1(\mathbf{R})\}$ were obtained by B. MUCKENHOUPT [2], [3], [9]. As a tool for proving some saturation theorems the author of present paper had to consider the above version dealing with the L^p space $\{f : f(y)e^{-y^2/2} \in L^p(\mathbf{R})\}$ in [7] and [8].

DEFINITION. Suppose $f(y)e^{-y^2/2} \in L^p(\mathbf{R})$, $1 \le p \le \infty$. The conjugate function $\tilde{f}$ of f is the function defined by

$$\tilde{f} \sim \sum_{n=1}^{\infty} a_n h_{n-1}. \tag{1}$$

We have seen above that for $1 < p < \infty$, $f(y)e^{-y^2/2} \in L^p(\mathbf{R})$ there exists such a (unique) $\tilde{f}$ for which $\tilde{f}(y)e^{-y^2/2} \in L^p(\mathbf{R})$ and condition (1) holds. In [3] is proved that if $1 < p < \infty$ and $f \in L^p_{e^{-y^2}}(\mathbf{R})$ then there exists a (unique) $\tilde{f} \in L^p_{e^{-y^2}}(\mathbf{R})$ satisfying (1) and the norm estimates, analogous to that of Theorem 2. b) and c) hold. The point a) has also its counterpart: if $f \in L^1_{e^{-y^2}}(\mathbf{R})$ and $E_\alpha := \left(\sup\limits_{x>o} |\tilde{f}(x,y)| > \alpha\right)$ where $\tilde{f}(x,y)$ is defined above, then we have

$$(2) \qquad \int\limits_{E_\alpha} e^{-y^2}\, \mathrm{d}y \le \frac{c}{\alpha} ||f||_{L^1_{e^{-y^2}}}.$$

MUCKENHOUPT proved also that for $1 \le p \le \infty$ there exists

$$(3) \qquad \lim_{x\to 0+} \tilde{f}(x,y) = \tilde{f}(y) \quad \text{a.e.}$$

REMARK. On the ground of formula (1) there are several ways to define the Hermite conjugation. For example, let $f \in L^1_{e^{-y^2}}$ (resp. $f(y)e^{-y^2/2} \in L^1(\mathbf{R})$) and suppose that there exist the coefficients

$$a_k = \int\limits_{-\infty}^{\infty} f(x)h_k(x)e^{-x^2}\, \mathrm{d}x \quad (k = 0, 1, 2, \ldots).$$

Then a function $\tilde{f} \in L^1_{e^{-y^2}}$ (resp. $\tilde{f}(y)e^{-y^2} \in L^1(\mathbf{R})$) is called the Hermite-conjugate of f if (1) holds. One can give other definition, based on the convergence of the right hand side of (1) in some sense (convergence a.e., in measure, in distribution sense etc.).

PROBLEM. Study the relations between the above defined conjugations and prove Alexits-type or other saturation theorems using these concepts.

To illustrate the question in the problem we investigate here a distributional approach of the conjugation. As it is well-known, the Schwartz class S of test functions is the space of all functions $\varphi \in C^\infty(\mathbf{R})$ such that $x^m\varphi^{(n)}(x) \to 0$ $(|x| \to \infty)$ for every fixed $n, m \in \mathbf{N}$. We say that $\varphi_k \to \varphi$ in S if for every fixed $n, m \in \mathbf{N}$

$$(4) \qquad x^m \left(\varphi_k^{(n)} - \varphi^{(n)}\right)(x) \to 0 \qquad (k \to \infty)$$

uniformly for $x \in \mathbf{R}$. The Schwartz (or tempered) distributions $f \in S'$ are the linear functionals on S, continuous with respect to the above defined convergence (see [5]). For $f_n,\ f \in S'$ we mean by $f_n \xrightarrow{S'} f$ $(n \to \infty)$ that $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ for all $\varphi \in S$.

LEMMA 1.

a) If $\varphi(x)e^{-x^2/2} \in S$ has the Hermite–Fourier series

$$\varphi(x)e^{-x^2/2} \sim \sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2} \tag{5}$$

$$\left(c_k = \int_{-\infty}^{\infty} \varphi(x)h_k(x)e^{-x^2}\,\mathrm{d}x \right),$$

then for every $s \in \mathbf{N}$ we have

$$k^s c_k \to 0 \qquad (k \to \infty). \tag{6}$$

b) If a sequence (c_k) is given for which (6) holds for every $s \in \mathbf{N}$, then the series

$$\sum c_k h_k(x)e^{-x^2/2} \tag{7}$$

converges in S to a function $\varphi(x)e^{-x^2/2}$ whose Hermite–Fourier series is (7).

PROOF.

a) Since $\varphi(x)e^{-x^2/2} \in L^2(\mathbf{R})$, its Hermite–Fourier series tends to $\varphi(x)e^{-x^2/2}$ in $L^2(\mathbf{R})$-norm and then

$$\varphi(x)e^{-x^2/2} = \sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2} \tag{8}$$

in S' sense. The series converging in distribution sense can be derivated term by term:

$$\varphi'(x)e^{-x^2/2} - x\varphi(x) = \sum_{k=1}^{\infty} c_k h_k(x)e^{-x^2/2} - x\sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2}$$

or, taking into account (8),

$$\varphi'(x)e^{-x^2/2} = \sum_{k=1}^{\infty} c_k h'_k(x)e^{-x^2/2}.$$

Analogously

$$\varphi^{(s)}(x)e^{-x^2/2} = \sum_{k=s}^{\infty} c_k h_k^{(s)}(x)e^{-x^2/2}.$$

for any $s \in \mathbb{N}$. Since $\varphi^{(s)}(x)e^{-x^2/2} \in L^2(\mathbf{R})$ its Hermite–Fourier coefficient is by

$$c_{k+s}\sqrt{2^s\frac{(k+s)!}{k!}} \to 0 \qquad \text{which proves (6).}$$

b) Let (c_k) satisfy (6) for all $s \in \mathbb{N}$. We know that there exist constants C_0, $D_0 > 0$ such that (see [4])

$$\text{(9)} \qquad |h_n(x)e^{-x^2/2}| \le C_0 \cdot e^{-D_0 x^2} \qquad \left(|x| \ge \sqrt{2n+1}, \quad n \in \mathbb{N}\right).$$

Consequently

$$\left\|h_n(x)e^{-x^2/2}\right\|_{L^1(\mathbf{R})} \le C \int_{-\sqrt{2n+1}}^{\sqrt{2n+1}} e^{-x^2/2}|h_n(x)|\,dx \le$$

$$\text{(10)} \qquad \le C\left(1 + n^{1/4}\|e^{-x^2/2}h_n(x)\|_{L^2(\mathbf{R})}\right) \le Cn^{1/4}.$$

Hence the series $\varphi(x)e^{-x^2/2} := \sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2}$ converges in $L^1(\mathbf{R})$. The expression $[(1+x^2)^s\varphi(x)e^{-x^2/2}]^{(r+1)}$, where the formal $r+1$ derivates are taken term by term in the sum defining $\varphi(x)$ will give a (formal) sum $\sum c_k' h_k(x)e^{-x^2/2}$ where (c_k') also fulfills (6), so this series also converges in $L^1(\mathbf{R})$. But then the series obtained by term by term differentiation in $[(1+x^2)^s\varphi(x)e^{-x^2/2}]^{(r)}$ will converge uniformly. Since r and s are arbitrary, the Proof of Lemma 1 is complete.

LEMMA 2. The formal sum $\sum_{k=0}^{\infty} a_k h_k(x)e^{-x^2/2}$ is convergent in S' if and only if there exists $r \in \mathbb{N}$ with

$$\text{(11)} \qquad a_k = O(k^r).$$

PROOF. Suppose (11) is fulfilled and define the function

$$e^{-x^2/2}g(x) = \sum_{k=0}^{\infty} b_k h_{k+2r+2}(x)e^{-x^2/2},$$

where

$$b_k := a_k \frac{1}{2^{r+1}} \sqrt{\frac{k!}{(k+2r+2)!}} .$$

Since $b_k = O(1/k)$, this series converges in $L^2(\mathbf{R})$ so it can be differentiated term by term in S'. Differentiating $2r+2$ times, we get

$$e^{-x^2/2} g'(x) = \sum b_k \sqrt{2(k+2r+2)} h_{k+2r+1}(x) e^{-x^2/2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$e^{-x^2/2} g^{(2r+2)}(x) = \sum a_k h_k(x) \cdot e^{-x^2/2}.$$

So the if part of Lemma 2 is proved. To see the only part, take a series

$$e^{-x^2/2} f(x) = \sum_{k=0}^{\infty} a_k h_k(x) e^{-x^2/2}$$

converging in S'. It is known (see [5, vol. II. Chapter VII]) that there exists a function $g(x)e^{-x^2/2} \in C(\mathbf{R})$, increasing at most polynomially for $|x| \to \infty$ and there exists an $r \in \mathbf{N}$ with $[g(x)e^{-x^2/2}]^{(r)} = f(x)e^{-x^2/2}$. Let

$$g(x)e^{-x^2/2} \sim \sum_{k=0}^{\infty} b_k h_k(x) e^{-x^2/2} \tag{12}$$

and suppose that $g(x) \cdot e^{-x^2/2} x^{-s} \in L^2(\mathbf{R})$. (Since $g(x)e^{-x^2/2}$ increases polynomially, such $s \in \mathbf{N}$ exists). Now

$$|b_k| = \left| \int_{-\infty}^{\infty} g(x) e^{-x^2/2} x^{-s} x^s h_k(x) e^{-x^2/2} \, \mathrm{d}x \right| \leq$$
$$\leq \| g(x) e^{-x^2/2} x^{-s} \|_{L^2(\mathbf{R})} \| x^s h_k(x) e^{-x^2/2} \|_{L^2(\mathbf{R})}.$$

Integrating by parts we obtain

$$\int_{-\infty}^{\infty} e^{-x^2/2} h_k^2(x) x^{2s} \, \mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2} [h_k^2 x \, x^{2s-1}]' \, \mathrm{d}x = \ldots = O(k^s),$$

hence $b_k = O(k^s)$. From Lemma 1 we see that the series (12) converges to $g(x)e^{-x^2/2}$ in S'. Indeed, if $\varphi(x)e^{-x^2/2} = \sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2} \in S$, then

$$\langle g(x)e^{-x^2/2}, \varphi(x)e^{-x^2/2}\rangle = \lim_{n\to\infty}\left\langle ge^{-x^2/2}, \sum_{k=0}^{\infty} c_k h_k(x)e^{-x^2/2}\right\rangle =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{\infty} b_k c_k = \lim_{n\to\infty}\left\langle \sum_{k=0}^{\infty} b_k h_k(x)e^{-x^2/2}, \varphi(x)e^{-x^2/2}\right\rangle$$

as we asserted. So (12) can be derived term by term in distribution sense. We know that

$$f(x)e^{-x^2/2} = [g(x)e^{-x^2/2}]^{(r)} = \sum_{j=0}^{r} g^{(j)}(x)\binom{r}{j}p_{r-j}(x)e^{-x^2/2}, \tag{13}$$

where $p_{r-j}(x)$ is a polynomial of order $\leq r-j$. Since

$$e^{-x^2/2}g^{(j)}(x) = \sum_{k=0}^{\infty} b_k h_k^{(j)}(x)e^{-x^2/2}$$

and since we can eliminate from (13) the polynomials p_{r-j} by repeated use of the rule (see [1]):

$$\sqrt{2}xh_h(x) = \sqrt{k+1}\,h_{k+1}(x) + \sqrt{k}\,h_{k-1}(x) \tag{14}$$

we get indeed that the coefficients a_k of f increase polynomially. Lemma 2 is proved.

Introduce some notations. Let

$$F(H) := \{e^{-x^2/2}f(x) \ : \ \text{there exist } a_k \in R \text{ such that}$$
$$e^{-x^2/2}f(x) = \sum_{k=0}^{\infty} a_k h_k(x)e^{-x^2/2} \text{ in } S' \text{ sense}\},$$

$L := \{(a_k) \subset R \ : \ \text{there exists } r \in \mathbb{N} \text{ with } k^{-r}a_k \to 0\}$,

$L_r := \{(a_k) \subset R \ : \ ||(a_k)||_r < \infty\}$, where $||(a_k)||_r := \sup_{k\in\mathbb{N}} k^{-r}|a_k|$.

Then L_r are Banach spaces and $L_r \subset L_{r+1}$, $L = \bigcup_{r=1}^{\infty} L_r$. Take on L the weak topology generated by the inclusions $L_r \subset L$, i.e. let $(a_k^{(n)}) \xrightarrow{L} (a_k)$

$(n \to \infty)$ if and only if there exists $r \in \mathbb{N}$ such that $\|(a_k^{(n)})\|_r < \infty$, $\|(a_k)\|_r < \infty$ and $\|(a_k^{(n)} - a_k)\|_r \to \infty$ $(n \to \infty)$.

The main result of the present paper is the following

THEOREM. *Consider the vector space $F(H) \subset S'$ endowed with the topology of S'. Then*

a) *The mapping $A : F(H) \to L$, $A[fe^{-x^2/2}] := (a_k)$ is a linear isomorphism between $F(H)$ and L.*

b) *The conjugation operator $\sim$: $F(H) \to F(H)$, $e^{-x^2/2} f \to e^{-x^2/2}\tilde{f}$ $\left(\sim \sum_{k=1}^{\infty} a_k h_{k-1}(x) e^{-x^2/2}\right)$ is continuous and in fact establishes an isomorphism between $F(H)$ and $F_0(H) := \{e^{-x^2/2} f \in F(H) \ : \ a_0 = 0\}$.*

PROOF.

a) Let $e^{-x^2/2} f_n \in F(H)$, $A[f_n e^{-x^2/2}] = (a_k^{(n)})$. To show the continuity of A it is enough to prove that $e^{-x^2/2} f_n \xrightarrow{S'} 0$ implies $(a_k^{(n)}) \xrightarrow{L} 0$. By definition, $e^{-x^2/2} f_n \xrightarrow{S'} 0$ means that

$$\sum_{k=0}^{\infty} a_k^{(n)} c_k \to 0 \qquad (n \to \infty), \tag{15}$$

for all $(c_k) = A[\varphi e^{-x^2/2}]$, $\varphi e^{-x^2/2} \in S$ (we used Lemma 1). This yields $a_k^{(n)} \to 0$ $(n \to \infty)$ for all fixed k. Suppose indirectly that for any $r \in \mathbb{N}$, $\sup_k |a_k^{(n)}| k^{-r} \not\to 0$ $(n \to \infty)$, i.e. for each fixed r there exists a subsequence $n_{j,r}$ such that $\sup_k |a_k^{(n_{j,r})}| k^{-r} > \delta_r$ for every j. Since $a_k^{(n)} \to 0$ $(n \to \infty)$ for all fixed k, hence, for large $n_{j,r}$ we can eliminate $k = 1, 2, \ldots, [\delta_r^{-1}]$ from the supremum, so for some $n(r)$ we have

$$\sup_k \left| a_k^{(n_{j,r})} \right| k^{-r+1} > 1 \text{ for every } r,\ j, \text{ and } n_{j,r} > n(r). \tag{16}$$

By (16) we can choose inductively a subsequence $r_i \to \infty$ (tending very rapidly to infinity) further subsequences j_{r_i}, k_{r_i} such that for all i

$$n_{j_{r_i}, r_i} > n(r_i) + r_i, \tag{17}$$

$$\left|a_{k_{r_i}}^{(n_{j r_i}, r_i)}\right| > k_{r_i}^{r_i - 1}, \tag{18}$$

$$\sum_{\ell < i} \left|a_{k_r}^{(n_{j r_i}, r_i)}\right| k_{r_\ell}^{1 - r_\ell} < \frac{1}{4}, \tag{19}$$

$$\sum_{k = k_{r_i}}^{\infty} \left|a_k^{(n_{j r_\ell}, r_\ell)}\right| k^{1 - r_i} < \frac{1}{4} 2^{-\ell} \quad \text{for all} \quad \ell < i. \tag{20}$$

Now we define

$$c_k := \begin{cases} k_{r_i}^{1 - r_i} \operatorname{sgn} a_{k_{r_i}}^{(n_{j r_i}, r_i)} & \text{if } k = k_{r_i}, \\ 0 & \text{if } k \neq k_{r_i}. \end{cases}$$

Then $(c_k) = A[\varphi e^{-x^2/2}]$ for some $\varphi(x) e^{-x^2/2} \in S$ but

$$\left|\sum_{k=0}^{\infty} a_{k_r}^{(n_{j r_i}, r_i)} c_k\right| \geq 1 - \sum_{\ell < i} \left|a_{k_r}^{(n_{j r_i}, r_i)}\right| k_{r_\ell}^{1 - r_\ell} - \sum_{\ell > i} \left|a_{k_r}^{(n_{j r_i}, r_i)}\right| k^{1 - r_\ell} >$$
$$> 1 - \frac{1}{4} - \frac{1}{4} \sum 2^{-\ell} \geq \frac{1}{2}.$$

This contradicts to (15). So the mapping A is continuous. It maps $F(H)$ onto L by Lemma 2 and A is obviously injective. The continuity of A is easy to see: if $\sup_k |a_k^{(n)}| k^{-r} \to 0$ $(n \to \infty)$ for some r, then

$$\left|\sum_{k=0}^{\infty} a_k^{(n)} c_k\right| \leq \sum_{k=0}^{\infty} |a_k^{(n)}| k^{-r} k^r c_k \leq$$
$$\leq \sup_k |a_k^{(n)}| k^{-r} \sum_{k=0}^{\infty} k^r c_k \to 0, \quad n \to \infty.$$

b) Using a) we can transform the conjugation operator to L, where it is a simple shift of indices $(a_k) \to (a_{k+1})$. The continuity of this transformation is obvious; it maps L to $L_0 := \{(a_k) \in L \ : \ a_0 = 0\}$ and the continuity of its inverse is also trivial. The Theorem is proved.

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ON SOME CLASSES OF SEMICONTINUOUS FUNCTIONS

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0. (Weakly) composition-closed function classes and related notions were introduced by Á. CSÁSZÁR [1] and J. R. ISBELL [4]. These properties and their connections to classes of continuous functions have been examined in [1], [2] and partially in [3] minutiously.

In the present paper we are going to investigate problems concerning classes of (upper) semicontinuous functions, similar to those which have been studied in [1] and [2]. Therefore we introduce the notion of semi-composition-closedness and some of its modified versions. Basic properties of these notions will be investigated, as well as some of their relations to each other and to the classes of (upper) semicontinuous functions.

Notation and terminology of [1] and [2] will be used; according to them any function class will be considered as a class of real functions containing every constant function. Throughout this paper "semicontinuous" will always mean "upper semicontinuous". $\mathbf{R}^*$ will denote the real line equipped with the left order topology i.e. non-trivial open sets in $\mathbf{R}^*$ are exactly those having the form $(-\infty, a)$ for some $a \in \mathbf{R}$. For any topological space X, the class of $f : X \to \mathbf{R}^*$ continuous, i.e. semicontinuous functions will be denoted by $\tilde{C}(X)$. For any function class Φ the class of functions belonging to Φ and being bounded above, bounded below and bounded, will be denoted by $\overline{\Phi}, \underline{\Phi}$ and Φ^*, respectively.

1. Let X be an arbitrary non-empty set. A function class Φ on X will be called a *cone-lattice*, if it is a function lattice having the property that $f, g \in \Phi$, $0 \le c \in \mathbf{R}$ imply $f + g \in \Phi$ and $cf \in \Phi$. Let the function class Φ be called *strongly self-adjoint*, if for any $f \in \Phi$ any of the inequalities $f > 0$ and $f < 0$ implies $-\frac{1}{f} \in \Phi$ and *weakly self-adjoint*, if for a suitable $c \in \mathbf{R}$ the inequalities $f \ge c > 0$ or $f \le c < 0$ imply the same.

A function class Φ on X is said to be *strongly semi-composition-closed* (ssc), *semi-composition-closed* (sc) or *weakly semi-composition-closed* (wsc), if the following holds: given a family $\{f_i : i \in I\} \subset \Phi$, consider the map

$$(1.1) \qquad h : X \to \mathbf{R}^{*^I}, h(x) = (f_i(x)) \qquad (x \in X),$$

and a function $k \in \tilde{C}(h(X))$, $k \in \tilde{C}(cl_{\mathbf{R}^{*^I}} h(X))$, or $k \in \tilde{C}(\mathbf{R}^{*^I})$, respectively; then $k \circ h \in \Phi$. Here $\mathbf{R}^{*^I}$ is equipped with the product topology, $h(X)$ and $cl_{\mathbf{R}^{*^I}} h(X)$ are considered as subspaces of $\mathbf{R}^{*^I}$.

If in the definition above I is restricted to be countable, then Φ is said to be *countably strongly semi-composition-closed* (cssc), *countably semi-composition-closed* (csc), and *countably weakly semi-composition-closed* (cwsc). Similarly, if I is supposed to be finite, Φ is said to be *finitely strongly semi-composition-closed* (fssc), *finitely semi-composition-closed* (fsc) and *finitely weakly semi-composition-closed* (fwsc), respectively.

The following implications evidently hold:

$$(1.2) \qquad \begin{array}{ccccc} \text{ssc} & \overset{1}{\Rightarrow} & \text{sc} & \overset{2}{\Rightarrow} & \text{wsc} \\ \Downarrow 7 & & \Downarrow 8 & & \Downarrow 9 \\ \text{cssc} & \overset{3}{\Rightarrow} & \text{csc} & \overset{4}{\Rightarrow} & \text{cwsc} \\ \Downarrow 10 & & \Downarrow 11 & & \Downarrow 12 \\ \text{fssc} & \overset{5}{\Rightarrow} & \text{fsc} & \overset{6}{\Rightarrow} & \text{fwsc} \end{array}$$

The following propositions will be used:

(1.3) *Let A be an arbitrary subspace of $\mathbf{R}^{*^I}$ for any index set I, $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ two elements of A with $x_i \le y_i$ $(i \in I)$ and $f \in \tilde{C}(A)$. Then $f(x) \le f(y)$.*

PROOF. Consider $S = \{x, y\}$ as a subspace of A. As $\mathbf{R}^{*^I}$ is a T_0 space, so is S, but it is not discrete, since every open set of $\mathbf{R}^{*^I}$ containing y contains x, too. Thus S is homeomorphic with the Sierpińsky space, having x as an open, and y as a closed point and $f|S$ is semicontinuous because of $f \in \tilde{C}(A)$. Hence $f(y) < f(x)$ is impossible. ∎

(1.4) *Let n be a positive integer. $f \in \tilde{C}(\mathbf{R}^{*^n})$ iff for any $x = (x_1, \dots, x_n) \in \mathbf{R}^{*^n}$*

$$f(x_1, \dots, x_n) = \inf\{f(y_1, \dots, y_n) : y_i > x_i, i \le n\}$$

holds.

PROOF. Suppose $f \in \tilde{C}(\mathbf{R}^{*^n})$ and let $x = (x_1, \ldots, x_n)$ be an arbitrary element of $\mathbf{R}^{*^n}$. If $y_i > x_i$ $(i \leq n)$, then $f(y_1, \ldots, y_n) \geq$ $\geq f(x_1, \ldots, x_n)$ by (1.3). Assume to find a number $c \in \mathbf{R}$ such that for any $y \in \mathbf{R}^{*^n}$ with $y_i > x_i$ $(i \leq n)$ $f(y) \geq c > f(x)$. In this case $x \in f^{-1}((-\infty, c))$, but for any $y_i > x_i$ $(i \leq n)$ $\times\{(-\infty, y_i) : i \leq$ $\leq n\} \subset f^{-1}((-\infty, c))$ fails to hold. Hence $f^{-1}((-\infty, c))$ cannot be open in R^{*^n}, in contradiction with $f \in \tilde{C}(\mathbf{R}^{*^n})$.

On the other hand taking an arbitrary $c \in \mathbf{R}$ suppose that

$$(x_1, \ldots, x_n) = x \notin f^{-1}([c, +\infty)).$$

Then

$$c > f(x) = \inf\{f(y_i, \ldots, y_n) : y_i > x_i, i \leq n\}$$

and thus $y_i > x_i$ $(i \leq n)$ can be found such that $f(y_1, \ldots, y_n) < c$. Hence for the neighbourghood $U = \times\{(-\infty, y_i) : i \leq n\}$ of x we have: $U \cap f^{-1}([c, +\infty)) = \emptyset$ i.e. $f^{-1}([c, +\infty))$ is closed in $\mathbf{R}^{*^n}$. Thus $f \in \tilde{C}(\mathbf{R}^{*^n})$. ∎

(1.5) $f \in \tilde{C}(\mathbf{R}^*)$ *iff* f *is nondecreasing and continuous from the right (in Euclidean sense).*

PROOF. This is a direct consequence of (1.4). ∎

2. In this section we are going to investigate the basic properties of different versions of semi-composition-closedness.

(2.1) *Let* Φ *be a* fwsc *function class on* X. *Then* Φ *is a weakly self-adjoint cone-lattice.*

PROOF. If $0 \leq c \in \mathbf{R}$, consider the following functions:

$$k_1(x) = cx \qquad (x \in \mathbf{R}^*),$$

$$\left.\begin{aligned} k_2(x,y) &= x + y \\ k_3(x,y) &= \max(x,y) \\ k_4(x,y) &= \min(x,y) \end{aligned}\right\} \quad \left((x,y) \in \mathbf{R}^{*^2}\right).$$

$k_1 \in \tilde{C}(\mathbf{R}^*)$, k_2, k_3, $k_4 \in \tilde{C}(\mathbf{R}^{*^2})$ by (1.5) and (1.4), respectively. Thus, Φ is a cone-lattice.

Now let c be a positive number and $f \in \Phi$ with $f(x) \geq c > 0$ $(x \in$ $\in X)$. Consider the function

$$k(x) = \begin{cases} -\frac{1}{x} & \text{if} \quad x \geq c \\ -\frac{1}{c} & \text{if} \quad x < c \end{cases} \qquad (x \in \mathbf{R}^*).$$

$k \in C(R^*)$ by (1.5), hence $-\frac{1}{f} = k \circ f \in \Phi$.

Similarly, if c is negative and $f(x) \leq c < 0$ $(x \in X)$, then the function

$$k'(x) = \begin{cases} -\frac{1}{c} & x \geq c \\ -\frac{1}{x} & x < c \end{cases} \qquad (x \in \mathbf{R}^*)$$

will do. ∎

(2.2) *If Φ is a fwsc class on X, f, $g \in \Phi$, $f(x) \geq 0$, $g(x) \geq 0$ $(x \in X)$, then $fg \in \Phi$.*

PROOF. Consider the function

$$k(a,b) = \begin{cases} a \cdot b & \text{if} \quad a \geq 0,\ b \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (a,b) \in \mathbf{R}^{*2}.$$

$k \in \tilde{C}(\mathbf{R}^{*2})$ by (1.4), hence $f \cdot g = k \circ (f, g) \in \Phi$. ∎

(2.3) *If Φ is fwsc on X, then for every $f \in \Phi$ the function*

$$\psi_f(x) = \begin{cases} 1 & \textit{if} \quad f(x) \geq 0 \\ 0 & \textit{if} \quad f(x) < 0 \end{cases} \qquad (x \in X)$$

belongs to Φ.

PROOF. It is an immediate consequence of (1.5). ∎

(2.4) *If Φ is fssc on X, then it is strongly self-adjoint.*

PROOF. The subspaces of $\mathbf{R}^*$ determined by $(0, +\infty)$ and $(-\infty, 0)$ will be denoted by $\mathbf{R}^*_+$ and $\mathbf{R}^*_-$, respectively. For $0 \neq c \in \mathbf{R}$ consider $k(c) = -\frac{1}{c}$. $k_1 = k|\mathbf{R}^*_+ \in \tilde{C}(\mathbf{R}^*_+)$ by (1.5) (there is an increasing homeomorphism $g : \mathbf{R}^*_+ \to \mathbf{R}^*$) and if $f(x) > 0$ $(x \in X)$, then $f(X) \subset \subset \mathbf{R}^*_+$, thus $-\frac{1}{f} = k_1 \circ f \in \Phi$. If $f(x) < 0$ $(x \in X)$, then consider $k_2 = k|\mathbf{R}^*_-$. ∎

(2.5) *Let Φ be a (c)wsc function class on X and I an arbitrary index set ($|I| = \omega$). If f is a function on X bounded below and having the form $f = \inf\{f_i : i \in I\}$ for some $f_i \in \Phi$ $(i \in I)$, then $f \in \Phi$.*

PROOF. Let m be a lower bound of f, then it is a lower bound of f_i $(i \in I)$, too. For every $y \in \mathbf{R}^{*I}$ let

$$k(y) = \begin{cases} \inf\{y_i : i \in I\} & \text{if} \quad y_i \geq m \ (i \in I), \\ m & \text{otherwise.} \end{cases}$$

On the other hand, denoting by π_i, as usual, the projection of $\mathbf{R}^{*I}$ to the i-th coordinate, we have

$$k = \inf\{\max(\pi_i, m) : i \in I\}.$$

Since for every $i \in I$, $\pi_i \in \tilde{C}(\mathbf{R}^{*I})$, so is k. Considering the map

$$h : X \to \mathbf{R}^{*I}, h(x) = (f_i(x)) \quad \text{for} \quad x \in X,$$

we have $f = k \circ h$. Thus $f \in \Phi$. ∎

(2.6) *Let Φ be a* (c)ssc *function class on X and I an arbitrary index set ($|I| = \omega$). If f is a function on X of the form $f = \inf\{f_i : i \in I\}$ with $f_i \in \Phi$ $(i \in I)$, then $f \in \Phi$.*

PROOF. Consider the map

$$h : X \to \mathbf{R}^{*I}, h(x) = (f_i(x)) \qquad \text{for} \qquad x \in X.$$

Let

$$k = \inf\{\pi_i | h(X) : \ i \in I\}. \tag{2.7}$$

For every $z \in h(X)$ there is an $x \in X$ with $z = h(x)$ i.e. $z_i = f_i(x) \geq \geq f(x)$ hence the set $\{\pi_i(z) : i \in I\}$ is bounded below. Definition (2.7) is therefore correct and since $\pi_i|h(X) \in \tilde{C}(h(X))$ $(i \in I)$ so is k. Thus $f = k \circ h \in \Phi$. ∎

(2.8) *Let Φ be a* cssc *function class on X, then Φ is complete.*

PROOF. Suppose $f_n \in \Phi$ and $f_n \to f$ uniformly. For any $k \in \mathbb{N}$ there can be found $n_k \in \mathbb{N}$ such that $|f_{n_k} - f| < \frac{1}{k}$. Now $f_{n_k} + \frac{1}{k} \in \Phi$, of course and $f = \inf\{f_{n_k} + \frac{1}{k} : k \in \mathbb{N}\}$. Thus $f \in \Phi$ by (2.6). ∎

REMARK the following:

(2.9) *The proof above is correct, if Φ is supposed to be a function class, having merely the following properties:*
(a) if $f \in \Phi$ and $c \in R$, then $f + c \in \Phi$,

(b) if $f = \inf\{f_n : n \in \mathsf{N}\}$ *with* $f_n \in \Phi$ $(n \in \mathsf{N})$, *then* $f \in \Phi$.

(2.10) *Let* Φ *be a* (f, c)wsc *function class on* X, *then* $\overline{\Phi}$, $\underline{\Phi}$, *and* Φ^* *have the same property.*

PROOF. Let I be an arbitrary (or finite, or countable) index set, $f_i \in \overline{\Phi}$ $(i \in I)$ and $h(x) = (f_i(x))$ $(x \in X)$. If $k \in \tilde{C}(\mathbf{R}^{*^I})$, then $k \circ h \in \Phi$, since Φ is (f, c)wsc. On the other hand, define M_i $(i \in I)$ and M by

$$M_i = \sup\{f_i(x) : x \in X\}(i \in I), M = (M_i) \in \mathbf{R}^{*^I}.$$

Now, for any $z \in h(X)$ we have $z_i \leq M_i$ $(i \in I)$ and therefore $k(z) \leq k(M)$ by (1.3). Thus $k(M)$ is an upper bound of $k \circ h$ i.e. $k \circ h \in \overline{\Phi}$.

The same statement for the case of $\underline{\Phi}$ can be proved similarly, while the statement concerning Φ^* is a direct consequence of the ones above.∎

(2.11) *Let* Φ *be a* (f, c)sc *function class on* X, *then* $\overline{\Phi}$ *has the same property.*

PROOF. With a notation similar to the one above we have $z_i \leq$ $\leq M_i$ $(i \in I)$ for every $z \in h(X)$, hence $M \in cl_{\mathbf{R}^{*^I}}\{z\} \subset cl_{\mathbf{R}^{*^I}} h(X)$. Now we have $k(M) \geq k(z)$ for any $k \in \tilde{C}(cl_{\mathbf{R}^{*^I}} h(X))$ by (1.3) and therefore $k(M)$ is an upper bound of $k \circ h$ again. ∎

3. In this section some counterexamples will be constructed for further examination of the implications in (1.2).

(3.1) Consider

$$\Phi = \{f \in \tilde{C}(\mathbf{R}^*) : \exists x_0 \in \mathbf{R}^*(x \geq x_0 \Rightarrow f(x) = f(x_0))\}.$$

Φ *is a* fssc *class*, since for given $f_1, \ldots, f_n \in \Phi$ considering the map

$$h : \mathbf{R}^* \to \mathbf{R}^{*^n}, h(x) = (f_1(x), \ldots, f_n(x)) \qquad (x \in X),$$

it will be a continuous one, hence we have $k \circ h \in \tilde{C}(\mathbf{R}^*)$ for any $k \in \tilde{C}(h(\mathbf{R}^*))$. On the other hand an $x_i \in \mathbf{R}^*$ can be found for every $f_i (i \leq n)$ such that $f_i(x) = f_i(x_i) = M_i$ for any $x \geq x_i$. Taking therefore $x_0 \geq \max\{x_i : i \leq n\}$, we have $k \circ h \in \Phi$, since for every $x \geq x_0$

$$k(h(x)) = k(f_1(x), \ldots, f_n(x)) = k(M_1, \ldots, M_n) = k(h(x_0)).$$

Φ *is not* cwsc, *however.* Let $n \in \mathbb{N}$ be arbitrary and consider the function

$$f_n(x) = \begin{cases} \arctan x & \text{for} \quad x < n, \\ \frac{\pi}{2} & \text{for} \quad x \geq n. \end{cases}$$

Now we have $f_n \in \Phi$ $(n \in \mathbb{N})$, since $f_n \in \tilde{C}(\mathbf{R}^*)$ by (1.5) and $f_n(x) = f_n(n) = \frac{\pi}{2}$ for any $x \geq n$, but $\arctan = \inf\{f_n : n \in \mathbb{N}\}$ is bounded below, without belonging to Φ. Thus, by (2.5) Φ is not cwsc. ∎

(3.2) Let Φ be the class of every bounded real function on $\mathbf{R}$. Since the class of all real functions on $\mathbf{R}$ is evidently ssc, Φ *must be* wsc by (2.10).

However, Φ *fails to be* fsc. Consider the function

$$f(x) = \begin{cases} -1 & \text{for} \quad x \leq -1 \\ x & \text{for} \quad -1 < x < 1 \\ 1 & \text{for} \quad x \geq 1 \end{cases} \qquad (x \in \mathbf{R}).$$

$f \in \Phi$ and $-f \in \Phi$ evidently hold. Define $h : \mathbf{R} \to \mathbf{R}^{*^2}$, $h(x) = (f(x), -f(x))$, then the map φ defined by

$$\varphi : h(\mathbf{R}) \to [-1, 1], \varphi((p, -p)) = p$$

is a homeomorphism. $h(\mathbf{R})$ is considered here as a subspace of $\mathbf{R}^{*^2}$, while the interval $[-1, 1]$ as a subspace of $\mathbf{R}$ (not $\mathbf{R}^*$!). The function

$$g(x) = \begin{cases} 1 + \frac{1}{x} & \text{for} \quad -1 \leq x < 0 \\ 0 & \text{for} \quad 0 \leq x \leq 1 \end{cases}$$

is semicontinuous, hence $g \circ \varphi \in \tilde{C}(h(\mathbf{R}))$. This function, which is not bounded below will be extended to be semicontinuous on $cl_{R^{*2}} h(\mathbf{R})$.

First for all

$$cl_{\mathbf{R}^{*2}} h(\mathbf{R}) = \{(p_1, p_2) \in \mathbf{R}^{*^2} : \exists x \in \mathbf{R}(p_1 \geq f(x), p_2 \geq -f(x))\}$$

obviously holds. The set $(-\infty, p_1] \times (-\infty, p_2] \subset \mathbf{R}^{*^2}$ will be denoted by L_z, for any $z = (p_1, p_2) \in \mathbf{R}^{*^2}$. Now we have $L_z \cap k(\mathbf{R}) \neq \emptyset$ for each $z \in cl_{\mathbf{R}^{*2}} h(\mathbf{R})$, and the function $g \circ \varphi$ is bounded above, thus the following definition is correct.

$$k(z) = \sup\{g(\varphi(y)) : y \in L_z \cap h(\mathbf{R})\} \qquad (z \in cl_{\mathbf{R}^{*2}} h(\mathbf{R})).$$

$k|h(\mathbf{R}) = g \circ \varphi$, since $L_z \cap h(\mathbf{R}) = \{z\}$ for every $z \in h(\mathbf{R})$. On the other hand, let $c \in \mathbf{R}$ be arbitrarily chosen. If $c > 0$ holds, then

$$\{z \in cl_{\mathbf{R}^{*2}} h(\mathbf{R}) : \ k(z) < c\} = cl_{\mathbf{R}^{*2}} h(\mathbf{R})$$

since $g \leq 0$ and therefore $k \leq 0$, while for $c \leq 0$ we have

$$\begin{aligned}
&\{z \in cl_{\mathbf{R}^{*2}} h(\mathbf{R}) : k(z) < c\} = \\
=&\{z \in cl_{\mathbf{R}^{*2}} h(\mathbf{R}) : \varphi(L_z \cap h(\mathbf{R})) \subset (\frac{1}{c-1}, 0)\} = \\
=&\{(p_1, p_2) \in cl_{\mathbf{R}^{*2}} h(\mathbf{R}) : p_1 < 0, p_2 < \frac{1}{1-c}\} = \\
=&cl_{\mathbf{R}^{*2}} h(\mathbf{R}) \cap ((-\infty, 0) \times (-\infty, \frac{1}{1-c}))
\end{aligned}$$

which is an open subspace of $cl_{\mathbf{R}^{*2}} h(\mathbf{R})$.

Thus we have a function k such that $k \in \tilde{C}(cl_{\mathbf{R}^{*2}} h(\mathbf{R}))$ and $k \circ h = = g \circ \varphi \circ h = g \circ f \notin \Phi$, since it is not bounded below. ∎

(3.3) Consider the class $\Phi = \overline{\tilde{C}(\mathbf{R}^*)}$ i.e. the class of real semicontinuous functions bounded above on $\mathbf{R}^*$. Since the class $\tilde{C}(\mathbf{R}^*)$ is obviously ssc, Φ *must be* sc by (2.11).

Define now the function

$$f(x) = \arctan x - \frac{\pi}{2} \qquad (x \in \mathbf{R}^*),$$

$f \in \Phi$, since $f \in \tilde{C}(\mathbf{R}^*)$ by (1.5) and $f(x) < 0$ $(x \in \mathbf{R}^*)$. On the other hand $\lim\limits_{x \to \infty} f(x) = 0$, hence $-\frac{1}{f}$, being not bounded above, cannot belong to Φ. Thus Φ *fails to be* fssc by (2.4). ∎

(3.4) Having constructed the counterexamples above, we can see at once, that implications (10), (11) and (12) in (1.2) *fail to be equivalences* by (3.1), implications (2), (4) and (6) by (3.2), while implications (1), (3) and (5) by (3.3). ∎

4. The following theorems show a complete analogy with Theorems 2., 3. and 4. in [2].

(4.1) *If Φ is a function class on X, the following statements are equivalent:*

(a) *There are a topological space Y and a surjective map $p : X \to Y$ such that $\Phi = \tilde{C}(Y) \circ p$.*

(b) *Φ is* ssc.

(c) *There is a topology on X such that $\Phi = \tilde{C}(X)$.*

(4.2) *If Φ is a function class on X, the following statements are equivalent:*

(a) *There are a topological space Y and a map $p : X \to Y$ such that $\Phi = \tilde{C}(Y) \circ p$.*

(b) *Φ is* wsc.

(c) *There are a set $Y \supset X$ and a topology on Y such that $\Phi = \tilde{C}(Y)|X$.*

(4.3) *If Φ is a function class on X, the following statements are equivalent:*

(a) *There are a topological space Y and a map $p : X \to Y$ such that $p(X)$ is dense in Y and $\Phi = \tilde{C}(Y) \circ p$.*

(b) *Φ is* sc.

(c) *There are a set $Y \supset X$ and a topology on Y such that X is dense in Y and $\Phi = \tilde{C}(Y)|X$.*

PROOF. We prove the three theorems simultaneously, denoting the details concerning only the case is ssc, wsc or sc classes by α, β and γ, respectively.

(a)$\Rightarrow$(b) Suppose $f_i \in \Phi$, then $f_i = g_i \circ p$ for some $g_i \in \tilde{C}(Y)$ $(i \in I)$. Define

$$f : X \to \mathbf{R}^{*I}, \qquad h(x) = (f_i(x)) \qquad (x \in X)$$

and

$$g : Y \to \mathbf{R}^{*I}, \qquad g(y) = (g_i(y)) \qquad (y \in Y).$$

Now $h = g \circ p$ and g is a continuous map.

(γ) Therefore $cl_{\mathbf{R}^{*I}} h(X) = cl_{\mathbf{R}^{*I}} g(p(X)) \supset g(cl_Y p(X)) = g(Y)$.

Thus, for any

(α) $k \in \tilde{C}(h(x)) = \tilde{C}(g(p(X))) = \tilde{C}(g(Y))$,

(β) $k \in \tilde{C}(\mathbf{R}^*)$,

(γ) $k \in \tilde{C}(cl_{\mathbf{R}^{*I}} h(X))$,

we have $k \circ g \in \tilde{C}(Y)$ and consequently $k \circ h \in \tilde{C}(Y) \circ p = \Phi$.

(b)$\Rightarrow$(c). Let I be an appropriate index set such that $\Phi = \{f_i : i \in I\}$ and consider the map

$$h : X \to \mathbf{R}^{*I}, \qquad h(x) = (f_i(x)) \qquad (x \in X).$$

Now define
(α) $Y = X$,
(β) $Y = X \cup (\mathbf{R}^{*I} - h(X))$,
(γ) $Y = X \cup (cl_{\mathbf{R}^{*I}}[h(X)] - h(X))$
and
(α) $\tilde{h} : Y \to h(X)$ by $\tilde{h} = h$,
(β) $\tilde{h} : Y \to \mathbf{R}^{*I}$ by $\tilde{h}|X = h$, $\tilde{h}|Y - X = id_{\mathbf{R}^{*I} - h(X)}$,
(γ) $\tilde{h} : Y \to cl_{\mathbf{R}^{*I}} h(X)$ by $\tilde{h}|X = h$, $\tilde{h}|Y - X = id_{cl_{\mathbf{R}^{*I}}[h(X)] - h(X)}$,
and equip Y with the initial topology corresponding $\tilde{h}$. Then (γ) X is obviously dense in Y and for any $i \in I$ we have $f_i = \pi_i \circ h = \pi_i \circ \tilde{h}|X \in \in \tilde{C}(Y)|X$ i.e. $\Phi \subset \tilde{C}(Y)|X$. On the other hand, if $g \in \tilde{C}(Y)$, let
(α) $k : h(X) \to \mathbf{R}^*$,
(β) $k : \mathbf{R}^{*I} \to \mathbf{R}^*$,
(γ) $k : cl_{\mathbf{R}^{*I}} h(X) \to \mathbf{R}^*$
be defined by $k(\tilde{h}(y)) = g(y) \qquad (y \in Y)$.

If there are y, $y' \in Y$ such that $\tilde{h}(y) = \tilde{h}(y')$, then they have the same neighbourhoods in Y, hence $g(y) = g(y')$, for $\mathbf{R}^*$ is a T_0-space. Thus, the definition of k is correct. Taking an open set $G \subset \mathbf{R}^*$, the set $g^{-1}(G) = \tilde{h}^{-1}(k^{-1}(G))$ is also open and so is the set $k^{-1}(G)$ since $\tilde{h}$ is an onto map. Therefore
(α) $k \in \tilde{C}(h(X))$,
(β) $k \in \tilde{C}(\mathbf{R}^{*I})$,
(γ) $k \in \tilde{C}(cl_{\mathbf{R}^{*I}} h(X))$,
and $g|X = k \circ \tilde{h}|X = k \circ h \in \Phi$. Hence $\Phi = \tilde{C}(Y)|X$.

(c)$\Rightarrow$(a). Obvious. ∎

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A REMARK ON THE FARTHEST POINT PROBLEMS II.

By

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Introduction

Let be $(Y, \|\cdot\|)$ a normed linear space, $K \subset Y$ a bounded set. We define

$$Q_K(y) = \{x \in K; \|y - x\| = \sup_{x' \in K} \|y - x'\|\}.$$

We call the set K as a uniquely farthest set, if $Q_K(y)$ is singleton for all $y \in Y$. Naturally, all singleton K sets are uniquely farthest. The following question seems to be open. Is all uniquely farthest sets are the singletons? There are some partial answers concerning this question. The answer is affirmative in finite dimensional spaces, in case of norm-compact K[1], in case of norm-continuous Q_K [2], there is an equivalent renorming of the space that in the new norm the answer is affirmative [3].

In this paper we prove a result, which means that the ℓ_1-product of two nontrivial such space has the same property, assuming that one of them is strictly convex.

The result

THEOREM. *Let be* $(X, \|\cdot\|)$ *and* $(Z, \|\cdot\|)$ *real normed linear spaces, in which the uniquely remotal sets are singletons,* $(X, \|\cdot\|)$ *is strictly convex,* $\dim X > 0$, $\dim Y > 0$. *Then in the* $Y = X \otimes Z$, $\|(x, y)\| = \|x\| + \|y\|$ *normed linear space the uniquely remotal sets are singletons.*

PROOF. Let $K \subset Y$ a uniquely remotal set. We denote by P and Q the projections onto $(X, O) \sim X$ and $(O, Z) \sim Z$ respectively. We have the following cases:

CASE I. There exists $x' \in X$, for which

$$P(Q_K(x', O)) = (x', 0).$$

CASE II. For arbitrary $x \in X$

$$P(Q_K(x,0)) \neq (x,0).$$

In Case I, we show that for arbitrary $x \in X$

(1) $$Q_K(x,0) = Q_K(x',0).$$

Clearly,

$$\begin{aligned}&\|(x,0) - Q_K(x,0)\| \leq \\ \leq &\|x - x'\| + \|(x',0) - Q_K(x,0)\| \leq \\ \leq &\|x - x'\| + \|(x',0) - Q_K(x',0)\|,\end{aligned}$$

on the other side, using $P(Q_K(x',0)) = (x',0)$ we have

$$\|x - x'\| + \|(x',0) - Q_K(x',0)\| = \|(x,0) - Q_K(x',0)\|,$$

so, by the definition of Q_K, we have (1).

In Case II, we define the mapping $f: x \to X$ on the following way:

$$f(x) = x + \frac{P(Q_K(x,0) - (x,0))}{\|P(Q_K(x,0) - (x,0))\|} \cdot \|(x,0) - Q_K(x,0)\|.$$

We can see instantly, that

(2) $$\begin{aligned}&\|x - f(x)\| = \|(x,0) - Q_K(x,0)\| = \\ = &\|Q(Q_K(x,0))\| + \|P((x,0) - Q_K(x,0))\|.\end{aligned}$$

Applying (2) in the definition of f,

(3) $$\|f(x) - P(Q_K(x,0))\| = \|Q(Q_K(x,0))\|.$$

So, for arbitrary $x_1, x_2 \in X$

(4) $$\|x_1 - f(x_2)\| \leq \|(x_1,0) - Q_K(x_2,0)\|,$$

because of

$$\begin{aligned}&\|x_1 - f(x_2)\| \leq \|x_1 - P(Q_K(x_2,0))\| + \|P(Q_K(x_2,0)) - f(x_2)\| = \\ = &\|x_1 - P(Q_K(x_2,0))\| + \|Q(Q_K(x_2,0))\| = \|(x_1,0) - Q_K(x_2,0)\|.\end{aligned}$$

(Here we have used (3).)

We show also that for $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$

$$\|x_1 - f(x_2)\| < \|x_1 - f(x_1)\|. \tag{5}$$

First, we prove (5) for $Q_K(x_1, 0) = Q_K(x_2, 0)$ indirectly. Using (2), (4) and the definition of Q_K, the indirect hypothesis can be formulated as

$$\|x_1 - f(x_2)\| = \|x_1 - f(x_1)\|. \tag{6}$$

Using (2), (3), (6) and the definition of f,

$$\begin{aligned} &\|x_1 - f(x_1)\| = \\ &= \|P(Q_K(x_1, 0)) - (x_1, 0)\| + \|P(Q_K(x_1, 0)) - f(x_1)\| \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\|x_1 - f(x_2)\| = \|(x_1, 0) - Q_K(x_1, 0)\| = \\ &= \|Q(Q_K(x_1, 0))\| + \|P((x_1, 0) - Q_K(x_1, 0))\| = \\ &= \|Q(Q_K(x_2, 0))\| + \|P(Q_K(x_1, 0)) - (x_1, 0)\| = \\ &= \|f(x_2) - P(Q_K(x_1, 0))\| + \|P(Q_K(x_1, 0)) - (x_1, 0)\|. \end{aligned} \tag{8}$$

(In the latter relation $Q_K(x_1, 0) = Q_K(x_2, 0)$ is used too.) Because of the definition of f, $x_1 - f(x_1)$ is parallel to $P(Q_K(x_1, 0)) - (x_1, 0)$. By (7) and (8), using the strict convexity of $(x, \|\cdot\|)$, $P(Q_K(x_1, 0)) - -f(x_1)$ is parallel to $P(Q_K(x_1, 0)) - f(x_2)$. Because of the same length $/\|Q(Q_K(x_1, 0))\|/$, $f(x_1) = f(x_2)$. Now, we have (5) in the case $Q_K(x_1, 0) = Q_K(x_2, 0)$. In the case $Q_K(x_1, 0) \neq Q_K(x_2, 0)$, using (2) and (4),

$$\begin{aligned} &\|x_1 - f(x_1)\| = \|(x_1, 0) - Q_K(x_1, 0)\| > \\ &> \|(x_1, 0) - Q_K(x_2, 0)\| \geq \|(x_1, 0) - f(x_2)\| \end{aligned}$$

so, we have (5). Using (5), $K_1 = f(X)$ is uniquely farthest in $(X, \|\cdot\|)$. This implies that K_1 is a singleton.

We prove indirectly that $Q_K((X, 0))$ is a singleton too. Let be $x_1, x_2 \in X$ such that $Q_K(x_1, 0) = k_1$, $Q_K(x_2, 0) = k_2$, $k_1 \neq k_2$. Using (2) and (4),

$$\begin{aligned} &\|x_1 - f(x_1)\| = \|(x_1, 0) - k_1\| > \\ &> \|(x_1, 0) - k_2\| \geq \|x_1 - f(x_2)\|, \end{aligned}$$

which contradicts $f(x_1) = f(x_2)$ (the latter comes from the singletonness of K_1). So, in Cases I and II, we proved that $Q_K((X, 0))$ is a singleton. We can have the same reasoning when beside of K we have $(0, z) + K$ for arbitrary $z \in Z$. This implies that $Q_K((X, z))$ is a singleton for arbitrary $z \in Z$. We denote this element by $k(z)$. So,

$$\sup_{k \in K} \|(x, z) - k\| = \|(x, 0) - P(k(z))\| + \|(0, z) - Q(k(z))\|,$$

and the supremum on the left side is exactly attained at $k(z)$.

We choose x so, that $(x, 0) = P(k(z))$. Now,

$$\sup_{k\in K} \|(x,z) - k\| \geq \|(0,z) - Q(k(z))\|$$

and because of the special choice,

$$\sup_{k\in K} \|(x,z) - k\| = \|(0,z) - Q(k(z))\|.$$

Again, the supremum on the left side is exactly attained only at $k(z)$. Using again the special choice, we can write

$$\sup_{k\in K} \|(0,z) - Q(k)\| = \|(0,z) - Q(k(z))\|.$$

This means that $Q(K)$ is uniquely remotal in the subspace $(Z, 0)$. So, $Q(K)$ is a singleton, implying that $k(z)$ does not depend on z. So K is a singleton.

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ON STOPPED RANDOM WALKS

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1. Introduction

Let $Y, Y_1, Y_2, Y_3, \dots$ be i.i.d. random variables. Consider the σ-fields $\mathcal{F}_n = \sigma(Y_1, Y_2, \dots, Y_n)$ $n \geq 1$. Let ν be a positive integer-valued stopping time with respect to the sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of σ-fields, for which $P(\nu < +\infty) = 1$. Consider the generalized random walk

$$S_0 = 0, \qquad S_n = Y_1 + Y_2 + \dots + Y_n \qquad n = 1, 2, \dots.$$

Then the limit

$$\lim_{n \to \infty} S_{\nu \wedge n} = S_\nu$$

exists and is finite a.s.

The following results are obtained by A. GUT and S. JANSON.

(1) Let $p \geq 1$, and suppose that $E\nu^p < \infty$. Then

$$E|S_\nu|^p < \infty \Longleftrightarrow E|Y|^p < \infty. \tag{1.1}$$

(2) Let $p \geq 1$ and suppose that $E\nu^{(p/2)\vee 1} < \infty$; $EY = 0$. Then

$$E|S_\nu|^p < \infty \Longleftrightarrow E|Y|^p < \infty. \tag{1.2}$$

(3) Let $p \geq 1$ and suppose that $EY = a \neq 0$. Then

$$E|S_\nu|^p < \infty \text{ and } E|Y|^p < \infty \Rightarrow E\nu^p < \infty. \tag{1.3}$$

In this paper we give an estimate for the desired moment of $\sup_{n \geq 1} |S_{\nu \wedge n}|$. We establish the necessary and sufficient conditions on the existence of the desired moment of $\sup_{n \geq 1} |S_{\nu \wedge n}|$. The obtained results improve a part of the results of paper [2] by A. GUT and S. JANSON.

2. The case $EY = a \neq 0$.

In this case we give a necessary and sufficient condition to ensure that

$$\sup_{n\geq 1} |S_{\nu\wedge n}|$$

belong to L_p. We prove the following

THEOREM 1. *Let $p \geq 1$ and suppose $EY = a \neq 0$. Then*

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty \Longleftrightarrow E|Y|^p < \infty \text{ and } E\nu^p < \infty.$$

PROOF. Suppose first $E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty$. Since

$$|Y_1| \leq \sup_{n\geq 1} |S_{\nu\wedge n}|$$

we have

$$E|Y|^p = E|Y_1|^p \leq E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty.$$

Next $S_\nu = \lim_{n\to\infty} S_{\nu\wedge n}$, therefore

$$|S_\nu|^p = \lim_{n\to\infty} |S_{\nu\wedge n}|^p \leq \sup_{n\geq 1} |S_{\nu\wedge n}|^p, \quad \text{so} \quad E|S_\nu|^p < \infty.$$

We apply (1.3) in the case $EY = a \neq 0$ to obtain $E\nu^p < \infty$.

Conversely, let $E|Y|^p < \infty$ and $E\nu^p < \infty$. Without the loss of the generality we can suppose now that $EY_1 = a > 0$. (The case $a < 0$ can be treated similarly by taking $\{-Y_i\}$ instead of $\{Y_i\}$ $i = 1, 2, \ldots$). Then $(S_{\nu\wedge n}, \mathcal{F}_n)$ is submartingale. Its Doob-decomposition has the form

$$S_{\nu\wedge n} = [S_{\nu\wedge n} - a(\nu \wedge n)] + a(\nu \wedge n),$$

where $S_{\nu\wedge n} - a(\nu \wedge n)$ is martingale and $a(\nu \wedge n) \leq a\nu$ for $n = 1, 2, \ldots$.

Consider two cases:

a) If $1 \leq p \leq 2$, then by the Burkholder-Davis-Gundy inequality we have

$$E\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu \wedge n)|^p \leq C_p E\left(\sum_{i=1}^{\infty} (Y_i - a)^2 \chi(\nu \geq i)\right)^{p/2} \leq$$

$$\leq C_p E\left[\sum_{i=1}^{\infty} |Y_i - a|^p \chi(\nu \geq i)\right] = C_p \sum_{i=1}^{\infty} E|Y - a|^p P(\nu \geq i) =$$

$$= C_p E|Y - a|^p E\nu \leq C_p E|Y - a|^p E\nu^p < \infty$$

so

$$E\left(\sup_{n\geq 1}|S_{\nu\wedge n}|\right)^p \leq 2^{p-1}\left[E\left(\sup_{n\geq 1}|S_{\nu\wedge n}-a(\nu\wedge n)|\right)^p + a^p E\nu^p\right] < \infty.$$

b) If $p \geq 2$, by the J. MOGYORÓDI inequality [3] we have

$$E\left(\sup_{n\geq 1}|S_{\nu\wedge n} - a(\nu\wedge n)|\right)^p \leq$$

$$\leq C_p\left\{E\left[\sum_{i=1}^{\infty} E\left((Y_i - a)^2|\mathcal{F}_i\right)\chi(\nu\geq i)\right]^{p/2} + \right.$$

$$\left. + \sum_{i=1}^{\infty} E\left(|Y_i - a|^p\chi(\nu\geq i)\right)\right\} =$$

$$= C_p\{E|Y-a|^p E\nu + (E|Y-a|^2)^{p/2} E\nu^{p/2}\} < \infty$$

so

$$E\left(\sup_{n\geq 1}|S_{\nu\wedge n}|\right)^p < \infty.$$

This proves the assertion.

REMARK. Since

$$|S_\nu| \leq \sup_{n\geq 1}|S_{\nu\wedge n}|,$$

so $\sup_{n\geq 1}|S_{\nu\wedge n}| \in L_p$ implies that $S_\nu \in L_p$ too. Therefore one direction of (1.1) is obtained immediately from Theorem 1.

COROLLARY. The conditions

$$E(\sup_{n\geq 1}|S_{\nu\wedge n}|)^p < \infty \quad \text{and} \quad \sup_{n\geq 1} E(|S_{\nu\wedge n}|)^p < \infty$$

are equivalent, provided that $EY = a \neq 0$.

Indeed, $\sup E|S_{\nu\wedge n}|^p < \infty$ is trivial from $E(\sup_{n\geq 1}|S_{\nu\wedge n}|)^p < \infty$.

Conversely, suppose that $\sup_{n\geq 1} E|S_{\nu\wedge n}|^p < \infty$. It is evident that $E|Y|^p < \infty$. On the other hand, by the Fatou lemma we have

$$E|S_\nu|^p = \int \liminf_{n\to\infty}|S_{\nu\wedge n}|^p dP \leq \liminf_{n\to\infty}\int |S_{\nu\wedge n}|^p dP \leq \sup_{n\geq 1} E|S_{\nu\wedge n}|^p < \infty.$$

By (1.3) it follows that $E\nu^p < \infty$. By Theorem 1 $E|Y|^p < +\infty$ and $E\nu^p < \infty$ together ensure the finiteness of

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|^p) < \infty.$$

REMARK. By (1.1) the conditions $E|Y|^p < \infty$ and $E\nu^p < \infty$ imply the finiteness of $E|S_\nu^*|^p < \infty$, where $S_\nu^* = \sum_{i=1}^{\nu} |Y_i|$ but conversely

$$E|S_\nu^*|^p < \infty \quad \text{implies} \quad E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty$$

so we can conclude that the following conditions are equivalent provided that $EY = a \neq 0$

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty \Longleftrightarrow \sup_{n\geq 1} E|S_{\nu\wedge n}|^p < \infty \Longleftrightarrow E|S_\nu^*|^p < \infty.$$

3. The case $EY = 0$

For $p \geq 2$ we have here a situation similar to Theorem 1, but the assumption on the moments of ν can be weakened.

THEOREM 2. *Let $p \geq 2$, and suppose that $EY = 0$. Then*

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p < \infty \Longleftrightarrow E|Y|^p < \infty \text{ and } E\nu^{p/2} < \infty.$$

PROOF. In the case $EY = 0$, $S_{\nu\wedge n} = \sum_{i=1}^{n} Y_i \chi(\nu \geq i)$ is martingale and its differences are $d_i = Y_i \chi(\nu \geq i)$ $i \geq 1$ where Y_i and $\chi(\nu \geq i)$ are independent.

Applying the inequality of J. MOGYORÓDI for $p \geq 2$, we have

$$c_p M \leq E(\sup_{n\geq 1} |S_{\nu\wedge n}|)^p \leq C_p M.$$

Where c_p and C_p are positive constants depending on p, and

$$\begin{aligned} M =& E\Big[\sum_{i=1}^{\infty} E(d_i^2|\mathcal{F}_{i-1})\Big]^{p/2} + \sum_{i=1}^{\infty} E|d_i|^p = \\ =& E\Big[\sum_{i=1}^{\infty} E(Y_i^2\chi(\nu \geq i)|\mathcal{F}_{i-1})\Big]^{p/2} + \sum_{i=1}^{\infty} E(|Y_i|^p\chi(\nu \geq i)) = \\ =& E\Big[\sum_{i=1}^{\infty} \sigma^2\chi(\nu \geq i)\Big]^{p/2} + \sum_{i=1}^{\infty} P(\nu \geq i)E|Y|^p = \\ =& \sigma^p E\nu^{p/2} + E|Y|^p E\nu. \end{aligned}$$

Here $\sigma^2 = EY^2$ and we have used that Y_i and $\mathcal{F}_{i-1}$ as well Y_i and $\chi(\nu \geq i)$ are independent.

From these inequalities we get that

$$E \sup_{n \geq 1} |S_{\nu \wedge n}|^p < \infty \Longleftrightarrow E|Y|^p < \infty \text{ and } E\nu^{p/2} < \infty$$

remarking that $E\nu \leq E\nu^{p/2}$ and $E|Y|^p > \sigma^2$.

This proves our assertion.

Unfortunately, we do not know whether Theorem 2 for the case $1 \leq p < 2$ holds true. But we have the following

THEOREM 3. *Let $1 \leq p < 2$ and suppose that $EY = 0$. Then*

$$E|Y|^p < \infty \text{ and } E\nu < \infty \Rightarrow E \sup_{n \geq 1} |S_{\nu \wedge n}|^p < \infty.$$

PROOF. $S_{\nu \wedge n}$ is a martingale as in the previous proof. By the Burkholder-Davis-Gundy inequality we have

$$E \sup |S_{\nu \wedge n}|^p \leq C_p E \left(\sum_{i=1}^{\infty} Y_i^2 \chi(\nu \geq i) \right)^{p/2} \leq C_p E|Y|^p E\nu < \infty.$$

REMARK. By Doob-inequality, it is evident that in the case $EY = 0$

for $p > 1$ $\qquad E(\sup |S_{\nu \wedge n}|)^p < \infty \Longleftrightarrow \sup E(|S_{\nu \wedge n}|)^p < \infty.$

However, the following example shows that for $p = 1$ this is not valid.

EXAMPLE. Consider a symmetric simple random walk, that is, suppose that $P(Y = 1) = P(Y = -1) = \frac{1}{2}$.

Let ν be the stopping time defined by the formula

$$\nu = \min\{n; S_n = 1\}.$$

Then

$$E(|S_{\nu \wedge n}|) = E(S_{\nu \wedge n}) + E(|S_{\nu \wedge n}|) =$$
$$= 2E(S^+_{\nu \wedge n}) = 2P(S^+_{\nu \wedge n} = 1) = 2P(\nu \leq n)$$

so

$$\sup_{n \geq 1} E|S_{\nu \wedge n}| = 2.$$

On the other hand, by the Burkholder-Davis-Gundy inequality we have

$$E(\sup|S_{\nu\wedge n}|) \geq c_p E\left(\sum_{i=1}^{\infty} Y_i^2 \chi(\nu \geq i)\right)^{1/2} =$$

$$= c_p E\left(\sum_{i=1}^{\infty} \chi(\nu \geq i)\right)^{1/2} = c_p E\nu^{1/2} = +\infty.$$

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ON THE STRONG LAW OF LARGE NUMBERS FOR AMARTS

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_n)_{n\geq 0}$ be a non-decreasing sequence of σ-fields of events. We say that the sequence $(X_n, \mathcal{F}_n)$, $n \geq 0$, of real-valued and adapted random variables forms an amart, if $E(|X_n|) < +\infty$ for every $n \geq 1$ and

$$\lim_{\tau \in T} E(X_\tau)$$

exists and is finite, where T denotes the family of all bounded stopping times for the sequence $(\mathcal{F}_n)_{n\geq 0}$.

The following assertion is the key observation for the proof of the strong law of large numbers.

THEOREM 1. *Let $(X_n, \mathcal{F}_n)_{n\geq 0}$ be a predictable amart (i.e. X_{n+1} is $\mathcal{F}_n$-measurable, $n \geq 0$). Then $X_n/n \to 0$ a.s. as $n \to +\infty$.*

For the proof of this theorem we recall the lemma of [2].

LEMMA 1. (see [2], Lemma 3.1. p.208.) Let $(Z_n, \mathcal{F}_n)_{n\geq 0}$ be an amart such that

$$\lim_{n\to+\infty} E(Z_n|\mathcal{F}_m) = 0 \quad \text{a.s.}$$

for all fixed $m \geq 1$. Then

(a) $\int \sup_{n\geq 1} |E(Z_n|\mathcal{F}_m)| < +\infty$ for every $m \geq 1$,

(b) $E(Z_n|\mathcal{F}_m) \to 0$ in L_1 for every $m \geq 1$,

(c) $\lim_{\tau\in T} \int |Z_\tau| = 0$,

(d) $Z_n \to 0$ a.s. and in L_1,

(e) $(Z_\tau)_{\tau\in T}$ is uniformly integrable.

PROOF OF THEOREM 1. Introduce the notation $Z_n = X_{n+1} - X_n$, $n \geq 1$. Then Z_n is $\mathcal{F}_n$-measurable for all $n \geq 1$ since (X_n) is predictable.

We show that $(Z_n, \mathcal{F}_n)$, $n \geq 1$, is an amart. For any bounded stopping time τ with respect to $(\mathcal{F}_n)_{n \geq 1}$ define $\sigma = \tau + 1$. Obviously σ is a bounded stopping time and we have

$$
\begin{aligned}
E(Z_\tau) &= \int\limits_{(\tau=n_1)} (X_{n_1+1} - X_{n_1})\,\mathrm{d}P + \ldots + \int\limits_{(\tau=n_k)} (X_{n_k+1} - X_{n_k})\,\mathrm{d}P = \\
&= \left[\int\limits_{(\sigma=n_1+1)} X_{n_1+1}\,\mathrm{d}P + \ldots + \int\limits_{(\sigma=n_k+1)} X_{n_k+1}\,\mathrm{d}P \right] - \\
&- \left[\int\limits_{(\tau=n_1)} X_{n_1}\,\mathrm{d}P + \ldots + \int\limits_{(\tau=n_k)} X_{n_k}\,\mathrm{d}P \right] = \\
&= E(X_\sigma) - E(X_\tau).
\end{aligned}
$$

This equality implies that

$$\lim_{\tau \in T} E(Z_\tau) = 0$$

since $(X_n, \mathcal{F}_n)$ is an amart. The integrability of (Z_n) is obvious. Consequently, $(Z_n, \mathcal{F}_n)$ is also an amart.

Using the Riesz decomposition of (X_n) (see [2], Theorem 3.2.) we can write X_m in the form $X_m = M_m + P_m$ and this decomposition is unique, here $(M_m, \mathcal{F}_m)$ is a martingale and

$$M_m = \lim_{n \to +\infty} E(X_n | \mathcal{F}_m) \quad \text{a.s.},$$

whilst $(P_m, \mathcal{F}_m)$ is an amart with the property that $P_m \to 0$ a.s. and in L_1. Thus for the amart $(Z_n, \mathcal{F}_n)$ we have

$$
\begin{aligned}
\lim_{n \to +\infty} E(Z_n | \mathcal{F}_m) = \lim_{n \to +\infty} E(X_{n+1} | \mathcal{F}_m) - \lim_{n \to +\infty} E(X_n | \mathcal{F}_m) &= \\
= M_m - M_m = 0 \quad &\text{a.s.}.
\end{aligned}
$$

Therefore, by Lemma 1 we have that $Z_n \to 0$ a.s. when $n \to +\infty$. This implies that

$$\frac{Z_1 + \ldots + Z_n}{n} \to 0 \quad \text{a.s.},$$

or in the formulation

$$\frac{(X_2 - X_1) + \ldots + (X_{n+1} - X_n)}{n} = \frac{X_{n+1}}{n} - \frac{X_1}{n} \to 0 \quad \text{a.s.}.$$

Consequently,

$$\frac{X_{n+1}}{n+1} = \frac{X_{n+1}}{n} \cdot \frac{n}{n+1} \to 0 \quad \text{a.s..}$$

This proves the assertion.

THEOREM 2. *Let $(X_n, \mathcal{F}_n)$ be an amart and denote by ΔX_n, $n \geq 1$, its difference sequence, where $\Delta X_0 = X_0 = 0$ a.s. Then $X_n/n \to 0$ a.s. as $n \to +\infty$, if*

(a) $1 \leq p \leq 2$ *and the series*

$$\sum_{n=1}^{\infty} n^{-p} E(|\Delta X_n|^p)$$

converges, or

(b) $2 \leq p < +\infty$ *and the series*

$$\sum_{n=1}^{\infty} n^{-(1+p/2)} E(|\Delta X_n|^p)$$

converges.

PROOF. Using the decomposition of Doob we have $X_n = Y_n + Z_n$ where

$$Y_1 = X_1, \quad Y_n - Y_{n-1} = X_n - E(X_n|\mathcal{F}_{n-1}), \qquad n = 2, 3, \ldots$$

and

$$Z_1 = 0, \quad Z_n - Z_{n-1} = E(X_n|\mathcal{F}_{n-1}) - X_{n-1}, \qquad n = 2, 3, \ldots .$$

Obviously, $(Y_n, \mathcal{F}_n)$ is a martingale. Consequently, $(Z_n, \mathcal{F}_n)$ is an amart and the sequence (Z_n) is predictable. Thus by Theorem 1 $n^{-1}Z_n \to 0$ a.s. as $n \to +\infty$.

To complete the proof it suffices to show that $n^{-1}Y_n \to 0$ a.s..

We have for all $p \geq 1$ the inequality

$$E(|\Delta Z_n|^p) = E(|E(\Delta X_{n-1}|\mathcal{F}_n)|^p) \leq E(|\Delta X_n)|^p).$$

This by the C_p-inequality implies that

$$E(|\Delta Y_n|^p) \leq E[2^{p-1}(|\Delta X_n|^p + |\Delta Z_n|^p)] \leq 2^p E(|\Delta X_n)|^p).$$

So, if $1 \leq p \leq 2$, we have

$$\sum_{n=1}^{\infty} n^{-p} E(|\Delta Y_n|^p) \leq 2^p \sum_{n=1}^{\infty} n^{-p} E(|\Delta X_n|^p),$$

which by our assumption converges. Now if $2 \leq p < +\infty$ then

$$\sum_{n=1}^{\infty} n^{-(1+p/2)} E(|\Delta Y_n|^p) \leq 2^p \sum_{n=1}^{\infty} n^{-(1+p/2)} E(|\Delta X_n|)^p) < +\infty.$$

By the SLLN for martingales (see e.g. [1]) in both cases we have

$$n^{-1} Y_n \to 0 \quad \text{a.s.}$$

as $n \to +\infty$.

Our assertion is therefore proved.

REMARKS. 1. For $2 \leq p < +\infty$ the result of Theorem 2 is given in [3], Theorem 2, without proof. Our proof seems to be other than the proof of the authors of the mentioned paper.

2. The SLLN for martingales immediately follows from the Burkholder-Davis-Gundy inequality. (see, [4].)

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AN INEQUALITY BETWEEN THE HELLY AND CARATHEODORY NUMBERS ($H \leq C$)

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Let X be an arbitrary set. Consider a mapping $[.,.] : X \times X \to$ $\to P(X)$. We say that $[x_1, x_2]$ is the segment between x_1 and x_2. The set $K \subset X$ is called convex if x_1, $x_2 \in K$ implies $[x_1, x_2] \subset K$. The convex hull $\langle H \rangle$ of a set $H \subset X$ is

$$\langle H \rangle := \bigcap \{K \ : \ H \subset K \subset K, \quad K \text{ is convex } \}.$$

Obviously $\langle H \rangle$ will be a convex set. We have thus defined a convexity structure on X.

DEFINITION. The Helly number of a convexity $X, [.,.]$ is the smallest integer $n \geq 0$ with the following property. If for a finite collection of convex subsets of X any $n+1$ sets have common point then all sets have a common point. The Caratheodory number of $X, [.,.]$ is the smallest integer $n \geq 0$ such that for any $H \subset X$

$$\langle H \rangle := \bigcup \{\langle F \rangle \ : \ F \subset A, \quad |F| \leq n+1\}.$$

In what follows we prove an inequality between Helly and Caratheodory numbers in a class of convexity spaces. Remark that for special convexities K. BEZDEK and I. JOÓ [3] further M. HORVÁTH [4] investigated the Helly and Caratheodory numbers.

We consider $g : X \times X \times [0,1] \to X$ and define the segment structure by $[x_1, x_2] := \{g(x_1, x_2, \lambda) \ : \ 0 \leq \lambda \leq 1\}$. A function $f : X \to \mathbf{R}$ is called convex if

(1) $f(x_3) \leq \lambda f(x_1) + (1-\lambda) \cdot f(x_2), \quad x_3 = g(x_1, x_2, \lambda), \quad 0 \leq \lambda \leq 1.$

We shall prove the following.

THEOREM *Let X be a compact T_1 topological space endowed with the above convexity and suppose that each convex set is the zero set* *of a convex function. Then the Helly number is not greater than the Caratheodory number: $H \le C$.*

PROOF. We have to show that $H \ge n$ implies $C \ge n$. Suppose $H \ge n$; this means that there is a finite collection of convex sets $K_1, \dots, K_{k+1}$ such that $k \ge n$ and

$$K_1 \cap \ldots \cap K_{k+1} = \emptyset \tag{2}$$

but for any $1 \le i \le k+1$ there exists an element

$$x_i \in K_1 \cap \ldots \cap K_{i-1} \cap K_{i+1} \cap \ldots \cap K_{k+1}. \tag{3}$$

We can suppose that $\widehat{K_i} := \langle \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+1}\} \rangle = K_i$ since the system $(\widehat{K_i})$ satisfies (2) and (3) as well. Introduce the notation $K := \langle \{x_1, \dots, x_{k+1}\} \rangle$. Take continuous convex functions $f_i : X \to \mathbf{R}$ for which

$$f_i^{-1}(\{0\}) = K_i. \tag{4}$$

Then the relation (2) implies by compactness that

$$c := \min_{x \in K} \max_{1 \le i \le k+1} f_i(x) > 0. \tag{5}$$

Now apply the same device as in [6] and [7]. Namely let $\varphi : K \to \mathbf{R}^{k+1}$, $\varphi(x) := (f_1(x) - c, \dots, f_{k+1}(x) - c)$. Then (6) means that $\varphi(K) \cap K^- = \emptyset$ where $K^- \subset \mathbf{R}^{k+1}$ denotes the cone of all vectors of $\mathbf{R}^{k+1}$ with negative coordinates. Using (1) we get that the convex hull of $\varphi(K)$ fulfills the $\operatorname{co} \varphi(K) \cap K^- = \emptyset$. Now the Hahn–Banach theorem guarantees a vector $0 \ne \mu \in \mathbf{R}^{k+1}$ such that

$$(\varphi(K), \mu) \ge 0, \qquad (K^-, \mu) \le 0. \tag{6}$$

The second inequality implies for $\mu = (\lambda_1, \dots, \lambda_{k+1})$ that $\lambda_i \ge 0$ and hence we can normalize μ to ensure $\sum \lambda_i = 1$. The first inequality of (6) then implies that

$$\sum \lambda_i f_i(x) \ge c, \qquad x \in K. \tag{7}$$

Take an $x_0 \in K$ such that $c = \max_i f_i(x_0)$, then we get by (7) that

$$\max_i f_i(x_0) \le \sum \lambda_i f_i(x_0). \tag{8}$$

Now we can prove that $C \geq n$. Suppose indirectly that $C \leq n-1(\leq \leq k-1)$, then

$$K = \bigcup_{i=1}^{k+1} K_i \tag{9}$$

consequently there exists an x_0 with

$$x_0 \in K_{i_0}. \tag{10}$$

We know from (10) that $f_{i_0} = 0$ and then $\lambda_{i_0} = 0$. On the other hand we know that $f_i(x_{i_0}) = 0$ for $i \neq i_0$ and then $\sum \lambda_i f_i(x_0) = \lambda_{i_0} \cdot f_{i_0}(x_{i_0}) = 0$ and this contradicts to (7). The Theorem is proved. ∎

REMARK. If X is a compact metric space and the metric is convex in the sense that

$$\begin{gathered} \varrho(x_3, y_3) \leq \lambda\varrho(x_1, y_1) + (1-\lambda)\varrho(x_2, y_2) \\ x_3 = g(x_1, x_2, \lambda), \quad y_3 = g(y_1, y_2, \lambda), \quad 0 \leq \lambda \leq 1, \end{gathered} \tag{11}$$

then the assumption of the Theorem holds and hence $H \leq C$ is valid. Indeed, for a convex set $K \subset X$ the distance function of the set K will be convex in the sense of (1). We formulate the following

PROBLEM. If X is a (non-metrizable) compact uniform space, how can we formulate a condition on the uniformity which ensures some analogon of (11)? We can investigate this problem for other generalizations of metric spaces too (see e.g. [5]).

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ON THE ORDER OF MEAN CONVERGENCE OF INTERPOLATING PROCESSES

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To the memory of professor PAUL TURÁN

1. The mean convergence of interpolating processes was first investigated in the famous work [1] of P. ERDŐS and P. TURÁN (1937) (see also [3]) where the weighted L^2-convergence of the Lagrange interpolation on a finite interval is proved if the nodes of the interpolation are the zeros of the polynomials orthogonal with respect to this weight, satisfying some conditions. The mean convergence of the Lagrange interpolation on infinite intervals presented more difficulties; it was solved only in 1961 by J. BALÁZS and P. TURÁN in [6]. The question of pointwise convergence on finite intervals were investigated by many authors. The deepest results in this question were obtained by O. SHISHA, R. BOJANIC, K. BALÁZS, G. FREUD, J. SZABADOS, P. VÉRTESI and others. The pointwise convergence on infinite interval was investigated by BALÁZS and TURÁN in [4] for Lagrange and Hermite–Fejér interpolation. Using the ideas of [4] K. BALÁZS obtained in [8], [9] some pointwise convergence theorems for Lagrange and Hermite–Fejér interpolation, based on the roots of the Laguerre polynomials.

If we consider arbitrary point system instead of the zeros of orthogonal polynomials, the resulting Hermite–Fejér interpolation operators will be positive, if the point system is normal. The concept of normal point system was introduced by L. FEJÉR for finite intervals (see [17]) and the analogous notion for infinite intervals is given in [11], where the stability of the corresponding interpolation is proved and an estimate is given for its pointwise convergence.

The above investigations were continued by many authors in three main directions:

a) interpolations for the largest possible family of weights are studied,

b) the L^p-convergence, generalizing the L^2-convergence,

c) the speed of convergence of the interpolating processes for various classes of functions; saturation theorems.

Several open problems on this topic can be found in Turán's nice paper [2].

For finite intervals the Erdős–Turán theorem gives estimate for the speed of convergence by E_n (see [17]). In [6] BALÁZS and TURÁN proved only the L^2-convergence for infinite intervals. In connection to the work [7] TURÁN raised the question of the estimate of rate of the L^2-convergence of the Lagrange interpolation for infinite intervals. Proving a Jackson type inequality, JOÓ and KY [20] proved the desired estimate if the interpolation is based on Laguerre nodes. In [15] JOÓ and SZABADOS proved an L^1-convergence theorem for interpolating processes based on Laguerre nodes, with estimate for the rate of convergence. Analogous estimates were proved also in [10].

2. Denote $L_n^\alpha(x) := x^{-\alpha}e^x\frac{1}{n!}\left[e^{-x}x^{n+\alpha}\right]^{(n)}$ the Laguerre polynomials ($\alpha > -1$) and $x_1\ x_2\ \ldots\ x_n$ the zeros of $L_n^{(\alpha)}(x)$ (in fact $x_k = x(k,n,\alpha)$; we simplify the notation). Define further $\ell_k(x) := := L_n^{(\alpha)}(x)/L_n^{(\alpha)'}(x_k)(x-x_k)$ the fundamental polynomials of the Lagrange interpolation based on the roots of $L_n^{(\alpha)}(x)$.

We shall investigate the convergence of the following operators:

$$L_n(f,x) := \sum_{k=1}^{n} f(x_k)\ell_k(x) \tag{1}$$

(Lagrange interpolations),

(2)

$$F_n(f,x) := \sum_{k=1}^{n}\left[f(x_k)\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}+f'(x_k)(x-x_k)\right]\ell_k^2(x),$$

(Hermite-operator)

$$H_n(f,x) := \sum_{k=1}^{n} f(x_k)\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}\ell_k^2(x), \tag{3}$$

(Hermite–Fejér operator)

$$P_n(f,x) = \sum_{k=1}^{n} f(x_k)\left(\frac{1+\alpha}{x_k}x-\alpha\right)\ell_k^2(x) \tag{4}$$

(it is defined in [11]), (generalized Hermite–Fejér operator)

$$G_n(f,x)=\sum_{k=1}^{n} f(x_k)\ell_k^2(x) \tag{5}$$

(Grünwald-operator, see [16]),

$$I_n(f,x)=\sum_{k=1}^{n} f(x_k)\frac{x}{x_k}\ell_k^2(x) \tag{6}$$

(Egerváry–Turán operator;
it is defined first in [4] and investigated its convergence in [5]).

Introduce the space $C(\lambda):=\{f\in C(0,\infty)\ :\ \lim\limits_{x\to\infty} f(x)e^{-\lambda x}=0\}$. For any $f\in C(\lambda)$, denote $F(x):=f(x^2)e^{-\lambda x^2}$ further let

$$w(F,\delta):=w_\infty(F,\delta):=\sup_{\substack{x,y\in\mathbf{R}\\|x-y|\le\delta}}|F(x)-F(y)|$$

the modulus of continuity of F. We shall prove the following

THEOREM. *Let $\alpha>-1$ and $f\in C(\lambda)$, then we have*

$$\left(\int_0^\infty x^\alpha e^{-x}|L_n(f,x)-f(x)|^2\,dx\right)^{1/2}\le c_\lambda^f w(F,1/\sqrt{n}), \tag{7}$$

$$\int_0^\infty x^\alpha e^{-x}|F_n(f,x)-f(x)|\,dx\le c_\lambda^f w(f'(x^2)e^{-\lambda x^2},1/\sqrt{n}), \tag{8}$$

$$\int_0^\infty x^\alpha e^{-x}|H_n(f,x)-f(x)|\,dx\le c_\lambda^f w(F,1/\sqrt{n}), \tag{9}$$

$$\int_0^\infty x^\alpha e^{-x}|P_n(f,x)-f(x)|\,dx\le c_\lambda^f w(F,1/\sqrt{n}), \tag{10}$$

$$\int_0^\infty x^\alpha e^{-x}|G_n(f,x)-f(x)|\,dx\le c_\lambda^f w(F,1/\sqrt{n}), \tag{11}$$

$$\int_0^\infty x^\alpha e^{-x}|I_n(f,x)-f(x)|\,dx\le c_\lambda^f w(F,1/\sqrt{n}). \tag{12}$$

For the proof we need some lemmas.

LEMMA 1. For $\alpha > -1$, $0 < \nu < 1$ we have

$$\left[L_n^{(\alpha)'}(x_k)\right]^{-2} \le cn^{-\alpha-1/2} x_k^{\alpha+3/2} e^{-\nu x_k}, \quad (k = 1, \dots, n), \tag{13}$$

where c denotes a constant depending only on α (everywhere in this paper).

PROOF. Introduce the notation $a \asymp b$; this means that $ca \le b \le Ca$ with some constants $0 < c < C < \infty$ which do not depend on x, x_k and n. From [13], (13) we know the relation

$$\lambda_n(x_k) \asymp \sqrt{\frac{x_k}{4n - x_k}} x_k^\alpha e^{-x_k}, \quad (1 \le x_k < n) \tag{14}$$

where $\lambda_n(x) := \left(\sum_{k=0}^{n-1} [\ell_k^{(\alpha)}(x)]^2\right)^{-1}$ is the Christoffel function formed by the normed Laguerre polynomials $\ell_k^{(\alpha)}(x) := (-1)^k \left[\Gamma(\alpha+1)\binom{k+\alpha}{k}\right]^{-1/2} \cdot L_n^{(\alpha)}(x)$. (see [18]). On the other hand [18], (15.3.5) states that

$$\lambda_n(x_k) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)} x_k^{-1} \cdot [L_n^{(\alpha)'}(x_k)]^{-2}, \quad (k = 1, \dots, n) \tag{15}$$

hence for $x_k < n$ we obtain from (14) and (15)

$$[L_n^{(\alpha)'}(x_k)]^{-2} \le cn^{-\alpha} x_k^{\alpha+1} \sqrt{\frac{x_k}{4n - x_k}} e^{-x_k} \le cn^{-\alpha-1/2} x_k^{\alpha+3/2} e^{-x_k}.$$

If $x_k \ge n$ then we apply from [19] Lemma 3.1.5 with $G(x) := xe^{\nu' x}$, $\nu < \nu' < 1$ to obtain $\lambda_n(x_k) x_k e^{\nu' x_k} \le \int_0^\infty x^{\alpha+1} e^{-(1-\nu)x}\, dx < \infty$ and then

$$[L_n^{(\alpha)'}(x_k)]^{-2} \le c x_k n^{-\alpha} \lambda_n(x_k) \le cn^{-\alpha} e^{-\nu' x_k} \le cn^{-\alpha-1/2} x_k^{\alpha+3/2} \cdot e^{-\nu x_k}.$$

Lemma 1 is proved. ∎

LEMMA 2. Let $\alpha > -1$, $0 < \tau$, $0 < \gamma$, then

$$\int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n \frac{[L_n^{(\alpha)}(x)]^2}{|x - x_k|} x_k^\gamma e^{-\tau x_k}\, dx \le cn^\alpha \quad (n = 1, 2, \dots) \tag{16}$$

with a constant $c > 0$ depending only on α and γ, τ but independent of n.

PROOF. Denote by I the integrand in (16). By [18], (6.31.11)

$$x_k \asymp \frac{k^2}{n}. \tag{17}$$

Using the estimate

$$|L_n^{(\alpha)}(x)| \le cn^\alpha, \quad 0 \le x \le x_1 \tag{18}$$

(see [18], (7.6.8)) we obtain

$$\int_0^{x_1/2} I\,\mathrm{d}x \le cn^{2\alpha} \sum_{k=1}^{n} \int_0^{x_1/2} x^\alpha \frac{1}{x_k - x} x_k^\gamma e^{-\tau x_k}\,\mathrm{d}x \le$$

$$\le cn^2 \sum_{k=1}^{n} x_1^{\alpha+1} x_k^{\gamma-1} e^{-\tau x_k}\,\mathrm{d}x \le cn^{\alpha-1} \left(\sum_{\substack{k \\ x_k \le 1}} \left(\frac{k^2}{n}\right)^{\gamma-1} + n \right) \le cn^\alpha.$$

Now suppose $\frac{x_1}{2} \le x \le n$ and consider the index $j = j(x)$ defined by $|x - x_j| = \min\limits_{1 \le k \le n} |x - x_k|$. As in [15], we get from [18] Problem 35 that

$$|x - x_k| \ge \sqrt{x}|\sqrt{x} - \sqrt{x_k}| \ge c\sqrt{x}|\sqrt{x_j} - \sqrt{x_k}| \ge \frac{c\sqrt{x}|j-k|}{\sqrt{n}}. \tag{19}$$

Denote for $x_j < n$ $x_j^* := \frac{x_{j-1}+x_j}{2}$, $x_1^* := \frac{x_1}{2}$, $x_j^{**} := \frac{x_j + x_{j+1}}{2}$, $x_j^{**} = n$ if $x_{j+1} \ge n$. It is clear from (17) that $x \asymp x_j$ if $x \in [x_j^*, x_j^{**}]$. Using the estimate ([18], (7.6.8) and Theorem 8.91.2)

$$|L_n^{(\alpha)}(x)| \le cx^{-\alpha/2-1/4} n^{\alpha/2-1/4} e^{x/2}, \quad \frac{x_1}{2} \le x \le n, \tag{20}$$

further the identity

$$\frac{[L_n^{(\alpha)}(x)]^2}{x - x_j} = 2L_n^{(\alpha)}(\xi) L_n^{(\alpha)'}(\xi) = -2L_n^{(\alpha)}(\xi) L_{n-1}^{(\alpha+1)}(\xi)$$

for some $\xi \in [x, x_j]$ (or $\xi \in [x_j, x]$) we get

$$\int_{x_j^*}^{x_j^{**}} I\,\mathrm{d}x = \int_{x_j^*}^{x_j^{**}} x^\alpha e^{-x} \sum_{k=1}^{n} \frac{[L_n^{(\alpha)}(x)]}{|x - x_k|} x_k^\gamma e^{-\tau x_k}\,\mathrm{d}x \le$$

$$\leq c \int_{x_j^*}^{x_j^{**}} x^\alpha e^{-x}\Big(x^{-\alpha-1/2}e^x n^{\alpha-1/2}\sum_{\substack{k\\ k\neq j}}\frac{1}{|x-x_k|}x_k^\gamma e^{-\tau x_k}+$$

$$+\xi^{-\alpha/2-1/4}\xi^{-(\alpha+1)/2-1/4}n^{\alpha/2+(\alpha+1)/2-1/2}e^\xi x_j^\gamma e^{-\tau x_j}\Big)\,\mathrm{d}x \leq$$

$$\leq c(x_j^{**}-x_j^*)x_j^{-1/2}n^{\alpha-1/2}\sum_{\substack{k\\ k\neq j}}\frac{\sqrt{n}}{\sqrt{x_j}}\frac{1}{|j-k|}x_k^\gamma e^{-\tau x_k}+$$

$$+cx_j^{-1}n^\alpha\int_{x_j^*}^{x_j^{**}} e^{-x}e^\xi\,\mathrm{d}x x_j^\gamma e^{-\tau x_j}.$$

It follows from [18], Problem 35 that $x_{j+1}-x_j=O(\sqrt{x_j})$ and hence

$$e^x \asymp e^{x_j}, \qquad x_j^*\leq x\leq x_j^{**}, \qquad 1<x_j \tag{21}$$

so we get

$$\int_{x_j^*}^{x_j^{**}} I\,\mathrm{d}x \leq cn^\alpha(x_j^{**}-x_j^*)x_j^{-1}\Big(x_j^\gamma e^{-\tau x_j}+\sum_{\substack{k\\ k\neq j}}\frac{1}{|j-k|}x_k^\gamma e^{-\tau x_k}\Big).$$

The estimate

$$x_{j+1}-x_j\asymp\sqrt{\frac{x_j}{n}} \qquad (x_j\leq n) \tag{22}$$

is known ([13], (9) and (10)), and then

$$\int_{x_j^*}^{x_j^{**}} I\,\mathrm{d}x \leq cn^{\alpha-1/2}x_j^{-1/2}\left(x_j^\gamma e^{-\tau x_j}+\sum_{\substack{k\\ k\neq j}}\frac{1}{|j-k|}x_k^\gamma e^{-\tau x_k}\right).$$

Now we have

a) $n^{\alpha-1/2}\sum\limits_{j\leq[\sqrt{n}]}x_j^{\gamma-1/2}\leq cn^{\alpha-1/2}\sum\limits_{j\leq[\sqrt{n}]}\left(\frac{j^2}{n}\right)^{\gamma-1/2}\leq cn^\alpha$

Newt we shall use the following formula of Euler

$$\sum_{a<k\leq b} f(k)=\int_a^b f(x)\,\mathrm{d}x-[\varrho(b)f(b)-\varrho(a)f(a)]+\int_a^b\varrho(x)f'(x)\,\mathrm{d}x \tag{23}$$

(see P. Szász [21], p. 685), where $\varrho(x) = x - [x] - 1/2$.

b) $n^{\alpha-1/2} \sum_{j \geq [\sqrt{n}]+1} x_k^{\gamma-1/2} e^{-\tau x_j} \leq cn^{\alpha-1/2} \sum_{j \geq [\sqrt{n}]+1} e^{-\tau x_j/2} \leq$

$\leq cn^{\alpha-1/2} \sum_{j \geq [\sqrt{n}]+1} e^{-c_0 j^2/n} =$

$$= cn^{\alpha-1/2} \left[\sum_{[\sqrt{n}]+1 \leq j \leq [\sqrt{n}\log n]} + \sum_{[\sqrt{n}\log n]+1 \leq j} \right] = cn^{\alpha-1/2} \left(\sum\nolimits_1 + \sum\nolimits_2\right).$$

Obviously $n^{\alpha-1/2} \sum_2 \leq cn^{\alpha+1/2} e^{-c_0 \log^2 n}$, on the other hand, applying (23) for $a = [\sqrt{n}] + 1/2$, $b = [\sqrt{n}\log n] + 1/2$, $f(x) = e^{-c_0 x^2/n}$ (hence $f'(x) = -(2c_0/n)xe^{-c_0 x^2/n}$) we get

$$\sum_{j=[\sqrt{n}]+1}^{\sqrt{n}\log n} e^{-c_0 j^2/n} = \int_a^b e^{-c_0 x^2/n}\,dx - \frac{2c_0}{n}\int_a^b x^2 e^{-c_0 x^2/n}\,dx +$$
$$+ \frac{2c_0}{n}\int_a^b x[x] e^{-c_0 x^2/n}\,dx + \frac{c_0}{n}\int_a^b x e^{-c_0 x^2/n}\,dx \leq$$
$$\leq \int_a^b e^{-c_0 x^2/n}\,dx + \frac{c_0}{n}\int_a^b x e^{-c_0 x^2/n}\,dx;$$

using the substitution $c_0 x^2/n = u/2$ ($dx/du = \sqrt{n/2c_0}$, $x = \sqrt{\frac{n}{2c_0}}u$, $a' = \sqrt{\frac{2c_0}{n}}a$, $b' = \sqrt{\frac{2c_0}{n}}b$) we obtain at last

$$\sum_{j=[\sqrt{n}]+1}^{[\sqrt{n}\log n]} e^{-c_0 j^2/n} \leq \sqrt{\frac{n}{2c_0}} \int_{a'}^{b'} e^{-x^2/2}\,dx + \frac{!}{2}\int_{a'}^{b'} x e^{-x^2/2}\,dx \leq c\sqrt{n}.$$

c) $$n^{\alpha-1/2} \sum_{j=1}^{n} x_j^{-1/2} \sum_{\substack{k \\ k \neq j}} \frac{1}{|j-k|} x_k^{\gamma} e^{-\tau x_k} =$$

$$= n^{\alpha-1/2} \sum_{k=1}^{n} x_k^{\gamma} e^{-\tau x_k} \sum_{\substack{j \\ j \neq k}} x_j^{-1/2} \frac{1}{|j-k|} \leq cn^{\alpha} \sum_{k=1}^{n} x_k^{\gamma} e^{-\tau x_k} \sum_{\substack{j \\ j \neq k}} j^{-1} \frac{1}{|j-k|},$$

d)
$$\sum_{k=1}^{n} x_k^{\gamma} e^{-\tau x_k} k^{-1} \le$$
$$\le \sum_{k=1}^{[\sqrt{n}]} \left(\frac{k^2}{n}\right)^{\gamma} k^{-1} + \sum_{k=[\sqrt{n}]+1}^{n} c/k \le n^{-\gamma} \sum_{k=1}^{[\sqrt{n}]} k^{2\gamma-1} + c \le c.$$

The above estimates imply $\int\limits_{x_1/2}^{n} I\,\mathrm{d}x = \sum \int\limits_{x_j^*}^{x_j^{**}} I\,\mathrm{d}x \le cn^{\alpha}$. Consider finally the case $n \le x$. Again let $|x - x_j| = \min\limits_{1 \le k \le n} |x - x_k|$ and use the estimate

$$|L_n^{(\alpha)}(x)| \le cn^{\alpha/2+2/3} x^{-\alpha/2-1} e^{x/2} \qquad (x \ge 1) \tag{24}$$

(see [18], Theorem 8.91.2); then

$$\int\limits_n^{\infty} I\,\mathrm{d}x \le \int\limits_n^{\infty} x^{\alpha} e^{-x} \sum_{k=1}^{n} \frac{[L_n^{(\alpha)}(x)]^2}{|x - x_k|} e^{-\tau x_k/2}\,\mathrm{d}x \le$$
$$\le \int\limits_n^{\infty} x^{\alpha} e^{-x} [L_n^{(\alpha)}(x)]^2 \left(\sum_{x_k \le n^{1/3}} \frac{2}{n} + \sum_{n^{1/3} < x_k \le \frac{n}{2}} \frac{2}{n} e^{-\tau n^{1/3}/2} \right) \mathrm{d}x +$$
$$+ e^{-\tau n/4} \int\limits_n^{x_n+1} x^{\alpha} e^{-x} \left\{ [L_n^{(\alpha)}]^2 \sum_{\substack{x_k \ge n/2 \\ k \ne j}} \frac{1}{|x - x_k|} + \left| L_n^{(\alpha)}(x) L_{n-1}^{(\alpha+1)}(\xi) \right| \right\} \mathrm{d}x +$$
$$+ e^{-\tau n/4} \int\limits_{x_n+1}^{\infty} x^{\alpha} e^{-x} [L_n^{(\alpha)}(x)]^2 \sum_{x_k \ge \frac{n}{2}} \frac{1}{|x - x_k|}\,\mathrm{d}x.$$

Here $\xi \in (x, x_j)$ and by (22) we also know that $|x - \xi| = O(1)$; consequently, using also (22) we get

$$\int\limits_n^{\infty} I\,\mathrm{d}x \le cn^{\alpha+4/3} \int\limits_n^{\infty} x^{-2} \left(n^{-1/3} + e^{-\tau n^{1/3}/2} \right) \mathrm{d}x +$$

$$+e^{-\tau n/4}\int_{n}^{x_n+1} x^{-2} n^{\alpha+4/3}\log n\,\mathrm{d}x+e^{-\tau n/4}\int_{n}^{x_n+1} x^{\alpha}e^{-x}n^{\alpha+11/6}x^{-\alpha-5/2}\,\mathrm{d}x+$$

$$+e^{-\tau n/4}\int_{x_n+1}^{\infty} x^{-2}n^{\alpha+4/3}\log n\,\mathrm{d}x \le cn^{\alpha}.$$

Finally we get

$$\int_0^{\infty} I\,\mathrm{d}x=\int_0^{x_1/2}+\int_{x_1/2}^{n}+\int_n^{\infty}\le cn^{\alpha}+cn^{\alpha}+cn^{\alpha}\le cn^{\alpha}.$$

Lemma 2 is proved. ∎

LEMMA 3. ([15], Lemma 1). Given $\lambda<\mu<1$ and $\frac{n}{2}>a>0$, to every $f\in C(\lambda)$ there exists a polynomial $p(x)\in\mathcal{P}_n$ such that

$$(25)\ |f(x)-p(x)|=O\left(e^{\mu x}\right)\left\{w(F,a/n)+\left(\frac{\lambda}{\mu}\right)^2\right\},\ (0\le x\le a^2),$$

$$(26)\int_0^{\infty} x^{\alpha}e^{-x}|f(x)-p(x)|\,\mathrm{d}x=O\left\{w(F,a/n)+\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{n}{a^2(1-\mu)}\right)^{n/2}\right\},$$

$$(27)\quad |p'(x)|=O\left(e^{\lambda x}\right)\left\{\frac{n}{a\sqrt{n}}w(F,a/n)+1\right\},\quad (0<x\le a^2/4).$$

We need also the inequality in [10]:

$$\int_0^{\infty} x^{\alpha}e^{-x}\sum_{k=1}^{n}e^{\mu x_k}\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}\ell_k^2(x)\,\mathrm{d}x\le\int_0^{\infty}x^{\alpha}e^{-(1-\mu)x}\,\mathrm{d}x,$$

and hence, taking into account the orthogonality of the Laguerre polynomials, a simple calculation gives

$$(29)\qquad \left(\int_0^{\infty} x^{\alpha}e^{-x}|L_n(f,x)|^2\,\mathrm{d}x\right)^{1/2}\le c\max_{1\le k\le n} f(x_k)e^{-\mu x_k},$$

$$(30)\int_0^{\infty} x^{\alpha}e^{-x}|F_n(f,x)|\,\mathrm{d}x\le c\max_{1\le k\le n} f(x_k)e^{-\mu x_k}+\bar{c}\max_{1\le k\le n} f'(x_k)e^{-\mu x_k},$$

$$(31) \qquad \int_0^\infty x^\alpha e^{-x}|A_n(f,x)|\,dx \le c \max_{1\le k\le n} f(x_k)e^{-\mu x_k},$$

where $0<\mu<1$, $f\in C(\mu)$ in case (29) and (31) further f, $f'\in C(\mu)$ in case if (30) and A_n denotes any of the operators H_n, P_n, G_n, I_n. For the proof of (7) we apply Lemma 3 for $a=c\sqrt{n}$, choose $p\in\mathcal{P}_n$ according to this lemma, taking into account (26), (29) and

$$\int_0^\infty x^\alpha e^{-x}|L_n(f,x)-f(x)|^2\,dx \le$$

$$2\int_0^\infty x^\alpha e^{-x}|L_n(f-p,x)|^2\,dx + 2\int_0^\infty x^\alpha e^{-x}|f(x)-p(x)|^2\,dx,$$

the desired estimate (7) follows. Remark that we use also in the following the fact: there exists $\delta_0=\delta_0(F)>0$ and $c_0=c_0(F)>$ >0 satisfying $w(F,\delta)\ge c_0\delta$ ($\delta<\delta_0$, $f\not\equiv 0$). Indeed, the indirect assumption would lead to $F=\text{const}$ but then $F=0$, $f=0$ and this case being trivial, can be excluded.

For the proof of (8) let $f\in C(\lambda)$, $\lambda<\mu<1$, apply Lemma 3 at $a=c\sqrt{n}$ (for some constant c; we take into account also in below that for the largest zero of $L_n^{(\alpha)}x$ we have $x_n\le c\sqrt{n}$). Choose for f' the polynomial $p'\in\mathcal{P}_n$ satisfying (25), (26), (27) (with f', p' in place of f and p). We may suppose $f(0)=p(0)$ and then

$$f(x)-p(x)=\int_0^x\left(f'(t)-p'(t)\right)\,dt=\int_0^x e^{-\mu t}e^{\mu t}\left(f'(t)-p'(t)\right)\,dt=$$

$$=O(e^{\mu x})\int_0^x e^{-\mu t}\left(f'(t)-p'(t)\right)\,dt=O(e^{\mu x})\max_{0\le t\le x}\left(f'(t)-p'(t)\right)e^{-\mu t}.$$

Hence by Lemma 3 and by (30) the desired estimate (8) follows.

Now prove (10) (and after (9)). Since we have for any $p\in\mathcal{P}_n$:

$$p(x)=F_n(p,x)=P_n(p,x)=\sum_{k=1}^n\left(p'(x_k)-p(x_k)\right)(x-x_k)\ell_k^2(x)$$

hence by Lemmas 1, 2 we obtain for any $0<\tau<\nu-\mu$

$$|p(x)-P_n(p,x)|\le\sum_{k=1}^n\left(|p'(x_k)|+|p(x_k)|\right)|x-x_k|\ell_k^2(x)\le$$

$$\leq c\sum_{k=1}^{n} e^{\mu x_k}\left(\frac{1}{\sqrt{x_k}}\sqrt{n}w\left(F,\frac{1}{\sqrt{n}}\right)+1\right)|x-x_k|\ell_k^2(x)\leq$$

$$\leq c\sum_{k=1}^{n} e^{(\mu-\nu)\cdot x_k} n^{-\alpha-1/2}\left(\sqrt{n}w\left(F,\frac{1}{\sqrt{n}}\right)+\sqrt{x_k}\right)x_k^{\alpha+1}\frac{[L_n^{(\alpha)}(x)]^2}{|x-x_k|}\leq$$

$$\leq cn^{-\alpha-1/2}\sum_{k=1}^{n} e^{-\tau x_k}\left(\sqrt{n}w\left(F,\frac{1}{\sqrt{n}}\right)+1\right)x_k^{\alpha+1}\frac{[L_n^{(\alpha)}(x)]^2}{|x-x_k|}\leq$$

$$\leq cn^{-\alpha}\sum_{k=1}^{n} e^{-\tau x_k} w\left(F,\frac{1}{\sqrt{n}}\right)\cdot x_k^{\alpha+1}\frac{[L_n^{(\alpha)}(x)]^2}{|x-x_k|}$$

and hence

$$\int_0^\infty x^\alpha e^{-x}|p(x)-P_n(p,x)|\,\mathrm{d}x\leq cw\left(F,\frac{1}{\sqrt{n}}\right)$$

and using also Lemma 3 we get

$$\int_0^\infty x^\alpha e^{-x}|f(x)-P_n(f,x)|\,\mathrm{d}x\leq\int_0^\infty x^\alpha e^{-x}|f(x)-p(x)|\,\mathrm{d}x+$$

$$+\int_0^\infty x^\alpha e^{-x}|p(x)-P_n(p,x)|\,\mathrm{d}x+\int_0^\infty x^\alpha e^{-x}|P_n(p-f,x)|\,\mathrm{d}x\leq cw\left(F,\frac{1}{\sqrt{n}}\right).$$

Now we prove (9). Observe that

$$H_n(f,x)=P_n(f,x)-\sum_{k=1}^{n} f(x_k)(x-x_k)\ell_k^2(x)$$

and hence

$$\int_0^\infty x^\alpha e^{-x}|H_n(f,x)-P_n(f,x)|\,\mathrm{d}x\leq c\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\lambda x_k}|x-x_k|\ell_k^2(x)\,\mathrm{d}x\leq$$

$$\leq cn^{-\alpha-1/2}\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{(\lambda-\nu)x_k}x_k^{\alpha+3/2}\frac{[L_n^{(\alpha)}(x)]}{|x-x_k|}\,\mathrm{d}x\leq\frac{c}{\sqrt{n}}.$$

For the proof of (6) it is enough to remark that

$$\int_0^\infty x^\alpha e^{-x}|P_n(f,x)-I_n(f,x)|\,\mathrm{d}x=$$

$$=\int_0^\infty x^\alpha e^{-x}\left|\alpha\sum_{k=1}^n f(x_k)\frac{x-x_k}{x_k}\ell_k^2(x)\right|\,\mathrm{d}x\le$$

$$\le c\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n e^{\lambda x_k}\frac{|x-x_k|}{x_k}\ell_k^2(x)\,\mathrm{d}x\le$$

$$\le c\int_0^\infty x^\alpha e^{-x}n^{-\alpha-1/2}\sum_{k=1}^n e^{(\lambda-\nu)x_k}x_k^{\alpha+1/2}\frac{[L_n^{(\alpha)}(x)]^2}{|x-x_k|}\,\mathrm{d}x\le c\frac{1}{\sqrt{n}},$$

finally, taking into consideration the identity

$$-\alpha G_n(f,x)=P_n(f,x)+(\alpha-1)I_n(f,x),$$

the estimate (12) follows easily. ∎

PROBLEM. Extend our theorems for L_p-norm.

REMARKS. 1. It is easy to define the Greedy expansion for any sequence of complex numbers (z_k) and for any complex number z, namely, let $\varepsilon_n(z)=1$ if there exists $\varepsilon_{n+1},\varepsilon_{n+2},\dots,\in\{0,1\}$ such that $\sum_1^{n-1}\varepsilon_k(z)z_k+z_n+\sum_{n+1}^\infty\varepsilon_kz_k=z$, and 0 in the opposite case. Let $\varepsilon_n^*(z):=1-\varepsilon_n\left(\sum z_k-z\right)$. Then it is easy to see that if $f(z)$ is an entire function, which is additive with respect to all Greedy expansions i.e. $f\left(\sum\varepsilon_kz_k\right)=\sum\varepsilon_kf(z_k)$, further there exists $C>0$ such that

$$C|z_n|\ge\max_{\varepsilon_n}\left|\sum_{k=n+1}^\infty\varepsilon_kz_k\right|$$

for every n, then $f(z)=c\cdot z$. Indeed, let $z=\sum_n\varepsilon_n(z)z_n$, $N_1:=\{n\in$ $\in N:\ \varepsilon_n(z)=1\}$. Fix an $n\in N_1$, consider the smallest k for which $\varepsilon_n(z-kz_n)=1$ but $\varepsilon_n(z-(k+1)z_n)=0$. Then $k\le C$ and the other ε-s in the Greedy expansion of $z-kz_n$ and $z-(k+1)z_n$ are the same, i.e. $z^{-1}[f(z-kz_n)-f(z-(k+1)z_n)]=f(z_n)/z_n$ $(n\in N_1)$; hence, taking into account $f(z-kz_n)=f(z)-kz_nf'(z)+O(|z_n|^2)$

and $f(z-(k+1)z_n) = f(z) - (k+1)z_n f'(z) + O(|z_n|^2)$ we obtain $z_n^{-1} f(z_n) = f'(z) + O(|z_n|)$ i.e. $\lim\limits_{n \in N_1} \frac{f(z_n)}{z_n} = f'(z)$. Hence, in the special case $z = z_0 = \sum\limits_1^\infty z_n$ we obtain $\lim\limits_{n\to\infty} \frac{f(z_n)}{z_n} = f'(z_0)$, consequently

$$f'(z) = \lim_{n \in N_1} \frac{f(z_n)}{z_n} = f'(z_0)$$

for every z, i.e. $f' = c$.

2. *Problem*: Suppose (z_n) is circle filling, f is totally additive on $H = \{\sum \varepsilon_n z_n \ : \ \varepsilon_n = 0 \text{ or } 1\}$, then is it true $f(z) = cz$ $(z \in H)$ without continuity assumption on f?

3. *Conjecture*: if $|z_n| > \sup \left| \sum\limits_{n+1}^\infty \varepsilon_k z_k \right|$ for every n, then every number z has at most one expansion $z = \sum \varepsilon_k z_k$.

4. *Problem*: if $|z_n| < \sup \left| \sum\limits_{n+1}^\infty \varepsilon_k z_k \right|$ for every n, then (z_n) is circle-filling or not?

5. *Problem*: If $H = \{\sum \varepsilon_n z_n \ : \ \varepsilon_n = 0 \text{ or } 1\}$ has inner point, then does it have 0 as an inner point?

6. *Problem*: is the set H closed?

7. *Problem*: $1 < x < 2$, $H := \left\{ \sum\limits_0^n \varepsilon_i x^i \ : \ \varepsilon_i = 0 \text{ or } 1 \right\} = \{y_n\}$, $y_1 \le y_2 \le \ldots$. Characterize those x-es, for which $y_{n+1} - y_n \to 0$ $(n \to \infty)$.

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DIRECTLY CARTESIAN CLOSED SUBCATEGORIES OF RELATIONAL CATEGORIES

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In relational categories, i.e. functor-structured categories whose types are coproducts of hom-functors of Set, we introduce a binary operation of direct exponentiation for objects. The operation introduced is a natural extension of the Birkhoff's cardinal exponentiation for posets.

It is well known that topological categories having well-behaved function spaces, i.e. being Cartesian closed, play important roles in many branches of mathematics. It seems therefore worthwhile to discover new Cartesian closed topological categories. For relational categories there arises a more special problem to determine subcategories in which the direct exponentiation is well-behaved. Such subcategories we call directly Cartesian closed. In the paper we deal with the problem mentioned above — we find topological axioms of Herrlich that give some directly Cartesian closed subcategories of relational categories.

Basic used notions from the category theory can be found e.g. in [1]. By a *relational category* we understand a category $S(F)$ where F is a coproduct of hom-functors of Set. Recall that for a given set-valued functor $F : \mathcal{K} \to$Set the symbol $S(F)$ denotes the functor-structured category of type F, i.e. the concrete category over $\mathcal{K}$ whose objects are pairs (X, ϱ) where X is a $\mathcal{K}$-object and $\varrho \subseteq FX$ is a subset, and whose morphism from (X, ϱ) into (Y, σ) are the $\mathcal{K}$-morphisms $f : X \to Y$ with $Ff(\varrho) \subseteq \sigma$. Each functor-structured category is initially complete and fibre small. H. HERRLICH [5] introduced the concept of *topological axioms* in functor-structured categories: Let (X, ϱ), (X, σ) be a pair of objects of $S(F)$ with $\varrho \subseteq \sigma$. An object (Z, τ) of $S(F)$ is said to satisfy the topological axiom

$$\varrho \vdash \sigma[X]$$

provided that for each $\mathcal{K}$-morphism $f : X \to Z$, if $f : (X, \varrho) \to (Z, \tau)$ is a morphism in $S(F)$, then $f : (X, \sigma) \to (Z, \tau)$ is a morphism in $S(F)$.

For a given set M the symbol H_M will denote the (covariant) hom-functor of Set with respect to M. By H we denote a coproduct of hom-functors of Set $H = \coprod_{i\in I} H_{M_i}$ where $I \neq \emptyset$ is a set and $\{M_i | i \in I\}$ is a family of nonvoid sets. Now we shall define a direct exponentiation for objects of $S(H)$ which extends the Birkhoff's cardinal exponentiation for posets [3]:

Let $A = (X, \varrho)$ and $B = (Y, \sigma)$ be objects of $S(H)$. By the *direct power* A^B of A and B (in $S(H)$) we mean the object (Z, τ) of $S(H)$ where $Z = \mathrm{Mor}(B, A)$ ($\mathrm{Mor}(B, A)$ denotes the set of all morphisms from B into A in $S(H)$) and $\tau \subseteq HZ$ is defined as follows: $t \in HZ$, $t \in \tau \Leftrightarrow$ for the index $i \in I$ with $t \in H_{M_i}Z$ and for each $y \in Y$ the map $t^y : M_i \to X$ given by $t^y(u) = t(u)(y)$ fulfils $t^y \in \varrho$.

DEFINITION. A full subcategory $\mathcal{R}$ of a relational category $S(H)$ is called a *directly Cartesian closed category* if it satisfies the following four conditions:

(i) in $\mathcal{R}$ all constant maps are morphisms,

(ii) $\mathcal{R}$ is closed w.r.t. formation of initial objects of sources in $S(H)$,

(iii) $\mathcal{R}$ is closed w.r.t. direct exponentiation in $S(H)$,

(iv) for any three objects A, B, C of $\mathcal{R}$ and any map $f : B \times C \to A$ the following conditions are equivalent:

(a) $f : B \times C \to A$ is a morphism in $\mathcal{R}$,

(b) $f^* : C \to A^B$, defined by $f^*(c)(b) = f(b, c)$, is a morphism in $\mathcal{R}$.

Thus, each directly Cartesian closed category is a Cartesian closed topological category (in the sense of [4]) in which powers coincidence with direct powers. In other words, in each directly Cartesian closed category the direct exponentiation is well-behaved, i.e. behaved analogically to the exponentiation in Set.

By $R(H)$ we denote the full subcategory of $S(H)$ whose objects are precisely those objects of $S(H)$ that satisfy the following two topological axioms:

$$\emptyset \vdash H\{\emptyset\}\ [\{\emptyset\}] \tag{1}$$

$$\pi \vdash \pi \cup \{id\}\ \left[\coprod_{i\in I} M_i \times M_i\right] \tag{2}$$

where $\pi = \coprod_{i\in I}\{f : M_i \to M_i \times M_i |$ there exists a constant map $f_0 : M_i \to M_i$ such that $f = (f_0, id_{M_i})$ or $f = (id_{M_i}, f_0)\}$ and $id = \coprod_{i\in I}\{(id_{M_i}, id_{M_i})\}$.

As an example, let the family $\{M_i | i \in I\}$ consist of finite sets only. Then $S(H)$ is the category of sets with systems of relations of finite arities. Objects of $R(H)$ are precisely those objects of $S(H)$ for which all relations $\varrho \cap H_{M_i}X$ $(i \in I)$ on X have the two properties "to be weakly reflexive" and "to have the diagonal property" introduced in [6]. Especially, if $\{M_i | i \in I\}$ consists of only one two-point set, then $S(H)$ is the category of sets with binary relations and $R(H)$ is the category of preordered sets.

THEOREM. *$R(H)$ is a directly Cartesian closed category.*

PROOF. (i) The axiom (1) immediately implies that all constant maps in $R(H)$ are morphisms.

(ii) Let $f_k : X \to (X_k, \varrho_k)$, $k \in K$, be a source in $R(H)$. Obviously, the object (X, ϱ) of $S(H)$ where $\varrho = \{r \in HX | Hf_k(r) \in \varrho_k$ for every $k \in K\}$ is the initial object of this source in $S(H)$. It can be easily seen that (X, ϱ) is an object of $R(H)$.

(iii) Let $A = (X, \varrho)$, $B = (Y, \sigma)$ be two objects of $R(H)$. Clearly, the direct power A^B in $S(H)$ satisfies the topological axiom (1). We are to prove that it satisfies also the axiom (2). On that account, denote $(T, \lambda) = A^B$. Let $f : (\coprod_{i\in I} M_i \times M_i, \pi) \to (T, \lambda)$ be a morphism in $R(H)$. For any $i \in I$ let $g_i \in H_{M_i}T$ be the map defined by $g_i(u) = f(u, u)$ whenever $u \in M_i$. For each $y \in Y$ and each $d \in T$ put $h_y(d) = d(y)$. Let $r \in \lambda$ and let $i \in I$ be the index with $r \in H_{M_i}T$. Then for each $y \in Y$ and each $u \in M_i$ there holds $Hh_y(r)(u) = h_y(r(u)) = r(u)(y) = r^y(u)$. Hence, for each $y \in Y$ and each $r \in \lambda$ we get $Hh_y(r) = r^y \in \varrho$. Therefore $h_y : (T, \lambda) \to (X, \varrho)$ is a morphism in $R(H)$ for every $y \in Y$. Consequently, $h_y \circ f : (\coprod_{i\in I} M_i \times M_i, \pi) \to (X, \varrho)$ is a morphism for each $y \in Y$. Since A fulfils axiom (2), it holds $H(h_y \circ f)(id_{M_i}, id_{M_i}) = = h_y \circ f \circ (id_{M_i}, id_{M_i}) \in \varrho$ for each $i \in I$. Next, for any $y \in Y$, $i \in I$ and $u \in M_i$ we have $g_i^y(u) = g_i(u)(y) = f(u, u)(y) = h_y(f(u, u))$. Thus $g_i^y = h_y \circ f \circ (id_{M_i}, id_{M_i})$ for each $y \in Y$ and $i \in I$. Therefore $g_i^y \in \varrho$ for any $y \in Y$ and $i \in I$, i.e. $g_i \in \lambda$ for all $i \in I$. Since $Hf(id_{M_i}, id_{M_i}) = g_i$ for every $i \in I$, the topological axiom (2) is satisfied for A^B.

(iv) Let $A = (X, \varrho)$, $B = (Y, \sigma)$, $C = (Z, \tau)$ be objects of $R(H)$ and $f : B \times C \to A$ a map. If $Y \times Z = \emptyset$, i.e. if $Y = \emptyset$ or $Z = \emptyset$, then $\mathrm{Mor}(C, A^B)$ has exactly one element. This element can be considered to be the map f^*. Then both f and f^* are constant maps, hence they are morphisms in $R(H)$ by (i). Therefore axiom (iv) is fulfiled. Suppose $Y \times Z \neq \emptyset$.

I. Let $f : B \times C \to A$ be a morphism in $R(H)$. Let $t \in \tau$ and let $i \in I$ be the index with $t \in H_{M_i}Z$. Put $(V, \mu) = B \times C$. Let $z \in Z$, $s \in \sigma \cap H_{M_i}Y$ and let $g : M_i \to V$ be the map given by $g(u) = (s(u), z)$ for each $u \in M_i$. Then $g \in \mu$ because $C = (Z, \tau)$ fulfils axiom (1). Since $Hf(\mu) \subseteq \varrho$, we have $H_i f(g) = f \circ g \in \varrho$. For each $u \in M_i$ there holds $f(g(u)) = f(s(u), z) = f^*(z)(s(u))$. Thus $f^*(z) \circ s = f \circ g$. Consequently, $f^*(z) \circ s \in \varrho$ and hence $f^*(z) \in$ $\in \mathrm{Mor(B,A)}$. Particularly, $f^*(t(u)) \in \mathrm{Mor}(B, A)$ for each $u \in M_i$. Next, let $y \in Y$ and let $h : M_i \to V$ be the map defined by $h(u) = (y, t(u))$ for each $u \in M_i$. Then $h \in \mu$ because (Y, σ) fulfils axiom (1). We have $(f^* \circ t)^y(u) = f^*(t(u))(y) = f(y, t(u)) = f(h(u))$ whenever $y \in Y$ and $u \in M_i$. Hence $(f^* \circ t)^y = f \circ h$ for all $y \in Y$. As $Hf(\mu) \subseteq \varrho$, there holds $Hf(h) = f \circ h \in \varrho$. Thus, $(f^* \circ t)^y \in \varrho$ for every $y \in Y$. Denoting $(T, \lambda) = A^B$ we get $f^* \circ t \in \lambda$. Therefore $Hf^*(\tau) \subseteq \lambda$, i.e. $f^* : C \to A^B$ is a morphism in $R(H)$.

II. Let $f^* : C \to A^B$ be a morphism in $R(H)$. Clearly, there holds $f = e \circ (id_Y \times f^*)$ where $e : B \times A^B \to A$ is the evaluation map defined by $e(y, g) = g(y)$. Thus, it is sufficient to show that e is a morphism in $R(H)$. On that account, denote $(W, \nu) = B \times A^B$. Let $p \in \nu$ and let $i \in I$ be the index with $p \in H_{M_i}W$. Then there exist $s \in \sigma \cap H_{M_i}Y$ and $r \in \lambda \cap H_{M_i}T$ (recall that $(T, \lambda) = A^B$) such that $p(u) = (s(u), r(u))$ for each $u \in M_i$. Put $\varrho_i = \varrho \cap H_{M_i}X$. Then $r^y \in \varrho_i$ for each $y \in Y$. Therefore $r^{s(u)} \in \varrho_i$ for any $u \in_{M_i}$. For each pair $u, v \in M_i$ put $p_v(u) = r^{s(u)}(v)$. Then $p_v(u) = r(v)(s(u))$ for all $u, v \in M_i$, i.e. $p_v = r(v) \circ s$ for each $v \in M_i$. Since $r(v)$ is a morphism from B into A in $S(H)$ for every $v \in M_i$, we have $p_v = r(v) \circ s =$ $= H(r(v))(s) \in \varrho_i$ for all $v \in M_i$. Put $(X \times X, \xi) = (X, \varrho_i) \times (X, \varrho_i)$. Let $g : M_i \times M_i \to X \times X$ be the map defined by $g(u, v) = (p_v(u), p_u(v))$ and let $f \in \pi_i = \pi \cap H_{M_i}(M_i \times M_i)$ (for the definition of π see the topological axiom (2)). Suppose $f = (f_0, id_{M_i})$ and let $u_0 \in M_i$ be the point with $f_0(u) = u_0$ for each $u \in M_i$. Then for any $u \in M_i$ there holds $g(f(u)) = g(u_0, u) = (p_u(u_0), p_{u_0}(u)) = (r^{s(u_0)}(u), p_{u_0}(u))$. Hence $H_{M_i}g(f) = g \circ f = (r^{s(u_0)}, p_{u_0})$. Otherwise, supposing $f = (id_{M_i}, f_0)$

we get $H_{M_i}g(f) = (p_{u_0}, r^{s(u_0)})$. Thus, in both the cases we have $H_{M_i}g(f) \in \xi$. Therefore g is a morphism of $(M_i \times M_i, \pi_i)$ into $(X \times X, \xi)$ in $S(H_{M_i})$. Obviously, since the object $(X, \varrho) \times (X, \varrho)$ fulfils axiom (2) in $S(H)$, the object $(X \times X, \xi)$ fulfils the axiom

$$\pi_i \vdash \pi_i \cup \{(id_{M_i}, id_{M_i})\}[M_i \times M_i]$$

in $S(H_{M_i})$. Thus, the map $w : M_i \to X \times X$ defined by $w(u) = g(u, u)$ fulfils $w \in \xi$. Consequently, the map $q : M_i \to X$ given by $q(u) = p_u(u)$ fulfils $q \in \varrho_i \subseteq \varrho$. For any $u \in M_i$ there holds $e(p(u)) = e(s(u), r(u)) = = r(u)(s(u)) = r^{s(u)}(u) = p_u(u) = q(u)$. This implies $e \circ p = q$ and hence $He(p) = e \circ p \in \varrho$. Therefore e is a morphism of $B \times A^B$ into A in $R(H)$. The proof is complete.

REMARK. Denote by $\cong$ the isomorphism in $S(H)$. One of the consequences of the directly Cartesian closedness of $R(H)$ is the validity of the law $(A^B)^C \cong A^{B \times C}$ for objects of $R(H)$. Let $R_1(H)$ or $R_2(H)$ respectively be the full subcategory of $S(H)$ obtained by applying only the topological axiom (1) or (2) respectively in the definition of $R(H)$. Of course, neither $R_1(H)$ nor $R_2(H)$ are directly Cartesian closed in general. Let A be an object of $R_2(H)$ and B, C be objects of $R_1(H)$. Then from the proof of Theorem it follows that the map $f \to f^*$ is a bijection of $A^{B \times C}$ onto $(A^B)^C$. But it can be easily shown that this bijection is an isomorphism in $S(H)$, i.e. that $(A^B)^C \cong A^{B \times C}$.

Now we are aiming to give an application of the Theorem. Let G be a small set-functor [1], i.e. such a functor G : Set$\to$Set for which there exists a set M such that $GX = \bigcup_{f:M \to X} Gf(GM)$ for any set X. Put $H = \coprod_{t \in GM} H_{M_t}$ where $M_t = M$ for each $t \in GM$, and for every set X let $\tau_X : HX \to GX$ be the map defined by $\tau_X(h) = Gh(t)$ whenever $h \in H_{M_t}X$ and $t \in GM$. Let $T(G)$ be the full subcategory of the functor-structured category $S(G)$ whose objects are precisely those objects (X, ϱ) of $S(G)$ for which $(X, \tau_X^{-1}(\varrho))$ satisfies both the topological axioms (1) and (2) in $S(H)$. Since τ_X is surjective for each set X and since $\tau : H \to G$ is a natural transformation, the map given by $(X, \varrho) \to (X, \tau_X^{-1}(\varrho))$ defines a full concrete embedding of $T(G)$ into $R(H)$. Thus, according to Theorem we get

COROLLARY. There exists a full concrete embedding of $T(G)$ into a directly Cartesian closed category.

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THE UNIQUE AMALGAMATION PROPERTY FOR LATTICES

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1. Introduction

Let A and B be lattices, and let S be a common sublattice of A and B. The lattice L *amalgamates* A and B over S iff L contains A and B as sublattices sharing S as a sublattice. Such a lattice L always exists; we express this by saying that the variety $\mathbf{L}$ of all lattices has the Amalgamation Property.

Of course, if L amalgamates A, B, S, so does every lattice containing L as a sublattice. However, even if we assume that L is generated by $A \cup B$, in general, there are very many lattices amalgamating A, B, S. It is not difficult to find an example of A, B, and S, all finite, such that there are infinitely many different lattices generated by $A \cup B$ amalgamating A and B over S.

A classical example of amalgamation where the result is unique and $L = A \cup B$ is the *gluing* of lattices (the special case when S is a dual ideal of A and an ideal of B). A recent series of papers (A. SLAVÍK [8], A. DAY and J. JEŽEK [1], E. FRIED and G. GRÄTZER [2] and [3] E. FRIED, G. GRÄTZER, and H. LAKSER [4]) investigate a generalization of gluing called pasting: Let L be a lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Then L *pastes* A and B together over S, if every amalgamation of A and B over S contains L as a sublattice.

In this paper, we investigate the Unique Amalgamation Property: Let A and B be lattices, and let S be a common sublattice of A and B. If there is a lattice L amalgamating A and B over S with the property that every lattice L' amalgamating A and B over S contains L as a sublattice, then we say that A, B, and S have the *Unique Amalgamation Property*. The actual definition is slightly more technical, see Section 2.

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For isntance, if L *pastes* A and B together over S, then A, B, and S have the Unique Amalgamation Property. In this note, we prove the converse for finite lattices. We also provide examples to show that the converse does not hold for infinite lattices.

2. Definitions

We start with some definitions:

DEFINITION 1. (see, e.g., G. GRÄTZER [5]). Let s_A and s_B be embeddings of the lattice S into the lattices A and B, respectively. The lattice L *amalgamates* A and B over S, if there are embeddings f_A and f_B of A and B into L, respectively, such that $s_A f_A = s_B f_B$ (see Figure 1).

$$\begin{tikzcd} S \arrow[r, "s_A"] \arrow[d, "s_B"'] & A \arrow[d, "f_A"] \\ B \arrow[r, "f_B"'] & L \end{tikzcd}$$

Fig. 1

$$\begin{tikzcd} & A \arrow[d, "f_A"'] \arrow[ddr, "g_A"] & \\ B \arrow[r, "f_B"] \arrow[drr, "g_B"'] & L \arrow[dr, dashed, "h"] & \\ & & K \end{tikzcd}$$

Fig. 2

DEFINITION 2. (G. GRÄTZER [5, Exercise 12 of Section V. 4], A. SLAVÍK [8]). Let L be a lattice. Let A, B,S be sublattices of L, $A \cap B = S$, $A \cup B = L$. Let f_A and f_B be the embeddings of A and B into L, respectively. Then L *pastes* A and B together over S if whenever g_A and g_B are embeddings of A and B into a lattice K, respectively, satisfying $xg_A = xg_B$ for all $x \in S$, then there is a homomorphism h of L into K satisfying $f_A h = g_A$ and $f_B h = g_B$ (see Figure 2).

Note that the homomorphism h is always an embedding; this follows from a result of E. FRIED and G. GRÄTZER [3].

We cannot hope for unique amalgamation unless we require that A and B sit properly in L. To formalize this, we need a definition (B. JÓNSSON [7]):

DEFINITION 3. Let A, B, and S be lattices, $A \cap B = S$. On $P = A \cup B$, we define a poset $P = P(A, B, S)$ as follows:

(i) For $x, y \in A$ (and for $x, y \in B$), $x \leq y$ in P iff $x \leq y$ in A (resp., $x \leq y$ in B).

(ii) For $x \in A$ and for $y \in B$, $x \leq y$ in P iff there exists an $s \in S$ with $x \leq s$ in A and $s \leq y$ in B; and dually, for $y \leq x$.

P contains A and B as subposets; in fact, if $a_1 \vee a_2 = a_3$ in A, then $\sup\{a_1, a_2\} = a_3$ in P, and dually for $\wedge$ and inf, and symmetrically for B. Now we are ready to define the Unique Amalgamation Property:

DEFINITION 4. Let s_A and s_B be embeddings of the lattice S into the lattices A and B, respectively. $\langle A, B, S, s_A, s_B \rangle$ has the *Unique Amalgamation Property* if there exists a lattice L and embeddings f_A and f_B of A and B into the lattice L, respectively, such that the following conditions hold:

(i) L amalgamates A and B over S, that is, $s_A f_A = s_B f_B$;

(ii) whenever g_A and g_B are embeddings of A and B into the lattice K satisfying $s_A g_A = s_B g_B$ and such that $g = g_A \cup g_B$ is an order-isomorphism of $P = A \cup B$ into K, then there is an embedding h of L into K satisfying $g_A = f_A h$ and $g_B = f_B h$ (see Figure 3).

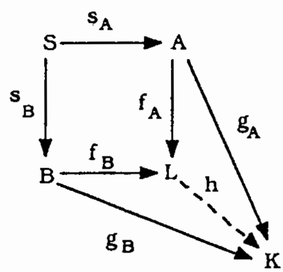

Fig. 3

For brevity, we shall say that A, B, S have the Unique Amalgamation Property if $\langle A, B, S, s_A, s_B \rangle$ has the Unique Amalgamation Property and s_A and s_B are understood, for instance, if S is a sublattice of A and B. If A, B, S have the Unique Amalgamation Property, and L is the lattice described in Definition 4, then we call L the *unique amalgam* of A, B, S.

At first it may appear that (ii) should have been formulated as follows:

(ii') whenever g_A and g_B are embeddings of A and B into the lattice K satisfying $s_A g_A = s_B g_B$, then there is an embedding h of L into K satisfying $g_A = f_A h$ and $g_B = f_B h$.

We shall show an example in the next section to illustrate why this more natural definition is not satisfactory. A related question is the following: if L is the unique amalgam of A, B, S, do the embeddings

f_A and f_B satisfy the assumptions of (ii)? It follows from the discussion in Section 4 that the answer is in the affirmative.

3. Examples

All the examples of pasting (see the references, especially, [1], [2], [3], [6], [8]) are examples of the Unique Amalgamation Property. Figure 4 provides a new type of example. In Figure 4, the black-filled elements from S, an infinite chain of type $\omega + \omega^*$. The lattice A is $S \cup \{a\}$, and B is $S \cup \{b\}$; both are chains. Then A, B, S have the Unique Amalgamation Property, and the unique amalgam is the lattice L of Figure 5.

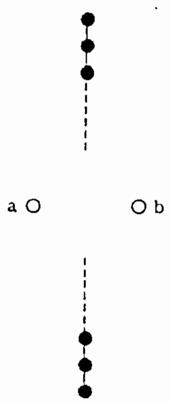

Fig. 4

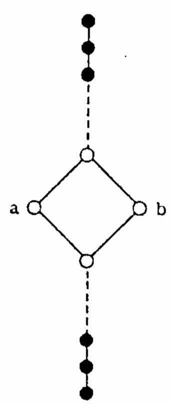

Fig. 5

Fig. 6

The lattice L' of Figure 6 illustrates why we need condition (ii) rather than (ii') in Definition 4. If, instead of (ii), we use (ii'), then L is not the unique amalgam of A, B, S; indeed, A and B have obvious embeddings into L' but L does not embed into L'. In fact, with (ii') replacing (ii), A, B, S does not have the Unique Amalgamation Property.

Figure 7 provides a different type of example of unique amalgamation. The black-filled elements form S; it is a chain made up of three copies of the chain of integers. The striped elements form B–A; the lattice B is constructed as follows: add a zero to S and form its direct product with C_2 (the two-element chain). Finally, A–S consists of the two white-filled elements.

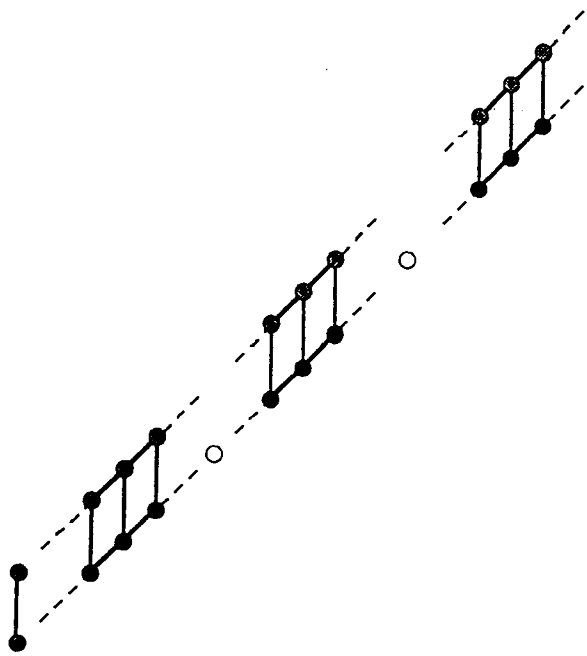

Fig. 7

The reader may find it instructive to reason precisely why Figures 4 and 7 depict pairs of lattices with the Unique Amalgamation Property. The reasons for the two situations are quite different.

4. Unique Amalgams

In this section we prove some properties of unique amalgams.

We shall utilize two well-known lattice constructions.

If P is a partially ordered set and $X \subseteq P$, then let X^u denote the set of all upper bounds of X in P and let X^1 denote the set of all lower bounds. $X^c = (X^u)^1$ is the *closure* of X. $X \subseteq P$ is *closed* if $X^c = X$; the closure of a set is closed. The *MacNeille completion* P^c of P consists of all the closed sets (see G. Grätzer [5], and the references therein).

P has a natural map ϕ^c into P^c by mapping p into $(p] = \{x \in \in P | x \leq p\}$. The MacNeille completion P^c of P is always a complete lattice; moreover, every sup in P becomes a (complete) join and every inf becomes a (complete) meet. Hence if $\sup\{x, y\} = z$ in P, then $x\phi^c \vee y\phi^c = z\phi^c$, and dually. ϕ^c is an embedding of the poset.

Now we can answer the question raised after Definition 4: If L is the unique amagam of A, B, S, do the embeddings f_A and f_B satisfy the assumptions of Definition 4 (ii)? Indeed, take the MacNeille completion P^c of $P = P(A, B, S)$ (see Definition 3). Then ϕ^c embeds A and B into P^c in a manner required by Definition 4 (ii); so f_A and f_B must also satisfy the assumptions of Definition 4 (ii).

For the next lattice construction, we need an additional definition:

DEFINITION 5. Let A, B, and S be lattices, $A \cap B = S$. On the set $P = A \cup B$, the partial algebra Part(A, B, S) with the partial binary operations $\wedge$ and $\vee$ is defined as follows:

(i) If $x \leq y$ in P, then $x \wedge y = x$ and $x \vee y = y$ in Part(A, B, S).

(ii) If $x, y, z \in A$ and $x \wedge y = z$ in A, then $x \wedge y = z$ in Part(A, B, S); and similarly for $x \vee y = z$ in A.

(iii) Same as (ii), for $x, y, z \in B$.

An *ideal* I of Part(A, B, S) is a subset I of P such that if $x \in I$, $y \in P$, $y \leq x$, then $y \in I$, and if x, y, $z \in I$ and $x \vee y = z$ in Part(A, B, S), then $z \in I$. For a subset X of P, $(X]$ will denote the smallest ideal containing X; if $X = \{x\}$, we write $(x]$ for $(X]$ and call it *principal.* The ideal I is *finitely generated* if $I = (X]$ for a finite set X.

Dual ideals are defined dually.

Let Id(Part(A, B, S)) (or Id(P), for short) denote the lattice of ideals of Part(A, B, S). Then Part(A, B, S) has a natural map ϕ^{Id} into Id(P) by mapping p into $(p]$ for $p \in P$. The map ϕ^{Id} is one-to-one and it preserves the partial operations of Part(A, B, S): if $x \vee y$ exists in Part(A, B, S), then $(x \vee y)\phi^{\mathrm{Id}} = x\phi^{\mathrm{Id}} \vee y\phi^{\mathrm{Id}}$; if $x \wedge y$ exists in Part(A, B, S), then $(x \wedge y)\phi^{\mathrm{Id}} = x\phi^{\mathrm{Id}} \wedge y\phi^{\mathrm{Id}}$. Note, however, that $(x \vee y)\phi^{\mathrm{Id}} = z\phi^{\mathrm{Id}}$ may hold event though $x \vee y = z$ does not hold in Part(A, B, S).

LEMMA 6. Let A, B, S have the Unique Amalgamation Property, $P = P(A, B, S)$. Form the lattices: P^c_{fd} and $\mathrm{Id}_{fd}(P)$, the sublattices of P^c and Id(P) generated by the images of P under ϕ^c and ϕ^{Id}, respectively. Then there exist an isomorphism α^{Id} from P^c_{fd} into $\mathrm{Id}_{fd}(P)$ such that $\phi^c\alpha^{\mathrm{Id}} = \phi^{\mathrm{Id}}$ (see Figure 8).

PROOF. Let L be the unique amalgam of A, B, S with the embeddings f_A and f_B of A and B into L, respectively. Now apply Definition 4 to $K = P^c_{fd}$ and the embeddings g_A: the restriction of ϕ^c to A, g_B: the restriction of ϕ^c to B. It is easy to prove that the assumptions of Definition 4 (ii) are satisfied for g_A and g_B. Hence, there is an embedding h^c of L into K. Since P^c_{fd} is generated by the image of P under ϕ^c, the map h^c is onto, hence it is an isomorphism.

We proceed in the same fashion with $\mathrm{Id}_{fd}(P)$; again, for the natural embeddings g_A and g_B, the assumptions of Definition 4 (ii) are satisfied. Thus we obtain an isomorphism h^{Id} of L with $\mathrm{Id}_{fd}(P)$. We can obviously define $\alpha^{\mathrm{Id}} = (h^c)^{-1}\, h^{\mathrm{Id}}$. The lemma now follows.

Lemma 6 is quite trivial, nevertheless, it has some interesting consequences.

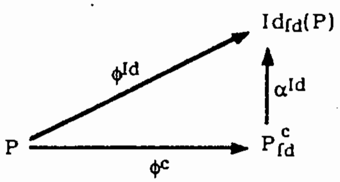

Fig. 8

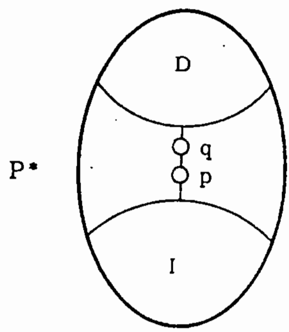

Fig. 9

THEOREM 7. *Let A, B, S have the Unique Amalgamation Property, $P = P(A, B, S)$. Let I be a finitely generated ideal, and let D be a finitely generated dual ideal of* Part(A, B, S), *I, $D \neq \emptyset$. If $I^u = D$ and $D^l = I$, then I is principal.*

PROOF. Let us assume that I is not principal. It follows that D is not principal either; indeed, if $D = [d)$, then $D^l = I$, a contradiction. We further claim that $I \cap D = \emptyset$. Indeed, if $u \in I \cap D$, then $u \in D$, hence u is an upper bound for all the elements of I; on the other hand, $u \in I$, hence $I = (u], a$ contradiction.

Now let P^* be P with two new elements p and q (see Figure 9). We define the partial ordering on P^* as follows: for u, $v \in P$, $u < v$ in P^* iff $u < v$ in P; $p < q$; $i < p$, q for all $i \in I$; p, $q < d$ for all $d \in D$. It is routine to check that P^* is a poset.

Moreover, if $u \vee v = w$ in Part(A, B, S), then $\sup\{u, v\} = w$ in P^*. Indeed, $\sup\{u, v\} = w$ in P. If p or q happens to be an upper bound for u and v in P^*, then u, $v \in I$, therefore $w \in I$, implying that $w \leq p$ and q.

Now let L be the unique amalgam of A, B, S with the embeddings f_A and f_B. Let K be the lattice $(P^*)^c$, with the embedding j of P^* into K defined by $uj = \{v | v \in P^* \text{ and } v \leq u\}$ $(= (u] \text{ in } P^*)$. Define h_A and h_B as the restrictions of j to A and B, respectively. By Definition 4, there is an embedding h of L into K (see Figure 3). Obviously, h maps the element af_A of $L (a \in A)$ into $\{a\}^c = (a]$ and bf_B of L $(b \in B)$ into $\{b\}^c = (b]$.

I and D are finitely generated, say, $I = (\{u_1, \ldots, u_n\}]$ and $D = [\{v_1, \ldots, v_m\})$; since I and D are not principal, n, $m > 1$. In P^*, $p = \sup\{u_1, \ldots, u_n\}$ and $q = \inf\{v_1, \ldots, v_m\}$, hence in K, $pj = u_1 j \vee \ldots \vee u_n j$ and $qj = v_1 j \wedge \ldots \wedge v_m j$.

We can repeat the same argument with the partially ordered set P^{**} which is defined exactly as P^* except that $p = q$. We obtain the embedding j' of P^{**} into $K' = (P^{**})^c$ and the embedding h' of L into K'. In K', $u_1 j' \vee \ldots \vee u_n j' = pj' = qj' = v_1 j' \wedge \ldots \wedge v_m j'$.

This is now a contradiction, since

$$\begin{aligned} pj &= u_1 j \vee \ldots \vee u_n j = u_1 j'(h')^{-1} h \vee \ldots \vee u_n j'(h')^{-1} h = \\ &= (u_1 j' \vee \ldots \vee u_n j')(h')^{-1} h = (v_1 j' \wedge \ldots \wedge v_m j')(h')^{-1} h = \\ &= v_1 j'(h')^{-1} h \vee \ldots \vee v_m j'(h')^{-1} h = v_1 j \wedge \ldots \wedge v_m j = qj \end{aligned}$$

but $pj < qj$. The proof of the theorem is complete.

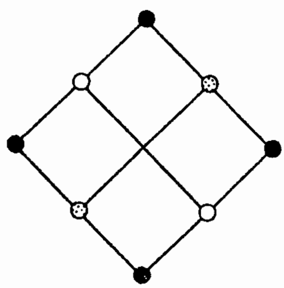

Fig. 10

A small example is shown in Figure 10. Let A and B be both isomorphic to $C_2 \times C_3$ and let S be $C_2 \times C_2$ (C_n is the n element chain, $n = 2, 3$). Figure 10 shows $P = P(A, B, S)$; the elements of S are black-filled, the elements of A–B are dot-filled, and the elements of B–A are white-filled. A, B, S fail the Unique Amalgamation Property. Indeed, let the ideal I be the bottom three elements, the dual ideal D be the top three elements. Then I and D satisfy the assumptions of Theorem 7, but I is not principal.

THEOREM 8. *Let A, B, S have the Unique Amalgamation Property, $P = P(A, B, S)$. Let I be a nonempty finitely generated ideal of Part(A, B, S). Then I is closed.*

PROOF. By Lemma 6. there is a lattice embedding ϕ from $\mathrm{Id}_{fd}(P)$ into P^c such that $(u]\phi = (u]$ for all $u \in P$. We claim that $I\phi = I$. Since I is nonempty and finitely generated, we can assume that $I = (\{u_1, \ldots, u_n\}]$ for some $u_1, \ldots, u_n \in P$ and $n \geq 1$. Then $I = (u_1] \vee \ldots \vee (u_n]$, and so $I\phi = (u_1]\phi \vee \ldots \vee (u_n]\phi = (u_1] \vee \ldots \vee (u_n]$, where the joins on the right side are formed in P^c, thus $(u_1] \vee \ldots \vee (u_n] = I^c$. We conclude that $I\phi = I^c$. Now if $x \in I^c$, then $(x] \subseteq I^c$, and so $(x]\phi^{-1} \subseteq I^c\phi^{-1}$, that is, $(x] \subseteq I$, and therefore $x \in I$. Thus we have proved that $I^c \subseteq I$, and so $I^c = I$, completing the proof.

5. The finite case

We now investigate the Unique Amalgamation Property for finite lattices.

THEOREM 9. *Let A, B, and S be finite lattices, $A \cap B = S$. If A, B, and S have the Unique Amalgamation Property and L is the unique amalgam, then L pastes A and B over S.*

PROOF. Let $P = P(A, B, S)$, and choose $a \in A - B$ and $b \in B - A$.

Firstly, we claim that if a and b have an upper bound in P, then they have a least upper bound $\sup\{a, b\}$ and $\sup\{a, b\} \in (\{a, b\}]$ (the ideal is formed in Part(A, B, S)).

To prove this, define $I = (\{a, b\}]$, $D = I^u$, and $J = D^l$. Since P is finite, J and D are finitely generated ideals; obviously, J, $D \neq \emptyset$. Thus Theorem 7 applies, J is principal, $J = (c]$. Obviously, $c = \sup\{a, b\}$. On the other hand, $I^c = J$, and by Theorem 8, I is closed, hence $I = J$. Thus $I = (c]$, which was to be proved.

Secondly, we claim that a and b have a common upper bound in P.

Indeed, if they do not, then the unit element a^* of A and b^* of B have no common upper bound in P. Obviously, $a^* \in A - B$ and $b^* \in B - A$; otherwise, $a^* \vee b^*$ exists and it is an upper bound for a and b.

Form $D = \{a^*, b^*\}$, Clearly, D is a dual ideal of Part(A, B, S).

Now define $I = D^l$. Take any $s \in S$. Then $s \leq a^*$, since $s \in A$. Similarly, $s \leq b$. Thus $s \in I$, and so $I \neq \emptyset$. Since A and B are finite, I and D are finitely generated. Therefore, we can apply Theorem 7 and conclude that D is principal, an obvious contradiction. This proves our second claim.

Now we quote a result from E. FRIED and G. GRÄTZER [3] (Lemma 8, part (VIII)): if $(\{a, b\}] = (c]$ in Part(A, B, S), and any pair of elements in P have a common lower bound (which holds by the dual of the second claim above), then there is an increasing sequence of elements of S: $x_1, \ldots, x_n$ satisfying $x_1 \leq a$, $x_2 \leq b \vee x_1$, $x_3 \leq a \vee x_2, \ldots$ and $a \vee x_n = c$ if n is even, $b \vee x_n = c$ if n is odd, or alternatively, $x_1 \leq b$, $x_2 \leq a \vee x_1$, $x_3 \leq b \vee x_2$, ... and $a \vee x_n = c$ if n is odd, $b \vee x_n = c$ if n is even.

By the two claims above (along with their duals), P as a poset is a lattice. P contains A and B as sublattices and $A \cup B$ equals P. Since A, B, and S have the Unique Amalgamation Property, we conclude that we can take P as the unique amalgam.

Finally, we apply Theorem 5 of E. FRIED and G. GRÄTZER [3] which states that, under these conditions, L is the pasting of A and B over S (condition (Ord) of [3] is satisfied by L by the construction of P, see Definition 3; condition (Seq) of [3] was verified in the last but one paragraph). This completes the proof of the theorem.

In conclusion we mention a problem: which lattice varieties are closed under the formation of unique amalgams? In E. FRIED and G. GRÄTZER [3] it is proved the the varieties **M** (of all modular lattices) and **D** (of all distributive lattices) are closed under pasting.

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EVERY FINITE LATTICE IS A COMPLETE CONGRUENCE LATTICE

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Introduction

For a complete lattice L, a congruence relation Θ of L is called a *complete congruence* iff it satisfies the condition:

$$x_i \equiv y_i(\Theta), i \in I, \text{implies that } \wedge x_i \equiv \wedge y_i(\Theta) \text{ and } \vee x_i \equiv \vee y_i(\Theta).$$

The complete congruence relations of L form a complete meet subsemilattice of the congruence lattice of L; hence they form a complete lattice, denoted by $\mathrm{Com}\, L$. It is well-known that the congruence lattice of a lattice is algebraic and distributive. However, this is not so in the case of the complete congruence lattice. K. Reuter and R. Wille [1] show that if D is a complete distributive lattice in which each element is the supremum of $\vee$-irreducibles, then D is isomorphic to $\mathrm{Com}\, L$ for some complete lattice L. They also give examples showing that $\mathrm{Com}\, L$ is not distributive for some complete lattice L. They ask the question whether every complete lattice K is isomorphic to $\mathrm{Com}\, L$ for some complete lattice L. We give an affirmative answer to this problem when K is finite.

Preliminaries

Let $[n]$ and ω be the set of first n natural numbers and the set of all natural numbers, respectively. Let γ be an ordinal and let $\{L_\alpha | \alpha < \gamma\}$ be a family of lattices. The *sum* $\sum(L_\alpha | \alpha < \gamma)$ is the lattice with underlying set $\cup(L_\alpha | \alpha < \gamma)$ and, besides the inherited order relations of each L_α, we have $x < y$ for all $x \in L_\alpha$, $y \in L_\beta$, with $\alpha < \beta < \gamma$. The *zero* and the *unit* of a lattice (if they exist) are denoted by O and I, respectively. Let L_1 and L_2 be lattices such the L_1 has a unit and

L_2 has a zero; then $L_1 \oplus L_2$ is the lattice obtained from $L_1 + L_2$ by identifying I_{L_1} with O_{L_2}. The *dual* of a lattice L is denoted by L^d.

We shall be considering chains which can be obtained form ω and $[n]$ by the operations $+$, $\oplus$, $()^d$. Thus it is appropriate for us to define, for a chain C, the *support* of C to be the set $\operatorname{supp} C = \{[x,y] | x \prec y \text{ in } C\}$. Let C_1 and C_2 be chains, we define the *support* of $C_1 \times C_2$ to be the set: $\operatorname{supp}(C_1 \times C_2) = \{[(x,y),(u,v)] | [x,u] \in C_1, [y,v] \in C_2\}$. A *valuation* of a chain C by a set R is a mapping $\varphi : \operatorname{supp} C \to R$. Let φ be a valuation of C, the *induced valuation* $\varphi \times \varphi : \operatorname{supp}(C_1 \times C_2) \to R \times R$ is the mapping $\varphi \times \varphi([(x,y),(u,v)]) = (\varphi[x,u], \varphi[y,v])$. The natural valuation of L^d obtained from φ is denoted by φ^d.

Let C be a chain and let φ be a valuation of C. We construct a lattice $\varphi^* C$ as follows: $\varphi^* C$ has the underlying set

$$(C \times C) \cup \{u_\alpha | \alpha \in \operatorname{supp}(C \times C), \varphi \times \varphi(\alpha) \in \Delta\},$$

where Δ is the diagonal of $R \times R$ and, besides the inherited order relations of $C \times C$, we define $x \prec u_\alpha \prec y$ for each $\alpha = [x,y]$.

Let $D = C^d \oplus C$ be a chain with valuation φ. We call the sets $\Delta(\varphi^* D) = \{x(r,r) | r \in D\}$ and $\Delta^+(\varphi^* D) = \{x(r,r) | r \in C\}$, the *diagonal* and *upper diagonal* of $\varphi^* D$ respectively.

The elements of ω will be named by $0, 1, 2, \ldots$ in the usual order. The element $x \in L_\alpha$ in $\sum(L_\alpha | \alpha < \gamma)$ will be written as x_α and the corresponding element of $x \in L$ in L^d will be denoted by x^d. We shall reserve the letters x and y for the labelling of the elements of $\varphi^* C$ in the following manners: Each $(.,\ .) \in C \times C$, we label by $x(.,\ .)$ and each $u_\alpha \prec x(.,\ .)$, we label by $y(.,\ .)$. An appropriate subscript will be attached to x and y whenever it is necessary.

Let L be a complete lattice. For a, $b \in L$, let $\Theta^*(a,b)$ be the principal complete congruence of L collapsing a and b. For $\Theta \in \operatorname{Com} L$, we have $\Theta = \bigvee(\Theta^*(a,b) | [a,b] \in I)$ where I ranges over all the closed interval $[a,b]$ collapsed by Θ. We say L is *con-discrete* if for each $\Theta \in \operatorname{Com} L$, the index set I can be restricted to the set of discrete intervals of L. Thus if C is a chain obtained from ω and $[n]$ by the operations $+$, $\oplus$, $()^d$, then C is condiscrete.

The Construction of L

Let K be a complete lattice with zero $\emptyset$ and unit $\imath$. Let $K^* = K - \{\emptyset\}$. Let the elements of K^* be listed in a fixed sequence $a_1, a_2, \ldots, a_n = \imath$. Let $K^{(1)} = K^* - \{\imath\}$ and $K^{(2)} = \{\{a,b\} | a, b \in K^* \text{ and } a, b \text{ are not comparable}\}$. We construct the following complete sublattices of L.

(i) The sublattice L_0.

Let C_0 be the chain $\omega + [1]$ and let the valuation $\varphi_0 : \operatorname{supp} C \to K$ be given by $\varphi_0[2k-1, 2k] = a_k$, for $k = 1, 2, \ldots, n-1$, and $\varphi_0[k, k+1] = \imath$, otherwise. Let $D_0 = C_0^d \oplus C_0$ and let $\psi_0 = \varphi_0^d \cup \varphi_0$ be the natural valuation of D_0. Let $L_0 = (\psi_0^* D_0) \cup \{z_0\}$ be given additional order relation $O_{\psi_0^* D_0} \prec z_0 \prec I_{\psi_0^* D_0}$. Then L_0 is a complete lattice. The elements of $\psi_0^* D_0$ will be labelled with a subscript 0.

(ii) The sublattice L_a, for $a \in K^{(1)}$.

Let the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ of $a_1, a_2, \ldots, a_n$ be a listing of $(a] - \{\emptyset\}$. Let C_a be the chain $\omega + [1]$ and let the valuation $\varphi_a : \operatorname{supp} C_a \to K$ be given by:

$$\varphi_a[x,y] = \begin{cases} a, & \text{if } [x,y] = [0,1] \text{ or } [r+1, r+2]; \\ a_{i_k}, & \text{if } [x,y] = [k, k+1] \text{ or} \\ & \quad [2k+r+1, 2k+r+2],\ k = 1, 2, \ldots, r; \\ \imath, & \text{otherwise.} \end{cases}$$

Let $D_a = C_a^d \oplus C_a$ and let $\psi_a = \varphi_a^d \cup \varphi_a$ be the natural valuation of D_a. Let $L_a = (\psi_a^* D_a) \cup \{w_a, z_a\}$ and let

$$O_{\psi_a^* D_a} \prec z_a \prec I_{\psi_a^* D_a} \quad \text{and } y((r+1)^d, (r+1)^d) \prec w_a \prec y((r+2), (r+2)).$$

Then L_a is a complete lattice. The elements of $\psi_a^* D_a$ will be labelled with a subscript a.

(iii) The sublattice L_α, for $\alpha \in K^{(2)}$, $\alpha = \{a, b\}$.

Let the subsequence $a_{i_1}, a_{i_2}, a_{i_3}$ of $a_1, a_2, \ldots, a_n$ be a listing of $\{a, b, a \vee b\}$. Let ω_1 and ω_2 be two copies of ω and let C_α be the chain $\omega_1 + \omega_2 + [1]$. Let the valuation φ_α be given as follows:

$$\varphi_\alpha[x,y] = \begin{cases} a \vee b, & \text{if } [x,y] = [0,1]; \\ a, & \text{if } [x,y] = [(k)_1, (k+1)_1] \text{ and } k \text{ is odd;} \\ b, & \text{if } [x,y] = [(k)_1, (k+1)_1] \text{ and } k \text{ is even;} \\ a_{i_k}, & \text{if } [x,y] = [(2k-1)_2, (2k)_2], \text{ for } k = 1, 2, 3; \\ \imath, & \text{otherwise.} \end{cases}$$

Let $D_\alpha = C_\alpha^d \oplus C_\alpha$ and let $\psi_\alpha = \varphi_\alpha^d \cup \varphi_\alpha$. Let $L_\alpha = (\psi_\alpha^* D_\alpha) \cup \cup\{w_\alpha, z_\alpha\}$ be given the additional order relations:

$$O_{\psi_\alpha^* D_\alpha} \prec z_\alpha \prec I_{\psi_\alpha^* D_\alpha} \quad \text{and } x(0_2^d, 0_2^d) \prec w_\alpha \prec x(0_2, 0_2).$$

Then L_α is a complete lattice. The elements of $\psi_\alpha^* D_\alpha$ will be labelled with a subscript α.

A sketch of the lattice L_0, L_a, and L_α are given in Figure 1(a), 1(b), and 1(c), respectively.

Let $L' = L_0 \cup \bigcup \left(L_b | b \in K^{(1)} \right) \cup \bigcup \left(L_\alpha | \alpha \in K^{(2)} \right)$. We identify all the zeros of L_0, L_b, L_α and all the units of L_0, L_b, L_α. Furthermore, let the following additional order relations be defined on L':

(A.i.) For each L_b, $b \in K^{(1)}$, $a_j = a_{i_k} \in (b] - \{\emptyset\}$, $k = 1, 2, \ldots, r$,
$x_0(2j-1, 2j-1) \prec x_b(2k+r+1, 2k+r+1)$,
$x_0(2j, 2j) \prec x_b(2k+r+2, 2k+r+2)$,
$x_0(2n-1, 2n-1) \prec x_b(3r+3, 3r+3)$,
$x_0(2n, 2n) \prec x_b(3r+4, 3r+4)$.

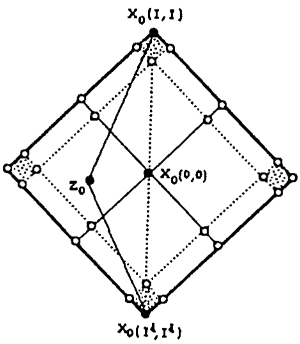

Fig. 1(a)

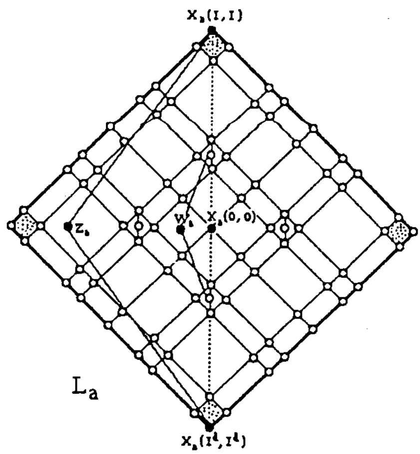

Fig. 1(b)

(A.ii.) For each L_α, $\alpha = \{b, c\} \in K^{(2)}$, $a_j = a_{i_k} \in \{b, c, b \vee c\}$, $k = 1, 2, 3$,
$x_0(2j-1, 2j-1) \prec x_\alpha((2k-1)_2, (2k-1)_2)$,
$x_0(2j, 2j) \prec x_\alpha((2k)_2, (2k)_2)$.

Let L be the resulting poset obtained from L'. For a subset S of L, we write

$$S = S_0 \cup \bigcup \left(S_b | b \in K^{(1)} \right) \cup \bigcup \left(S_\alpha | \alpha \in K^{(2)} \right),$$

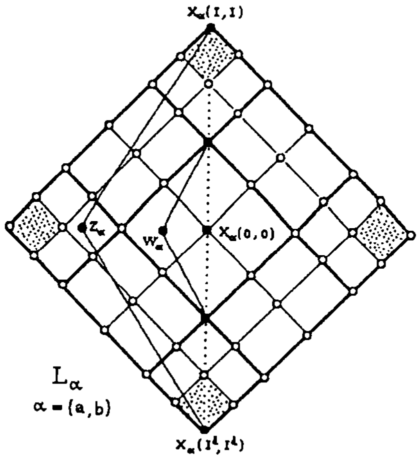

Fig. 1(c)

where $S_0 = S \cap L_0$, $S_b = S \cap L_b$ and $S_\alpha = S \cap L_\alpha$. By observing that the additional order relations (A.i) and (A.ii) are given along the upper diagonals of L_0, L_b and L_α. We have

(B.i) For each S_i, $i \in K^{(1)} \cup K^{(2)}$, there is a largest element $p_0 \in \Delta(\psi_0^* D_0)$ such that $p_0 \leq \wedge_i S_i \in L_i$. Hence $\wedge S_i$ in L is equal to $\wedge_i S_i$ in L_i. Obviously $\wedge S_0$ is equal to $\wedge_0 S_0$ in L_0.

(B.ii) There is a least element $q_i \in \Delta^+(\psi_i^* D_i)$ such that $q_i \geq \vee_0 S_0 \in$ $\in L_0$. Thus $\vee S_0$ in L is equal to $\vee_0 S_0$ in L_0. Clearly $\vee S_i$ in L is equal to $\vee_i S_i$ in L_i for $i \in K^{(1)} \cup K^{(2)}$.

CLAIM 1. *L is a complete lattice.*

PROOF. In view of (B.i) and (B.ii), we need only verify that any two elements of L have a join and a meet. Let r, $s \in L$; up to symmetry, we have the following possibilities:

(i) r, $s \in L_0$ (or r, $s \in L_i$, $i \in K^{(1)} \cup K^{(2)}$)

In this case, the join and the meett of r and s are respectively the join and meet that already exist in L_0 (or L_i).

(ii) $r \in L_i$, $s \in L_j$, $i, j \in K^{(1)} \cup K^{(2)}$, $i \neq j$.

In this case, $r \vee s$ is always the unit of L and $r \wedge s = p_0^1 \wedge_0 p_0^2$, where $p_0^1 \leq r$ and $p_0^2 \leq s$ are as defined in (B.i).

(iii) $r \in L_0$, $s \in L_i$.

Let $P_0 \leq s$ and $q_i \geq r$ be as defined in (B.i) and (B.ii) respectively. Then $r \wedge s = p_0 \wedge_0 r$ and $r \vee s = q_i \vee_i s$.

CLAIM 2. $\mathrm{Com}\, L \cong K$.

PROOF. It is not difficult to see that L_0 and all L_i, $i \in K^{(1)} \cup \cup K^{(2)}$ are condiscrete. Indeed, $\mathrm{Com}\, L_i$, $i = 0$ or $i \in K^{(1)} \cup K^{(2)}$, is generated by those intervals in $[\Delta^+(\psi_i^* D_i)]^2 \cap \mathrm{supp}(D_i \times D_i)$. For $\Theta \in \mathrm{Com}\, L$, if there exist r and s such that $r \in L_i$, $s \in L_j$, $L_i \neq L_j$, and $r \equiv s(\Theta)$, then $r \wedge s \equiv r \vee s(\Theta)$. This implies that Θ must collapse some interval $[x, y]$ having value $(1, 1)$ and $\Theta = 1$. Thus, if $\Theta \neq 1$, every congruence class of Θ must be contained in some L_i. By (A.i) and (A.ii), we have that every $\Theta \in \mathrm{Com}\, L$ is generated by those intervals in $[\Delta^+(\psi_0^* D_0)]^2 \cap \mathrm{supp}(D_0 \times D_0)$. For each $a \in K^*$, and each $i = 0$ or $i \in K^{(1)} \cup K^{(2)}$, it is not difficult to see that there is an $\Theta_i(a) \in \mathrm{Com}\, L_i$ such that:

$[x, y] \in [\Delta^+(\psi_i^* D_i)]^2 \cap \mathrm{supp}(D_i \times D_i)$ and $x \equiv y$ $(\Theta_i(a))$ if and only if $\psi_i \times \psi_i[x, y] = (b, b)$, $b \leq a$.

Let $\Theta(a) = \cup(\Theta_i(a) | i = 0$, or $i \in K^{(1)} \cup K^{(2)})$, then $\Theta(a) \in \mathrm{Com}\, L$. The substitution property can be verified by using the formula for the join and the meet of elements of L as stated in claim 1. For each $\Theta \in \mathrm{Com}\, L$, let

$$\pi(\Theta) = \{a | a \in K, [x, y] \in [\Delta^+(\psi_0^* D_0)]^2, x \equiv y(\Theta), \psi_0 \times \psi_0[x, y] = (a, a)\}.$$

If $a, b \in \pi(\Theta)$, then $a \vee b \in \pi(\Theta)$ by $L_{\{a,b\}}$. If $b \leq a$ and $a \in \pi(\Theta)$, then $b \in \pi(\Theta)$ by L_a. Clearly, we have $\pi(\Theta(a)) = (a]$. Thus the mapping $\Theta \to \pi(\Theta)$ is an isomorphism of $\mathrm{Com}\, L$ with the principal ideals of K. Hence $\mathrm{Com}\, L \cong K$.

Thus, we have proved:

THEOREM. *Every finite lattice is the complete congruence lattice of a complete lattice.*

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NONLOCAL AND NONLINEAR FIRST BOUNDARY VALUE PROBLEM FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

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We shall consider the problem of the form

$$(1) \qquad Q(u) := \sum_{i,j=1}^{n} a_{ij}(x,u,\partial u)\partial_i\partial_j u + b(x,u,\partial u) = 0 \quad \text{in} \quad \Omega$$

$$(2) \qquad u(x) = h(x, u(\Phi(x))), \quad x \in \partial\Omega$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain,

a) $\Phi : \partial\Omega \to \overline{\Omega}$ is a continuous mapping, $h : \partial\Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function with the property $\sup|\partial_2 h| < 1$.

We shall prove existence and uniqueness of the solution of problem (1), (2).

Linear elliptic equations with nonlocal boundary conditions have been considered firstly by T. CARLEMAN [1] and then by several authors, e.g. F. E. BROWDER [2], A. V. SAMARSKIJ and A. V. BITSADZE [3], [4], A. L. SKUBACHEVSKIJ [5]. Nonlinear elliptic equation with nonlocal boundary conditions have been studied by L. SIMON [6], [7]. Assume that Q satisfies the conditions of the following comparison principle of [8] (see Theorem 10.1.):

THEOREM 1. *Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Qu \geq Qv$ in Ω, $u \leq v$ on $\partial\Omega$, where*

(i) the operator Q is elliptic;

(ii) the coefficients $a_{ij}(x,z,p)$ are independent of z;

(iii) the coefficient $b(x,z,p)$ is nonincreasing in z for each $(x,p) \in \in \Omega \times \mathbf{R}^n$;

(iv) the coefficients a_{ij}, b are continuously differentiable with respect to the variable p in $\Omega \times \mathbf{R} \times \mathbf{R}^n$. Then it follows that $u \leq v$ in Ω.

In [8] there are formulated conditions such that the Dirichlet problem $Q(u) = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for any $\varphi \in C(\partial\Omega)$ (see theorem 15.18 of [8]).

THEOREM 2. *Assume that conditions* a) *and* (i)–(iv) *are fulfilled with hypothesis of theorem* 15.18 *of* [8]. *Then there exists a unique solution of the problem* (1), (2).

PROOF. Denote by $G(\varphi)$ the solution u of the Dirichlet problem $Q(u) = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ for all $\varphi \in C(\partial\Omega)$. Furthermore, define operator B by

$$[B(\varphi)](x) := h[x, (G(\varphi))(\Phi(x))],$$

then $B : C(\partial\Omega) \to C(\partial\Omega)$ is a nonlinear mapping, $C(\partial\Omega)$ is a complete metric space with the metric

$$\varrho(\varphi_1, \varphi_2) := \sup_{\partial\Omega} |\varphi_1 - \varphi_2|.$$

If $\varphi \in C(\partial\Omega)$ is a fixed point of B, i.e. $B(\varphi) = \varphi$, then $u = G(\varphi)$ is a solution of the problem (1), (2).

Therefore to prove existence of (1), (2) it is sufficient to show that B has a fixed point. This will be a consequence of Banach's fixed point theorem. Since B has exactly one fixed point, then we shall obtain that the solution of (1), (2) is unique.

Now we show that $B : C(\partial\Omega) \to C(\partial\Omega)$ is a contraction in $C(\partial\Omega)$, i.e. for any φ_1, $\varphi_2 \in C(\partial\Omega)$

$$\varrho(B(\varphi_1), B(\varphi_2)) = \sup |B(\varphi_1) - B(\varphi_2)| \leq q \cdot \varrho(\varphi_1, \varphi_2),$$

where q is a nonnegative number < 1. By

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) = h[x, (G(\varphi_1))(\Phi(x))] - h[x, (G(\varphi_2))(\Phi(x))],$$

further by using Lagrange's mean value theorem and the notation $b_j :=$ $:= (G(\varphi_j))(\Phi(x))$, $(j = 1, 2)$ we find that

$$[B(\varphi_1)](x) - [B(\varphi_2)](x) = \partial_2 h(x, b_2 + \tau b_1)(b_1 - b_2) =$$
$$\partial_2 h(x, b_2 + \tau b_1)[G(\varphi_1))(\Phi(x)) - (G(\varphi_2))(\Phi(x))].$$

Consequently

$$|[B(\varphi_1)](x) - [B(\varphi_2)](x)| \leq \sup |\partial_2 h| \, |G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))|.$$

If we can prove that

$$|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| \leq \varrho(\varphi_1, \varphi_2)$$

and $q := \sup|\partial_2 h| < 1$ is satisfied, then we shall have

$$\varrho(B(\varphi_1), B(\varphi_2)) \leq q\varrho(\varphi_1, \varphi_2).$$

This means that B is a contraction in $C(\partial\Omega)$.

By using the comparison principle we want to prove that $\forall y \in \Omega$

$$|[G(\varphi_1)](y) - [G(\varphi_2)](y)| \leq \sup_{\partial\Omega}|\varphi_1 - \varphi_2|. \tag{3}$$

For $u_1 := G(\varphi_1)$, $u_2 := G(\varphi_2)$ we have $Q(u_1) = Q(u_2) = 0$ in Ω and

$$u_1\Big|_{\partial\Omega} = \varphi_1, \qquad u_2\Big|_{\partial\Omega} = \varphi_2.$$

By using notation $\varepsilon := \sup_{\partial\Omega}|\varphi_1 - \varphi_2|$ we may write $\varphi_1 - \varepsilon \leq \varphi_2 \leq$ $\leq \varphi_1 + \varepsilon$.

Consider the functions $u := u_2$, $v := u_1 + \varepsilon$ then $Q(u) = Q(u_2) = 0$ and

$$Q(v) = Q(u_1 + \varepsilon) =$$
$$= \sum_{i,j=1}^{n} a_{ij}(x, \partial(u_1 + \varepsilon)) \cdot \partial_i(u_1 + \varepsilon)\partial_j(u_1 + \varepsilon) + b(x, u_1 + \varepsilon, \partial(u_1 + \varepsilon)) \leq$$
$$\leq \sum_{i,j=1}^{n} a_{ij}(x, \partial u_1)(\partial_i u_1)(\partial_j u_1) + b(x, u_1, \partial u_1) = Q(u_1) = 0.$$

So we have

$$Q(v) = Q(u_1 + \varepsilon) \leq 0 = Q(u_2) = Q(u) \qquad \text{in } \Omega$$

i.e. $Q(v) \leq Q(u)$ in Ω, and $v = u_1 + \varepsilon = \varphi_1 + \varepsilon \geq \varphi_2 = u_2 = u$ on $\partial\Omega$. It means that conditions of comparison principle are fulfilled, this implies that $u \leq v$ in Ω, i.e. $\forall y \in \Omega$ $u_2(y) \leq u_1(y) + \varepsilon$.

Similarly can be proved that $\forall y \in \Omega \quad u_1(y) - \varepsilon \leq u_2(y)$, thus

$$|u_2(y) - u_1(y)| \leq \varepsilon = \sup_{\partial\Omega}|\varphi_1 - \varphi_2|$$

i.e. we have shown (3) which completes the proof of Theorem 2.

THEOREM 3. *Assume that Q satisfies the conditions of theorem 15.18 of* [8] *and $\Phi : \partial\Omega \to \partial\Omega$ is a continuous mapping, h is a continuous function with property* $\sup|\partial_2 h| < 1$, *then there exists a solution of the problem* (1), (2).

PROOF. The proof of Theorem 3. is similar to the proof of Theorem 2. except of the proof of (3).

If $\Phi : \partial\Omega \to \partial\Omega$, then $x \in \partial\Omega$ implies that $\Phi(x) \in \partial\Omega$, and so $G(\varphi_1)(\Phi(x)) = \varphi_1(\Phi(x))$, $G(\varphi_2)(\Phi(x)) = \varphi_2(\Phi(x))$ and thus

$$|G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| = |\varphi_1(\Phi(x)| - \varphi_2(\Phi(x))| \leq \varrho(\varphi_1, \varphi_2).$$

So the proof can be continued in the same way.

REMARK. If condition $\sup|\partial_2 h| < 1$ is not fulfilled then the nonlocal boundary value problem may have no solution or it may have several solutions.

EXAMPLE 1. $n = 1$

(1) $u'' = 1$ in $(0, 1)$

(2) $\begin{cases} u(0) = a_0 u(\alpha) + b_0 \\ u(1) = a_1 u(\beta) + b_1 \end{cases}$

where a_j, b_j are constants, α, $\beta \in [0, 1]$, $(j = 1, 0)$.

We know that all solutions of equation (1) can be given by $u(x) = cx + d$. u satisfies (1) iff c, d satisfy the following system of equations:

$$(a_0\alpha)c + (a_0 - 1)d = -b_0, \qquad (a_1\beta - 1)c + (a_1 - 1)d = -b_1.$$

We show that if the conditions $|a_0| < 1$, $|a_1| < 1$ are not fulfilled ($\sup|\partial_2 h| < 1$ is not satisfied), then α, $\beta \in [0, 1]$ can be chosen such that the determinant of the above system will be 0. In this case the system may have no solution or it may have infinite number of solutions.

Special cases:

(i) $a_j = 1$, $(j = 0, 1)$,

$$h_0(z) = z + b_0,\ h_0'(z) = 1,\ h_1(z) = z + b_1,\ h_1'(z) = 1$$

and the determinant is 0 for any α, $\beta \in [0, 1]$.

(ii) Similarly can be shown that if $a_j > 1$ then α, β can be chosen such that the determinant is 0.

EXAMPLE 2. $n = 2$

(1) $\Delta u + cu = 0$ in $B_{1,2}$, $c \leq 0$ where $B_{1,2} := \{x \in \mathbf{R}^2 : 1 < |x| < 2\}$ and the boundary condition on $S_j := \{x \in \mathbf{R}^2 : |x| = j\}$ $(j = 1, 2)$ is:

(2) $\begin{cases} u(x) = \beta_1 y(\gamma_1 x) + \delta_1, & x \in S_1 \\ u(x) = \beta_2 u(\gamma_2 x) + \delta_2, & x \in S_2 \end{cases}$

where $1 \leq \gamma_1 \leq 2$, $\frac{1}{2} \leq \gamma_2 \leq 1$.

In this example also we shall consider cases when $|\beta_j| \geq 1$ $(j = 1, 2)$, i.e. when the condition $\sup|\partial_2 h| < 1$ is not fulfilled. Introduce polar coordinates r, φ in $\mathbf{R}^2$: $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$.

So we get new unknown function

$$U(r,\varphi) := u(x_1,x_2) = u(r\cos\varphi, r\sin\varphi).$$

Define function V by

$$V(r) := \int_0^{2\pi} U(r,\varphi)d\varphi.$$

Assuming that V is a solution of (1), (2) we have

(3) $V'' + \frac{1}{r}V' + cV = 0.$

Denote a fundamental system of this equation by V_1, V_2, then all solutions of (3) can be written in the form

$$V = d_1V_1 + d_2V_2, \qquad d_1,\ d_2 \quad \text{are constants.}$$

From the boundary condition (2) we obtain conditions

(4) $\begin{cases} V(1) = \beta_1 V(\gamma_1) + \delta_1 \\ V(2) = \beta_2 V(\gamma_2) + \delta_2. \end{cases}$

Thus we have shown that if u is a solution of (1), (2) then V satisfies (3), (4), where (4) is system of linear algebraic equations for d_1, d_2.

If the conditions $|\beta_1| < 1$, $|\beta_2| < 1$ are not satisfied, it is not difficult to choose constants β_j, γ_j such that the determinant of the above mentioned system is 0. In this case problem (3), (4) may have no solution or it may have several solutions. Then problem (1), (2) has no solution, respectively it has several solutions.

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BIRKHOFF INTERPOLATION BY QUINTIC SPLINES

By

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1. Introduction

Let

$$\Delta : 0 = x_0 < x_1 < \ldots < x_n = 1$$

be a partition of the interval $I = [0,1]$, $h_k = x_{k+1} - x_k$ and let $f(x) \in C^3(I)$. We denote by $S_{n,5}^{(2)}$ the class of quintic splines $s(x)$ such that

(i) $s(x) \in C^2(I)$,

(ii) $s(x)$ is a quintic polynomial in each interval $[x_k, x_{k+1}]$. J. PRASAD and A. K. VARMA [1] have obtained error bounds for this class of splines when it interpolates a special lacunary data namely, the following:

$$s_n(x_k) = f(x_k),\ k = O(1)n;$$

$$s_n^{(p)}\left(\frac{x_k + x_{k+1}}{2}\right) = f^{(p)}\left(\frac{x_k + x_{k+1}}{2}\right),\ k = O(1)n-1,\ p = 0,3;$$

$$s_n'(x_0) = f_0'(x_0),\ s_n'(x_n) = f_n'(x_n).$$

Recently GUO ZHU-RUI [2] has studied cubic splines of class C^1 which interpolate a given function $f(x)$ at the knots and whose second derivative interpolate $f''(x)$ midway between the knots and with only one end condition. Such spline functions may be used to obtain approximate solution of initial value problems in non-linear differential equations. Our object here is to consider a related problem by the splines of class $S_{n,5}^{(2)}$ and obtain error bounds when they are interpolants of a certain function $f(x)$. We shall return elsewhere with an application of our method in approximating the solution of Cauchy's initial value problems.

We now state the main results to be proved.

THEOREM 1. *Given* Δ *and the numbers* f_k, f'_k, $k = O(1)n$, $f'''_{k+\frac{1}{2}}$, $k = O(1)n-1$, f''_0, f'''_0, *there exist a unique* $s_\Delta(x) \in S^{(2)}_{n,5}$ *such that:*

$$\begin{aligned} s_\Delta(x_k) &= f_k; \\ s'_\Delta(x_k) &= f'_k, \quad k = O(1)n; \\ s'''_\Delta(x_{k+\frac{1}{2}}) &= f'''_{k+\frac{1}{2}}, \quad k = O(1)n-1; \\ s''_\Delta(x_0) &= f''_0 \quad or \quad s'''_\Delta(x_0) = f'''_0. \end{aligned} \tag{1.1}$$

Here $x_{k+\frac{1}{2}} := \frac{x_k + x_{k+1}}{2}$.

THEOREM 2. *Let* $f \in C^\ell(I)$, $\ell = 5, 6$. *Then for the unique spline* $s_\Delta(x)$ *of Theorem 1 associated with* f, *we have*

$$\|s^{(5)}_\Delta(x) - f^{(5)}(x)\| \le \begin{cases} O(\omega_5(H)) & \text{if } f \in C^5(I) \\ \frac{3H}{2}\|f^{(6)}\| + \\ \qquad +O(H\omega_6(H)) & \text{if } f \in C^6(I), \end{cases} \tag{1.2}$$

and if $\frac{\max h_k}{\min h_k} \le \lambda < \infty$, *then*

$$\|s^{(q)}_\Delta(x) - f^{(q)}(x)\| \le \begin{cases} O(H^{4-q}\omega_5(H)) & \text{if } f \in C^5(I) \\ \frac{3}{2}H^{6-q}\|f^{(6)}\| + \\ \qquad +O(H^{5-q}\omega_6(H)) & \text{if } f \in C^6(I), \end{cases} \tag{1.3}$$

$q = O(1)4$. *Here* $\omega_\ell(\cdot)$ *denotes the modulus of continuity of* $f^{(\ell)}$, $H = \max(x_{k+1} - x_k)$, $k = O(1)n-1$ *and* $\|\cdot\|$ *is the uniform norm.*

2. Proof of Theorem 1

We shall prove the theorem with the initial condition $s''_\Delta(x_0) = f''_0$ only. For the condition $s'''_\Delta(x_0) = f'''_0$ a similar proof follows.

If we set

$$s_\Delta(x) = \begin{cases} s_0(x), & \text{when } x_0 \le x \le x_1 \\ s_k(x), & \text{when } x_k \le x \le x_{k+1},\ k = 1(1)n-1, \end{cases} \tag{2.1}$$

then

$$s_0(x) = f_0 + (x - x_0)f'_0 + \frac{(x-x_0)^2}{2!}f''_0 + \frac{(x-x_0)^3}{3!}a_{0,3} + \tag{2.2}$$

$$+\frac{(x-x_0)^4}{4!}a_{0,4}+\frac{(x-x_0)^5}{5!}a_{0,5},$$

$$s_k(x)=f_k+(x-x_k)f'_k+\frac{(x-x_k)^2}{2!}a_{k,2}+\frac{(x-x_k)^3}{3!}a_{k,3}+ \tag{2.3}$$
$$+\frac{(x-x_k)^4}{4!}a_{k,4}+\frac{(x-x_k)^5}{5!}a_{k,5},$$

where the coefficients are determined by the interpolatory conditions (1.1) and the continuity requirement that $s_\Delta(x)\in C^2(I)$. If we apply these conditions, we have the following three sets of equations:

$$\begin{aligned} f_1&=f_0+h_0f'_0+\frac{h_0^2}{2!}f''_0+\frac{h_0^3}{3!}a_{0,3}+\frac{h_0^4}{4!}a_{0,4}+\frac{h_0^5}{5!}a_{0,5},\\ f'_1&=f'_0+h_0f''_0+\frac{h_0^2}{2!}a_{0,3}+\frac{h_0^3}{3!}a_{0,4}+\frac{h_0^4}{4!}a_{0,5},\\ f'''_{\frac12}&=a_{0,3}+\frac{h_0}{2}a_{0,4}+\frac{h_0^2}{8}a_{0,5}; \end{aligned} \tag{2.4}$$

$$\begin{aligned} f_{k+1}&=f_k+h_kf'_k+\frac{h_k^2}{2!}a_{k,2}+\frac{h_k^3}{3!}a_{k,3}+\frac{h_k^4}{4!}a_{k,4}+\frac{h_k^5}{5!}a_{k,5},\\ f'_{k+1}&=f'_k+h_ka_{k,2}+\frac{h_k^2}{2!}a_{k,3}+\frac{h_k^3}{3!}a_{k,4}+\frac{h_k^4}{4!}a_{k,5},\\ f'''_{k+\frac12}&=a_{k,3}+\frac{h_k}{2}a_{k,4}+\frac{h_k^2}{8}a_{k,5}; \end{aligned} \tag{2.5}$$

$$a_{k+1,2}=a_{k,2}+h_ka_{k,3}+\frac{h_k^2}{2!}a_{k,4}+\frac{h_k^3}{3!}a_{k,5},\quad k=1(1)n-2, \tag{2.6}$$
$$a_{1,2}=f''_0+h_0a_{0,3}+\frac{h_0^2}{2!}a_{0,4}+\frac{h_0^3}{3!}a_{0,5}.$$

Solving (2.4) we get

$$a_{0,3}=\frac{6}{h_0^2}(f'_1-f'_0-h_0f''_0)-2f'''_{\frac12}, \tag{2.7}$$

$$a_{0,4}=\frac{120}{h_0^4}\left(f_1-f_0-h_0f'_0-\frac{h_0^2}{2}f''_0\right)- \tag{2.8}$$
$$-\frac{72}{h_0^3}(f'_1-f'_0-h_0f''_0)+\frac{16}{h_0}f'''_{\frac12},$$

$$a_{0,5}=-\frac{480}{h_0^5}\left(f_1-f_0-h_0f'_0-\frac{h_0^2}{2}f''_0\right)+ \tag{2.9}$$

$$+\frac{240}{h_0^4}(f_1'-f_0'-h_0f_0'')-\frac{40}{h_0^2}f_{\frac{1}{2}}'''.$$

From (2.5) we have

$$a_{k,3}=-\frac{6}{h_k}a_{k,2}+\frac{6}{h_k^2}(f_{k+1}'-f_k')-2f_{k+\frac{1}{2}}''', \tag{2.10}$$

$$a_{k,4}=\frac{12}{h_k^2}a_{k,2}+\frac{120}{h_k^4}(f_{k+1}-f_k-h_kf_k')- \tag{2.11}$$

$$-\frac{72}{h_k^3}(f_{k+1}'-f_k')+\frac{16}{h_k}f_{k+\frac{1}{2}}''',$$

$$a_{k,5}=-\frac{480}{h_k^5}(f_{k+1}-f_k-h_kf_k')+ \tag{2.12}$$

$$+\frac{240}{h_k^4}(f_{k+1}'-f_k')-\frac{40}{h_k^2}f_{k+\frac{1}{2}}'''.$$

From (2.6), (2.7), (2.8) and (2.9) we get

$$a_{1,2}=f_0''-\frac{20}{h_0^2}\left(f_1-f_0-h_0f_0'-\frac{h_0^2}{2}f_0''\right)+ \tag{2.13}$$

$$+\frac{10}{h_0}(f_1'-f_0'-h_0f_0'')-\frac{2}{3}h_0f_{\frac{1}{2}}''',$$

$$a_{k+1,2}-a_{k,2}=\frac{-20}{h_k^2}(f_{k+1}-f_k-h_kf_k')+ \tag{2.14}$$

$$+\tfrac{10}{h_k}(f_{k+1}'-f_k')-\tfrac{2}{3}h_kf_{k+\frac{1}{2}}''' \quad , \quad k=1(1)n+2.,$$

The coefficient matrix of the system of equations (2.13) and (2.14) in the unknowns $a_{k,2}, k=1(1)n-1$, is non-singular matrix and hence they are uniquely determined and so are, therefore, the coefficients $a_{k,3}$ and $a_{k,4}$, $a_{k,5}$.

3. Auxiliary Lemmas

In this section we prove three lemmas which will be used in the next section to obtain the proof of Theorem 2.

LEMMA 1. Let $A_{k,2} := a_{k,2} - f''_k$. Then we have for $k = 1(1)n-1$,

$$|A_{k,2}| = \begin{cases} O\Big(\sum\limits_{\nu=0}^{k-1} h_\nu^3 \omega_5(h_\nu)\Big), & \text{if } f \in C^5(I) \\ O\Big(\sum\limits_{\nu=0}^{k-1} h_\nu^4 \omega_6(h_\nu)\Big), & \text{if } f \in C^6(I). \end{cases}$$

PROOF. From (2.14) we have

$$A_{k+1,2} - A_{k,2} := (a_{k+1,2} - f''_{k+1}) - (a_{k,2} - f''_k) = \tag{3.1}$$

$$= -\frac{20}{h_k^2}(f_{k+1} - f_k - h_k f'_k) + \frac{10}{h_k}(f'_{k+1} - f'_k) - (f''_{k+1} - f''_k) - \frac{2}{3} h_k f'''_{k+\frac{1}{2}} := \alpha_k,$$

$$k = 1(1)n-2.$$

If $f \in C^5(I)$, then by Taylor's formula

$$\alpha_k = -\frac{20}{h_k^2}\left[\frac{h_k^2}{2} f''_k + \frac{h_k^3}{6} f'''_k + \frac{h_k^4}{24} f_k^{(4)} + \frac{h_k^5}{120} f_k^{(5)} + O(h_k^{(5)} \omega_5(h_k))\right] +$$

$$+ \frac{10}{h_k}\left[h_k f''_k + \frac{h_k^2}{2} f'''_k + \frac{h_k^3}{6} f_k^{(4)} + \frac{h_k^4}{24} f_k^{(5)} + O(h_k^4 \omega_5(h_k))\right] -$$

$$- \left[h_k f'''_k + \frac{h_k^2}{2} f_k^{(4)} + \frac{h_k^3}{6} f_k^{(5)} + O(h_k^3 \omega_5(h_k))\right] -$$

$$- \frac{2}{3} h_k \left[f'''_k + \frac{h_k}{2} f_k^{(4)} + \frac{h_k^2}{8} f_k^{(5)} + O(h_k^2 \omega_5(h_k))\right] =$$

$$= O(h_k^3 \omega_5(h_k)). \tag{3.2}$$

Similarly if $f \in C^6(I)$, then

$$\alpha_k = O(h_k^4 \omega_6(h_k)). \tag{3.3}$$

Also from (2.13) we have

$$|A_{1,2}| := |a_{1,2} - f''_1| = \begin{cases} O(h_0^3 \omega_5(h_0)), & \text{if } \ f \in C^5(I) \\ O(h_0^4 \omega_6(h_0)), & \text{if } \ f \in C^6(I). \end{cases} \tag{3.4}$$

So from (3.1) using (3.2), (3.3) and (3.4) we have

$$|A_{k,2}| = \begin{cases} O\Big(\sum\limits_{\nu=0}^{k-1} h_\nu^3 \omega_5(h_\nu)\Big), & \text{if} \quad f \in C^5(I) \\ O\Big(\sum\limits_{\nu=0}^{k-1} h_\nu^4 \omega_6(h_\nu)\Big), & \text{if} \quad f \in C^6(I), \ k = 1(1)n-1. \end{cases}$$

This proves the assertion of the lemma.

Using Lemma 1 we prove

LEMMA 2. Let

$$A_{k,4} := a_{k,4} - f^{(4)},$$

and

$$\frac{\max h_k}{\min h_k} \le \lambda < \infty, \ H = \max_{0 \le k \le n-1} h_k.$$

Then we have for $k = O(1)n-1$,

$$|A_{k,4}| \le \begin{cases} O(\omega_5(H)), & \text{if} \quad f \in C^5(I) \\ \frac{3}{2} H^2 \|f^{(6)}\| + O(H\omega_6(H)), & \text{if} \quad f \in C^6(I). \end{cases}$$

PROOF. From (2.8) and (2.11), observing that $A_{0,2} = 0$, we have

$$\begin{aligned} A_{k,4} : &= a_{k,4} - f_k^{(4)}, \qquad k = O(1)n-1 \\ &= \frac{12}{h_k^2} A_{k,2} + \beta_k, \text{where} \end{aligned} \tag{3.5}$$

$$\beta_k := \frac{120}{h_k^4}\Big(f_{k+1} - f_k - h_k f_k' - \frac{h_k^2}{2} f_k''\Big) - \frac{72}{h_k^3}\big(f_{k+1}' - f_k' - h_k f_k''\big) + \\ + \frac{16}{h_k} f'''_{k+\frac{1}{2}} - f_k^{(4)}.$$

If $f \in C^5(I)$, then as in Lemma 1, by Taylor's formula

$$\beta_k = O(h_k \omega_5(h_k)), \tag{3.6}$$

and if $f \in C^6(I)$, then

$$\beta_k = -\frac{h_k^2}{10} f_k^{(6)} + O(h_k^2 \omega_6(h_k)). \tag{3.7}$$

Now using Lemma 1, we have for $k = O(1)n-1$,

$$|A_{k,4}| = \begin{cases} O\Big(\frac{1}{h_k^2}\sum_{\nu=0}^{k-1} h_\nu^3\omega_5(h_\nu)\Big) + O(h_k\omega_5(h_k)), & \text{if } f \in C^5(I) \\ O\Big(\frac{1}{h_k^2}\sum_{\nu=0}^{k-1} h^4\omega_6(h_\nu)\Big) - \frac{h_k^2}{10} f_k^{(6)} + O(h_k^2\omega_6(h_k)), & \text{if } f \in C^6(I). \end{cases}$$

The result obviously holds for $k = 0$. Hence if

$$\frac{\max h_k}{\min h_k} \le \lambda < \infty \quad \text{and} \quad H = \max_{0\le k\le n-1} h_k,$$

we have from (3.5)–(3.7)

$$|A_{k,4}| \le \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ \frac{3}{2}H^2\|f^{(6)}\| + O(H\omega_6(H)), & \text{if } f \in C^6(I) \end{cases}$$

$k = O(1)n-1$, which proves the theorem.

LEMMA 3. *Let $A_{k,5} := a_{k,5} - f_k^{(5)}$. Then we have for $k = O(1)n-1$,*

$$|A_{k,5}| \le \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ \frac{H}{2}\|f^{(6)}\| + O(H\omega_6(H)), & \text{if } f \in C^6(I). \end{cases}$$

PROOF. From (2.12) we have

$$\text{(3.8)}\quad A_{k,5} = -\frac{480}{h_k^5}(f_{k+1} - f_k - h_k f_k') + \frac{240}{h_k^4}(f_{k+1}' - f_k') - \frac{40}{h_k^2} f_{k+\frac{1}{2}}''' - f_k^{(5)}$$
$$:= \nu_k, \quad k = O(1)n-1.$$

If $f \in C^5(I)$, then as in Lemma 1, by Taylor's formula

$$\text{(3.9)}\qquad \nu_k = O(\omega_5(H))$$

and if $f \in C^6(I)$, then

$$\text{(3.10)}\qquad \nu_k = \frac{h_k}{2} f_k^{(6)} + O(h_k\omega_6(h_k)).$$

Hence from (3.8)–(3.10) we have

$$|A_{k,5}| \le \begin{cases} O(\omega_5(H)), & \text{if } f \in C^5(I) \\ \frac{H}{2}\|f^{(6)}\| + O(H\omega_6(H)), & \text{if } f \in C^6(I). \end{cases}$$

This proves the lemma.

4. Proof of Theorem 4

Let $x \in [x_k, x_{k+1}]$, $k = O(1)n-1$. From (2.2) and (2.3) we have

$$s_k^{(5)}(x) = a_{k,5}, \tag{4.1}$$

and

$$s_k^{(4)}(x) = a_{k,4} + (x - x_k)a_{k,5}. \tag{4.2}$$

Therefore

$$\begin{aligned}|s_k^{(5)}(x) - f^{(5)}(x)| =&|a_{k,5} - f_k^{(5)} + f_k^{(5)} - f^{(5)}(x)| \le \\ \le&|a_{k,5} - f_k^{(5)}| + |f_k^{(5)} - f^{(5)}(x)|.\end{aligned}$$

If $f \in C^5(I)$, then owing to Lemma 3 we obviously have

$$|s_k^{(5)}(x) - f^{(5)}(x)| = O(\omega_5(H)). \tag{4.3}$$

However, if $f \in C^6(I)$, then

$$\begin{aligned}|s_k^{(5)}(x) - f^{(5)}(x)| &= |a_{k,5} - f_k^{(5)} - (x - x_k)f^{(6)}(\xi_k)|, \quad x_k < \xi_k < x \\ &\le \frac{3H}{2}||f^{(6)}|| + O(H\omega_5(H)).\end{aligned} \tag{4.4}$$

Further if $f \in C^5(I)$, then from (4.2)

$$\begin{aligned}s_k^{(4)}(x) - f^{(4)}(x) &= (a_{k,4} - f_k^{(4)}) + (x - x_k)(a_{k,5} - f_k^{(5)}) - \\ &\quad - [f^{(4)}(x) - f_k^{(4)} - (x - x_k)f_k^{(5)}] \\ &= A_{k,4} + (x - x_k)A_{k,5} - (x - x_k)[f^{(5)}(\eta_k) - f_k^{(5)}],\end{aligned} \tag{4.5}$$

$x_k < \eta_k < x$. Thus

$$|s_k^{(4)}(x) - f^{(4)}(x)| \le |A_{k,4}| + H|A_{k,5}| + H\omega_5(H),$$

which on using Lemma 2 and Lemma 3 yields

$$|s_k^{(4)}(x) - f^{(4)}(x)| = O(\omega_5(H)) + O(H\omega_5(H)) = O(\omega_5(H)). \tag{4.6}$$

Now

$$\begin{aligned}|s_k'''(x) - f'''(x)| &= \left| \int_{x_{k+1/2}}^{x} [s_k^{(4)}(t) - f^{(4)}(t)] \mathrm{d}\, t \right| \le \\ &\le |x - x_{k+1/2}|\, |s_k^{(4)}(x) - f^{(4)}(x)| = O(H\omega_5(H)).\end{aligned} \tag{4.7}$$

Set $h(x) := s_k'(x) - f'(x)$. Then by (1.1)

$$h(x_k) = h(x_{k+1}) = 0,$$

and so by Rolle's Theorem, there exists a μ_k $(x_k < \mu_k < x_{k+1})$ such that

$$h'(\mu_k) = s_k''(\mu_k) - f''(\mu_k) = 0,$$

from which we obtain

$$|s_k''(x) - f''(x)| = \left| \int_{\mu_k}^{x} [s_k'''(t) - f'''(t)] \mathrm{d}\, t \right| \leq$$

$$\leq |x - \mu_k|\, |s_k'''(x) - f'''(x)| = O(H^2 \omega_5(H)). \tag{4.8}$$

Again

$$|s_k'(x) - f'(x)| = \left| \int_{x_k}^{x} [s_k''(t) - f''(t)] \mathrm{d}\, t \right| = O(H^3 \omega_5(H)), \tag{4.9}$$

and

$$|s_k(x) - f(x)| = \left| \int_{x_k}^{x} [s_k'(t) - f'(t)] \mathrm{d}\, t \right| = O(H^4 \omega_5(H)). \tag{4.10}$$

Thus (4.3) and (4.6)-(4.10) complete the proof of Theorem 2 when $f \in C^5(I)$.

Now let $f \in C^6(I)$. Then from (4.5)

$$s_k^{(4)}(x) - f^{(4)}(x) = A_{k,4} + (x - x_k) A_{k,5} - \frac{(x - x_k)^2}{2} f^{(6)}(\xi_k),\ x_k < \xi_k < x.$$

Using again, Lemma 2 and Lemma 3 we have

$$|s_k^{(4)}(x) - f^{(4)}(x)| \leq \frac{3H^2}{2} ||f^{(6)}|| + O(H \omega_6(H)). \tag{4.11}$$

From (4.11) on using the method of successive integration we at once have

$$|s_k^{(q)}(x) - f^{(q)}(x)| \leq \frac{3}{2} H^{6-q} ||f^{(6)}|| + O(H^{5-q} \omega_6(H)),\ q = 1(1)4. \tag{4.12}$$

Thus $(4,4)$ and (4.12) prove the theorem when $f \in C^6(I)$.

5. Concluding Remark

We find that the interpolation problem:

$$
(5.1)\qquad
\begin{aligned}
s_\Delta(x_k) &= f_k \\
s''_\Delta(x_k) &= f''_k, \qquad k = O(1)n \\
s'''_\Delta(x_{k+1/2}) &= f'''_{k+1/2}, \qquad k = O(1)n-1 \\
s'_\Delta(x_0) = f'_0 \quad &\text{or} \quad s'''_\Delta(x) = f'''_0
\end{aligned}
$$

has unique solution in the class $S^{(2)}_{n,5}$ and that it has similar error estimates as in Theorem 2. Since the proofs for the problem (5.1) can be carried on the same pattern as for the problem (1.1), we do not give the details here.

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A NEW APPROXIMATION FOR THE FOURIER TRANSFORM I.

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In [1], [2], [3] there are given approximative expressions for the Fourier transform of functions, using the values of there functions only in some discrete points. The method of K. BALÁZS is very nice and convenient also from point of view of numerical analysis. She gave estimate for the error, and this shows that her method is a good approximation.

In the present note we give a seemingly new approximation for the Fourier transform and give an estimate for the error.

Let f be an r times continuously differentiable function on $\mathbf{R}$ ($r \geq 1$ is integer) and suppose: $|f^{(i)}(t)| \leq k_1 e^{-k_2|t|}$, ($k_1$, $k_2 > 0$, $t \in \mathbf{R}$, $i = 0, 1, \ldots, r$).

Denote

$$F(f,\omega) := \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \mathrm{d}\, t$$

(the Fourier transform of f) and approximate this by the expression:

$$F_n(f,\omega) :=$$

$$= \frac{\pi}{n} \sum_{k=1}^{n} \cos^{-2}\left(\frac{2k-1-n}{2n}\pi\right) f\left(\operatorname{tg} \frac{2k-1-n}{2n}\pi\right) \exp\left\{-i\omega \operatorname{tg} \frac{2k-1-n}{2n}\pi\right\}$$

THEOREM. *Under the assumptions above we have*

$$(1) \qquad \Delta_n := F(f,\omega) - F_n(f,\omega) = \overline{\overline{o}}(n^{-r}) \qquad (n \to \infty)$$

uniformly in ω on compact intervals of $\mathbf{R}$.

PROOF. Introduce the notations:

$$g(t) := f(t) \cdot \exp\{-i\omega t\},$$

$$g^*(\varphi) := g(\operatorname{tg}\varphi).$$

It follow immediately from the assumptions on f that

$$g^*(\varphi) \in C^r_\pi(-\pi/2, \pi/2),$$

i.e. $g^*(\varphi)$ as the function of φ is r times continuously differentiable and

$$\frac{d^i}{d\varphi^i} g^*(-\pi/2+0) = \frac{d^i}{d\varphi^i} g^*(\pi/2-0)(=0) \qquad (i = 0, 1, \ldots, r).$$

Hence, for the Fourier coefficients of g we obtain easily (integrating by parts r times): [1)]

$$c_n(g^*) := \pi^{-1/2} \int_{-\pi/2}^{\pi/2} g^*(\varphi)\exp\{-i2n\varphi\}\mathrm{d}\,\varphi = \overline{\overline{o}}(|n|^{-r}). \tag{2}$$

On the other hand we have for $r > 1$ [2)]

$$\text{(3)}\quad F(f,\omega) - F_n(f,\omega) = \int_{-\pi/2}^{\pi/2} g^*(\varphi)\mathrm{d}\,\varphi - \frac{\pi}{n}\sum_{k=1}^{n} g^*\left(\frac{2k-1-n}{2n}\pi\right) =$$

$$= \pi^{1/2}c_0(g^*) - \left(\frac{\pi^{1/2}}{n}\right)\sum_{k=1}^{n}\lim_{m\to\infty}\sum_{j=-m}^{m} c_j \exp\left\{\frac{i2j\pi(2k-1-n)}{2n}\right\} =$$

$$= -(\pi^{1/2}/n)\sum_{k=1}^{n}\lim_{m\to\infty}\sum_{\substack{j=-m\\ j\neq 0}}^{m} c_j\exp\{i2j\pi(2k-1-n)/2n\} =$$

$$= -(\pi^{1/2}/n)\lim_{m\to\infty}\sum_{j=-m}^{m} c_j\sum_{k=1}^{n}\exp\{i2j\pi(2k-1-n)/2n\} =$$

$$= -\pi^{1/2}\lim_{m\to\infty}\sum_{\substack{s=-m\\ s\neq 0}}^{m}\exp\{-i\pi(n-1)s\}c_{ns}(g^*) =$$

$$= -\pi^{1/2}\lim_{m\to\infty}\sum_{\substack{s=-m\\ s\neq 0}}^{m}(-1)^{s(n-1)}c_{ns}(g^*) = \overline{\overline{o}}(n^{-r})$$

---

1) We expand with respect to the system $\{\pi^{-1/2}\cdot\exp(i2nx)\}_{-\infty}^{\infty}$ on $(-\pi/2, \pi/2)$; In fact it is enough assume on $g^{*(r)}$ that it is of bounded variation or less, namely that $g^{*(r)}$ is periodic and $c_n(g^{*(r)}) = \overline{\overline{o}}(n^{-1})$.

2) If $r = 1$ then the method of [4] p. 76. gives: $\Delta_n = \overline{\overline{o}}(n^{-1})$

(taking into account (2)).

The uniformity in ω on compact intervals of $\mathbf{R}$ follows from the easily verificable fact that for any compact interval $I \subset \mathbf{R}$ the set

$$\{f(\operatorname{tg}\varphi)\exp(-i\omega\operatorname{tg}\varphi) : \omega \in I\} = \{g(\varphi) : \omega \in I\}$$

is compact set in $C_\pi^r[-\pi/2, \pi/2]$ and taking into account the well know fact that if $L_n : B_1 \to B_2$ is any sequence of continuous linear operators (B_1, B_2 are any Banach spaces) converging in every $x \in B_1$, then the convergence is uniform on every compact subset of B_1.

PROBLEM 1. We don't know if for any fixd $\omega \in \mathbf{R}$ the estimates (1) and (2) are equivalent or not.

The implication (2)$\Longrightarrow$(1) is proved above.

The reverse, i.e. (1)$\Longrightarrow$(2) is open.

But we can say something in this direction. Namely, if in (3) we use the points $((2k-n)/2n)\pi$ in place of $((2k-1-n)/2n)\pi$, then for any power type, π periodic $g^* \in C_\pi^r[-\pi/2, \pi/2]$, we have (formally):

$$\tilde{\Delta}_n = \int\limits_{-\pi/2}^{\pi/2} g^*(\varphi)\mathrm{d}\varphi - \frac{\pi}{n}\sum_{k=1}^{n} g^*\left(\frac{2k-n}{2n}\pi\right) = -\pi^{1/2}\sum_{s=1}^{\infty}(-1)^{sn}c_{ns}(g)$$

and

$$-\pi^{1/2}(-1)^n c_n(g) = \sum_{m=1}^{\infty} \mu(m)\tilde{\Delta}_{nm} \tag{4}$$

(here μ is the Moebius function) and hence:

$$\tilde{\Delta} = \overline{\overline{o}}(n^{-r}) \Longleftrightarrow c_n = \overline{\overline{o}}(n^{-r})$$

($r > 1$ is any real number). If $r = 1$, then what can we say? The Moebius function is applied in approximation theory for other problems e.g. in the excellent book [3] of P. J. DAVIS and J. RABINOWITZ.

Our argument works obviously also for multidimension. (We omit it because it is trivial repetition of the argument above).

PROBLEM 2. Can we state (1) uniformly in ω on $\mathbf{R}$?

PROBLEM 3. Give the estimate (1) in terminology of the modulus of continuity of f.

PROBLEM 4. If f is not power type then does it follow from (1) the estimate $c_n = \overline{\overline{o}}(|n|^{-r})\,(n \in \mathbf{Z}, |n| \to \infty)$?

PROBLEM 5. What can we say for complex ω?

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ON REALIZATIONS AND POSITIVITY OF LINEAR INTEGRAL OPERATORS

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1. Introduction

Define a linear integral operator L by the requirement

$$(1.1)\qquad (\mathsf{L}\Psi)(t)=\int\limits_Y g(t,s)L(\Psi(s))\mathrm{d}\,\nu(s),\quad \mu\text{-a.e. } t\in X.$$

Here $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{A}'\nu)$ are σ-finite, complete measure spaces. The function $g: X\times X\to C$ belongs to $L_2(X\times X,\overline{\mathcal{A}\times\mathcal{A}'},\overline{\mu\times\nu})$, where $(X\times Y,\overline{\mathcal{A}\times\mathcal{A}'},\overline{\mu\times\nu})$ is the completion of the product of $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{A}',\nu)$. The operator L is a linear operator from the Schwartz class S into S and Ψ is assumed to be a finitely-valued function $Y\to S$. Assuming that the formal transpose $L':S\to S$ of L exists, one obtains that the formal transpose L' of L exists, that is,

$$(1.2)\qquad (\mathsf{L}\Psi,\Phi):=\int\limits_X((\mathsf{L}\Psi)(t),\Phi(t))\mathrm{d}\,\mu=(\Psi,\mathsf{L}'\Phi),$$

for all $\Psi\in D(Y,S):=\{\Psi|\Psi:Y\to S \text{ is finitely-valued }\}$ and $\Phi\in$ $\in D(X,S)$, where we denoted $(\varphi,\psi):=\int\limits_{\mathbf{R}^n}\varphi(x)\psi(x)\mathrm{d}\,x$ (cf. Theorem 2.5).

Let $L_2(X,B_{p,k})$ denote the Bochner μ-square integrable functions $X\to B_{p,k}$. Here $B_{p,k}$, $p\in[1,\infty[$, $k\in K'$ (K' is an appropriate class of weight functions $\mathbf{R}^n\to\mathbf{R}$) is essentially the completion of S with respect to the norm $\|\varphi\|_{p,k}:=\|(F\varphi)k\|_p$. The existence of L' makes it possible to define the minimal closed extension $\mathsf{L}^{\sim}_{p,k,h}$ and the maximal closed extension $\mathsf{L}'^{\#}_{p,k,h}$ of L from $L_2(Y,B_{p,k})$ to $L_2(X,B_{p,h})$, when $k,h \in K'$ and $p \in]1,\infty[$. One has always $\mathsf{L}^{\sim}_{p,k,h} \subset \mathsf{L}'^{\#}_{p,k,h}$. The minimal and maximal closed extensions $L^{\sim}_{p,k,h}$ and $L'^{\#}_{p,k,h}$ from $B_{p,k}$ to $B_{p,h}$ can be also defined. One knows several kind of criteria for the equality $L^{\sim}_{p,k,h} = L'^{\#}_{p,k,h}$, when L is a pseudo-differential operator (cf. [11], [5], [14], [4] and [12]). In this contribution we are interested (among others) in obtaining analogous criteria for $\mathsf{L}^{\sim}_{p,k,h} = \mathsf{L}'^{\#}_{p,k,h}$.

We characterize the solvability of the maximal equation (cf. Theorem 3.1). One sees that the solutions of $\mathsf{L}'^{\#}_{p,k,h}u = f$ are exactly the solutions for

$$L'^{\#}_{p,k,h}\Big(\int_Y g(t,s)u(s)\mathrm{d}\,\nu(s)\Big) = f(t), \qquad \mu\text{-a.e. } t \in X. \tag{1.3}$$

Furthermore, one obtains that the solutions of $\mathsf{L}^{\sim}_{p,k,h}u = f$ obey the relation

$$L^{\sim}_{p,k,h}\Big(\int_X g(t,s)u(s)\mathrm{d}\,\nu(s)\Big) = f(t), \qquad \mu\text{-a.e. } t \in X. \tag{1.4}$$

In the case when L is a pseudo-differential operator with the symbol $L(\xi)$, an algebraic characterization of solvability of the equation $\mathsf{L}'^{\#}_{p,k,h}u = f$ is given (cf. Theorem 3.4). In addition, in this case the equality $\mathsf{L}^{\sim}_{p,k,h} = L'^{\#}_{p,k,h}$ is established.

Assuming that L is a pseudo-differential operator of the Beals and Fefferman type, that is the symbol $L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}$, we verify that $\mathsf{L}^{\sim}_{2,q,k_0} = \mathsf{L}'^{\#}_{2,q,k_0}$ with a suitable $q \in S^{M-1,m-1}_{\Phi,\phi}$. Here $k_0 \equiv 1$ (and so $B_{2,k_0} = L_2$) (cf. Theorem 4.2).

Finally, a criterion for the positivity of L in $L_2(X,L_2)$ is given, when $X = Y = \mathbf{R}^{n'}, \mu = \nu =$ Lebesgue measure, $g(t,s)$ is of the form $a(t-s)b(t)b(s)$ and L is a pseudo-differential operator of the Beals and Fefferman type (cf. Theorem 5.2). The positivity of L implies the positivity of $\mathsf{L}^{\sim} := \mathsf{L}^{\sim}_{2,k_0,k_0}$ and so in the case when $\mathsf{L}^{\sim} = \mathsf{L}'^{\#} :=$ $:= \mathsf{L}'^{\#}_{2,k_0,k_0}$, $\mathsf{L}'^{\#}$ is a positive operator. The positivity of $\mathsf{L}'^{\#}$ and $\mathsf{L}^{\sim}$ is often used property in the study of linear Volterra equations (cf. [8], [2], [3], [9] and [13]). The operators $\mathsf{L}'^{\#}_{p,k,h}$ and $\mathsf{L}^{\sim}_{p,k,h}$ are called realizations of L.

2. Notations and definitions

2.1 . Denote by K' the totality of continuous positive weight functions $k : \mathbf{R}^n \to \mathbf{C}$ such that

$$(2.1) \qquad k(\xi+\eta) \le C_k k_{N_k}(\xi)k(\eta) \qquad \text{for all} \qquad \xi,\eta \in \mathbf{R}^n,$$

where $C_k > 0$ and $N_k \ge 0$ are constants (here we denote $k_s(\xi) := (1 + |\xi|^2)^{s/2}$). Let $B_{p,k}; p \in [1,\infty[$, $k \in K'$ be the completion of C_0^∞ with respect to the norm $\|\cdot\|_{p,k}$ defined by

$$(2.2) \qquad \|\varphi\|_{p,k} = \left((2\pi)^{-n} \int\limits_{\mathbf{R}^n} |(F\varphi)(\xi)k(\xi)|^p \mathrm{d}\,\xi \right)^{1/p},$$

where F denotes the Fourier transform from the Schwartz class S into S. The dual of S is denoted by S' (that is, S' is the space of all tempered distributions) and the Fourier transform from S' into S' is also denoted by F. One knows that $B_{p,k}$ is essentially that subspace of S' for whose elements u it holds: $Fu\,k \in L_p := L_p(\mathbf{R}^n)$. In addition, one has $\|u\|_{p,k} = (2\pi)^{-n/p}\|Fu\,k\|_p$. When $p = 2$ we write $H_k = B_{2,k}$. H_k is a Hilbert space and the scalar product is given by

$$(2.3) \qquad (u,v)_k = (2\pi)^{-n} \int\limits_{\mathbf{R}^n} (Fu)(\xi)\overline{(Fv)(\xi)}k^2(\xi)\mathrm{d}\,\xi.$$

2.2 . Let $L : S \to S$ be a continuous linear operator so that the formal transpose $L' : S \to S$ exists, that is, the operators L and L' obey the relation

$$(2.4) \qquad (L\varphi,\psi) := \int\limits_{\mathbf{R}^n} (L\varphi)(x)\psi(x)\mathrm{d}\,x = (\varphi, L'\psi) \qquad \text{for} \qquad \varphi,\psi \in S.$$

Then $L' : S \to S$ is continuous (cf. the Closed Graph Theorem) and so we can define the continuous extension $\overline{L} : S' \to S'$ of L with the requirement

$$(2.5) \qquad (\overline{L}u)(\varphi) = u(L'\varphi) \qquad \text{for} \qquad u \in S' \qquad \text{and} \qquad \varphi \in S$$

(in S' one uses the weak dual topology). In addition, for any $p \in [1,\infty[$, k, $h \in K'$ one is able to define the dense operators $L_{p,k,h}$ and $L'^{\#}_{p,k,h} : B_{p,k} \to B_{p,h}$ by

$$(2.6) \qquad \begin{cases} D(L_{p,k,h}) = S \\ L_{p,k,h}\varphi = L\varphi \qquad \text{for} \quad \varphi \in S \end{cases}$$

and

$$(2.7)\quad \begin{cases} D(L'^{\#}_{p,k,h}) = \{u \in B_{p,k} | & \text{there exists } f \in B_{p,h} \text{ such that} \\ & u(L'\varphi) = f(\varphi) \quad \text{for all } \varphi \in S\} \\ L'^{\#}_{p,k,h} u = f. & \end{cases}$$

One sees that $L_{p,k,h}$ is closable, $L'^{\#}_{p,k,h}$ is closed and that $L_{p,k,h} \subset L'^{\#}_{p,k,h}$. The smallest closed extension of $L_{p,k,h}$ from $B_{p,k}$ to $B_{p,h}$ is denoted by $L^{\sim}_{p,k,h}$ (cf. [15], pp. 76–78).

REMARK 2.1. A. Suppose that $L(\cdot) \in C^{\infty}(\mathbf{R}^n)$ so that

$$(2.8)\quad |(D^{\beta}_{\xi} L)(\xi)| \le C_{\beta} k_{\mu_\beta}(\xi) \qquad \text{for all} \qquad \xi \in \mathbf{R}^n,$$

where $C_\beta > 0$ and $\mu_\beta \in \mathbf{R}$ (here we denote $k_s(\xi) := (1 + |\xi|^2)^{s/2}$). Then the pseudo-differential operator $L(D)$ defined by

$$(2.9)\quad (L(D)\varphi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} L(\xi)(F\varphi)(\xi) e^{i(\xi,x)} \mathrm{d}\,\xi$$

maps S continuously into S and the formal transpose $L'(D) : S \to S$ exists ($L'(D)$ is defined by (2.9) where $L(\xi)$ is replaced with $L(-\xi)$).

B. Suppose that $L(\cdot,\cdot) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ belongs to the Beals and Fefferman class $S^{M,m}_{\Phi,\phi}$ of symbols (cf. [1]). Then the pseudo-differential operator $L(x, D)$ defined by

$$(2.10)\quad (L(x,D)\varphi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} L(x,\xi)(F\varphi)(\xi) e^{i(\xi,x)} \mathrm{d}\,\xi$$

maps S continuously into S. The formal transpose $L'(x, D) : S \to S$ exists and the symbol $L'(\cdot,\cdot)$ of $L'(x, D)$ belongs to $S^{M,m}_{\Phi^\vee,\phi^\vee}$ (where $\Phi^\vee(x,\xi) = \Phi(x,-\xi), \phi^\vee(x,\xi) = \phi(x,-\xi)$).

2.3 . Choose Θ from C_0^∞ so that supp $\Theta \subset \overline{B}(0,2) := \{x \in \mathbf{R}^n | \, |x| \le \le 2\}$ and that $\Theta(x) = 1$ for $x \in B(0,1)$. Define functions $\Theta_j \in C_0^\infty$ and $\psi_j \in S$ by

$$(2.11)\quad \Theta_j(x) = \Theta(x/j) \quad \text{and} \quad \psi_j = (2\pi)^{-n}(F\Theta_j{}^\vee).$$

Then one sees that for any $p \in [1, \infty[$, $k \in K'$ and $u \in B_{p,k}$ one has

$$(2.12)\quad \|\Theta_j u - u\|_{p,k} \to 0$$

and

$$(2.13)\quad \|\psi_j * u - u\|_{p,k} \to 0 \qquad \text{with } j \to \infty.$$

The pseudo-differential operators (2.9) with the symbols

$$F\psi_j = (2\pi)^{-n} F(F\Theta_j{}^{\vee}) = \Theta_j$$

is denoted by $\widehat{\psi}_j(D)$. One sees that for any $u \in S'$, $\varphi \in S$

$$(\overline{\widehat{\psi}}_j(u))(\varphi) = u(\widehat{\psi}_j(D)\varphi) = u(\psi_j * \varphi) = (\psi_j * u)(\varphi)$$

and so by (2.13)

$$\|\overline{\widehat{\psi}}_j(u) - u\|_{p,k} \to 0 \qquad \text{with } j \to \infty. \tag{2.14}$$

We remark that

$$\overline{\widehat{\psi}}_j(u) \in \bigcap_{k \in K'} B_{p,k}, \qquad \text{for } u \in \bigcup_{k \in K'} B_{p,k}. \tag{2.15}$$

2.4 . Let $(X, \mathcal{A}, \mu)$ be a complete, σ-finite measure space. Denote by $L_2(X, B_{p,k})$ (and by $L_1(X, B_{p,k})$, resp.) the linear space of all Bochner μ-square integrable functions $u : X \to B_{p,k}$ (and of all Bochner μ-integrable functions $u : X \to B_{p,k}$, resp.). For the preliminaries of the Bochner integrals cf. [15], pp. 130–134. The space $L_2(X, B_{p,k})$ is a Banach space and the corresponding norm is defined by

$$|||u|||_{p,k,X} = \left(\int_X \|u(t)\|^2_{p,k} \mathrm{d}\,\mu \right)^{1/2}. \tag{2.16}$$

Furthermore, the space $L_2(X, H_k)$ is a Hilbert space with the following scalar product

$$[u, v]_{k,X} = \int_X (u(t), v(t))_k \mathrm{d}\,\mu \tag{2.17}$$

In the sequel we denote by $D(X, S)$ the linear space of all finitely-valued functions $\Psi : X \to S$, that is, for any Ψ there exist disjoint sets $B_i \in \mathcal{A}\,(i = 1, \ldots, m)$ and $u_i \in S$ so that $\mu(B_i) < \infty$, $\Psi(x) = u_i$ for $x \in B_i$ and $\Psi(x) = 0$ for $x \in X \setminus \bigcup_{i=1}^{m} B_i$. One sees that $D(X, S)$ is a subset of $L_2(X, B_{p,k})$ (since any finitely-valued function $u : X \to B_{p,k}$ lies in $L_2(X, B_{p,k})$). In addition, we have

LEMMA 2.2. *The space $D(X, S)$ is dense in $L_2(X, B_{p,k})$ for any $p \in [1, \infty[$ and $k \in K'$.*

PROOF. Let Θ_j and ψ_j be defined by (2.11). The space $D(X, B_{p,k})$ of all finitely-valued functions is dense in $L_2(X, B_{p,k})$. Let u be in $D(X, B_{p,k})$. Then one sees that the function $v_j(t) := \psi_j * (u(t))$ is finitely-valued and that $v_j(t) \in \bigcap_{k\in K'} B_{p,k} \subset C^\infty(\mathbf{R}^n)$. In addition, one has

$$\|v_j(t) - v(t)\|_{p,k} \underset{j}{\to} 0 \qquad \text{for any } t \in X$$

and

$$\|v_j(t)\|_{p,k} \leq C\|u(t)\|_{p,k} \qquad \text{for } t \in X.$$

Hence the Dominated Convergence Theorem implies that

$$|||v_j - u|||_{p,k,X} \to 0 \qquad \text{with } j \to \infty,$$

and so $D(X, \bigcap_{k\in K'} B_{p,k})$ is dense in $L_2(X, B_{p,k})$.

Choose $w \in D(X, \bigcap_{k\in K'} B_{p,k})$. Then the function $w_j(t) := \Theta_j w(t)$ is finitely-valued and $w_j(t) \in C_0^\infty$ for any $t \in X$. Furthermore, one sees that $|||w_j - w|||_{p,k,X} \to 0$ with $j \to \infty$. This finishes the proof. ∎

REMARK 2.3. A. As the proof of Lemma 2.2 shows, the space $D(X, C_0^\infty)$ of all finitely-valued functions $X \to C_0^\infty$ is dense in $L_2(X, B_{p,k})$.

B. The space $L_2(X, B_{p,k})$ is essentially the completion of $D(X, C_0^\infty)$ with respect to the norm $||| \cdot |||_{p,k,X}$.

In the sequel we use the following notations

$$(\Psi, \Phi)_X := \int_X (\Psi(t), \Phi(t)) \mathrm{d}\,\mu \qquad \text{for } \Psi, \Phi \in D(X, S), \tag{2.18}$$

where we denoted

$$(\psi, \phi) = \int_{\mathbf{R}^n} \psi(x)\phi(x) \mathrm{d}\, x \qquad \text{for } \psi, \phi \in S.$$

Let $p' \in]1, \infty]$ with $1/p + 1/p' = 1$ and let $k^\vee \in K'$ so that $k^\vee(\xi) = k(-\xi)$. Then one sees that the inequality

$$\begin{aligned} |(\Psi, \Phi)_X| &\leq \int_X |(\Psi(t), \Phi(t))| \mathrm{d}\,\mu \leq \\ &\leq \int_X \|\Psi(t)\|_{p,k} \|\Phi(t)\|_{p',1/k^\vee} \mathrm{d}\,\mu \leq |||\Psi|||_{p,k,X} |||\Phi|||_{p',1/k^\vee,X} \end{aligned} \tag{2.19}$$

holds for any $\Psi, \Phi \in D(X,S)$ (for $p' = \infty$ we define

$$|||\Phi|||_{\infty,k,X} := \left(\int_X \|\Phi(t)\|^2_{\infty,k} \mathrm{d}\,\mu \right)^{1/2},$$

where $\|\Phi(t)\|_{\infty,k} := \sup_{\xi \in \mathbf{R}^n} |F(\Phi(t))(\xi)k(\xi)|$).

2.5 . Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{A}', \nu)$ be two complete, σ-finite measure spaces. Denote by $(X \times Y, \overline{\mathcal{A} \times \mathcal{A}'}, \overline{\mu \times \nu})$ the completion of the product measure space $(X \times Y, \mathcal{A} \times \mathcal{A}', \mu \times \nu)$. Choose a function $g : X \times Y \to \mathbf{C}$ from $L_2(X \times Y, \overline{\mathcal{A} \times \mathcal{A}'}, \overline{\mu \times \nu}, \mathbf{C})$. Then one finds subsets $A_1 \in \mathcal{A}$ and $A_2 \in \mathcal{A}'$ so that $\mu(A_1) = \nu(A_2) = 0$ and that

$$g(\cdot, s) \in L_2(X, \mathbf{C}) =: L_2(X) \qquad \text{for } s \in Y \setminus A_2$$

and

$$g(t, \cdot) \in L_2(Y) \qquad \text{for } t \in X \setminus A_1$$

(cf. [6], pp. 384–388).

Let L be a linear operator $S \to S$ so that $L' : S \to S$ exists. We define a linear operator $\mathbf{L}$ in $D(Y,S)$ by

$$(2.20) \qquad (\mathbf{L}\Psi)(t) = \begin{cases} \int_Y g(t,s) L(\Psi(s)) \mathrm{d}\,\nu, & t \in X \setminus A_1, \\ 0, & t \in A_1. \end{cases}$$

Since Ψ is finitely-valued, one sees that the function $L(\Psi(\cdot))$ is also finitely-valued and so $L(\Psi(\cdot)) \in D(Y,S) \subset L_2(Y, B_{p,k})$. Since for any $t \in X \setminus A_1$

$$\|(\mathbf{L}\Psi)(t)\|_{p,k} \leq \int_Y |g(t,s)| \, \|L(\Psi(s))\|_{p,k} \mathrm{d}\,\nu \leq$$

$$\leq \left(\int_Y |g(t,s)|^2 \right)^{1/2} |||L(\Psi(\cdot))|||_{p,k,Y},$$

one gets that $\mathbf{L}\Psi \in L_2(X, B_{p,k})$ and that

$$(2.21) \quad |||\mathbf{L}\Psi|||_{p,k,X} \leq \left(\int_{X \times Y} |g(t,s)|^2 \mathrm{d}\, \overline{\mu \times \nu} \right)^{1/2} |||L(\Psi(\cdot))|||_{p,k,Y},$$

for any $k \in K'$, $p \in [1, \infty[$. Hence we have

LEMMA 2.4. The operator L is a mapping from $D(X,S)$ into $\bigcap_{p,k} L_2(X,B_{p,k})$ such that

$$(\mathsf{L}\Psi)(t)\in S \qquad \text{for any} \quad t\in X. \tag{2.22}$$

PROOF. We have to show (2.22). Choose $\Psi=\sum_{i=1}^{m} u_i\chi_{B_i}$ (here χ_{B_i} is the characteristic function of B_i). Then one has for $t\in X\setminus A_1$

$$\begin{aligned}(\mathsf{L}\Psi)(t) &= \int_Y g(t,s)\left(\sum_{i=1}^{m}(Lu_i)\chi_{B_i}(s)\right)\mathrm{d}\,\nu = \\ &= \sum_{i=1}^{m}(Lu_i)\int_Y g(t,s)\chi_{B_i}(s)\mathrm{d}\,\nu,\end{aligned} \tag{2.23}$$

where $\int_Y g(t,s)\chi_{B_i}(s)\mathrm{d}\,\nu\in\mathbf{C}$. Since $(\mathsf{L}\Psi)(t)=0$ for $t\in A_1$, the proof is ready. ∎

2.6 . Let $D_{p,k}(X,S):=\{u\in L_2(X,B_{p,k})|u(t)\in S \text{ for any } t\in X\}$. Due to Lemma 2.4 one sees that L maps $D(Y,S)$ into $D_\infty(X,S):= := \bigcap_{p,k} D_{p,k}(X,S)$. We say that $\mathsf{L}':D(X,S)\to D_\infty(Y,S)$ is a formal transpose of L if the relation

$$\begin{aligned}(\mathsf{L}\Psi,\Phi)_X :&= \int_X ((\mathsf{L}\Psi)(t),\Phi(t))\,\mathrm{d}\,\mu = \\ &= \int_Y (\Psi(s),(\mathsf{L}'\Phi)(s))\,\mathrm{d}\,\nu = (\Psi,\mathsf{L}'\Phi)_Y\end{aligned} \tag{2.24}$$

for all $\Psi\in D(Y,S)$ and $\Phi\in D(X,S)$. We have

THEOREM 2.5. *The operator* L' *defined on* $D(X,S)$ *by*

$$(\mathsf{L}'\Phi)(s)=\begin{cases}\int_X g(t,s)L'(\Phi(t))\mathrm{d}\,\mu, & s\in Y\setminus A_2\\ 0, & s\in A_2\end{cases} \tag{2.25}$$

is a formal transpose of L.

PROOF. Since L' maps S into S, one sees that $L'(\Phi(\cdot))\in D(X,S)$ for any $\Phi\in D(X,S)$. Hence we obtain as above that (cf. (2.21) and (2.23)) $\mathsf{L}'\Phi\in D_\infty(Y,S)$. Define

$$\tilde g(t,s):=\begin{cases} g(t,s) & \text{for } (t,s)\in X\setminus A_1\times Y\setminus A_2\\ 0 & \text{for } (t,s)\in A_1\times Y\cup X\times A_2.\end{cases}$$

The function $(t,s) \to \tilde{g}(t,s)(L(\Psi(s)),\Phi(t))$ is $\overline{\mu\times\nu}$-measurable and one gets

$$
\begin{aligned}
&\int_X \left(\int_Y |\tilde{g}(t,s)(L(\Psi(s)),\Phi(t))| \mathrm{d}\,\nu \right) \mathrm{d}\,\mu \leq \\
&\leq \int_X \left(\int_Y |\tilde{g}(t,s)| \|L(\Psi(S))\|_0 \mathrm{d}\,\nu \right) \|\Phi(t)\|_0 \mathrm{d}\,\mu \leq \\
&\leq \int_X \left(\int_Y |\tilde{g}(t,s)|^2 \right)^{1/2} |||L(\Psi(\cdot))|||_{0,Y} \|\Phi(t)\|_0 \mathrm{d}\,\mu \leq \\
&\leq \left(\int_{X\times Y} |\tilde{g}(t,s)|^2 \mathrm{d}\,\overline{\mu\times\nu} \right)^{1/2} |||L(\Psi(\cdot))|||_{0,Y} |||\Phi|||_{0,X} < \infty,
\end{aligned}
\tag{2.26}
$$

where we denoted

$$
\|\varphi\|_0 = \|\varphi\|_{2,k_0} = \|\varphi\|_{L_2} \quad \text{and} \quad |||\Psi|||_{0,Y} = \left(\int_Y \|\Psi(t)\|_0^2 \mathrm{d}\,\nu \right)^{1/2}
$$

(recall that $L_2 = B_{2,k_0}$).

The functional $T_f : L_2 \to \mathbf{C}$ defined by

$$
T_f w = (w,f) := \int_{\mathbf{R}^n} w(x) f(x) \mathrm{d}\,x
$$

is bounded on L_2 for any $f \in L_2$. Hence one has (cf. [15], p. 134): For any $u \in L_1(Y, L_2)$, the function $T_f(u(\cdot))$ belongs to $L_1(Y)$ and

$$
\int_Y T_f(u(s)) \mathrm{d}\,\nu = T_f \left(\int_Y u(s) \mathrm{d}\,\nu \right).
$$

Thus we have

$$
\begin{aligned}
&\int_Y (u(s),f) \mathrm{d}\,\nu = \int_Y T_f(u(s)) \mathrm{d}\,\nu = \\
&= T_f \left(\int_Y u(s) \mathrm{d}\,\nu \right) = \left(\int_Y u(s) \mathrm{d}\,\nu, f \right).
\end{aligned}
\tag{2.27}
$$

Specifically, for any $t \in X \setminus A_1$ we obtain

$$\int_Y \tilde{g}(t,s)(L(\Psi(s)),\Phi(t))\mathrm{d}\,\nu = \left(\int_Y \tilde{g}(t,s)L(\Psi(s))\mathrm{d}\,\nu, \Phi(t)\right). \tag{2.28}$$

Using the Fubini Theorem and the relations (2.26) and (2.28) we finally get

$$\begin{aligned}
(\mathbf{L}\Psi,\Phi)_X &= \int_X \left(\int_Y \tilde{g}(t,s)L(\Psi(s))\mathrm{d}\,\nu, \Phi(t)\right)\mathrm{d}\,\mu = \\
&= \int_X \left(\int_Y \tilde{g}(t,s)(L(\Psi(s)),\Phi(t))\mathrm{d}\,\nu\right)\mathrm{d}\,\mu = \\
&= \int_Y \left(\int_X \tilde{g}(t,s)(L(\Psi(s)),\Phi(t))\mathrm{d}\,\mu\right)\mathrm{d}\,\nu = \\
&= \int_Y \left(\int_X \tilde{g}(t,s)(\Psi(s),L'(\Phi(t)))\mathrm{d}\,\mu\right)\mathrm{d}\,\nu = \\
&= \int_Y \left(\Psi(s), \int_X \tilde{g}(t,s)L'(\Phi(t))\mathrm{d}\,\mu\right)\mathrm{d}\,\nu = (\Psi,\mathbf{L}'\Phi)_Y,
\end{aligned} \tag{2.29}$$

as required. ∎

2.7 . Suppose that $u \in L_2(Y, B_{p,k})$ and that $v \in D_\infty(Y,S)$. Then the function $q : s \to u(s)(v(s))$ is ν-measurable and one gets

$$|q(s)| \le \|u(s)\|_{p,k}\|v(s)\|_{p',1/k^{\vee}}.$$

Hence $q \in L_1(Y)$ and

$$\int_Y |q(s)|\mathrm{d}\,\nu \le |||u|||_{p,k,Y}|||V|||_{p',1/k^{\vee},Y}. \tag{2.30}$$

We define $u(v) := \int_Y u(s)(v(s))\mathrm{d}\,\nu$. The estimate (2.30) yields

$$|u(v)| \le |||u|||_{p,k,Y}|||v|||_{p',1/k^{\vee},Y}. \tag{2.31}$$

Denote the dual space of $L_2(X, B_{p,k})$ by $L_2(X,B_{p,k})^*$. We obtain the following characterization for $p \in]1,\infty[$.

LEMMA 2.6. Suppose that l^* belongs to $L_2(X, B_{p,k})^*$. Then there exists an unique element $l \in L_2(X, B_{p',1/k^\vee})$ so that

$$l^*(\Psi) = l(\Psi) := \int_X l(s)(\Psi(s))\mathrm{d}\,\mu \qquad \text{for all} \quad \psi \in D_\infty(X, S). \tag{2.32}$$

Conversely, assume that $l \in L_2(X, B_{p',1/k^\vee})$. Then the linear form $\Psi \to l(\Psi)$ can be (uniquely) continuously extended on $L_2(X, B_{p,k})$.

In addition, the linear mapping

$$\lambda_{p,k,X} : L_2(X, B_{p,k})^* \to L_2(X, B_{p',1/k^\vee})$$

such that

$$\lambda_{p,k,X}(l^*) = l \tag{2.33}$$

is an isometrical isomorphism.

PROOF. Choose l^* from $L_2(X, B_{p,k})^*$. Then there exists an unique element $\tilde{l} \in L_2(X, B^*_{p,k})$ such that

$$l^*(u) = \int_X \tilde{l}(t)(u(t))\mathrm{d}\,\mu \qquad \text{for all} \quad u \in L_2(X, B_{p,k}), \tag{2.34}$$

and

$$\|l^*\| = \|\tilde{l}\| := \left(\int_X \|\tilde{l}(t)\|^2 \mathrm{d}\,\mu\right)^{1/2} \tag{2.35}$$

(cf. [10]). Furthermore, there exists an isometrical isomorphism $j_{p,k}$ from $B^*_{p,k}$ onto $B_{p',1/k^\vee}$ so that

$$[j_{p,k}(f^*)](\varphi) = f^*(\varphi) \qquad \text{for all} \quad \varphi \in S$$

(cf. [7], p. 42). Thus we get

$$l^*(\Psi) = \int_X \tilde{l}(t)(\Psi(t))\mathrm{d}\,\mu = \int_X [j_{p,k}(\tilde{l}(t))](\Psi(t))\mathrm{d}\,\mu. \tag{2.36}$$

The function $j_{p,k}(\tilde{l}(\cdot))$ is μ-measureable and

$$\|j_{p,k}(\tilde{l}(t))\|_{p',1/k^\vee} = \|\tilde{l}(t)\|.$$

Hence $l := j_{p,k}(\tilde{l}(\cdot)) \in L_2(X, B_{p',1/k^\vee})$ and one has

$$|||l|||_{p',1/k^\vee} = |||\tilde{l}||| = |||l^*|||. \tag{2.37}$$

The converse part of the assertion follows from (2.31) and by (2.37) one sees that $\lambda_{p,k,X}$ is isometric. This completes the proof. ∎

REMARRK 2.7. Lemma 2.6 implies the following fact: Suppose that $f \in L_2(X, B_{p,k})$ such that $f(\Psi) = 0$ for any $\Psi \in D(X,S)$. Then $f = 0$. This can be seen as follows: Choose f from $L_2(x, B_{p,k})$. Then one has

$$[\lambda^{-1}_{p',1/k^\vee,X}(f)](\Psi) = f(\Psi) = 0$$

and so $\lambda^{-1}_{p',1/k^\vee,X}(f) = 0$ (since $D(X,S)$ is dense in $L_2(X, B_{p',1/k^\vee})$). Thus $f = 0$.

Choose p from $]1,\infty[$ and choose the functions k and h from K'. Let L be defined by (2.20). We define dense linear operators $\mathsf{L}_{p,k,h}$ and $\mathsf{L}'^{\#}_{p,k,h}$ from $L_2(Y, B_{p,k})$ to $L_2(X, B_{p,h})$ by

$$(2.38)\qquad \begin{cases} D(\mathsf{L}_{p,k,h}) = D(X,S) & \\ \mathsf{L}_{p,k,h}\Psi = \mathsf{L}\Psi & \text{for } \Psi \in D(X,S), \end{cases}$$

$$(2.39)\qquad \begin{cases} D(\mathsf{L}'^{\#}_{p,k,h}) = \{u \in L_2(Y, B_{p,k})| & \text{there exists } f \in L_2(X, B_{p,k}) \\ & \text{such that } u(\mathsf{L}'\Psi) = f(\Psi) \\ & \text{for } \Psi \in D(X,S)\} \\ \mathsf{L}'^{\#}_{p,k,h} u = f. & \end{cases}$$

In virtue of (2.31) and Remark 2.7 one sees that $\mathsf{L}_{p,k,h}$ is closable, that is, if there exists a sequence $\{\Psi_n\} \subset D(Y,S)$ so that

$$|||\Psi_n|||_{p,k,Y} + |||\mathsf{L}\Psi_n - f|||_{p,h,X} \underset{n}{\to} 0,$$

then $f = 0$ (cf. [14], pp. 77–79). Furthermore, one sees that $\mathsf{L}'^{\#}_{p,k,h}$ is closed and that $\mathsf{L}_{p,k,h} \subset \mathsf{L}'^{\#}_{p,k,h}$. Let $\mathsf{L}^{\sim}_{p,k,h} : L_2(Y, B_{p,k}) \to L_2(X, B_{p,h})$ be the smallest closed extension of $\mathsf{L}_{p,k,h}$. Then the relation

$$(2.40)\qquad \mathsf{L}^{\sim}_{p,k,h} \subset \mathsf{L}'^{\#}_{p,k,h}$$

is always true.

3. Characterization of solutions

3.1 . In this character we shall assume that $p \in]1,\infty[$ and that L is a linear operator from S into S such that $L' : S \to S$ exists and that one finds constants $C > 0$ and $N \in \mathbb{N}$ with which

$$(3.1)\qquad \|L\varphi\|_{p,hk_{-N}} \le C\|\varphi\|_{p,k} \qquad \text{for all} \quad \varphi \in S.$$

At first we characterize the solvability of the equation $\mathsf{L}'^{\#}_{p,k,h}u = = f; u \in L_2(Y, B_{p,k}), f \in L_2(X, B_{p,h})$.

THEOREM 3.1. *The function u is a solution of $\mathsf{L}'^{\#}_{p,k,h}u = f$ if and only if $\int_Y g(t,s)u(s)\mathrm{d}\,\nu \in D(L'^{\#}_{p,kh})$ μ-a.e. $t \in X$ and*

$$L'^{\#}_{p,k,h}\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right) = f(t), \qquad \mu\text{-a.e. } t \in X. \tag{3.2}$$

PROOF. A. Choose u from $D(\mathsf{L}'^{\#}_{p,k,h})$ and let $\mathsf{L}'^{\#}_{p,k,h}u = f$. For any $\Psi \in D(X,S)$, the function $(t,s) \to g(t,s)u(s)(L'(\Psi(t)))$ is $\overline{\mu \times \nu}$-measurable and one has

$$\int_X \left(\int_Y |\tilde{g}(t,s)u(s)(L'(\Psi(t)))|\mathrm{d}\,\nu\right)\mathrm{d}\,\mu \leq \tag{3.3}$$

$$\leq \int_X \left(\int_Y |\tilde{g}(t,s)|^2 \mathrm{d}\,\nu\right)^{1/2} \|L'(\Psi(t))\|_{p',1/k^\vee}|||u|||_{p,k,Y} =$$

$$= \left(\int_{X\times Y} |\tilde{g}(t,s)|^2 \mathrm{d}\,\overline{\mu \times \nu}\right)^{1/2} |||u|||_{p,k,Y}|||L'(\Psi(\cdot))|||_{p',1/k^\vee,X} < \infty.$$

In Virtue of (3.1) the operator $L'^{\#}_{p,k,hk_{-N}}$ is bounded from $B_{p,k}$ to $B_{p,hk_{-N}}$. Hence one gets:

$$G(t) := \int_Y \tilde{g}(t,s)L'^{\#}_{p,k,hk_{-N}}(u(s))\mathrm{d}\,\nu \in B_{p,hk_{-N}}$$

and the relation

$$\int_Y \tilde{g}(t,s)L'^{\#}_{p,k,hk_{-N}}(u(s))\mathrm{d}\,\nu = L'^{\#}_{p,k,hk_{-N}}\left(\int_Y \tilde{g}(t,s)u(s)\mathrm{d}\,\nu\right) \tag{3.4}$$

holds.

From the Fubini Theorem we obtain (note that $G \in L_2(X, B_{p,hk_{-N}})$)

$$
\begin{aligned}
G(\Psi) &= \int_X \left(\int_Y \tilde{g}(t,s) L'^{\#}_{p,k,hk_{-N}}(u(s)) \mathrm{d}\,\nu \right) (\Psi(t)) \mathrm{d}\,\mu = \\
&= \int_X \left(L'^{\#}_{p,k,hk_{-N}} \left(\int_Y \tilde{g}(t,s) u(s) \mathrm{d}\,\nu \right) (\Psi(t)) \right) \mathrm{d}\,\mu = \\
&= \int_X \left(\int_Y \tilde{g}(t,s) u(s) \mathrm{d}\,\nu \right) (L'(\Psi(t)) \mathrm{d}\,\mu = \\
&= \int_X \left(\int_Y \tilde{g}(t,s) u(s) (L'(\Psi(t))) \mathrm{d}\,\nu \right) \mathrm{d}\,\mu = \\
&= \int_Y \left(\int_X u(s) (\tilde{g}(t,s) L'(\Psi(t))) d\mu \right) \mathrm{d}\,\nu = \\
&= \int_Y u(s) \left(\int_X \tilde{g}(t,s) L'(\Psi(t)) \mathrm{d}\,\mu \right) \mathrm{d}\,\nu = u(\mathsf{L}'\Psi) = f(\Psi),
\end{aligned} \tag{3.5}
$$

for all $\Psi \in D(X, S)$. Here we used the relation

$$
\int_X u(s)(\tilde{g}(t,s) L'(\Psi(t))) \mathrm{d}\,\mu = \int_X \tilde{g}(t,s) u(s) (L'(\Psi(t))) \mathrm{d}\,\mu, \tag{3.6}
$$

which is valid, since the functional $T_u : B_{p',1/k^\vee} \to \mathrm{C}$ defined by $T_u(v) = (j_{p',1/k^\vee} u)(v)$ is bounded. The fourth step in (3.5) is similarly seen.

Due to Remark 2.7 we see that $G = f$ in $L_2(X, B_{p,hk_{-N}})$, that is, $G(t) = f(t)$ μ-a.e. $t \in X$. Since $\int_Y \tilde{g}(t,s) u(s) \mathrm{d}\,\nu \in B_{p,k}$ and since $f(t) \in B_{p,k}$ μ-a.e. $t \in X$, one gets the relation (3.2) from (3.4).

B. Suppose that (3.2) is valid. Then one sees that $G = f$ and so by (3.5) u belongs to $D(\mathsf{L}'^{\#}_{p,k,h})$ and $\mathsf{L}'^{\#}_{p,k,h} u = f$, as desired. ∎

REMARK 3.2. As the above proof shows, the function $u \in$ $\in L_2(Y, B_{p,k})$ is a solution of $\mathsf{L}'^{\#}_{p,k,h} u = f$, $f \in L_2(X, B_{p,h})$ if and only if

$$\int_Y g(t,s) L'^{\#}_{p,k,hk_{-N}}(u(s)) \mathrm{d}\,\nu = f(t), \qquad \mu\text{-a.e. } t \in X. \tag{3.7}$$

In virtue of the definition of $\mathsf{L}^{\sim}_{p,k,h}$ one gets: The function $u \in D(\mathsf{L}^{\sim}_{p,k,h})$ and $\mathsf{L}^{\sim}_{p,k,h} u = f$ if and only if there exists a sequence $\{\Psi_n\} \subset D(Y,S)$ such that

$$\int_Y \|\Psi_n(t) - u(t)\|^2_{p,k} \mathrm{d}\,\nu + \int_X \|(\mathsf{L}\Psi_n)(t) - f(t)\|^2_{p,h} \mathrm{d}\,\mu \underset{n}{\to} 0. \tag{3.8}$$

From (3.8) we obtain

THEOREM 3.3. *Suppose that the function u is a solution of $\mathsf{L}^{\sim}_{p,k,h} u = f$. Then $\int_Y g(t,s)u(s)\mathrm{d}\,\nu \in D(L^{\sim}_{p,k,h})$, μ-a.e. $t \in X$ and the relation*

$$L^{\sim}_{p,k,h}\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right) = f(t) \quad \mu\text{-a.e. } t \in X \tag{3.9}$$

holds.

PROOF. Let u be in $D(\mathsf{L}^{\sim}_{p,kh})$ and let $\{\Psi_n\} \subset D(Y,S)$ be a sequence such that (3.8) holds. In virtue of (3.8) one finds a subsequence $\{\Psi_{n_j}\}$ of $\{\Psi_n\}$ such that $\|(L\Psi_{n_j})(t) - f(t)\|_{p,k} \underset{j}{\to} 0$ and $|||\Psi_{n_j} - u|||_{p,k,Y} \underset{j}{\to} 0$ μ-a.e. in X, say in $X \setminus A$ (cf. [6], pp. 192-193). Since for $t \in X \setminus A_1$

$$\|\int_Y g(t,s)\Psi_{n_j}(s)\mathrm{d}\,\nu - \int_Y g(t,s)u(s)\mathrm{d}\,\nu\|_{p,k} \leq$$

$$\leq \left(\int_Y |g(t,s)|^2\right)^{1/2} |||\Psi_{n_j} - u|||_{p,k,Y} \underset{j}{\to} 0,$$

and since

$$(\mathsf{L}\Psi_{n_j})(t) = \int_Y g(t,s) L(\Psi_{n_j}(s))\mathrm{d}\,\nu = L\left(\int_Y g(t,s)\Psi_{n_j}(s)\mathrm{d}\,\nu\right),$$

one sees that $\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu \in D(L^{\sim}_{p,k,h})$ for $t \in X \setminus (A \cup A_1)$ and that $L^{\sim}_{p,k,h}\left(\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu\right) = f(t)$ for $t \in X \setminus (A \cup A_1)$, This finishes the proof. ∎

3.2. In this section we assume that L is a pseudo-differential operator $L(D)$ defined by (2.9), where $L(\cdot)$ obeys (2.8). One sees easily that

$$\|L(D)\varphi\|_{p,hk_{-N_0-N_{h/k}}} \le C_0 C_{h/k}\|\varphi\|_{p,k}, \tag{3.10}$$

and so (3.1) holds. The next characterization is obtained

THEOREM 3.4. *Let L be defined by (2.9). Then the function $u \in L_2(Y, B_{p,k})$ is a solution of $L'^{\#}_{p,k,h}u = f$, $f \in L_2(X, B_{p,h})$ if and only if μ-a.e. $t \in X$ the relation*

$$L(\xi)F\left(\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu\right)(\xi) = F(f(t))(\xi), \qquad \text{m-a.e. } \xi \in \mathbf{R}^n \tag{3.11}$$

holds, where m denotes the Lebesgue measure in $\mathbf{R}^n$.

PROOF. By Theorem 3.1 the relation $L'^{\#}_{p,k,h}u = f$ holds if and only if one has μ-a.e. $t \in X$ (say $t \in X \setminus A$)

$$L'^{\#}_{p,k,h}\left(\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu\right) = f(t). \tag{3.12}$$

The relation

$$L'^{\#}_{p,k,h}w = v$$

holds if and only if

$$w(L'(D)\varphi) = v(\varphi) \qquad \text{for all } \varphi \in S$$

and so if and only if

$$w(L'(D)(F\varphi)) = v(F\varphi) = (Fv)(\varphi) \qquad \text{for all } \varphi \in S. \tag{3.13}$$

Since

$$w(L'(D)(F\varphi)) = (2\pi)^{-n}F(Fw)((L'(D)(F\varphi))^{\vee}) =$$
$$= (2\pi)^{-n}Fw((F(L'(D)(F\varphi)))^{\vee}) = (Fw)(L(\cdot)\varphi) = (L(\cdot)Fw)(\varphi),$$

one sees that $L'^{\#}_{p,k,h}w = v$ holds if and only if

$$L(\xi)(Fw)(\xi) = (Fv)(\xi) \qquad \text{m-a.e. } \xi \in \mathbf{R}^n. \tag{3.14}$$

Thus $\mathsf{L}'^{\#}_{p,k,h}u = f$ holds if and only if for any $t \in X \setminus A$ one has

$$L(\xi)F\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right)(\xi) = F(f(t))(\xi) \quad \text{m-a.e. } \xi \in \mathbf{R}^n. \tag{3.15}$$ ∎

REMARK 3.5. A. Let $L_{p,k}$ be the linear subspace of $L_1^{\mathrm{loc}}(\mathbf{R}^n)$ whose elements u satisfy that $u \cdot k \in L_p$. Then $L_{p,k}$ equipped with the norm $\|u\|^{\sim}_{p,k} := \|u \cdot k\|_p$ is a Banach space and the Fourier transform $F : B_{p,k} \to L_{p,k}$ is an isometry. Hence for any $t \in X \setminus A_1$, the function $s \to g(t,s)F(u(s))$ belongs to $L_2(Y, L_{p,k})$ and one has

$$\int_Y g(t,s)F(u(s))\mathrm{d}\,\nu = F\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right), \text{ for } u \in L_2(Y, B_{p,k}). \tag{3.16}$$

This implies that in the characterization of Theorem 3.4, the function $F\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right)$ can be replaced with $\int_Y g(t,s)F(u(s))\mathrm{d}\,\nu$.

B. Suppose that $u \in L_2(Y, L_1)$. For any $v \in L_1$, the Fourier transform Fv is a continuous function $\mathbf{R}^n \to \mathbf{C}$ and one has

$$|(Fv)(x)| \leq \|v\|_{L_1}.$$

Hence the linear functional $T_x v := (Fv)(x)$ is bounded on L_1, which yields for any $\xi \in \mathbf{R}^n$

$$\begin{aligned}\left(\int_Y g(t,s)F(u(s))\mathrm{d}\,\nu\right)(\xi) &= T_\xi\left(\int_Y g(t,s)F(u(s))\mathrm{d}\,\nu\right) = \\ &= \int_Y g(t,s)T_\xi(F(u(s)))\mathrm{d}\,\nu = \int_Y g(t,s)F(u(s))(\xi)\mathrm{d}\,\nu.\end{aligned} \tag{3.17}$$

Theorem 3.4 implies immediately

COROLLARY 3.6. Let L be defined by (2.9). Then the equality

$$\mathsf{L}^{\sim}_{p,k,h} = \mathsf{L}'^{\#}_{p,k,h},\ p \in]1,\infty[,\ k,h \in K' \tag{3.18}$$

holds.

PROOF. A. We have to show $\mathsf{L}'^{\#}_{p,k,h} \subset \mathsf{L}^{\sim}_{p,k,h}$. Choose u from $D(\mathsf{L}'^{\#}_{p,k,h})$ and let $\mathsf{L}'^{\#}_{p,k,h}u = f$. Let $\widehat{\psi}_j(D)$ be defined as in 2.3. Then $\overline{\widehat{\psi}}_j : B_{p,k} \to B_{p,k}$ is bounded, $\overline{\widehat{\psi}}_j v \in \bigcap\limits_{p,k} B_{p,k}$ and $\|\overline{\widehat{\psi}}_j v - v\|_{p,k} \underset{j}{\to} 0$ for any $v \in B_{p,k}$. Let A be a set fo μ-measure zero such that $u(s) \in B_{p,k}$ for any $s \in Y \setminus A$, and define $u_j(s) := \begin{cases} \overline{\widehat{\psi}}_j(u(s)), & s \in Y \setminus A \\ 0, & s \in A \end{cases}$. Then one sees that

$$\|u_j(s)\|_{p,k} \le C'\|u(s)\|_{p,k}$$

and

$$\|u_j(s) - u(s)\|_{p,k} \underset{j}{\to} 0 \qquad \text{for any} \quad s \in Y \setminus A$$

and so by the Dominated Convergence Theorem one has

$$|||u_j - u|||_{p,k,Y} \underset{j}{\to} 0. \tag{3.19}$$

In addition, one gets

$$\|u_j(s)\|_{p,k_{N_0}h} \le \|\overline{\widehat{\psi}}_j(u(s)\|_{p,k_{N_0}h}$$
$$||\psi_j * u(s)||_{p,k_{N_0}h} \le \|\psi_j\|_{\infty,k_{N_0}(h/k)}\|u(s)\|_{p,k}$$

and so

$$u_j \in L_2(Y, B_{p,k_{N_0}h}). \tag{3.20}$$

We obtain

$$\begin{aligned} |||\mathsf{L}\Psi|||^2_{p,h,X} &= \int\limits_X \|(\mathsf{L}\Psi)(t)\|^2_{p,h} \mathrm{d}\,\mu \le \\ &\le \int\limits_X \left(\int\limits_Y |\tilde{g}(t,s)|\, \|L(\Psi(s))\|_{p,h} \mathrm{d}\,\nu \right)^2 \mathrm{d}\,\mu \le \\ &\le \left(\int\limits_{X\times Y} |\tilde{g}(t,s)|^2 \mathrm{d}\,\overline{\mu\times\nu} \right) |||L(\Psi(\cdot))|||^2_{p,h,Y} \le \\ &\le C_0 \left(\int\limits_{X\times Y} |\tilde{g}(t,s)|^2 \mathrm{d}\,\overline{\mu\times\nu} \right) |||\Psi|||^2_{p,k_{N_0}h,Y} \end{aligned}$$

and so

$$L_2(Y, B_{p,k_{N_0}h}) \subset D(\mathsf{L}^{\sim}_{p,k,h}). \tag{3.21}$$

B. To complete the proof we have to verify that

$$|||\mathsf{L}^{\sim}_{p,k,h}u_j - f|||_{p,h,X} \underset{j}{\to} 0. \tag{3.22}$$

One sees by (3.11) that μ-a.e. $t \in X$

$$\begin{aligned} F((\mathsf{L}^{\sim}_{p,k,h}u_j)(t))(\xi) &= F((\mathsf{L}'^{\#}_{p,k,h}u_j)(t))(\xi) = \\ &= L(\xi)F\left(\int_Y g(t,s)u_j(s)\mathrm{d}\,\nu\right)(\xi) = \\ &= L(\xi)F\left(\int_Y g(t,s)(\overline{\widehat{\psi}}_j u)(s)\mathrm{d}\,\nu\right)(\xi) = \\ &= L(\xi)F\left(\overline{\widehat{\psi}}_j\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right)\right)(\xi) = \\ &= L(\xi)(F\psi_j)(\xi)F\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right)(\xi) = \\ &= (F\psi_j)(\xi)F(f(t))(\xi) = \left(\overline{\widehat{\psi}}_j(f(t))\right)(\xi), \quad \text{m-a.e. } \xi \in \mathbf{R}^n \end{aligned} \tag{3.23}$$

and so

$$(\mathsf{L}^{\sim}_{p,k,h}u_j)(t) = \overline{\widehat{\psi}}_j(f(t)) \quad \mu\text{-a.e.} \quad t \in X. \tag{3.24}$$

This implies that (3.22) holds, which completes the proof. ∎

4. On the equality of realizations for $L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}$

We recall that $S^{M,m}_{\Phi,\phi}$ denotes the Beals and Fefferman class of symbols (cf. Remark 2.1.B.). In the sequel we shall consider the case where $p = 2$, $h \equiv 1$ and $k = q$, where q belongs to $S^{M-1,m-1}_{\Phi,\phi}$. The operators $\widehat{\psi}_j(D)$ are again defined as in 2.3. Define $L_i(\cdot,\cdot) := \Theta_i L(\cdot,\cdot)$. We need

LEMMA 4.1. Suppose that $L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}$. Furthermore, assume that $q(\cdot) \in S^{M-1,m-1}_{\Phi,\phi}$ such that with $c > 0$,

$$q(\xi) \geq c\Phi^{M-1}(x,\xi)\phi^{m-1}(x,\xi) \qquad \text{for} \quad x,\xi \in \mathbb{R}^n. \tag{4.1}$$

Then for any $j \in \mathbb{N}$ there exists $R_j(\cdot,\cdot) \in S^{M-1,m-1}_{\Phi,\phi}$ such that

$$\overline{\widehat{\psi}}_j(\overline{L}_i u) = \overline{L}_i\left(\overline{\widehat{\psi}}_j u\right) + \overline{R}_{i,j} u \tag{4.2}$$

and

$$\|\overline{R}_{i,j}u\|_0 := \|R_{i,j}u\|_{L_2} \leq C\|u\|_{2,q} =: C\|u\|_q \tag{4.3}$$

for all $u \in H_q := B_{2,q}$ and $i,j \in \mathbb{N}$.

PROOF. A. In view of the Taylor formula one gets that any $q(\cdot) \in$ $\in S^{M-1,m-1}_{\Phi,\phi}$, which obeys (4.1) belongs to K'. In the sequel we denote $(L \circ P)(x,D) := L(x,D) \circ P(x,D)$.

B. One sees that $\{L_i(\cdot,\cdot)\}$ is a bounded sequence in $S^{M,m}_{\Phi,\phi}$. Furthermore, one sees that $\{\widehat{\psi}_j(\cdot)\} = \{\Theta_j\}$ is a bounded sequence in $S^{0,0}_{\Phi,\phi}$ (for the definition of the topology in $S^{M,m}_{\Phi,\phi}$ cf. [1]). Hence one has

$$(\widehat{\psi}_j \circ L_i)(\cdot,\cdot) = (L_i \circ \widehat{\psi}_j)(\cdot,\cdot) + R_{i,j}(\cdot,\cdot) \tag{4.4}$$

where $\{R_{i,j}(\cdot,\cdot)\}_{i,j}$ is a bounded set in $S^{M-1,m-1}_{\Phi,\phi}$ (cf. the proof of Lemma 1 in [1]). Since $\{R_{i,j}(\cdot,\cdot)q^{-1}(\cdot)\}_{i,j}$ is a bounded set in $S^{0,0}_{\Phi,\phi}$ there exists a constant $C > 0$ such that

$$\|R_{i,j}(x,D)\varphi\|_0 \leq C\|\varphi\|_q \qquad \text{for all } \varphi \in S \tag{4.5}$$

(cf. the proof of Theorem 2 in [1]).

From (4.4) we obtain that

$$\begin{aligned}
\left(\left(\overline{\widehat{\psi}_j \circ L_i}\right) u\right)(\varphi) &= u((L_i' \circ \widehat{\psi}_j')(x,D)\varphi) = \\
&= u((\widehat{\psi}_j' \circ L_i')(x,D)\varphi) + u(R_{i,j}'(x,D)\varphi) = \\
&= \left(\left(\overline{L_i \circ \widehat{\psi}_j}\right) u + \overline{R}_{i,j}u\right)(\varphi)
\end{aligned} \tag{4.6}$$

for $u \in S'$ and $\varphi \in S$ and so

$$\overline{\widehat{\psi}}_j(\overline{L}_i u) = \overline{L}_i\left(\overline{\widehat{\psi}}_j u\right) + \overline{R}_{i,j}u.$$

Due to (4.5) one gets the estimate (4.3). This completes the proof. ∎

In the following we denote $L_q^\sim := L_{2,q,k_0}^\sim, L_q'^\# := L_{2,q,k_0}'^\#$, $\mathsf{L}_q^\sim :=$ $:= \mathsf{L}_{2,q,k_0}^\sim$ and $\mathsf{L}_q'^\# := \mathsf{L}_{2,q,k_0}'^\#$. We recall that $L_2(X;H_k)$ is a Hilbert space and so for any bounded sequence $\{u_n\}$ of $L_2(X,H_k)$ one finds a subsequence $\{u_{n_j}\}$ so that

$$|||(1/r)\left(\sum_{j=1}^{r} u_{n_j}\right) - u|||_{k,X} \underset{r}{\to} 0 \tag{4.7}$$

with some $u \in L_2(X,H_k)$ (cf. The Banach-Saks Theorem). We use the notations $|||\cdot|||_{k_0,X} = |||\cdot|||_{0,X}$ and $\|\cdot\|_{k_0} = \|\cdot\|_0 (= \|\cdot\|_{L_2})$.

THEOREM 4.2. *Suppose that $L(\cdot,\cdot)$ belongs to $S_{\Phi,\phi}^{M,m}$ and that there exists a symbol $q(\cdot) \in S_{\Phi,\phi}^{M-1,m-1}$ so that (4.1) holds. Then the equality*

$$\mathsf{L}_q^\sim = \mathsf{L}_q'^\# \tag{4.8}$$

is valid.

PROOF. A. Choose u from $D(\mathsf{L}_q'^\#)$ and denote $\mathsf{L}_q'^\# u = f$. Define functions $u_j : Y \to H_q$ (as above) by

$$u_j(s) := \begin{cases} \overline{\widehat{\psi}}_j(u(s)), & s \in Y \setminus A, \\ 0, & s \in A \end{cases}$$

where A is a subset of Y such that $\nu(A) = 0$ and $u(s) \in H_q$ for any $s \in Y \setminus A$. Then we have (cf. the proof of Corollary 3.6)

$$|||u_j - u|||_{q,Y} \underset{j}{\to} 0 \tag{4.9}$$

and

$$u_j \in L_2(Y, H_{k_N+q}) \qquad \text{for any } N \in \mathbb{N}. \tag{4.10}$$

B. Since the weight functions Φ and ϕ obey the property (i) (cf. [1]) one gets that $k_{-N}(\cdot) \in S_{\Phi,\phi}^{-M,-m}$ for $N \in \mathbb{N}$ large enough. Hence the symbol $(L \circ k_{-N})(\cdot,\cdot)$ belongs to $S_{\Phi,\phi}^{0,0}$ and so

$$\|((L \circ k_{-N})(x,D)\varphi)\|_0 \le C\|\varphi\|_0 \qquad \text{for all } \varphi \in S.$$

This implies that

$$\|L(x,D)\varphi\|_0 \le C\|\varphi\|_{k_N} \le C\|\varphi\|_{k_N+q} \qquad \text{for all } \varphi \in S. \tag{4.11}$$

Similarly, as in the proof of Corollary 3.6 one gets from (4.11) that for all $\Psi \in D(Y,S)$

$$\|\mathsf{L}\Psi\|_{0,X} \le C\left(\int\limits_{X\times Y} |g(t,s)|^2\right)^{1/2} |||\Psi|||_{k_N+q,Y}. \tag{4.12}$$

Hence the inclusion

$$L_2(Y, H_{k_N+q}) \subset D(\mathsf{L}_q^{\sim}) \tag{4.13}$$

holds. Specifically, any u_j belongs to $D(\mathsf{L}_q^{\sim})$.

C. One sees that $\overline{L}_i v = \Theta_i \overline{L} v$ for any $v \in S'$. Hence we obtain

$$\|\overline{L}_i v - L_q'^{\#} v\|_0 \underset{i}{\to} 0 \qquad \text{for any } v \in D(L_q'^{\#}). \tag{4.14}$$

Since $\overline{\widehat{\psi}}_j : L_2 \to L_2$ is bounded (note that $\widehat{\psi}_j(\cdot) \in S^{0,0}_{\Phi,\phi}$) we observe by (4.2)–(4.3) that

$$\|\overline{\widehat{\psi}}_j(L_q'^{\#} v) - L_q'^{\#}\left(\overline{\widehat{\psi}}_j v\right)\|_0 \le C\|v\|_q \qquad \text{for any } v \in D(L_q'^{\#}). \tag{4.15}$$

Due to Theorem 3.1 we know that there exists a subset $A \subset X$ of μ-measure zero so that $\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu \in D(L_q'^{\#})$ and that

$$L_q'^{\#}\left(\int\limits_Y g(t,s)u(s)d\nu\right) = f(t) \qquad \text{for any } t \in X \setminus A. \tag{4.16}$$

Hence we get by (4.15) that for $t \in X \setminus (A \cup A_1)$

$$\begin{aligned} &\|L_q'^{\#}\left(\overline{\widehat{\psi}}_j\left(\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu\right)\right)\|_0 \le \\ &\le \|\overline{\widehat{\psi}}_j(f(t))\|_0 + C\|\int\limits_Y g(t,s)u(s)\mathrm{d}\,\nu\|_q \le \\ &\le C'\|f(t)\|_0 + C\left(\int\limits_Y |g(t,s)|^2 \mathrm{d}\,\nu\right)^{1/2} |||u|||_{q,Y}. \end{aligned} \tag{4.17}$$

In virtue of Theorem 3.1 μ-a.e. $t \in X$

$$(\mathsf{L}_q^{\sim} u_j)(t) = (\mathsf{L}_q'^{\#} u_j)(t) = L_q'^{\#}\left(\int_Y g(t,s)u_j(s)\mathrm{d}\,\nu\right) =$$

$$= L_q'^{\#}\left(\int_Y g(t,s)\overline{\widetilde{\psi}}_j(u(s))\mathrm{d}\,\nu\right) = L_q'^{\#}\left(\overline{\widetilde{\psi}}_j\left(\int_Y g(t,s)u(s)\mathrm{d}\,\nu\right)\right),$$

where the last step is a consequence of the boundedness of $\overline{\widetilde{\psi}}_j : H_q \to H_q$. Hence by (4.17)
(4.18)

$$|||\mathsf{L}_q u_j||_{0,X}^2 \leq 2C'^2|||f|||_{0,X}^2 + 2C^2\left(\int_{X\times Y} |g(t,s)|^2\mathrm{d}\,\overline{\mu\times\nu}\right)|||u|||_{q,Y}^2.$$

The estimate (4.18) implies that there is a subsequence $\{u_{j_l}\}$ so that with some $e \in L_2(X, L_2)$

$$|||\mathsf{L}_q^{\sim}\left((1/r)\left(\sum_{l=1}^{r} u_{j_l}\right) - e\right)|||_{0,X} \underset{r}{\to} 0.$$

Since also

$$|||(1/r)\left(\sum_{l=1}^{r} u_{j_l}\right) - u|||_{0,Y} \underset{r}{\to} 0$$

we see that $u \in D(\mathsf{L}_q^{\sim})$ and that $\mathsf{L}_q^{\sim} u = e = f$, as desired. ∎

In the case when $M = m = 1$ we get

COROLLARY 4.3. Suppose that $L(\cdot,\cdot) \in S_{\Phi,\phi}^{1,1}$. Then the relation

$$\mathsf{L}^{\sim} := \mathsf{L}_{k_0}^{\sim} = \mathsf{L}'^{\#} \tag{4.19}$$

holds.

PROOF. We have only to find that $q(\cdot) = k_0(\cdot)$ obeys (4.11) with $M = m = 1$. ∎

REMARK 4.4. The Corollary 4.3 implies that $\mathsf{L}^{\sim} = \mathsf{L}'^{\#}$, when $L(x, D)$ is any first-order smooth linear partial differential operator $\sum\limits_{|\sigma|\leq 1} a_\sigma(x)D^\sigma$ such that

$$\sup_{x\in\mathbf{R}^n} |D_x^\alpha a_\sigma(x)| \leq C_\alpha, \quad \alpha \in \mathbf{N}_0^n. \tag{4.20}$$

5. On the positivity of L

5.1. In this chapter we consider the case where $X = Y = \mathbf{R}^{n'}, \mu = = \nu = m$ (= the Lebesgue measure in $\mathbf{R}^{n'}$ and where $g(t,s)$ is of the special form $g(t,s) = a(t-s)b(t)b(s)$ with $a \in L_\infty$, $b \in L_2$. Furthermore, we are interested in paying our attention only to the case $p = 2$ and $k = k_0$ (that is, $B_{p,k} = L_2$). Thus we are studying operators of the form

$$(5.1) \quad (\mathsf{L}\Psi)(t) = \int\limits_{\mathbf{R}^{n'}} a(t-s)b(t)b(s)L(\Psi(s))\,\mathrm{d}\,m, \qquad \Psi \in D(\mathbf{R}^{n'}, S).$$

Our aim is to give sufficient criteria for the estimate

$$(5.2) \qquad \mathrm{Re}[\mathsf{L}\Psi, \Psi]_0 := \mathrm{Re}[\mathsf{L}\Psi, \Psi]_{k_0,X} \geq -c|||\Psi|||_0^2$$

for all $\Psi \in D(\mathbf{R}^{n'}, L_2)$.

The next lemma is needed

LEMMA 5.1. Suppose that $L(\cdot,\cdot)$ belongs to $S_{\Phi,\phi}^{M,m}$ so that

$$(5.3) \qquad \mathrm{Re}\, L(x,\xi) \geq c\Phi^M(x,\xi)\phi^m(x,\xi) \qquad \text{for all } x, \xi \in \mathbf{R}^n.$$

Then for each $N \in \mathbf{N}$ there exist $P_N(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M,(1/2)m}$ and $R_N(\cdot,\cdot) \in S_{\Phi,\phi}^{M-N,m-N}$ such that

$$(5.4) \qquad L(x,D) + L^*(x,D) = 2(P_N^* \circ P_N)(x,D) - R_N(x,D),$$

where $L^*(x,D)$ (and $P_N^*(x,D)$) is the formal adjoint of $L(x,D)$ (and of $P_N(x,D)$, resp.).

PROOF. A. We shall verify that for any $N \in \mathbf{N}$ there exist

$$Q_j(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M-(j-1),(1/2)m-(j-1)}; j = l, \ldots, N$$

and $R_N(\cdot,\cdot) \in S_{\Phi,\phi}^{M-N,n-N}$ such that

(5.5)

$$2\left(\left(\sum_{j=1}^N Q_j^*\right)\left(\sum_{j=1}^N Q_j\right)\right)(x,D) = L^*(x,D) + L(x,D) + R_N(x,D),$$

which implies the assertion.

B. Assume that $N = 1$. Define $Q_1(\cdot,\cdot) := (L_{\mathrm{Re}}(\cdot,\cdot))^{1/2}$, where $L_{\mathrm{Re}}(\cdot,\cdot) := \operatorname{Re} L(\cdot,\cdot)$. Then by (5.3) one sees that $Q_1(\cdot,\cdot) \in$ $\in S_{\Phi,\phi}^{(1/2)M,(1/2)m}$. The formal adjoint $Q^*(x,D)$ is generated by the symbol $Q_1^*(x,\xi) := \overline{Q_1'(x,-\xi)}$ and so $Q_1^*(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M,(1/2)m}$. applying the symbolic calculus expressed in [1], we observe that there exist symbols $R_{1,1}(\cdot,\cdot) \in S_{\Phi,\phi}^{M-1,m-1}$ and $R_{1,2}(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M-1,(1/2)m-1}$ so that

$$
\begin{aligned}
(Q_1^* \circ Q_1)(x,\xi) &= Q_1^*(x,\xi)Q_1(x,\xi) + R_{1,1}(x,\xi) = \\
&= Q_1^2(x,\xi) + Q_1(x,\xi)R_{1,2}(x,\xi) + R_{1,1}(x,\xi).
\end{aligned} \tag{5.6}
$$

Writing $R_{1,3}(x,\xi) = Q_1(x,\xi)R_{1,2}(x,\xi) + R_{1,1}(x,\xi)$ and noting that $Q_1^2(x,\xi) = L_{\mathrm{Re}}(x,\xi)$, we see that

$$
\begin{aligned}
2(Q_1^* \circ Q_1)(x,D) &= (Q_1^* \circ Q_1)(x,D) + (Q_1^* \circ Q_1)^*(x,D) = \\
&= L_{\mathrm{Re}}(x,D) + L_{\mathrm{Re}}^*(x,D) + R_{1,3}(x,D) + R_{1,3}^*(x,D).
\end{aligned} \tag{5.7}
$$

Finally, we find that there exist $R_{1,4}(\cdot,\cdot)$ and $R_{1,5}(\cdot,\cdot) \in S_{\Phi,\phi}^{M-1,m-1}$ so that

$$
\begin{aligned}
L_{\mathrm{Re}}(x,\xi) + L_{\mathrm{Re}}^*(x,\xi) &= L_{\mathrm{Re}}(x,\xi) + \overline{L_{\mathrm{Re}}(x,\xi)} + R_{1,4}(x,\xi) = \\
&= 2L_{\mathrm{Re}}(x,\xi) + R_{1,4}(x,\xi)
\end{aligned} \tag{5.8}
$$

and

$$
\begin{aligned}
L(x,\xi) + L^*(x,\xi) &= L(x,\xi) + \overline{L(x,\xi)} + R_{1,5}(x,\xi) = \\
&= 2L_{\mathrm{Re}}(x,\xi) + R_{1,5}(x,\xi).
\end{aligned} \tag{5.9}
$$

Thus (5.5) is valid with $Q_1(\cdot,\cdot) = L_{\mathrm{Re}}(\cdot,\cdot)^{1/2}$ and $R_1(\cdot,\cdot) := R_{1,3}(\cdot,\cdot) +$ $+ R_{1,3}^*(\cdot,\cdot) + R_{1,4}(\cdot,\cdot) - R_{1,5}(\cdot,\cdot) \in S_{\Phi,\phi}^{M-1,m-1}$.

C. Suppose that (5.5) holds with $N = l$. Then there exist $Q_j(\cdot,\cdot) \in$ $\in S_{\Phi,\phi}^{(1/2)M-(j-1),(1/2)m-(j-1)}$, $j = 1,\dots,l$ and $R_l(\cdot,\cdot) \in S_{\Phi,\phi}^{M-l,m-l}$ so that

(5.10)

$$
2\left(\left(\sum_{j=1}^{l} Q_j^*\right) \circ \left(\sum_{j=1}^{l} Q_j\right)\right)(x,D) = L^*(x,D) + L(x,D) + R_l(x,D).
$$

To complete the induction one has to find $Q_{l+1}(.,.) \in S_{\Phi,\phi}^{(1/2)M-l,(1/2)m-l}$ and $R_{l+1}(.,.)S_{\Phi,\phi}^{M-(l+1),m-(l+1)}$ so that

(5.11)
$$2\left(\left(\sum_{j=1}^{l+1} Q_j^*\right) \circ \left(\sum_{j=1}^{l+1} Q_j\right)\right)(x,D) = L^*(x,D)+L(x,D)+R_{l+1}(x,D).$$

From (5.10) we see that there exists $R_{l+1,1}(\cdot,\cdot) \in S_{\Phi,\phi}^{M-(l+1),m-(l+1)}$ such that

$$\begin{aligned} &4\left(\left(\sum_{j=1}^{l} Q_j^*\right) \circ \left(\sum_{j=1}^{l} Q_j\right)\right)(x,D) = \\ &= 2(L^*(x,D) + L(x,D)) + R_l(x,D) + R_l^*(x,D) = \\ &= 2(L^*(x,D) + L(x,D)) + 2(R_l)_{\mathrm{Re}}(x,D) + R_{l+1,1}(x,D). \end{aligned} \tag{5.12}$$

Define $Q_{l+1}(\cdot,\cdot) := -(1/4)Q_1^{-1}(\cdot,\cdot)R_l(\ ,\ )$. Then we find by (5.12) that

$$\begin{aligned} &2\left(\left(\sum_{j=1}^{l+1} Q_j^*\right)\left(\sum_{j=1}^{l+1} Q_j\right)\right)(x,D) = L^*(x,D) + L(x,D)+ \\ &+ (R_l)_{\mathrm{Re}}(x,D) + (1/2)R_{l+1,1}(x,D) + 2(Q_{l+1}^* \circ Q_1)(x,D)+ \\ &+ 2(Q_1^* \circ Q_{l+1})(x,D) + 2\left(Q_{l+1}^* \circ \left(\sum_{j=2}^{l} Q_j\right)\right)(x,D)+ \\ &+ \left(\left(\sum_{j=2}^{l} Q_j^*\right) \circ Q_{l+1}\right)(x,D) + 2(Q_{l+1}^* \circ Q_{l+1})(x,D). \end{aligned} \tag{5.13}$$

There exist symbols $R_{l+1,2}(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M-(l+1),(1/2)m-(l+1)}$ and $R_{l+1,3}(\cdot,\cdot) \in S_{\Phi,\phi}^{M-(l+1),m-(l+1)}$ such that

$$\begin{aligned} &(Q_{l+1}^* \circ Q_1)(x,\xi) = Q_{l+1}^*(x,\xi)Q_1(x,\xi) + R_{l+1,3}(x,\xi) = \\ &= \overline{Q_{l+1}(x,\xi)}Q_1(x,\xi) + R_{l+1,2}(x,\xi)Q_1(x,\xi) + R_{l+1,3}(x,\xi) \end{aligned} \tag{5.14}$$

(note that $Q_{l+1}(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M-l,(1/2)m-l}$). Hence we have

$$(Q^*_{l+1} \circ Q_1)(x,\xi) = -(1/4)\overline{R_l(x,\xi)} + R_{l+1,4}(x,\xi), \tag{5.15}$$

where

$$R_{l+1,4}(\cdot,\cdot) := R_{l+1,2}(\cdot,\cdot)Q_1(\cdot,\cdot) + R_{l+1,3}(\cdot,\cdot) \in S^{M-(l+1),m-(l+1)}_{\Phi,\phi}.$$

Similarly, one sees that with a symbol $R_{l+1,5}(\cdot,\cdot) \in S^{M-(l+1),m-(l+1)}_{\Phi,\phi}$

$$(Q^*_1 \circ Q_{l+1})(x,\xi) = -(1/4)R_l(x,\xi) + R_{l+1,5}(x,\xi). \tag{5.16}$$

Furthermore, one observes that for any $j = 2,\dots,l+1$

$$(Q^*_{l+1} \circ Q_j(\cdot,\cdot) \in S^{M-l-(j-1),m-l-(j-1)}_{\Phi,\phi} \subset S^{M-(l+1),m-(l+1)}_{\Phi,\phi} \tag{5.17}$$

and

$$(Q^*_j \circ Q_{l+1})(\cdot,\cdot) \in S^{M-(l+1),m-(l+1)}_{\Phi,\phi}. \tag{5.18}$$

Combining (5.13), (5.15), (5.16), (5.17) and (5.18) we see that

$$2\left(\left(\sum_{j=1}^{l+1} Q^*_j\right) \circ \left(\sum_{j=1}^{l+1} Q_j\right)\right)(x,D) = L^*(x,D) + L(x,D) + \tag{5.19}$$
$$+(1/2)R_{l+1,1}(x,D) + 2R_{l+1,4}(x,D) + 2R_{l+1,5}(x,D) +$$
$$+2\left(Q^*_{l+1} \circ \left(\sum_{j=2}^{l} Q_j\right)\right)(x,D) + 2\left(\left(\sum_{j=2}^{l} Q^*_j\right) \circ Q_{l+1}\right)(x,D) +$$
$$+2(Q^*_{l+1} \circ Q_{l+1})(x,D) =: L^*(x,D) + L(x,D) + R_{l+1}(x,D),$$

where $R_{l+1}(\cdot,\cdot) \in S^{M-(l+1),m-(l+1)}_{\Phi,\phi}$. This completes the proof. ∎

5.2. We assume that there exists $N \in \mathbb{N}$ so that

$$S^{M-N,m-N}_{\Phi,\phi} \subset S^{0,0}_{\Phi,\phi}. \tag{5.20}$$

From the previous lemma we obtain

THEOREM 5.2. *Suppose that* $L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}$ *so that the estimate* (5.3) *is valid. In addition, assume that the inclusion* (5.20) *holds with some* $N \in \mathbb{N}$. *Let* L *be defined by* (5.1) *where* $a \in L_1 \cap L_\infty$ *and* $b \in L_2$ *such that*

$$(Fa)(\lambda) \geq 0,\ m\text{-a.e. } \lambda \in \mathbb{R}^{n'},\ a(x) = \overline{a(-x)} =: \overline{a^\vee(x)},\ b(x) = \overline{b(x)}. \tag{5.21}$$

Then there exists a constant $c \geq 0$ *so that*

$$\operatorname{Re}[\mathsf{L}\Psi,\Psi]_0 \geq -c|||\Psi|||^2_0 \quad \text{for all } \Psi \in D(\mathbb{R}^{n'},S). \tag{5.22}$$

PROOF. A. Let N be a natural number such that (5.20) holds. Due to Lemma 5.1 we find symbols

$$P_N(\cdot,\cdot) \in S_{\Phi,\phi}^{(1/2)M,(1/2)m} \text{ and } R_N(\cdot,\cdot) \in S_{\Phi,\phi}^{M-N,m-N} \subset S_{\Phi,\phi}^{0,0}$$

so that

$$L^*(x,D) + L(x,D) = 2(P_N^* \circ P_N)(x,D) - R_N(x,D).$$

A direct computation yields (we denote $dm(t) = \mathrm{d}\,t$)

$$[\mathsf{L}\Psi,\Psi]_0 = \int_{\mathbf{R}^{n'}} ((\mathsf{L}\Psi)(t),\Psi(t))_0 \mathrm{d}\,t =$$

$$(5.23) \qquad = \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} a(t-s)b(t)b(s)L(\Psi(s))\mathrm{d}\,s, \Psi(t) \right)_0 \mathrm{d}\,t =$$

$$= \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} a(t-s)(L((b\Psi)(s)),(\bar{b}\Psi)(t))_0 \mathrm{d}\,s \right) \mathrm{d}\,t$$

and

$$[\Psi,\mathsf{L}\Psi]_0 = \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} \overline{a(t-s)}((\bar{b}\Psi)(t), L((b\Psi)(s)))_0 \mathrm{d}\,s \right) \mathrm{d}\,t =$$

$$(5.24) \qquad = \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} \overline{a(t-s)}(L^*(\bar{b}\Psi)(t),(b\Psi)(s))_0 \mathrm{d}\,s \right) \mathrm{d}\,t =$$

$$= \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} \overline{a(t-s)}(L^*(\bar{b}\Psi)(t),(b\Psi)(s))_0 \mathrm{d}\,t \right) \mathrm{d}\,s =$$

$$= \int_{\mathbf{R}^{n'}} \left(\int_{\mathbf{R}^{n'}} \overline{a^{\vee}(t-s)}(L^*(\bar{b}\Psi)(t),(b\Psi)(s))_0 \mathrm{d}\,s \right) \mathrm{d}\,t.$$

Here we applied (2.27) and in the third step wee used the Fubini Theorem, which is legitimate, since the function

$$(s,t) \to a(t-s)(L^*(\bar{b}\Psi)(t),(b\Psi)(s))_0 = \overline{a(t-s)}\bar{b}(t)\bar{b}(s)(L^*\Psi(t),\Psi(s))_0$$

belongs to $L_1(\mathbf{R}^{n'} \times \mathbf{R}^{n'})$. Noting that $b = \overline{b}$ and $a = \overline{a^\vee}$ we obtain by the Fubini Theorem and by (5.23), (5.24) that

$$
\begin{aligned}
2\,\mathrm{Re}[\mathbf{L}\Psi, \Psi]_0 &= [\mathbf{L}\Psi, \Psi]_0 + [\Psi, \mathbf{L}\Psi]_0 = \\
&= \int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} a(t-s)((L+L^*)((b\Psi)(s), (b\Psi)(t)))_0 = \\
&= 2 \int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} a(t-s)(P_N((b\Psi)(s)), P_N((b\Psi)(t)))_0 - \\
&\quad - \int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} a(t-s)(R_N((b\Psi)(s)), (b\Psi)(t))_0.
\end{aligned} \tag{5.25}
$$

C. For the rest term in (5.25) we get

$$
\left| \int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} a(t-s)(R_N((b\Psi)(s)), (b\Psi)(t))_0 \right| \leq \tag{5.26}
$$

$$
\leq \int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} |a(t-s)|\, \|R_N(b\Psi)(s))\|_0 \|(b\Psi)(t)\|_0 \leq C\|a\|_{L_\infty} \|b\|_0^2 |||\Psi|||_0^2,
$$

since $\overline{R}_N : L_2 \to L_2$ is bounded.

D. The function

$$
(t, s, x) \to a(t-s) P_N((b\Psi)(s))(x) \overline{P_N((b\Psi)(t))(x)}
$$

belongs to $L_1(\mathbf{R}^{n'} \times \mathbf{R}^{n'} \times \mathbf{R}^n)$ (note that $P_N(\Psi(s))(x)$ is of the form $\sum\limits_{l=1}^{m} (P_N \varphi_l)(x) \chi_{B_l}(s)$, since Ψ is finitely-valued $R^{n'} \to S$). Hence the Fubini Theorem, the Parseval identity and (5.21) yield

$$
\int\limits_{\mathbf{R}^{n'}} \int\limits_{\mathbf{R}^{n'}} a(t-s)(P_N((b\Psi)(s)), P_N((b\Psi)(t)))_0 \mathrm{d}\,t \mathrm{d}\,s = \tag{5.27}
$$

$$
= \int\limits_{\mathbf{R}^n} \left(\int\limits_{\mathbf{R}^{n'}} \left(\int\limits_{\mathbf{R}^{n'}} a(t-s) P_N((b\Psi)(s))(x) \overline{P_N((b\Psi)(t))(x)} \mathrm{d}\,s \right) \mathrm{d}\,t \right) \mathrm{d}\,x =
$$

$$= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^{n'}} (a * P_N((b\Psi)(\cdot))(x))(t) \overline{P_N((b\Psi)(t))(x)} \mathrm{d}\, t \right) \mathrm{d}\, x =$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^{n'}} F(a * P_N((b\Psi)(\cdot))(x))(\lambda) \overline{F(P_N((b\Psi)(\cdot))(x))(\lambda)} \mathrm{d}\, \lambda \right) \mathrm{d}x =$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^{n'}} (Fa)(\lambda) |F((P_N(b\Psi)(\cdot))(x))(\lambda)|^2 \mathrm{d}\, \lambda \right) \mathrm{d}\, x \geq 0.$$

Thus the assertion follows from (5.25)–(5.27). ∎

REMARK 5.3. A. In the case when $M = m$, the inclusion (5.20) is always valid (with $N = M$). The inclusion (5.20) is also valid (with some $N \in \mathbf{N}$) when there exists a constant $\varepsilon > 0$ such that $\Phi\phi \geq ck_\varepsilon$.

B. The proof of Theorem 5.2 yields also the following fact: Suppose that $L(\cdot)$ satisfies (2.8) and that

$$\operatorname{Re} L(\xi) \geq 0. \tag{5.28}$$

Let $a \in L_1 \cap L_\infty$, $b \in L_2$ such that (5.21) holds. Then the operator L defined by

$$(\mathsf{L}\Psi)(t) = \int_{\mathbf{R}^{n'}} a(t-s)b(t)b(s)L(D)(\Psi(s))\mathrm{d}\, s \tag{5.29}$$

satisfies

$$\operatorname{Re}[\mathsf{L}\Psi, \Psi]_0 \geq 0 \qquad \text{for all } \Psi \in D(\mathbf{R}^{n'}, S). \tag{5.30}$$

We have only to note that $L(\xi) + L^*(\xi) = \operatorname{Re} L(\xi) = ((L_{\mathrm{Re}}(\xi))^{1/2})^2$, and then the proof runs similarly as in Theorem 5.2.

C. Suppose that $n' = 1$. Let T be a positive number. We can choose $b = \chi_{[0,T]}$. Then the operator L is of the form

$$(\mathsf{L}\Psi)(t) = \begin{cases} \int_0^T a(t-s)L(\Psi(s))\mathrm{d}\, s, & t \in [0,T]. \\ 0, & t \notin [0,T]. \end{cases}$$

The function a can be chosen to be an extension of a continuous function $A : [-T, T] \to \mathbf{R}$ such that $a(t) = a(-t)$, $a \in L_1 \cap L_\infty$ and that $(Fa)(\lambda) \geq 0$ a.e. $\lambda \in \mathbf{R}$.

D. Suppose that $b = \chi_{[0,T]}$ and that A is a continuous function $[0,T] \to \mathbf{R}$. Let a be the null-extension $\mathbf{R} \to \mathbf{R}$ of A. Then one has

$$(\mathsf{L}\Psi)(t) := \int_0^t A(t-s)L(\Psi(s))\,\mathrm{d}\,s = \int_0^T a(t-s)L(\Psi(s))\,\mathrm{d}\,s$$

and

$$(\mathsf{L}^*\Psi)(t) = \int_0^T a(s-t)L^*(\Psi(s))\,\mathrm{d}\,s = \int_t^T a(s-t)L^*(\Psi(s))\,\mathrm{d}\,s$$

and so

$$((\mathsf{L}+\mathsf{L}^*)\Psi)(t) = \int_{\mathbf{R}} \tilde{a}(t-s)b(t)b(s)(L+L^*)(\Psi(s))\,\mathrm{d}\,s,$$

where

$$\tilde{a}(t) = \begin{cases} a(t), & \text{for } t \geq 0 \\ a(-t), & \text{for } t \leq 0, \end{cases}$$

and where L^* and L^* are the formal adjoints of L and L, resp.. Since $2\,\mathrm{Re}[\mathsf{L}\Psi,\Psi]_0 = [(\mathsf{L}+\mathsf{L}^*)\Psi,\Psi]_0$, the positivity of L can be examined with the help of $\mathsf{L}+\mathsf{L}^*$, which is of the considered form.

E. Let $k(\cdot)$ be in $S^{M',m'}_{\Phi,\phi}$ so that $k \geq c\Phi^{M'}\phi^{m'}$. Applying the symbolic calculus on $\bigcup\limits_{M,m} S^{M,m}_{\Phi,\phi}$, similar criteria as in Theorem 5.2 can be obtained also for positivity of $\mathsf{L}^{\sim}_k : L_2(\mathbf{R}^{n'}, H_k) \to L_2(\mathbf{R}^{n'}, H_k)$.

F. We remark that in the proof of Theorem 5.2 we verified the estimate

$$\begin{aligned}\mathrm{Re}[\mathsf{L}\Psi,\Psi]_0 \geq & (2\pi)^{-n}\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^{n'}}(Fa)(\lambda)F(P_N(b\Psi)(\cdot))(x)(\lambda)|^2\,\mathrm{d}\,\lambda\Big)\mathrm{d}\,x - \\ & - C\|a\|_{L_\infty}\|b\|_0^2|||\Psi|||_0^2.\end{aligned}$$

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ON RANGES OF ADJOINT OPERATORS

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For a densely defined operator A in the (complex) Hilbert space H there is an interesting recent characterization ([4], Theorem 1) of the range of A^* i.e. a necessary and sufficient condition for the solution of the equation

$$A^*z = y \tag{1}$$

where y is a given element of H and z is of course the solution of (1) searched for.

The method of proof by SEBESTYÉN gives a solution of a problem of DOUGLAS [2] concerning the factorization of unbounded operators, especially with a bounded cofactor. A characterization of the range of a contraction (i.e. an operator on H with norm not greater than one) is given in ([3], Theorem 4.1) recalling that the result is implicit in the work of DE BRANGES and ROVNYAK ([1], p. 23) and that it seems to have been first proved by R. C. DOUGLAS.

Here we give a form of characterization of the range of adjoint, not necessarily bounded operator of the latter kind in (ii) of the theorem below, where (iii) is Sebestyén's result.

THEOREM. *Let A be a densely defined operator with domain $D(A)$ in a (complex) Hilbert space H and y in H. The following assertions are equivalent:*

(i) *There is z in H such that $A^*z = y$,*

(ii) *There is a constant $c \geq 0$ such that*

$$2\mathrm{Re}\langle x, y\rangle - \|Ax\|^2 \leq c \qquad (x \in D(A)),$$

(iii) *There is $m \geq 0$ such that*

$$|\langle x, y\rangle \leq m\|Ax\| \qquad (x \in D(A)).$$

PROOF. (i) $\Longrightarrow$ (ii). Using (i) we have

$$2\mathrm{Re}\langle x,y\rangle - \|Ax\|^2 = 2\mathrm{Re}\langle x,A^*z\rangle - \|Ax\|^2 = 2\mathrm{Re}\langle Ax,z\rangle - \|Ax\|^2 = \\ = \|z\|^2 - \|Ax-z\|^2 \le \|z\|^2 = c \qquad (x\in D(A))$$

(ii) $\Longrightarrow$ (iii). If x_0 is an arbitrary fixed element of $D(A)$, taking $\Theta_0 \in [0,2\pi)$ such that $\langle x_0,y\rangle e^{i\Theta_0} = |\langle x_0,y\rangle|$. We have from (ii), with $x_t = e^{i\Theta_0}tx_0$ ($t\in\mathbf{R}$) that

$$2t|\langle x_0,y\rangle| - t^2\|Ax_0\|^2 = 2\mathrm{Re}\langle x_t,y\rangle - \|Ax_t\|^2 \le c \qquad (t\in\mathbf{R})$$

and then that

$$2t|\langle x_0,y\rangle| - t^2\|Ax_0\|^2 \le c \qquad (t\in\mathbf{R}). \tag{2}$$

It is easy to see from (2) that $Ax_0 = 0$ implies $|\langle x_0,y\rangle| = 0$ and then that

$$|\langle x_0,y\rangle| = 0 = \|Ax_0\|. \tag{$*$}$$

Then we can assume, without losing generality that $Ax_0 \ne 0$. In this case taking $t_0 = \frac{|\langle x_0,y\rangle|}{\|Ax_0\|^2}$ in (2), we have

$$\frac{|\langle x_0,y\rangle|^2}{\|Ax_0\|^2} = 2t_0|\langle x_0,y\rangle| - t_0^2\|Ax_0\|^2 \le c.$$

Since x_0 in $D(A)$ is arbitrary it follows from this and ($*$) that

$$|\langle x,y\rangle| \le \sqrt{c}\|Ax\| \qquad (x\in D(A))$$

(iii) $\Longrightarrow$ (i). As in the proof of ([4], Theorem 1), let $\varphi : R(A) \to \mathbf{C}$ be given on the range $R(A)$ of A by

$$\varphi(Ax) = \langle x,y\rangle \qquad (x\in D(A)).$$

φ is continuous. Therefore φ can be extended in a unique way to a continuous linear functional on the closed subspace $\overline{R(A)}$ of H. Here $\overline{R(A)}$ is a Hilbert space in its own right, so the celebrated Riesz representation theorem gives a uniquely determined z in $\overline{R(A)}$ with $\varphi(w) = \langle w,z\rangle$ ($w\in R(A)$). In particular we have

$$\langle x,y\rangle = \varphi(Ax) = \langle Ax,z\rangle \qquad (x\in D(A)).$$

It follows from this that, by the definition of the adjoint,

$$z\in D(A^*), \quad A^*z = y$$

The proof is complete.

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SPANNING TREES OF A GRAPH AND NETWORK RELIABILITY

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1. Introduction

An undirected graph may be considered as a model of network for transmission of material flows. The lines of the network may be unreliable. One of the characteristics of the network reliability is the number of spanning trees of the corresponding graph.

It is natural to consider two types of problems on network reliability:

(P1) a problem of analysis: find the number of spanning trees of a graph (or obtain formulae for the number of spanning trees for some classes of graphs) and

(P2) a problem of synthesis: construct a graph with given number of vertices and edges which has the maximal (or minimal) number of spanning trees (or, moreover, describe "optimal" graphs of some types).

Many papers are devoted to the first problem (see for example [1], [2], [3]). One approach to this problem is based on the fact that the number of trees of a graph equals a principal minor of a certain matrix of the graph [4]. In many papers this fact was used to derive formulae for the number of spanning trees of a graph.

An investigation of the characteristic polynomial of the above matrix (we call it the polynomial of the graph) gave an algorithm for obtaining formulae for polynomials and for their roots (and, as a byproduct, for the number of spanning trees) of graphs decomposable with respect to certain operations onto graphs with known polynomials [5].

Many formulae which were derived elsewhere may be easily obtained by the method described in [5] since the graphs for which these formulae were deduced may be constructed by using the above operations from graphs whose polynomials are known or may be easily found.

Another way to find the number of spanning trees of a graph is to use the principle of inclusion and exclusion [6]. By this method formulae for the number of trees in some special cases were also obtained. As it was shown in [7] the basic formula for the number of spanning trees of a graph derived in [6] by the principle of inclusion and exclusion is one of the relations between the coefficients of the polynomial of a graph and its complement stated in [8]. This shows the relationship between these approaches.

Note that the problem of counting spanning trees in a graph is a part of a more general problem of determination of the probability of connectivity of a graph with randomly deleted edges [9], [12].

Much fewer papers are devoted to the construction of graphs with given number of vertices and edges and the extremal number of spanning trees (see e.g. [7], [8], [13–15]). As A. K. KELMANS has noted in his publications, there are different directions for generalizing the above problem. One of the generalizations is to find a subset X of m vectors of a set A from the n-dimensional space K^n (K is a field or a ring) such that X has the maximal number of bases of X.

Another generalization is to construct a graph with a given number of vertices and edges and with the maximal probability of connectivity (see e.g. [11], [16], [17]).

In [18] a coding of trees is given providing a natural correspondence between the set of codes of spanning trees of a graph and the set of codes of spanning trees of the so called extension of the graph. This correspondence provides a simple relation between the so called volumes of a graph and its extension (and in particular a simple relation between the spanning tree numbers of a graph and its regular extension). The results of this paper can be used to obtain simple combinatorial proofs of many previous results on the number of spanning trees of a graph and to obtain formulae for the number of spanning trees of graphs of some new classes.

This paper is based on [18]. Using the above mentioned results we obtain formulae for the number of spanning trees of graphs of some types. By means of these formulae we find graphs with given number of vertices and edges which have the extremal number of spanning trees among the considered classes of graphs.

2. Some notions and notations

We consider undirected graphs without parallel edges but possibly with loops [4]. Let VG and EG denote the set of vertices and edges of a graph G, respectively. Let $d(v, G)$ denote the number of edges of G incident to a vertex v (loops are counted once).

Let $\tau(G)$ and $t(G)$ denote the set and the number of spanning trees of a graph G, respectively.

Let $\pi(G) = \pi\{d(v, G) : v \in VG\}$.

Let $g(n, m)$ denote the set of graphs with n vertices and m edges.

3. Previous results

1. Let G be an undirected graph. Let X be a finite set of elements and $X_v \subseteq X$ for $v \in VG$. Put $X(v) = \{vx : x \in X_v\}$ for $v \in VG$. Put $\chi = \{X(v) : v \in VG\}$.

Let us construct the new graph $\Gamma = G(\chi)$ as follows:

(a1) $V\Gamma = \cup\{X(v) : v \in VG\}$ and

(a2) for a pair of vertices v_1x_1 and v_2x_2 of Γ

$$(v_1x_1, v_2x_2) \in E\Gamma \text{ iff } (v_1, v_2) \in EG.$$

We call $G(\chi)$ the *χ-extension* of G.

Put $|VG| = n$, $|X(v)| = k(v)$ and $|VG(\chi)| = s$ so that $\sum\{k(v) : v \in VG\}$.

2. Let $f : VG \to K$ be an arbitrary function where K is a commutative ring.

Suppose first that G is a tree T. Put

$$R(T, f) = \pi\{f(v)^{d(v,T)-1} : v \in VT\}.$$

For an arbitrary graph G put

$$R(G, f) = \sum\{R(T, f) : T \in \tau G\}.$$

We call $R(G, f)$ the *spanning tree volume of a weighted graph* (G, f) (or simply a *volume* of (G, f)). Put

$$f^{\chi}(v) = \sum\{f(vx) : vx \in X(v),\ v \in VG\}.$$

3. THEOREM [18].

$$R([G]^{\chi}, f) = Q(G, \chi, f^{\chi}) \cdot R(G, f^{\chi}),$$

where

$$Q(G, \chi, f^{\chi}) = \pi\Big\{\Big[\sum\{f^{\chi}(a) : a \in VG$$

and

$$(a, v) \in EG\}\Big]^{k(v)-1} : v \in VG\Big\}.$$

4. Consider a particular case when $f \equiv 1$. Then $R(G(\chi), f) = t(G(\chi))$. Suppose also that all the sets $X(v)$ in $\chi = \{X(v) : v \in VG\}$ have the same cardinality: $|X(v)| = k$ for any $v \in VG$.

Let $[G]^k$ denote the graph $G(\chi)$ in this case. We say that $[G]^k$ is the *k-regular extension* of G.

COROLLARY [18].

$$t([G]^k) = k^{nk-2}\pi\{d(v,G) : v \in VG\}^{k-1} \cdot t(G)$$

Thus we have a formula for the number of spanning trees of the k-regular extension of a graph G in terms of the number of spanning trees and the vertex degrees of G.

5. Let F^m be a graph with m edges (so that P^m and S^m are the chain and the star with m edges). Let $m \cdot P^1$ be the graph-matching with m edges.

THEOREM [7], [13], [14]. *Then*

$$t(K_n - mP^1) > t(K_n - F^m) > t(K_n - S_n^m)$$

for any $n \geq 2m$ and for any graph F^m with m edges distinct from mP^1 and S^m.

6. THEOREM [13], [14].

$$t(K_n - P^m) > t(K_n - F^m) > t(K_n - S^m)$$

for any $n \geq m+1$ and any connected graph F^m with m edges distinct from P^m and S^m.

7. Let C^m be the cycle with m edges.

THEOREM [13].

$$t(K_n - C^m) > t(K_n - F^m)$$

for any $n \geq m$ and any 2-edge-connected graph F^m with m edges.

4. Comparison of graphs by the number of spanning trees of their extensions

1. Let τ_n denote the set of trees with n vertices. We would like to compare n-vertex trees by the number of spanning trees of their extensions. Put $\pi(G) = \pi\{d(v,G) : v \in VG\}$

1.1. Let A and B be disjoint undirected graphs, $a \in VA$ and $b \in VB$. Let $(Aa \circ bB)$ denote the graph obtained from A and B by identifying vertices a from A and b from B with a new vertex $(a \circ b) = z$.

LEMMA. Let a_1, $a_2 \in VA$ and $d(a_1, A) \geq d(a_2, A)$. Then $\pi(Aa_1 \circ \circ bB) \leq \pi(Aa_2 \circ bB)$. The proof is easy, hence omitted.

1.2. Obviously $t(Aa_1 \circ bB) = t(Aa_2 \circ bB)$. Therefore Corollary III. 4. and Lemma 1.1. imply

LEMMA. Let a_1, $a_2 \in VA$. Suppose that $d(a_1, A) \geq (>) d(a_2, A)$. Then $t([Aa_2 \circ bB]^k) \geq (>)\ t[Aa_1 \circ bB]^k)$ for any $k = 2, 3, \ldots$.

1.3. Let T^1 and T^2 be trees with the same number of vertices: T^1, $T^2 \in \tau_n$. We write $T^1 \preceq T^2$ if $T^1 = (Aa_1 \circ bB)$, $T^2 = (Aa_2 \circ bB)$ and $d(a_1, A) \geq d(a_2, A)$ for some A and B.

Let P_n and S_n denote the chain and the star with n vertices, respectively (so that P_n, $S_n \in \tau_n$). One can easily prove.

LEMMA.

$$S_n \preceq T_n \preceq P_n \text{ for any } T_n \in \tau_n.$$

1.4. From Lemma 1.2 we have

LEMMA. Let T^1, $T^2 \in \tau_n$. If $T^1 \leq T^2$ then $\pi(T^1) \leq \pi(T^2)$.

1.5. Corollary III. 4, Lemmata 1.3 and 1.4 imply

THEOREM.

$$t([P_n]^k) = k^{nk-2} \cdot 2^{n-2}, \tag{a1}$$

$$t([S_n]^k) = k^{nk-2} \cdot (n-1), \tag{a2}$$

$$t([S_n]^k) \leq t([T_n]^k) \leq t([P_n]^k) \tag{a3}$$

for any $T_n \in \tau_n$ *and for any* $k = 1, 2, \ldots$.

2. From Corollary III. 4. we have:

THEOREM. *Let G be a graph with n vertices and m edges.*

$$\frac{t([G]^k)}{t(G)} = k^{nk-2}\pi\{d(v, G) : v \in VG\}^{k-1} \leq k^{nk-2}\left(\frac{2 \cdot m}{n}\right)^{k-1}$$

and equality holds iff G is a regular graph.

3. Let as above $g(n, m)$ be the set of graphs with n vertices and m edges. Different authors (e.g. Bedrosian, Kelmans) independently have the following

CONJECTURE. There exists $G^* \in g(n, m)$ such that $t(G^*) \geq t(G)$ and $\pi(G^*) \geq t(G)$ for any $G \in g(n, m)$.

In particular if $2m/n = d$ is an integer then there exists a regular graph $G^* \in g(n, m)$ such that $t(G^*) \geq t(G)$ for any $G \in g(n, m)$.

4. Corollary III. 4. implies

PROPOSITION. Suppose that Conjecture 3 holds for a pair (n,m), and that G^* is a corresponding graph from $g(n,m)$. Then

$$t([G^*]^k) \geq t([G]^k) \text{ for any } G \in g(n,m) \text{ and } k = 1,2,\ldots.$$

5. Consider the set of graphs with n vertices and n edges. Let C_n be the cycle with n vertices and R_n be the graph obtained from the star S_n with n vertices by adding a new edge between two end vertices of S_n.

THEOREM.

(a1) $$t([C_n]^k) = k^{nk-2} \cdot 2^{k-1} \cdot n,$$

(a2) $$t([R_n]^k) = k^{nk-2} \cdot 12(n-1),$$

(a3) $$t([R_n]^k) \leq t([G]^k) \leq t([C_n]^k)$$

for any $G \in g(n,m)$ *and* $k = 1,2,\ldots$.

The proof of the theorem is based on Corollary III. 4. In order to prove the theorem we introduce some operations on graphs which "improve" graphs just the same way as we did it for trees (see Sec. 1). The upper bound can also be proved by means of the Proposition.

6. Consider graphs from $g(n,n+1)$. Let B_n denote the graph which consists of three internally disjoint chains P_1, P_2, P_3 with the common end vertices such that

if $n+1 = 3k$ then each P_i has k edges,

if $n+1 = 3k+1$ then P_1 and P_2 has k edges and P_3 has $k+1$ edges,

if $n+1 = 3k+2$ then P_1 and P_2 has $k+1$ edges and P_3 has k edges.

Obviously $B_n \in g(n,n+1)$. A. K. Kelmans proved before that

$$t(B_n) > t(G) \text{ for } G \in g(n,n+1) - \{B_n\}.$$

It is easy to see that

$$\pi(B_n) > \pi(G) \text{ for any } G \in g(n,n+1).$$

Thus for $g(n,n+1)$ Conjecture 3 holds and $G^* = B_n$. Therefore from Proposition 4 we have

THEOREM.

$$t([B_n]^k) > t([G]^k) \textit{ for any } G \in g(n,n+1) - \{B_n\}.$$

7. Using Proposition 4, Corollary III. 4. we obtain from Theory III. 5. – III. 7.

THEOREM.

(a1) $$t([K_n - mP^1]^k) > t([K_n - F^m]^k) > t([k_n - S^m]^k)$$

for any $n \geq 2m$, $k = 1, 2, \ldots$ *and for any graph* F^m *with* m *edges distinct from* mP^1 *and* S^m.

(a2) $$t([K_n - P^m]^k) > t([K_n - F^m]^k)$$

for any $n \geq m + 1$, $k = 1, 2, \ldots$ *and for any connected graph* F^m *with* m *edges distinct from* P^m.

(a3) $$t([K_n - C^m]^k) > t([K_n - F^m]^k)$$

for any $n \geq m$, $k = 1, 2, \ldots$ *and for any 2-edge-connected graph* F^m *distinct from* C^m.

5. Conclusion

In general it is not clear what is the relation between the number of spanning trees of a graph and its set of the vertex degrees. It seems to be natural that (as pointed out in Conjecture 3) among the graphs with the maximal number of spanning trees (and with given number of vertices and edges) there should be a rather regular graph with respect to its vertex degrees. But now this conjecture is proved only in some particular cases. The importance of Corollary 4 is just the fact that it gives the directed interconnection between the set of the vertex degrees of a graph and the number of spanning trees of its extension. This makes it possible to use previous results on graphs with the maximal number of spanning trees for obtaining new constructions of optimal graphs under some restrictions.

In this paper we gave new constructions of graphs with the extremal number of spanning trees among the graphs obtained by a regular extension from graphs with fixed number of vertices and edges. The above results prove in particular that the conjecture on regularity of optimal graphs holds for the cases sonsidered.

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NOTE ON A THEOREM OF J. L. MAUCLAIRE AND LEO MURATA ON MULTIPLICATIVE FUNCTIONS

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An arithmetic function $f(n) \not\equiv 0$ is said to be multiplicative if $(n,m)=1$ implies that

$$f(nm) = f(n)f(m)$$

and it is completely multiplicative if the above relation holds for all n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of complex-valued multiplicative and completely multiplicative functions, respectively.

In 1980 J. L. MAUCLAIRE and LEO MURATA [3] proved that if $f \in \mathcal{M}$ satisfies the properties

$$|f(n)| = 1 \qquad (n = 1, 2, \ldots)$$

and

$$\frac{1}{x}\sum_{n\leq x} |f(n+1) - f(n)| \to 0 \qquad (x \to \infty),$$

then $f \in \mathcal{M}^*$. By using the method of K. H. INDLEKOFER and I. KÁTAI [1]–[2] one can prove that for any fixed positive integer B the conditions

$$f \in \mathcal{M}, \quad |f(n)| = 1 \qquad (n = 1, 2, \ldots),$$
$$\frac{1}{x}\sum_{n\leq x} |f(n+B) - f(n)| \to 0 \qquad (x \to \infty)$$

imply that

$$f(p^k) = (f(p))^k \qquad (k = 1, 2, \ldots)$$

for each prime p coprime to B.

The aim of this note is to prove the following

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THEOREM. *Let A, B be positive integers and let C be a non-zero complex number. If $f \in \mathcal{M}$ satisfies the conditions*

$$(1) \qquad |f(n)| = 1 \qquad (n = 1, 2, \ldots)$$

$$(2) \qquad \frac{1}{x} \sum_{n \le x} |f(An + B) - Cf(n)| \to 0 \qquad (x \to \infty).$$

then

$$(3) \qquad f(p^k) = (f(p))^k \qquad (k = 1, 2, \ldots)$$

for each prime p coprime to $2AB$. If $(2, AB) = 1$, then (1) and (2) imply that

$$f(2^k) = \left(\frac{f(A)}{C}\right)^{k-1} (f(2))^k \qquad (k = 1, 2, \ldots)$$

and

$$f(A)^2 = C^2.$$

PROOF. Let A, B be fixed positive integers. For an arbitrary positive integer n, let $B(n)$ be the product of prime factors of B composed from the prime divisors of n, i.e. $B(n)|B$, $(B(n), B/B(n)) = 1$ and every prime divisor of $B(n)$ is a divisor of n.

For each positive integer Q we define the sequence $R = R(AQ) = \{R_k\}_{k=0}^{\infty}$ by the initial term $R_0 = 0$ and by the formula

$$(4) \qquad R_k = 1 + \ldots + (AQ)^{k-1}$$

for $k \ge 1$. Moreover, let

$$(5) \qquad T_k(n, Q) = (AQ)^k B(Q) n + B R_k(AQ).$$

First we show that $f \in \mathcal{M}$ with the property (2) implies that

(6)

$$\sum_{n \le x} \left| f(T_k(n, Q)) - C^k \left(\frac{f(QB(Q))}{f(B(Q))}\right)^{k-1} f(QB(Q)n) \right| = o(x) \ (x \to \infty)$$

holds for every positive integer k, Q. By using (2) and (5), it is obvious that (6) is true for $k = 1$. Let $k > 1$ and assume that (6) holds for $k - 1$. By using (4) and (5), we have

$$T_k(n, Q) = AQT_{k-1}(n, Q) + B$$

and

$$(QB(Q), T_{k-1}(n, Q)/B(Q)) = 1.$$

Thus, by (2) and using the fact $f \in \mathcal{M}$, we get

$$(7)\quad \sum_{n\le x}\left|f(T_k(n,Q))-C\frac{f(QB(Q))}{f(B(Q))}f(T_{k-1}(n,Q))\right|=o(x)\quad (x\to\infty).$$

Since (6) holds for $k-1$, (7) implies that (6) also holds for k. So (6) has been proved.

Let $f\in\mathcal{M}$ satisfying the properties (1) and (2). We shall prove that if the positive integers k and Q satisfy the conditions

$$(8)\qquad (R_k(AQ),B(Q)+B)=1$$

then

$$(9)\qquad f(A^{k-1}Q^kB(Q))=C^{k-1}\frac{(f(QB(Q)))^k}{(f(B(Q)))^{k-1}}.$$

Assume that (8) holds. Let $R_k=R_k(AQ)$. Then, we have

$$(10)\quad (R_k,(AQ)^kB(Q)(AQR_km+1)+B)=(R_k,B(Q)+B)=1$$

for every integer m. Considering

$$n=R_k(AQR_km+1)$$

and taking into account (6), using (1) and (10), we get

$$\sum_{m\le x}\left|f((AQ)^kB(Q)(AQR_km+1)+B)-\right.$$
$$\left.-C^k\left(\frac{f(QB(Q))}{f(B(Q))}\right)^{k-1}f(QB(Q)(AQR_km+1))\right|=o(x)$$

and so, by (2)

$$\left|Cf(A^{k-1}Q^kB(Q))-C^k\frac{f(QB(Q))^k}{f(B(Q))^{k-1}}\right|\sum_{m\le x}|f(AQR_km+1)|=o(x)$$

as $x\to\infty$. This with (1) implies (9).

Let p be prime for which $(p,2AB)=1$. It is obvious that (8) holds for every positive integer k if $Q=2B$ or $Q=2Bp$. In this case $B(Q)=B$ and by (9) we obtain

$$(11)\qquad f(A^{k-1}2^kB^{k+1})=C^{k-1}f(2B^2)^k/f(B)^{k-1}$$

and

$$f(A^{k-1}2^kB^{k+1}p^k)=C^{k-1}f(2B^2p)^k/f(B)^{k-1}$$

which, using the fact $f\in\mathcal{M}$ and $(p,2AB)=1$, give

$$(12)\qquad f(p^k)=(f(p))^k$$

for every positive integer k. So we have proved (3).

Finally, assume that $(2, AB) = 1$. In this case (8) holds for every odd integer $k \geq 1$ if $B|Q$. Thus, by applying (9) in the case $Q = B$, we have

$$f(A^{k-1}B^{k+1}) = C^{k-1}f(B^2)^k/f(B)^{k-1},$$

and so, by (11)

$$f(2^k) = (f(2))^k \tag{13}$$

for each odd integer $k \geq 1$. The proof of the theorem has been finished if we show that

$$f(2^k) = (f(A)/C)^{k-1}(f(2))^k \tag{14}$$

holds for each positive integer k, since by (13), (14) we have $f(A)^2 = C^2$.

Let $B = B(A)B_0$. By Dirichlet's theorem there is a positive integer Q_0 such that

$$AQ_0 \equiv 1 \pmod{4B_0} \quad \text{and} \quad (Q_0, 2AB) = 1. \tag{15}$$

Let $R = R_2(AQ_0) = 1 + AQ_0$. Using (15), we have $(R/2, 2AB) = 1$ and by (12)

$$f(R^k) = f(2^k)f(R/2)^k \quad \text{and} \quad f(Q_0^k) = f(Q_0)^k \tag{16}$$

for every positive integer k. Since

$$\begin{aligned} &Cf(R)f(AQ_0^2R^km) - C^2f(Q_0)f(Q_0R^{k+1}m) = \\ &\quad = \{f((AQ_0)^2R^{k+1}m + BR) - C^2f(Q_0)f(Q_0R^{k+1}m)\} - \\ &\qquad\qquad -\{f(R)f((AQ_0)^2R^km + B) - Cf(R)f(AQ_0^2R^km)\}. \end{aligned}$$

by (6), using (1), we get

$$\begin{aligned} &\sum_{\substack{m \leq x \\ (m, AQ_0R)=1}} |f(R)f(AQ_0^2R^km) - Cf(Q_0)^2f(R^{k+1}m)| = \\ &\quad = |f(R)f(AQ_0^2R^k) - Cf(Q_0)^2f(R^{k+1})| \sum_{\substack{m \leq x \\ (m, AQ_0R)=1}} |f(m)| = o(x) \end{aligned}$$

and

$$f(R)f(AQ_0^2R^k) = Cf(Q_0)^2f(R^{k+1}). \tag{17}$$

By (16) and (17) it follows that

$$f(2^{k+1}) = \frac{f(A)}{C}f(2)f(2^k), \tag{18}$$

which proves (14). This completes the proof of the theorem.

REMARKS. (i) Similarly to the proof of (18), using (6) and (12) one can deduce that if $f \in \mathcal{M}$ with the properties (1) and (2) then

$$f(P^k) = \left(\frac{f(A^{P-1})}{C^{P-1}}\right)^{k-1} (f(P))^k$$

for each positive integer P coprime to AB. Thus, from our theorem, we have

$$f(A^{P-1}) = C^{P-1}$$

for each positive integer P coprime to $2AB$.

(ii) Using (8), (9) and (12) one can prove that if $(2, AB) = 1$, then

$$f(A^k 2^{k+1}) = C^k f(2)^{k+1}$$

and so

$$f(A^k) f(A)^k = C^{2k}$$

for each positive integer k.

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MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY II.

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1. An arithmetic function $f(n) \not\equiv 0$ is said to be multiplicative if $(n,m)=1$ implies that

$$f(nm) = f(n)f(m) \tag{1}$$

and it is completely multiplicative if (1) holds for all n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of integer-valued multiplicative and completely multiplicative functions, respectively.

M. V. SUBBARAO [3] proved that if $f \in \mathcal{M}$ satisfies the relation

$$f(n+m) \equiv f(n) \pmod{m} \tag{2}$$

for every positive integer n and m, then $f(n)$ is a power of n with non-negative integer exponent. In [1] among others we extended this result proving that if (2) holds for every positive integer n and every prime m, then $f(n)$ also is of the same form.

In the space of the sequences $\{x_1, x_2, \ldots\}$ let $E, I, \triangle$ denote the operators defined by following relations

$$Ex_n = x_{n+1}, \quad Ix_n = x_n, \quad \triangle x_n = x_{n+1} - x_n.$$

for any integer $k \geq 0$ let $\triangle^k = (E-I)^k$ with $\triangle^0 = I$.

The aim of this paper is to prove the following

THEOREM. *Let $k \geq 0$ be an integer. If $f \in \mathcal{M}$ and $\triangle^k f(n)$ satisfies the relation*

$$\triangle^k f(n+p) \equiv \triangle^k f(n) \pmod{p} \tag{3}$$

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for every positive integer n and every prime p, then $f(n) = n^a$, where a is a non-negative integer.

2. LEMMAS. We need some lemmas for the proof of our theorem. Note that in Lemma 1–2, we shall not use the multiplicativity of $f(n)$.

LEMMA 1. Let $f(n)$ be an integer-valued arithmetic function and k, Q be positive integers. If $\Delta^k f(n)$ satisfies the relation

$$\Delta^k f(n+Q) \equiv \Delta^k f(n) \pmod{Q} \tag{4}$$

for every positive integer n, then for $s = 1, \dots, k$

$$\Delta^{k-s} f(n+tQ) - \Delta^{k-s} f(n) \equiv \sum_{j=0}^{s-1} \binom{n-1}{j} \Delta_f^{k-s+j}(Q,t) \pmod{Q} \tag{5}$$

holds for every integer $n \geq 1$ and $t \geq 0$, where

$$\Delta_f^i(Q,t) = \Delta^i f(1+tQ) - \Delta^i f(1) \quad (i = 0, 1, \dots).$$

PROOF. Let t be a fixed non-negative integer. Obviousily, (5) holds for $n = 1$. Let $n > 1$. We shall prove (5) by induction on s.

Using (4), we have

$$\sum_{i=1}^{n-1} \Delta^k f(i+tQ) \equiv \sum_{i=1}^{n-1} \Delta^k f(i) \pmod{Q},$$

and so

$$\Delta^{k-1} f(n+tQ) - \Delta^{k-1} f(1+tQ) \equiv \Delta^{k-1} f(n) - \Delta^{k-1} f(1) \pmod{Q},$$

which proves (5) in the case $s = 1$.

Assume that $s < k$ and (5) holds for s. By using (5), we have

$$\Delta^{k-(s+1)} f(n+tQ) - \Delta^{k-(s+1)} f(1+tQ) - \\ - \Delta^{k-(s+1)} f(n) + \Delta^{k-(s+1)} f(1) =$$

$$= \sum_{i=1}^{n-1} \left\{ \Delta^{k-s} f(i+tQ) - \Delta^{k-s} f(i) \right\} \equiv$$

$$\equiv \sum_{i=1}^{n-1} \left\{ \sum_{j=0}^{s-1} \binom{i-1}{j} \Delta_f^{k-s+j}(Q,t) \right\} =$$

$$= \sum_{j=0}^{s-1} \left\{ \Delta_f^{k-s+j}(Q,t) \sum_{i=1}^{n-1} \binom{i-1}{j} \right\} =$$

$$= \sum_{j=0}^{s-1} \left\{ \Delta_f^{k-s+j}(Q,t) \binom{n-1}{j+1} \right\} =$$

$$= \sum_{j=1}^{s} \binom{n-1}{j} \Delta_f^{k-(s+1)+j}(Q,t) \pmod{Q},$$

and so

$$\Delta^{k-(s+1)} f(n+tQ) - \Delta^{k-(s+1)} f(n) \equiv$$

$$\equiv \sum_{j=0}^{s} \binom{n-1}{j} \Delta_f^{k-(s+1)+j}(Q,t) \pmod{Q},$$

which proves that (5) holds for $(s+1)$ and so Lemma 1 is proved.

LEMMA 2. Let $f(n)$ be an integer-valued arithmetic function, k be a positive integer and Q be a prime. If $\Delta^k f(n)$ satisfies the relation (4) for every positive integer n, then for $s = 1, \dots, k$

$$\Delta_f^{k-s}(Q,t) \equiv \sum_{j=0}^{\left[\frac{s-1}{Q}\right]} \binom{t}{j+1} \Delta_f^{k-s+jQ}(Q,1) \pmod{Q}, \tag{6}$$

holds for every positive integer t, where $[x]$ denotes the largest integer not exceeding x.

PROOF. Applying (5) in the case $n = 1 + iQ$ and $t = 1$, we have

$$\Delta^{k-s} f(1+(i+1)Q) - \Delta^{k-s} f(1+iQ) \equiv$$

$$\equiv \sum_{j=0}^{s-1} \binom{iQ}{j} \Delta_f^{k-s+j}(Q,1) \equiv$$

$$\equiv \sum_{j=0}^{\left[\frac{s-1}{Q}\right]} \binom{iQ}{jQ} \Delta_f^{k-s+jQ}(Q,1) \equiv \tag{7}$$

$$\equiv \sum_{j=0}^{\left[\frac{s-1}{Q}\right]} \binom{i}{j} \Delta_f^{k-s+jQ}(Q,1) \pmod{Q},$$

because it is well-known that for a prime Q we have $\binom{iQ}{j} \equiv 0 \pmod{Q}$ if $(Q,j)=1$ and $\binom{iQ}{jQ} \equiv \binom{i}{j} \pmod{Q}$. From (7) it follows that

$$\begin{aligned}
&\Delta^{k-s} f(1+tQ) - \Delta^{k-s} f(1) = \\
&= \sum_{i=0}^{t-1} \left\{ \Delta^{k-s} f(1+(i+1)Q) - \Delta^{k-s} f(1+iQ) \right\} \equiv \\
&\equiv \sum_{i=0}^{t-1} \sum_{j=0}^{[(s-1)/Q]} \binom{i}{j} \Delta_f^{k-s+jQ}(Q,1) = \\
&= \sum_{j=0}^{[(s-1)/Q]} \left\{ \Delta_f^{k-s+jQ}(Q,1) \sum_{i=0}^{t-1} \binom{i}{j} \right\} = \\
&= \sum_{j=0}^{[(s-1)/Q]} \binom{t}{j+1} \Delta_f^{k-s+jQ}(Q,1) \pmod{Q},
\end{aligned}$$

which proves Lemma 2.

LEMMA 3. Let α be a positive integer. If $f \in \mathcal{M}$ satisfies the relation

$$f(n+p^{\alpha}) \equiv f(n) \pmod{p} \tag{8}$$

for every positive integer n and every prime p, then $f \in \mathcal{M}^*$ and for each prime q

$$f(q) = q^{a(q)}, \tag{9}$$

where $a(q) \geq 0$ is an integer.

PROOF. Assume that $f \in \mathcal{M}$ satisfies the relation (8) for every positive integer n and every prime p. We first note that by (8) it follows that

$$f(n+hp^{\alpha}) \equiv f(n) \pmod{p} \tag{10}$$

for every integer $n \geq 1$, $h \geq 0$ and every prime p.

We shall prove that for a prime p and a positive integer m

$$p \mid f(m) \quad \text{implies} \quad p \mid m. \tag{11}$$

Assume indirectly that for a prime p

$$p \mid f(m) \quad \text{and} \quad p \nmid m.$$

Thus $(p,m) = 1$ and so by Dirichlet's theorem there exist positive integers x, y such that

$$(m,x) = 1 \quad \text{and} \quad mx = 1 + p^\alpha y. \tag{12}$$

Applying (10), by using (12), we have

$$0 \equiv f(m)f(x) = f(mx) = f(1+p^\alpha y) \equiv f(1) = 1 \pmod{p},$$

which is a contradiction, since p is a prime. So we proved (11).

We now prove that $f \in \mathcal{M}$ with (8) implies $f \in \mathcal{M}^*$. We show it by proving that for each prime q and each positive integer s

$$f(q^s) = (f(q))^s. \tag{13}$$

We use induction on s. Let q be a fixed prime. Obviously, (13) holds for $s = 1$. Assume that (13) holds for s. Let $p > q$ be an arbitrary prime. For each prime p there exist positive integers $u = u(p)$, $v = v(p)$ such that

$$(u,q) = 1 \quad \text{and} \quad q^s u = 1 + p^\alpha v. \tag{14}$$

Since $p \nmid u$, then, as we showed above

$$f(u) \not\equiv 0 \pmod{p}. \tag{15}$$

Using (10) and (14), we have

$$f(q^s u) = f(1+p^\alpha v) \equiv f(1) = 1 \pmod{p}$$

and

$$f(q^{s+1}u) = f(q + p^\alpha qv) \equiv f(q) \pmod{p}.$$

From these, by using (14) and (15), we get

$$f(q^{s+1}) \equiv f(q)f(q^s) \pmod{p}. \tag{16}$$

Since $p > q$ is an arbitrary prime, from (13) and (16)

$$f(q^{s+1}) = f(q)f(q^s) = (f(q))^{s+1}$$

follows, which proves that (13) holds for $s+1$. Thus $f \in \mathcal{M}^*$.

Finally, by (11) it follows that for a prime q

$$f(q) = \pm q^{a(q)}, \tag{17}$$

where $a(q) \geq 0$ is an integer. We shall prove that

$$f(q) > 0 \tag{18}$$

for every prime q.

Let $t > 6$ be an odd integer. Clearly there exists an odd prime p such that $q^t \equiv 1 \pmod{p}$ and so

$$q^{t \cdot p^{\alpha-1}} \equiv 1 \pmod{p^\alpha}. \tag{19}$$

If $f(q) < 0$, then by (17) $f(q) = -q^{a(q)}$ follows. Applying (10) in the case $n = 1$ and $hp^\alpha = q^{t\cdot p^{\alpha-1}} - 1$, we have

$$f\left(q^{t\cdot p^{\alpha-1}}\right) = f(1 + hp^\alpha) \equiv f(1) \pmod p$$

and

$$f\left(q^{t\cdot p^{\alpha-1}}\right) = (f(q))^{t\cdot p^{\alpha-1}} = -q^{a(q)t\cdot p^{\alpha-1}} \equiv -1 \pmod{p^\alpha},$$

because $t \cdot p^{\alpha-1}$ is an odd integer, $f \in \mathcal{M}^*$ and (19) holds. Then we have $2 \equiv 0 \pmod p$ in contradiction with $p > 2$. Thus we proved (18). This completes the proof of Lemma 3.

3. Proof of the theorem. Assume that $f \in \mathcal{M}$ satisfies the relation

$$\Delta^k f(n+p) \equiv \Delta^k f(n) \pmod p \tag{20}$$

for every positive integer n and every prime p. For $k = 0$ the assertion follows from the result of [1]. In the following let $k \geq 1$. We shall prove that $f \in \mathcal{M}$ and (20) imply $f \in \mathcal{M}^*$.

Using Lemma 1 and Lemma 2, it follows by (20) that for $s = 1, \dots, k$

$$\Delta^{k-s} f(n+tp) - \Delta^{k-s} f(n) \equiv \sum_{j=0}^{s-1} \binom{n-1}{j} \Delta_f^{k-s+j}(p,t) \pmod p \tag{21}$$

and

$$\Delta_f^{k-s}(p,t) \equiv \sum_{i=0}^{[(s-1)/p]} \binom{t}{i+1} \Delta_f^{k-s+ip}(p,1) \pmod p \tag{22}$$

hold for every positive integer n, t and every prime p. Let $\alpha = \alpha(k)$ be an intger for which $k < 2^\alpha$. It can be easily seen that

$$\binom{p^\alpha}{1} \equiv \dots \equiv \binom{p^\alpha}{k} \equiv 0 \pmod p \tag{23}$$

for every prime p. Thus, using (21) and (22), we have

$$\Delta_f^0(p,p^\alpha) \equiv \dots \equiv \Delta_f^{k-1}(p,p^\alpha) \equiv 0 \pmod p$$

and

$$f(n+p^{\alpha+1}) - f(n) \equiv \sum_{j=0}^{k-1} \binom{n-1}{j} \Delta_f^j(p,p^\alpha) \equiv 0 \pmod p \tag{24}$$

for every positive integer n and every prime p. Using Lemma 3, (24) implies that $f \in \mathcal{M}^*$ and

$$f(q) = q^{a(q)} \tag{25}$$

for every prime q, where $a(q) \geq 0$ is an integer.

Now we prove that for distinct primes q, r we have $a(q) = a(r)$, and so the proof of the theorem will be finished.

Applying (22) in the case $k = s$, we have

$$f(1+tp) - 1 \equiv t\{f(p+1) - 1\} \pmod p \tag{26}$$

for every positive integer t and every prime $p \geq k$, because $[(k-1)/p] = 0$ for $p \geq k$. Applying (26) with $t = p+2$, we obtain that $(f(p+1)-1)^2 \equiv 0 \pmod p$ and so by (26)

$$f(1+tp) \equiv 1 \pmod p \tag{27}$$

holds for every integer $t \geq 0$ and every prime $p \geq k$.

Let q, r be distinct primes and let $a(q) \geq a(r)$. Then there is a prime p such that

$$p > \max(k, q^{a(q)-a(r)}) \text{ and } qr^s - 1 \equiv 0 \pmod p$$

for some positive integer s. (For the existence of such p, see, e.g. [2]). Using (27), we have $f(qr^s) \equiv f(1) = 1 \pmod p$ and

$$f(qr^s) = f(q)(f(r))^s = q^{a(q)}r^{a(r)s} \equiv q^{a(q)-a(r)} \pmod p,$$

which imply $a(q) = a(r) = a$. Hence $f(n) = n^a$ for every positive integer n. This completes the proof of our theorem.

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NOTE ON MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY

By

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Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of integer-valued multiplicative and completely multiplicative functions, respectively.

In 1966 M. V. SUBBARAO [2] proved that if $f \in \mathcal{M}$ and f satisfies the relation

$$f(n+m) \equiv f(m) \pmod n \tag{1}$$

for every positive integer n and m, then

$$f(n) = n^\alpha \quad (n = 1, 2, \ldots), \tag{2}$$

where α is a non-negative integer. In [1] A. IVÁNYI extended this result proving that if $f \in \mathcal{M}^*$ and (1) holds for a fixed m and for every positive integer n, then $f(n)$ also is of the same form (2).

The purpose of this note is to generalize the results of SUBBARAO and IVÁNYI mentioned above. We shall prove the following theorem.

THEOREM. *Let M be a positive integer and let $f \in \mathcal{M}$. If $f(M) \neq 0$ and*

$$f(n+M) \equiv f(M) \pmod n \tag{3}$$

for every positive integer n, then

$$f(n) = n^\alpha \quad (n = 1, 2, \ldots),$$

where α is a non-negative integer.

We shall use the following result in the proof of our theorem.

LEMMA. Let A be a positive integer and let $f \in \mathcal{M}$. If

$$f(1 + Am) \equiv 1 \pmod m \tag{4}$$

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for every positive integer m, then there is a non-negative integer α such that

$$|f(n)| = n^{\alpha} \tag{5}$$

for all positive integers n which are prime to A.

PROOF. We first prove that

$$f(ab) = f(a)f(b) \tag{6}$$

holds for all positive integers a, b which are prime to A.

Let a, b be positive integers for which $(ab, A) = 1$. Let p be a prime which satisfies the condition

$$(p, ab) = 1. \tag{7}$$

Then there are positive integers x, y, u and v such that

$$ax = 1 + Apy,\ (x, ab) = 1, \qquad bu = 1 + Apv,\ (u, abx) = 1.$$

From these we have

$$abxu = (1 + Apy)(1 + Apv) = 1 + ApT,$$

where $T = y + v + Apyv$. We now apply (4) with m given by py, pv and pT, respectively. Then we have

$$f(a)f(x) = f(ax) \equiv 1 \pmod{p}, \tag{8}$$

$$f(b)f(u) = f(bu) \equiv 1 \pmod{p} \tag{9}$$

and

$$f(ab)f(x)f(u) = f(abxu) \equiv 1 \pmod{p}. \tag{10}$$

From (8) and (10) we get that $f(x) \not\equiv 0 \pmod{p}$ and

$$f(ab)f(u) \equiv f(a) \pmod{p},$$

which with (9) imply that

$$f(a)f(b) \equiv f(ab)f(b)f(u) \equiv f(ab) \pmod{p}.$$

This shows that (6) holds, since there are infinitely many primes p satisfying (7). So (6) is proved.

It can be easily seen by (7) and (8) that if $(a, A) = 1$, then

$$f(a) \not\equiv 0 \pmod{p} \quad \text{if } (a, p) = 1. \tag{11}$$

From (11) it follows that for each prime Q coprime to A we have

$$f(Q) = \pm Q^{\alpha(Q)}, \tag{12}$$

where $\alpha(Q)$ is a non-negative integer. By using (6) and (12), in order to prove (5) it is enough to show that

$$\alpha(P) = \alpha(Q) \tag{13}$$

for all primes P, Q which are prime to A.

Let P, Q be distinct primes with $(PQ, A) = 1$ and let $\alpha(P) \geq \alpha(Q)$. Then there is a positive integer s such that

$$H = H(s) := \frac{(PQ^s)^{2\varphi(A)} - 1}{A} > P^{2(\alpha(P)-\alpha(Q))\varphi(A)}, \tag{14}$$

where φ denotes the Euler's totient function. If follows from Euler's Theorem that H is a positive integer and

$$(PQ^s)^{2\varphi(A)} = 1 + AH. \tag{15}$$

Applying (4) with m given by H we have

$$f\left[(PQ^s)^{2\varphi(A)}\right] = f(1 + AH) \equiv 1 \pmod{H}. \tag{16}$$

On the other hand, using (6), (12), (15) and the fact $(PQ, A) = 1$ we have

$$\begin{aligned} f\left[(PQ^s)^{2\varphi(A)}\right] &= f(P)^{2\varphi(A)} f(Q)^{2s\varphi(A)} = \\ &= P^{2\sigma(P)\varphi(A)} Q^{2s\alpha(Q)\varphi(A)} \equiv P^{2(\alpha(P)-\alpha(Q))\varphi(A)} \pmod{H}, \end{aligned} \tag{17}$$

which with (14) and (16) implies $\alpha(P) = \alpha(Q)$. So we have proved that (13) holds.

By this the lemma has been proved.

Proof of the Theorem. Assume that M, f satisfy the conditions of the theorem and (3) holds for every positive integer n.

We apply now (3) with $n = M^2|f(M)|m$. Since $f(M) \neq 0$, we have $f(1 + M|f(M)|m) \equiv 1 \pmod{m}$ for every positive integer m. Applying the lemma with $A = M|f(M)|$ we have

$$f(ab) = f(a)f(b) \tag{18}$$

and

$$|f(n)| = n^\alpha \tag{19}$$

for all positive integers a, b, n which are prime to $Mf(M)$, where α is a non-negative integer.

We shall deduce from (18) and (19) that

$$f(nM) = n^\alpha f(M) \quad (n = 1, 2, \ldots). \tag{20}$$

Let n be a positive integer and let Q be a prime for which

$$(Q, nMf(M)) = 1.$$

Then from (3) we get that

$$f(nQ^{2s}M) = f[M + (nQ^{2s} - 1)M] \equiv f(M) \pmod{(nQ^{2s} - 1)} \tag{21}$$

holds for every positive integer s.

Since $(Q, nMf(M)) = 1$, it follows from (18) that

$$f(nQ^{2s}M) = f(Q)^{2s} f(nM),$$

which with (19) and (21) implies

$$f(nM) \equiv n^\alpha Q^{2\alpha s} f(nM) \equiv n^\alpha f(M) \pmod{(nQ^{2s} - 1)}$$

for every positive integer s. This shows that (20) holds.

We get directly from (20) that

$$f(n) = n^\alpha \quad \text{if } (n, M) = 1. \tag{22}$$

By using (22), in order to prove our theorem it is enough to show that if $p|M$ then

$$f(p^k) = p^{\alpha k} \quad (k = 1, 2, \ldots). \tag{23}$$

Let d be a positive divisor of M. Then there are infinitely many positive integers m which satisfy

$$\left(m + \frac{M}{d}, Md\right) = 1. \tag{24}$$

By using (3), (22) and (24) we have

$$f(M) \equiv f(md + M) = f(d) f\left(m + \frac{M}{d}\right) =$$
$$= f(d)\left(m + \frac{M}{d}\right)^\alpha \equiv f(d)\left(\frac{M}{d}\right)^\alpha \pmod{m},$$

which implies

$$\frac{f(d)}{d^\alpha} = \frac{f(M)}{M^\alpha}. \tag{25}$$

Applying (25) in the case $d = 1$ we get that $f(M) = M^\alpha$, and so

$$f(d) = d^\alpha \quad \text{for all } d|M. \tag{26}$$

It can be easily seen by (26) that

$$f(p^k) = p^{\alpha k} \quad \text{if } k \le k_0, \tag{27}$$

where $p^{k_0}\|M$. If $k > k_0$, then it follows from (20) that

$$f(p^{k-k_0}M) = (p^{k-k_0})^\alpha f(M),$$

which with (27) implies

$$f(p^k) = (p^{k-k_0})^\alpha f(p^{k_0}) = (p^{k-k_0})^\alpha (p^{k_0})^\alpha = p^{\alpha k}. \tag{28}$$

From (27) and (28) we have proved that (23) holds.

Finally, from (22) and (23) the proof of our theorem is finished.

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ON RIESZ BASES II.

By

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In this paper we shall generalize a result of [1] connected with exponential Riesz bases.

Given a Hilbert space H the system $\{\varphi_n\}_1^\infty \subset H$ is called Riesz basis in H if there exists an isomorphism A of H onto itself such that $\{A\varphi_n\}$ is an orthonormal basis in H. If $H = L^2(0,a)$, $(a < \infty)$ or $H = L^2(0,\infty)$ and (φ_n) is of the form $e^{i\lambda_n x}$, $\lambda_n \in \mathbb{C}$, then we call (φ_n) exponential basis in H. We shall prove the following

THEOREM. *Let $(e^{i\lambda_n x})$ $(n \in Z)$ be a Riesz basis in $L^2(0,a)$ $(a < \infty)$ and let $0 < a' < a$. Then the system $(e^{i\lambda_n x})$ contains a subsystem which is Riesz basis in $L^2(0,a')$.*

REMARK. In the important special case $\lambda_n = n$ this result was obtained by the second author in [1] as a byproduct of his investigations in control theory [2].

For the proof we introduce first the notion of sine type function, used first by B. JA. LEVIN [3] (see also [4]).

An exponential type entire function F is called of sine type if

(a) Its zeros λ_n are in the horizontal strip $0 < \delta \le \mathrm{Im}\lambda_n \le C < \infty$,

(b) $0 < \inf\limits_{x\in\mathbf{R}} |F(x)|, \quad \sup\limits_{x\in\mathbf{R}} |F(x)| < \infty.$

We shall use the following characterization of exponential bases.

LEMMA 1 ([5]). Let $0 < \inf\limits_n \mathrm{Im}\lambda_n$, $\sup\limits_n \mathrm{Im}\lambda_n < \infty$ and suppose that $\inf\limits_{n\ne k} |\lambda_n - \lambda_k| > 0$. Then the following statement are equivalent:

(A) $\{e^{i\lambda_n x}\}$ $(n \in Z)$ is Riesz basis in $L^2(0,a)$,

(B) There exists a sine type function of exponential type with zero set $\{\mu_n\}$ $(n \in Z)$ (counted with multiplicity) such that for some $0 < d < 1/4$ we have

$$(1) \qquad d\,\mathrm{Re}(\lambda_{n-1} - \lambda_n) \le \mathrm{Re}(\mu_n - \lambda_n) \le d\,\mathrm{Re}(\lambda_{n+1} - \lambda_n).$$

LEMMA 2. Let F be such a sine type function with indicator diagram $[-i\sigma, i\sigma]$ and let $\{\lambda_n\}$ $(n \in Z)$ be its zero set, each zero counted with multiplicity. Then we have

$$(2) \qquad \frac{n(r+T) - n(r)}{T} \to \frac{\sigma}{\pi} \quad (T \to \infty)$$

uniformly in $r \in \mathbf{R}$, where

$$n(r) := \sum_{\substack{n \\ \mathrm{Re}\lambda_n \in [0,r]}} 1.$$

PROOF. It is enough to prove (2) for $r > 0$. We know ([4]) that there exist N, $p_N > 0$ such that

$$(3) \qquad n(r+N) - n(r) \le p_N \qquad (r > 0).$$

Let $\varepsilon > 0$ be fixed. By (3) it is enough to prove that for $T > T_0(\varepsilon)$ the values r, T can be shifted by $O(1)$ such that the modified values, denoted again by r and T, satisfy

$$(4) \qquad \left| \frac{n(r+T) - n(r)}{T} - \frac{\sigma}{\pi} \right| < c\varepsilon,$$

where $c = c(F)$ is independent of ε, r and T. In what follows we shall suppose that

$$(5) \qquad r \ge T$$

because for $r \le T$ (4) follows from [7], p. 323. Consider the segment $[re^{i\vartheta}, (r+T)e^{i\vartheta}]$ where the angle $0 < \vartheta < \frac{\pi}{2}$ is defined by

$$(6) \qquad r \sin \vartheta = 2 \sup_k \mathrm{Im}\lambda_k.$$

In [4], Lemma 2 and 3 it is proved that

$$(7) \qquad \int_r^{r+T} \frac{\mathrm{d}}{\mathrm{d}t} \arg F(te^{i\vartheta})\mathrm{d}t = -\pi[n(r+T) - n(r)] + O(1)$$

where the constant in $O(1)$ does not depend on r, T and ϑ. By the Cauchy–Riemann equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\arg F(te^{i\vartheta}) = -\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\ln|F(te^{i\vartheta})|.$$

Fix $k > 0$ arbitrarily, then (7) implies

$$\text{(8)} \qquad \int\limits_r^{r+T} \frac{1}{t}\frac{\ln|F(te^{i(\vartheta+k)})| - \ln|F(te^{i\vartheta})|}{k}\mathrm{d}t = \pi[n(r+T)-n(r)]+O(1).$$

We shall prove that (5) and (6) yield

$$\text{(9)} \qquad \sup_{t\in[r,r+T]}\left|\frac{\ln|F(te^{i\vartheta})|}{t} - \sigma\sin\vartheta\right| \to 0 \quad (r\to\infty)$$

and the same is true with $\vartheta + k$ instead of ϑ whenever $0 < k < \vartheta$ and $r > r_0$ (this ensures that $2\vartheta < \pi/2$). First we verify (4) using (9). Choose $k > 0$ so small that

$$\left|\frac{\sin(\vartheta+k)-\sin\vartheta}{k} - \cos\vartheta\right| < \varepsilon.$$

If $T > T_0(\varepsilon)$ then by (5) $1-\cos\vartheta < \varepsilon$ and

$$\left|\frac{\ln|F(te^{i\vartheta})|}{t} - \cos\vartheta\right| < \varepsilon,$$

$$\left|\frac{\ln|Fte^{i(\vartheta+k)})|}{t} - \sigma\sin(\vartheta+k)\right| < \varepsilon k$$

and then

$$\frac{1}{T}\int\limits_r^{r+T} \frac{1}{t}\frac{\ln|F(te^{i(\vartheta+k)})| - \ln|F(te^{i\vartheta})|}{k}\mathrm{d}t = \sigma + O(\varepsilon).$$

Comparing this with (8) we get (4). So it remains to prove (9). The following factorization of F is known [7], p. 311:

$$\text{(10)} \qquad \ln\left|F(te^{i\vartheta})\right| = \frac{1}{\pi}\mathrm{Im}\int\limits_{\mathbf{R}}\frac{\ln|F(\tau)|}{\tau - te^{i\vartheta}}\mathrm{d}\tau + \sigma t\sin\vartheta + \ln|B(te^{i\vartheta})|$$

where $B(z)$ is the Blaschke product corresponding to the zeros of F, namely

$$B(z) = \prod_{n\in Z}\left(1-\frac{z}{\lambda_n}\right)\left(1-\frac{z}{\overline{\lambda}_n}\right)^{-1}.$$

From the boundedness of $\ln|F(t)|$ we get

$$\left|\frac{1}{\pi}\mathrm{Im}\int_{\mathbf{R}}\frac{\ln|F(\tau)|}{\tau-te^{i\vartheta}}\mathrm{d}\tau\right|\le\frac{c}{\pi}\mathrm{Im}\int_{\mathbf{R}}\frac{1}{\tau-te^{i\vartheta}}\mathrm{d}\tau=c. \tag{11}$$

We estimate the Blaschke product using the inequality

$$\ln|1+u|\le|u|\qquad(-1\ne u\in\mathbb{C}).$$

We have

$$0<-\ln|B(z)|=\sum_{k\in Z}\ln\left|1+2iz\frac{\mathrm{Im}1/\lambda_k}{1-\frac{z}{\lambda_k}}\right|\le$$

$$\le 2r\sum_{k\in Z}\frac{|\lambda_k|}{|\lambda_k-z|}\left|\mathrm{Im}\frac{1}{\lambda_k}\right|\le cr\sum_{k\in Z}\frac{1}{|\lambda_k-z|\cdot|\lambda_k|}$$

since $\left|\mathrm{Im}\frac{1}{\lambda_k}\right|=\left|\mathrm{Im}\frac{\overline{\lambda}_k}{|\lambda_k|^2}\right|\le c/|\lambda_k|^2$.

Let now $z=te^{i\vartheta}$, $t\in[r,r+T]$, $z=x+iy$. By (3), (5), (6) we can write the following estimates for $r>r$

$$\sum_{|x-\mathrm{Re}\lambda_k|<y}\frac{1}{|\lambda_k-z||\lambda_k|}\le\sum_{|x-\mathrm{Re}\lambda_k|}\frac{1}{yx}<c/x;$$

$$\sum_{x+y<\mathrm{Re}\lambda_k<2x}\frac{1}{|\lambda_k-z||\lambda_k|}\le c/x\sum_{x+y<\mathrm{Re}\lambda_k<2x}\frac{1}{|\mathrm{Re}\lambda_k-x|}\le c\frac{\ln x}{x};$$

$$\sum_{\mathrm{Re}\lambda_k\ge 2x}\frac{1}{|\lambda_k-z||\lambda_k|}\le c\sum_{\mathrm{Re}\lambda_k\ge 2x}\frac{1}{|\mathrm{Re}\lambda_k|^2}\le c/x;$$

$$\sum_{\frac{x}{2}\le\mathrm{Re}\lambda_k\le x-y}\frac{1}{|\lambda_k-z||\lambda_k|}\le c/x\sum_{\frac{x}{2}\le\mathrm{Re}\lambda_k\le x-y}\frac{1}{|\mathrm{Re}\lambda_k-x|}\le c\frac{\ln x}{x};$$

$$\sum_{-\frac{x}{2}\le\mathrm{Re}\lambda_k\le\frac{x}{2}}\frac{1}{|\lambda_k-z||\lambda_k|}\le(c/x)\sum_{-\frac{x}{2}\le\mathrm{Re}\lambda_k\le\frac{x}{2}}\frac{1}{|\mathrm{Re}\lambda_k|+1}\le c\frac{\ln x}{x};$$

$$\sum_{\mathrm{Re}\lambda_k\le-\frac{x}{2}}\frac{1}{|\lambda_k-z||\lambda_k|}\le c\sum_{\mathrm{Re}\lambda_k\le-\frac{x}{2}}\frac{1}{|\mathrm{Re}\lambda_k|^2}\le c/x.$$

Consequently for $z=te^{i\vartheta}$, $t\in[r,r+T]$ we get

$$0<-\ln|B(z)|\le cr\frac{\ln(t\cos\vartheta)}{t\cos\vartheta}. \tag{12}$$

With (11) this gives (9) and so (4). Lemma 2 is proved.

Lemma 3 ([6]). Let $\{\lambda_n\}$ $(n \in Z)$ be a sequence of complex numbers satisfying

$$0 < \inf_n \mathrm{Im}\lambda_n, \quad \sup_n \mathrm{Im}\lambda_n < \infty, \quad \inf_{n \neq k} |\lambda_n - \lambda_k| > 0$$

and the condition (2). Then for every $0 < a' < a$ the system $\{e^{i\lambda_n x}\}$ which is assumed to be a Riesz basis in $L^2(0, a)$ contains a Riesz basis in $L^2(0, a')$.

From the lemmas 1–3 the Theorem follows immediately.

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