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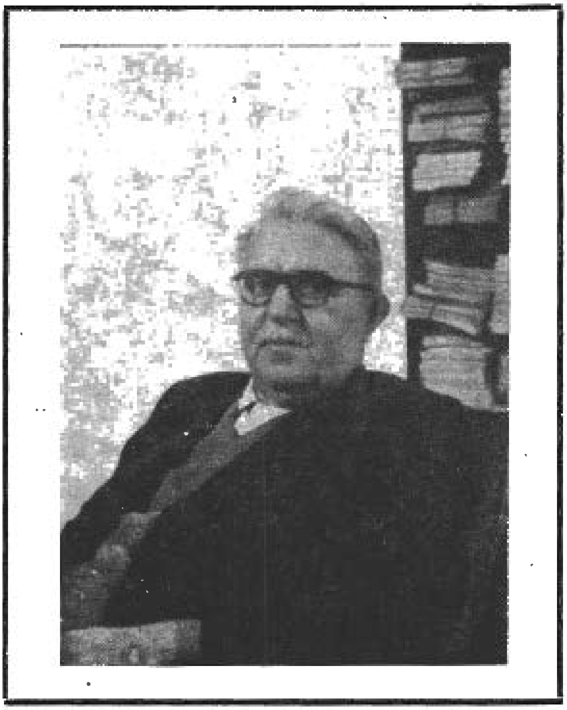

PÁL SZÁSZ
(1901–1978)

Professor P. SZÁSZ died the 12th of February, 1978 after a very long and grave illness. His death is an irreparable loss not only for the editorial board of our review but also for the international mathematical science and especially for the scientific life in Hungary.

О СУЩЕСТВОВАНИИ И ЕДИНСТВЕННОСТИ РЕШЕНИЯ НЕКОТОРЫХ СИСТЕМ ИНТЕГРОДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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I. Постановка задачи

Рассмотрим начальную задачу

$$(1)\qquad \dot{x}(t)=f\Big\{t,x(t),\bar{x}(t),\tilde{x}(t),x(\tau_0^{x(t)}),\bar{\dot{x}}(t),\tilde{\dot{x}}(t),\dot{x}(\Delta_0^{x(t)}),$$

$$\int_0^t \mathcal{F}\,[t,s,x(s),\bar{x}(s),\tilde{x}(s),x(\tau_0^{x(s)}),\dot{x}(s),\bar{\dot{x}}(s),\tilde{\dot{x}}(s),\dot{x}(\Delta_0^{x(s)})]\,ds\Big\},\quad t\ge 0,$$

$$(2)\qquad x(t)=\varphi(t),\quad \dot{x}(t)=\dot{\varphi}(t),\quad t\in I_0=[\alpha_0,0],$$

где $x=(x_1,\ldots,x_n)$, $f=(f_1,\ldots,f_n)$, $\mathcal{F}=(\mathcal{F}_1,\ldots,\mathcal{F}_l)$ (n и l-натуральные числа) при каждом $t\in I_T=[0,T]$ $(T>0)$, $\varphi=(\varphi_1,\ldots,\varphi_n)$ — начальная функция, определенная и непрерывно дифференцируемая на сегменте I_0, $\dot{x}(t)=\dfrac{dx(t)}{dt}$ (под $\dot{x}(0)$ понимается правая производная),

$$\bar{x}(t)=\sup_{u\in[0,t]}x(u),\quad \tilde{x}(t)=\sup_{u\in[t-d,t]}x(u);\quad d>0.$$

Преобразованные аргументы $\tau_0^{x(t)}$ и $\Delta_0^{x(t)}$ определяются при помощи рекуррентных соотношений

$$\tau_k^{x(t)}=\tau_k\big(t,x(t),\bar{x}(t),\tilde{x}(t),x(\tau_{k+1}^{x(t)}),\dot{x}(t),\bar{\dot{x}}(t),\tilde{\dot{x}}(t),\dot{x}(\Delta_{k+1}^{x(t)})\big),$$

$$\Delta_k^{x(t)}=\Delta_k\big(t,x(t),\bar{x}(t),\tilde{x}(t),x(\tau_{k+1}^{x(t)}),\dot{x}(t),\bar{\dot{x}}(t),\tilde{\dot{x}}(t),\dot{x}(\Delta_{k+1}^{x(t)})\big),$$

$$k=0,1,\ldots,m-1\quad(m\ge 1),$$

$$\tau_m^{x(t)}=\tau_m\big(t,x(t),\bar{x}(t),\tilde{x}(t),\dot{x}(t),\bar{\dot{x}}(t),\tilde{\dot{x}}(t)\big),$$

$$\Delta_m^{x(t)}=\Delta_m\big(t,x(t),\bar{x}(t),\tilde{x}(t),\dot{x}(t),\bar{\dot{x}}(t),\tilde{\dot{x}}(t)\big).$$

Предположим, что функция $f(t, \xi_1, \xi_2, \xi_3, \xi_4^0, \eta_2, \eta_3, \eta_4^0, u)$ определена по t на сегменте I_T, а по остальным аргументам – некотором множестве $G_1 \subset R^n \times R^n \times R^n \times R^n \times R^n \times R^n \times R^n \times R^l$ (R-вещественная ось); функция $\mathcal{F}(t, s, \xi_1, \xi_2, \xi_3, \xi_4^0, \eta_1, \eta_2, \eta_3, \eta_4^0)$ определена по t и s на множестве $I_T \times I_T$ $(0 \leq s \leq t \leq T)$, а по остальным аргументам – на множестве $G_2 \subset R^n \times R^n \times$ $\times R^n \times R^n \times R^n \times R^n \times R^n \times R^n$; функция $\tau_k(t, \xi_1, \xi_2, \xi_3, \xi_4^k, \eta_1, \eta_2, \eta_3, \eta_4^k)$ и $\Delta_k(t, \xi_1, \xi_2, \xi_3, \xi_4^k, \eta_1, \eta_2, \eta_3, \eta_4^k)$ $(k = 0, 1, \ldots, m-1)$ определены по t на сегменте I_T, а по остальным аргументам – на G_2; функции $\tau_m(t, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ и $\Delta_m(t, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ определены по t на сегменте I_T, а по остальным аргументам – на множестве $G_3 \subset R^n \times R^n \times R^n \times R^n \times R^n \times R^n$;

$$\alpha_0 = \min \Big\{ -d, \min_k \inf_{\zeta_k \in I_T \times G_2} \tau_k(\zeta_k), \min_k \inf_{\zeta_k \in I_T \times G_2} \Delta_k(\zeta_k),$$

$$\inf_{\zeta_m \in I_T \times G_3} \tau_m(\zeta_m), \inf_{\zeta_m \in I_T \times G_3} \Delta_m(\zeta_m) \Big\}$$

$$(k = 0, 1, \ldots, m-1).$$

Начальная задача (1), (2) является обобщением некоторых задач, которые впервые появились в автоматическом регулировании при анализе так называемых систем с насыщением [1]. Для частного случая, когда f зависит только от t, $x(t)$, $\bar{x}(t)$ и $\tilde{x}(t)$, задача (1), (2) исследована в [2] и [3].

Пусть $F = F(t)$ – скалярная, неотрицательная и интегруемая на сегменте I_T функция.

Определим следующие множества из R^n:

$$\omega_T = \left\{ \xi : |\xi| \leq |\varphi(0)| + \int_0^T F(t)\, dt \right\},$$

$$\Omega_T = \{\eta : |\eta| \leq F^* = \sup_{t \in I_T} F(t)\},$$

$$\tilde{\omega} = \bigcup_{s \in I_0} \{\varphi(s)\}, \quad \tilde{\Omega} = \bigcup_{s \in I_0} \{\dot{\varphi}(s)\}$$

$(|\cdot|$ – некоторая норма в соответствующем конечномерном пространстве).

Пусть G_1, G_2 и G_3 определены следующим образом:

$$G_1 = \omega^{(1)} \times \omega^{(2)} \times \omega^{(3)} \times \omega^{(4)} \times \Omega^{(2)} \times \Omega^{(3)} \times \Omega^{(4)} \times R^l,$$

$$G_2 = \omega^{(1)} \times \omega^{(2)} \times \omega^{(3)} \times \omega^{(4)} \times \Omega^{(1)} \times \Omega^{(2)} \times \Omega^{(3)} \times \Omega^{(4)},$$

$$G_3 = \omega^{(1)} \times \omega^{(2)} \times \omega^{(3)} \times \Omega^{(1)} \times \Omega^{(2)} \times \Omega^{(3)},$$

где

$$\omega^{(1)} = \omega^{(2)} = \omega_T, \quad \omega^{(3)} = \omega^{(4)} = \omega_T \cup \tilde{\omega},$$

$$\Omega^{(1)} = \Omega^{(2)} = \Omega_T, \quad \Omega^{(3)} = \Omega_T \cup \tilde{\Omega}, \quad \Omega^{(4)} = \Omega_{T-\Delta} \cup \tilde{\Omega},$$

а $\Delta > 0$ – некоторое число, которое будет определено ниже.

Всюду дальше принимаем, что выполнены следующие условия, обозначенные через (А):

А1. В области $Q_T = I_T \times G_1$ функция f непрерывна по t, удовлетворяет неравенству

$$|f(t, \xi_1, \xi_2, \xi_3, \xi_4^0, \eta_2, \eta_3, \eta_4^0)| \leq F(t)$$

и условию Липшица по всем аргументам кроме первого с константами соответственно $L_1, L_2, L_3, L_4, M_2, M_3, M_4, \alpha$.

А2. В области $\overline{Q}_T = I_T \times G_2$ функция $\mathcal{F}$ непрерывна по t и s и удовлетворяет условую Липшица по всем ангументам кроме первого и второго с константами соответственно $N_1, N_2, N_3, N_4, P_1, P_2, P_3, P_4$.

А3. В области $\overline{\overline{Q}}_T = I_T \times G_2$ функции τ_k и Δ_k $(k = 0, 1, \ldots, m-1)$ непрерывны по t и удовлетворяют условию Липшица по всем аргументам кроме первого с константами соответственно $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3, \mu_4$ и ограничениям:

$$\min_k \inf_{\zeta_k \in \overline{\overline{Q}}_T} \{t - \tau_k(\zeta_k)\} \geq 0\,, \quad \min_k \inf_{\zeta_k \in \overline{\overline{Q}}_T} \{t - \Delta_k(\zeta_k)\} \geq \Delta > 0\,;$$

в области $\overline{\overline{\overline{Q}}} = I_T \times G_3$ функции τ_m и Δ_m непрерывны по t, удовлетворяют условию Липшица по всем аргументам кроме первого с константами соответственно $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ и ограничениям

$$\inf_{\zeta_m \in \overline{\overline{\overline{Q}}}_T} \{t - \tau_m(\zeta_m)\} \geq 0\,, \quad \inf_{\zeta_m \in \overline{\overline{\overline{Q}}}_T} \{t - \Delta_m(\zeta_m)\} \geq \Delta > 0\,.$$

А4. На интервале I_0 функции $\varphi(t)$ и $\dot{\varphi}(t)$ удовлетворяют условиям Липшица с константами соответственно B и β.

А5. Выполнено условие согласования:

$$\dot{\varphi}(0) = f(0, \varphi(0), \varphi(0), \tilde{\varphi}(0), \varphi(\tau_0^{\varphi(0)}), \dot{\varphi}(0), \tilde{\dot{\varphi}}(0), \dot{\varphi}(\Delta_0^{\varphi(0)}), 0)\,.$$

Пусть $h = \min\{T, \Delta\}$ и $I_h = [\alpha_0, h]$. Обозначим через C пространство функций $y: I_h \to R^n$, определенных и непрерывных на сегменте I_h, с метрикой, порожденной нормой [4]

$$\|y\| = \sup\{|y(t)|\, e^{-\varrho t} : t \in I_h\}\,, \tag{3}$$

где

$$\varrho > \varrho_0 = \frac{2a}{-b + \sqrt{b^2 + 4a(1-c)}}\,,$$

а

$$a = \alpha N_0 + \lambda_0 (N_4 \Phi + P_4 \beta) \alpha q\,,$$

$$b = L_0 + \lambda_0 q(L_4 \Phi + M_4 \beta) + P_0 \alpha + \mu_0 q\, \alpha(N_4 \Phi + P_4 \beta)\,,$$

$$c = M_0 + \mu_0 q(L_4 \Phi + M_4 \beta)\,,$$

$$L_0 = L_1 + L_2 + L_3 + L_4\,,$$

$$M_0 = M_2 + M_3\,,$$
$$N_0 = N_1 + N_2 + N_3 + N_4\,,$$
$$P_0 = P_1 + P_2 + P_3\,,$$
$$\lambda_0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\,,$$
$$\mu_0 = \mu_1 + \mu_2 + \mu_3\,,$$
$$\Phi = \max\{B, F^*\}\,,$$
$$q = \frac{1-(\lambda_4\Phi + \mu_4\beta)^{m+1}}{1-(\lambda_4\Phi + \mu_4\beta)}\,.$$

Можно доказать, что C — полное метрическое пространство.

II. Теоремы существования и единственности решения начальной задачи

Теорема 1. *Пусть выполнены условия* (A) . *Пусть кроме того*

$$1 - [M_0 + \mu_0 q(L_4\Phi + M_4\beta)] > 0\,. \tag{4}$$

Тогда начальная задача (1), (2) *имеет единственное решение на интервале I_h в классе непрерывно дифференцируемых функций $x(t)$, производная $\dot{x}(t)$ которых удовлетворяет условию: $|\dot{x}(t)| \le F(t)$, $t \in I_h = [0, h]$.*

Доказательство. Пусть оператор Π действует в C по формуле

$$\Pi y(t) = \begin{cases} f\Big\{t, x(t), \bar{x}(t), \tilde{x}(t), x(\tau_0^{x(t)}), \bar{y}(t), \tilde{y}(t), y(\Delta_0^{x(t)}), \int_0^t \mathcal{F}[t, s, x(s), \\ \bar{x}(s), \tilde{x}(s), x(\tau_0^{x(s)}), y(s), \bar{y}(s), \tilde{y}(s), y(\Delta_0^{x(s)})]\,ds\Big\}, \; t \in I_h \\ \dot{\varphi}(t), \; t \in I_0 \end{cases} \tag{5}$$

где

$$x(t) = \varphi(t_0) + \int_0^t y(s)\,ds\,,$$

и пусть Y_h — множество функций $y \in C$, удовлетворяющие условиям:

$$|y(t)| \le F(t), \quad t \in I_h\,,$$
$$y(t) = \dot{\varphi}(t), \quad t \in \mathcal{I}_0\,.$$

Легко видно, что $\Pi\, Y_h \subset Y_h$.

На основе (5) и A4 получаем

$$|x(t) - x(\bar{t})| \le \Phi\, |t - \bar{t}| \quad \text{при} \quad t, \bar{t} \in \mathcal{I}_h,\; y \in Y_h\,.$$

Пусть $y_1, y_2 \in Y_h$. Тогда

$$|y_1(t)-y_2(t)| \leq \|y_1-y_2\| e^{\varrho t}$$

а для соответствующих x_1, x_2 имеем

$$|x_1(t)-x_2(t)| \leq \frac{1}{\varrho} \|y_1-y_2\| e^{\varrho t}$$

т.е.

(6)
$$\|x_1-x_2\| \leq \frac{1}{\varrho} \|y_1-y_2\| .$$

Из условий A1 – A4, (5) и (3) следует

$$|\Pi y_1(t)-\Pi y_2(t)| \leq L_1 |x_1(t)-x_2(t)| + L_2 |\bar{x}_1(t)-\bar{x}_2(t)| +$$
$$+ L_3 |\tilde{x}_1(t)-\tilde{x}_2(t)| + L_4 |x_1(\tau_0^{x_1(t)})-x_2(\tau_0^{x_2(t)})| + M_2 |\bar{y}_1(t)-\bar{y}_2(t)| +$$
$$+ M_3 |\tilde{y}_1(t)-\tilde{y}_2(t)| + M_4 |y_1(\Delta_0^{x_1(t)})-y_2(\Delta_0^{x_2(t)})| +$$
$$+ \int_0^t \{N_1 |x_1(s)-x_2(s)| + N_2 |\bar{x}_1(s)-\bar{x}_2(s)| + N_3 |\tilde{x}_1(s)-\tilde{x}_2(s)| +$$
$$+ N_4 |x_1(\tau_0^{x_1(s)})-x_2(\tau_0^{x_2(s)})| + P_1 |y_1(s)-y_2(s)| + P_2 |\bar{y}_1(s)-\bar{y}_2(s)| +$$
$$+ P_3 |\tilde{y}_1(s)-\tilde{y}_2(s)| + P_4 |y_1(\Delta_0^{x_1(s)})-y_2(\Delta_0^{x_2(s)})|\} ds \leq$$
$$\leq \frac{L_0}{\varrho} \|y_1-y_2\| e^{\varrho t} + M_0 \|y_1-y_2\| e^{\varrho t} + L_4 \Phi |\tau_0^{x_1(t)}-\tau_0^{x_2(t)}| +$$
$$+ M_4 \beta |\Delta_0^{x_1(t)}-\Delta_0^{x_2(t)}| + \alpha \int_0^t \left\{ \frac{N_0}{\varrho} \|y_1-y_2\| e^{\varrho s} + P_0 \|y_1-y_2\| e^{\varrho s} + \right.$$

(7)
$$\left. + N_4 \Phi |\tau_0^{x_1(s)}-\tau_0^{x_2(s)}| + P_4 \beta |\Delta_0^{x_1(s)}-\Delta_0^{x_2(s)}| \right\} ds .$$

Обозначая через γ_k какую-нибудь из функции τ_k и Δ_k $(k = 0, 1, \ldots, m)$ из A3 получаем

$$|\gamma_k^{x_1(t)}-\gamma_k^{x_2(t)}| \leq \frac{\lambda_0}{\varrho} \|y_1-y_2\| e^{\varrho t} + \mu_0 \|y_1-y_2\| e^{\varrho t} + \lambda_4 \Phi |\tau_{k+1}^{x_1(t)}-\tau_{k+1}^{x_2(t)}| +$$

(8)
$$+ \mu_4 \beta |\Delta_{k+6}^{x_1(t)}-\Delta_{k+1}^{x_2(t)}|, \quad k = 0, 1, \ldots, m-1 ;$$

(9)
$$|\gamma_m^{x_1(t)}-\gamma_m^{x_2(t)}| \leq \frac{\lambda_1+\lambda_2+\lambda_3}{\varrho} \|y_1-y_2\| e^{\varrho t} + \mu_0 \|y_1-y_2\| e^{\varrho t} .$$

Из (8) и (9) следует

$$|\gamma_0^{x_1(t)} - \gamma_0^{x_2(t)}| \leq \left(\frac{\lambda_0}{\varrho} + \mu_0\right) q \, \|y_1 - y_2\| \, e^{\varrho t}. \tag{10}$$

Подставляем (10) в (7) и получаем

$$|\Pi \, y_1(t) - \Pi \, y_2(t)| \leq \varkappa(\varrho) \, \|y_1 - y_2\| \, e^{\varrho t}$$

где положено

$$\varkappa(\varrho) = a \frac{1}{\varrho^2} + b \frac{1}{\varrho} + c.$$

Имея ввиду (3), получаем

$$\|\Pi \, y_1 - \Pi \, y_2\| \leq \varkappa(\varrho) \, \|y_1 - y_2\|.$$

Из условия (4) следует $\varkappa(\varrho) < \varkappa(\varrho_0) = 1$. Следовательно Π — оператор сжатия на множестве Y_h. Тогда существует единственное решение операторного уравнения $y = \Pi y$, которое может быть найдено последовательными приближениями по схеме $y_{n+1} = \Pi y_n$, $n = 0, 1, \ldots$, если только $y_0 \in Y_h$. Отсюда следует утверждеине Теоремы 1.

Теорема 2. *Пусть выполнены условия* (A) *при* $T = \infty$ *и в области* $\overline{Q}_\infty$

$$|\mathcal{F}(t, s, \xi_1, \xi_2, \xi_3, \xi_4^0, \eta_1, \eta_2, \eta_3, \eta_4^0)| \leq W \quad (W = \text{const} > 0).$$

Пусть кроме того функции f, $\mathcal{F}$, γ_k $(k = 0, 1, \ldots, m-1)$ *и* γ_m *соответственно в областях* Q_∞, $\overline{Q}_\infty$, $\overline{\overline{Q}}_\infty$, $\overline{\overline{\overline{Q}}}_\infty$ *удовлетворяют условию Липшица по* t *с константами* E, Θ, e, e. *Пусть наконец*

$$L_2 = L_3 = M_2 = M_3 = \lambda_2 = \lambda_3 = \mu_2 = \mu_3 = 0,$$

$$\lambda_4 \Phi + \mu_4 \tilde{\beta} < 1, \quad c_3 > 0, \quad c_2 \geq 0, \quad c_3 - c_2 \geq \sqrt{2 c_1 c_4}, \tag{11}$$

где

$$c_1 = [E + L_1 \Phi + \alpha(W + \Theta \Delta)](1 - \lambda_4 \Phi) + L_4 \Phi(e + \lambda_1 \Phi),$$

$$c_2 = M_4(e + \lambda_1 \Phi) - \mu_4 [E + L_1 \Phi + \alpha(W + \Theta \Delta)],$$

$$c_3 = 1 - \lambda_4 \Phi - \mu_1 L_4 \Phi,$$

$$c_4 = \mu_4 + \mu_1 M_4,$$

$$\tilde{\beta} = \frac{1}{2 c_4} \left(c_3 - c_2 + \sqrt{(c_3 - c_2)^2 - 4 c_1 c_4}\right).$$

Тогда, если $\beta \le \tilde{\beta}$, то начальная задача (1), (2) *имеет единственное решение на интервале $I_\infty = [\alpha_0, +\infty)$ в классе непрерывно дифференцируемых функции $x(t)$, производная $\dot{x}(t)$ которых уддвлетворяет условию:*

$$|\dot{x}(t)| \le F(t), \quad t \in I_\infty = [0, +\infty).$$

Доказательство. Рассмотрим шаговый процесс построения решения с шагом Δ. Покажем, что полученное таким образом решение будет обладать производной, удовлетворяющей условию Липшица с константой $\tilde{\beta}$. Действительно, легко проверить, что из условий теоремы 2 следуют условия теоремы 1. Следовательно, существует единственное решение $x(t)$ начальной задачи (1), (2) на интервале $I_\Delta = [\alpha_0, \Delta]$.

Пусть $t, t^* \in I_\Delta$. Тогда из условий теоремы 2 следует

$$\begin{aligned} &|\dot{x}(t) - \dot{x}(t^*)| \le E\,|t-t^*| + L_1 \Phi\,|t-t^*| + L_4 \Phi\,|\tau_0^{x(t)} - \tau_0^{x(t^*)}| + \\ (12)\qquad &+ M_4 \beta\,|\Delta_0^{x(t)} - \Delta_0^{x(t^*)}| + \alpha\,[W\,|t-t^*| + \Theta\Delta\,|t-t^*|]; \\ &|\gamma_k^{x(t)} - \gamma_k^{x(t^*)}| \le e\,|t-t^*| + \lambda_1 \Phi\,|t-t^*| + \lambda_4 \Phi\,|\tau_{k+1}^{x(t)} - \tau_{k+1}^{x(t^*)}| + \\ &+ \mu_1\,|\dot{x}(t) - \dot{x}(t^*)|| + \mu_4 \beta\,|\Delta_{k+1}^{x(t)} - \Delta_{k+1}^{\dot{x}(t^*)}|, \quad k = 0, 1, \ldots, m-1, \\ &|\gamma_m^{x(t)} - \gamma_m^{x(t^*)}| \le e\,|t-t^*| + \lambda_1 \Phi\,|t-t^*| + \mu_1\,|\dot{x}(t) - \dot{x}(t^*)| \end{aligned}$$

и следовательно

$$(13)\qquad |\gamma_0^{x(t)} - \gamma_0^{x(t^*)}| \le \frac{(e + \lambda_1 \Phi)\,|t-t^*| + \mu_1\,|\dot{x}(t) - \dot{x}(t^*)|}{1 - (\lambda_4 \Phi + \mu_4 \beta)}.$$

Подставляя (13) в (12) и пользуясь (11), получаем

$$|\dot{x}(t) - \dot{x}(t^*)| \le \frac{c_1 + c_2 \tilde{\beta}}{c_3 - c_4 \tilde{\beta}}\,|t-t^*| = \tilde{\beta}\,|t-t^*|.$$

Допустим, что при помощи N $(N \ge 2)$ шагов построено решение начальной задачи на интервале $I_{N\Delta}$, причем производная этого решения удовлетворяет условию Липшица с постоянной $\beta_N \le \tilde{\beta}$. Тогда это решение может быть продолжено на интервал $I_{(N+1)\Delta}$. Аналогичным образом, как это делалось для $N = 1$, можно показать, что производная продолженного решения будет удовлетворять условию Липшица с постоянной

$$\beta_{N+1} \le \max\left\{\beta_N, \frac{c_1 + c_2 \tilde{\beta}}{c_3 - c_4 \tilde{\beta}}\right\} = \tilde{\beta},$$

т.е. индукцией по N показано, что рассматриваемое решение обладает производной, удовлетворяющей условию Липшица с постоянной $\tilde{\beta}$ (на каждом интервале $I_{N\Delta}$).

Шаговый процесс построения решения можно продолжить до бесконечности. Полученное решение будет соответствовать утверждению теоремы 2.

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SETS OF DIVERGENCE IN THE GROUP OF INTEGERS OF A p-ADIC OR p-SERIES FIELD

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1. Introduction

Let G denote the group of integers of a p-adic field Q_p or a p-series field K_p, p prime. Thus an element x of G can be represented as $x = (x_0, x_1, \ldots)$ with $0 \leq x_i < p$ for each $i \geq 0$. The subgroups G_n of G are defined by $G_0 = G$ and for $n \geq 1$ by

$$G_n = \{x \in G; x_0 = x_1 = \ldots = x_{n-1} = 0\}.$$

The elements of the dual group $\hat{G}$ of G can be described as follows. For $k \geq 0$ and $x \in G$ let

$$(1) \qquad \varphi_k(x) = \begin{cases} \exp(2\pi i\, x_k/p) & \text{if } G \subset K_p, \\ \exp(2\pi i(x_0 + \ldots + x_k p^k)/p^{k+1}, & \text{if } G \subset Q_p. \end{cases}$$

Next, for $n \geq 0$ with $n = a_0 p^0 + \ldots + a_k p^k$, $0 \leq a_i < p$ for each i, let

$$\chi_n(x) = (\varphi_0)^{a_0}(x) \cdot \ldots \cdot (\varphi_k)^{a_k}(x).$$

Then the χ_n, $n \geq 0$, are precisely the elements of $\hat{G}$.

REMARK. If G is the set of integers of the 2-series field K_2 then G is the dyadic group, $G = 2^\omega$, and $\hat{G}$ is the group of Walsh funtions ordered by Paley's method.

Let dx or m denote the normalized Haar measure on G. For $f \in L_1(G)$ its Fourier series is defined by

$$\sum_{k=0}^{\infty} \hat{f}(k)\chi_k(x), \quad \text{where} \quad \hat{f}(k) = \int_G f(t)\overline{\chi_k(t)}\,dt.$$

Its partial sums will be denoted by $S_n(f; x)$ and we set $S^*(f; x) = \sup\{S_n(f; x);\ n \geq 1\}$. Additional information on groups G as considered here, and their Fourier theory can be found in [7, Chapter II. 6].

Many results in the Fourier theory for G are similar to results in the Fourier theory on the circle group T. The present paper is a contribution to the question whether a well-known theorem of KAHANE and KATZNELSON on sets of divergence in T can be extended to G. The theorem in question states that every subset of measure zero in T is a set of divergence for $C(T)$, see [3] or [5, Chapter II. 3.4]. It is still unknown whether the same holds for $C(G)$. In [6] SCHIPP showed that there exist uncountable sets of divergence for $C(2^\omega)$. Additional results for the case $G=2^\omega$ were recently obtained by HARRIS and WADE [2], who proved, among others, that every subset of 2^ω of measure zero is a set of divergence for $L_r(2^\omega)$, $1\leq r<\infty$, and that certain subsets of measure zero in 2^ω are sets of divergence for $C(2^\omega)$. In the following we present some new results on this problem in the more general context of groups G as described here. Our first theorem shows that the result of HARRIS and WADE for $L_r(2^\omega)$ can easily be extended to $L_r(G)$.

THEOREM 1. *If $E\subset G$ is a subset of Haar measure zero, then E is a set of divergence for $L_r(G)$, $1\leq r<\infty$.*

In order to formulate our main result for $C(G)$ we need the following definition.

DEFINITION 1. A subset E of G is of logarithmic Hausdorff measure zero, $H(E)=0$, if for each $\varepsilon>0$ there exists a collection of disjoint cosets of certain G_n in G, say $\{x_i+G_{n(i)}\}_{i=1}^{\infty}$, such that

(i) $$E\subset\bigcup_{i=1}^{\infty}(x_i+G_{n(i)}),$$

(ii) $$\sum_{i=1}^{\infty}|\log m(x_i+G_{n(i)})|^{-1}<\varepsilon.$$

THEOREM 2. *If $E\subset G$ satisfies $H(E)=0$, then E is a set of bounded divergence for $C(G)$.*

For trigonometric Fourier series a similar result was obtained by ERDŐS, HERZOG and PIRANIAN, see [1].

2. Proofs of the Theorems

Since the proof of Theorem 1 is to a large extend the same as corresponding proofs in [2], [4] or [5] we shall only present an outline here. First we observe that the following holds.

LEMMA 1. A subset $E\subset G$ is a set of divergence for a homogeneous Banach space $B\subset L_1(G)$ if and only if there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ on G such that

(i) $$\sum_{n=1}^{\infty}\|P_n\|_B<\infty$$

and

(ii) $$\sup\{S^*(P_n; x); n \geq 1\} = \infty$$

for all $x \in E$.

Next, if $D_n(x) = \sum_{i=0}^{n-1} \chi_i(x)$ denotes a Dirichlet kernel on G then, unlike the Dirichlet kernels on T, we have, see [7, Chapter II. 6],

(2) $$D_{p^k}(x) = \begin{cases} p^k, & \text{if } x \in G_k, \\ 0, & \text{if } x \in G \setminus G_k. \end{cases}$$

Thus, $p^{-k} D_{p^k}(x)$ is precisely the characteristic function of the subset G_k in G.

PROOF OF THEOREM 1. Let $E \subset G$ be a set of measure zero. Then there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of subsets of G with the following properties.

(i) Each E_n is the union of finitely many cosets of some $G_{k(n)}$, say

$$E_n = \bigcup_{i=1}^{N(n)} x_{i,n} + G_{k(n)},$$

(ii) $m(E_n) < 2^{-n}$,
(iii) each $x \in E$ belongs to infinitely many E_n's.

If Φ_n is the characteristic function of E_n, then (2) implies that

$$\Phi_n(x) = p^{-k(n)} \sum_{i=1}^{N(n)} D_{p^k}(x - x_{i,n}).$$

Thus Φ_n is a polynomial on G and if $\psi_n(x) = n\Phi_n(x)$, then ψ_n is also a polynomial on G and we have

(i) $\|\psi_n\|_r = n(m(E_n))^{1/r} \leq n\, 2^{-n/r}$ and
(ii) $\psi_n(x) = n$ for $x \in E_n$.

Therefore, applying Lemma 1 we obtain the result stated in Theorem 1.

In order to prove Theorem 2 we start with a technical lemma. For k, $l \geq 0$ we define $k \dotplus l$ by $\chi_{k \dotplus l} = \chi_k \cdot \chi_l$. Also, to simplify our notation, for each $n \geq 1$ we set $\bar{n} = [(n-1)/2]$,

$$m(n) = (p-1) \sum_{j=0}^{\bar{n}} p^{2j} \quad \text{and} \quad H(n) = \{j;\ p^{n-1} \leq j < p^n\}.$$

LEMMA 2. For each $n \geq 1$ we have

(a) if $j \in H(2s)$ for some s with $2 \leq 2s \leq n$, then

$$m(n) < p^{2s-1} + (p-1) \sum_{j=s}^{\bar{n}} p^{2j} \leq m(n) \dotplus j < p^{2s+1} + (p-1) \sum_{j=s+1}^{\bar{n}} p^{2j},$$

(b) If $j \in H(2s+1)$ for some s with $1 \leq 2s+1 \leq n$, then

$$(p-1)\sum_{j=s+1}^{\bar{n}} p^{2j} \leq m(n) \dotplus j < (p-1)\sum_{j=s}^{\bar{n}} p^{2j} \leq m(n).$$

Proof. (a) Let $j = \sum_{i=0}^{2s-1} a_i p^i$ with $0 \leq a_i < p$ for each i and $a_{2s-1} \neq 0$. Then

$$\chi_{m(n) \dotplus j} = (\varphi_0)^{a_0+p-1} \cdot (\varphi_1)^{a_1} \cdot \ldots \cdot (\varphi_{2s-1})^{a_{2s-1}} (\varphi_{2s})^{p-1} \cdot \ldots \cdot (\varphi_{2\bar{n}})^{p-1}.$$

Hence,

$$m(n) \dotplus j < p^{2s} + (p-1)(p^{2s} + \ldots + p^{2\bar{n}}) =$$

$$= p^{2s+1} + (p-1)\sum_{j=s+1}^{\bar{n}} p^{2j}.$$

Also,

$$m(n) \dotplus j \geq a_{2s-1} p^{2s-1} + (p-1)(p^{2s} + \ldots + p^{2\bar{n}}) \geq$$

$$\geq p^{2s-1} + (p-1)\sum_{j=s}^{\bar{n}} p^{2j} > m(n).$$

(b) Let $j = \sum_{i=0}^{2s} a_i p^i$ with $0 \leq a_i < p$ for each i and $a_{2s} \neq 0$. Then

$$\chi_{m(n) \dotplus j} = (\varphi_0)^{a_0+p-1} \cdot (\varphi_1)^{a_1} \cdot \ldots \cdot (\varphi_{2s-1})^{a_{2s-1}} \cdot (\varphi_{2s})^{a_{2s}+p-1} \cdot \ldots \cdot (\varphi_{2\bar{n}})^{p-1}.$$

Thus

$$m(n) \dotplus j = b_0 p^0 + \ldots + b_{2s-1} p^{2s-1} + b_{2s} p^{2s} + (p-1)\sum_{j=s+1}^{\bar{n}} p^{2j},$$

for certain b_i with $0 \leq b_i < p$ for each i and $b_{2s} \neq p-1$. Therefore,

$$(p-1)\sum_{j=s+1}^{\bar{n}} p^{2j} \leq m(n) \dotplus j < (p-1)\sum_{j=s}^{\bar{n}} p^{2j} \leq m(n).$$

Now we define certain polynomials on G which are a generalization of the polynomials Q_n defined by Schipp in [6]. Let $c_0 = 0$ and if $n \in H(k+1)$ for some $k \geq 0$, let $c_n = (-1)^k p^{-k}$.

Lemma 3. For $l, n \geq 1$ and $n > l+1$, let $Q(l, n)$ be defined by

$$Q(l, n)(x) = \chi_{m(n)}(x) \sum_{k=p^l}^{p^n-1} c_k \chi_k(x).$$

Then the following holds.

(a) $Q(l, n)(x) = 0$ if $x \notin G_l$,

(b) $\|Q(l, n)\|_\infty \leq C$,

(c) $S^*(Q(l,n)) = \sup_k \|S_k(Q(l,n))\|_\infty \le C(n-l)$,
(d) $S_{m(n)}(Q(l,n);x) \ge C(n-l)$ if $x \in G_n$,
(e) $S^*(Q(l,n);x) \le C$ if $x \notin G_l$,

where C denotes a certain constant independent of l and n, which is not necessarily the same each time.

PROOF. (a) We first observe that $Q(l,n)(x)$ can be represented as

$$Q(l,n)(x) = \chi_{m(n)}(x) \sum_{j=l}^{n-1} \sum_{k=p^j}^{p^{j+1}-1} c_k \chi_k(x) =$$

$$= \chi_{m(n)}(x) \sum_{j=l}^{n-1} (-1)^j p^{-j} (D_{p^{j+1}}(x) - D_{p^j}(x)).$$

Since, according to (2), $D_{p^j}(x) = 0$ if $j \ge l$ and $x \notin G_l$, we see immediately that (a) holds.

(b) Next, for each $x \in G$, $x \ne 0$, there exists an $r \ge 0$ such that $x \in G_r \setminus G_{r+1}$. For such an x we have

$$|Q(l,n)(x)| = \left| \sum_{j=l}^{n-1} (-1)^j p^{-j} (D_{p^{j+1}}(x) - D_{p^j}(x)) \right| \le$$

$$\le \left| \sum_{j=l}^{r-1} (-1)^j p^{-j} (p^{j+1} - p^j) \right| + |(-1)^r p^{-r} (-p^r)| \le$$

$$\le (p-1) \left| \sum_{j=l}^{r-1} (-1)^j \right| + 1 \le p,$$

which proves (b).

(c) Now we observe that for each $r \ge 1$ we have

$$S_r(Q(n,l);x) \le \sum_{k=p^l}^{p^n-1} |c_k| \le \sum_{j=l}^{n-1} \sum_{k=p^j}^{p^{j+1}-1} p^{-j} = \sum_{j=l}^{n-1} (p-1) = (p-1)(n-l),$$

which implies (c).

(d) According to Lemma 2 we have

$$H(l,n) \equiv \{j;\ p^l \le j < p^n \text{ and } j \dotplus m(n) < m(n)\} = \bigcup_{l < 2s+1 \le n} H(2s+1).$$

Therefore, since $\chi_j(0) = 1$ for all $j \ge 0$ and since for $j \in H(2s+1)$ we have $c_j = p^{-2s}$, we see that

$$S_{m(n)}(Q(l,n);0) = \sum_{j \in H(l,n)} c_j \chi_{j \dotplus m(n)}(0) =$$

$$= \sum_{l < 2s+1 \le n} p^{-2s}(p^{2s+1} - p^{2s}) \ge \frac{1}{3}(p-1)(n-l).$$

Also, $Q(l, n)$ is a polynomial of degree $< p^n$, thus, since such polynomials are constant on cosets of G_n, we see that for all $x \in G_n$ we have

$$S_{m(n)}\, Q((l, n); x) \geq C(n-l).$$

(e) We first observe that (a) implies that for each $k >$ degree $Q(l, n)$ and $x \notin G_l$ we have $S_k(Q(l, n); x) = 0$. Next consider k such that $k \leq$ degree $Q(l, n)$. We observe that Lemma 2 implies a certain ordering of the elements belonging to the sets $\{m(n) \dotplus j;\ j \in H(r)\}$ with $l < r \leq n$. To be precise, we have: if $j_i \in H(2s+i)$, $i = 0, 1, 2, 3$, with $2s+3 \leq n$, then $m(n) \dotplus j_3 < m(n) \dotplus j_1 < m(n) \dotplus j_2 < m(n) \dotplus j_0$. Also, for $x \notin G_l$ and $l < s \leq n$ we have

$$\sum_{j \in H(s)} c_j \chi_{m(n) \dotplus j}(x) = \chi_{m(n)}(x)(-1)^{s-1} p^{-(s-1)} \left(D_{p^s}(x) - D_{p^{s-1}}(x)\right) = 0.$$

So, choosing $k \leq$ degree $Q(l, n)$ and assuming that $k-1 = m(n) \dotplus j$ for some $j \in H(r)$ with $l < r \leq n$, we find for $x \notin G_l$

$$|S_k(Q(l, n); x)| \leq \left| \sum_{j=p^{r-1}}^{p^r-1} c_j \chi_{m(n) \dotplus j}(x) \right| \leq \sum_{j=p^{r-1}}^{p^r-1} |c_j| = p-1.$$

Similarly, if $k-1 \neq m(n) \dotplus j$ for all $j \in H(r)$ and all r with $l < r \leq n$, then $S_k(Q(l, n); x) = 0$. This completes the proof of (e).

By choosing $l = 0$ in Lemma 3 we obtain polynomials $Q(0, n) = Q_n$ which are the analogue on G of the classical Fejér polynomials. For these polynomials we have

COROLLARY 1. For each $n > 1$ there exists a polynomial Q_n on G of degree $< p^n$ and such that

(i) $\|Q_n\|_\infty \leq C$,
(ii) $|S_{m(n)}(Q_n; x)| \geq Cn$ for $x \in G_n$,
(iii) $\|S^*(Q_n)\|_\infty \leq Cn$.

PROOF OF THEOREM 2. Assume $E \subset G$ has logarithmic Hausdorff measure equal to zero. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive real numbers such that $\sum_{k=1}^\infty \varepsilon_k = 1$. According to Definition 1 for each $k \geq 1$ there exists a sequence of disjoint subsets of G, say $\{x_{l,k} + G_{n(l,k)}\}_{l=1}^\infty$ for which

(i) $E \subset \bigcup_{l=1}^\infty (x_{l,k} + G_{n(l,k)})$,

(ii) $\sum_{l=1}^\infty |\log m(x_{l,k} + G_{n(l,k)})|^{-1} = \sum_{l=1}^\infty (n(l, k) \log p)^{-1} < \varepsilon_k$.

Let the function f be defined by (compare (1) for the definition of the functions φ_i)

$$f(x) = \sum_{k=1}^\infty \sum_{l=1}^\infty (n(l, k))^{-1} Q_{n(l,k)}(x - x_{l,k})\, \varphi_{q(l,k)}(x), \tag{2}$$

where the $q(l, k)$ are chosen in such a way that no two different polynomials of the type $\varphi_{q(l,k)} Q_{n(l,k)}$ overlap, that is to say, contain the same character χ_i. It is obvious that $f \in C(G)$. Next, let $\{M(n)\}_{n=1}^{\infty}$ be the sequence obtained by arranging the numbers $q(l, k)$, $1 \leq k, l < \infty$, in increasing order, let $\{N(n)\}_{n=1}^{\infty}$ be the sequence of corresponding numbers $n(l, k)$ and let $x_n = x_{l,k}$ be defined similarly. Then

$$f(x) = \sum_{n=1}^{\infty} (N(n))^{-1} \varphi_{M(n)}(x) Q_{N(n)}(x - x_n).$$

For a given $k \geq 1$ we distinguish the following two cases.

(a) If $p^{M(r)} + p^{N(r)} \leq k < p^{M(r+1)}$ for some $r \geq 1$, then, according to Corollary 1 (i) and the assumption that $\sum_{k=1}^{\infty} \varepsilon_k = 1$, we have for each $x \in G$

$$S_k(f; x) = \left| \sum_{n=1}^{r} (N(n))^{-1} \varphi_{M(n)}(x) Q_{N(n)}(x - x_n) \right| \leq$$

$$\leq \sum_{n=1}^{r} (N(n))^{-1} \|Q_{N(n)}\|_{\infty} \leq C.$$

(b) If $p^{M(r)} \leq k < p^{M(r)} + p^{N(r)}$ for some $r \geq 1$, then it follows from Corollary 1 (iii) that for each $x \in G$ we have

$$|S_k(f; x)| \leq \sum_{n=1}^{r-1} (N(n))^{-1} \|Q_{N(n)}\|_{\infty} + (N(r))^{-1} C\, N(r) \leq C.$$

Thus, $|S^*(f; x)| \leq C$ for all $x \in G$. Finally, it follows from Corollary 1 (ii) and the definition of $f(x)$ in (3) that

$$|S_{p^{q(l,k)} + m(n(l,k))}(f; x) - S_{p^{q(l,k)}}(f; x)| =$$

$$= |(n(l, k))^{-1} S_{m(n(l,k))}(Q_{n(l,k)}; x - x_{l,k})| \geq C(n(l, k))^{-1} n(l, k) = C,$$

whenever $x - x_{l,k} \in G_{n(l,k)}$, that is to say, whenever $x \in x_{l,k} + G_{n(l,k)}$. Since each $x \in E$ belongs to infinitely many of the sets $x_{l,k} + G_{n(l,k)}$, we see that $\lim_{k \to \infty} S_k(f; x)$ does not exist for $x \in E$, which shows that E is a set of bounded divergence for $C(G)$.

As an additional consequence to Lemma 2 we present another sufficient condition for a set to be a set of divergece for $C(G)$.

THEOREM 3. Let $\{l_i\}_{i=1}^{\infty}$ *and* $\{n_i\}_{i=1}^{\infty}$ *be two increasing sequences such that* $n_i > l_i + 1$ *for each* $i \geq 1$ *and* $\sum_{i=1}^{\infty} (n_i - l_i)^{-1} < \infty$. *For each* $i \geq 1$ *let* E_i *be the union of certain cosets of* G_{n_i} *with the property that each of the cosets of* G_{n_i} *belongs to a different coset of* G_{l_i}. *If* $E = \bigcap_{i=1}^{\infty} E_i$, *then* E *is a set of bounded divergence for* $C(G)$.

PROOF. For $i \geq 1$, let $E_i = \bigcup_{k=1}^{N(i)} (x_{k,i} + G_{n_i})$ and define f_i by

$$f_i(x) = \sum_{k=1}^{N(i)} \varphi_{\varkappa(k)}(x)\, Q(l_i, n_i)(x - x_{k,i}),$$

where the $\varkappa(k)$, $1 \leq k \leq N(i)$, are chosen so that the polynomials in the sum defining f_i do not overlap. Next, define f by

$$f(x) = \sum_{i=1}^{\infty} (n_i - l_i)^{-1} \varphi_{\lambda(i)}(x) f_i(x),$$

where the $\lambda(i)$, $i \geq 1$, are chosen so that the polynomials in the sum defining f do not overlap. Since, according to Lemma 3 (a), the support of $Q(l_i, n_i)$ is contained in G_{l_i} and since $x_{r,i} + G_{l_i} \cap x_{s,i} + G_{l_i} = \emptyset$ for $r \neq s$ and $1 \leq r, s \leq N(i)$, Lemma 3 (b) implies that $\|f_i\|_\infty \leq C$ for each $i \geq 1$. Consequently, $f \in C(G)$. Also, an argument like that used in the proof of Theorem 2 shows that $S^*(f; x) < C$ for all $x \in G$ and that $\{S_n(f; x)\}_{n=1}^{\infty}$ does not converge for $x \in E$, that is to say, E is a set of bounded divergence for $C(G)$.

By choosing $n_i = l_{i+1} - 1$ in Theorem 3 we easily obtain the following corollary, which shows again that there exist uncountable subsets of G which are sets of bounded divergence for $C(G)$.

COROLLARY 2. Let $\{l_i\}_{i=1}^{\infty}$ be a monotone increasing sequence with $l_{i+1} > l_i + 2$ for each i and $\sum_{i=1}^{\infty} (l_{i+1} - l_i)^{-1} < \infty$. Let $E = \{x \in G;\ 0 \leq x_k < p$ if $k = l_i$ for some $i \geq 1$, and $x_k = 0$ otherwise$\}$. Then E is a set of bounded divergence for $C(G)$.

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РАВНОМЕРНЫЕ ОЦЕНКИ СХОДИМОСТИ К ПОКАЗАТЕЛЬНОМУ РАСПРЕДЕЛЕНИЮ СУММ СЛУЧАЙНОГО ЧИСЛА СЛУЧАЙНЫХ ВЕЛИЧИН

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Пусть

(1) $$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

— последовательность независимых, одинаково распределенных случайных величин с конечным математическим ожиданием $M\xi_1 = a \neq 0$ и ν_n — последовательность положительных, целочисленных случайных величин. Й. Модьороди [1] доказал, что если для некоторой последовательности положительных чисел α_n, $\alpha_n \to \infty$ при $n \to \infty$, существует функция распределения $G(x)$ такая, что при $n \to \infty$

$$P\left\{\frac{\nu_n}{\alpha_n} < x\right\} \to G(x), \quad G(+0) = 0,$$

тогда

(2) $$\lim_{n\to\infty} P\left\{\frac{\xi_1+\xi_2+\ldots+\xi_{\nu_n}}{a\,\alpha_n} < x\right\} = G(x).$$

Введем обозначения:

$$\sigma^2 = D\,\xi_1, \ \beta = M|\xi_1 - a|^3, \ S_n = \xi_1+\xi_2+\ldots+\xi_n,$$

$$\Delta_n(G) = \sup_x \left| P\left\{\frac{S_{\nu_n}}{a\,\alpha_n} < x\right\} - G(x)\right|,$$

$$d_n(G) = \sup_x \left| P\left\{\frac{\nu_n}{\alpha_n} < x\right\} - G(x)\right|,$$

$$Q(h, G) = \sup_x\,[G(x+h) - G(x)], \ h > 0,$$

$$E(x) = \begin{cases} 0 & \text{при } x \le 0 \\ 1 - e^{-x} & \text{при } x > 0. \end{cases}$$

Отметим, что если ν_n имеет геометрическое распределение, т.е.

$$P\{\nu_n = k\} = p_n(1-p_n)^{k-1}, \quad k = 1, 2, \ldots, \quad 0 < p_n < 1 \tag{3}$$

и $\lim_{n\to\infty} p_n = 0$, то полагая $\alpha_n = M\nu_n = \frac{1}{p_n}$, легко проверить, что $G(x) = E(x)$. И такая схема суммирования случайных величин с геометрически распределенным случайным индексом, часто встречается в задачах теории массового обслуживания и теории надежности. Известно [2] также, что обстоятельно исследована скорость сходимости в предельном соотношении (2), когда ν_n имеет геометрическое распределение. При этом предполагалось что ν_n не зависит или же имеет определенный вид зависимости от случайных величин последовательности (1). В [3] получены некоторые равномерные оценки скорсоти сходимости непосредственно в (2), без допольнительных ограничений на зависимость ν_n и $\{\xi_n\}$. А именно доказана следующая теорема.

Теорема 1. *Пусть $Q(h, G) \le k_1 h$ при всех $h > 0$, $\alpha_n = M\nu_n$ и $D\nu_n \le k_2 \alpha_n^{\delta}$ для всех n и при некотором δ, $2 \le \delta < 4$. Если $D\xi_1 < \infty$, тогда*

$$\Delta_n(G) \le d_n(G) + \left[2 + k_2 + k_1 \frac{\sigma}{|a|}\right] \alpha_n^{-4-\delta/7}, \tag{4}$$

а в случае существования конечного третьего момента случайной величины ξ_1

$$\Delta_n(G) \le d_n(G) + c\left[\frac{\sigma}{|a|} + \frac{\beta^2}{\sigma^6}\right] \alpha_n^{-3/17(4-\delta)}, \tag{5}$$

где c — постоянная зависящая только от k_1 и k_2.

В данной статье доказана возможность некоторых улучшений по порядку сходимости полученной в теореме 1 в предельном соотношении (2), при этом не налагая какие-либо моментные условия на случайный индекс ν_n. Так как во многих задачах теории массового окслуживания и надежности экспоненциалный закон часто вступает в качестве предельного распределения, поэтому результаты сформулируем и докажем для $G(x) = E(x)$.

Теорема 2. *Пусть $D\xi_1 < \infty$. Тогда для любого $\delta > 0$*

$$\Delta_n(E) \le 2d_n(E) + \left(1 + \frac{\sigma}{|a|}\right) \alpha_n^{-1/3(1-\delta)} + e^{-\alpha_n^{\delta}}. \tag{6}$$

Теорема 3. *Пусть $M|\xi_1 - a|^3 < \infty$. Тогда для любого $\delta > 0$ существуют абсолютные постоянные c_1 и c_2 такие, что*

$$\Delta_n(E) \le 2d_n(E) + \left(\frac{\sigma}{|a|} + c_1 \frac{\beta}{\sigma^3}\right) \alpha_n^{-(1/2-\delta)} + c_2 e^{-\frac{1}{4}\alpha_n^{\delta}}. \tag{7}$$

Следствие. Пусть ν_n имеет геометрическое распределение. Если $D\xi_1 < \infty$, тогда для любого $\delta > 0$

$$\Delta_n(E) \leq \left(c_3 + \frac{\sigma}{|a|}\right) \alpha_n^{-1/3(1-\delta)} + e^{-\alpha_n^\delta}, \tag{8}$$

а в случае $M|\xi_1 - a|^3 < \infty$

$$\Delta_n(E) \leq \left(\frac{\sigma}{|a|} + c_4 \frac{\beta}{\sigma^3}\right) \alpha_n^{-(1/2-\delta)} + c_2 e^{-\frac{1}{4}\alpha_n^\delta}, \tag{9}$$

где $\alpha_n = M\nu_n = \dfrac{1}{p_n}$ и c_3, c_4 — абсолютные постоянные.

Утверждения (8) и (9) получаются из (6) и (7), соответственно, при помощи следующего нетрудно проверяемого факта [2]: существует абсолютная постаянная c_5 такая, что

$$d_n(E) = \sup_x |P\{p_n \nu_n < x\} - E(x)| \leq c_5 p_n = c_5 \alpha_n^{-1}.$$

Замечания. (а) Показательные слагаемые $\exp\{-\alpha_n^\delta\}$ и $\exp\left\{-\dfrac{1}{4}\alpha_n^\delta\right\}$ для любого $\delta > 0$ могут быть оценены следующим образом:

$$\exp\{-\alpha_n^\delta\} \leq c_1(\delta)\, \alpha_n^{-1/3(1-\delta)},$$

$$\exp\left\{-\frac{1}{4}\alpha_n^\delta\right\} \leq c_2(\delta)\, \alpha_n^{-(1/2-\delta)},$$

где $c_1(\delta)$, $c_2(\delta)$ — постоянные зависящие только от δ, явный вид которых легко определить элементарным способом,

(б) для сравнения отметим, например, что в случае (3) из (4), (5) соответственно имеем (см. [3])

$$\Delta_n(E) \leq \left(c_6 + \frac{\sigma}{|a|}\right) p_n^{2/7},$$

$$\Delta_n(E) \leq c_7 \left[\frac{\sigma}{|a|} + \frac{\beta^2}{\sigma^6}\right] p_n^{6/17},$$

где c_6, c_7 — абсолютные постоянные. А из (8) и (9) следует, что если $\delta < \dfrac{1}{7}$ то $\dfrac{2}{7} < \dfrac{1}{3}(1-\delta)$ и если $\delta < \dfrac{5}{34}$ то $\dfrac{6}{17} < \left(\dfrac{1}{2} - \delta\right)$.

Прежде чем приступить к доказательству теорем 2 и 3 сформулируем одну лемму из [4] (лемма 3, стр. 27) в следующем виде.

Лемма. Пусть ξ и η — случайные величины, $\zeta = \xi + \eta$ и $G(x)$ — произвольная функция распределения. Тогда для любого $\varepsilon > 0$

$$\sup_x |P\{\zeta < x\} - G(x)| \leq \sup_x |P\{\xi < x\} - G(x)| + P\{|\eta| \geq \varepsilon\} + Q(\varepsilon, G).$$

Доказательства теорем 2 и 3. Очевидно, что

$$\Delta_n(E) = \sup_x \left| P\left\{\left(\frac{S_{\nu_n}}{a\,\nu_n} - 1\right)\frac{\nu_n}{\alpha_n} + \frac{\nu_n}{\alpha_n} < x\right\} - E(x)\right| .$$

Используем лемму, полагая

$$\eta = \left(\frac{S_{\nu_n}}{a\,\nu_n} - 1\right)\frac{\nu_n}{\alpha_n} .$$

Тогда имеем

$$\Delta_n(E) \le d_n(E) + Q(\varepsilon, E) + P\left\{\left|\frac{S_{\nu_n} - a\,\nu_n}{a\,\nu_n}\right| \ge \varepsilon\right\},$$

здесь $\varepsilon > 0$ — пока произвольное, подлежащее оптимальному выбору. Теперь оценим вероятности

$$p_n = P\left\{\left|\frac{S_{\nu_n} - a\,\nu_n}{a\,\alpha_n}\right| \ge \varepsilon\right\}.$$

Нетрудно проверить, что для любого $\varrho > 0$

$$p_n \le P\{\nu_n > \varrho\} + P\{|S_{\nu_n} - a\,\nu_n| \ge |a|\,\varepsilon\alpha_n,\ \nu_n \le \varrho\} \le$$
$$\le P\{\nu_n > \varrho\} + P\left\{\max_{1\le k\le \varrho} |S_k - ak| \ge |a|\,\varepsilon\alpha_n\right\}.$$

При помощи $d_n(E)$ легко оценить, что

$$P\{\nu_n > \varrho\} \le d_n(E) + \exp\left\{-\frac{\varrho}{\alpha_n}\right\}.$$

Таким образом

$$\Delta_n(E) \le 2d_n(E) + \varepsilon + \exp\left\{-\frac{\varrho}{\alpha_n}\right\} + P\left\{\max_{1\le k\le \varrho} |S_k - ak| \ge |a|\,\varepsilon\alpha_n\right\},$$

так как $Q(\varepsilon, E) \le 1 - e^{-\varepsilon} \le \varepsilon$, $\varepsilon > 0$. Для оценки вероятности

$$\overline{P}_n = P\left\{\max_{1\le k\le \varrho} |S_k - ak| \ge |a|\,\varepsilon\alpha_n\right\}$$

воспользуемся неравенством А. Н. Колмогорова. Тогда будем иметь, что

$$\overline{P}_n \le \frac{\varrho\sigma^2}{(a\,\varepsilon\alpha_n)^2} .$$

Теперь для любого $\delta > 0$ положим

$$\varepsilon = \frac{\sigma}{|a|}\alpha_n^{-1/3(1-\delta)}, \quad \varrho = \alpha_n^{1+\delta} .$$

Тогда $\overline{P_n} \leq \alpha_n^{-1/3(1-\delta)}$. Следовательно

$$\Delta_n(E) \leq 2d_n(E) + \frac{\sigma}{|a|} \alpha_n^{-1/3(1-\delta)} + \exp\{-\alpha_n^\delta\} + \alpha_n^{-1/3(1-\delta)},$$

что и доказывает теорему 2.

Чтобы доказать теорему 3, для оценки вероятности $\overline{P_n}$ воспользуемся следующим результатом из [5] (замечание б):

$$\sup_x \left| P\left\{\max_{1\leq k\leq n} (S_k - ak) > x\sigma\sqrt{n}\right\} - \frac{\sqrt{2}}{\pi} \int_x^\infty e^{-u^2/2}\, du \right| \leq \frac{c_8 \beta}{\sigma^3 \sqrt{n}},$$

где c_8 — абсолютная постоянная.

Отсюда легко вывести, что

$$\overline{P_n} \leq \frac{2c_8 \beta}{\sigma^3 \sqrt{\varrho}} + \frac{4}{\sqrt{2\pi}} \int_{T_n}^\infty e^{-u^2/2}\, du,$$

где

$$T_n = \frac{|a|\, \varepsilon \alpha_n}{\sigma \sqrt{\varrho}}.$$

Также легко проверить, что

$$\overline{P_n} \leq 2c_8 \frac{\beta}{\sigma^3 \sqrt{\varrho}} + 2\sqrt{2} \exp\left\{-\frac{1}{4} T_n^2\right\}.$$

Теперь если для любого $\delta > 0$ положим

$$\varepsilon = \frac{\sigma}{|a|} \alpha_n^{-(1/2-\delta)}, \quad \varrho = \alpha_n^{1+\delta}$$

то имеем

$$\overline{P_n} \leq 2c_8 \frac{\beta}{\sigma^3} \alpha_n^{-1+\delta/2} + 2\sqrt{2} \exp\left\{-\frac{1}{4} \alpha_n^\delta\right\}.$$

Следовательно

$$\Delta_n(E) \leq 2d_n(E) + \exp\{-\alpha_n^\delta\} + \frac{\sigma}{|a|} \alpha_n^{-(1/2-\delta)} +$$
$$+ 2c_8 \frac{\beta}{\sigma^3} \alpha_n^{-1+\delta/2} + 2\sqrt{2} \exp\left\{-\frac{1}{4} \alpha_n^\delta\right\}.$$

Откуда и получаем доказаательство теоремы 3.

Проследив ход доказательства теорем 2 и 3, нетрудно убедиться в том что для произвольной $G(x)$ теоремы 2 и 3 могуть быть обобщены сленующим образом

Теорема 4. *Пусть $Q(h, G) \leq k, h$ при всех $h > 0$. Если $D\xi_1 < \infty$ тогда для любого $\delta > 0$*

$$\Delta_n(G) \leq 2d_n(G) + \bar{G}(\alpha_n^\delta) + \left(1 + k_1 \frac{\sigma}{|a|}\right) \alpha_n^{-1/3(1-\delta)},$$

а в случае существования конечного третьего момента случайной величины ξ_1, для любого $\delta > 0$

$$\Delta_n(G) \leq 2d_n(G) + \bar{G}(\alpha_n^\delta) + \left(k_1 \frac{\sigma}{|a|} + c_1 \frac{\beta}{\sigma^3}\right) \alpha_n^{-(1/2-\delta)} + c_2 e^{-1/4\alpha_n^\delta},$$

где $\bar{G}(x) = 1 - G(x)$ и c_1, c_2 — абсолютные постоянные.

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TWO CHARACTERIZATIONS OF HILBERT SPACE BY APPROXIMATION THEORETICAL PROPERTIES

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Introduction. In this paper we shall prove for real Banach spaces of dimension ≥ 3 the following: if any countable family of finite-dimensional subspaces can be transformed by arbitrary small renorming into Chebyshev subspaces so that the corresponding metric projections are linear, then this Banach space is a Hilbert space (Theorem 2).

This result resembles the JAMES—RUDIG—SCHMIDT theorem ([1] p. 249). We mention that the restriction on the dimension is essential, because all subspaces of an arbitrary strictly convex (essentially strictly convex in the terminology of [2]) two-dimensional Banach space are Chebyshev subspaces with linear metric projections.

We also prove that for all real Banach spaces it is possible to transform a countable family of finite-dimensional subspaces to Chebyshev subspaces by arbitrary small renorming (Theorem 1).

Theorem 3 characterizes Hilbert spaces by a symmetry property of metric projections.

Preliminaries. If $(X, \|\cdot\|)$ is a Banach space and $H' \subset H \subset X$, we denote by H^L the linear closure of H, by $co\, H$ the convex closure of H, by $\overline{H}$ the topological closure of H (all topological concepts refer to the norm topology), and by $\mathrm{mar}_H\, H'$ the boundary of H' in the relative topology of H.

We denote by P_L the metric projection to the Chebyshev subspace L. More generally, P_H denotes the metric projection $P_{\overline{H^L}}$ to the Chebyshev subspace $\overline{H^L}$.

For $x, y \in X$; $r \in \mathbf{R}$, $r \geq 0$, we set

$$S_{\|\cdot\|}(x, r) = \{z; z \in X, \|x-z\| \leq r\},$$

$$F_{\|\cdot\|}(x, r) = \{z; z \in X, \|x-z\| = r\},$$

$$[x, y] = \{\lambda x + (1-\lambda) y;\ 0 \leq \lambda \leq 1\},$$

$$(x, y) = \{\lambda x + (1-\lambda) y;\ 0 < \lambda < 1\},$$

$$d(x, H) = d(H, x) = \inf_{h \in H} \|x-h\|.$$

DEFINITION. We say the sequence $\{F_n\}_{n\in\mathbf{N}}$ of two dimensional planes $F_n\subset X$ converges to the plane $F\subset X$ if there exist $a_1, a_2, a_3\in F$ which are non--collinear and such that for any $\varepsilon>0$ we have $d(a_1, F_n)<\varepsilon$, $d(a_2, F_n)<\varepsilon$, $d(a_3, F_n)<\varepsilon$ provided n is sufficiently large.

Clearly, if $\{F_n\}_{n\in\mathbf{N}}$ converges to F, and $x\in\bigcap_{n=1}^{\infty} F_n$, then $x\in F$.

DEFINITION. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Let $\varepsilon>0$ be fixed. We say that $\|\cdot\|_1$ is ε-equivalent to $\|\cdot\|_2$ if either

$$(1-\varepsilon)\,\|x\|_1\leq\|x\|_2\leq(1+\varepsilon)\,\|x\|_1 \quad \text{for all} \quad x\in X$$

or

$$(1-\varepsilon)\,\|x\|_2\leq\|x\|_1\leq(1+\varepsilon)\,\|x\|_2 \quad \text{for all} \quad x\in X.$$

We shall often use the following properties of the metric projection P_L of the Chebyshev subspace L (for the proofs, see [1]):

(a) $P_L(tx) = tP_L(x) \quad (t\in\mathbf{R})$,

(b) $P_L(x+l) = P_L(x)+l \quad (l\in L)$,

(c) $\|P_L(x)\|\leq 2\,\|x\|$.

For the rest of the terminology see [2].

THEOREM 1. *Let $(X, \|\cdot\|)$ be a real Banach space, and $\{M_i\}_{i\in\mathbf{N}}$ a countable family of finite-dimensional subspaces in X. Then for any $\varepsilon>0$ there exists a norm $|||\cdot|||$ on X, which is ε-equivalent to $\|\cdot\|$ and such that M_i is a Chebyshev subspace of $(X, |||\cdot|||)$ for all $i\in\mathbf{N}$.*

LEMMA 1. Let $(X, \|\cdot\|')$ be a real Banach space and let M be a finite--dimensional subspace of X. If the relations $[x, y]\subset F_{\|\cdot\|'}(0, 1)$, $y\neq x$ imply $y-x\notin M$, then M is a Chebyshev subspace of $(X, \|\cdot\|')$.

PROOF OF LEMMA 1. M is proximinal in $(X, \|\cdot\|')$, because M is finite--dimensional. If some $x\in X$ has two different best approximants $x_1, x_2\in M$ then all elements of $[x_1, x_2]$ are best approximants of x in M (triangle inequality). So all elements of $[x_1, x_2]$ are at the same distance from x, and $x_1-x_2\in M$.

PROOF OF THEOREM 1. Using the Hahn-Banach theorem, we find closed subspaces K_i such that

$$X = M_i\oplus K_i \quad (i = 1, 2, \ldots). \tag{1}$$

Let $e^i_j\,(j = 1, \ldots, \dim M_i)$ be a normed basis of M_i. Then for each $x\in X$ and $i\in\mathbf{N}$ there is a unique representation of the form

$$x = x_i+\sum_{j=1}^{\dim M_i} a^i_j e^i_j,$$

where $x_i\in K_i$, $a^i_j\in\mathbf{R}$ $(j = 1, \ldots, \dim M_i)$.

Set

$$p_i(x) = \sqrt{\sum_{j=1}^{\dim M_i} |a^i_j|^2} \quad (x\in X;\ i = 1, 2, \ldots).$$

p_i is a seminorm on X. Let $S_i = \{x \in M_i;\ p_i(x) = 1\}$. S_i is compact in X because M_i is finite-dimensional. Further, p_i and $\|\cdot\|$ are equivalent on S_i.

We have $d_i = d(S_i, K_i) > 0$. Otherwise there exist $\{y_j\}_{j\in\mathbf{N}} \subset S_i$ and $\{z_j\}_{j\in\mathbf{N}} \subset K_i$ with $\lim_j \|y_i - z_j\| = 0$. Taking a convergent subsequence

$$\{y_{n_j}\}_{j\in\mathbf{N}} \subset \{y_j\}_{j\in\mathbf{N}},\ \lim_j y_{n_j} = y,$$

we obtain $\lim_j \|y - z_{n_j}\|$. Thus $y \in S_i \cap K_i$ and $y \neq 0$, which contradicts (1).

Let $x \in X$, $x = x_i + z_i$, $x_i \in K_i$, $z_i \in M_i$, $z_i \neq 0$. Then

$$0 < d_i = d(S_i, K_i) = \inf_{u\in K_i, v\in S_i} \{\|u - v\|\} \leq \left\| \frac{x_i}{p_i(z_i)} - \frac{-z_i}{p_i(z_i)} \right\| = \frac{\|x\|}{p_i(z_i)},$$

so that

(2) $$p_i(z_i)\, d_i \leq \|x\|.$$

This is fulfilled also in the case $z_i = 0$. But $p_i(x) = p_i(z_i)$. Hence

(3) $$0 \leq d_i\, p_i(x) \leq \|x\|.$$

Set

$$|||x||| = \sqrt{\|x\|^2 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} (d_i\, p_i(x))^2}.$$

$|||\cdot|||$ is a norm on X (the triangle inequality comes from the Cauchy – Buniakovsky – Schwarz inequality). Using (3) we get

(4) $$\|x\| \leq |||x||| \leq (1+\varepsilon)\, \|x\|$$

for all $x \in X$.

We demonstrate that $[x, y] \subset F_{|||\cdot|||}(0, 1)$, $x \neq y$, $x - y \in M_i$ is impossible for any $i \in \mathbf{N}$.

Otherwise there exist $x, y \in X$ and $l \in \mathbf{N}$ such that $x \neq y$, $x - y \in M_l$, and for $0 \leq \lambda \leq 1$

$$1 = |||\lambda x + (1-\lambda) y||| = |||\lambda x + (1-\lambda) y|||^2 =$$

(5) $$= \|\lambda x + (1-\lambda) y\|^2 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \big(d_i\, p_i(\lambda x + (1-\lambda) y)\big)^2.$$

Consider the relations

(6) $$\left\| \frac{x+y}{2} \right\|^2 \leq \left(\frac{\|x\| + \|y\|}{2} \right)^2 \leq \frac{\|x\|^2 + \|y\|^2}{2},$$

(7) $$p_j^2\left(\frac{x+y}{2}\right) \leq \left(\frac{p_j(x) + p_j(y)}{2} \right)^2 \leq \frac{p_j^2(x) + p_j^2(y)}{2} \quad (j = 1, 2, \ldots).$$

Summing them and applying (5) to $\lambda=0, \frac{1}{2}, 1$ it follows that in (6) and (7) we always have the case of equality. Therefore $\|x\|=\|y\|$ and $p_j(x)= = p_j(y)$ for all j. In particular, $p_l(x)=p_l(y)$ and

$$(8) \qquad p_l^2\left(\frac{x+y}{2}\right)=\left(\frac{p_l(x)+p_l(y)}{2}\right)^2.$$

But this is impossible since $x-y\in M_l$ and p_l is strictly convex on M_l.

So Lemma 1 and relation (4) yield the desired conclusion.

THEOREM 2. *Let $(X, \|\cdot\|)$ be a real Banach space with* $\dim X\geq 3$. *If for every countable family $\{M_i\}_{i\in\mathbf{N}}$ of finite-dimensional subspaces of X and every $\varepsilon>0$ there exists a norm $|||\cdot|||$, ε-equivalent to $\|\cdot\|$ on X, such that M_i is a Chebyshev subspace of $(X, |||\cdot|||)$ and P_{M_i} is a linear operator for all $i\in\mathbf{N}$, then $(X, \|\cdot\|)$ is a Hilbert space.*

LEMMA 2. Let $(T, \|\cdot\|')$ be a real Banach space, $x\in Y$, $x\neq 0$. If $\{x\}^L$ is a Chebyshev subspace and P_x is linear, then the function

$$g_y:\mathbf{R}\to\mathbf{R}, \quad g_y(\lambda)=\|y+\lambda x\|$$

has strict global minimum at 0 for all $y\in P_x^{-1}(0)$.

PROOF. By the definition of P_x and y,

$$\min_\lambda g_y(\lambda)=\min_\lambda\|y+\lambda x\|'=\|y-P_x(y)\|'=\|y\|'.$$

Moreover, this is a strict minimum, because $\{x\}^L$ is a Chebyshev subspace.

PROOF OF THEOREM 2. We shall demonstrate that $(G, \|\cdot\|)$, where G is any three-dimensional subspace of X, is a Hilbert space.

Let $\bigcup_{i\in\mathbf{N}}\{a_i\}^L$ be dense in $(G, \|\cdot\|)$, and $|||\cdot|||$ be a norm, ε-equivalent to $\|\cdot\|$, such that $\{a_i\}^L$ is a Chebyshev subspace in $(G, |||\cdot|||)$ and P_{a_i} is linear for all $i\in\mathbf{N}$. It means no loss of generality if we assume

$$(9) \qquad |||a_i|||=1 \quad a\in\{a_i\}_{i\in\mathbf{N}}\Rightarrow -a\in\{a_i\}_{i\in\mathbf{N}}.$$

Further, let us denote $P_{a_i}\equiv P_i$.

Our next aim is to show that $|||\cdot|||$ is strictly convex.

Otherwise two cases are possible:

(A) There exists a supporting plane H of the unit ball of $(G, |||\cdot|||)$ whose intersection with the unit ball is an interval.

(B) There exists a supporting plane H of the unit ball of $(G, |||\cdot|||)$ whose intersection with the unit ball contains a disc.

In the case (A), let

$$(10) \qquad [a, b]=F_{|||\cdot|||}(0,1)\cap H,$$

$$(11) \qquad h\in H, \quad |||h-b|||=\frac{|||a-b|||}{100}=\frac{c}{100}>0,$$

(12) $$d(h, b+\{a-b\}^L) = \frac{c}{100}.$$

Choose a subfamily $\{a_{i_n}\}_{n\in\mathbf{N}} \subset \{a_i\}_{i\in\mathbf{N}}$ such that

(13) $$(a+\{a_{i_n}\}^L) \cap S_{|||\cdot|||}\left(h, \frac{c}{1000}\right) \neq \emptyset, \ \lim_n a_{i_n}$$

exists, and

(14) $$\lim_n \left(\sup_{u\in[a-a_{i_n}, a+a_{i_n}]} d(H, u)\right) = 0.$$

This is possible because of (9) and the density of

$$\bigcup_{i\in\mathbf{N}} \{a_i\}^L.$$

Set

(15) $$c_k = \max_{\mu, \nu} \{|||x-y|||;\ x = a+\mu a_{i_k},\ y = a+\nu a_{i_k},\ \mu, \nu\in\mathbf{R},\ |||x|||\leq 1,\ |||y|||\leq 1\},$$

(16) $$d_k = \max_{\mu, \nu} \{|||x-y|||;\ x = b+\mu a_{i_k},\ y = b+\nu a_{i_k};\ \mu, \nu\in\mathbf{R},\ |||x|||\leq 1,\ |||y|||\leq 1\}$$

and denote by $\bar{x}_k$, $\bar{y}_k$, $\bar{\bar{x}}_k$ and $\bar{\bar{y}}_k$, respectively, the points where these maxima are attained. We are going to prove the relations

(17) $$\lim_k c_k = 0,$$

(18) $$\lim_k d_k = 0.$$

Suppose that (17) fails. The triangle inequality and (15) yield

(19) $$0 \leq c_k \leq 2.$$

So there is a subfamily

$$\{c_{k_j}\}_{j\in\mathbf{N}} \subset \{c_k\}_{k\in\mathbf{N}}$$

such that the limits

$$\lim_j c_{k_j} = f,\ \lim_j \bar{x}_{k_j} = x^*,\ \lim_j \bar{y}_{k_j} = y^*$$

exist and $f\neq 0$. On account of (15),

(20) $$|||x^*||| = |||y^*||| = 1.$$

From (14), (15) and (19) it follows that $x^*, y^* \in H$. Hence, in view of (20) and (10), x^* and y^* belong to $[a, b]$. On the other hand, $x^*-y^* = f\lim_k a_{i_k}$ or $x^*-y^* = -f\lim_k a_{i_k}$. Therefore

$$\lim_k a_{i_k} \in \{b-a\}^L .$$

This, however, contradicts the relations (11)–(13).

The proof of (18) is similar.

Now let K be so large that

$$0 \leq c_K < \frac{c}{100}, \quad 0 \leq d_K < \frac{c}{100}.$$

By (15), (16) and Lemma 2

$$P_{i_K}^{-1}(0) \cap [\bar{y}_K, \bar{x}_K] \neq \emptyset, \quad P_{i_K}^{-1}(0) \cap [\bar{\bar{x}}_K, \bar{\bar{y}}_K] \neq \emptyset .$$

Consequently

(21) $$d(a, P_{i_K}^{-1}(0)) < \frac{c}{100},$$

(22) $$d(b, P_{i_K}^{-1}(0)) < \frac{c}{100}.$$

By (13) there exists $\beta \in \mathbf{R}$ with

$$|||a + \beta a_{i_K} - h||| < \frac{c}{1000}.$$

Hence, in view of (9), (11) and the triangle inequality,

$$\frac{989c}{1000} \leq |\beta| \leq \frac{1011c}{1000}.$$

Let a_{i_K} be chosen so that $\beta > 0$. Then

$$|||a + ca_{i_K} - h||| < \frac{12c}{1000}.$$

Making use of (11) and (22) we obtain

$$d(a + ca_{i_K}, P_{i_K}^{-1}(0)) \leq \frac{32c}{1000}.$$

So there exists $l \in P_{i_K}^{-1}(0)$ with

$$|||a + ca_{i_K} - l||| \leq \frac{33c}{1000}.$$

By the linearity of P_{i_K} and property (c) of the metric projection

(23) $$|||P_{i_K}(a + ca_{i_K})||| = |||P_{i_K}(a + ca_{i_K} - l)||| \leq \frac{66c}{1000}.$$

On the other hand, by (21) and the same property (c)

$$\text{(24)} \qquad |||P_{i_K}(a)||| \leq \frac{21c}{1000}.$$

Further on account of (b)

$$P_{i_K}(a+ca_{i_K})-P_{i_K}(a) = ca_{i_K}.$$

So using (9), (23) and (24) we find

$$c = |||P_{i_K}(a+ca_{i_K})-P_{i_K}(a)||| \leq \frac{87c}{1000}.$$

By (11) this is a contradiction.

In the case (B), let e_0 be an extreme point of $F_{|||\cdot|||}(0, 1) \cap H$ (such an e_0 exists by virtue of the Krein-Milman theorem). Let $b \in F_{|||\cdot|||}(0, 1) \cap H$,

$$\delta > 0,\ H \cap S_{|||\cdot|||}(b, \delta) \subset H \cap F_{|||\cdot|||}(0, 1),$$

$$\text{(25)} \qquad b_1, b_2 \in H \cap S_{|||\cdot|||}\left(b, \frac{\delta}{2}\right),\ \ \frac{b_1+b_2}{2} = b,\ b_1, b_2 \notin [b, e_0],$$

$$\text{(26)} \quad \lambda_i > 0,\ e_i = b_i + \lambda_i (e_0 - b),\ e_i \in \operatorname{mar}_H (H \cap F_{|||\cdot|||}(0, 1)) \quad (i = 1, 2)$$

(such e_i exist because of the boundedness of $H \cap F_{|||\cdot|||}(0, 1)$).

Choose a subfamily

$$\{a_{i_K}\}_{K \in \mathbf{N}} \subset \{a_i\}_{i \in \mathbf{N}}$$

such that a_{i_K} tends to

$$\text{(27)} \qquad \frac{e_0 - b}{|||e_0 - b|||},\quad b + a_{i_K} \notin H;$$

0 and $b + a_{i_K}$ lie on the same side of H. This is possible because of (9) and the density of $\bigcup_{i \in \mathbf{N}} \{a_i\}^L$.

Let

$$\text{(28)} \qquad m_K^{(l)} = \max\{\mu;\ |||e_l + \mu\, a_{i_K}||| \leq 1,\ \mu \geq 0\} \quad (l = 0, 1, 2).$$

Then

$$\text{(29)} \qquad \lim_K m_K^{(l)} = 0.$$

In fact, otherwise some subsequence of $\{m_K^{(l)}\}$ has a limit $S_l > 0$, so that (28) and the definition of $\{a_{i_K}\}$ yield $e_l + S_l (e_0 - b) \in H \cap F_{|||\cdot|||}(0, 1)$, which contradicts (25), (26) and the convexity of $H \cap F_{|||\cdot|||}(0, 1)$.

As H is a supporting plane of the unit ball, (27) implies that

$$|||e_l + \lambda\, a_{i_K}||| > 1 \quad \text{if} \quad \lambda < 0.$$

So by (28) and Lemma 2

(30) $$P_{i_K}^{-1}(0)\cap[e_l, e_l+m_K^{(l)}a_{i_K}]\neq\emptyset\quad (l=0,1,2).$$

As e_0 is an extreme point of $H\cap F_{|||\cdot|||}(0,1)$, the points e_0, e_1, e_2 are not contained in a line (because (25) and (26) imply that e_1 and e_2 are on different sides of the line in H containing e_0 and b). Therefore, in view of (29) and (30) the planes $P_{i_K}^{-1}(0)$ are converging to H. This, however, contradicts the relations $0\in\cap P_{i_K}^{-1}(0)$, $0\notin H$.

Now we shall prove that for any $x\in G$, $x\neq 0$, the metric projection P_x is linear (by the strict convexity of $|||\cdot|||$ just established, $\{x\}^L$ is a Chebyshev subspace in $(G, |||\cdot|||)$). Let $\{a_{i_K}\}_{K\in\mathbf{N}}\subset\{a_i\}_{i\in\mathbf{N}}$ satisfy

$$\lim_K a_{i_K}=\frac{x}{|||x|||}.$$

We claim that

(31) $$\lim_K P_{i_K}(y)=P_x(y).$$

Assume the converse. Clearly

$$\lim_K d(y, [-2a_{iK}|||y|||, 2a_{iK}|||y|||])=d\left(y, \left[\frac{-2x}{|||x|||}|||y|||, \frac{2x}{|||x|||}|||y|||\right]\right).$$

Hence by property (c),

(32) $$\lim_K |||y-P_{i_K}(y)|||=|||y-P_x(y)|||.$$

From (c), $\{P_{i_K}(y)\}_{K\in\mathbf{N}}$ is bounded in $(G, |||\cdot|||)$. So we may assume, there exists

$$\lim_K P_{i_K}(y)\neq P_x(y).$$

Then

$$\lim_K d(y, [-2a_{i_K}|||y|||, 2a_{i_K}|||y|||])=\lim_K |||y-P_{i_K}(y)|||=\left|\left|\left|y-\lim_K P_{i_K}(y)\right|\right|\right|.$$

From this, using (32) and that $\{x\}^L$ is a Chebyshev subspace, we obtain contradiction.

As a consequence of (31), P_x is linear. Therefore, by the James—Rudin—Schmidt theorem ([1], p. 249), $(G, |||\cdot|||)$ is a Hilbert space. So, for arbitrary $x, y\in G$.

$$2|||x|||^2+2|||y|||^2=|||x+y|||^2+|||x-y|||^2.$$

If $\varepsilon\to 0$, then $|||z|||\to\|z\|$ for any $z\in G$. Thus

$$2\|x\|^2+2\|y\|^2=\|x+y\|^2+\|x-y\|^2.$$

THEOREM 3. *The real Banach space X with strictly convex norm $\|\cdot\|$ is a Hilbert space iff for all pairs $a, b \in X$, $\|a\| = \|b\| = 1$, we have*

$$\|P_a(b)\| = \|P_b(a)\|. \tag{33}$$

PROOF. If $(X, \|\cdot\|)$ is a Hilbert space, then

$$\|P_a(b)\| = \|a\langle a, b\rangle\| = \|b\langle a, b\rangle\| = \|P_b(a)\|.$$

To prove the converse, assume that (33) is valid.

Choose any $a \in X$, $a \neq 0$, and let $b \in X$, $b \neq 0$, be Riesz-orthogonal to $\{a\}^L$. Then a is Riesz-orthogonal to $\{b\}^L$ since by (33) the relation $P_a(b) = 0$ implies $P_b(a) = 0$. It is clear that $a + b \neq 0$. By property (a) of the metric projection

$$\|P_{a+b}(a)\| = \|a\| \left\| P_{a+b/\|a+b\|}\left(\frac{a}{\|a\|}\right) \right\|, \tag{34}$$

$$\|P_{a+b}(b)\| = \|b\| \left\| P_{a+b/\|a+b\|}\left(\frac{b}{\|b\|}\right) \right\|. \tag{35}$$

By properties (a) and (b)

$$P_{a+b}(a) = a + b - P_{a+b}(b), \tag{36}$$

that is,

$$P_{a+b}(a) + P_{a+b}(b) = a + b. \tag{37}$$

Using (b), $P_a(a+b) = a + P_a(b) = a$, $P_b(a+b) = P_b(a) + b = b$. So (a) yields

$$\left\| P_a\left(\frac{a+b}{\|a+b\|}\right) \right\| = \left\| P_{a/\|a\|}\left(\frac{a+b}{\|a+b\|}\right) \right\| = \frac{\|a\|}{\|a+b\|}, \tag{38}$$

$$\left\| P_b\left(\frac{a+b}{\|a+b\|}\right) \right\| = \left\| P_{b/\|b\|}\left(\frac{a+b}{\|a+b\|}\right) \right\| = \frac{\|b\|}{\|a+b\|}. \tag{39}$$

Replacing a and b of (33) by $\dfrac{a}{\|a\|}$ and $\dfrac{a+b}{\|a+b\|}$, respectively, then applying (34) and (38) we obtain

$$\|P_{a+b}(a)\| = \|a\| \frac{\|a\|}{\|a+b\|}. \tag{40}$$

Similarly, from (33) $\left(\text{applied to } \dfrac{b}{\|b\|} \text{ and } \dfrac{a+b}{\|a+b\|}\right)$, (35) and (39) we deduce

$$\|P_{a+b}(b)\| = \|b\| \frac{\|b\|}{\|a+b\|}. \tag{41}$$

By (37), (40) and (41), either

$$\|a+b\|^2 = \|a\|^2+\|b\|^2 \tag{42}$$

or

$$\|a+b\|^2 = |\|a\|^2-\|b\|^2| . \tag{43}$$

But a is Riesz-orthogonal to $\{b\}^L$, and b is Riesz-orthogonal to $\{a\}^L$, so that $\|a+b\| \geq \max\{\|a\|, \|b\|\}$; hence (43) is impossible.

Let x, y be linearly independent, and let the vector $z\in\{x, y\}^L$, $z\neq 0$, be Riesz-orthogonal to $\{x\}^L$. There is a unique $c\in\mathbf{R}$ with $y-cz\in\{x\}^L$, i.e., $y=cz+dx$, $x-y=-cz+(1-d)x$, $x+y=cz+(1+d)x$. Since cz is Riesz-orthogonal to $\{x\}^L$, (42) yields

$$\|y\|^2 = \|cz\|^2+\|dx\|^2 ,$$

$$\|x-y\|^2 = \|-cz\|^2+\|(1-d)x\|^2 ,$$

$$\|x+y\|^2 = \|cz\|^2+\|(1+d)x\|^2 .$$

From the last theree formulas we obtain

$$2\|x\|^2+2\|y\|^2 = \|x+y\|^2+\|x-y\|^2 .$$

This holds also when x, y are linearly dependent. The proof is complete.

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A REMARK ON THE LINEARITY OF METRIC PROJECTIONS

By

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Let $(X, \|\cdot\|)$ be a Banach space, M a subspace of it. Let us define

$$P_M(x) = \left\{y;\ y\in M,\ \|x-y\| = \inf_{m\in M} \|x-m\|\right\}.$$

If for $\forall x\in X\ P_M(x)\neq\emptyset$, we call M a proximinal subspace, and P_M is called the metric projection to M. We say P_M has a linear selection if $\exists A,\ A: X\to M$, A linear, such that for $\forall x\in X\ A(x)\in P_M(x)$.

When for $\forall x\in X\ P_M(x)$ has one and only one element, M is called a Chebyshev subspace of $(X, \|\cdot\|)$.

In this paper we prove an analogon of the JAMES—RUDIN—SCHMIDT theorem (see [1], p. 159). The latter says:

Let X be a Banach space, $\dim X\geq K+2$, $K\in\mathbf{N}$ fixed. Then X is a Hilbert space iff every K-dimensional subspace of X is a Chebyshev subspace with a linear metric projection.

If $H\subset X$, denote by $[H]$ the linear span of H. For other notations, definitions and theorems to be used see [1] and [2].

THEOREM. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed linear space. Then the following condition* (C) *is sufficient, and in the case of X a Banach space also necessary, for X being an inner product space:*

(C) *Given $a\in X$, $a\neq 0$, there exist reflexive Chebyshev subspaces $X_1\supset X_2\supset\ldots\supset X_n\supset\ldots$ with linear metric projections such that*

$$\bigcap_{n=1}^{\infty} X_n = [a].$$

PROOF. If X is a Banach inner product space, that is, a Hilbert space, then the Riesz-orthogonal projection theorem yields (C).

For the rest we need a

LEMMA. If $Y \subset Z$ are Chebyshev subspaces with linear metric projection, then $P_Y \circ P_Z = P_Y$.

PROOF OF THE LEMMA. It is clear that

$$X = Z \oplus P_Z^{-1}(0), \quad Z = Y \oplus (P_Y^{-1}(0) \cap Z).$$

So for $x \in X$ we have $x = y(x) + y_0(x) + z_0(x)$, where

$$y(x) \in Y, \quad y_0(x) \in P_Y^{-1}(0) \cap Z, \quad z_0(x) \in P_Z^{-1}(0).$$

From the definition and linearity of P_Y and P_Z we obtain:

$$P_Y(x) = P_Y(y(x) + y_0(x) + z_0(x)) = y(x) = P_Y(y(x) + y_0(x)) = P_Y(P_Z(x)). \quad \blacksquare$$

Let us return to the proof of the theorem. Let us write $P_{X_i} \equiv P_i$. Consider an element $x \in X$. Obviously (see [1] p. 159)

$$\|P_i(x)\| \leq 2\|x\|.$$

So, $\exists n_1 < n_2 < \ldots < n_i < \ldots$ $(n_i \in \mathbf{N})$ such that weak

$$\lim_i P_{n_i}(x) = x_0$$

exists. It is clear that $x_0 \in [a]$.

By Mazur's theorem, for $\forall \varepsilon > 0$ $\exists K$, c_j $(j = 1, \ldots, K)$,

$$0 < c_j \leq 1, \quad \sum_{j=1}^{K} c_j = 1,$$

and $i_1, \ldots, i_K$ such that

$$\left\| \sum_{j=1}^{K} c_j P_{i_j}(x) - x_0 \right\| < \varepsilon. \tag{1}$$

Using our lemma in the case $n \geq \max\{i_1, \ldots, i_K\}$, we have

$$P_n\left(\sum_{j=1}^{K} c_j P_{i_j}(x)\right) = \sum_{j=1}^{K} c_j P_n(P_{i_j}(x)) = \sum_{j=1}^{K} c_j P_n(x) = P_n(x).$$

Hence, in view of (1) and $x_0 \in [a] \subset X_n$,

$$\|P_n(x) - x_0\| = \left\| P_n\left(\sum_{j=1}^{K} c_j P_{i_j}(x)\right) - x_0 \right\| = \left\| P_n\left(\sum_{j=1}^{K} c_j P_{i_j}(x) - x_0\right) \right\| \leq 2\varepsilon.$$

Consequently, $\lim_n P_n(x) = x_0$. Therefore

$$\lim_n P_n(x) \in P_{[a]}(x),$$

so that $P_{[a]}$ has a linear selection. Using [3], our theorem is proved.

The same proof yields the following

PROPOSITION. If $(X, \|\cdot\|)$ is a normed linear space, and $X_1 \supset X_2 \supset \dots \supset X_n \supset \dots$ are reflexive Chebyshev subspaces of X with linear metric projections, then $\bigcap_{n=1}^{\infty} X_n$ is a proximinal subspace of X, its metric projection has a linear selection, and P_{X_n} converges pointwise to this selection.

The next example shows that in the latter proposition the reflexivity of X_n is essential.

Let

$$x = (x_1, x_2, \dots, x_n, \dots) \in l_\infty ,$$

$$\|x\| = \sup_n |x_n| + \sum_{n=1}^{\infty} \frac{1}{n^2} |x_n| ,$$

$$e_n = (\underbrace{0}_{1}, \underbrace{0}_{2}, \dots, 0, \underbrace{1}_{n}, 0, \dots) .$$

It can be proved that if X_n is the closure of $[e_1, e_n, e_{n+1}, \dots]$, then X_n is a Chebyshev subspace with linear metric projection in $(l_\infty, \|\cdot\|)$, but P_{X_n} does not converge to $P_{[e_1]}$. (For

$$x = \sum_{n=1}^{\infty} e_n$$

we have

$$P_{X_n}(x) = e_1 + \sum_{i=n}^{\infty} e_i, \quad (n > 1), \quad P_{[e_1]}(x) = e_1 ,$$

$$\|P_{X_n}(x) - P_{[e_1]}(x)\| > 1 .)$$

I should like to thank J. BOGNÁR for useful advice.

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ON A MAXIMAL FUNCTION

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Introduction

We introduce the concept of a maximal function to the theory of the Vilenkin-Fourier series. This is a modification of the well-known Hardy-Littlewood maximal function [3]. It is shown that our maximal operator M has weak type $(1, 1)$, for all $1<p\leq\infty$ has type (p, p) and $\|Mf\|_1$ is bounded for all $f\in L(\log^+ L)$. For application of this see [2].

1. We introduce some notations and definitions. Let

$$m=(m_0, \dots, m_k, \dots) \quad (2\leq m_k, m_k\in \mathbf{N}:=\{0, 1, \dots\})$$

be a sequence of natural numbers and denote by Z_{m_k} $(k\in\mathbf{N})$ the m_kth discrete cyclic group. If we define the group G_m as the direct product of the groups Z_{m_k}, then G_m is a compact Abelian group.

Let μ be the normalized Haar measure on G_m. Each element of G_m we can consider as a sequence $(x_k, k\in\mathbf{N})$ with $0\leq x_k<m_k$ $(k\in\mathbf{N})$. Set $M_0:=1$,

$$M_k:=\prod_{i=0}^{k-1} m_i \quad (k\in\mathbf{P}:=\{1, 2, \dots\}).$$

We can identify G_m with the interval $(0, 1)$ as follows:

$$G_m\ni(x_k, k\in\mathbf{N})\to\sum_{k=0}^{\infty}\frac{x_k}{M_{k+1}}\in[0, 1].$$

This mapping is "almost" one-one (except the set of m_k-rationals), onto and measure preserving.

Let $(I_k, k\in\mathbf{N})$ be a sequence of subgroups of G_m defined by

$$I_0:=G_m, \quad I_k:=\{z=(0, \dots, 0, x_k, \dots)/x\in G_m\} \quad (k\in\mathbf{P}).$$

For all $x\in G_m$ and $k\in\mathbf{N}$ we define the set $I_k(x)$ as the coset of I_k by x, i.e.: $I_k(x):=\{x\}\dotplus I_k$ ($\dotplus$ denote the group operation on G_m). Let σ_k $(k\in\mathbf{N})$ be the σ-algebra generated by the cosets of I_k. We remark that on the interval (0, 1) atoms of σ_k are the intervals of the form

$$\left[\frac{i}{M_k}, \frac{i+1}{M_k}\right) \quad (i=0, \ldots, M_k-1, k\in\mathbf{N}).$$

The set $I\subset G_m$ is called "interval", if exist $x\in G_m$ and $k\in\mathbf{N}$ such that I is a proper subset of $I_k(x)$, $I\in\sigma_{k+1}$ and I is an interval on $I_k(x)$, if $I_k(x)$ is considered as a circle. We denote the set of intervals with $\mathcal{J}$.

Furthermore define the maximal function Mf of $f\in L^1(G_m)$ as follows

$$Mf(x) := \sup_{\substack{x\in I\\ I\in\mathcal{J}}} \frac{1}{\mu(I)} \int_I |f| .$$

M is evidently sublinear.

$C>0$ will denote an absolute, although not always the same, constant.

2. We have the following analogue of a well-known theorem [3].

THEOREM. *There are absolute constants C and $C_p>0$ such that*

(i) $$\mu\{x: Mf(x)>y\} \leq C\cdot\frac{\|f\|_1}{y} \quad (f\in L^1(G_m),\ y>0),$$

(ii) $$\|Mf\|_p \leq C_p\|f\|_p \quad \left(f\in L^p(G_m),\ 1<p\leq\infty,\ C_p = O\left(\frac{1}{p-1}\right)(p\to 1)\right),$$

(iii) $$\|Mf\|_1 \leq C\cdot\|f\cdot\log^+|f|\|_1 + C \quad (f\in L(\log^+ L)).$$

In the case $p=\infty$ the statement (ii) is trivial. For $1<p<\infty$ follows the part (ii) from the weak type (1, 1) inequality (i) and from the case $p=\infty$ by the Marcinkiewicz interpolation theorem [3]. Applying the part (ii) and an extrapolation theorem [3] then yields the part (iii).

For the proof of the inequality (i) we need a modified form of the Calderon-Zygmund decomposition lemma [1].

LEMMA ([2], Lemma). Given $f\in L^1(G_m)$ and $y\geq\|f\|_1$. Then there exist decompositions $G_m = F\cup\overline{F}$ $(\overline{F} := G_m\setminus F)$, $f=f_0+w$ such that

(a) $F = \bigcup_{k\in\mathbf{P}} H_k$, where $H_k\in\mathcal{J}$ is a proper interval, $H_k\cap H_j=\emptyset$ $(k\neq j)$,

(b) $w = \sum_{k\in\mathbf{P}} f_k$, $\operatorname{supp} f_k\subset H_k$, $\int_{H_k} |f_k| \leq 12\mu(H_k)$, $\int_{H_k} f_k = 0$,

$\int_{H_k} |f| \geq y\cdot\mu(H_k)$ $(k\in\mathbf{P})$,

(c) $\|f_0\|_\infty \leq 6y$, $\|f_0\|_1 \leq 2\|f\|_1$, $\mu(F)\leq\dfrac{\|f\|_1}{y}$.

PROOF OF THE PART (i) OF THEOREM. For the proof of (i) we note that we can evidently assume $\|f\|_1 \leq y$. Decompose f as in Lemma and we use the following notation. Let $H_k \in \sigma_{j_k+1}$ and H_k contained in the coset $I_{j_k}(\xi^{(k)})$ of I_{j_k} ($k \in \mathbf{N}$, $j_k \in \mathbf{N}$, $\xi^{(k)} \in G_m$). We consider $I_{j_k}(\xi^{(k)})$ as a circle, and let $DH_k \in \mathcal{J}$ denote the interval inside $I_{j_k}(\xi^{(k)})$ which contains H_k at its center and $\mu(DH_k) = 3 \cdot \mu(H_k)$. When this is not possible, then let DH_k denote $I_{j_k}(\xi^{(k)})$. Let $DF := \bigcup_{k \in \mathbf{P}} DH_k$, then by Lemma

$$\mu(DF) \leq 3 \cdot \frac{\|f\|_1}{y} \tag{1}$$

holds. We prove the following inequality:

$$Mw(x) \leq C \cdot y \quad (x \notin DF,\ C > 6). \tag{2}$$

To prove of (2) let $x \notin DF$ be arbitraly fixed and let $I \in \mathcal{J}$ be an interval such that $x \in I$, I is a proper subset of $I_k(x)$, $I \in \sigma_{k+1}$ for certain $k \in \mathbf{N}$. Applying Lemma, we have the following estimation:

$$\frac{1}{\mu(I)} \int_I |w| = \frac{1}{\mu(I)} \sum_{H_i \subset I_k(x)} \int_{H_i \cap I} |f_i| =$$

$$= \frac{1}{\mu(I)} \left[\sum_{H_i \subset I} \int_{H_i} |f_i| + \sum_{\substack{H_i \not\subset I \\ H_i \cap I \neq \emptyset}} \int_{H_i} |f_i| \right] =: \Sigma_1 + \Sigma_2 .$$

For Σ_1 we have easily an estimation: the sets H_i are by pairs disjoint, therefore

$$\Sigma_1 = \frac{1}{\mu(I)} \sum_{H_j \subset I} \int_{H_j} |f_j| \leq C \cdot y \cdot \frac{1}{\mu(I)} \sum_{H_j \subset I} \mu(H_j) \leq C \cdot y .$$

Furthermore, Lemma (a) implies that Σ_2 contains not more than two member, e.g. H_i and H_j. By definition of $\mathcal{J}$ we have in this case that H_i and H_j are measurable with respect to σ_{k+1}. Since $x \notin DF$, thus we have

$$\mu(I) \geq \mu(H_i) + \mu(H_j) .$$

Therefore, the following inequality holds:

$$\Sigma_2 \leq \frac{1}{\mu(I)} \left[\int_{H_i} |f_i| + \int_{H_j} |f_j| \right] \leq C \cdot y \cdot \frac{1}{\mu(I)} [\mu(H_i) + \mu(H_j)] \leq C \cdot y .$$

Hence, we have proved that for all $x \notin DF$ and for all interval $I \in \mathcal{J}$ which contains x, the following inequality holds:

$$\frac{1}{\mu(I)}\int_I |w| \leq C\cdot y.$$

From this, by definition of M, easily follows for the above x that $Mw(x)\leq C\cdot y$.

Now, we can prove the inequality of the part (i) of Theorem as follows. Since by Lemma (c) $\|f_0\|_\infty\leq 6\cdot y$, we have

$$|Mf_0(x)\leq 6\cdot y \quad (x\in G_m). \tag{3}$$

Hence, applying the inequalities (1), (2) and (3), the following estimation holds (C denote the constant in the inequality (2)):

$$\mu\{x: Mf(x)>2Cy\}\leq \mu\{x: Mf_0(x)>Cy\}+\mu\{x: Mw(x)>Cy\} =$$
$$=\mu\{x: Mf_0(x)>Cy\}+\mu\{x: x\in DF,\ Mw(x)>Cy\}+$$
$$+\mu\{x: x\notin DF,\ Mw(x)>Cy\} = \mu\{x: x\in DF,\ Mw(x)>Cy\}\leq 3\cdot\frac{\|f\|_1}{y}.$$

From this easily follows that

$$\mu\{x: Mf(x)>y\}\leq 6\cdot C\cdot\frac{\|f\|_1}{y}$$

holds.

This concludes the proof of Theorem.

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ON DIFFERENCE SETS OF SEQUENCES OF INTEGERS, II.

By

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1. Throughout this paper, we use the following notations:

We write $e^x = \exp x$. $C, C_1, C_2, \dots$ denote absolute constants, $p, p_1, p_2, \dots$ (positive) prime numbers. We denote the least (positive) quadratic non-residue modulo p by $q(p)$:

$$\left(\frac{1}{p}\right) = \left(\frac{2}{p}\right) = \dots = \left(\frac{q(p)-1}{p}\right) = +1, \quad \left(\frac{q(p)}{p}\right) = -1$$

$\left\{\text{where } \left(\frac{n}{p}\right) \text{ denotes the Legendre symbol}\right\}$. If p is a prime number, n is an integer then we denote the least non-negative residue of n modulo p by $r(n, p)$, i.e., $r(n, p)$ is defined by

$$r(n, p) \equiv n \pmod p,$$

$$0 \leqq r(n, p) \leqq p-1.$$

A set of positive integers $u_1 < u_2 < \dots$ will be called an $\mathcal{A}$-set if its difference set does not contain the square of a positive integer; in other words, if

$$u_x - u_y = z^2$$

(where x, y, z are integers) implies that $x = y$, $z = 0$. Let $A(x)$ denote the greatest number of integers that can be selected from $1, 2, \dots, x$ to form an $\mathcal{A}$-set and let us write

$$a(x) = \frac{A(x)}{x}.$$

2. In Part I of this series I proved (starting out from a conjecture of L. LOVÁSZ) that

$$a(x) = O\left(\frac{(\log\log x)^{2/3}}{(\log x)^{1/3}}\right)$$

(see [2]). The aim of this paper is to investigate the problem that how far is this estimate from the best possible?

I conjecture that for any $\varepsilon > 0$,

(1) $$a(x) = O(x^{-1/2+\varepsilon}).$$

Theorem 1 will show that it is near hopeless to prove this conjecture at the present time.

On the other hand, it can be shown easily that there exists an absolute constant $C_1 > 0$ such that for $x \geqq 4$,

(2) $$a(x)x^{1/2} > C_1.$$

Namely, let p denote the greatest prime number satisfying $p \leqq x^{1/2}$. Let us investigate the set

(3) $$p, 2p, \ldots, p^2.$$

If $1 \leqq i < j \leqq p$ then obviously,

$$p/jp - ip = (j-i)p$$

and

$$0 < jp - ip = (j-i)p < p^2,$$

hence $p^2 \nmid jp - ip$. These facts imply that the set (3) forms an $\mathcal{A}$-set, selected from $1, 2, \ldots, x$. Thus

$$a(x) = \frac{A(x)}{x} \geqq \frac{p}{x} > C_1 x^{-1/2}$$

$\left(\text{since } p > \frac{1}{2}x^{1/2} \text{ by Bertrand's postulate and } x \geqq 4\right)$ and (2) is proved.

P. Erdős raised the problem (oral communication) whether

$$\lim_{x \to +\infty} a(x)x^{1/2} = +\infty?$$

In Theorem 2, we shall prove (4) in a sharper form. ((4) could be deduced also from the simple construction given in Theorem 1. The aim of Theorem 2 is to give a better lower estimate for $a(x)x^{1/2}$ than the one which could be deduced in a relatively complicated way from Theorem 1.)

3. In this section, we will prove the following theorem:

Theorem 1. *Let x be a positive integer, p a prime number such that*

(5) $$\frac{1}{2}x^{1/2} < p \leqq x^{1/2}$$

and

(6) $$p \equiv 1 \pmod 4$$

(provided that such a prime number p exists at all). Then

(7) $$a(x)\, x^{1/2} > \frac{1}{2} q(p)\,.$$

PROOF. Let s denote a fixed integer satisfying

(8) $$\left(\frac{s}{p}\right) = -1\,.$$

For $i = 0, 1, 2, \dots$, let $\mathcal{B}_i$ denote the set of the integers $ip+r(s, p)$, $ip+r(2s, p)$, $\dots$, $ip+r(q(p)\,s, p)$. Finally, let

$$\mathcal{B} = \bigcup_{i=0}^{p-1} \mathcal{B}_i\,.$$

We are going to show at first that

(9) $$\mathcal{B} \subset \{1, 2, \dots, x\}\,.$$

(8) implies that

(10) $$(s, p) = 1\,.$$

Furthermore, if $1 \leqq j \leqq q(p)$ then $(j, p) = 1$. Thus if $1 \leqq j \leqq q(p)$ then $r(js, p) > 0$, consequently, the elements of $\mathcal{B}$ are positive.

On the other hand, $b \in \mathcal{B}$ implies that b can be written in form

(11) $$b = ip+r(js, p) \quad \text{where} \quad 0 \leqq i \leqq p-1 \quad \text{and} \quad 1 \leqq j \leqq q(p)\,.$$

Hence

$$b = ip+r(js, p) < (p-1)\,p+p = p^2 \leqq x$$

which completes the proof of (9).

Now we are going to show that $\mathcal{B}$ is an $\mathcal{A}$-set. To show this, we have to prove that if b_1, b_2, z are integers satisfying

(12) $$b_1 \in \mathcal{B}, \quad b_2 \in \mathcal{B}$$

and

(13) $$b_1 - b_2 = z^2$$

then

(14) $$b_1 = b_2$$

must hold.

By (12), b_1 and b_2 can be written in form (11). Let

$$b_k = i_k p + r(j_k s, p) \quad \text{for} \quad k = 1, 2$$

where

(15) $$0 \leqq i_k \leqq p-1 \quad (\text{for} \quad k = 1, 2)$$

and

(16) $$1 \leqq j_k \leqq q(p) \quad (\text{for} \quad k = 1, 2).$$

If

(17) $$p \nmid b_1 - b_2$$

then (13) implies that

$$\left(\frac{b_1 - b_2}{p}\right) = \left(\frac{z^2}{p}\right) = +1.$$

On the other hand, it follows from (17) that $j_1 \neq j_2$, thus with respect to (16),

(18) $$0 < |j_1 - j_2| < q(p).$$

By (6), (8), (17) and (18),

$$\left(\frac{b_1 - b_2}{p}\right) = \left(\frac{(i_1 p + r(j_1 s, p)) - (i_2 p + r(j_2 s, p))}{p}\right) =$$

$$= \left(\frac{j_1 s - j_2 s}{p}\right) = \left(\frac{s}{p}\right)\left(\frac{j_1 - j_2}{p}\right) = -\left(\frac{|j_1 - j_2|}{p}\right) = -1.$$

Thus (17) leads to a contradiction which proves that $p / b_1 - b_2$. By (13), this implies also

$$p^2 / b_1 - b_2 = (i_1 p + r(j_1 s, p)) - (i_2 p + r(j_2 s, p)) =$$

$$= (i_1 - i_2) p + (r(j_1 s, p) - r(j_2 s, p)).$$

By (15), this implies that $i_1 = i_2$ and $r(j_1 s, p) = r(j_2 s, p)$ which proves (14), thus $\mathcal{B}$ is an $\mathcal{A}$-set.

Summarizing: $\mathcal{B}$ is an $\mathcal{A}$-set, selected from $1, 2, \ldots, x$ (by (9)). Thus with respect to (5),

$$A(x) \geqq \sum_{b \in \mathcal{B}} 1 = \sum_{i=0}^{p-1} \left(\sum_{b \in \mathcal{B}_i} 1\right) = \sum_{i=0}^{p-1} q(p) = pq(p) > \frac{1}{2} x^{1/2} q(p).$$

Dividing by $x^{1/2}$, we obtain (7).

By Theorem 1, conjecture (1) (for *all* $\varepsilon > 0$) would imply that

$$q(p) = O(p^{\varepsilon}) \quad \text{for} \quad p \equiv 1 \pmod 4;$$

but, in fact, it is hopeless to prove even this consequence at the present time.

4. To prove Theorem 2, we shall use the same idea as in the proof of Theorem 1. The proof will be based on Lemma 2.

LEMMA 1 (P. ERDŐS and G. SZEKERES). *Let K, N be positive integers and G_K a graph of K vertices. If*

$$K \geqq \binom{2N-2}{N-1} \tag{19}$$

then either G_K or the complement of G_K contains a complete subgraph of N vertices.

For this lemma and its proof, see [1].

COROLLARY 1. *Let K be a positive integer. For each graph G_K of K vertices, either G_K or the complement of G_K contains a complete subgraph of $\left[\frac{\log K}{\log 4}+1\right]$ vertices.*

PROOF. Let us put

$$N = \left[\frac{\log K}{\log 4}+1\right].$$

Then

$$2^{2N-2} = 2^{2\left[\frac{\log K}{\log 4}\right]-2} \leqq 2^{2\frac{\log K}{\log 4}} = K.$$

Combining this with the trivial inequality

$$\binom{2N-2}{N-1} \leqq 2^{2N-2},$$

we obtain (19) thus Lemma 1 yields the corollary.

For any prime number p, a set of integers $v_1, v_2, \ldots, v_t$ will be called an $\mathcal{M}_p$ set if

$$(v_i, p) = 1 \quad \text{for} \quad 1 \leqq i \leqq t,$$

$$v_i \not\equiv v_j \pmod p \quad \text{dor} \quad 1 \leqq i < j \leqq t$$

and

$$\left(\frac{v_i - v_j}{p}\right) = -1 \quad \text{for} \quad 1 \leqq i < j \leqq t.$$

Let M_p denote the greatest number of integers that can be selected from $1, 2, \ldots, p-1$ to form an $\mathcal{M}_p$-set.

LEMMA 2. *If the prime number p satisfies*

$$p \equiv 1 \pmod 4 \tag{20}$$

then

$$M_p \geqq \left[\frac{\log(p-1)}{\log 4}+1\right].$$

PROOF. Let us define the graph G_{p-1} of $p-1$ vertices $P_1, P_2, \ldots, P_{p-1}$ in the following way:

The vertices P_i, P_j (where $i \neq j$) should be connected if and only if

$$\left(\frac{i-j}{p}\right) = -1$$

$\left\{\text{by (20), } \left(\frac{j-i}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{i-j}{p}\right) = \left(\frac{i-j}{p}\right), \text{ i.e., the graph is not directed}\right\}$.

Let us write

$$T = \left[\frac{\log(p-1)}{\log 4} + 1\right].$$

Applying Corollary 1, we obtain that either G_{p-1} or the complement of G_{p-1} contains a complete subgraph of T vertices let us denote the vertices of this subgraph by $P_{i_1}, P_{i_2}, \ldots, P_{i_T}$. In the first case, the integers $i_1, i_2, \ldots, i_T$ form an $\mathcal{M}_p$ set while in the second case, the integers $r(i_1 s, p)$, $r(i_2 s, p), \ldots, r(i_T s, p)$ do where s denotes a fixed integer satisfying (8). In either case, we have an $\mathcal{M}_p$-set of T integers thus

$$M_p \geqq T = \left[\frac{\log(p-1)}{\log 4} + 1\right]$$

which completes the proof of Lemma 2.

THEOREM 2. *For any* $\varepsilon > 0$ *and* $x > X_0(\varepsilon)$,

$$a(x) > x^{-1/2} \exp\left\{\left(\frac{1}{2} - \varepsilon\right) \frac{\log x \log\log\log x}{\log\log x}\right\}. \tag{21}$$

PROOF. Let us denote the n^{th} prime number of form $4k+1$ by

$$p_n : p_1 = 5,\ p_2 = 13,\ p_3 = 17,\ p_4 = 29, \ldots.$$

Let x be a large integer and let us define the positive integer R by

$$\prod_{n=1}^{R} p_n \leqq x^{1/2} < \prod_{n=1}^{R+1} p_n \tag{22}$$

and let

$$P = \prod_{n=1}^{R} p_n.$$

By the number theorem of the arithmetic progressions,

$$\sum_{p_n \leqq y} 1 \sim \frac{1}{2} \frac{y}{\log y}, \tag{23}$$

$$p_n \sim 2n \log n, \tag{24}$$

$$\log \prod_{n=1}^{k} p_n \sim k \log k. \tag{25}$$

For $n = 1, 2, \dots, R$, let S_n be a maximal $\mathcal{M}_{p_n}$-set selected from $1, 2, \dots, p_n - 1$. (Obviously, if certain integers form an $\mathcal{M}_{p_n}$-set then also their least non-negative residues do; thus we may select an $\mathcal{M}_{p_n}$-set also from $1, 2, \dots, p_n - 1$.)

For $i = 0, 1, 2, \dots$, let $\mathcal{B}_i$ denote the set of the integers of form $iP + j$ where

(26) $$r(j, p_n) \in S_n \quad \text{for} \quad n = 1, 2, \dots, R$$

and

(27) $$0 < j < P.$$

Finally, let

$$\mathcal{B} = \bigcup_{i=0}^{P-1} \mathcal{B}_i.$$

Obviously, the elements of $\mathcal{B}$ are positive. Furthermore, by (22) and (27), $b = iP + j \in \mathcal{B}$ implies that

$$b = iP + j < (P-1)P + P = P^2 \leqq x$$

thus

(28) $$\mathcal{B} \subset \{1, 2, \dots, x\}.$$

We are going to show that $\mathcal{B}$ is an $\mathcal{A}$-set. In other words, we have to prove that if b_1, b_2, z are integers satisfying

(29) $$b_1 \in \mathcal{B}, \quad b_2 \in \mathcal{B}$$

and

(30) $$b_1 - b_2 = z^2$$

then

(31) $$b_1 = b_2$$

must hold.

(29) implies that for $k = 1, 2$, b_k can be written in form

$$b_k = i_k P + j_k$$

where

(32) $$0 \leqq i_k \leqq P - 1 \quad (\text{for} \quad k = 1, 2)$$

and (in view of (27))

(33) $$0 < j_k < P \quad \text{for} \quad k = 1, 2.$$

If

(34) $$p_n \nmid b_1 - b_2$$

for some n (where $1 \leqq n \leqq R$) then (30) yields that

$$\left(\frac{b_1-b_2}{p_n}\right) = \left(\frac{z^2}{p_n}\right) = +1$$

while (26) (with j_1 and j_2 in place of j) implies that

$$\left(\frac{b_1-b_2}{p_n}\right) = \left(\frac{(i_1P+j_1)-(i_2P+j_2)}{p_n}\right) = \left(\frac{j_1-j_2}{p_n}\right) = -1.$$

Thus (34) leads to a contradiction; consequently,

$$p_n / b_1 - b_2 \quad \text{for} \quad n = 1, 2, \ldots, R.$$

By (30), this implies also

$$p_n^2 / b_1 - b_2 \quad \text{for} \quad n = 1, 2, \ldots, R,$$

hence

$$P^2 / b_1 - b_2 = (i_1P + j_1) - (i_2P + j_2) = (i_1 - i_2)P + (j_1 - j_2).$$

By (32) and (33), this implies that $i_1 = i_2$ and $j_1 = j_2$ which proves (31).

Summarizing: $\mathcal{B}$ is an $\mathcal{A}$-set selected from $1, 2, \ldots, x$ (in view of (28)). Thus

$$\text{(35)} \qquad A(x) \geqq \sum_{b \in \mathcal{B}} 1 = \sum_{i=0}^{p-1} \left(\sum_{b \in \mathcal{B}_i} 1 \right).$$

The inner sum is equal to the number of the integers j satisfying (26) and (27). If $r(j, p_n)$ is fixed for $n = 1, 2, \ldots, R$ then (26) and (27) determine j uniquely. Furthermore, for fixed n, $r(j, p_n)$ in (26) can be chosen in M_{p_n} ways. Thus for fixed i,

$$\sum_{b \in \mathcal{B}_i} 1 = \prod_{n=1}^{R} M_{p_n}.$$

Putting this into (35) and using Lemma 2, we obtain that

$$\text{(36)} \qquad A(x) \geqq \sum_{i=0}^{p-1} \left(\prod_{n=1}^{R} M_{p_n} \right) = P \sum_{n=1}^{R} M_{p_n} \geqq P \sum_{n=1}^{R} \left[\frac{\log (p_n - 1)}{\log 4} + 1 \right].$$

With respect to (23), (24) and (25),

$$\text{(37)} \qquad \log \prod_{n=1}^{R} \left[\frac{\log (p_n - 1)}{\log 4} + 1 \right] = \sum_{n=1}^{R} \log \left[\frac{\log (p_n - 1)}{\log 4} + 1 \right] \sim$$

$$\sim \sum_{n=1}^{R} \log \log p_n = \sum_{\sqrt{R} < n \leqq R} \log \log p_n + \sum_{1 \leqq n \leqq \sqrt{R}} \log \log p_n =$$

$$= (1 + o(1)) \sum_{\sqrt{R} < n \leqq R} \log \log R + O(\sqrt{R} \log \log R) =$$

$$= (1 + o(1)) R \log \log R + O(\sqrt{R} \log \log R) \sim R \log \log R.$$

(22), (24) and (25) imply that

$$\log P = \log\left(\prod_{n=1}^{R} p_n\right) \sim R\log R \sim \log x^{1/2} = \frac{1}{2}\log x$$

hence

$$R \sim \frac{1}{2}\frac{\log x}{\log\log x}, \tag{38}$$

furthermore, for $x > X_1$,

$$P > \frac{x^{1/2}}{p_{R+1}} > \frac{x^{1/2}}{3R\log R} > \frac{x^{1/2}}{4\cdot\frac{1}{2}\log x} = \frac{1}{2}\frac{x^{1/2}}{\log x}. \tag{39}$$

(37) and (38) yield that

$$\log \prod_{n=1}^{R}\left[\frac{\log(p_n-1)}{\log 4}+1\right] \sim \frac{1}{2}\frac{\log x}{\log\log x}\log\log\log x. \tag{40}$$

Combining (36), (39) and (40), we obtain for $x > X_2(\varepsilon)$ that

$$A(x) > \frac{1}{2}\frac{x^{1/2}}{\log x}\exp\left\{\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\frac{\log x\log\log\log x}{\log\log x}\right\} >$$

$$> x^{1/2}\exp\left\{\left(\frac{1}{2}-\varepsilon\right)\frac{\log x\log\log\log x}{\log\log x}\right\}.$$

Dividing by x, we obtain (21).

5. In Part III of this series, we will prove the solvability of the equations

$$u_x - u_y = z^2 - 1$$

and

$$u_x - u_y = p - 1,$$

respectively, for sequences $u_1 < u_2 < \dots$ of positive upper density. (We remark that e.g.

$$u_x - u_y = z^2 + 1,$$

$$u_x - u_y = p$$

need not be solvable.)

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ON THE GEOMETRIC CHARACTERIZATION OF DIFFERENTIATION IN NORMED VECTOR SPACES

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In the mathematical analysis the differentiability of a real function f can be interpreted by the tangent line to the graph of f.

One may ask how the notions of tangent lines can be generalized to the graph of F if F is a function from X into Y where X, Y are normed spaces, and whether the several concepts of the differentiability of functions can be characterized by these generalized notions.

In this paper we consider this problem.

Let X, Y be normed vector spaces, and let Γ_W, Γ_C and Γ_S denote the following subsets of X:

$$\Gamma_W = \{\{x\};\quad x\in X\}$$
$$\Gamma_C = \{\Omega\subset X;\quad \Omega \text{ is compact}\}$$
$$\Gamma_S = \{\Omega\subset X;\quad \Omega \text{ is bounded}\}.$$

For convenience, in all the following definitions Γ is considered as common notation for Γ_W, Γ_C and Γ_S.

Let F be a function from a subset of X into Y. We denote the domain, graph and the interior of the domain of F by $D(F)$, $G(F)$ and $\operatorname{Int} D(F)$, respectively.

The function $F: X\to Y$ is said to be

(i) *Γ-continuous* at $x_0\in \operatorname{Int} D(F)$ if $F(x_0+tx)$ converges to $F(x_0)$ as $t\to 0$ uniformly in $x\in\Omega$, on each $\Omega\in\Gamma$.

(ii) *Γ-differentiable* at $x_0\in \operatorname{Int} D(F)$ if there exists a continuous linear operator $A\in\mathcal{L}(X, Y)$, such that

$$\lim_{\substack{t\to 0\\ t\neq 0}}\left(\frac{F(x_0+tx)-F(x_0)}{t} - Ax\right) = \Theta$$

holds uniformly in x, on each $\Omega\in\Gamma$. If such an operator A exists, it is unique, and we call it the differential of F at x_0, and we denote it by $d_\Gamma F(x_0)$.

Analogously to differentiation at an interior point of $D(F)$, following NASHED [2] we say that F is *Γ-differentiable at* ∞ (or asymptotically differentiable), if

$$\{x;\ \|x\| \geq R\} \subset D(F) \quad \text{for any} \quad R > 0,$$

i.e. $\infty \in \operatorname{Int} D(F)$, and there exists an $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{t \to \infty} \left(\frac{F(tx)}{t} - Ax \right) = \Theta, \quad x \in \Omega$$

holds uniformly on Ω, for all $\Omega \in \Gamma$.

Again, if such an operator A exists, it is unique, and we call it the differential of F at ∞, and also denote it by $d_\Gamma F(\infty)$. These definitions of differentiability coincide with differentiability by Gâteaux, compact and Fréchet, respectively. It is an important question, whether the different types of differentiability defined above, can "geometrically" be characterized by appropriate tangent cones to the graph of F.

This question was positively settled by FLETT [1] in the case when both X and Y are finite dimensional normed vector spaces and the assumption on differentiability is that of Fréchet differentiability.

This question in the infinite dimensional case is open, NASHED [2].

In this paper we extend the previously mentioned finite-dimensional result to the case when X is an infinite dimensional normed vector space, but Y is still assumed to be finite dimensional.

In addition, our discussion has enabled us to show that this result remains in force for such types of differentiability which can be defined by means of sequences converging is some sense, i.e. for Gâteaux, compact and Fréchet differentiability. In order to show that the extension of this result fails in the case when Y is an infinite dimensional space, we give an example in Hilbert space.

The definitions of tangent cones for the different types of differentiability which we use, are described below. It is helpful to introduce a common terminology for three types of well-known convergences of the sequence $\{x_n\}$ in X. A sequence $\{x_n\}$ in X is said to be

(i) *Γ_W-convergent* to x, (denoted by $x_n \xrightarrow{\Gamma_W} x$) if $x_n = x$ for all n;

(ii) *Γ_C-convergent* to x, (denoted by $x_n \xrightarrow{\Gamma_C} x$) if x_n converges to x in norm;

(iii) *Γ_S-convergent* to x, (denoted by $x_n \xrightarrow{\Gamma_S} x$) if x_n converges to x weakly.

It will often happen later we should state and prove three similar theorems differing only in the types of convergence. When we say that a sequence $\{x_n\}$ is Γ-convergent to x then we conceive that the sequence $\{x_n\}$ converges to x in one of the previously defined types of convergence.

Let $G(F)$ be the graph of F. We define the Γ-tangent *cone to* $G(F)$ *at* $x_0 \in \operatorname{Int} D(F)$ as the set of all $(x, y) \in X \times Y$ with the property that there exists a sequence $\{x_n\}$ in $D(F)$, converging to x_0 in norm, and a sequence of non-negativ numbers $\{\lambda_n\}$ such that $\lambda_n(x_n - x_0)$ is Γ-convergent to x; further $\{F(x_n)\}$ and $\{\lambda_n(F(x_n) - F(x_0))\}$ converge to $F(x_0)$ and y in norm, respectively, i.e. we have the formulae

$$T_\Gamma = T\big(G(F);\ (x_0,\ F(x_0))\big)$$

$$(1)\qquad = \Big\{(x, y);\ \text{there exist sequences } \{x_n\} \text{ in } D(F)\ \{\lambda_n\},\ \lambda_n \geq 0, \text{ such that } x_n \xrightarrow{\Gamma_C} x_0,\ \lambda_n \to +\infty,\ F(x_n) \xrightarrow{\Gamma_C} F(x_0),\ \lambda_n(F(x_n - F(x_0)) \xrightarrow{\Gamma_C} y,\ \lambda_n(x_n - x_0) \xrightarrow{\Gamma} x\Big\},$$

where $T_\Gamma\big(G(F);\ (x_0,\ F(x_0))\big)$ or its abbreviation T_Γ denotes the Γ-tangent cone to $G(F)$ at $(x_0,\ F(x_0))$. It is easy to check that T_Γ is a cone, and

$$T_{\Gamma_W} \subset T_{\Gamma_C} \subset T_{\Gamma_S}.$$

Analogously to the Γ-tangent cone at $(x_0,\ F(x_0))$, we can introduce the notion of the Γ-tangent cone at ∞ in the following way:

$$(2)\qquad T_\Gamma^\infty = T_\Gamma^\infty(G(F)) = \Big\{x, y) \in X \times Y;\ \text{there exist sequences } \{x_n\} \in D(F) \text{ and } \{\lambda_n\} \in R \text{ such that } \lambda_n \to +\infty,\ \lambda_n^{-1} F(x_n) \xrightarrow{\Gamma_C} y,\ \lambda_n^{-1} x_n \xrightarrow{\Gamma} x\Big\}.$$

It is evident that T_Γ^∞ are cones, too.

In order to reveal the relationship between Γ-differentiability and the Γ-tangent cone, we have to reformulate the definitions of Γ-differentiability by using Γ-convergent sequences instead of defining uniform convergence on a Γ-set. More precisely it can be shown easily that the function F is Γ-differentiable, $\Gamma = \Gamma_W$ or Γ_C at an interior point x_0 of $D(F)$ or at ∞ if and only if there exists a linear operator A such that

$$(3)\qquad \lim_{\substack{t_n \to 0 \\ t_n \neq 0 \\ x_n \xrightarrow{\Gamma} x}} \frac{F(x_0 + t_n x_n) - F(x_0)}{t_n} = Ax$$

or

$$(4)\qquad \lim_{\substack{t_n \to +\infty \\ x_n \xrightarrow{\Gamma} x}} \frac{F(t_n x_n)}{t_n} = Ax$$

holds for all Γ-convergent sequences, where lim stands for convergence in norm. This statement remains true also in the case $\Gamma = \Gamma_S$ if X is a reflexive Banach space and Y is a finite dimensional normed vector space. FLETT

[1] has shown that if X, Y are finite dimensional spaces and F is continuous at x_0, then F is Fréchet-differentiable at x_0 if and only if there exists an $A \in \mathcal{L}(X, Y)$ such that the Γ_S-tangent cone to $G(F)$ at $(x_0, F(x_0))$ is contained in $G(A)$, and then A is equal to $d_{\Gamma_S} F(x_0)$.

Generalizations of this result for the infinite dimensional case cannot be made under the same conditions, as we show by an example below. Nevertheless, we can prove a similar theorem by only assuming that the range of the function is finite dimensional.

More exactly we have the following

THEOREM 1. *Let X be a Banach space and let Y be a finite dimensional normed vector space, further let $F: X \to Y$ be a function from $D(F)$ into Y which is Γ-continuous at $x_0 \in \operatorname{Int} D(F)$. Then*

(i) *if $\Gamma = \Gamma_W$ or $\Gamma = \Gamma_C$, then F is Γ-differentiable at x_0 if and only if there exists an $A \in \mathcal{L}(X, Y)$ such that*

$$T_\Gamma\big(G(F); (x_0, F(x_0))\big) \subset G(A)$$

and then A is equal to $d_\Gamma F(x_0)$.

(ii) *If X is a reflexive Banach space, then* (i) *remains valid in the case $\Gamma = \Gamma_S$ too.*

(iii) *The statement* (i) *and* (ii) *are true in the case $x_0 = \infty$ too, without assuming continuity. The following example shows that the theorem is not true in general in the case Y is an infinite dimensional normed vector space.*

Let $X = R$ and $H = Y$ be a Hilbert space and $\{\varphi_n\} \subset H$ be a complete orthonormal system. Further, let F be defined by

$$F(t) = \begin{cases} t\varphi_n & \text{if } \dfrac{1}{n} < |t| < \dfrac{1}{n-1} \quad n = 1, 2, \dots \\ 0 & \text{if } t = 0, \text{ or } t = \dfrac{1}{n} \quad n = 1, 2, \dots \end{cases}.$$

Since $\|F(t)\| \leq |t|$ for all t, the function F is continuous at 0.

However, F is not differentiable at 0, because

$$\frac{F(t_n) - F(0)}{t_n} = \varphi_n \quad \text{if} \quad t_n = \frac{2n-1}{2n(n-1)}.$$

Next we consider the graph of F

$$G(F) = \begin{cases} (t, t\varphi_n) & \text{if } \dfrac{1}{n} < |t| < \dfrac{1}{n-1} \quad n = 1, 2, \dots \\ (t, \Theta) & \text{if } |t| = \dfrac{1}{n}, \text{ or } t = 0 \quad n = 1, 2, \dots \end{cases}.$$

There is no difficulty in showing that

$$T\big(G(F), (0, F(0))\big) = R\times\{\Theta\}$$

where $R\times\{\Theta\}$ is evidently the graph of the zero operator from X into H. Furthermore, since the dimension of the space X is equal to 1, we have

$$T = T_{\Gamma_W} = T_{\Gamma_C} = T_{\Gamma_S}.$$

For Gâteaux-differentiability we have a theorem in the case of K convex function.

Let X, Y be real normed vector spaces, and assume that Y is an order complete vector lattice with normal order cone K [3].

The space Y is said to be an order complete vector lattice if

(i) Y is an ordered vector space with the positive cone

$$K = \{y\in Y;\ y\geq 0\}$$

(K is a pointed convex closed cone.),

(ii) Y is a vector lattice, i.e. $\inf\{y_1, y_2\}$ exists for all $y_1, y_2\in Y$,

(iii) Y is order complete i.e. $\inf \Omega$ exists for each nonempty subset Ω of Y such that Ω is order bounded from below.

The cone K is said to be normal, if the space Y has the following property, if a sequence $\{y_n\}\subset Y$ is monotonously non-decreasing and bounded from above, then it is convergent. The function $F: X\to Y$ is said to be K-convex, if

(i) $D(F)$ is nonempty convex subset of X,

(ii) $\alpha F(x)+(1-\alpha)F(z)-F\big(\alpha x+(1-\alpha)z\big)\in K$ for all $x, z\in D(F)$ and $\alpha\in[0, 1]$.

THEOREM 2. *Let $F: X\to Y$ be a K-convex function which is Γ_C-continuous at $x_0\in \operatorname{Int} D(F)$. The function F is Gâteaux-differentiable at x_0 if and only if there exists an $A\in\mathcal{L}(X, Y)$ such that*

$$T_{\Gamma_W}\big(G(F); (x_0, F(x_0))\big)\subset G(A)$$

and then A is equal to $d_{\Gamma_W} F(x_0)$.

Before proving the Theorem 1, we formulate two lemmas.

LEMMA 1. Let X, Y be normed vector spaces, let F be a function from a subset of X into Y.

(i) If F is Γ-differentiable at an interior point x_0 of $D(F)$, then

$$T_\Gamma = T_\Gamma\big(G(F); (x_0, F(x_0))\big)\subset G\big(d_\Gamma F(x_0)\big).$$

(ii) If F is Γ-differentiable at ∞, then

$$T_\Gamma^\infty\big(G(F)\big)\subset G\big(d_\Gamma F(\infty)\big).$$

PROOF OF LEMMA 1. In case (i), let (x, y) be a point in T_Γ. On account of the definition of T_Γ, there exist sequences $\{\lambda_n\}$ and $\{x_n\}$ such that

$$\lambda_n \to +\infty, \quad x_n \xrightarrow{\Gamma_C} x_0, \quad F(x_n) \xrightarrow{\Gamma_C} F(x_0)$$

and

$$\lambda_n (F(x_n) - F(x_0)) \xrightarrow{\Gamma_C} y,$$

$$\lambda_n (x_n - x_0) \xrightarrow{\Gamma} x \quad \text{as} \quad n \to +\infty .$$

From this it follows that

$$y = \lim_n \lambda_n (F(x_n) - F(x_0)) =$$

$$= \lim_n \frac{F\left(x_0 + \dfrac{\lambda_n (x_n - x_0)}{\lambda_n}\right) - F(x_0)}{\lambda_n^{-1}} = d_\Gamma F(x_0) x$$

hence $(x, y) \in G(d_\Gamma F(x_0))$ and this gives (i) immediately.

To prove (ii), let (x, y) be a point in $T_\Gamma^\infty (G(F))$. On account of the definition of T_Γ^∞ there are two sequences $\{\lambda_n\}$ and $\{x_n\}$ such that

$$\lambda_n \to +\infty, \quad \lambda_n^{-1} F(x_n) \xrightarrow{\Gamma_C} y$$

and

$$\lambda_n^{-1} x_n \xrightarrow{\Gamma} x \quad \text{as} \quad n \to +\infty .$$

Since

$$y = \lim_n \frac{F(x_n)}{\lambda_n} = \lim_n \frac{F\left(\dfrac{\lambda_n x_n}{\lambda_n}\right)}{\lambda_n} = d_\Gamma F(\infty) x ,$$

this implies that $(x, y) \in G(d_\Gamma F(\infty))$, whence the case (ii) follows.

To the second lemma we introduce some notations.

Let $F: X \to Y$ be a function, $x_0 \in \operatorname{Int} D(F)$, further let $\{x_n\} \subset X$, $\{t_n\} \subset R$ be two sequences and $x \in X$ with the following properties

$$x_n \xrightarrow{\Gamma} x \tag{5}$$

$$|t_n| > |t_{n+1}|, \quad t_n \to 0 . \tag{6}$$

If $x_0 = +\infty$, then

$$t_n < t_{n+1}, \quad t_n \to +\infty . \tag{7}$$

By means of the triples $(\{x_n\}, \{t_n\}, x)$ defined above, we construct functions f and g from R into X by

$$f(t) = \begin{cases} F(x_0 + t_n x_n) & \text{if} \quad t_n = t \\ F(x_0 + tx) & \text{if} \quad t_n \neq t \end{cases} \qquad n = 1, 2, \ldots \tag{8}$$

and

$$(9) \qquad g(t) = \begin{cases} F(t_n x_n) & \text{if} \quad t_n = t \\ F(tx) & \text{if} \quad t_n \neq t \end{cases} \quad n = 1, 2, \ldots .$$

Now let F be Γ-differentiable at $x_0 \in \operatorname{Int} D(F)$ (resp. at ∞). Then it is easy the check that f (resp. g) is differentiable at 0 (resp. ∞) and further

$$\left.\frac{df}{dt}\right|_{t=0} = d_\Gamma F(x_0) x, \quad \left.\frac{dg}{dt}\right|_{t=\infty} = d_\Gamma F(\infty) x .$$

LEMMA 2. Let X, Y be normed vector spaces, let F be a function from a subset of X into Y, and $x_0 \in \operatorname{Int} D(F)$, ($x_0 \in X$ or $x_0 = \infty$). If there exists a linear operator A such that

$$T_\Gamma\big(G(F); (x_0, F(x_0))\big) \subset G(A)$$

(resp. $T_\Gamma^\infty \subset G(A)$), then for any triple $(\{x_n\}, \{t_n\}, x)$ with the properties (5), (6) (resp. (5), (7)), and the function f (resp. g) associated with the triple, we have

$$T\big(G(f); (0, f(0))\big) \subset G(Ax)$$

and

$$T^\infty\big(G(g)\big) \subset G(Ax)$$

respectively, where Ax is considered a linear function from R into Y with the property $(Ax)(t) = t(Ax)$.

PROOF OF LEMMA 2. Since the arguments are similar in both cases, we consider only the first case. Let $(\{x_n\}, \{t_n\}, x)$ be a triple with the properties (5) and (6), and let f be the function associated with the triple in the sense, mentioned above. Now take any (s, u) in $T\big(G(f); (0, f(0))\big)$. By definition, there exist two sequences $\{s_n\}, \{\mu_n\} \in R$ such that

$$s_n \to 0, \quad \mu_n \to +\infty, \quad f(s_n) \to f(0)$$

$$\mu_n s_n \to s \quad \text{and} \quad \mu_n\big(f(s_n) - f(0)\big) \to u \quad \text{as} \quad n \to +\infty .$$

Put

$$z_n = \begin{cases} x_0 + t_m x_m & \text{if} \quad s_n = t_m \quad \text{for any } m, \quad m = 1, 2, \ldots \\ x_0 + s_n x & \text{if} \quad s_n \neq t_m \qquad\qquad\qquad m = 1, 2, \ldots \end{cases}$$

where $\{x_m\}$ and $\{t_m\}$ are the sequences used for defining f. The definition of f shows that

$$F(z_n) = f(s_n)$$

hence

$$(z_n, F(z_n)) \in G(F),$$

and

$$\mu_n\big(F(z_n) - F(x_0)\big) = \mu_n\big(f(s_n) - f(0)\big) \xrightarrow[n \to \infty]{} u .$$

The sequence $\{z_n\}$ converges to x_0, and we have

$$\mu_n(z_n - x_0) = \begin{cases} \mu_n t_m x_n = \mu_n s_n x_m & \text{if} \quad s_n = t_m \\ \mu_n s_n x & \text{if} \quad s_n \neq t_m \end{cases} \quad m = 1, 2, \ldots .$$

This implies that

$$\mu_n(z_n - x_0) \xrightarrow{\Gamma} sx$$

and so

$$(s, x, u) \in T_\Gamma .$$

Now using the condition $T_\Gamma \subset G(A)$ we have

$$sAx = u .$$

Since (s, u) was an arbitrary point of

$$T\big(G(f); (0, f(0))\big)$$

we finally obtain

$$T\big(G(f); (0, f(0))\big) \subset G(Ax) .$$

PROOF OF THE THEOREM 1. It will suffice to prove the theorem in the case $x_0 \in \operatorname{Int} D(F)$, because the proof for the asymptotical case is similar. Lemma 1 shows, that if F is Γ-differentiable at $x_0 \in \operatorname{Int} D(F)$, then

$$T_\Gamma\big(G(F); (x_0, F(x_0))\big) \subset G(d_\Gamma F(x_0)) .$$

Let x be an arbitrary point in X, and let $\{x_n\}$, $\{t_n\}$ be two sequences with the property that

$$(10) \qquad x_n \xrightarrow{\Gamma} x, \quad t_n \to 0 \quad |t_n| > |t_{n+1}| .$$

To prove the theorem, it suffices to show that the differential quotient

$$\frac{F(x_0 + t_n x_n) - F(x_0)}{t_n}$$

converges to Ax in norm, for every triples $(\{x_n\}, \{t_n\}, x)$ which satisfy the condition (10). The triple $(\{x_n\}, \{t_n\}, x)$ determines a function f by (8) and on account of lemma 2.

$$T\big(G(f); (0, f(0))\big) \subset G(Ax) .$$

Now we define a sequence $\{u_n\}$ in the space $R \times Y$.

$$u_n = \frac{z_n}{\|z_n\|} = \left(\frac{t_n}{|t_n| + \|f(t_n) - f(0)\|}, \frac{f(t_n) - f(0)}{|t_n| + \|f(t_n) - f(0)\|} \right) .$$

Since $\|u_n\| = 1$ for all n, there exists a subsequence of $\{u_n\}$, converging to an element of the unit sphere in $R \times Y$.

Let us denote this element by (s, y), and suppose that the subsequence is indexed in the same way as the original one. Put

$$\lambda_n = \frac{1}{\|z_n\|} = (|t_n| + \|f(t_n) - f(0)\|)^{-1}.$$

The function F is Γ-continuous at x_0, and so f is continuous at 0, therefore

$$\lambda_n \to +\infty .$$

By definition

$$\lambda_n (f(t_n) - f(0)) \to y$$

and

$$\lambda_n t_n \to s ,$$

which means

$$(s, y) \in T\big(G(f), (0, f(0))\big) .$$

By lemma 2.

$$(s, y) \in G(Ax)$$

i.e.

(11) $$sAx = y .$$

From this and the condition $|s| + \|y\| = 1$ it follows that $s \neq 0$, therefore

$$\lim_n \frac{f(t_n) - f(0)}{t_n} = \lim_n \frac{f(t_n) - f(0)}{\|z_n\|} \cdot \frac{\|z_n\|}{t_n} = \frac{1}{s} y .$$

Let us consider the differential quotient of F

$$\frac{F(x_0 + t_n x_n) - F(x_0)}{t_n} .$$

By definition $f(t_n) = F(x_0 + t_n x_n)$ for all n, from which we get

$$\lim_n \frac{F(x_0 + t_n x_n) - F(x_0)}{t_n} = \frac{1}{s} y$$

and according to (11) $sAx = y$, i.e.

$$\lim_n \frac{F(x_0 + t_n x_n) - F(x_0)}{t_n} = Ax$$

which means F is Γ-differentiable at x_0 and

$$d_\Gamma F(x_0) = A .$$

PROOF OF THE THEOREM 2. To prove the theorem, it is sufficient to show that, if

$$T_{\Gamma W}\big(G(F); (x_0, F(x_0))\big) \subset G(A)$$

then F is Gâteaux-differentiable at x_0.

It is known, that if y is an order complete vector lattice with normal order cone, then the directional derivate of F at x_0 in the direction x exists for all $x \in X$, and we have

$$\lim_{t \to 0+} \frac{F(x_0+tx)-F(x_0)}{t} = V_+ F(x_0; x).$$

From the definition of T_{Γ_W} it follows that

$$(x, V_+ F(x_0; x)) \in T_{\Gamma_W} \subset G(A),$$

hence

$$V_+ F(x_0; x) = Ax$$

for all $x \in X$. This means that the directional derivate of F at x_0 is linear and continuous i.e. F is Gâteaux-differentiable at x_0 and

$$d_{\Gamma_W} Fx_0 = A.$$

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CONSTRUCTIONS AND COMPOSITIONS OF RADICAL AND SEMISIMPLE CLASSES

By

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§ 1. Introduction

In the present paper we construct largest radical and semisimple classes contained in a given class of rings (Theorems 1, 2 and 3). Starting from radical classes α and β we define a composition $(\alpha \circ \beta)$. If α and β are hereditary radicals, then the class $(\alpha \circ \beta)$ is hereditary and has the inductive property as well as the extension property (Theorem 4), though $(\alpha \circ \beta)$ is generally neither a radical nor a semisimple class. Theorem 4 can be regarded as a generalization of a theorem of SNIDER [9]. We also discuss the interrelations between the class $(\alpha \circ \beta)$ and those studied in [9]. If α and β are radicals and $\mathcal{S}\alpha$ is a homomorphically closed semisimple class, then $(\alpha \circ \beta)$ is a radical (Theorem 5). If α is any hereditary radical and β is a radical — semisimple class, then $(\alpha \circ \beta)$ is a semisimple class (Theorem 6). Our results of constructing semisimple classes are based on a recent characterization of semisimple classes given by SANDS [8] and VAN LEEUWEN — ROOS — WIEGANDT [7].

We shall work with associative (or alternative) rings. For the fundamental concepts and results of the radical theory we refer to the textbooks [4], [11] and [12]. Nevertheless, we recall some definitions and results.

A class $\mathbf{C}$ of rings is said to be *hereditary*, if $B \triangleleft A \in \mathbf{C}$ implies $B \in \mathbf{C}$. A class $\mathbf{C}$ of rings has the *inductive* property, if for every ring A and every ascending chain $I_1 \subset \ldots \subset I_\lambda \subset \ldots$ of $\mathbf{C}$-ideals of A it follows $\bigcup I_\lambda \in \mathbf{C}$. Dually, a class $\mathbf{C}$ has the *co-inductive property,* if for any descending chain $A \supset \supset I_1 \supset \ldots \supset I_\lambda \supset \ldots$ of ideals of A such that $A/I_\lambda \in \mathbf{C}$ it follows $A/\bigcap I_\lambda \in \mathbf{C}$. We say that a class $\mathbf{C}$ is *closed under extensions* (or has the *extension property*) if $B \in \mathbf{C}$ and $A/B \in \mathbf{C}$ imply $A \in \mathbf{C}$. We shall apply frequently the *Anderson — Divinsky — Suliński — Theorem* (cf. [2] or [12] Theorem 5.2):

For any radical $\mathbf{R}$ $I \triangleleft A$ implies $\mathbf{R}(I) \triangleleft A$.

Also *Andrunakievič' Lemma* will be applied:

* Temporarily at the *Science Centre, University of Alexandria, Egypt.* Most of the results were obtained during a visit of the second named author at the first named one.

If $K \triangleleft I \triangleleft A$ then $\bar{K}^3 \subseteq K$ whenever $\bar{K}$ denotes the ideal of A generated by K.

AMITSUR [1] has proved that *a class* **R** *of rings is a radical class if and only if* **R** *is homomorphically closed, and has the inductive property and the extension property.*

The dual statement characterizing semisimple classes, was recently discovered (SANDS [8] and VAN LEEUWEN – ROOS – WIEGANDT [7]): *A class* **S** *of rings is a semisimple class if and only if* **S** *is hereditary, has the co-inductive property and the extension property.*

§ 2. Constructions of radical and semisimple classes

We define two operators a and h acting on classes **X** of rings by

$a\mathbf{X} = \{A \mid$ every accessible subring of A is in $\mathbf{X}\}$

and

$h\mathbf{X} = \{A \mid$ every homomorphic image of A is in $\mathbf{X}\}$.

Obviously $a\mathbf{X}$ (and $h\mathbf{X}$) are the largest hereditary (and homomorphically closed) subclasses of **X**.

PROPOSITION 1. *If a class* **B** *of rings has the inductive property, then also* $h\mathbf{B}$ *has the inductive property.*

PROOF. Let $I_1 \subset \ldots \subset I_\lambda \subset \ldots$ be an ascending chain of ideals of a ring A such that $I_\lambda \in h\mathbf{B}$ for every λ. We claim that $\bigcup_\lambda I_\lambda = B \in h\mathbf{B}$. Consider an arbitrary ideal L of B and take the chain

$$(I_1+L)/L \subset \ldots \subset (I_\lambda+L)/L \subset \ldots.$$

We have $(I_\lambda+L)/L \cong I_\lambda/(I_\lambda \cap L) \in \mathbf{B}$, for $I_\lambda \in h\mathbf{B}$. Since **B** has the inductive property, we have $B/L = \bigcup_\lambda (I_\lambda+L)/L \in \mathbf{B}$ for every ideal L of B. Thus $B \in h\mathbf{B}$. ∎

PROPOSITION 2. *If* **B** *is closed under extensions, then also* $h\mathbf{B}$ *is closed under extensions.*

PROOF. Suppose $B \in h\mathbf{B}$ and $A/B \in h\mathbf{B}$, and consider an arbitrary ideal $K \triangleleft A$. We want to see that $A/K \in \mathbf{B}$. Since $h\mathbf{B}$ is homomorphically closed, $A/B \in h\mathbf{B}$ implies $A/(B+K) \in h\mathbf{B}$. Further,

$$(B+K)/K \cong B/(B \cap K) \in h\mathbf{B}$$

for $B \in h\mathbf{B}$. Hence by

$$A/K/(B+K)/K \cong A/(B+K) \in h\mathbf{B}$$

the extension property of **B** yields $A/K \in \mathbf{B}$. ∎

THEOREM 1. *Let* **C** *be a class of rings having the inductive property and being closed under extensions. Then* $h\mathbf{C}$ *is a radical class, the largest radical class contained in* **C**.

PROOF. It is obvious in view of Propositions 1 and 2.

PROPOSITION 3. *If* **B** *is a hereditary class of rings which is closed under extensions and which contains all nilpotent rings, then $h\mathbf{B}$ is hereditary.*

PROOF. Suppose $B \lhd A \in h\mathbf{B}$. We have to show that for each ideal K of B also $B/K \in \mathbf{B}$ holds. Let $\bar{K}$ denote the ideal of A generated by K. Then $B/\bar{K} \lhd A/\bar{K} \in \mathbf{B}$, for $A \in h\mathbf{B}$. Hence the hereditariness of **B** implies $B/\bar{K} \in \mathbf{B}$. On the other hand, $\bar{K}/K$ is nilpotent by *Andrunakievič' Lemma*, so $\bar{K}/K \in \mathbf{B}$. Since

$$\frac{B/K}{\bar{K}/K} \cong B/\bar{K} \in \mathbf{B}$$

the extension property of **B** implies $B/K \in \mathbf{B}$. ▌

COROLLARY 1. *If* **C** *is a hereditary class having the inductive property and the extension property and if* **C** *contains all nilpotent rings, then $h\mathbf{C}$ is the smallest supernilpotent radical contained in* **C**. ▌

PROPOSITION 4. *If* **B** *is a hereditary class such that* **B** *has the inductive property and the extension property and if* **B** *consists of idempotent rings, then $h\mathbf{B}$ is hereditary.*

PROOF. Using the notations of Proposition 3, we have to see that $B/K \in \mathbf{B}$. Since $\bar{K} \lhd A \in \mathbf{B}$ and **B** is hereditary, so $\bar{K} \in \mathbf{B}$, moreover, $\bar{K}$ is idempotent. Hence $\bar{K} = \bar{K}^3 \subseteq K \subseteq \bar{K}$ yields $K \in \mathbf{B}$. Thus $B/K \lhd A/K \in \mathbf{B}$ and the hereditariness of **B** implies $B/K \in \mathbf{B}$. ▌

COROLLARY 2. *If* **C** *is a hereditary class having the inductive property as well as the extension property and* **C** *consists of idempotent rings, then $h\mathbf{C}$ is the smallest subidempotent radical containing* **C**. ▌

PROPOSITION 5. *If the class* **B** *has the co-inductive property, then also $a\mathbf{B}$ has the co-inductive property.*

PROOF. Let A be a ring having a descending chain $A \supset I_1 \supset \ldots \supset I_\lambda \supset \ldots$ of ideals such that $A/I_\lambda \in a\mathbf{B}$ for every λ. Since $a\mathbf{B} \subseteq \mathbf{B}$, we have $A \in \mathbf{B}$. The assertion will be proved, if we proceed to show that every accessible subring $J/\cap I_\lambda$ of $A/\cap I_\lambda$ is again in **B**. Consider the descending chain

$$J \supset J \cap I_1 \supset \ldots \supset J \cap I_\lambda \supset \ldots .$$

Now we have $J/(J \cap I_\lambda) \cong (J + I_\lambda)/I_\lambda$. The right hand side is an accessible subring of $A/I_\lambda \in a\mathbf{B}$. Hence $J/(J \cap I_\lambda) \in \mathbf{B}$. Moreover, $\cap (J \cap I_\lambda) \subseteq \cap I_\lambda$. Thus $J/\cap I_\lambda \in \mathbf{B}$ since **B** has the co-inductive property. ▌

PROPOSITION 6. *If the class* **B** *is closed under extensions, then $a\mathbf{B}$ is again closed extensions, too.*

PROOF. Assume $B \in a\mathbf{B}$ and $A/B \in a\mathbf{B}$. Then by $a\mathbf{B} \subseteq \mathbf{B}$ and by the hypothesis we have $A \in \mathbf{B}$. Let us consider an arbitrary accessible subring J of A. Since $(J+B)/B$ is an accessible subring of $A/B \in a\mathbf{B}$, we have $J/(J \cap B) \cong$

$\cong (J+B)/B \in \mathbf{B}$. Further $J \cap B$ is an accessible subring of $B \in a\mathbf{B}$, hence $J \cap B \in \mathbf{B}$ holds. Since $\mathbf{B}$ is closed under extensions, we get $J \in \mathbf{B}$. Thus $A \in a\mathbf{B}$. ▌

THEOREM 2. *If* $\mathbf{C}$ *is a class if rings having the co-inductive property and being closed under extensions, then* $a\mathbf{C}$ *is a semisimple class, moreover,* $a\mathbf{C}$ *is the largest semisimple class contained in* $\mathbf{C}$.

PROOF. Apply Propositions 5 and 6. ▌

In the proof of Theorem 2 we used that the considered rings are associative or alternative (cf. [7]).

As usual, let $\mathcal{S}$ denote the semisimple operator

$$\mathcal{S}\mathbf{X} = \{A \mid A \text{ has no nonzero ideal in } \mathbf{X}\}.$$

PROPOSITION 7. *If* $\mathbf{B}$ *is a homomorphically closed class, then* $\mathcal{S}\mathbf{B}$ *is closed under subdirect sums, hence* $\mathcal{S}\mathbf{B}$ *has the co-inductive property.*

PROOF. Let A be a subdirect sum of rings $A_\lambda \in \mathcal{S}\mathbf{B}$, $\lambda \in \Lambda$. Consider an arbitrary $\mathbf{B}$-ideal I of A and the projection π_λ of A onto A_λ. Since $\mathbf{B}$ is homomorphically closed, we have $\pi_\lambda(I) \in \mathbf{B}$ and also $\pi_\lambda(I) \triangleleft A_\lambda \in \mathcal{S}\mathbf{B}$. Hence $\pi_\lambda(I) = 0$ for every $\lambda \in \Lambda$. Thus $I = 0$ $A \in \mathcal{S}\mathbf{B}$ holds. ▌

PROPOSITION 8. *If* $\mathbf{B}$ *is a homomorphically closed class, then* $\mathcal{S}\mathbf{B}$ *is closed under extensions.*

PROOF. Suppose $B \in \mathcal{S}\mathbf{B}$ and $A/B \in \mathcal{S}\mathbf{B}$, and take a $\mathbf{B}$-ideal I of A. We have $I/(I \cap B) \cong (I+B)/B \triangleleft A/B \in \mathcal{S}\mathbf{B}$.

Since $\mathbf{B}$ is homomorphically closed, it follows $I/(I \cap \mathbf{B}) \in \mathbf{B}$. Hence $I/(I \cap B) = 0$, that is $I \subseteq B \in \mathcal{S}\mathbf{B}$. Since $I \in \mathbf{B}$, so necessarily $I = 0$. Thus $A \in \mathcal{S}\mathbf{B}$ holds. ▌

THEOREM 3. *If* $\mathbf{C}$ *is a homomorphically clossed class, then* $a\mathcal{S}\mathbf{C}$ *is a semisimple class, the largest semisimple class contained in* $\mathcal{S}\mathbf{C}$, *a* $\mathcal{S}\mathbf{C}$ *is the semisimple class of the lower radical class determined by* $\mathbf{C}$. ▌

PROOF. Apply Propositions 7 and 8 and Theorem 2. ▌

§ 3. A composition of radical classes

In this section we are going to define a composition of two radical classes. This composition is dual to that investigated by SNIDER [9].*

Let α and β be two radical classes. We define

$$(\alpha \circ \beta) = \{A \mid \alpha(I) \subseteq \beta(I) \text{ for every } I \triangleleft A\}.$$

PROPOSITION 9. *If* $\alpha \subseteq \beta$, *then* $(\alpha \circ \beta)$ *is the class of all rings. If* α *is the class of all rings, and* β *is hereditary, then* $(\alpha \circ \beta) = \beta$. *If* $\beta = 0$, *then* $(\alpha \circ \beta) = \mathcal{S}\alpha$. ▌

* Also N. DIVINSKY and A. SULIŃSKI studied the dual composition in their preprint *Radical pairs,* Oct. 1976.

PROPOSITION 10. *If β is a hereditary radical, then $(\alpha \circ \beta)$ is hereditary and $\beta \subseteq (\alpha \circ \beta)$.*

PROOF. Suppose $I \lhd B \lhd A \in (\alpha \circ \beta)$. Now we have $\alpha(I) \subseteq \alpha(B) \subseteq \beta(B)$, since $A \in (\alpha \circ \beta)$. Further $\alpha(I) \subseteq \beta(B) \cap I = \beta(I)$, for β is hereditary.

Hence $B \in (\alpha \circ \beta)$. The second assertion is trivial. ∎

PROPOSITION 11. *If α is a hereditary radical, then $(\alpha \circ \beta)$ has the inductive property.*

PROOF. First, consider a ring C which is the union $C = \bigcup J_\lambda$ of ideals $J_\lambda \lhd C$. Using the hereditariness of α we have

$$\alpha(\bigcup J_\lambda) = (\bigcup J_\lambda) \cap \alpha(C) = \bigcup (J_\lambda \cap \alpha(C)) = \bigcup \alpha(J_\lambda).$$

Next, let $I_1 \subset \ldots \subset I_\lambda \subset \ldots$ be an ascending chain of ideals of a ring A such that $I_\lambda \in (\alpha \circ \beta)$ for each λ and put $B = \bigcup_\lambda I_\lambda$. We want to prove that $B \in (\alpha \circ \beta)$ that is $\alpha(K) \subseteq \beta(K)$ for every $K \lhd B$. For each λ we have $\alpha(K \cap I_\lambda) \subseteq \beta(K \cap I_\lambda)$ and so $\alpha(K) = \alpha(K \cap (\bigcup_\lambda I_\lambda)) = \alpha(\bigcup_\lambda (K \cap I_\lambda))$.

By the above remark we get

$$\alpha(K) = \alpha(\bigcup_\lambda (K \cap I_\lambda)) = \bigcup_\lambda \alpha(K \cap I_\lambda) \subseteq \bigcup_\lambda \beta(K \cap I_\lambda) \subseteq$$
$$\subseteq \beta(\bigcup_\lambda (K \cap I_\lambda)) = \beta(K \cap (\bigcup_\lambda I_\lambda)) = \beta(K).$$

Thus $B \in (\alpha \circ \beta)$. ∎

PROPOSITION 12. *If α and β are hereditary radicals, then $(\alpha \circ \beta)$ is closed under extensions.*

PROOF. Suppose that $I \in (\alpha \circ \beta)$ and $A/I \in (\alpha \circ \beta)$. We claim that $\alpha(A) \subseteq \beta(A)$. Since $(I + \alpha(A))/I \lhd A/I$, we have

$$\alpha\left((I + \alpha(A))/I\right) \subseteq \beta\left((I + \alpha(A))/I\right).$$

Since

$$(I + \alpha(A))/I \cong \alpha(A)/(I \cap \alpha(A)) \in \alpha$$

we get

$$\alpha\left((I + \alpha(A))/I\right) = (I + \alpha(A))/I$$

and so

$$(I + \alpha(A))/I \subseteq \beta\left((I + \alpha(A))/I\right)$$

which means that

$$\alpha(A)/(I \cap \alpha(A)) \cong (I + \alpha(A))/I \in \beta.$$

Since α is hereditary and $I \in (\alpha \circ \beta)$, we have $I \cap \alpha(A) = \alpha(I) \subseteq \beta(I)$. Using the hereditariness of β we get $I \cap \alpha(A) \in \beta$. The extension property of β yields that $\alpha(A) \in \beta$, hence $\alpha(A) \subseteq \beta(A)$.

Taking an arbitrary ideal $J \lhd A$, the hereditariness of β implies

$$\alpha(J) = \alpha(A) \cap J \subseteq \beta(A) \cap J = \beta(J).$$

Thus $A \in (\alpha \circ \beta)$ holds. ∎

THEOREM 4. *If α and β are hereditary radicals, then the class $(\alpha \circ \beta)$ is a hereditary class having the inductive property as well as the extension property.*

PROOF. Apply Propositions 10, 11 and 12. ∎

For hereditary radicals α and β the obtained class $(\alpha \circ \beta)$ is, so to say, close to radical classes as well as to semisimple classes: $(\alpha \circ \beta)$ possesses namely two out of the three characterizing properties of radical classes and semisimple classes, respectively. But $(\alpha \circ \beta)$ is generally neither a radical class (it need not be homomorphically closed), nor a semisimple class (it need not have the co-inductive property).

PROPOSITION 13. *Let α and γ be hereditary radicals and β any radical. Then $\gamma \cap \alpha \subseteq \beta$ if and only if $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$ holds for every ring A.*

PROOF. $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$ for every ring A implies $\gamma \cap \alpha \subseteq \beta$ without any assumptions. Suppose $\gamma \cap \alpha \subseteq \beta$ and take a ring A. Then by the hereditariness of γ and α we have $\gamma(A) \cap \alpha(A) \in \beta$. Hence $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$. ∎

Following SNIDER [9], let $\gamma = (\beta : \alpha)$ denote the largest hereditary radical class such that $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$, further let $\mathbf{R}$ denote the class

$$\mathbf{R} = \{A \mid \alpha(A/I) \subseteq \beta(A/I) \text{ for every } I \lhd A\}.$$

In view of Proposition 10 clearly $\mathbf{R} \subseteq (\alpha \circ \beta)$ whenever β is hereditary. SNIDER has proved that for any radical class δ the class $a\delta$ is the largest hereditary radical class in δ ([9] Proposition 24).

COROLLARY 3. *If α and β are hereditary radicals, then $\delta = h(\alpha \circ \beta)$ is a radical class, the largest radical such that $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$ for every ring A and also the largest radical class in $(\alpha \circ \beta)$. Moreover, $h(\alpha \circ \beta) = (\beta : \alpha)$ if and only if $h(\alpha \circ \beta)$ is hereditary.*

PROOF. By Theorems 1 and 4 $h(\alpha \circ \beta)$ is the largest radical class contained in $(\alpha \circ \beta)$. For any ring A we have $\gamma(A) \cap \alpha(A) \subseteq \beta(A)$. If δ is a radical class such that $\delta \setminus \gamma \neq \emptyset$, then by Theorem 1 there is a ring $A \in \delta \setminus (\alpha \circ \beta)$. Hence there is an ideal I of A such that $\alpha(I) \subsetneqq \beta(I)$. By the hereditariness of α and β we have $\alpha(A) \cap I \subsetneqq \beta(A) \cap I$ which implies $\alpha(A) \subsetneqq \beta(A)$. Hence by the hereditariness of α and β $\delta(A) \cap \alpha(A) = A \cap \alpha(A) \subsetneqq \beta(A)$. ∎

Let us remark that for hereditary radicals α and β

$$\gamma = h(\alpha \circ \beta) = \{A \mid A/I \in (\alpha \circ \beta) \text{ for every } I \lhd A\}$$

and by Proposition 9 this means

$$\gamma = \{A \mid \alpha(A/I) \subseteq \beta(A/I) \text{ for every } I \lhd A\} = \mathbf{R}.$$

Hence the statement of Corollary 3 is that of SNIDER [9] Theorem 26. Our Theorem 4 can be considered, in fact, as a generalization of ANDRUNAKIEVIČ [3] Theorem 2 and of SNIDER [9] Theorem 26. Hence we have also

COROLLARY 4. *If α and β are hereditary radicals, then $ah(\alpha \circ \beta)$ is the largest hereditary radical calss in $h(\alpha \circ \beta)$ as well as in $(\alpha \circ \beta)$. Moreover, $ah(\alpha \circ \beta) = (\beta : \alpha)$, $\mathbf{R} = h(\alpha \circ \beta)$ and $\mathbf{R} \subseteq (\alpha \circ \beta)$.* ∎

Analogous proof yields the following generalization of Corollary 3:

COROLLARY 5. *If α and β are hereditary radicals, then $\varphi = (\alpha \circ \beta)$ is the largest class such that for any ring A the relation $\varphi(A) \cap \alpha(A) \subseteq \beta(A)$ holds where $\varphi(A)$ denotes any maximal (not necessarily unique) φ-ideal of A.* ▮

In this context we present also

PROPOSITION 14. *Let α and β be arbitrary radicals. A hereditary class ψ is contained in $(\alpha \circ \beta)$ if and only if $\psi \cap \alpha \subseteq \beta$.*

PROOF. Assume that $\psi \subseteq (\alpha \circ \beta)$ and take a ring $A \in \psi \cap \alpha$. Then $A \in \psi$ implies $A \in (\alpha \circ \beta)$, so $\alpha(A) \subseteq \beta(A)$. But $A \in \alpha$, so $A = \alpha(A) \subseteq \beta(A)$ and $A \in \beta$.

Conversely, suppose that $\psi \cap \alpha \subseteq \beta$. Take a ring $B \in \psi$ and an ideal $I \lhd B$. Then the hereditariness of ψ implies $\alpha(I) \in \psi \cap \alpha \subseteq \beta$. Thus $\alpha(I) \subseteq \beta(I)$ and so $B \in (\alpha \circ \beta)$ holds. ▮

PROPOSITION 15. *For any radicals α and β a homomorphically closed hereditary class δ is contained in $(\alpha \circ \beta)$ if and only if $\delta \cap \mathcal{S}\beta \subseteq \mathcal{S}\alpha$.*

PROOF. If $\delta \subseteq (\alpha \circ \beta)$, then $\delta \cap \mathcal{S}\beta \subseteq (\alpha \circ \beta) \cap \mathcal{S}\beta \subseteq \mathcal{S}\alpha$.

If $\delta \not\subseteq (\alpha \circ \beta)$, then there is a ring $A \in \delta \setminus (\alpha \circ \beta)$. Hence there exists an ideal $I \lhd A$ such that $\alpha(I) \not\subseteq \beta(I)$ and by the hereditariness of δ we have $\alpha(I) \in \delta$. Take the ring $\alpha(I)/(\alpha(I) \cap \beta(I)) \neq 0$. Since δ is also homomorphically closed, it follows

$$0 = \alpha(I)/(\alpha(I) \cap \beta(I)) \in \delta \cap \alpha \subseteq \delta \setminus \mathcal{S}\alpha .$$

On the other hand,

$$\alpha(I)/(\alpha(I) \cap \beta(I)) \cong (\alpha(I) + \beta(I))/\beta(I) \lhd I/\beta(I) \in \mathcal{S}\beta .$$

Thus

$$0 \neq \alpha(I)/(\alpha(I) \cap \beta(I)) \in (\delta \cap \mathcal{S}\beta) \setminus \mathcal{S}\alpha . \text{ ▮}$$

Using the operator

$$\mathcal{U}\mathbf{X} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathbf{X}\}$$

we can easily construct a semisimple class containing $(\alpha \circ \beta)$ and excluding $\mathcal{U}(\alpha \circ \beta)$.

PROPOSITION 16. *If α and β are radicals and β is hereditary, then $\sigma = \mathcal{S}\mathcal{U}(\alpha \circ \beta)$ is a semisimple class such that $\sigma \cap \mathcal{U}(\alpha \circ \beta) = 0$. Moreover, σ is the smallest semisimple class containing β and $\mathcal{S}\alpha$.*

PROOF. By Proposition 10 $(\alpha \circ \beta)$ is hereditary, so $\mathcal{U}(\alpha \circ \beta)$ is a radical class, which proves the first statement. As β is hereditary, we get $\beta \subseteq (\alpha \circ \beta) \subseteq \sigma$ (Proposition 10). Obviously $\mathcal{S}\alpha \subseteq (\alpha \circ \beta)$ without any assumptions, so $(\alpha \circ \beta) \subseteq \sigma$ implies $\mathcal{S}\alpha \subseteq \sigma$.

Take a ring $A \in (\alpha \circ \beta)$. Then $\alpha(A) \subseteq \beta(A) \in \beta$ and the hereditariness of β implies $\alpha(A) \in \beta$. Further $A/\alpha(A) \in \mathcal{S}\alpha$. Thus every ring $A \in (\alpha \circ \beta)$ is an extension of a β-ring by an $\mathcal{S}\alpha$-ring. Hence $(\alpha \circ \beta)$ is contained in every semisimple class containing β and $\mathcal{S}\alpha$. Let δ be a semisimple class such that $\beta \subseteq \delta$, $\mathcal{S}\alpha \subseteq \delta$. Then $(\alpha \circ \beta \subseteq \delta$. But $\sigma = \mathcal{S}\mathcal{U}(\alpha \circ \beta)$ is the semisimple closure of $(\alpha \circ \beta)$ (cf. WIEGANDT [12] Theorems 6.2 and 7.4), hence $\sigma \subseteq \delta$. ▮

§ 4. Criterions for $(\alpha \circ \beta)$ to be a radical or a semisimple class

As we have already seen, $(\alpha \circ \beta)$ is generally neither a radical, nor a semisimple class. In this section we impose conditions upon α and β implying that $(\alpha \circ \beta)$ becomes a radical and a semisimple class, respectively. We start with

PROPOSITION 17. *Let α and β be hereditary radicals. The following assertions are equivalent:*

(i) $(\alpha \circ \beta)$ *is radical class*
(ii) $(\alpha \circ \beta) = h(\alpha \circ \beta)$;
(iii) $(\alpha \circ \beta) = (\beta : \alpha)$;
(iv) $(\alpha \circ \beta) = ah(\alpha \circ \beta)$;
(v) $(\alpha \circ \beta) = \mathbf{R}$.

The proof follows straightforward from Corollary 4. ▌

Let us remark that $h(\alpha \circ \beta) = (\beta : \alpha)$ does not imply that $(\alpha \circ \beta)$ is a radical class. If $(\alpha \circ \beta)$ is not hereditary, then $(\beta : \alpha) \subsetneqq h(\alpha \circ \beta) \subsetneqq (\alpha \circ \beta)$.

Omitting the hereditariness of β but imposing the restrictive requirement that $\mathcal{S}\alpha$ is homomorphically closed, we can prove $(\alpha \circ \beta) = h(\alpha \circ \beta)$, that is, $(\alpha \circ \beta)$ is a radical. We shall make use of

PROPOSITION 18. *The semisimple class $\mathcal{S}\alpha$ of a radical α is homomorphically closed if and only if $\alpha(A/I) = (\alpha(A) + I)/I$ for all ideals I in A and for all rings A.*

PROOF. $(\alpha(A) + I)/I \subseteq \alpha(A/I)$ is true without any assumptions. Assume that $\mathcal{S}\alpha$ is homomorphically closed. Then

$$\frac{A}{\alpha(A)+I} \cong \frac{A/\alpha(A)}{(\alpha(A)+I)/\alpha(A)} \in \mathcal{S}\alpha$$

for any ring A and $I \lhd A$. But then

$$\frac{\alpha(A/I)}{(\alpha(A)+I)/I} \lhd \frac{A/I}{(\alpha(A)+I)/I} \cong A/(\alpha(A)+I) \in \mathcal{S}\alpha$$

whereas $\alpha(A/I) \in \alpha$. Thus $\alpha(A/I) \subseteq (\alpha(A) + I)/I$. Hence $\alpha(A/I) = (\alpha(A) + I)/I$.

Conversely, let $\alpha(A/I) = (\alpha(A) + I)/I$ for every ring A and ideal $I \lhd A$. Take a ring $A \in \mathcal{S}\alpha$. Then $\alpha(A/I) = (\alpha(A) + I)/I = 0$. Hence $A/I \in \mathcal{S}\alpha$. ▌

PROPOSITION 19. *If a semisimple class* $\mathbf{S}$ *is homomorphically closed, then its upper radical $\mathcal{U}\mathbf{S}$ is hereditary. In particular, if $\mathcal{S}\alpha$ is homomorphically closed, then the radical α is hereditary.*

PROOF. As is well-known (cf. STEWART [10], GARDNER [5] and GARDNER—STEWART [6]), any $\mathbf{S}$-ring is a subdirect sum of finite fields. Hence $\mathcal{U}\mathbf{S}$ is the upper radical class of finite fields (rings with unity) and so $\mathcal{U}\mathbf{S}$ is hereditary (cf. WIEGANDT [12] Theorem 15.4). ▌

PROPOSITION 20. *If α and β are radicals and $\mathcal{S}\alpha$ is homomorphically closed, then $(\alpha \circ \beta)$ is homomorpphically closed.*

PROOF. Suppose $A\in(\alpha\circ\beta)$ and $I, K\lhd A$ such that $I\subseteq K$. We have to show that $\alpha(K/I)\subseteq\beta(K/I)$. Since $(\beta(K)+I)/I\cong\beta(K)/(\beta(K)\cap I)\in\beta$, Proposition 18 and $A\in(\alpha\circ\beta)$ yield

$$\alpha(K/I)=(\alpha(K)+I)/I\subseteq(\beta(K)+I)/I=\beta(K/I). \blacksquare$$

PROPOSITION 21. *If α and β are radicals and $\mathcal{S}\alpha$ is homomorphically closed, then $(\alpha\circ\beta)$ has the inductive property.*

PROOF. Propositions 19 and 11 prove the assertion. ▮

PROPOSITION 22. *If α and β are radicals and $\mathcal{S}\alpha$ is homomorphically closed, then $(\alpha\circ\beta)$ is closed under extensions.*

PROOF. Assume $I\in(\alpha\circ\beta)$ and $A/I\in(\alpha\circ\beta)$. First we show that $A/\alpha(I)\subseteq$ $\subseteq(\alpha\circ\beta)$. Take an arbitrary ideal $K/\alpha(I)$ of $A/\alpha(I)$. Since $\mathcal{S}\alpha$ is homomorphically closed, so α is hereditary and by $\alpha(I)\subseteq K$ we get $\alpha(I)\subseteq\alpha(K)$ and

$$\alpha(K)\cap I=\alpha(A)\cap K\cap I=\alpha(I)\cap K=\alpha(I)=\alpha(I)\cap\alpha(K).$$

Hence by Proposition 18 we have

$$\alpha(K/\alpha(I))=(\alpha(K)+\alpha(I))/\alpha(I)=\alpha(K)/(\alpha(K)\cap I)\cong(\alpha(K)+I)/I$$

and $(\alpha(K)+I)/I\in\alpha$. Since $\alpha(K)\lhd A$ and $A/I\in(\alpha\circ\beta)$, so

$$(\alpha(K)+I)/I=\alpha\left((\alpha(K)+I)/I\right)\subseteq\beta\left((\alpha(K)+I)/I\right)\subseteq(\alpha(K)+I)/I.$$

Thus $\alpha(K/\alpha(I))\cong(\alpha(K)+I)/I\in\beta$ holds which means $\alpha(K/\alpha(I))\subseteq\beta(K/\alpha(I))$ is valid for every ideal $K/\alpha(I)$ of $A/\alpha(I)$. Hence $A/\alpha(I)\in(\alpha\circ\beta)$.

Let J be an arbitrary ideal of A. Since $A/\alpha(I)\in(\alpha\circ\beta)$ and

$$(J+\alpha(I))/\alpha(I)\lhd A/\alpha(I),$$

we have

$$\alpha\left((J+\alpha(I))/\alpha(I)\right)\subseteq\beta\left((J+\alpha(I))/\alpha(I)\right) \tag{*}$$

and also

$$\alpha\left(J/(J\cap\alpha(I))\right)\subseteq\beta\left(J/(J\cap\alpha(I))\right).$$

By the hereditariness of α we have

$$\alpha(J)=J\cap\alpha(A)\supseteq J\cap\alpha(I),$$

so Proposition 17 gives

$$\alpha(J)/(\alpha(I)\cap J)\cong\left(\alpha(J)+(\alpha(I)\cap J)\right)/(\alpha(I)\cap J)=\alpha\left(J/(\alpha(I)\cap J)\right).$$

Thus by (*) we obtain

$$\alpha(J)/(\alpha(I)\cap J)\subseteq\beta\left(J/(\alpha(I)\cap J)\right). \tag{**}$$

Since $\alpha(I)\cap J\lhd I\in(\alpha\circ\beta)$, the hereditariness of α and $(\alpha\circ\beta)$ yields

$$\alpha(I)\cap J=\alpha(\alpha(I)\cap J)\subseteq\beta(\alpha(I)\cap J).$$

Hence $\alpha(I)\cap J=\beta(\alpha(I)\cap J)\in\beta$. Further, there exists an ideal $T\lhd J$ such that $T/(\alpha(I)\cap J)=\beta(J/(\alpha(I)\cap J))$. Now by $\alpha(I)\cap J\in\beta$ it follows $T\in\beta$. Hence $T\lhd J$ gives $T\subseteq\beta(J)$. Then by (**)

$$\alpha(J)/(\alpha(I)\cap J)\subseteq T/((\alpha(I)\cap J)\subseteq\beta(J)/(\alpha(I)\cap J))$$

holds whence $\alpha(J)\subseteq\beta(J)$. Since J was arbitrarily chosen, we have established $A\in(\alpha\circ\beta)$. ▮

THEOREM 5. *Let α and β be radicals. If $\mathcal{S}\alpha$ is homomorphically closed, then $\gamma=(\alpha\circ\beta)$ is a radical class staisfying $\gamma(A)\cap\alpha(A)\subseteq\beta(A)$ for every ring A.*

PROOF. Propositions 20, 21 and 22 imply that $(\alpha\circ\beta)$ is a radical class. By the hereditariness of α we have $\alpha(\gamma(A))=\alpha(A)\cap\gamma(A)$ and $\gamma(A)\in(\alpha\circ\beta)$ for every ring A. Hence $\gamma(A)\cap\alpha(A)=\alpha(\gamma(A))\subseteq\beta(\gamma(A))\subseteq\beta(A)$. ▮

COROLLARY 6: *Let α and β be radicals. If $\mathcal{S}\alpha$ is homomorphically closed, then $(\alpha\circ\beta)\subseteq\mathbf{R}$.*

PROOF. Assume $A\in(\alpha\circ\beta)$. Since $(\alpha\circ\beta)$ is homomorphically closed, we have $\alpha(A/I)\subseteq\beta(A/I)$ for every $I\lhd A$. ▮

Generally $(\alpha\circ\beta)\neq\mathbf{R}$ even if $\mathcal{S}\alpha$ is homomorphically closed. Consider, for instace, the lower radical $\beta=\mathcal{L}(Z(p^\infty))$, where $Z(p^\infty)$ denotes the zero--ring over the quasicyclic group $C(p^\infty)$, and let $\mathcal{S}\alpha$ be the homomorphically closed semisimple class of all boolean rings ($a^2=a$ for all $a\in A$). β is not hereditary, for $Z(p)\lhd Z(p^\infty)$ but $Z(p)\notin\mathcal{L}(Z(p^\infty))$. Furthermore, $Z(p)\notin\mathcal{S}\alpha$ implies $Z(p)\in\alpha$, for $Z(p)$ is simple. Hence $Z(p)=\alpha(Z(p))\nsubseteq\beta(Z(p))=0$ implies $Z(p^\infty)\notin(\alpha\circ\beta)$. On the other hand, obviously $Z(p^\infty)\in\mathbf{R}$.

COROLLARY 7. *If $\mathcal{S}\alpha$ is homomorphically closed and β is hereditary, then $(\alpha\circ\beta)$ is a hereditary radical and $(\alpha\circ\beta)=h(\alpha\circ\beta)=(\beta:\alpha)=\mathbf{R}$. $\gamma=(\alpha\circ\beta)$ is the largest hereditary radical such that $\gamma\cap\alpha\subseteq\beta$ as well as $\gamma\cap\mathcal{S}\beta\subseteq\mathcal{S}\alpha$.*

PROOF. By Theorem 5 and Proposition 10 $(\alpha\circ\beta)$ is a hereditary radical. Corollaries 4 and 6 and Proposition 15 imply the remainders. ▮

Next, let us suppose that β is a homomorphically closed semisimple class. Since every homomorphically closed semisimple class is a hereditary radical (cf. WIEGANDT [12] Corollary 32.2), $(\alpha\circ\beta)$ makes sense. It may be remarked that, if we take $\beta=0$ in Theorem 5, $(\alpha\circ\beta)=\mathcal{S}\alpha$ is a radical class (Proposition 9). So, by Theorem 5, a homomorphically closed semisimple class is a radical class.

PROPOSITION 23. *If β is a hereditary class having the co-inductive property, then $(\alpha\circ\beta)$ has the co-inductive property.*

PROOF. Assume that a ring A has a descending chain $A\supset I_1\supset\ldots\supset\supset I_\lambda\supset\ldots$ of ideals such that $A/I_\lambda\in(\alpha\circ\beta)$ for every λ. Consider $\alpha(A/\cap I_\lambda)=$ $=L/\cap I_\lambda$. Then $(L+I_\lambda)/I_\lambda\lhd A/I_\lambda\in(\alpha\circ\beta)$. Since by Proposition 10 $(\alpha\circ\beta)$ is hereditary, we get $L/I_\lambda\cap L\cong(L+I_\lambda)/I_\lambda\in(\alpha\circ\beta)$ and also $\alpha(L/(I_\lambda\cap L))\subseteq$ $\subseteq\beta(L/(I_\lambda\cap L))$. Since

$$\frac{L}{I_\lambda \cap L} \cong \frac{L/\cap I_\lambda}{(I_\lambda \cap L)/\cap I_\lambda} \in \alpha$$

so

$$L/(I_\lambda \cap L) = \alpha(L/(I_\lambda \cap L)) \subseteq \beta(L/(I_\lambda \cap L))$$

implies $L/(I_\lambda \cap L) \in \beta$ for every λ. Now

$$L \supset (L \cap I_1) \supset \ldots \supset (L \cap I_\lambda) \supset \ldots$$

is a descending chain of ideals of L with $L/(I_\lambda \cap L) \in \beta$ for every λ. Since β has the co-inductive property, it follows $L/\cap I_\lambda = L/\cap(I_\lambda \cap L) \in \beta$. Thus $\alpha(A/\cap I_\lambda) = L/\cap I_\lambda \subseteq \beta(A/\cap I_\lambda)$. Further, by the hereditariness of β for any ideal $K/\cap I_\lambda$ of $A/\cap I_\lambda$ we have

$$\alpha(K/\cap I_\lambda) \subseteq \alpha(A/\cap I_\lambda) \cap (K/\cap I_\lambda) \subseteq \beta(A/\cap I_\lambda) \cap (K/\cap I_\lambda) = \beta(K/\cap I_\lambda).$$

Hence $A/\cap I_\lambda \in (\alpha \circ \beta)$. ▌

THEOREM 6. *Let α be a hereditary radical and β a homomorphically closed semisimple class. Then $(\alpha \circ \beta)$ is a semisimple class, moreover $(\alpha \circ \beta)$ is the smallest semisimple class containing $\mathcal{S}\alpha$ and β.*

PROOF. Since β is also a hereditary radical, the first statement is a consequence of Propositions 10, 23 and 12. Now $(\alpha \circ \beta) = \mathcal{S}\mathcal{U}\ (\alpha \circ \beta)$ and Proposition 16 implies the second statement. ▌

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REMARK ON A THEOREM OF J. NEVEU

By

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1. J. NEVEU in [1] (Appendix, Proposition A–3–4) formulates the following theorem:

Let $\varphi(t)$ be a Young function with $\varphi(0)=0$ and let us consider the corresponding Orlicz space $L^\varphi(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A}, P)$ is a probability space. If $(X_n, \mathcal{F}_n)$ is a martingale, defined on this space, such that $\sup_n \|X_n\|_\varphi < +\infty$ then $(X_n, \mathcal{F}_n)$ is regular and converges in L^φ-norm to its almost sure limit

$$\lim_{n\to+\infty} X_n = X_\infty .$$

The purpose of this note is to give a counterexample which shows that the above statement is false for general Young functions. Namely, the convergence in L^φ-norm of X_n to X_∞ fails to be true.

Conversely, if we impose on φ the additional growth condition

$$\varphi(2x) \leqq d\,\varphi(x), \quad x \geqq 0,$$

where $d>0$ is a constant, then the assertion remains valid.

Throughout the present note we use the notions and results of [1].

2. Let $\varphi(t)=e^t-1-t$, $t\geqq 0$. Then $\varphi(t)$ is a Young function. Let $\Omega = [0, 1)$, $\mathcal{A}$ the σ-field of Borel-measurable sets of $[0, 1)$ and let P be the corresponding Lebesgue measure. The random variable

$$X(\omega) = \log \frac{1}{1-\omega}, \quad \omega \in [0, 1)$$

is exponentially distributed with parameter 1 and its L^φ-norm is $\frac{1+\sqrt{5}}{2}$. In fact, for $C>1$ we have

$$E(e^{X/C}) = E\left(\frac{1}{(1-\omega)^{1/C}}\right) = \int_0^1 \frac{1}{(1-\omega)^{1/C}}\, d\omega = \frac{C}{C-1}.$$

From this

$$E\left(\varphi\left(\frac{X}{C}\right)\right) \leqq 1,$$

if $C \geqq \frac{1+\sqrt{5}}{2}$. Thus $\|X\|_\varphi = \frac{1+\sqrt{5}}{2}$. We remark that for $0 < C \leqq 1$ trivially we have

$$E\left(\varphi\left(\frac{X}{C}\right)\right) = +\infty.$$

Let $\mathcal{F}_n$ be the σ-field generated by the n-th dyadic partition of $[0, 1)$, $n = 0, 1, 2, \ldots$. The atoms of $\mathcal{F}_n$ are

$$\left[\frac{k-1}{2n}, \frac{k}{2n}\right), \quad k = 1, 2, \ldots, 2^n.$$

These σ-fields increase with n and the σ-field $\mathcal{F}_\infty$ generated by $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ coincides with $\mathcal{A}$. Consequently, the regular martingale

$$X_n = E(X|\mathcal{F}_n)$$

converges almost surely to X. Since $\varphi(t) = e^t - 1 - t$ is convex, by Jensen's inequality for $C \geqq \frac{1+\sqrt{5}}{2}$ we have

$$E\left(\varphi\left(\frac{X_n}{C}\right)\right) = E\left(\varphi\left(E\left(\frac{X}{C}\middle|\mathcal{F}_n\right)\right)\right) \leqq E\left(E\left(\varphi\left(\frac{X}{C}\right)\middle|\mathcal{F}_n\right)\right) =$$

$$= E\left(\varphi\left(\frac{X}{C}\right)\right) \leqq 1.$$

Thus $\|X_n\|_\varphi \leqq \frac{1+\sqrt{5}}{2}$ and the conditions of Neveu's theorem are satisfied.

We remark that on the atom

$$\left[\frac{2^n-1}{2^n}, 1\right)$$

the value of X_n is

$$2^n \int_{1-1/2^n}^{1} \log \frac{1}{1-y}\, dy = 1 + n \log 2.$$

Put $C = \frac{1}{2}$ and remark that

$$E\left(\varphi\left(\frac{|X-X_n|}{C}\right)\right) \geqq E\left(e^{2(X-X_n)}\right)-1 .$$

We show that

$$E\left(\exp\left(2\left(X-X_n\right)\right)\right)=+\infty$$

for every n. In fact,

$$E\left(\exp\left(2\left(X-X_n\right)\right)\right)=\int_0^1 e^{2\left(\log \frac{1}{1-y}-X_n\right)} d y \geqq$$

$$\geqq \int_{1-1/2^n}^1 e^{2\left(\log \frac{1}{1-y}-1-n \log 2\right)} d y=e^{-2(1+n \log 2)} \int_{1-1/2^n}^1 \frac{1}{(1-y)^2} d y=+\infty .$$

Thus X_n does not converge in L^φ-norm to $X=\lim\limits_{n \to +\infty} X_n$.

3. We prove the following assertion.

THEOREM. *Let $\varphi(t)$ be a Young function with $\varphi(0)=0$ and suppose that for every $t \geqq 0$ we have*

$$\varphi(2t) \leqq d\,\varphi(t),$$

where $d>0$ is a constant. Consider the corresponding Orlicz space $L^\varphi(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A}, P)$ is a probability space. If $(X_n, \mathcal{F}_n)$ is a martingale defined on $(\Omega, \mathcal{A}, P)$ such that $\sup\limits_n \|X_n\|_\varphi<+\infty$ then $\{X_n\}$ is regular and X_n converges in L^φ to its almost sure limit $\lim\limits_{n \to +\infty} X_n=X_\infty$.

PROOF. The proof of the regularity of $\{X_n\}$ is the same as that of Neveu. Regularity implies that the almost sure limit X_∞ of the sequence $\{X_n\}$ exists and

$$X_n=E(X_\infty \mid \mathcal{F}_n) .$$

If X_∞ is bounded by a constant, i.e. if $|X_\infty| \leqq a$ almost surely with $a>0$ then $|X_n| \leqq a$ almost surely and for arbitrary $c>0$ we have

$$\varphi\left(\frac{|X_n-X_\infty|}{c}\right) \leqq \varphi\left(\frac{2a}{c}\right),$$

further

$$\varphi\left(\frac{|X_n-X_\infty|}{c}\right)$$

converges to $\varphi(0)=0$ almost surely. Thus the Lebesgue dominated convergence theorem implies that $\|X_n-X_\infty\|_\varphi \to 0$ as $n \to +\infty$.

In the general case let $\sigma=\sup\limits_n \|X_n\|_\varphi$. Then by our condition $\sigma<+\infty$ and without loss of the generality we can suppose that $\sigma>0$. Let further $a>0$ be arbitrary and take

$$X_\infty^* = \begin{cases} X_\infty, & \text{if} \quad |X_\infty| \leqq a \\ 0, & \text{if} \quad |X_\infty| > a \end{cases}.$$

Put $X_\infty^{**} = X_\infty - X_\infty^*$. Then we have

$$X_n - X_\infty = E(X_\infty^* | \mathcal{F}_n) - X_\infty^* + E(X_\infty^{**} | \mathcal{F}_n) - X_\infty^{**}.$$

By the triangle inequality for the norms

$$\|X_n - X_\infty\|_\varphi \leqq \|E(X_\infty^* | \mathcal{F}_n) - X_\infty^*\|_\varphi + \|E(X_\infty^{**} | \mathcal{F}_n) - X_\infty^{**}\|_\varphi .$$

We have proved that the first term on the right hand side tends to 0 as $n \to +\infty$, since $|X_\infty^*|$ is bounded by a. By the Jensen inequality the conditional expectation contracts the norm. Thus we have the following estimate for the second term

$$\|E(X_\infty^{**} | \mathcal{F}_n) - X_\infty^{**}\|_\varphi \leqq 2 \|X_\infty^{**}\|_\varphi .$$

We prove that the growth condition

$$\varphi(2t) \leqq d\, \varphi(t), \quad t \geqq 0 ,$$

implies that $2\|X_\infty^{**}\|_\varphi$ becomes as small as we please if $a > 0$ is chosen suitably. In fact, for arbitrary $C > 0$ we have

$$E\left(\varphi\left(\frac{|X_\infty^{**}|}{C}\right)\right) = \int\limits_{\{|X_\infty| > a\}} \varphi\left(\frac{|X_\infty|}{C}\right) dP = \int\limits_{\{|X_\infty| > a\}} \varphi\left(\frac{\sigma}{C} \frac{|X_\infty|}{\sigma}\right) dP .$$

If $\dfrac{\sigma}{C} \leqq 1$ then by the monotonicity of φ we have

$$E\left(\varphi\left(\frac{|X_\infty^{**}|}{C}\right)\right) \leqq \int\limits_{\{|X_\infty| > a\}} \varphi\left(\frac{|X_\infty|}{\sigma}\right) dP \leqq 1 ,$$

since $\|X_\infty\|_\varphi \leqq \sigma$. If $\dfrac{\sigma}{C} \geqq 1$ then there exists a positive integer k such that $2^{k-1} \leqq \dfrac{\sigma}{C} < 2^k$. From this by the growth condition imposed on φ

$$E\left(\varphi\left(\frac{|X_\infty^{**}|}{C}\right)\right) \leqq d^k \int\limits_{\{|X_\infty| > a\}} \varphi\left(\frac{|X_\infty|}{\sigma}\right) dP .$$

Finally, since

$$\int\limits_{\Omega} \varphi\left(\frac{|X_\infty|}{\sigma}\right) dP \leqq 1 ,$$

it follows by the Beppo Levi theorem that

$$\lim_{a\uparrow+\infty} \int\limits_{\{|X_\infty|>a\}} \varphi\left(\frac{|X_\infty|}{\sigma}\right) dP = 0 .$$

This means that for arbitrary $C>0$ we have

$$E\left(\varphi\left(\frac{|X_\infty^{**}|}{C}\right)\right) \leqq 1 ,$$

if $a>0$ is chosen suitably. Consequently, $\|X_\infty^{**}\|_\varphi \leqq C$ for suitable $a>0$. This proves the theorem.

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ON THE ABSOLUTE PEANO DERIVATIVES

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1. Introduction. Let $f(x)$ be a function defined in a neighbourhood of x_0. The numbers a_i $(0 \leqq i \leqq n)$ are said to be the Peano derivatives of $f(x)$ at x_0 (in symbols $f_{(i)}(x_0) = a_i$) if $a_0 = f(x_0)$ and, in the case $n > 0$,

$$f(x) - \sum_{i=0}^{n} a_i \frac{(x-x_0)^i}{i!} = o((x-x_0)^n)$$

holds as $x \to x_0$ (see [3]). Suppose that $f_{(n-1)}(x)$ exists in a neighbourhood of x_0. We shall prove in Theorem 1 that the function $f_{(n-1)}(x)$ determines the number $f_{(n)}(x_0)$ i.e. whenever $g_{(k-1)}(x) = f_{(n-1)}(x)$ holds for a function $g(x)$ and a natural number k in a neighbourhood of x_0 and $g_{(k)}(x_0)$ exists then $g_{(k)}(x_0) = f_{(n)}(x_0)$. We interpret this result as follows: $f_{(n)}(x_0)$ can be regarded as a derivative of the function $f_{(n-1)}(x)$ in a generalized sense. This generalized derivative will be called absolute Peano derivative.

DEFINITION. Let $f(x)$ be defined in a neighbourhood of x_0. We say that the absolute Peano derivative (a.P.d.) of $f(x)$ at x_0 exists and equals to A (in symbols $f^*(x_0) = A$) if there are a function $g(x)$, a non-negative integer n and $\delta > 0$ such that

(i) $g_{(n)}(x) = f(x)$ holds in $(x_0 - \delta, x_0 + \delta)$ and
(ii) $g_{(n+1)}(x_0) = A$.

In this paper we are going to study the properties of the a.P.d. $f^*(x)$. Making use of some theorems of M. E. COROMINAS, H. W. OLIVER, C. E. WEIL and S. VERBLUNSKY we shall prove that $f^*(x)$ is a Darboux Baire 1 function (Theorems 5 and 7) and possesses the following property well-known to hold for the Peano derivatives: if $f^*(x)$ is bounded from above or below in $[a, b]$, then $f'(x)$ exists in $[a, b]$ and $f^*(x) = f'(x)$ (Theorem 6).

The existence of $f'(x_0)$ obviously implies the existence of $f^*(x_0)$ (we can choose $g(x) = f(x)$ and $n = 0$ in the Definition). More generally, if $f_{(n)}(x)$ exists everywhere then we have $(f_{(n-1)}(x))^* = f_{(n)}(x)$. On the other hand, we shall prove that there exists a continuous function $f(x)$ having a.P.d. $f^*(x)$ everywhere in $[0, 1]$ such that $f^*(x)$ is not of the form $g_{(n)}(x)$ $(n \geqq 1)$ (Theorem 9).

The following example shows that $f^*(x)$ may exist everywhere even if $f(x)$ is discontinuous.

EXAMPLE 1. Let $f(x) = \sin \frac{1}{x^2}$ for $x \neq 0$, $f(0) = 0$. Then $f^*(x)$ is defined everywhere and $f^*(0) = 0$.

In fact, let

$$g(x) = \int_0^x f(t)\, dt,$$

then we have

$$g(x) = \int_0^x \sin \frac{1}{t^2}\, dt = \frac{1}{2} \int_{x^{-2}}^{\infty} \frac{\sin u}{u^{3/2}}\, du = \frac{1}{2} \cdot x^3 \cdot \int_{x^{-2}}^{\xi} \sin u\, du$$

by the second mean-value theorem of the calculus. Hence $|g(x)| \leqq |x^3|$ everywhere from which $g_{(1)}(0) = g_{(2)}(0) = 0$. Consequently $g(x)$ satisfies (i) and (ii) in the Definition with $n = 1$ and we have

$$f^*(0) = g_{(2)}(0) = 0.$$

2. We introduce the following notation.

$$\psi_{k,n}(f, x_0, h) = \frac{(n-k+1)!}{h^{n-k+1}} \left(f_{(k)}(x_0+h) - \sum_{i=k}^{n} f_{(i)}(x_0) \frac{h^{i-k}}{(i-k)!} \right).$$

LEMMA. Let $0 \leqq k < l \leqq n$ and suppose that $f_{(l)}(x)$ is defined in $[a, b]$ and $f_{(n)}(x)$ is defined at $x_0 \in [a, b]$. Then for every $h \neq 0$, $x_0 + h \in [a, b]$ there exists $0 < \alpha < 1$ such that

$$\psi_{k,n}(f, x_0, h) = \psi_{l,n}(f, x_0, \alpha h).$$

PROOF. Let $h > 0$ (the case $h < 0$ is similar). Suppose first $l = k+1$ and let

$$g(x) = f_{(k+1)}(x) - \sum_{i=k+1}^{n} f_{(i)}(x_0) \frac{(x-x_0)^{i-k-1}}{(i-k-1)!} - \psi_{k,n}(f, x_0, h) \cdot \frac{(x-x_0)^{n-k}}{(n-k)!}.$$

We have to show that $g(x)$ has a zero in (x_0, x_0+h). It is easy to see that $g(x) = s_{(k+1)}(x)$ where

$$s(x) = f(x) - \sum_{i=0}^{n} f_{(i)}(x_0) \frac{(x-x_0)^i}{i!} - \psi_{k,n}(f, x_0, h) \cdot \frac{(x-x_0)^{n+1}}{(n+1)!}.$$

It follows that $g(x)$ has the property of Darboux (see [3], Theorem 2, p. 450). If $g(x) \neq 0$ in (x_0, x_0+h), then $g(x) > 0$ or $g(x) < 0$ in (x_0, x_0+h). Hence, by a theorem of M. E. COROMINAS and H. W. OLIVER (see [1], p. 221 or [3], Theorem 3, p. 452), $s^{(k+1)}(x)$ exists in (x_0, x_0+h). Thus $s_{(k)}(x) = s^{(k)}(x)$ is strictly monotone in (x_0, x_0+h) moreover, by the Darboux property of $s_{(k)}(x)$, in $[x_0, x_0+h]$ (if $k=0$, then $s_{(k)}(x)=s(x)$ is continuous since $f_{(1)}(x)=f'(x)$ exists in $[a, b]$). However, we have $s_{(k)}(x_0)=0$ and

$$s_{(k)}(x_0+h)=f_{(k)}(x_0+h)-\sum_{i=k}^{n} f_{(i)}(x_0)\frac{h^{i-k}}{(i-k)!}-$$
$$-\psi_{k,n}(f, x_0, h)\cdot\frac{h^{n-k+1}}{(n-k+1)!}=0,$$

a contradiction.

Finally, if $0\leqq k<l\leqq n$ are arbitrary, then we have

$$\psi_{k,n}(f, x_0, h)=\psi_{k+1,n}(f, x_0, \alpha_1 h)=\ldots=\psi_{l,n}(f, x_0, \alpha_1\alpha_2\cdot\ldots\cdot\alpha_{l-k}\cdot h)$$

with suitable $0<\alpha_1, \alpha_2, \ldots, \alpha_{l-k}<1$ and the proof is complete.

We recall the definition of the upper and lower Peano derivatives. Let $f(x)$ be defined in a neighbourhood of x_0 and suppose that $f_{(i)}(x_0)$ exists for $0\leqq i\leqq n-1$. Then $\bar{f}_{(n)}(x_0)$ and $\underline{f}_{(n)}(x_0)$ are defined by

$$\bar{f}_{(n)}(x_0)=\overline{\lim_{h\to 0}}\,\psi_{0,n-1}(f, x_0, h) \quad\text{and}\quad \underline{f}_{(n)}(x_0)=\underline{\lim_{h\to 0}}\,\psi_{0,n-1}(f, x_0, h)$$

(see [4]).

THEOREM 1. *Let $0\leqq m\leqq n$ and suppose that $f_{(n)}(x)=g_{(m)}(x)$ holds in $[a, b]$ for the functions $f(x)$, $g(x)$. Then*

(i) *There exists a polynomial $p(x)$ of degree $<m$ such that $g(x)=f_{(n-m)}(x)+p(x)$ in $[a, b]$.*

(ii) *If $k\geqq 0$ and $g_{(m+k)}$ exists at $x_0\in[a, b]$, then so does $f_{(n+k)}(x_0)$ and $f_{(n+k)}(x_0)=g_{(m+k)}(x_0)$.*

(iii) *We have*

$$\underline{g}_{(m+1)}(x)\leqq\underline{f}_{(n+1)}(x)\leqq\bar{f}_{(n+1)}(x)\leqq\bar{g}_{(m+1)}(x)$$

for every $x\in[a, b]$.

PROOF. Let $n>0$ be fixed and let $m=0$. First we prove (ii) by induction. The case $k=0$ is trivial. Let $k>0$ and suppose that (ii) is true for $0, 1, \ldots, k-1$ and that

$$g_{(k)}(x_0)=\lim_{h\to 0}\psi_{0,k-1}(g, x_0, h)$$

exists. Then, by the inductive hypothesis, $f_{(n+j)}(x_0)=g_{(j)}(x_0)$ holds for $j=0, 1, \ldots, k-1$ and hence

$$\psi_{0,k-1}(g, x_0, h)=\psi_{n,n+k-1}(f, x_0, h).$$

Let (h_i) be an arbitrary sequence tending to zero, then by the Lemma there are $\alpha_i\in(0, 1)$ such that

$$\psi_{0,n+k-1}(f, x_0, h_i)=\psi_{n,n+k-1}(f, x_0, \alpha_i h_i) \quad (i=1, 2, \ldots).$$

This implies

$$\lim_{i\to\infty} \psi_{0,n+k-1}(f, x_0, h_i) = \lim_{i\to\infty} \psi_{n,n+k-1}(f, x_0, \alpha_i h_i) =$$

$$= \lim_{i\to\infty} \psi_{0,k-1}(g, x_0, \alpha_i h_i) = g_{(k)}(x_0).$$

Hence we have $f_{(n+k)}(x_0) = g_{(k)}(x_0)$ and (ii) is proved for every k.

Next, let $0 < m \leqq n$ be arbitrary. Since $g_{(1)}(x) = g'(x)$ exists in $[a, b]$, $g(x)$ is continuous in $[a, b]$. Hence there exists a function $h(x)$ defined in $[a, b]$ with $h^{(n-m)}(x) = g(x)$. Then $h_{(n-m)}(x) = h^{(n-m)}(x) = g(x)$ and we can apply (ii) replacing f, n, m, k by h, $n-m$, 0 and m, respectively. We get $h_{(n)}(x) = f_{(n)}(x)$, that is $(h(x)-f(x))_{(n)} \equiv 0$ in $[a, b]$. It follows (see [3], Corollary of Theorem 3, p. 454) that $h(x)-f(x)$ is a polynomial of degree $< n$. Let $p(x) = (h(x)-f(x))^{(n-m)}$, then we have

$$g(x) = h_{(n-m)}(x) = p(x) + f_{(n-m)}(x)$$

which proves (i).

Let $k \geqq 0$ be arbitrary and suppose that $g_{(m+k)}(x_0)$ exists. Since $p_{(m+k)}(x) \equiv 0$, it follows from (i) that the $(m+k)$th Peano derivative of $f_{(n-m)}$ exists at x_0 and equals to $g_{(m+k)}(x_0)$. Thus we can apply (ii) replacing g, n, m, k by $f_{(n-m)}$, $n-m$, 0 and $m+k$, respectively, and we get $f_{(n+k)}(x_0) = g_{(m+k)}(x_0)$, which is the assertion of (ii).

Finally we prove (iii). It easily follows from (i) and (ii) that

$$g_{(i)}(x) = f_{(n-m+i)}(x) + p_{(i)}(x)$$

holds for every $x \in [a, b]$ and $i = 0, 1, \ldots, m$. Thus we have

$$\psi_{0,m}(g, x, h) = \psi_{n-m,n}(f, x, h) + \psi_{0,m}(p, x, h).$$

Since $p(x)$ is of degree $< m$, hence

$$\lim_{h\to 0} \psi_{0,m}(p, x, h) = p_{(m+1)}(x) = 0$$

and

$$\bar{g}_{(m+1)}(x) = \overline{\lim_{h\to 0}}\, \psi_{0,m}(g, x, h) = \overline{\lim_{h\to 0}}\, \psi_{n-m,n}(f, x, h).$$

Now, by the Lemma, for every $h \neq 0$ there is an $\alpha \in (0, 1)$ such that

$$\psi_{0,n}(f, x, h) = \psi_{n-m,n}(f, x, \alpha h).$$

This easily implies

$$\bar{g}_{(m+1)}(x) = \overline{\lim_{h\to 0}}\, \psi_{n-m,n}(f, x, h) \geqq \overline{\lim_{h\to 0}}\, \psi_{0,n}(f, x, h) = \bar{f}_{(n+1)}(x).$$

Applying this result to the functions $-f(x)$ and $-g(x)$ we get

$$\underline{g}_{(m+1)}(x) \leqq \underline{f}_{(n+1)}(x)$$

which completes the proof of Theorem 1.

We mention that the assertion (ii) is known in the special case $m=0$ ([1], Théorème XIV).

The definition of a.P.d. and Theorem 1 (ii) obviously imply

THEOREM 2. *If $f^*(x_0)$ exists then it is unambiguously defined.*

THEOREM 3. *If $f^*(x)$ exists in $[a, b]$ then there exist a function $g(x)$ and a natural number k such that $g_{(k)}(x)=f(x)$ holds for every $x\in[a, b]$.*
If $f(x)$ is bounded in $[a, b]$ then $f(x)$ is an ordinary derivative.

PROOF. Suppose first that $f(x)$ is bounded and let

$G=\{x\in[a, b]: f(x)$ has a primitive in a neighbourhood of $x\}$.

Then $G\cap(a, b)$ is open and $[a, b]\setminus G=F$ is closed. Let $x_0\in F$, then there exist $n\geqq 0$ and $h(x)$ such that $h_{(n)}(x)=f(x)$ in a neighbourhood of x_0 and $h_{(n+1)}(x_0)=f^*(x_0)$. Since $f(x)$ is bounded, we have $h^{(n)}(x)=h_{(n)}(x)=f(x)$ in a neighbourhood of x_0. (See [1], Théorème XV, p. 216 or [3], Theorem 3, p. 452.) Now $x_0\notin G$ implies $n=0$ from which $f^*(x_0)=f_{(1)}(x_0)=f'(x_0)$. That is, $f(x)$ is differentiable at every point of F.

This implies that $f(x)$ is a Baire 1 function and

$$P(x)=\int_a^x f(t)\,dt$$

is a primitive of $f(x)$ in $[a, b]$. In fact, for $x_0\in F$, $P'(x_0)=f(x_0)$ follows from the continuity of $f(x)$ at x_0. If $x_0\in G$ and $H(x)$ is a primitive of $f(x)$ in $(x_0-\delta, x_0+\delta)$, then

$$P(x)=c+\int_{x_0}^x f(t)\,dt=c+H(x)-H(x_0)$$

for every $x\in(x_0-\delta, x_0+\delta)$ and thus $P'(x_0)=H'(x_0)=f(x_0)$.

Now let $f(x)$ be arbitrary. We put $H=\{a\}\cup\{x_0$: there exist $g(x)$ and $k\geqq 1$ such that $g_{(k)}(x)=f(x)$ for every $x\in[a, x_0]\}$ and $c=\sup H$.

By the definition of $f^*(c)$, there are $h(x)$, $n\geqq 0$ and $\delta>0$ such that $h_{(n)}(x)=f(x)$ whenever $x\in[c-\delta, c+\delta]\cap[a, b]$ and $h_{(n+1)}(c)=f^*(c)$.

We can suppose $n\geqq 1$. Indeed, if $n=0$ then $f(x)$ is differentiable at c and hence, $f(x)$ is bounded in a neighbourhood of c. By the preceding argument, $f(x)$ has a primitive in this neighbourhood and we can choose $n=1$, $h(x)$ and $\delta>0$ such that $h_{(n)}(x)=h'(x)=f(x)$ holds in $[c-\delta, c+\delta]\cap[a, b]$. This implies $c>a$.

Let $x_0\in H$, $x_0>c-\delta$, then there are $g(x)$ and $k\geqq 1$ with $g_{(k)}(x)=f(x)$ in $[a, x_0]$. Let $m=\max(k, n)$. By the continuity of $h(x)$ and $g(x)$ there are two functions $r(x)$ and $s(x)$ such that

$$r^{(m-k)}(x)=g(x) \quad \text{if} \quad x\in[a, x_0]$$

and

$$s^{(m-n)}(x)=h(x) \quad \text{if} \quad x\in[c-\delta, c+\delta]\cap[a, b].$$

By Theorem 1, (i) there exists a polynomial $p(x)$ of degree $<m$ with $s(x)=r(x)+p(x)$ in $[c-\delta, x_0]$. Thus for the function

$$t(x)=\begin{cases} r(x) & \text{if} \quad x\in[a,x_0] \\ s(x)-p(x) & \text{if} \quad x\in[x_0, \min(c+\delta, b)] \end{cases}$$

we have $t_{(m)}(x)=f(x)$ in the interval $[a, \min(c+\delta, b)]$. By the definition of H this implies $\min(c+\delta, b)\in H$ whence (by the definition of c) we get $c=b$ and $b\in H$. This proves Theorem 3.

COROLLARY. If $f^*(x)$ exists in $[a, b]$ then $f(x)$ is a Darboux Baire 1 function.

THEOREM 4. *If $f^*(x)>0$ holds in $[a, b]$ then $f(x)$ is continuous and increasing in $[a, b]$.*

PROOF. By Theorem 3 there are $g(x)$ and $k\geqq 1$ such that $g_{(k)}(x)=f(x)$ holds in $[a, b]$. Let $x_0\in[a, b]$ be arbitrary, then by the definition of $f^*(x_0)$ there exist $h(x)$ and $n\geqq 0$ such that $h_{(n)}(x)=f(x)$ in a neighbourhood of x_0 and $h_{(n+1)}(x_0)=f^*(x_0)$. If $n\leqq k$ then by Theorem 1, (ii), $g_{(k+1)}(x_0)$ exists and equals to $h_{(n+1)}(x_0)=f^*(x_0)>0$. If $n>k$ then by Theorem 1, (iii), we have

$$\bar{g}_{(k+1)}(x_0)\geqq h_{(n+1)}(x_0)=f^*(x_0)>0.$$

In both cases $\bar{g}_{(k+1)}(x)>0$ holds for every $x\in[a, b]$. Hence, by a theorem of S. VERBLUNSKY ([4], Theorem 1, p. 314), $g_{(k)}(x)=f(x)$ is increasing and continuous in $[a, b]$, q.e.d.

THEOREM 5. *If $f^*(x)$ is defined in $[a, b]$ then $f^*(x)$ is Baire 1 in $[a, b]$.*

PROOF. We shall prove that the level-set

$$H_c=\{x: f^*(x)>c\}$$

is F_σ for every c. We put

$$Q_n(u, x, h)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}u(x+ih).$$

It is well-known that

$$\lim_{h\to 0}\frac{1}{h^n}Q_n(u, x, h)=u_{(n)}(x) \tag{1}$$

whenever $u_{(n)}(x)$ exists.

By Theorem 3, there exists a function $g(x)$ and a natural number k such that $g_{(k)}(x)=f(x)$ holds in $[a, b]$. Since $g(x)$ is continuous, there exists a sequence of differentiable functions $G_n(x)$ satisfying

$$G_k(x)=g(x), \quad G'_{n+1}(x)=G_n(x) \quad (n=k, k+1, \dots). \tag{2}$$

We show that for every $x_0\in[a, b]$ there exists an $N\geqq k$ such that $G_{n(n+1)}(x_0)=f^*(x_0)$ holds for $n\geqq N$. By the definition of $f^*(x_0)$, there are

$h(x)$, $m \geqq 0$ and $\delta > 0$ such that $h_{(m)}(x) = f(x)$ for every $x \in (x_0 - \delta, x_0 + \delta)$ and $h_{(m+1)}(x_0) = f^*(x_0)$. Let $N = \max(m, k)$. If $n \geqq N$ then by (2) we have

$$G_{n(n-k)}(x) = G_n^{(n-k)}(x) = g(x)$$

and hence, by Theorem 1, (ii),

$$G_{n(n)}(x) = g_{(k)}(x) = f(x) = h_{(m)}(x).$$

Applying Theorem 1, (ii) again we get

$$G_{n(n+1)}(x_0) = h_{(m+1)}(x_0) = f^*(x_0).$$

We prove

$$H_c = \bigcup_{n=k}^{\infty} \left\{ x;\ \underline{\lim}_{h \to 0} \frac{1}{h^{n+1}} Q_{n+1}(G_n, x, h) > c \right\}. \tag{3}$$

Let B denote the set in the right hand side of (3). Let $x_0 \in H_c$ and let $n > k$ be such that $G_{n(n+1)}(x_0) = f^*(x_0)$ holds. Then by (1) we have

$$\lim_{h \to 0} \frac{1}{h^{n+1}} Q_{n+1}(G_n, x_0, h) = f^*(x_0) > c$$

and hence $x_0 \in B$.

Now let $x_0 \in B$, then there are $m \geqq k$ and $\varepsilon > 0$ satisfying

$$\underline{\lim}_{h \to 0} \frac{1}{h^{m+1}} Q_{m+1}(G_m, x_0, h) = c + \varepsilon. \tag{4}$$

Suppose indirectly that $x_0 \notin H_c$, i.e.

$$f^*(x_0) \leqq c. \tag{5}$$

Let $n > m$ be chosen with $G_{n(n+1)}(x_0) = f^*(x_0)$ and let

$$k(x) = G_n(x) - \sum_{i=0}^{n+1} G_{n(i)}(x_0) \frac{(x - x_0)^i}{i!}.$$

Then we have

$$k(x_0) = k_{(1)}(x_0) = \ldots = k_{(n+1)}(x_0) = 0 \tag{6}$$

and

$$k^{(n-m)}(x) = G_m(x) - \sum_{i=n-m}^{n+1} G_{n(i)}(x_0) \frac{(x - x_0)^{i-n+m}}{(i - n + m)!}$$

since $G_n^{(n-m)}(x) = G_m(x)$ by (2). Hence

$$Q_{m+1}(k^{(n-m)}, x, h) = Q_{m+1}(G_m, x, h) - G_{n(n+1)}(x_0) \cdot h^{m+1}.$$

By (4) and (5),

$$\lim_{h\to 0} \frac{1}{h^{m+1}} Q_{m+1}(k^{(n-m)}, x_0, h) \geqq c+\varepsilon-c = \varepsilon .$$

Hence there exists an $\eta>0$ such that

$$[Q_{m+1}(k, x_0, h)]^{(n-m)} = Q_{m+1}(k^{(n-m)}, x_0, h) > \frac{\varepsilon}{2} h^{m+1} \tag{7}$$

for every $h\in(0, \eta)$. We put

$$s(h) = Q_{m+1}(k, x_0, h) - \frac{\varepsilon}{2} \frac{h^{n+1}}{(n+1)n \dots (m+2)},$$

then (6) implies

$$s(0) = s'(0) = \dots = s^{(n-m)}(0) = 0 \tag{8}$$

and (7) implies

$$s^{(n-m)}(h) > 0 \quad \text{for every} \quad 0<h<\eta . \tag{9}$$

Now (8) and (9) easily imply that

$$s(h) = Q_{m+1}(k, x_0, h) - \frac{\varepsilon}{2} \frac{h^{n+1}}{(n+1)n \dots (m+2)} > 0$$

for every $h\in(0, \eta)$. Hence

$$\lim_{h\to 0} \frac{1}{h^{n+1}} Q_{m+1}(k, x_0, h) \geqq \frac{\varepsilon}{2(n+1)n \dots (m+2)} . \tag{10}$$

On the other hand, by (6) we have

$$\lim_{h\to 0} \frac{1}{h^{n+1}} \cdot k(x_0+h) = 0 .$$

It follows that the left hand side of (10) vanishes, which is a contradiction. This proves $H_c = B$.

For every fixed $n\geqq k$ the function

$$q_n(x, h) = \frac{1}{h^{n+1}} Q_{n+1}(G_n, x, h)$$

is continuous on the set

$$\{(x, h) : x\in[a, b],\ x+h\in[a, b],\ h\neq 0\} .$$

It follows that the set

$$\left\{x : \overline{\lim_{h\to 0}}\, q_n(x,h) > c\right\} =$$

$$= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \left\{x : q_n(x,h) \geqq c + \frac{1}{i} \quad \text{for every} \quad |h| \leqq \min\left(\frac{1}{j},\, x-a,\, b-x\right)\right\}$$

is an F_σ set for every n. Consequently, by (3), H_c is F_σ, as we stated above.

Applying this result to $-f^*(x)$ it follows that $H^c = \{x : f^*(x) < c\}$ is F_σ for every c as well and hence $f^*(x)$ is Baire 1, q.e.d.

THEOREM 6. *If $f^*(x)$ is defined in $[a,b]$ and $f^*(x)$ is bounded from above or below, then $f'(x)$ exists and $f^*(x) = f'(x)$ holds in $[a,b]$.*

PROOF. We can suppose $f^*(x) > 0$ in $[a,b]$. Then, by Theorem 4, $f(x)$ is increasing on $[a,b]$. Let $x_0 \in [a,b]$ be arbitrary and let n be the smallest natural number for which there is a function $g(x)$ such that $g_{(n-1)}(x) = f(x)$ in a neighbourhood of x_0 and $g_{(n)}(x_0) = f^*(x_0)$. We have to prove $n = 1$.

Since $f(x)$ is bounded in $[a,b]$, $g^{(n-1)}(x)$ exists and equals to $g_{(n-1)}(x) = f(x)$ in a neighbourhood of x_0. Let

$$h(x) = g(x) - \sum_{i=0}^{n-1} g^{(i)}(x_0) \frac{(x-x_0)^i}{i!},$$

then we have

$$(11) \qquad h(x_0) = h'(x_0) = \ldots = h^{(n-1)}(x_0) = 0$$

and

$$h_{(n)}(x_0) = f^*(x_0) > 0.$$

Since $h^{(n-1)}(x) = g^{(n-1)}(x) - g^{(n-1)}(x_0)$ is increasing, (11) implies that $h^{(i)}(x)$ is increasing in $[x_0, x_0+\delta]$, for every $0 \leqq i \leqq n-1$.

Suppose that $n \geqq 2$. By a theorem of S. VERBLUNSKY ([4], Lemma 2, p. 315), the monotonicity of $h'(x)$ and the conditions in (11) imply that the $(n-1)$th Peano derivative of the function $h'(x)$ exists at x_0 and equals to $h_{(n)}(x_0) = f^*(x_0)$. It follows that $(g')_{(n-1)}(x_0) = f^*(x_0)$. Since

$$(g')_{(n-2)}(x) = g^{(n-1)}(x) = f(x),$$

this contradicts the minimality of n. This contradiction proves $n = 1$, q.e.d.

THEOREM 7. *If $f^*(x)$ is defined in $[a,b]$ then*

(i) *$f^*(x)$ is a Darboux function and*

(ii) *$f^*(x)$ possesses property A, i.e. for every $c < d$ the set $\{x : c < f^*(x) < d\}$ is empty or has positive measure.*

PROOF. By Theorems 5 and 6, $f^*(x)$ is Baire 1 and possesses Darboux property in every interval in which $f^*(x)$ is bounded from above or below.

H. W. OLIVER proved in [3] that these conditions imply that $f^*(x)$ is Darboux in the whole interval $[a,b]$ (see [3], the proof of Theorem 2, p. 451).

The proof of (ii) is similar: by a theorem of C. E. WEIL ([5], Theorem 1), it is enough to verify that $f^*(x)$ is Baire 1 and possesses property A in every interval in which $f^*(x)$ is bounded from above or below. Since every (ordinary) derivative has property A (see [2]), Theorems 5 and 6 imply that $f^*(x)$ satisfies these conditions.

3. Miscellaneous counterexamples. First we show that the existence of $f^*(x_0)$ and $g^*(x_0)$ does not imply that the function $f(x)+g(x)$ has a.P.d. at x_0.

EXAMPLE 2. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}, \quad g(x) = \begin{cases} \sin \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and let $h(x) = f(x)+g(x)$. Then $f^*(0) = g^*(0) = 0$ and $h^*(0)$ does not exist.

Indeed, we have $f^*(0) = f'(x) = 0$ and $g^*(0) = 0$ (see Example 1). If $s_{(k)}(x) = h(x)$ holds in a neighbourhood of 0 then $k = 0$ since $h(x)$ is not Baire 1 in any interval. On the other hand $s_{(k+1)}(0) = s'(0) = h'(0)$ does not exist since $h(x)$ is discontinuous at 0. This proves the non-existence of $h^*(0)$.

THEOREM 8. *If $f^*(x)$ and $g^*(x)$ exist* e v e r y w h e r e *in $[a, b]$ then $(f+g)^*$ exists everywhere in $[a, b]$ and $(f+g)^*(x) = f^*(x)+g^*(x)$ holds.*

PROOF. For every $x_0 \in [a, b]$ there exist two functions $F(x)$ and $G(x)$ and $k \geqq 0$, $m \geqq 0$, $\delta > 0$ such that $F_{(k)}(x) = f(x)$ and $G_{(m)}(x) = g(x)$ hold in $(x_0-\delta, x_0+\delta)$ and $F_{(k+1)}(x_0) = f^*(x_0)$, $G_{(m+1)}(x_0) = g^*(x_0)$. Let $n > \max(k, m)$, then there exists a function $U(x)$ such that $U^{(n-k)}(x) = F(x)$ in a neighbourhood of x_0. In fact, this is obvious if $k \geqq 1$ since then $F(x)$ is continuous. If $k = 0$ then $f(x)$ is differentiable at x_0, consequently $f(x)$ is bounded in a neighbourhood of x_0. By Theorem 3 $f(x)$ has a primitive in this neighbourhood. If we choose $U(x)$ such that $U^{(n-k-1)}(x)$ is the primitive of $f(x)$ then we have $U^{(n-k)}(x) = f(x) = F(x)$.

We similarly get a function $V(x)$ with $V^{(n-m)}(x) = G(x)$ in a suitable neighbourhood of x_0. By Theorem 1, (ii) we have

$$U_{(n)}(x) = F_{(k)}(x) = f(x), \quad V_{(n)}(x) = G_{(m)}(x) = g(x),$$

$$U_{(n+1)}(x_0) = f^*(x_0), \quad V_{(n+1)}(x_0) = g^*(x_0).$$

Hence $(U+V)_{(n)}(x) = f(x)+g(x)$ and

$$(U+V)_{(n+1)}(x_0) = f^*(x_0)+g^*(x_0)$$

which proves

$$(f+g)^*(x_0) = f^*(x_0)+g^*(x_0).$$

In the following example we construct a function $f(x)$ which is not an ordinary derivative but $f^*(x)$ exists everywhere. At the same time, $f(x)$ is a (second) Peano derivative (cf. Theorem 3).

EXAMPLE 3. Let

$$g(x) = \begin{cases} x^4 \sin \frac{1}{x^3} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} g''(x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

Then $f(x) = g_{(2)}(x)$, $f^*(x)$ exists everywhere $(f^*(0) = g_{(3)}(0) = 0)$ and $f(x)$ is not an ordinary derivative. In fact, suppose indirectly that $F(x)$ is a primitive of $f(x)$. Since $g''(x) = f(x)$ holds for $x \neq 0$ hence we have

$$F(x) = \begin{cases} g'(x) + c & \text{if} \quad x > 0 \\ g'(x) + d & \text{if} \quad x < 0 \end{cases}$$

with suitable constants c and d. However, no function of this form can be continuous at 0, a contradiction.

The following theorem shows that the Peano derivatives form a proper subclass of the class of a.P.d. functions.

THEOREM 9. *There exists a continuous function $f(x)$ defined in $[0, 1]$ such that $f^*(x)$ is defined in $[0, 1]$ and $f^*(x)$ is not of the form $g_{(k)}(x)$ for any $k \geq 1$.*

PROOF. We shall prove that for every natural n there exists a function $g_n(x)$ with the following properties.

(12) $g_n^{(n)}(x)$ is continuous in $[0, 1]$, $g_n^{(n)}(0) = g_n^{(n)}(1) = 0$;

(13) $g_n^{(n+1)}(x)$ is continuous in $(0, 1]$, $g_n^{(n+1)}(1) = 0$;

(14) $g_{n(n+1)}(0) = 0$

and

(15) the function $g_n'(x)$ does not have nth Peano derivative at $x = 0$.

Then we construct the function $f(x)$ as follows. Let $M_n = \max_{0 \leq x \leq 1} |g_n^{(n)}(x)|$ and let

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \quad \text{or} \quad \frac{1}{2n} \leq x \leq \frac{1}{2n-1} \\ \frac{1}{n^2 M_n} g_n^{(n)}(2n(2n+1)x - 2n) & \text{if} \quad \frac{1}{2n+1} \leq x \leq \frac{1}{2n} \end{cases}$$

$$(n = 1, 2, \ldots).$$

First we show that $f(x)$ possesses the desired properties. (12) easily implies that $f(x)$ is continuous. Since

$$\lim_{x \to +0} \frac{f(x)}{x} = 0$$

we have $f^*(0)=f'(0)=0$. If $x=\frac{1}{2n}$ or $x=\frac{1}{2n+1}$ then $f^*(x)=0$ follows from $g_n^{(n+1)}(1)=0$ and $g_{n(n+1)}(0)=0$, respectively. If

$$\frac{1}{i+1}<x<\frac{1}{i}$$

then the existence of $f^*(x)$ follows by (13).

Suppose that $k\geqq 1$ and $f^*(x)=F_{(k)}(x)$ holds in $[0,1]$. Then for

$$\frac{1}{2k+1}\leqq x\leqq\frac{1}{2k}$$

we have

$$F_{(k)}(x)=f^*(x)=\frac{1}{k^2 M_k}g_{k(k+1)}(2k(2k+1)x-2k).$$

By Theorem 1, (i) there exists a polynomial $p(x)$ such that

$$F(x)=\frac{1}{k^2 M_k}g_k'(2k(2k+1)x-2k)+p(x)$$

for every

$$\frac{1}{2k+1}\leqq x\leqq\frac{1}{2k}.$$

This implies that $g_k'(x)$ has a kth Peano derivative at $x=0$, which contradicts (15).

Now we turn to the construction of the functions $g_n(x)$. Let n be a fixed natural number and let

$$s(x)=x^n(1-x)^n\sin(2\pi x).$$

Then $s^{(n)}(x)$ is continuous in $[0,1]$, $s^{(n)}(0)=s^{(n)}(1)=0$ and

$$\int_0^1 s(x)\,dx=0. \tag{16}$$

We put

$$h(x)=\begin{cases}0 & \text{if } x=0\\ i^{-n^2}\cdot s\left(\dfrac{x-(i+1)^{-n}}{i^{-n}-(i+1)^{-n}}\right) & \text{if } (i+1)^{-n}\leqq x\leqq i^{-n}\quad (i=1,2,\dots)\end{cases}$$

and

$$g_n(x)=\int_0^x h(t)\,dt.$$

It is easy to see that

(17) $h^{(n)}(x)$ is continuous in $(0, 1]$ and $h^{(n-1)}(1) = h^{(n)}(1) = 0$.

We prove

(18) $$\lim_{x\to+0} h^{(k)}(x) = 0 \quad \text{for} \quad 0 \leqq k \leqq n-1.$$

Let

$$K = \max\{|s^{(k)}(x)| : x \in [0, 1],\ 0 \leqq k \leqq n-1\}.$$

Then for every $(i+1)^{-n} \leqq x \leqq i^{-n}$ we have

$$|h^{(k)}(x)| = \left| s^{(k)}\left(\frac{x-(i+1)^{-n}}{i^{-n}-(i+1)^{-n}}\right)\right| i^{-n^2}\cdot\left(i^{-n}-(i+1)^{-n}\right)^{-k} \leqq$$

$$\leqq K\cdot i^{-n^2}\cdot\left(i^{-n}-(i+1)^{-n}\right)^{-(n-1)} \leqq K\cdot i^{-n^2}\cdot\left[n\cdot(i+1)^{-n-1}\right]^{-n+1} \leqq$$

$$\leqq K\,\frac{(i+1)^{n^2-1}}{i^{n^2}} \to 0 \quad \text{as} \quad i\to\infty.$$

Now, $h(0) = 0$ and

$$\lim_{x\to+0} h'(x) = 0$$

imply $h'(0) = 0$. $h'(0) = 0$ and

$$\lim_{x\to+0} h''(x) = 0$$

imply $h''(0) = 0$ etc., finally we get

(19) $$h^{(k)}(0) = 0 \quad \text{for} \quad 0 \leqq k \leqq n-1.$$

(17), (18) and (19) obviously imply (12) and (13). In order to prove (14) we show that $g_n(x) = o(x^{n+1})$ $(x\to 0)$. Let $(i+1)^{-n} \leqq x \leqq i^{-n}$. By (16)

$$g_n(x) = \int_0^x h(t)\,dt = \int_{(i+1)^{-n}}^x h(t)\,dt.$$

Hence

$$\left|\frac{g_n(x)}{x^{n+1}}\right| \leqq x^{-n-1} \int_{(i+1)^{-n}}^x |h(t)|\,dt \leqq (i+1)^{n(n+1)}\cdot\left(i^{-n}-(i+1)^{-n}\right)\cdot i^{-n^2} \leqq$$

$$\leqq (i+1)^{n(n+1)}\cdot n i^{-n-1}\cdot i^{-n^2} = n\,\frac{(i+1)^{n^2+n}}{i^{n^2+n+1}} \to 0$$

as $i\to\infty$. This proves (14).

Finally, suppose indirectly that $(g'_n)_{(n)}(x)=h_{(n)}(x)$ exists at $x=0$. Then by (19) we have

$$h_{(n)}(0)=\lim_{x\to+0}\frac{h(x)}{x^n/n!}.$$

Since $h(i^{-n})=0$ for $i=1,2,\dots$ hence $h_{(n)}(0)=0$.

Now let

$$x_i=(i+1)^{-n}+\frac{1}{4}\left(i^{-n}-(i+1)^{-n}\right),$$

then $\lim_{i\to\infty} x_i=0$ and

$$h(x_i)=i^{-n^2}s\left(\frac{1}{4}\right)=i^{-n^2}\cdot\frac{1}{4^n}\cdot\left(\frac{3}{4}\right)^n.$$

Thus

$$\frac{h(x_i)}{x_i^n}\geqq i^{-n^2}\cdot\frac{1}{4^n}\cdot\left(\frac{3}{4}\right)^n\cdot i^{n^2}=\left(\frac{3}{16}\right)^n$$

from which

$$h_{(n)}(0)\geqq n!\cdot\left(\frac{3}{16}\right)^n,$$

a contradiction. Consequently (15) holds and Theorem 9 is proved.

Finally we show that the concept of a.P.d. cannot be extended to approximative Peano derivatives. In other words: if we replace the Peano derivatives by approximative Peano derivatives in the Definition in section 1 then this definition becomes contradictory because the approximative Peano derivative $f_{(n),a}(x)$ does not determine the number $f_{(n+1),a}(x_0)$.

EXAMPLE 4. We construct the functions $f(x)$ and $g(x)$ such that $f'(x)=$ $=g(x)$ in $[0,1]$ and $f_{(2),a}(0)=1$, $g_{(1),a}(0)=0$.

Let

$$f(x)=\begin{cases}0 & \text{if } x=0\\ 2^{-1}(n+1)^{-2} & \text{if } \dfrac{1}{n+1}\leqq x\leqq\dfrac{1}{n}-\dfrac{1}{n^3} \quad (n=2,3,\dots)\\ 1/2 & \text{if } 1/2\leqq x\leqq 1\end{cases}$$

and let $f(x)$ be increasing and differentiable in the intervals $\left[\dfrac{1}{n}-\dfrac{1}{n^3},\dfrac{1}{n}\right]$ with

$$f'\left(\frac{1}{n}-\frac{1}{n^3}\right)=f'\left(\frac{1}{n}\right)=0 \quad (n=2,3,\dots).$$

Then

(i) $$f'(x) = g(x) \text{ exists everywhere } (f'(0) = 0),$$

(ii) $$f_{(2),a}(0) = f_{(2)}(0) = 1 \quad \text{since} \quad \lim_{x\to+0} \frac{f(x)}{x^2/2} = 1$$

and

(iii) $$g_{(1),a}(0) = 0.$$

In fact, 0 is a density point of the set

$$\{x : g(x) = 0\} \supseteqq \bigcup_{n=2}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n} - \frac{1}{n^3}\right]$$

and hence $$\lim_{x\to+0} \operatorname{app} \frac{g(x)-g(0)}{x-0} = 0.$$

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ON COMPLEXES

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To the memory of Professor P. TURÁN

We shall consider in this paper two questions about complexes.

The first one can be formulated as follows: How far is it suitable to generalize the concept of the geometric simplicial complex so that the notion of the supporting cell and that of the subdivision should preserve the essence of its geometrical meaning?

And the second question: How is it possible to determine the class of the geometric simplicial complexes as the intersection of two classes, where the first class of the complexes is defined by its combinatorical properties and the second by its geometrical ones (convexity etc.)?

To this purpose we shall regard three special classes of complexes: the class of the general simplicial complexes, the class of the general cell complexes and the class of the convex complexes. Showing the special properties of these classes we obtain the following result:

A complex is geometric simplicial iff it is convex and general simplicial.

As to the class of the general cell complexes, this is in a certain sense a common generalization of the CW complexes, that of the fundamental system of subcomplexes and of other similat structures.

We start with the general concept of the complex. In [1] these are the abstract complexes.

1. Complexes

We recall here some basic notions and properties of the complexes.

1.1. Definition: A *complex* K is a partially ordered set, ro each of whose elements s is assigned a non-negative integer $\dim s$, the *dimension* of the element s, such that $s_1 < s_2$ implies $\dim s_1 < \dim s_2$.

K denotes in this section always a complex.

1.2. Definition: The set of the elements of the complex K will be denoted by $|K|$. We say that $|K|$ is the *underlying set* of K.

1.3. For $s \in K$ the fact that $\dim s = t$ will be indicated by s^t.

1.4. DEFINITION: For $s, s' \in K$ s is said to be a *face* of s' (in K) if $s \leqq s'$ (that means: either $s < s'$ or $s = s'$). s is said to be a *proper face* of s' it $s < s'$.

1.5. DEFINITION: The maximum of the dimensions of the elements of K, provided that the maximum exists will be called the *dimension* of K and will be denoted by $\dim K$. If $K \neq \emptyset$ and this maximum does not exists then write $\dim K = \infty$. If $K = \emptyset$, then $\dim K = -1$.

1.6. DEFINITION: The complex K_1 is said to be a *subcomplex* of K – and this fact is indicated by $K_1 < K$ – if the following conditions are satisfied:

(a) $|K_1| \subset |K|$.

(b) For any $s \in K_1$ the dimension of s in K_1 is the same as the dimension of s in K.

(c) The partial ordering in K_1 is the restriction of the ordering in K.

Roughly speaking K_1 is a substructure of K.

1.7. Each subset M of $|K|$ is the underlying set of a unique subcomplex of K. This subcomplex will be denoted by M_K.

1.8. Note that $|M_K| = M$ for any $M \subset |K|$ and $|K_1|_K = K_1$ for any $K_1 < K$.

1.9. NOTATIONS. We shall use the symbols

$$K_1 \cup K_2,\ K_1 \cap K_2,\ K_1 \setminus K_2,\ \cup \{K_\alpha;\ \alpha \in A\},\ \cap \{K_\alpha;\ \alpha \in A\}$$

for

$$(|K_1| \cup |K_2|)_K;\ (|K_1| \cap |K_2|)_K,\ (|K_1| \setminus |K_2|)_K,$$

$$(\cup \{|K_\alpha|;\ \alpha \in A\})_K,\ (\cap \{|K_\alpha|;\ \alpha \in A\})_K$$

respectively, where K_1, K_2 and the K_α-s are subcomplexes of K.

In such a way the notion of the union, intersection and difference is defined in the family of subcomplexes of a given complex K.

Further we say that K_1 contains the subset M of $|K|$ if $|K_1|$ does it $(K_1 < K)$.

1.10. DEFINITION: The subsomplex K_1 of K is said to be a *closed subcomplex* of K, if each face (in K) of any element of K_1 belongs to K_1.

The system $\mathcal{F}$ of the underlying sets of the closed subcomplexes of K determines a T_0 – space K_t the underlying set of which is $|K|$ and the system of the closed sets in K_t is identical with $\mathcal{F}$.

1.11. DEFINITION: The subcomplex K_1 of K is said to be an *open subcomplex* of K if the set $|K_1|$ is open in K_t.

1.12. The subcomplex K_1 of K is evidently an open subsomplex of K iff $s, s' \in K$, $s < s'$ and $s \in K_1$ imply $s' \in K_1$.

1.13. Note that the union and the intersection of a family of open (closed) sets of K_t is an open (closed) set in K_t.

1.14. DEFINITION: For $M \subset |K|$ there is a unique minimal open subcomplex $K_1 < K$ and a unique minimal closed one $K_2 < K$ containing M. We say that K_1 is the *star* (in K) of M and $\mathrm{St}\, M$ or more precisely $\mathrm{St}^K M$

is written for K_1. On the other hand K_2 is the *closure* (in K) of M and $\overline{M}$ or more precisely $\overline{M}^K$ is written for K_2.

$\mathrm{St}\, M$ is the intersection of all open subcomplexes of K and $\overline{M}$ is the intersection of all closed ones containing M.

For $K^* < K$ write $\mathrm{St}\, K^*$ or $\mathrm{St}^K K^*$ instead of $\mathrm{St}^K |K^*|$ and $\overline{K^*}$ or $\overline{K^*}^K$ instead of $\overline{|K^*|}^K$. $\mathrm{St}\, K^*$ is the *star* and $\overline{K^*}$ the *closure* of K^*.

1.15. Definition: Let $s \in K$. Write $\mathrm{St}\, s$ or $\mathrm{St}^K s$ instead of $\mathrm{St}^K \{s\}$ and $\bar{s}$ or $\bar{s}^K$ instead of $\overline{\{s\}}^K$: $\mathrm{St}^K s$ is the *star* of s and $\bar{s}^K$ the *closure* of s (in K).

Evidently

$$|\mathrm{St}\, s| = \{s' \in K;\ s \leqq s'\}$$

and

$$|\bar{s}| = \{s' \in K;\ s' \leqq s\}.$$

1.16. Definition: For $s \in K$ $|\bar{s}| \setminus |\mathrm{St}\, s| = |\bar{s}| \setminus \{s\}$ is called the *frontier* of s. It will be denoted by $\mathrm{Fr}\, s$ or more precisely by $\mathrm{Fr}_K\, s$. Clearly

$$\mathrm{Fr}\, s = \{s' \in K;\ s' < s\}.$$

1.17. Definition: Let m be a non-negative integer and let us consider the subcomplex K^m of K consisting of all elements of K with the dimension not greater then m, i.e.:

$$|K^m| = \{s \in K;\ \dim s \leqq m\}.$$

K^m is called the *m-skeleton* of K. K^m is evidently a closed subcomplex of K.

1.18. Definition: Let $s \in K$. We say that $|\bar{s}| \cap |K^0|$ is the *0-skeleton* of s. This set $|\bar{s}| \cap |K^0|$ will be denoted by $d(s)$ or more precisely by $d_K(s)$.

$d(s)$ is the set of the 0-dimensional faces of s. If K_1 is a closed subcomplex of K and $s \in K_1$, then there is no difference between $d_K(s)$ and $d_{K_1}(s)$.

1.19. Note that $s \in K^0$ implies $d(s) = \{s\}$.

1.20. $s' < s$ implies evidetly $d(s') \subset d(s)$ for $s, s' \in K$.

2. General simplicial complexes

The notion of the general simplicial complexes in this paper corresponds to the unrestricted simplicial complexes in [1]. In this book of Alexandrov the unrestricted simplicial complexes are defined as abstract complexes being isomorphic to the unrestricted skeleton complexes. We shall give here a direct definition for the general simplicial complexes.

2.1. Definition: Let K be a complex. We say that K is a *general simplicial complex* if it satisfies the conditions:

(a) $d(s)$ has the cardinality $1+\dim s$ (i.e.: $\mathrm{Card}\ d(s) = 1+\dim s$) for every $s \in K$.

(b) $d(s) = d(s')$ implies $s = s'$ for every $s, s' \in K$.

(c) If $s \in K$ and if M is a nonempty subset of $d(s)$ $(\emptyset \neq M \subset d(s))$, then there exists a face s' of s in K such that $d(s') = M$.

2.2. Let K be a general simplicial complex and $s, s' \in K$. Then

$$s' \leqq s \Leftrightarrow d(s') \subset d(s).$$

In fact, 1.20. shows

$$s' \leqq s \Rightarrow d(s') \subset d(s).$$

Conversely if $d(s') \subset d(s)$ then $d(s') \neq \emptyset$ by 2.1. (a), hence there exists by 2.1. (c) an $s'' \in K$ such that $s'' \leqq s$ and $d(s'') = d(s')$. In view of 2.1. (b) this yields $s'' = s'$ and thus $s' \leqq s$.

2.3. It is immediate form 2.1. (b) that for any $s, s' \in K$

$$s' = s \Leftrightarrow d(s') = d(s),$$

where K is an arbitrary general simplicial complex.

2.4. For any subcomplex K_1 of a general simplicial complex K K_1 is evidently a general simplicial complex iff K_1 is a closed subcomplex of K.

2.5. Proposition: *Let K be a general simplicial complex, $s, s' \in K$, and suppose that $s \cap \bar{s}' \neq \emptyset$. Then there exists an $s'' \in K$ such that $s \cap \bar{s}' = \bar{s}''$.*

Proof. $\bar{s} \cap \bar{s}' \neq \emptyset$, 1.20. and 2.1. (a) imply $d(s) \cap d(s') \neq \emptyset$. Hence because of 2.1. (c) there exists an $s'' \in K$ such that $d(s'') = d(s) \cap d(s'')$. The proposition follows now immediately from 2.2. ∎

3. General cell complexes

Let X be a CW complex with respect to a family of cells S (see [3] p. 41). We shall first study the system of the carriers of the cells of S.

In particular attach to each (closed) cell $s \in S$ its carrier $\tilde{s}$ in (X, S) (i.e. the underlying set of the least subcomplex of (X, S) containing s). It is possible to describe the complex (X, S) through these carriers as well. Namely

$$s = \overline{\tilde{s} \setminus \bigcup \{\overset{\curvearrowright}{s'};\ s' \in S,\ \overset{\curvearrowright}{s'} \subset \tilde{s} \text{ and } \overset{\curvearrowright}{s'} \neq \tilde{s}\}}$$

holds for each $s \in S$. Observe that the condition: "X has the weak topology with respect to S" can be replaced by the following one: "X has the weak topology with respect to $\tilde{S} = \{\tilde{s} : s \in S\}$".

For two cells s, s' of X s' is the face of s if $s' \subset s$ (see [3] p. 79). This definition is very suitable if X is normal, i.e. if each closed cell is a subcomplex in X, but in the general case it is more convenient to use the following one: For two cells s, s' of the CW complex X s' is a face of s if $s' \subset \tilde{s}$, or — what is the same — s' is the face of s — $s' \leqq s$ — if $\overset{\curvearrowright}{s'} \subset \tilde{s}$. In such a way the concept of the face is also expressible through the carriers of the cells. Let $s' < s$ denote that $s' \neq s$ and s' is a face of s. Then one can obtain the interior of any cell s through the formula

$$s \setminus \dot{s} = \tilde{s} \setminus \bigcup \{\overset{\curvearrowright}{s'};\ s' < s\}.$$

Hence taking the system $\tilde{S}$ of the carriers of the cells of a CW complex X, setting $\overset{\frown}{s'} < \tilde{s}$ if $\overset{\frown}{s'} \subset \tilde{s}$ and $\overset{\frown}{s'} \neq \tilde{s}$, and assigning to every $\tilde{s} \in \tilde{S}$ a dimension number, which is the same as the dimension of $\tilde{s} \setminus \cup\{\overset{\frown}{s'};\ \overset{\frown}{s'} < \tilde{s}\}$ we get a complex K_1, the elements of which are certain subsets of the given space X.

A similar situation arises in connection with the fundamental system of subcomplexes of a complex K. In fact, let K be a triangulation (geometric simplicial complex). K is a complex in the usual sense and thus K determines a topological space K_t (see 1.10.). Let U_i^r be a system of subcomplexes of K of various dimensions satisfying the following conditions:

1° All the U_i^r are disjoint simple pseudomanifolds and their union is K.

2° For every r-dimensional U_i^r the subcomplex $B_i^{r-1} = \overline{U_i^r} \setminus U_i^r$ (the closure operation is meant in K_t) is $r-1$-dimensional and B_i^{r-1} is the union of certain pseudomanifolds U_j^s, $0 \leqq s \leqq r-1$ ([2] p. 129).

Setting $U_j^s < U_i^r$ if $|U_j^s| \subset |B_i^{r-1}|$ (see. 1.2.), and assigning to each U_i^r a dimension number, which is the same as its dimension as a complex, we obtain a new complex K'. This is the fundamental system of subcomplexes of K.

Let us consider now the sets $|\overline{U_i^r}|$ in K_t. Setting $|\overline{U_j^s}| < |\overline{U_i^r}|$ if $|\overline{U_j^s}| \subset |\overline{U^r}|$ and $|\overline{U_j^s}| \neq |\overline{U_i^r}|$, we see that the set of all $|\overline{U_i^r}|$ is partially ordered. It is now possible to express the U_i^r-s in the following way

$$|U_i^r| = |\overline{U_i^r}| \setminus \cup\{|\overline{U_j^s}|;\ |\overline{U_j^s}| < |\overline{U_i^r}|\}.$$

Assigning to each $|\overline{U_i^r}|$ the dimension number $\dim U_i^r = r$ we obtain a new complex $K_2 \cdot K_2$ is not a CW complex, but its elements are also sets in a space, namely in the T_0-space K_t.

We are going now to describe a family of complexes such that both of the complexes K_1 and K_2 belong to this family.

3.1. Definition: Let R be a topological or Euclidean space. By an *abstract cell complex* K with the *supporting space* $R = R(K)$ we understand a nonempty complex (see 1.1.), the elements of which are triples of the type $b = (R, Q, t)$ where Q is a nonempty subset of R and t – the *dimension* of b – is a non-negative integer: $t = \dim b$. Furthermore the partial order in K should be defined in the following way;

$$b = (R, Q, t) \leqq b' = (R, Q', t') \quad \text{iff} \quad Q \subset Q';$$
$$b < b' \quad \text{iff} \quad b \leqq b' \quad \text{and} \quad b \neq b'.$$

3.2. Definition: Let K be an abstract cell complex with the supporting space R and $b = (R, Q, t) \in K$. The *underlying set* of b – written "$|b|$" – is the set Q ($|b| = Q$). By the set b is unterstood also $|b|$. Thus

$$b \leqq b' \Leftrightarrow |b| \subset |b'| \quad \text{for} \quad b, b' \in K.$$

3.3. Remark: To present an abstract cell complex K it is enough to take its elements. The elements determine namely their dimensions and the partial order in K.

In order to get a complex indeed we must suppose that for $b, b' \in K$

(a) If $|b| = |b'|$ then $\dim b = \dim b'$, and thus $b = b'$.
(b) If $|b| \subset |b'|$ and $|b| \neq |b'|$, then $\dim b < \dim b'$.

3.4. Definition: Let K be an abstract cell complex with the supporting space R. We say that the topological subspace of R with the underlying set $\cup\{|b|; b \in K\}$ is the *body* of K (or of $|K|$) and we denote it by $[K]$ (or by $[|K|]$). We shall use also the symbols $[\emptyset]$ or $|[\emptyset]|$ for $\emptyset$, where $\emptyset$ can be regarded as a subcomplex of K (but not as an abstract cell complex).

If $b \in K$, then we call the subspace of R with the underlying set $|b|$ the *body* of the element b. $[b]$ will denote the body of b.

The underlying set of $[K]$ will be denoted by $|[K]|$.

$$|[K]| = \cup\{|b|;\ b \in K\}.$$

3.5. Remark: Any nonempty subcomplex K_1 of the abstract cell complex K is also an abstract cell complex. The supporting spaces of K_1 and K are the same. The body of K_1 is a subspace of $[K]$.

3.6. Let us observe that distinct subcomplexes of an abstract cell complex K may have the same body. In fact we can find evidently to each subcomplex K_1 of K a maximal subcomplex K_2 of K such that $[K_1] = [K_2]$. This K_2 is always a closed subcomplex of K and it contains K_1 as its subcomplex.

The body of a subcomplex is obviously the same as the body of its closure (see 1.14.). Moreover if $b \in K$ then evidently $[b] = [\bar{b}]$ (see 1.15.) and thus $|[\bar{b}]| = |b|$.

3.7. Definition: Let K be an abstract cell complex and $b \in K$. Then $\mathrm{int}_K\, b$ will denote the set $|b| \setminus |[\mathrm{Fr}_K\, b]|$ (see 1.16.). We call $\mathrm{int}_K\, b$ the *interior* of b rel. K.

3.8. Remark: $\{\mathrm{int}_K\, b;\ b \in K\}$ is obviously a covering of the body of the abstract cell complex K.

3.9. Remark: If K_1 is a nonempty closed subcomplex of the abstract cell complex K and $b \in K_1$, then evidently

$$\mathrm{int}_K\, b = \mathrm{int}_{K_1}\, b\,.$$

Now we shall turn to the definition of the general cell complexes.

3.10. Definition: Let K be an abstract cell complex with the supporting space R. K is said to be a *general cell complex* – or shortly a GC complex – if the following conditions are satisfied:

(a) The underlying sets of the elements of K are closed sets in R.
(b) $[K]$ has the weak topology with respect to the system $\{[b];\ b \in K\}$.
(c) Each element of K has only a finite number of faces.
(d) $\mathrm{int}_K\, b \neq \emptyset$ for each $b \in K$.
(e) For any two elements b, b' of K the set $|b| \cap |b'|$ is the union of the underlying sets of certain elements of K; i.e.

$$|b| \cap |b'| = \cup\{|b''|;\ b'' \leqq b \ \text{ and } \ b'' \leqq b'\} = |[\bar{b} \cap \bar{b}']|\,.$$

The elements of a GC complex are said to be the *cells* of this complex.

3.11. REMARK: For 3.10. (b) it is sufficient that 3.10. (a) should be satisfied and that the system of the underlying sets of the elements of K should be a locally finite system with respect to $[K]$. However this second condition is not necessary.

We show some properties of the general cell complexes and get acquainted meanwhile with some new notions too.

3.12. THEOREM: *Let K be a GC complex and K_1 a nonempty closed subcomplex of K. Then K_1 is also a GC complex and $[K_1]$ is a closed subspace of $[K]$.*

PROOF. Let $b \in K$. Because of 3.10. (e)

$$|b| \cap |[K_1]| = |b| \cap (\cup\{|b'|;\ b' \in K_1\}) = \cup\{|b| \cap |b'|;\ b' \in K_1\} =$$

$$(1) \qquad = \cup\{|[\bar{b} \cap \bar{b}']|;\ b' \in K_1\} = |[\bar{b} \cap (\cup\{\bar{b}';\ b' \in K_1\})]| = |[\bar{b} \cap \overline{K_1}]|\,.$$

b has only a finite number of faces. Their underlying sets are closed in $R(K)$ and hence also in $[K]$. Therefore (1) shows that $|b| \cap |[K_1]|$ is closed in $[K]$ and hence also in $[b]$. But since $|[K_1]| \cap |b|$ is closed for each $b \in K$, 3.10. (b) shows, that $[K_1]$ is indeed a closed subspace of $[K]$.

We are going to show that K_1 is a GC complex. The conditions 3.10. (a) and (c) are obviously satisfied. Since K_1 is closed (e) is also true. 3.9. shows that K_1 satisfies 3.10. (d) too. Hence we only need to verify that K_1 satisfies also 3.10. (b).

Indeed let M be a set in $[K_1]$ such that $|b''| \cap M$ should be closed in $[b'']$ for each $b'' \in K_1$. Let $b \in K$. Then by (1) and since K_1 is a closed subcomplex of K

$$|b| \cap M = |b| \cap M \cap |[K_1]| = |[\bar{b} \cap \overline{K_1}]| \cap M = |[\bar{b} \cap K_1]| \cap M =$$

$$(2) \qquad = \cup\{|b''| \cap M,\ b'' \in K_1 \ \text{ and } \ b'' \leqq b\}\,.$$

b has only a finite number of faces and the body of each face is closed in $[b]$. Hence by (2) $|b| \cap M$ is closed in $[b]$ and this is true for each $b \in K$. Therefore K being a GC complex M is closed in $[K]$ and hence also in $[K_1]$ as required. ▌

In the remainder of this section let K be always a GC complex and R its supporting space. The cells of K should be called *K-cells*.

3.13. DEFINITION: A subset M of R is said to be *K-compatible* if it satisfies the following conditions:

(a) $M \neq \emptyset$.

(b) There is a cell b of K containing $M : M \subset |b|$.

(c) If M is covered by a finite system of K-cells, then M is contained in a single cell of this covering:

$$(M \subset \cup\{|b_i|;\ i = 1, \ldots, k\} \Rightarrow \exists_j : 1 \leqq j \leqq k \ \text{ and } \ M \subset |b_j|)\,.$$

We shall look at first at a strengthening of the condition (c).

3.14. Proposition: *For any K-compatible set M and for any covering of M by K-cells there is a cell of this covering, containing M.*

Proof. Suppose that M is K-compatible and let

$$M \subset \cup\{|b_\gamma|;\ \gamma\in\Gamma\}. \tag{3}$$

Let

$$M \subset |b|;\ b\in K. \tag{4}$$

3.13. (b) shows the existence of such a b. In view of 3.10. (e) (3) and (4) yield

$$M \subset \cup\{|b| \cap |b_\gamma|;\ \gamma\in\Gamma\} = \cup\{|b_{\gamma,i}|;\ b_{\gamma,i} \leqq b_\gamma,\ b_{\gamma,i} \leqq b \text{ and } \gamma\in\Gamma\}. \tag{5}$$

By 3.10. (c) b has only a finite number of faces and this implies by (5) and 3.13. (c) the existence of a b_{γ_0, i_0} which contains M. For this γ_0 it is clearly $M \subset |b_{\gamma_0}|$ as required. ∎

To prepare the notion of the supporting cell we make here some preliminary remarks.

3.15. Let M be a K-compatible set and b a minimal dimensional cell of K containing M. Because of 3.13. (b) there exists such a b. In view of 3.3. (b) M is not contained in any proper face of b; i.e. $M \subset |b''|$ and $b'' \leqq b$ imply $b'' = b$.

3.16. Let M and b be the same as in 3.15. Then $M \subset |b_1|$ and $b_1 \in K$ imply $b \leqq b_1$.

Indeed by 3.13. (c) and by $M \subset |b| \cap |b_1| = |[\bar{b} \cap \bar{b}_1]|$ there is a $b_2 \in K$ such that $M \subset |b_2|$, $b_2 \leqq b$ and $b_2 \leqq b_1$. Because of 3.15. $b_2 = b$, and thus it holds $b \leqq b_1$.

3.17. Given a K-compatible set M there is only one minimal dimensional cell in K containing M.

For otherwise there would exist two distinct cells of this kind b_1 and b_2. Hence by 3.16. $b_1 \leqq b_2$ and $b_2 \leqq b_1$ would hold simultaneously, but this is impossible.

3.18. Definition: If M is a K-compatible set, then the minimal dimensional K-cell containing M is said to be the *supporting cell* of M in K, and it is denoted by $S_K(M)$.

Because of 3.17. $S_K(M)$ is a well defined cell.

3.19. If M is a K-compatible set and $M \subset |b|$ for some $b \in K$, then 3.16. shows that $S_K(M) \leqq b$.

3.20. Let M and N be K-compatible sets such that $M \subset N$. Then $M \subset |S_K(N)|$ and thus by 3.19. $S_K(M) \leqq S_K(N)$.

The supposting cell has the following useful characterization:

3.21. Proposition: *If M is a K-compatible set and $b \in K$, then $b = S_K(M)$ holds iff $M \subset |b|$ and $M \cap \operatorname{int}_K b \neq \emptyset$.*

Proof. Because of $M \subset |S_K(M)|$ we only need to show that for $M \subset |b|$, $b \in K$

$$b \neq S_K(M) \Leftrightarrow \operatorname{int}_K b \cap M = \emptyset.$$

Suppose now $M \subset |b|$, $b \in K$ and $b \neq S_K(M)$. Then by 3.19. $S_K(M) < b$, consequently

$$M \cap \operatorname{int}_K b = M \cap (|b| \setminus |[\operatorname{Fr}_K b]|) \subset (M \cap (|b| \setminus |S_K(M)|)) \subset$$
$$\subset (M \cap (|b| \setminus M)) = \emptyset .$$

Conversely suppose $M \subset |b|$, $b \in K$ and $M \cap \operatorname{int}_K b = \emptyset$. Then $M \subset \subset |[\operatorname{Fr}_K b]|$ and thus by 3.14. there is a $b'' < b$ for which $M \subset |b''|$. Since $\dim b'' < \dim b$ we get $b \neq S_K(M)$. ▌

We shall consider now the singletons (one point sets) of $[K]$. They are evidently K-compatible sets.

3.22. Definition: For $p \in [K]$ $S_K(\{p\})$ is called the *supporting cell* of p and we write also $S_K(p)$ instead of $S_K(\{p\})$.

In view of 3.19., 3.21. and 3.9. one easily checks the following statements:

3.23. If $p \in |b|$ $(b \in K)$ then $S_K(p) \leqq b$.
3.24. It holds $p \in \operatorname{int}_K S_K(p)$ for any $p \in [K]$.
3.25. If $p \in \operatorname{int}_K b$ $(b \in K)$, then $b = S_K(p)$.
3.26. If K_1 is a closed subcomplex of K and $p \in |[K_1]|$, then $S_{K_1}(p) = S_K(p)$.

As for the K-compatible sets, we shall give here in connection with 3.21. a characterization of them.

3.27. Proposition: *A subset M of R is a K-compatible set iff there exists a cell b of K such that $M \subset |b|$ and $M \cap \operatorname{int}_K b \neq \emptyset$.*

Proof. If M is a K-compatible set then 3.21. shows that for $b = S_K(M)$ $M \subset |b|$ and $M \cap \operatorname{int}_K b \neq \emptyset$.

Suppose conversely the existence of a $b \in K$ such that $M \subset |b|$ and $M \cap \cap \operatorname{int}_K b \neq \emptyset$. The conditions 3.13. (a) and 3.13. (b) are then clearly satisfied for the set M. Let us consider now a covering of M by a finite system of cells of $K: M \subset \bigcup \{|b_i|; b_i \in K; i = 1, \ldots, m\} = N$. Let $p \in M \cap \operatorname{int}_K b$. By 3.25. $S_K(p) = b$. Because $p \in N$ there exists an i such that $p \in |b_i|$. 3.23. shows then $S_K(p) \leqq b_i$, and since $M \subset |b|$ it holds $M \subset |b_i|$ too. Hence condition 3.13. (c) is also satisfied for the set M, M is indeed a K-compatible set. ▌

3.28. Recall from 3.8. that $\{\operatorname{int}_K b;\ b \in K\}$ is a covering of $[K]$ and note that this covering consists of pairwise disjoint sets.

In fact let b, b' be cells of K and suppose that $p \in \operatorname{int}_K b \cap \operatorname{int}_K b'$. Then 3.25. shows that $b = S_K(p) = b'$.

Let us regard some properties of the closed subcomplexes. The next proposition is of importance.

3.29. Proposition: *Let K_1 be a closed subcomplex of K and let $b \in K$. Then the following statements are equivalent:*

(a) $b \in K_1$,
(b) $\operatorname{int}_K b \subset |[K_1]|$,
(c) $\operatorname{int}_K b \cap |[K_1]| \neq \emptyset$.

PROOF. (a)⇒(b). It is obvious. (b)⇒(c). Because $\mathrm{int}_K b \neq \emptyset$. (c)⇒(a). Let $p \in \mathrm{int}_K b \cap |[K_1]|$. Since $\mathrm{int}_K b \cap |[K_1]| \neq \emptyset$ there exists such a point p. Because of 3.25. $b = S_K(p)$ and by 3.26. $S_K(p) = S_{K_1}(p)$. Consequently $b = S_{K_1}(p)$, $b \in K_1$. ▮

3.30. It may be noted that according to 3.29. $[K_1]$ determines uniquely the cells of the closed subcomplex K_1. That is, for two closed subcomplexes K_1 and K_2 of K, $[K_1] = [K_2]$ implies $K_1 = K_2$ and $|[K_1]| \subset |[K_2]|$ implies $K_1 < K_2$.

3.31. Observe that in view of 3.6. and 3.30. $[K_1] = [K_2]$ holds for two subcomplexes K_1 and K_2 of K iff $\overline{K_1}^K = \overline{K_2}^K$ (see 1.14.). Notice also that by 3.10. (c) the finiteness of any subcomplex K_1 of K implies the finiteness of its closure $\overline{K_1}^K$.

In connection with K-compatible sets according to 3.27., 3.29., 3.21. and 3.9. we can assert the following statement:

3.32. Let K_1 be a closed subcomplex of K and $M \subset |[K_1]|$. Then M is K_1-compatible iff it is K-compatible and in this case $S_K(M) = S_{K_1}(M)$.

3.33. Apply 3.29. to the case when K_1 is a subcomplex of the form $\overline{b'}^K$ where $b' \in K$ (see 1.15). Then $[K_1] = [b']$, $|[K_1]| = |b'|$ (see 3.6.) and $b \in K_1$ holds iff $b \leqq b'$. Whence we may conclude that for any b, $b' \in K$ the statements

(a) $b \leqq b'$,
(b) $\mathrm{int}_K b \subset |b'|$,
(c) $\mathrm{int}_K b \cap |b'| \neq \emptyset$

are equivalent.

Let us show some consequences of this statement above.

3.34. If $\dim b' < \dim b$ for b, $b' \in K$ then $\mathrm{int}_K b \cap |b'| = \emptyset$.

For otherwise $b \leqq b'$ by 3.33., hence $\dim b \leqq \dim b'$ would hold contradicting our assumption.

Consequently

3.35. For b, $b' \in K$ $\dim b' < \dim b$ implies

$$|b'| \cap |b| = |[\mathrm{Fr}_K b]| \cap |b'| .$$

3.36. If $\dim b = \dim b'$ and $b \neq b'$ for b, b' $b' \in K$, then $\mathrm{int}_K b \cap |b'| = \emptyset$.

For otherwise $b \leqq b'$ by 3.33. and since $\dim b = \dim b'$ this would yield $b = b'$ contradicting our assumption.

We can state now evidently the following remark:

3.37. If $\dim b = \dim b'$ and $b \neq b'$ for b, $b' \in K$ then

$$|b| \cap |b'| = |[\mathrm{Fr}_K b]| \cap |[\mathrm{Fr}_K b']| .$$

3.38. We take for each subcomplex K_1 of K the union of the interiors of its cells and we denote this subset of $|[K]|$ by $\langle K_1 \rangle$ or more precisely by $\langle K_1 \rangle_K$:

$$\langle K_1 \rangle = \langle K_1 \rangle_K = \bigcup \{\mathrm{int}_K b;\ b \in K_1\} .$$

Some elementary relations follow immediately from 3.28.
3.39.

$$\langle K_1 \cup K_2 \rangle = \langle K_1 \rangle \cup \langle K_2 \rangle ,$$
$$\langle K_1 \cap K_2 \rangle = \langle K_1 \rangle \cap \langle K_2 \rangle ,$$
$$\mathrm{int}_K b \cap \langle K_1 \rangle \neq \emptyset \Leftrightarrow b \in K_1 ,$$
$$\langle K_1 \backslash K_2 \rangle = \langle K_1 \rangle \backslash \langle K_2 \rangle ,$$
$$\langle \cup \{K_\alpha ;\ \alpha \in A\} \rangle = \cup \{\langle K_\alpha \rangle ;\ \alpha \in A\}$$
$$\langle \cap \{K_\alpha ;\ \alpha \in A\} \rangle = \cap \{\langle K_\alpha \rangle ;\ \alpha \in A\}$$

where K_1, K_2 and the K_α-s are arbitrary subcomplexes of K and $b \in K$.

3.40. Taking 3.10. (d) into account we obtain

$$K_1 < K_2 \Leftrightarrow \langle K_1 \rangle \subset \langle K_2 \rangle$$

for any two subcomplexes K_1 and K_2 of K. Consequently

$$K_1 = K_2 \Leftrightarrow \langle K_1 \rangle = \langle K_2 \rangle .$$

In particular

$$K_1 = \emptyset \Leftrightarrow \langle K_1 \rangle = \emptyset ,$$

and this yields

$$K_1 \cap K_2 = \emptyset \Leftrightarrow \langle K_1 \rangle \cap \langle K_2 \rangle = \emptyset .$$

3.41. If $K_1 = \{b\}_K$ for $b \in K$ (see 1.7.) then $\langle K_1 \rangle$ is the same as $\mathrm{int}_K b$. If $K_1 = K$ then $\langle K_1 \rangle = |[K]|$.
Furthermore $\langle K_1 \rangle \subset [K]$ holds for each subcomplex K_1 of K.

3.42. Let K_1 be a subcomplex of a closed subcomplex K_2 of K. Then by 3.9. $\langle K_1 \rangle_{K_2} = \langle K_1 \rangle_K$.

3.43. Let K_1 be a closed subcomplex of K. Then by 3.41. $\langle K_1 \rangle_K = \langle K_1 \rangle_{K_1} = |[K_1]|$. Consequently by 3.12. $\langle K_1 \rangle$ is a closed set in $[K]$.

3.44. If K_1 and K_2 are closed subcomplexes of K, then evidently

$$|[K_1]| \cap |[K_2]| = \langle K_1 \rangle \cap \langle K_2 \rangle = \langle K_1 \cap K_2 \rangle = |[K_1 \cap K_2]| .$$

3.45. Let K_1 be an open subcomplex of K. Then $K \backslash K_1$ is closed and hence by 3.43. $\langle K \backslash K_1 \rangle$ is a closed set in $[K]$. Consequently

$$\langle K_1 \rangle = \langle K \rangle \backslash \langle K \backslash K_1 \rangle = |[K]| \backslash \langle K \backslash K_1 \rangle$$

is an open set in the space $[K]$.

3.46. Let t be a positive integer. The t and $t-1$ dimensional skeletons K^t and K^{t-1} of K (see 1.17.) are closed subcomplexes of K. Therefore if $K^t \neq \emptyset$ then $|[K^t]| \backslash |[K^{t-1}]|$ is an open set in $[K^t]$.

3.47. Let b be a maximal element of $K (b \leq b' \Rightarrow b' = b)$. Then $\{b\}_K$ is an open subcomplex of K, consequently $\langle \{b\} \rangle_K = \mathrm{int}_K b$ is open in $[K]$. In particular for any $b \in K^t \backslash K^{t-1}$ the set $\mathrm{int}_K b$ is open in $[K^t]$.

We shall now construct two open coverings of $\langle K_1 \rangle$ where K_1 is an open subcomplex of K.

3.48. DEFINITION: Let $p \in [K]$. The *star* of p in K – denoted by $\mathrm{St}^K(p)$ – is then defined as the star of the supporting cell of p:

$$\mathrm{St}^K(p) = \mathrm{St}^K(S_K(p)).$$

3.49. 3.23. shows that

$$b \in \mathrm{St}^K(p) \Leftrightarrow p \in |b|$$

for $p \in [K]$ and $b \in K$.

3.50. Let K_1 be an open subcomplex of K and $p \in \langle K_1 \rangle$. Then $\mathrm{St}^K(p) < K_1$. In fact since $p \in \langle K_1 \rangle$ it holds $S_K(p) \in K_1$ and thus

$$\mathrm{St}^K(p) = \mathrm{St}^K(S_K(p)) < K_1.$$

3.51. Notice as a direct consequence of 3.50. that for any open subcomplex K_1 of K

$$K_1 = \cup \{\mathrm{St}^K(p);\ p \in \langle K_1 \rangle\}$$

and thus

$$\langle K_1 \rangle = \cup \{\langle \mathrm{St}^K(p) \rangle;\ p \in \langle K_1 \rangle\}.$$

3.52. Consider now an open subcomplex K_1 of K. Then

$$\Sigma_K(\langle K_1 \rangle) = \{\langle K_2 \rangle;\ K_2 \text{ is open in } K \text{ and } K_2 < K_1\}$$

and

$$\sigma_K(\langle K_1 \rangle) = \{\langle \mathrm{St}^K(p) \rangle;\ p \in \langle K_1 \rangle\}$$

are open (in $[K]$) coverings of $\langle K_1 \rangle$. $\sigma_K(\langle K_1 \rangle)$ is clearly a subsystem of $\Sigma_K(\langle K_1 \rangle)$ and each element $\langle K_2 \rangle$ of $\Sigma_K(\langle K_1 \rangle)$ is the union of those elements of $\sigma_K(\langle K_1 \rangle)$ which are contained in $\langle K_2 \rangle$.

Observe that the intersection and the union of any subsystem of $\Sigma_K(\langle K_1 \rangle)$ belong to $\Sigma_K(\langle K_1 \rangle)$.

Note that if K_1 is finite then $\Sigma_K(\langle K_1 \rangle)$ and $\sigma_K(\langle K_1 \rangle)$ are clearly finite coverings of $\langle K_1 \rangle$.

Now we shall study the subdivisions of a GC complex.

3.53. DEFINITION: A GC complex K' is called a *subdivision* of K – and this relation is denoted by $K' (\sqsubseteq K$ – if

(a) $R(K') = R(K)$,

(b) to every $b' \in K'$ there exists a $b \in K$ such that $|b'| \subset |b|$,

(c) to every $b \in K$ there exists a finite system $b'_1, \ldots, b'_m$ of cells of K' such that

$$|b| = |b'_1| \cup \ldots \cup |b'_m|.$$

3.54. Let us consider some direct consequences of the difinition:

$K'' (\sqsubseteq K''$ is true for every GC complex K''.

If $K'' (\sqsubseteq K'$ and $K' (\sqsubseteq K$ then $K'' (\sqsubseteq K$.

If $K' (\sqsubseteq K$ then $[K'] = [K]$.

In the remainder of this section let K' be always a GC complex such that $K' (\sqsubseteq K$.

3.55. PROPOSITION: *The underlying sets of the elements of K' are K-compatible sets.*

PROOF. Let $b' \in K'$ and let $p \in \mathrm{int}_{K'} b'$. 3.10. (d) shows the existence of such a point p. Let $b_1 = S_K(p)$ (see 3.22.). Then by 3.24. $p \in \mathrm{int}_K b_1 \subset |b_1|$. Choose $b'_1 \in K'$ such that $p \in |b'_1| \subset |b_1|$. Because of 3.53. (c) there exists such a b'_1. $\mathrm{int}_{K'} b' \cap |b'_1| \neq \emptyset$ implies then $b' \leqq b'_1$ (see 3.33.) consequently $|b'| \subset |b_1|$, and since $\mathrm{int}_K b_1 \cap |b'| \neq \emptyset$ 3.27. shows that $|b'|$ is indeed a K-compatible set. ∎

3.56. Because of 3.55. each $b' \in K'$ determines the supporting cell $S_K(|b'|)$ of $|b'|$ in K. For this cell we shall also use the symbol $S_K(b')$.

3.57. REMARK: If $b' \in K'$, $b \in K$ and $|b'| \subset |b|$ then 3.19. shows that $S_K(b') \leqq b$.

3.58. If b'_1, $b' \in K'$ and $b'_1 \leqq b'$ then by 3.20. $S_K(b'_1) \leqq S_K(b')$.

3.59. Notice that $|S_{K'}(p)| \subset |S_K(p)|$ holds for any $p \in [K'] = [K]$.

In fact by 3.53. (c) we can find to each $p \in [K]$ a cell $b' \in K'$ such that $p \in |b'| \subset |S_K(p)|$. This in turn yields

$$p \in |S_{K'}(p)| \subset |b'| \subset |S_K(p)| \,.$$

3.60. Because of 3.59. 3.19. and 3.20. it holds evidently

$$S_K(S_{K'}(p)) = S_K(p)$$

for any $p \in [K]$.

3.61. PROPOSITION: *Let $b' \in K'$. Then $\mathrm{int}_{K'} b' \subset \mathrm{int}_K S_K(b')$.*

PROOF. Let $p \in \mathrm{int}_{K'} b'$. Then $b' = S_{K'}(p)$ by 3.25. consequently

$$S_K(b') = S_K(S_{K'}(p)) = S_K(p) \,.$$

Thus $p \in \mathrm{int}_K S_K(p)$ (see 3.24.) and this yields the required inclusion

$$\mathrm{int}_{K'} b' \subset \mathrm{int}_K S_K(b') \,. \;∎$$

3.62. It may be noted as a consequence of 3.61. that the interior rel. K' of any cell b' of K' is contained in the interior rel. K of one and evidently only one cell b of K and this cell b is the supporting cell of $|b'|$ in K.

3.63. Because of $[K'] = [K]$ we can state now the following formula:

$$\mathrm{int}_K b = \bigcup \{\mathrm{int}_{K'} b';\ S_K(b') = b\}$$

for $b \in K$.

3.64. It follows from 3.62. that for any subdivison K'' of K' and for any $b'' \in K''$

$$S_K(S_{K'}(b'')) = S_K(b'') \,.$$

3.65. DEFINITION: Let K_1 be a subcomplex of K. Let K'_1 be the subcomplex of K' for which

$$|K'_1| = \{b' \in K';\ S_K(b') \in K_1\} \,.$$

K'_1 is a complex. We shall call it the *subdivision* of K_1 *induced by* K' and it shall be denoted by $K_1^{(K')}$.

The following statements are evidently clear:

3.66. The subdivision of K induced by K' is K' itself.
3.67. If $K_1=\emptyset$ then $K_1^{(K')}=\emptyset$.
If $K_1\neq\emptyset$ then in view of 3.63. and 3.10. (d) $K_1^{(K')}\neq\emptyset$.
3.68. $(K\setminus K_1)^{(K')}=K'\setminus K_1^{(K')}$ for any $K_1<K$.
3.69. If $K_2<K_1<K$ then

$$K_2^{(K')}<K_1^{(K')}.$$

3.70. Let K'' be a GC complex which is a subdivision of K' and let $K_1<K$. Then by 3.64.

$$(K_1^{(K')})^{(K'')}=K_1^{(K'')}.$$

3.71. Taking 3.63. into account we can state

$$\langle K_1^{(K')}\rangle_{K'}=\langle K_1\rangle_K \quad \text{for any} \quad K_1<K.$$

3.72. Let K_1 be a subcomplex of K and let

$$K_1'=K_1^{(K')}. \quad \text{Then} \quad |[K_1']|\subset|[K_1]|.$$

In fact let $p\in|b'|$ where $b'\in K_1'$. Then $p\in|b'|\subset|S_K(b')|$ where $S_K(b')\in\in K_1$. Hence indeed $|[K_1']|\subset|[K_1]|$.

3.73. Let K_1 be a finite subcomplex of K and let $K_1'=K_1^{(K')}$. Then K_1' is also finite.

In fact because of 3.53. (c) we can find a finite subcomplex K_2' of K' such that $[K_2']=[K_1]$. In view of 3.31. $\overline{K_2'}=\overline{K_2'}^{K'}$ is also finite and $[\overline{K_2'}]==[K_2']=[K_1]$. On the other hand because of 3.31. and 3.72. $|[\overline{K_1'}]|==|[K_1']|\subset|[K_1]|$ where $\overline{K_1'}=\overline{K_1'}^{K'}$. Hence $|[\overline{K_1'}]|\subset|[\overline{K_2'}]|$ and this yields by 3.30. $\overline{K_1'}<\overline{K_2'}$. Consequently $K_1'<\overline{K_2'}$, K_1' is also finite as required.

3.74. Let K_1 be a nonempty closed subcomplex of K. Then because of 3.67. and 3.58. the subdivision K_1' of K_1 induced by K' is a nonempty closed subcomplex of K'. Therefore by 3.12. K_1' is a GC complex. We show that K_1' is a subdivision of K_1 ($K_1'(\sqsubseteq K_1)$).

3.53. (a) and 3.53. (b) are obviously satisfied. Let $b\in K_1$. Then $|b|==|b_1'|\cup\ldots\cup|b_m'|$, where $b_i'\in K'$ for $i=1,\ldots,m$. Since $|b_i'|\subset|b|$ we get by 3.19. $S_K(b_i')\leqq b$, and therefore since K_1 is closed $S_K(b_i')\in K_1$ and thus $b_i'\in K_1'$. Hence 3.53. (c) is also satisfied, K_1' is indeed a subdivision of K_1.

3.75. Let K_1 be an open subcomplex of K. Then $K_2=K\setminus K_1$ is closed and therefore by 3.68., 3.67. and 3.74. $K_1^{(K')}=K'\setminus K_2^{(K')}$ is an open subcomplex of K'.

3.76. Proposition: *Let* $p\in[K]=[K']$. *Then* $\langle \mathrm{St}^{K'}(p)\rangle_{K'}\subset\langle \mathrm{St}^K(p)\rangle_K$ (see 3.38. and 3.48.).

Proof. Let $q\in\langle \mathrm{St}^{K'}(p)\rangle_{K'}$ and $b'=S_{K'}(q)$. Then by 3.24. $q\in\mathrm{int}_{K'}b'$ therefore $b'\in\mathrm{St}^{K'}(p)$ (see 3.39.) and thus $S_{K'}(p)\leqq b'$, $p\in|b'|$. Let $b=S_K(b')$. Then $|b'|\subset|b|$ and by 3.61. $\mathrm{int}_{K'}b'\subset\mathrm{int}_K b$, consequently $p\in|b|$ and $q\in\in\mathrm{int}_K b$. $p\in|b|$ implies by 3.23. $S_K(p)\leqq b$, that means $b\in\mathrm{St}^K(S_K(p))==\mathrm{St}^K(p)$ and this yields $\mathrm{int}_K b\subset\langle \mathrm{St}^K(p)\rangle_K$, $q\in\langle \mathrm{St}^K(p)\rangle_K$. Thus $\langle \mathrm{St}^{K'}(p)\rangle_{K'}\subset\subset\langle \mathrm{St}^K(p)\rangle_K$ as required. ▌

3.77. Let K_1 be an open subcomplex of K and let K_1' be the subdivision of K_1 induced by K'. Then besides the two open coverings $\Sigma_K(\langle K_1\rangle)$ and $\sigma_K(\langle K_1\rangle)$ of $\langle K_1\rangle_K$ (see 3.52.) we have two further open coverings, namely $\Sigma_{K'}(\langle K_1'\rangle)$ and $\sigma_{K'}(\langle K_1'\rangle)$ of $\langle K_1\rangle_K = \langle K_1'\rangle_{K'}$ (see 3.71.) where

$$\Sigma_{K'}(\langle K_1'\rangle) = \{\langle K_2'\rangle_{K'};\quad K_2' \text{ is open in } K' \text{ and } \quad K_2' < K_1'\}$$

and

$$\sigma_{K'}(\langle K_1'\rangle) = \{\langle \mathrm{St}^{K'}(p)\rangle_{K'};\ p \in \langle K_1'\rangle_{K'}\}.$$

In view of 3.76. $\sigma_{K'}(\langle K_1'\rangle)$ is a refinement of $\sigma_K(\langle K_1\rangle)$. $\Sigma_{K'}(\langle K_1'\rangle)$ is also a refinement of $\Sigma_K(\langle K_1\rangle)$. For each $\langle K_2'\rangle_{K'} \in \Sigma_{K'}(\langle K_1'\rangle)$ it holds namely that

$$\langle K_2'\rangle_{K'} \subset \langle K_1'\rangle_{K'} = \langle K_1\rangle_K \in \Sigma_K(\langle K_1\rangle).$$

We shall show now that $\Sigma_K(\langle K_1\rangle) \subset \Sigma_{K'}(\langle K_1'\rangle)$.

In fact let $\langle K_2\rangle_K \in \Sigma_K(\langle K_1\rangle)$. K_2 is an open subcomplex of K and $K_2 < K_1$. Therefore by 3.75. and 3.69. the subdivision K_2' of K_2 induced by K' is an open subcomplex of K' and $K_2' < K_1'$. Thus because of 3.71. $\langle K_2\rangle_K = \langle K_2'\rangle_{K'} \in \Sigma_{K'}(\langle K_1'\rangle)$ as required.

4. Convex complexes

4.1. Before the beginning of the discussion let us summarize some basic notions and facts about convex sets (see [1]).

Let R^n be the Euclidean n-space. This should be fixed in this section as well as in the next one.

(a) Let M be a nonempty convex set in R^n. The least dimensional plane R^s in R^n containing M is uniquely defined. We say that R^s is the *carrying plane* of M. s is called the *dimension* of M, it is denoted by dim M. int M denotes the set of the interior points of M with respect to its carrying plane. int $M \neq \emptyset$. The set of the boundary points of M with respect to its carrying plane will be denoted by $M^\cdot$. $M^\cdot$ is called the *boundary* of M.

If M is bounded then every halfline issuing from a point of int M and lying in the carrying plane of M contains one and only one boundary point of M. Therefore each compact convex set M of dimension $\geqq 1$ is the closed convex hull of its boundary; furthermore $M = \mathrm{int}\, M \cup M^\cdot$, $M = \overline{\mathrm{int}\, M}$.

Note that if M and M' are convex sets in R^n such that $M' \subset M^\cdot$ then dim $M' <$ dim M.

In fact let R^s be the carrying plane of M. Since $M^\cdot$ and thus M' do not contain any interior point with respect to R^s, therefore the plane R^s containing M' is not the carrying plane of M'. Hence the carrying plane of M' is lower dimensional than R^s, consequently dim $M' <$ dim $M = s$.

(b) Let M be a subset of R^n. The closed convex hull of M will be denoted by $C(M)$.

The points $a_0, a_1, \ldots, a_r$ of R^n are said to be *linearly independent* if they are not contained in any plane in R^n of dimension $< r$. If $a_0, a_1, \ldots, a_r$ are linearly independent points then $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ are also linearly independent for each

$$0 \leqq i_0 < i_1 < \ldots < i_k \leqq r.$$

Let $a_0, \ldots, a_r$ be linearly independent points of R^n. Then the closed convex hull of the set $\{a_0, \ldots, a_r\}$ is called the *simplex* σ^r with *vertices* $a_0, \ldots, a_r$:

$$\sigma^r = \sigma^r(a_0, \ldots, a_r) = C(\{a_0, \ldots, a_r\}).$$

σ^r determines its vertices. σ^r is an r-dimensional compact convex set.

The simplex σ^p in R^n is a *face* of the simplex σ^r in R^n if each vertex of σ^p is simultaneously a vertex of σ^r. σ^p is a *proper face* of σ^r if σ^p is a face of σ^r but $\sigma^p \neq \sigma^r$. Note that the boundary $\sigma^{\cdot}$ of a simplex σ is the union of its proper faces. Furthermore for any two distinct faces σ' and σ'' of the simplex σ we have $\operatorname{int}\sigma' \cap \operatorname{int}\sigma'' = \emptyset$.

4.2. Definition: A GC complex K (see 3.10.) is called a *convex complex* if the following conditions are satisfied:

(a) The supporting space of K is the Euclidean n-space R^n.

(b) The underlying sets of the elements of K are (nonempty) compact convex sets in R^n.

(c) For any $b \in K$, $\dim b = \dim |b|$ (see 3.1., 3.2. and 4.1. (a)).

(d) For any $b \in K$, $|[\mathrm{Fr}_K b]| = |b|^{\cdot}$ and thus $\operatorname{int}_K b = \operatorname{int}|b|$ (see 3.7. and 4.1. (a)).

We shall show some simple properties of the convex complexes.

4.3. Let K be a convex complex and K_1 a nonempty closed subcomplex of K. Then K_1 is evidently also a convex complex (see also 3.12.).

4.4. Proposition: *Let K be a convex complex. Then the system of the underlying sets of the cells of K is locally finite in* $[K]$.

Proof. We argue by contradiction. Let us suppose the existence of a $p \in [K]$ such that each neighbourhood of p intersects an infinite number of cells of K. Consequently in view of 4.2. (b) and 4.2. (d) there would exist an infinite sequence $p_1, \ldots, p_k, \ldots$ of points different from p with the limit point p such that each pair of points of the sequence is contained in the interior of different cells of K, i.e.: $i \neq j \Rightarrow S_K(p_i) \neq S_K(p_j)$. Let $M = \{p_1, \ldots, p_k, \ldots\}$ and $b \in K$. If $p_i \in |b|$ then $S_K(p_i) \leqq b$ (see 3.23.) and thus by 3.10. (c) $|b| \cap M$ is finite, consequently closed in $[b]$. Hence beceuse of 3.10. (b) M is closed in $[K]$ contradicting the assertion that p is a limit point of $M = M \setminus \{p\}$. ∎

It is immediate from 4.4.:

4.5. The set of the cells of any convex complex is always countable.

4.6. The body $[K]$ of a convex complex K is compact iff K is finite.

4.7. Proposition: *Let K be a convex complex and $b \in K$. Then $|b|$ is the closed convex hull of* $\bigcup\{|b^0|;\ b^0 \leqq b\}$: $|b| = C(|[d(b)]|)$ (see 1.18.). Roughly speaking: b is the closed convex hull of its 0-skeleton.

Proof. If $\dim b = 0$, then the only 0-dimensional face of b is b itself. Hence $\bigcup\{|b^0|;\ b^0 \leqq b\} = |b|$ and thus by the convexity of $|b|$ $C(\bigcup\{|b^0|;\ b^0 \leqq b\}) = |b|$.

Suppose now that $\dim b > 0$, and that the assertion is true for any b' satisfying the condition $\dim b' < \dim b$. Let $M = \bigcup\{|b^0|;\ b^0 \leqq b\}$. Since M is a subset of $|b|$ and $|b|$ is convex, the closed convex hull $C(M)$ of M is contained in $|b| : C(M) \subset |b|$.

Let $b' < b$ and $M' = \bigcup\{|b^0|;\ b^0 \leqq b'\}$. Then $\dim b' < \dim b$, $M' \subset M$ and therefore by the assumption above $|b'| = C(M') \subset C(M)$. This implies by 4.2. (d) $|b|^{\cdot} \subset C(M)$, and thus $|b| \subset C(M)$ (see 4.1. (a)). Hence $C(M) = |b|$ as required. ∎

4.8. Note as a direct consequence of 4.7. that for any convex complex K and for each $b \in K$ $d_K(b) \neq \emptyset$.

4.9. A 0-dimensional convex complex K is uniquely determined by its supporting space R^n and by the underlying set of its body. It holds namely

$$|K| = \{(R^n, \{p\}, 0);\ p \in |[K]|\}.$$

4.10. Remark as a consequence of 4.9. that for any convex complex K and for each $b \in K$ $d(b)$ is determined by $|[d(b)]|$. $d(b)$ is namely the underlying set of a 0-dimensional (see 4.8.) closed subcomplex of K. Hence by 4.3. $d(b)$ is the underlying set of a 0-dimensional convex complex.

4.11. Let K be a convex complex and $b, b' \in K$. Then clearly

$$|[d(b)]| \subset |[d(b')]| \Leftrightarrow d(b) \subset d(b').$$

4.12. Let K be a convex complex, $b \in K$ and $N \subset |[d(b)]|$. Then there exists obviously a unique $M \subset d(b)$ such that $|[M]| = N$. It is namely

$$M = \{(R^n, \{p\}, 0);\ p \in N\}.$$

Now we shall consider some special convex complexes the cells of which belong to a given family of compact convex sets.

4.13. Definition: Let $\mathcal{K}_0 = \{Q_\gamma;\ \gamma \in \Gamma\}$ be a nonvoid system of nonempty compact convex sets of R^n equipped with a partial order $<'$. $\mathcal{K}_0$ is called a *complexogenic structure* in R^n if for any $Q \in \mathcal{K}_0$

$$(6) \qquad Q^{\cdot} = \bigcup\{Q';\ Q' <' Q\}.$$

In the case $Q' <' Q$ Q' is said to be the *face* with respect to $\mathcal{K}_0$ of Q.

4.14. The system $\mathcal{K}_0^{\sigma, n}$ of the simplexes of R^n — equipped with the partial order: $\sigma' <' \sigma$ with respect to $\mathcal{K}_0^{\sigma, n}$ if σ' is a proper face of σ in the usual sense (see 4.1. (b)) — forms obviously a complexogenic structure.

In the remainder of this section let $\mathcal{K}_0$ be always a complexogenic structure in R^n.

4.15. Note that if $Q, Q' \in \mathcal{K}_0$ and $Q' <' Q$ then $Q' \subset Q^{\cdot}$ and thus $\dim Q' < \dim Q$ (see 4.1. (a)).

4.16. Definition: Two members Q and Q' of $\mathcal{K}_0$ are said to be *properly joined* with respect to $\mathcal{K}_0$ if either $Q \cap Q' = \emptyset$ or $Q \cap Q' \in \mathcal{K}_0$ and $Q \cap Q' \leqq' Q$; $Q \cap Q' \leqq' Q'$.

4.17. Let $Q, Q' \in \mathcal{K}_0$ and suppose that Q and Q' are properly joined with respect to $\mathcal{K}_0$. Then $Q' \subset Q$ implies $Q' = Q' \cap Q$ and thus $Q' \leqq' Q$.

4.18. Let $\mathcal{W}$ be a subset of $\mathcal{K}_0$ satisfying the condition: For any Q, $Q'\in\mathcal{W}$, Q and Q' are properly joined with respect to $\mathcal{K}_0$ Then by 4.17. and 4.13. (6)

$$Q'\subset Q \Leftrightarrow Q' \leqq' Q$$

for Q, $Q'\in\mathcal{W}$.

Now we shall consider convex complexes composed from certain members of $\mathcal{K}_0$.

4.19. Proposition: *Let $\mathcal{W}$ be a nonempty subset of $\mathcal{K}_0$ satisfying the conditions:*

(a) *$\mathcal{W}$ is a locally finite covering of $\bigcup\{Q;\, Q\in\mathcal{W}\}$.*
(b) *$Q\in\mathcal{W}$ and $Q'\leqq' Q$ imply $Q'\in\mathcal{W}$.*
(c) *For Q, $Q'\in\mathcal{W}$, Q and Q' are properly joined* (with respect to $\mathcal{K}_0$).

Let $Q^* = (R^n, Q, \dim Q)$ for $Q\in\mathcal{W}$. Then the abstract cell complex K consisting of the elements Q^* ($Q\in\mathcal{W}$) is a convex complex.

Proof. First of all observe that by (c), 4.18. and 4.15. Q, $Q'\in\mathcal{W}$, $Q'\subset Q$ and $Q'\neq Q$ imply $\dim Q' < \dim Q$. Hence the conditions (a) and (b) of 3.3. are satisfied, K is indeed an abstract cell complex.

Let us note that $|Q^*| = Q$ for $Q^*\in K$.

We are going to show that K is a GC complex.

Each $Q\in\mathcal{W}$ is a compact set in the Hausdorff space R^n. Hence 3.10. (a) is satisfied.

By (a) and 3.11. 3.10. (b) is also satisfied.

3.10. (c) holds, since $\mathcal{W}$ is locally finite and the members of $\mathcal{W}$ are compact sets.

By (b), (c) 4.18. and 4.13. (6) $\mathrm{int}_K Q^* = \mathrm{int}\, Q$ for $Q\in\mathcal{W}$, and therefore $\mathrm{int}_K Q^*\neq\emptyset$, 3.10. (d) is also satisfied.

Let Q, Q' be elements of $\mathcal{W}$ satisfying the condition $Q\cap Q'\neq\emptyset$. Then by (c) and (b) $Q\cap Q'\in\mathcal{W}$. Hence 3.10. (e) also holds.

K is a convex complex. 4.2. (a), 4.2. (b) and 4.2. (c) follow namely directly from the definition of K. By (b), (c), 4.18. and 4.13. (6) 4.2. (d) is also satisfied.

The proposition has been proved. ▌

5. Geometric simplicial complexes

5.1. Definition: Let $\mathcal{K}_0$ be the family of simplexes in R^n equipped with the usual partial order, namely $\sigma' <' \sigma$ if σ' is a proper face of σ. $\mathcal{K}_0$ is a complexogenic structure in R^n (see 4.14.).

Assign to each $\sigma\in\mathcal{K}_0$ the triple $\sigma^* = (R^n, \sigma, \dim\sigma)$. The cells σ^* obtained in this way should be called *simplicial cells.*

5.2. Definition: A nonempty set K of simplicial cells is said to be a *geometric simplicial complex* if the following conditions are satisfied:

(a) $\{\sigma;\, \sigma^*\in K\}$ is a locally finite covering of $\bigcup\{\sigma;\, \sigma^*\in K\}$.
(b) $\sigma^*\in K$, $\sigma'\in\mathcal{K}_0$ and $\sigma' <' \sigma$ imply $\sigma'^*\in K$.
(c) If σ^*, $\sigma'^*\in K$, then σ and σ' are properly joined with respect to $\mathcal{K}_0$.

5.3. As 4.19. shows each geometric simplicial complex is a convex complex.

5.4. THEOREM: *A complex K is a geometric simplicial complex iff it is convex and general simplicial* (see 2.1.).

PROOF. Suppose that K is a geometric simplicial complex. By 4.19. K is a convex complex, and obviously $|\sigma^*| = \sigma$ for $\sigma^* \in K$.

α) First of all we show that for

$$\sigma^* = (R^n, \sigma^r(a_0, \ldots, a_r), r) \in K$$

$$d(\sigma^*) = \{(R^n, \{a_i\}, 0); \quad i = 0, \ldots, r\}. \tag{7}$$

In fact let $\sigma^{*0} \in d(\sigma^*)$. Then $\sigma^{*0} \leqq \sigma^*$ in K and thus by 4.18.

$$|\sigma^{*0}| \leqq' |\sigma^*|. \tag{8}$$

Moreover $\dim |\sigma^{*0}| = \dim \sigma^{*0} = 0$, consequently $|\sigma^{*0}|$ is a singleton and thus by (8) $|\sigma^{*0}| = \{a_i\}$,

$$\sigma^{*0} = (R^n, \{a_i\}, 0) \quad \text{for some} \quad i.$$

Conversely for $i = 0, 1, \ldots, r$ $\{a_i\} \leqq' |\sigma^*|$ therefore by 5.2. (b) $(R^n, \{a_i\}, 0) \in K$. Moreover $(R^n, \{a_i\}, 0) \leqq \sigma^*$ in K and since $\dim \{a_i\} = \dim (R^n, \{a_i\}, 0) = 0$, we have $(R^n, \{a_i\}, 0) \in d(\sigma^*)$ as required.

β) Note that from 5.2. (c) and 4.18. we get immediately the formula

$$\sigma^{*\prime} \leqq \sigma^* \quad \text{in} \quad K \Leftrightarrow |\sigma^{*\prime}| \leqq' |\sigma^*|$$

for $\sigma^*, \sigma^{*\prime} \in K$.

γ) Now 2.1 (a) and 2.1. (b) follow immediately from α) and from the fact that each simplex in R^n is uniquely determined by the set of its vertices.

We are going to show that 2.1. (c) is also satisfied.

In fact let $\sigma^* \in K$ where $|\sigma^*| = \sigma^r(a_0, \ldots, a_r)$ and let M be a nonempty subset of $d(\sigma^*)$. Then by α) $M = \{(R^n, \{a_{i_j}\}, 0); j = 0, 1, \ldots, k\}$ where $0 \leqq \leqq i_0 < i_1 < \ldots < i_k \leqq r$. Let $\sigma'^k = \sigma'^k(a_{i_0}, \ldots, a_{i_k})$. Then $\sigma'^k \leqq' |\sigma^*|$ and thus by 5.2. (b) $\sigma'^{k*} \in K$. Because of β) $\sigma'^{k*} \leqq \sigma^*$ in K. Simultaneously by (7) $d(\sigma'^{k*}) = M$ as required.

Hence K is indeed convex and general simplicial.

δ) Now suppose in the remainder of the proof that K is a complex which is simultaneously convex and general simplicial.

Let $b \in K$ and let $\dim b = r$. Then Card $d(b) = r+1$. Since each 0-dimensional convex set is a singleton therefore $|[d(b)]| = \cup\{|b^0|; b^0 \leqq b\}$ consists of r+1 = $\dim |b| + 1$ points: $a_0, \ldots, a_r$ and by 4.7. $|b|$ is the closed convex hull of $M = \{a_0, \ldots, a_r\}$. $a_0, \ldots, a_r$ must be linearly independent since otherwise $|b| = C(\{a_0, \ldots, a_r\})$ would lie in a plane R^t for which $t < r$. Hence $|b| = C(M) \subset R^t$ would hold and this yields $\dim |b| \leqq t < r$ contradicting the assumption $\dim b = \dim |b| = r$.

Thus $a_0, \ldots, a_r$ are linearly independent and $|b| = C(M) = \sigma^r(a_0, \ldots, a_r)$, $|b|$ is a simplex with the vertices $a_0, \ldots, a_r$. The set of the vertices of $|b|$ is $|[d(b)]|$.

ε) We show finally that K is a geometric simplicial complex.

As we have seen in δ) the elements of K are simplicial cells. Because of 4.4., 5.2. (a) is satisfied.

To prove 5.2. (b) let $b\in K$ and $\sigma' \leqq' |b|$. Let N be the set of the vertices of σ'. Since the set of the vertices of $|b|$ is $|[d(b)]|$ (see δ)) therefore $\sigma' \leqq' |b|$ implies $\emptyset \neq N \subset |[d(b)]|$. Let M be the subset of $d(b)$ for which $|[M]| = N$ (see 4.12.). Evidently $M \neq \emptyset$. Choose $b' \in K$ such that $b' \leqq b$ and $d(b') = M$. 2. (c) shows the existence of such a b'. $|b'|$ and σ' have the same vertices. Consequently $|b'| = \sigma'$. Hence $\sigma'^* = b'$, $\sigma'^* \in K$ and 5.2. (b) is satisfied.

We need only to prove 5.2. (c).

Let b, $b' \in K$. If $|b| \cap |b'| = \emptyset$ then $|b|$ and $|b'|$ are evidently properly joined with respect to $\mathcal{K}_0$. Suppose $|b| \cap |b'| \neq \emptyset$. Then by 3.10. (e) $|b| \cap |b'| = |[\bar{b} \cap \bar{b}']|$ (see also 1.15.) and $\bar{b} \cap \bar{b}' \neq \emptyset$. Whence 2.5. shows the existence of a $b'' \in K$ such that $\bar{b} \cap \bar{b}' = \bar{b}''$. Consequently

$$(9) \qquad |b| \cap |b'| = |[\bar{b} \cap \bar{b}']| = |[\bar{b}'']| = |b''|$$

(see (3.6.) and simultaneously $b'' \leqq b$ and $b'' \leqq b'$. Therefore by 1.20. $d(b'') \subset d(b)$ and $d(b'') \subset d(b')$. Hence $|[d(b'')]| \subset |[d(b)]|$ and $|[d(b'')]| \subset |[d(b')]|$. Thus by δ) the set of the vertices of $|b''|$ is a subset of the vertices of $|b|$ and of $|b'|$, that is $|b''|$ is the face of $|b|$ and of $|b'|$ (see 4.1. (b)), $|b''| \leqq' |b|$ and $|b''| \leqq' |b'|$ and this yields by (9) $|b| \cap |b'| \leqq' |b|$, $|b| \cap |b'| \leqq' |b'|$ so that $|b|$ and $|b'|$ are properly joined with respect to $\mathcal{K}_0$ as required.

The theorem has been proved. ∎

5.5. To characterize the class of the geometric simplicial complexes as a subclass of the convex complexes it is enough to suppose that the cells are simplicial and 5.2. (c) is satisfied.

PROPOSITION: *Let K be a convex complex. K is geomertic simplicial iff*

(a) *For $b\in K$, b is a simplicial cell;*
(b) *For b, $b' \in K$ $|b|$ and $|b'|$ are properly joined with respect to $\mathcal{K}_0$.*

PROOF. The necessity of (a) and (b) is obvious.

To prove the sufficiency 4.4. shows that 5.2. (a) is satisfied. Thus we only need to prove 5.2. (b).

Let $b\in K$ and $\sigma' \leqq' |b|$. Let $q \in \operatorname{int} \sigma'$. Then $q \in |b| \subset [K]$. Let $b' = S_K(q)$ (see 3.22.). Then by 3.24. and 4.2. (d) $q \in \operatorname{int}_K b' = \operatorname{int} |b'|$. Thus $\operatorname{int}_K b' \cap |b| \neq \emptyset$ and therefore by 3.33. $b' \leqq b$, $|b'| \subset |b|$. Because of (b) and 4.17. this implies $|b'| \leqq' |b|$. Hence $|b'|$ and σ' are faces of the same simplex $|b|$ and they have a common interior point q. Consequently $|b'| = \sigma'$ (see 4.1. (b)), $b' = \sigma'^*$, $\sigma'^* \in K$ as required. ∎

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ON A FUNCTION CONCERNING SECOND ORDER RECURRENCES

By

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Let A and B be fixed rational integers statisfying the conditions $B\neq 0$ and $A^2\neq 4B$, and let R be a second order recurrence defined by $R_0=0$, $R_1=1$ and $R_{n+2}=A\cdot R_{n+1}-B\cdot R_n$. It is well known that the n^{th} term of the sequence R may be expressed in the form

$$(1)\qquad R_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\sqrt{D}}$$

where α and β are the roots of the equation

$$(2)\qquad x^2-Ax+B=0$$

and $D=A^2-4B$ is the discriminant of this equation.

We define the rank and period for a prime p in the sequence R. The integer $r=r(p)>0$ is called the rank of the occurrence of the prime p in the sequence R if $p|R_r$ and $p\nmid R_n$ for $0<n<r$, furthermore $\bar{r}=\bar{r}(p)>0$ is called the period of the sequence R modulo p if $\bar{r}$ is the smallest positive integer for which $R_{\bar{r}}\equiv 0$ and $R_{\bar{r}+1}\equiv 1 \bmod p$. It is well known that if $p\nmid B$ then $r(p)$ and $\bar{r}(p)$ exist (see e.g. in [1] and [2] respectively), and

$$(3)\qquad r(p)|(p-(D/p)),$$

$$(4)\qquad \bar{r}(p)|2d\cdot(p-(D/p)),$$

where (D/p) is Legendre's symbol in the case $p\nmid D$, $(D/p)=0$ if $p|D$ and d is the order of B mod p (i.e. d is the smallest positive integer for which $B^d\equiv 1$ mod p). (3) was proved by H. J. A. DUPARC [3] and (4) by P. BUNDSCHUH and J. S. SHIUE [2].

Let us introduce the functions

$$g(p)=\frac{p-(D/p)}{r(p)}\quad\text{and}\quad \bar{g}(p)=\frac{2d\cdot(p-(D/p))}{\bar{r}(p)}$$

for sepuences R, for which α/β is not a root of unity. These have meaning for the prime p if $p \nmid B$ and their values are integers on account of (3) and (4). D. JARDEN studied in [4] the function $g(p)$ related to the Fibonacci sequence and he proved that, for the special sequence R in which $A = -B = 1$, the function $g(p)$ is unbounded.

In this paper we generalize D. Jarden's theorem, we show that it also holds for the general sequence R, furthermore we give an upper estimate for the values of $g(p)$ and we prove that the function $\bar{g}(p)$ has the same properties as the function $g(p)$.

THEOREM.* *The function*

$$g(p) = \frac{p-(D/p)}{r(p)} \text{ is unbounded and } g(p) < c \cdot \frac{p}{\log p}$$

or all sufficiently large primes p where c is a constant depending only on A and B.

PROOF OF THE THEOREM. Let us suppose that $g(p)$ is a bounded function and so it has only a finite number of distinct values: $k_1, k_2, \ldots, k_s$. Let $n_0 > 1$ be any integer and let us consider the integer

$$(5) \qquad n = (n_0 k_1 - 1)\cdot(n_0 k_1 + 1)\cdot \ldots \cdot(n_0 k_s - 1)\cdot(n_0 k_s + 1) + n_0$$

for which clearly $n > n_0$. A. SCHINZEL [5] proved that there is an integer h such that for every integer $m > h$ there exists a prime p for which $r(p) = m$. Let us choose the integer n_0 satisfying the conditions $n_0 > h$ and $n_0 > |D|$. Now, by the theorem of SCHINZEL, there is a prime p for which $r(p) = n$ and so

$$g(p) = \frac{p-(D/p)}{n}.$$

From this follows

$$p = n \cdot g(p) + (D/p).$$

But $p \geqq n - 1 > n_0 > |D|$ implies $(D/p) = 1$ or $(D/p) = -1$, furthermore on account of our supposition $g(p) = k_1, g(p) = k_2, \ldots$ or $g(p) = k_s$, therefore

$$p = n \cdot k_i + 1 \quad \text{or} \quad p = n \cdot k_i - 1$$

for some integer i $(0 < i \leqq s)$. However, neither $n \cdot k_i + 1$ nor $n \cdot k_i - 1$ can be a prime because by (5), they are divisible by $n_0 k_i + 1$ and $n_0 k_i - 1$ respectively. This is a contradiction and so the function $g(p)$ cannot be bounded. Thus the first part of the statement of the Theorem is true.

* Note that after the manuscript of this paper had been given in, there appeared a paper by C. L. STEWART [On divisors of Fermat, Fibonacci, Lucas and Lehmer numbers, *Proc. London Math. Soc.*, **35** (1977), 425–447], from which the first part of our Theorem can also be deduced.

Now we prove that $g(p) < c \cdot \frac{p}{\log p}$ for all sufficiently large primes p.
Let $p \geqq 3$. If $r = r(p)$ then $p | R_r$ which implies

$$\frac{\alpha^r - \beta^r}{\sqrt{D}} \geqq p .$$

But α and β are the roots of the equation (2) therefore

$$|\alpha|, \ |\beta| \leqq \frac{|A| + |\sqrt{D}|}{2} = c_1 .$$

From this follows that

$$p \leqq \frac{|\alpha|^r + |\beta|^r}{|\sqrt{D}|} \leqq \frac{2c_1^r}{|\sqrt{D}|} \leqq 2c_1^r$$

since D is a rational integer. From this we have (as $p \geqq 3$, and so $c_1 > 1$)

$$r \geqq \frac{\log \frac{p}{2}}{\log c_1}$$

and

$$g(p) = \frac{p - (D/p)}{r} \leqq \frac{p+1}{\frac{\log \frac{p}{2}}{\log c_1}} = \log c_1 \cdot \frac{p+1}{\log \frac{p}{2}} < c \cdot \frac{p}{\log p}$$

where we can shoose

$$c = (1+\varepsilon) \log c_1 = (1+\varepsilon) \log \frac{|A| + |\sqrt{D}|}{2}$$

for $p > p_0(\varepsilon)$.

COROLLARY. If $(D/p) = -1$, then

$$\bar{g}(p) \leqq 4g(p) < c' \cdot \frac{p}{\log p} .$$

PROOF OF THE COROLLARY. By (3) and by the definition of d we have $r(p) | (p+1)$ and $d | (p-1)$, and so

(6) $$(r(p), d) \leqq (p+1, p-1) = 2 .$$

On the other hand O. WYLER [6] has proved that if $p \nmid B$ then $\bar{r}(p) = k \cdot [d, r(p)]$ where $k = 1$ or 2, and $[d, r(p)]$ denotes the least common multiple of d and $r(p)$. This implies together with (6) and the Theorem the inequality

$$\bar{g}(p) = \frac{2d(p+1)}{\bar{r}(p)} = \frac{2d(p+1)}{k\,[d, r(p)]} =$$

$$= \frac{2d(p+1)\,(d, r(p))}{kdr(p)} \leqq \frac{4(p+1)}{r(p)} = 4g(p) < c' \frac{p}{\log p}$$

which proves our statement.

Finally we note that in the case $(D/p) = 1$, the trivial inequality

$$\bar{g}(p) = \frac{2d(p-1)}{k\,[d, r(p)]} \leqq 2(p-1)$$

holds.

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ON THE THEORY OF FROBENIUS-GROUPS

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In this work we deal with Frobenius-groups. The theory of these groups is now well known enough. Burnside's fundamental result on the structure of the complement and Thompson's theorem on the nilpotency of the kernel represent some of the most significant features of the theory. In our paper we give a characterization of Frobenius-groups with abelian kernel and abelian complement. The method used is elementary, and leads to a complete characterization of the group-type considered. Some other result of similar kind are also included. Our notations are standard. See GORENSTEIN, [1] or HUPPERT [2].

In this paper all groups are finite.

THEOREM 1. *G is a Frobenius-group with abelian complement and abelian kernel if and only if:*

(i) *If $\beta \notin C_G(\alpha)$, then $C_G(\alpha) \cap C_G(\beta) = E$ for all $\alpha, \beta \in G$ with $\alpha \neq e \neq \beta$.*

(ii) *If $C_G(\alpha) \cap C_G(\beta) = E$, and $C_G(\alpha)$ is not a normal subgroup in G, $C_G(\beta)$ is not a normal subgroup in G then $N_G(C_G(\alpha)) \cap \cap N_G(C_G(\beta)) = E$.*

(iii) $Z(G) = E$.

For the proof we need the following Lemma, which presents known material

LEMMA. In a group G there are no proper subgroups H and K with following properties: $H \cap H^x = E$ for all $x \in G \setminus H$, $K \cap K^y = E$ for all $y \in G \setminus K$, and $H^x \cap K^y = E$ for all $x, y \in G$.

PROOF OF THEOREM 1. We first suppose that G is satisfying the properties (i), (ii), (iii). We show, G is a Frobenius-group with abelian complement and abelian kernel.

STATEMENT 1. In G there are no elements α, β with $\alpha \neq e \neq \beta$, $\alpha \notin C_G(\beta)$, and $C_G(\alpha) \lhd G$, $C_G(\beta) \lhd G$.

PROOF. Suppose the contrary. By (i) $C_G(\alpha)\cap C_G(\beta)=E$, so $C_G(\alpha)\cdot C_G(\beta)=C_G(\alpha)\times C_G(\beta)$, hence $\beta\in C_G(\alpha)$ which is a contradiction.

STATEMENT 2. If $\beta\in G$, $C_G(\beta)$ is not a normal subgroup in G, then $N_G(C_G(\beta))=C_G(\beta)$.

PROOF. Suppose the contrary. There exists an element x with $x\in N_G(C_G(\beta))$ and $x\notin C_G(\beta)$. By (i) $C_G(x)\cap C_G(\beta)=E$ holds.

(a) If $C_G(x)$ is not a normal subgroup in G, by (ii) $N_G(C_G(x))\cap N_G(C_G(\beta))=E$ but $x\neq e$, which is a contradiction.

(b) If $C_G(x)\lhd G$, let D denote $D\overset{\text{def}}{=}C_G(x)\cap N_G(C_G(\beta))$, $D\lhd N_G(C_G(\beta))$, $x\in D$, $C_G(\beta)\lhd N_G(C_G(\beta))$, $D\cap C_G(\beta)=E$, so $D\cdot C_G(\beta)=D\times C_G(\beta)$, hence $x\in C_G(\beta)$ which is a contradiction.

STATEMENT 3. If $\beta\in G$, $C_G(\beta)$ is not a normal subgroup in G, and $N_G(C_G(\beta))=C_G(\beta)$, then $C_G(\beta)^y\cap C_G(\beta)=E$ for all $y\in G\setminus C_G(\beta)$.

PROOF. We know $C_G(\beta)^y=C_G(\beta^y)$.

(a) If $\beta^y\notin C_G(\beta)$, then by (i) $C_G(\beta^y)\cap C_G(\beta)=E$.

(b) If $\beta^y\in C_G(\beta)$, since $y\notin C_G(\beta)=N_G(C_G(\beta))$ there exists an element b with $b\in C_G(\beta^y)$ and $b\notin C_G(\beta)$. We consider $C_G(b)$, $\beta^y\in C_G(b)$ and $\beta^y\in C_G(\beta)$, but by (i) $C_G(b)\cap C_G(\beta)=E$ which is a contradiction.

We now return to the proof of Theorem 1.

Let β be an element of G with $C_G(\beta)$ is not a normal subgroup in G. By (iii) and by Statement 1, β exists.

Put

$$T\overset{\text{def}}{=}\bigcup_{g\in G} C_G(\beta)^g.$$

(a) If there exists an element δ with $\delta\in G\setminus T$ and $C_G(\delta)$ is not a normal subgroup in G, then we show, that $C_G(\beta)^a\cap C_G(\delta)^b=E$ holds for all $a, b\in G$. By $\delta\notin T$, $C_G(\beta)ab^{-1}\cap C_G(\delta)=E$. The Statements 2, 3, are applicable for $C_G(\beta)$ and $C_G(\alpha)$. A contradiction by the Lemma.

(b) If $C_G(\delta)\lhd G$ for all $\delta\in G\setminus T$, consider an arbitrary $\delta\in G\setminus T$. We show $T\cup C_G(\delta)=G$. Suppose the contrary. Then there exists a γ with $\gamma\in G\setminus(T\cup C_G(\delta))$, so $C_G(\gamma)\lhd G$. By Statement 1, this is a contradiction.

$T\cap C_G(\delta)=E$ holds by (ii) and by the choice of δ. By application of Statements 2, 3,

$$|C_G(\delta)|=|G|-|T|+1=\frac{|G|}{|C_G(\beta)|}.$$

Using (ii) it follows: G is a Frobenius-group with kernel $C_G(\delta)$ and complement $C_G(\beta)$.

It remains to prove that, $C_G(\beta)$ and $C_G(\delta)$ are abelian. It is sufficient to show that, $C_G(a)$ is abelian for all $a\in G$, $a\neq e$.

Suppose the contrary. There exists an element a of G for which $C_G(a)$ is non-abelian. Then there exists an element b with $b \in C_G(a)$ and $C_G(b) \neq \neq C_G(a)$, but $C_G(b) < C_G(a)$, so there is a g with $g \in C_G(a)$ and $g \notin C_G(b)$. By (ii) $C_G(g) \cap C_G(b) = E$ follows. But $a \neq e$, which is a contradiction.

The converse statement is trivial. So the proof is complete.

THEOREM 2. *Let G be a group with the following properties: Let J denote: $J \stackrel{\text{def}}{=} \{i/1 \leq i \leq n\}$. Let a_i be an element of G for all $i \in J$. Put $H_k \stackrel{\text{def}}{=} C_G(a_k)$ for all $k \in J$.*

Suppose:

(i) $\bigcup_{k=1}^{n} H_k = G,\ H_i \cap H_j = E\ i \neq j,$

(ii) *$C_G(u_k) \leq C_G(a_k)$ for all $u_k \in C_G(a_k)$, and for all $k \in J$,*

(iii) *If H_i is not a normal subgroup in G, H_j is not a normal subgroup in G, $i, j \in J$, $i \neq j$, then $N_G(H_i) \cap N_G(H_j) = E$,*

(iv) $Z(G) = E$.

Then there exist $l, k \in J$ with $H_l \lhd G$, $G = H_l \cdot N_G(H_k)$ and $H_l \cap N_G(H_k) = = E$, and G is a Frobenius-group with kernel H_l and complement $N_G(H_k)$.

PROOF. $n > 1$ holds, by (i) and by (iv).

STATEMENT 1. If $H_i \lhd G$, $H_j \lhd G$ for some $i, j \in J$ then $i = j$.

PROOF. See the Statement 1, of Theorem 1.

STATEMENT 2. If H_k is not a normal subgroup in G, then $H_k \cap H_k^y = E$ for all $y \in G \backslash N_G(H_k)$.

PROOF. $H_k = C_G(a_k)$, by (i) and by (ii) $C_G(a_k^y) \leq H_l$ holds for some $l \in J$. If $l = k$, then $C_G(a_k^y) \leq H_k$, hence $H_k^y = H_k$. But $y \notin N_G(H_k)$. A contradiction. So $l \neq k$. By (i) $H_l \cap H_k = E$, hence $H_k \cap H_k^y = E$.

Statement 2. has a consequence.

STATEMENT 3. If $N_G(H_k) = H_k$ for some $k \in J$, then $H_k \cap H_k^z = E$ for all $z \in G \backslash H_k$.

STATEMENT 4. If H_k is not a normal subgroup in G for some $k \in J$, and $N_G(H_k) \neq H_k$ then $N_G(H_k) \cap N_G(H_k)^x = E$ for all $x \in G \backslash N_G(H_k)$.

PROOF. Let z be an element with $z \in N_G(H_k) \cap N_G(H_k)^x$, by Statement 2. $C_G(a_k) \cap C_G(a_k)^x = E$. Hence by (i) and by (ii) $C_G(a_k)^x \leq H_l$ for some $l \in J$, so $H_l \cap H_k = E$.

(a) If $z \notin N_G(H_l)$, by Statement 2, $H_l \cap H_l^z = E$. But $C_G(a_k^x) \leq H_l$, and $z \in N_G(H_k)^x$, a contradiction.

(b) If $z \in N_G(H_k) \cap N_G(H_l)$ and H_k is not a normal subgroup in G, then H_l is not a normal subgroup in G. Namely $C_G(a_k^x) \leq H_l$ $C_G(a_k^x) x^{-1} = H_k$ and $H_k \cap H_l = E$ imply the assertion. So $N_G(H_k) \cap N_G(H_l) = E$, hence $z = e$.

We now return to the proof of Theorem 2.

Let $k \in J$ be, with H_k is not a normal subgroup in G.
Define

$$T \overset{\text{def}}{=} \bigcup_{g \in G} N_G(H_k)^g .$$

(a) If there exists a b with $b \in G \setminus T$ and $b \in H_l$ for some $l \in J$, H_l is not a normal subgroup in G, we show:

$$(**) \qquad A \overset{\text{def}}{=} N_G(H_l)^{g_1} \cap N_G(H_k)^{g_2} = E$$

holds for all $g_1, g_2 \in G$.
Suppose $t \in A$, define

$$v \overset{\text{def}}{=} t g_2^{-1}, \ g_3 \overset{\text{def}}{=} g_1 g_2^{-1} \quad \text{so} \quad v \in N_G(H_l)^{g_3} \cap N_G(H_k) .$$

By (iii) $a_l^{g_3} \in H_j$ and $C_G(a_l^{g_3}) \leq H_j$ for some $j \in J$.

(α) If $v \notin N_G(H_j)$, by Statement 2, $H_j \cap H_j^v = E$. But $v \in N_G(C_G a_l^{g_3})$. A contradiction.

(β) If $v \in N_G(H_j) \cap N_G(H_k)$, $j \neq k$ then H_j is not a normal subgroup in G, H_j is not a normal subgroup in G since $C_G(a_l^{g_3}) \leq H_j$, $C_G(a_l^{g_3})^{g_3^{-1}} = H_l$ and H_l is not a normal subgroup in G. Hence by (iii) $N_G(H_k) \cap N_G(H_j) = E$. Thus $v = e$. Hence $t = e$ follows.

By Statement 4. and by (**), the Lemma of Theorem 1. is applicable for $N_G(H_k)$ and $N_G(H_j)$. A contradiction.

(b) If for all $b \in G \setminus T$, $H_l \lhd G$ holds for the suffix l with $b \in H_l$, so consider an arbitrary b with $b \in G \setminus T$. We show: $T \cup H_l = G$.

Suppose the contrary. Then there exists a c with $c \in G \setminus (T \cup H_l)$. By i, $c \in H_m$ for some $m \in J$ $m \neq l$, so $H_m \cap H_l = E$, $H_m \lhd G$, which is impossible by Statement 1.

We prove: $T \cap H_l = E$.

Suppose the contrary. Then there exists a k with $k \in J$ and $H_l \cap \cap N_G(H_k) \overset{\text{def}}{=} K \neq E$. By $K \leq N_G(H_k)$, we get, that $L \overset{\text{def}}{=} H_k K$ is a subgroup of G. $H_k \lhd L$ and $K \lhd L$ hold. By $H_l \lhd G$ and by $H_k \cap H_l = E$, using $K \leq H_l$. $L = K \times H_k$ follows. Hence $K \leq C_G(a_k)$ holds. So $K \leq H_k$ follows. Thus $K \leq \leq H_k \cap H_l$. A contradiction. Now it follows at once, that G is a Frobenius-group with kernel H_l, and complement $N_G(H_k)$. So the proof is complete.

It is known, that in a Frobenius-group the complement is a π-Hall subgroup.

THEOREM 3. *Let G be a Frobenius-group with the π-Hall complement H, and with kernel F. We prove, that G satisfies the π-Sylow property.*

PROOF. Suppose the contrary. Let G be a counter-example of smallest order. Then there is a subgroup L, with $|L| / |H|$ and $L \leq H^x$ is false for all $x \in G$. Choose such an L with the smallest possible order.

Let L_0 be a maximal subgroup of L. $|L_0|/|H|$, hence by the minimality of L, $L_0 \leq H^b$ for some $b \in G$. Without loss of the generality we may assume, that $L_0 \leq H$. $L_0 \leq H \cap L$ $L \neq H \cap L$. Thus $L_0 = H \cap L$. We show, that L is a Frobenius-group with complement $H \cap L$.

Let $a \in L$ be with $a \notin H \cap L$. Using $(H \cap L)^a \leq H^a$ and $H \cap H^a = E$, $(H \cap L) \cap (H \cap L)^a = E$ follows. Put

$$F_0 \stackrel{\text{def}}{=} L \setminus \bigcup_{a \in L} (H \cap L)^{\#a}.$$

Choose b with $b \in F_0$, $b \neq e$. $|\langle b \rangle|/|L|$ hence $|\langle b \rangle|/|H|$, by the minimality of L, $\langle b \rangle \leq H^c$ holds for some $c \in G \setminus H$. We show $H^c \cap L \subset F_0$ (as a subset). Suppose the contrary. Then there exists a $d \neq e$ with $d \notin F_0$, $d \in H^c \cap L$. Thus $d \in \in (H \cap L)^a$ for some $a \in L$, $a \notin H$. $(H \cap L)^a = H^a \cap L$, $d \in H^c \cap L \cap H^a$. Thus $d^{a^{-1}} \neq e$ and $d^{a^{-1}} \in H \cap H^{ca^{-1}}$. Hence $H = H^{ca^{-1}}$, $b \in H^a \cap L$. Consequently $b \in \in F_0 \cap (H^a \cap L)$. A contradiction.

One finds easily, that $(H^c \cap L) \cap (H^c \cap L)^a = E$ for all $a \in L \setminus (H^c \cap L)$. Observing, that F_0 is an invariant subset, the Lemma of Theorem 1. is applicable for L with $H \cap L$ and $H^c \cap L$. A contradiction.

REMARK. In this proof we did not use the fact, that the kernel is a normal subgroup.

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References

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LINEARLY COMPACT ALGEBRAS

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Introduction

In the first part of the present paper we generalize some results, known in the case of linearly compact modules and rings. We define the class of K-compact universal algebras of a fixed type, as a natural generalization of lineraly compact rings, groups, etc.

After the preparations we shall prove that a K-compact algebra is isomorphic and homeomorphic with the inverse limit of discrete algebras.

In the second part of this paper, using the results of **§1**, we study some properties of K-compact semigroups. We shall characterize those K-compact semigroups wich are inverse limit of torsion groups. It will be proved that a K-compact semigroup is ω-nilpotent if and only if it is the inverse limit of its nilpotent factor-semigroups.

The notion of linearly compact vector-spaces was first introduced by LEFSCHETZ [3]. ZELINSKY [9] [10] has discussed some properties of linearly compact modules and rings. H. LEPTIN [4] [5] has proved, among others, a generalization of the Wedderburn-Artin's theorem: The semisimple linearly compact rings are exactly the directe sums of endomorphism rings of vector-spaces. R. WIEGANDT [6] [7] [8] has investigated the semisimple, the transfinitly nilpotent and the regular linearly compact rings.

Finally I wish to express my thanks to R. WIEGANDT and ZS. NAGY for theirs remarks and valuable advices.

§1.

In the following an *algebra* means an element of a variety of algebras of a fixed type. A topological space is always a Hausdorff-space. We say, that $K\subseteq Q$ if $a\equiv b\ (K)$ implies $a\equiv b\ (Q)$, where K and Q are congruences of an algebra A. The 0-element of the lattice of congruences of an algebra A is the equality, denoted by O.

Denote $P(x)$ the set of all subsets of the set x.

A set S forms an *uniform base* over A if

(i) $S \subseteq P(A \times A)$,

(ii) for every $S_\alpha \in S : (x, x) \in S_\alpha$ and $(x, y) \in S_\alpha$ if and only if $(y, x) \in S_\alpha$,

(iii) for every S_α, $S_\beta \in S$ there exists S_γ such that $S_\gamma \subseteq S_\alpha \cap S_\beta$ (S is a filter-base),

(iv) for every S_α there exists S_β such that $\{(x, y) | \exists z : (x, y) \in S_\beta$ and $(z, y) \in S_\beta\} \subseteq S$.

If S is an uniform base over A, then S generates a topology on A. This topology is totally regular. (CSÁSZÁR [1]).

Let A be an algebra and $K = \{K_\alpha | \alpha \in \Gamma\}$ a filter-base, where K_α-s are congruences of A.

THEOREM 1. *If $\cap K = O$, then the sets $K' = \{(x, y) | y \equiv x(K)\}$ form an uniform base over A, the generated topology is T_2, the operations of A are (uniformly) continouos.*

PROOF. Since every K is a congruence, the set $K' = \{K'_\alpha | \alpha \in \Gamma\}$ is a subset of $P(A \times A)$, for every $x \in A$ $(x, x) \in K'_\alpha$, and $(x, y) \in K'_\alpha$ if and only if $(y, x) \in K'$. K is a filter-base, consequently (iii) is satisfied. Since every K_α is a transitive relation, also (iv) is satisfied: For every K_α the set $\{(x, y) | \exists z : (x, y)$ and $(z, y) \in K'_\alpha\}$ is a subset of K_α.

The sets $V_\alpha(x) = \{y | y \equiv x(K_\alpha)\}$ are open neighbourhoods of the point x in the topology generated by K. The set $\{V_\alpha(x) | \alpha \in \Gamma\}$ is a base of neighbourhoods of x. Since $\cap K = O$, there exists to every $x \neq y$ an index γ, such that $V_\gamma(x) \cap V_\gamma(y) = \emptyset$. Thus we proved, that the space is T_2.

Let F be an operation of the algebra A and let $F(x_1, x_2, \ldots, x_n)$ equal to y. If $x'_j \equiv x_j(K)$ $(j = 1, 2, \ldots, n)$, then $y \equiv F(x'_1, x'_2, \ldots, x'_n)(K)$.

Thus theorem 1 is completely proved.

DEFINITION. Let the set $K = \{K_\alpha | \alpha \in \Gamma\}$ be a subset of the set of all congruences of A. Let K be a filter-base, and we define the set $K^* = \{K'_\alpha | K'_\alpha = \{(x, y) | x \equiv y(K_\alpha)\} \alpha \in \Gamma\}$. The topological algebra A is called *K-topological*, if K', as an uniform base over A, generates the topology of A.

It follows immediatelly from the definition that a K-topological algebra is totally regular, and, if it is M_2, then it is metrisable.

Every set $V_\alpha(x)$ is open and closed. They are open namely for any $y \in V_\alpha(x)$ is $V_\alpha(y) = V_\alpha(x)$. Since

$$V_\alpha(x) = A \setminus \bigcup_{z \notin V_\alpha(x)} V_\alpha(z),$$

they are closed. A K-topological algebra is a 0-dimensional space.

The proofs of the general topological statements can be found in the cited book of CSÁSZÁR.

REMARK 1. A module A is *linearly topological* if 0 has a base neighbourhoods $\{M_\alpha\}$ consisting of submodules of M. Clearly the linearly topological modules are exactly the modules among the K-topological algebras.

A filter R is called *Cauchy-filter*, if to every K_α, there exist $r_\alpha \in R$ such that $(x, y) \in r_\alpha$ implies $(x, y) \in K_\alpha$ that is — in this case — $(x, y) \in r_\alpha$ implies $x \equiv y$ (K_α).

A topological space is called complete, if every Cauchy-filter $\{F_\alpha\}$ has the property: $\cap \overline{F}_\alpha \neq \emptyset$, where $\overline{F}_\alpha$ denote the topological closure of the set F_α.

If A' is a subalgebra or a factoralgebra of A, let the topology of A' the topology generated by the topology of A in the general topological meaning. In the first case it is the topology generated by the congruences K'_α, where K'_α is the congruence of A' induced by K_α. In the second case it is the finest topology on A', such that the natural homorphism $f: A \mapsto A'$ is continuous. It can be also characterized by the following property: Let N and N' be the family of the open sets in A and A' respectivelly. N′ generates the above topology on A', if $N = f^{-1}(N')$, that is $n' \in N'$ if and only if $f^{-1}(n') \in N$.

Definition. The congruence Q is *open (closed)*, if every class of Q is an open (closed) set.

As is well known that A/Q is a Hausdorff space in the case that Q is closed, and A/Q is discrete, if Q is open. In particular, if A is a K-topological algebra, then every A/K_α is discrete.

Finally, if F is an open-continuous homomorphism of A, then $F(A)$ is isomorphic and homeomorphic to $A/\mathrm{Ker}F$. (If $A/\mathrm{Ker}F$ has the topology generated by the topology of A, then the natural homomorphism is open--continuous).

Remark 2. A congruence is not nesseserly closed or open. If a class of the congruence Q is open (or closed). Q may be not open (not closed).

Example 1. Let S be a K-topological semigroup with 0, let K denote a set of congruences induces by Rees-factors. In particular, let S be the semigroup of the non-negative integers with the multiplication, and

$$Q = \{Q_n = \{x \mid x \geq n \quad \text{or} \quad x = 0\} \ n \in \omega\},$$

and let K be the set of the Rees-congruences induced by Q_n. The equality is a non-open congruence of S, but every point $x \neq 0$ is open.

Let A_σ be K_σ-topological algebra for $\sigma \in \Sigma$. The direct product of the A_σ-s can be defined as their algebraic direct product endowed with the Tychonoff-topology. This topology can be generated by congruences, too.

Let Γ be a directed set of indices. If a family of topological algebras A_γ, $\gamma \in \Gamma$ is given, and continuous homomorphisms π_α^β, $\alpha, \beta \in \Gamma$ with the following properties:

(i) $\pi_\beta^\alpha : A_\beta \mapsto A_\alpha$ for $\beta \geq \alpha$,

(ii) $\pi_\beta^\gamma \circ \pi_\alpha^\beta = \pi_\alpha^\gamma$ for $\gamma \geq \beta \geq \alpha$,

(iii) $\pi_\alpha^\alpha =$ the identity of A_α

then the set of all vectors $(\dots x_\alpha, \dots x_\beta, \dots)$ with the property $x_\alpha = \pi_\beta^\alpha(x_\beta)$ (for $\beta \geq \alpha$), is called the *inverse limit* of the *inverse system* $[A_\gamma, \pi_\alpha^\beta, \alpha, \beta, \gamma \in \Gamma] = \Omega$. The inverse limit will be denoted by $\lim\limits_{\leftarrow} \Omega$. (c. f. Grätzer [11] p. 131.)

It is not hard to see, that $\lim\limits_{\leftarrow} \Omega$ is a closed subalgebra of the direct product of the A_γ-s.

Let A be a K-topological algebra with $K=\{K_\alpha\}$, let Π_α be the natural homomorphism $A\mapsto A/K_\alpha$, Π_α^β the natural homomorphism $\Pi_\alpha\circ\Pi_\beta^{-1}$ if $\beta\geq\alpha$ that is $K_\alpha\geq K_\beta$. The inverse system $A/K_\alpha, \Pi_\alpha^\beta$ is called the *natural inverse system* belonging to the K-topological algebra A, and will be denoted by $T_{A,K}$.

Lemma 1. If the K-topological algebra A is complete, then $A\cong\cong\varprojlim T_{A,K}$ is valid in algebraical and topological sense.

Proof. Let $x_\alpha=\Pi_\alpha(x)$, for every $x\in A$.

The vector $(\dots x_\alpha,\dots)$ is an element of the inverse limit. The mapping $F:x\mapsto(\dots x_\alpha,\dots)$ is a homomorphism of A into $\varprojlim T_{A,K}$.

$F(x)=F(y)$ implies, that $x\equiv y\ (K_\alpha)$ for every K_α, that is $x=y$. We proved, that $\operatorname{Ker} F=0$.

Let $\mathbf{x}=(\dots x_\alpha,\dots)\in\varprojlim T_{A,K}$,

$$C_\alpha(\mathbf{x})=\{y\,|\,\Pi_\alpha(y)=x_\alpha\}=\Pi_\alpha^{-1}(x).$$

$C_\alpha(\mathbf{x})$ is a filter-base and every $C_\alpha(\mathbf{x})$ is a closed class of the congruence K_α. Hence $C_\alpha(\mathbf{x})$ is a Cauchy-filter, by the assumtion we have $\cap C_\alpha(\mathbf{x})\neq\emptyset$. Let x an element of the intersection. $x\in\cap C_\alpha(\mathbf{x})$ implies $F(x)=\mathbf{x}$, proving, that F is an isomorphism.

The algebras A/K_α are discrete, a base of neighbourhoods of $\mathbf{x}$ is the family of the sets $D_\alpha(\mathbf{x})=\{(\dots x_\alpha,\dots)\,|\,x_\gamma=\Pi_\gamma(x);\ \gamma\leq\alpha\}$. It is easy to see, that $F(V_\alpha(x))=D_\alpha(\mathbf{x})$. Hence F is a homeomorphism.

Remark 3. One can easily verify, that a K-topological algebra can be embedded into $\varprojlim T_{A,K}$, and $\overline{A}$ (the closure of A) is isomorphic and homeomorphic to this inverse limit.

Definition. The K-topological algebra A is K-compact, if every filter-base $R=\{r_\alpha\}$, which consists of classes of congruences of K, has the property: $\cap R\neq\emptyset$.

A is called linearly compact, if every filter-base, consisting of closed classes of congruences has this property.

Obviously an algebra, which is linearly compact and K-topological is always K-compact, but the converse statement is not true.

Example 2. Let S be the semigroup of the positive integers with the multiplication and with the K-topology generated by $K=\{O\}$. Let $x\equiv\equiv y(Q_n)$ if and only if $x,y\geq n$ or $x=y$. Q_n is a congruence of S for every positive integers n. The classes $Q_n(n)$ are closed, but $\cap Q_n(n)=\emptyset$. S is clearly K-compact, but it is not linearly compact.

Lemma 2. If A is K-compact, then it is complete.

Proof. Let $\{r_\alpha\}=R$ a Cauchy-filter, let the set $Q=\{K_\alpha(x_\alpha)\,|\,\exists r_\alpha\subseteq\subseteq K_\alpha(x_\alpha)\}$. Since R is a Cauchy-filter, there exists a class for every K_α such, that $K_\alpha(x_\alpha)\in Q$.

We prove, that Q is a filter-base:

Let $r_\alpha \subseteq K_\alpha(x_\alpha)$, $r_\beta \subseteq K_\beta(x_\beta)$. There exists a K_γ such that $K_\gamma \subseteq K_\alpha \cap K_\beta$, and the assumption that R is a Cauchy-filter, implies that $r_\gamma \subseteq K_\gamma(x_\gamma)$ for some x_γ and $r_\gamma \in R$. Let x an element of $r_\gamma \cap r_\alpha \cap r_\beta$. For this element we have $K_\alpha(x_\alpha) = K_\alpha(x)$, $K_\beta(x_\beta) = K_\beta(x)$ and $K_\gamma(x_\gamma) = K_\gamma(x)$. Thus $K_\gamma(x) \supseteq r_\gamma$, consequently $K_\gamma(x_\gamma) = K_\gamma(x) \in Q$ and Q is a filter-base.

The elements of Q are classes of congruences belonging to K, such that $\cap Q$ is not empty. Let z be an element of the intersection. We can write every element of Q in the form $K_\alpha(z)$.

Since Q contains at least one class of every K_α, every neighbourhood of z contains an element of R. This implies, that every $\bar{r}_\alpha$ contains the point z.

Corollary 1. If A is K-compact, then $A \cong \varprojlim T_{A,K}$ in algebraical and topological sense.

This fact is an immediate consequence of Lemmas 1 and 2.

Remark 4. If A is a linearly compact algebra, and the topology is generated by the set $K = \{K_\alpha\}$, then the algebras A/K_α are linearly compact. But the converse of Corollary 1 (if A is inverse limit of $T_{A,K}$, the algebras in $T_{A,K}$ are linearly compact, then A is linearly compact) is an unsolved problem.

Lemma 3. If the K-topological algebra A has the minimal-condition — that is every descending chain of congruences is finite —, then it is discrete and linearly compact.

Proof. The fact, that there exists a minimal element in the set $K = \{K_\alpha\}$, implies, that A is discrete. Let $K_\alpha\{x_\alpha\}$ be a closed filter-base in A. Among the congruences of this filter-base there exists a minimal K_γ. It is easy to see, that every element of $\{K_\alpha(x_\alpha)\}$ contains the class of this minimal congruence. Hence the assertion follows trivially.

Definition. The K-compact algebra A is *K-compact in the narrow sense,* if the factoralgebra A/Q satisfies the minimal-condition for every open congruence Q.

Using Lemma 1 and 2, as immediate consequence of the definition can we have the next:

Corollary 2. If A is a linearly compact algebra in the narrow sense, then $A \cong \varprojlim T_{A,K}$, (in algebraic and topological sense) and the algebras in $T_{A,K}$ satisfy the minimal-condition.

§ 2. K-compact semigroups

Let S be a semigroup with 0. We define $S^0 = S$, $S^{\gamma \dotplus 1} = S^0 \cdot S^\gamma$ and $S^\gamma = \bigcap_{\alpha<\gamma} S^\alpha$ whenever γ is a limitordinal. Clearly there exists an ordinal β such, that $S^\beta = S^{\beta \dotplus 1}$.

Definition. The semigroup S with 0 is called *β-nilpotent* if $S^\beta = 0$.

TEHOREM 2. *The semigroup S with 0, is ω-nilpotent if and only if there exists a set of congruences $K = \{K_\alpha\}$, such that S is K-compact and every S/K_α is nilpotent.*

PROOF. Let the semigroup S with 0 ω-nilpotent. Let $K = \{S^n\}$. S^n is ideal of S, $\bigcap S^n = 0$, consequently the Rees-factors belonging to this ideals generates a K-topology. It is easy to see, that S, equipped with this K-topology, is K-compact: If every element of the filter-base $K_\alpha(x_\alpha)$ contains at least two element, then every $K_\alpha(x_\alpha)$ is an ideal of S and contains the 0. If there exists a class consisting of one element of S, then every $K_\alpha(x_\alpha)$ contains this element.

On the other hand, let us suppose, that S is not ω-nilpotent and there exists a set $K = \{K_\alpha\}$ such, that S is K-compact and every factor S/K_α is nilpotent. Since S is K-topological, there exists a congruence K_α, such, that $K_\alpha(0) \not\supseteq S^n$, for every natural number n; By the assumption is S/K_α nilpotent, that is $(S/K_\alpha)^m = 0$ for a suitable m. Consequently $K_\alpha(0) \supseteq S^m$, and this is a contradiction.

COROLLARY 3. If the semigroup S with 0 is ω-nilpotent, then S is the inverse limit of nilpotent semigroups S_n. More precisely, we can create an inverse system $\Omega = [S_n, \Pi_n^m]$, such that every S_n is nilpotent, $|\{S_n\}| = \aleph_0$ and $S = \varprojlim \Omega$.

DEFINITION. A K-topological semigroup is Rees-topological if K consists of Rees-factors.

REMARK 5. Such a semigroup is K-compact if and only if it has a 0-element. We have proved the if part of the statement in the proof of theorem 2.

Let us suppose, that S is a Rees-topological semigroup, let I a set of ideals, and let the topology of S generated by the Rees-factors belonging to I. Since the intersection of two ideals of a semigroup contains the product of this ideals, this intersection is non-empty. Since the set of the Rees-congruences generated by I is a filter-base, the set I is also a filter-base. Clearly every I is a closed set, consequently $\bigcap I \neq \emptyset$. $\bigcap I$ has at most one element, (the space is T_2), say $\{x\} = \bigcap I$.

For any $y \in S$ $yx = y \bigcap I \subseteq \bigcap yI \subseteq \bigcap I = x$ and analogously $yx = x$, thus x is a 0-element of S.

DEFINITION. A Rees-compact semigroup S is *Rees-compact in narrow sense* if it satisfies one of the next equivalent statements:

(i) If I is an open ideal of S, then the Rees-factor S/I satisfies the minimal-condition for two-sided ideals.

(ii) If I is an open ideal of S, then the natural homomorphism $\mathcal{S}: S \mapsto S/I$ is open, for every Rees-topology of S/I.

We have to prove, that (i) and (ii) are equivalent.

Since S/I has a minimal ideal, it is — if it is Rees-topological — discrete, analogously to Lemma 3. Consequently the natural homomorphism is always open-continouos.

On the other hand, let $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \ldots$ a descending chain of ideals of S/I. Since I is open, we can choose I such that $\bigcap Q_n = \mathbf{0}$. The set $\{Q_n\}$ generatees a Rees-topology on S/I; this topology is not finer, than that induced by the topology of S, consequently the natural homomorphism is continouos, considering the topology generated by Q. According to the assumption, this homomorphism is open, thus the topology generated by Q is discrete, that is, some Q is equal to $\mathbf{0}$. (The set $\{O\}$ is an open neighbourhood of the point 0, thus it contains any Q_n).

REMARK 7. It is not hard to see, that a Rees-topological semigroup which is linearly compact in the narrow sense, is always Rees-compact in the narrow sense, The converse statement is not true, it is easy to create a counter-example.

Let us consider a semigroup with 0, S; $S^\omega = \bigcap S^n$ is a closed twosided ideal of S, and the Rees-factor semigroup S/S^ω is clearly ω-nilpotent.

S^ω contains every regular element (in the sense of von Neumann) of S; if we assume, that every element of S^ω is regular, we obtain that S has an idempotent ideal I, such that I is union of groups and S/I is inverse limit of countable nilpotent semigroups.

We can consider a Semigroup S not only as a binary algebra, but as an unary algebra with the operations f_s, where f_s is the left (or rigth) multiplication with the element s. We can define the left Rees-factor to any left ideal L, as the congruence: $a \equiv b$ if and only if $a = b$ or $a, b \in L$. It is not hard to see, that if S is K-compact in the topology generated by the left Rees-factors $\{L_\alpha\} = L$, and I is a subset of L consisting of two-sided ideals (Rees-factors), then S is I-compact in the Rees-topology generated by I if I generates a topology. Analogous statement holds for the property of K-compactness in the narrow sense.

DEFINITION. Let S a semigroup, $K = \{K_\alpha\}$ a set of left (right, two-sided) congrugences, and let S K-topological as unary (or binary) algebra. S is called hereditally left- (right- or two-sided) cancellative if every factor S/K is left-(right or two-sided) cancellative. In place of "two-sided cancellative" we shall write "cancellative".

LEMMA 4. If the semigroup S has the minimal-condition for left ideals and it is right cancellative, then it is a left group.

PROOF. Sa^k is a left ideal of the semigroup S and $Sa^k \supseteq Sa^{k+1}$. Since the set of left ideals $\{Sa^k | k \in \omega\}$ has a minimal element, $Sa^k = Sa^{k+1}$ holds for some k. The equality $xa^{k+1} = ba^k$ is solvable for every $a \neq 0$. (In a right cancellative semigroup $a^k = 0$ implies $a = 0$.) Since S is right cancellative, the equality $a \cdot x = b$ is also solvable for every $a \neq 0$.

COROLLARY 4. A cancellative semigroup S, which has the minimal condition for left and right ideals, is a group. (May be a group with 0.)

Let S be a commutative, hereditally cancellative semigroup, which is K-compact in the narrow sense. We know, that in this case there exists an inverse system $\Omega = [S_\alpha, \Pi_\alpha^\beta]$ such that every S_α has the minimal-condition for congruences, and every S_α is commutative and cancellative. In account of

Lemma 4 we can see, that every S_α is a group (wich may possess a 0-element). It is not hard to see, that every S_α or $S_\alpha \backslash 0$ is an Abelian group with minimal condition for subgroups.

By a theorem of Kurosh (FUCHS [2] p. 65.) such a group S_α can be written in the following form:

$$S_\alpha = \sum_{i=1}^{k} \oplus C(p_i^{n_i}), \quad \text{where} \quad 0 < n_i \leq \infty \quad \text{and for} \quad n_i < \infty \;\; C(p_i^{n_i})$$

is the cyclis group with order n_i, for $n_i = \infty$ $C(p_i^{n_i})$ is the Prüfer-group. (FUCHS [2]).

Thus we have

THEOREM 3. *If the commutative semigroup S is K-compact in narrow sense and hereditally cancellative, then there exists an inverse system $[S_\alpha, \Pi_\alpha^\beta] = \Omega$ such, that $S = \varprojlim \Omega$ and every S_α is an abelian group (or a group with 0), with the following structure: S_α (or $S_\alpha \backslash 0$) is the direct product of a finite collection of groups $C(p_i^{n_i})$, where $0 < n_i < \infty$ and $C(p_i^{n_i})$ is as above.*

THEOREM 4. *Let S be a K-compact semigroup with unity and satisfying the following property:*

$$(*) \qquad \forall\, U_\alpha(e), \quad x \in S$$

there exists an n such that $x^n \in U_\alpha(e)$, where $U_\alpha(e)$ is a neighbourhlod of the unit. Such a semigroup is always a group, there is an inverse system $\Omega = [G_\alpha, \Pi_\alpha^\beta]$ such, that every G_α is a torsion group.

PROOF. S is K-compact, consequently there exists an inverse system $\Omega = [G_\alpha, \Pi_\alpha^\beta]$ such, that $S = \varprojlim \Omega$. It is easy to see, that every G_α is a group, and every element of G_α has finite order. ($x^n \in U_\alpha(e)$ implies $K_\alpha(x^n) = K_\alpha(e)$, and $K_\alpha(e)$ is clearly the unity of G_α). A semigrouphomomorphism of a group is always a grouphomomorphism, consequently S is a group.

REMARK 7. S may have elements of infinite order.

REMARK 8. It is not hard to see, that a K-compact semigroup which is inverse limit of torsion groups, has the property (*).

THEOREM 5. *Let S be a commutative semigroup. If S has property (*) and it is K-compact in the narrow sense, then there exists an inverse system $\Omega = [G_\alpha, \Pi_\alpha^\beta]$ such that every G_α is a group of the form $G_\alpha = \sum_{i=1}^{k} \oplus C(p_i^{n_i})$, where $0 < n_i \leq \infty$ and the groups $C(p_i^{n_i})$ are defined as in Theorem 4.*

PROOF. Used Corollary 2 and Theorem 4 we can obtain, that there exists an inverse system $\Omega = [G_\alpha, \Pi_\alpha^\beta]$ such, that G_α-s are Abelian groups with minimal-condition. According to the cited theorem of KUROSH, every G_α has the form $G_\alpha = \sum_{i=1}^{k} \oplus C(p_i^{n_i})$, where the meaning of $C(p_i^{n_i})$ is determined in the Theorem 4.

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LEAST-SQUARES SOLUTION FOR N-PERSON MULTICRITERIA DIFFERENTIAL GAMES

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In this paper N-person games are considered in which each player has several performance indices. In other words, at the same time, several N-person games are played on the same strategy sets. The single N-person games have, in general, different equilibrium points. In [8] solution concepts were introduced for antagonistic two-person multicriteria games. In this note one of these concepts is generalized. This solution consists essentially in minimization of the mean-squares deviation of the single pay-off functions from their equilibrium values. This way of defining a solution for multicriteria games is a game-theoretic adaptation of a scalarization process used in multicriteria optimization ([5]).

First, in **§ 1.** the notion of the least-squares solution is introduced and its basic properties are established. In **§ 2.**, as an illustration, certain N-person linear multicriteria differential games are considered. An existence theorem, necessary conditions and bang-bang principle are obtained concerning the least-squares solution of these games. In **§ 3.** the problem of finding the least-squares solution for the considered differential games is reduced to the solving of a non-linear equation system.

For the brevity the proofs of some theorems are omitted.

§ 1. Least-squares solution. Existence and basic properties

Let $\mathcal{U}_k$ be an arbitrary nonempty set, considered as strategy sets for the k^{th} player $(k = 1, \ldots, N)$,

$$I_k^i : \bigtimes_{k=1}^{N} \mathcal{U}_k \to \mathbf{R}^1 \quad (k = 1, \ldots, N)$$

a given function, the pay-off function of the k^{th} player in the N-person game

$$\Gamma_i := \left(\mathop{\times}_{k=1}^{N} \mathcal{U}_k, I^i \right),$$

where $I^i := (I_1^i, \ldots, I_N^i)$. Define $I_k := (I_k^1, \ldots, I_k^n)$, and $I := (I_1, \ldots, I_N)$. The collection

$$\Gamma := \left(\mathop{\times}_{k=1}^{N} \mathcal{U}_k, I \right)$$

is said to be an *N-person multicriteria game* with the strategy sets $\mathcal{U}_k$ and vector-valued pay-off functions I_k for the k^{th} player ($k = 1, \ldots, N$).

The existence of an equilibrium point (in the sense of Nash) for the scalar games Γ_i is garanteed by the following classical theorem (see e.g. [6], § 23., Th. 2.)

THEOREM 1. *Let $\mathcal{U}_1, \ldots, \mathcal{U}_N$ be nonempty compact and convex strategy sets in a locally convex Hausdorff linear topological space, and*

$$f_k : \mathop{\times}_{k=1}^{N} \mathcal{U}_k \to R^1$$

continuous pay-off functions such that the function $u_k \to f_k(u_1, \ldots, u_k, \ldots, u_N)$ $(u_k \in \mathcal{U}_k)$ is quasi-concave for every fixed $u_j \in \mathcal{U}_j$, $j \neq k$. Then denoting $f := (f_k, \ldots, f_k)$, the game

$$\left(\mathop{\times}_{k=1}^{N} \mathcal{U}_k, f \right)$$

has an equilibrium point.

For the definition of the solution of the game Γ suppose that for all Γ_i $(i = 1, \ldots, n)$ an equilibrium point $(u_1^i, \ldots, u_N^i)$ is given. Denote $\boldsymbol{I}_k^i := I_k^i(u_1^i, \ldots, u_N^i)$ and $\boldsymbol{I}_k := (I_k^1, \ldots, I_k^n)$, and define the functions

$$\varphi_k^i(u_k) := I_k^i(u_1^i, \ldots, u_{k-1}^i, u_k, u_{k+1}^i, \ldots, u_N^i) \quad (u_k \in \mathcal{U}_k)$$

and $\varphi_k := (\varphi_k^1, \ldots, \varphi_k^n)$ $(k = 1, \ldots, N)$. Obviously $\max_{u_k \in \mathcal{U}_k} \varphi_k^i(u_k) = \boldsymbol{I}_k^i$.

DEFINITION. The point $(\bar{u}_1, \ldots, \bar{u}_N \in \mathop{\times}_{k=1}^{N} \mathcal{U}_k$ is said to be a *least-squares* (L–S) *solution* for the game Γ, if it minimizes the function

$$J(u_1, \ldots, u_N) := \sum_{k=1}^{N} |\varphi_k(u_k) - \boldsymbol{I}_k|^2, \quad \left((u_1, \ldots, u_N) \in \mathop{\times}_{k=1}^{N} \mathcal{U}_k \right). \tag{1}$$

REMARK 1. The L–S solution depends on the choice of the equilibrium point of the games Γ_i. We shall see that for a certain class of linear differential games these equilibrium points are unique.

The basic properties of the L–S solution are established in the following theorem, the proof of which is omitted.

THEOREM 2.

(a) *If each* Γ_i $(i=1, \dots, n)$ *satisfies the conditions of Theorem 1., then* Γ *has a* L–S *solution.*

(b) *For any* L–S *solution* $(\bar{u}_1, \dots, \bar{u}_N)$ *of* Γ $\bar{u}_k$ *is Pareto optimal (in the sense of maximization) for* φ_k $(k=1, \dots, N)$.

(c) *If* $(\bar{u}_1, \dots, \bar{u}_N)$ *is an equilibrium point for each* Γ_i $(i=1, \dots, n)$, *then* $(\bar{u}_1, \dots, \bar{u}_N)$ *is a* L–S *solution for* Γ.

§ 2. Differential games. Existence and bang-bang principle

The following statement can be proved using the standard tools of the functional analysis.

LEEMMA. Let $U \subset \boldsymbol{R}^m$ be a nonempty compact, convex set, $L_2^m := L_2([0, T], \boldsymbol{R}^m)$ and define

(2) $$\mathcal{U} := \{u \in L_2^m : u(t) \in U \quad \text{for a.e.} \quad t \in [0, T]\}.$$

Then $\mathcal{U}$ is weakly compact in the space L_2^m.

Let A be an $n \times n$ matrix, B_k $n \times m_k$ matrices $(k=1, \dots, N)$, and $0 \neq d_k^i \in \boldsymbol{R}^n$ given vectors $(i=1, \dots, n;\ k=1, \dots, N)$. For given nonempty compact, convex sets $U_k \subset \boldsymbol{R}^{m_k}$ define the strategy sets

$$\mathcal{U}_k \subset L_2^{m_k} := L_2\ ([0, T], \boldsymbol{R}^{m_k})$$

as in (2):

(3) $$\mathcal{U}_k := \{u_k \in L_2^{m_k} : u_k(t) \in U_k \quad \text{for a.e.} \quad t \in [0, T]\}.$$

On the fixed time interval $[0, T]$ consider the following N-person multicriteria differential game:

(4) $$\dot{x} = Ax + \sum_{k=1}^{N} B_k u_k,$$

(5) $$x(0) = 0,$$

(6) $$I_k^i(u_1, \dots, u_N) := \langle d_k^i, x(T) \rangle, (u_1, \dots, u_N) \in \mathop{\times}_{k=1}^{N} \mathcal{U}_k,$$

where x denotes the trajectory corresponding to the control strategies $(u_1, \dots, u_N)$. Further on, taking in account the notations of **§ 1.** and (4)–(6), denote

$$\Gamma_i := \left(\mathop{\times}_{k=1}^{N} \mathcal{U}_k, I^i \right) \quad \text{and} \quad \Gamma := \left(\mathop{\times}_{k=1}^{N} \mathcal{U}_k, I \right).$$

By means of the Lemma the following existence theorem can be proved.

THEOREM 3. *The game* Γ *has a* L–S *solution.*

It is not hard to prove the following theorem concerning the existence, uniqueness bang-bang principle for the games Γ_i.

THEOREM 4. *Let $U_k \subset \boldsymbol{R}^{m_k}$ in* (3) *be convex polyhedra such that triples A, B_k, U_k satify the condition of general position for $k=1, \dots, N$.* ([1], § 17.) *Then*

(a) *each game Γ_i $(i=1, \dots, n)$ has a unique equilibrium point $(u_1^i, \dots, \dots, u_N^i)$ satisfying the condition*

$$\langle \psi_k^i(t), B_k u_k^i(t) \rangle = \max_{u_k \in U_k} \langle \psi_k^i(t), B_k u_k \rangle \quad \text{for a.e.} \quad t \in [0, T],$$

where ψ_k^i denotes the solution of the adjoint system $\dot{\psi} = -A' \psi$ with the end-point condition $\psi_k^i(T) = d_k^i$. $(i=1, \dots, n;\ k=1, \dots, N)$;

(b) *the equilibrium point $(u_1^i, \dots, u_N^i)$ consists of piece-wise constant controls u_k^i $(i=1, \dots, n;\ k=1, \dots, N)$;*

(c) *if, besides, $U_k = \mathop{\times}\limits_{j=1}^{m_k} [\alpha_j^k, \beta_j^k]$, and A is symmetric, then the j^{th} coordinate of the equilibrium control u_k^i takes only the values α_j^k and β_j^k with at most $n-1$ switching points $(i=1, \dots, n;\ k=1, \dots, N)$.*

§ 3. Finding L – S solutions for linear differential games

For the simplicity we shall consider only two-person (nonzero-sum) games, with scalar controls.

Let $U_1 = U_2 = [-1, 1]$, and let

$$\mathcal{U}_1 = \mathcal{U}_2 = \{u \in L_2([0, T], \boldsymbol{R}^1),\ R_u \subset [-1, 1]\}.$$

Let A be an $n \times n$ matrix, $B_1, B_2 \in \boldsymbol{R}^n$, $d_1^i, d_2^i \in \boldsymbol{R}^n$ $(i=1, \dots, n)$ given column vectors. Denote by Γ the following two-person multicriteria differential game.

(7) $$\dot{x} = Ax + B_1 u_1 + B_2 u_2,$$

(8) $$x(0) = 0,$$

(9) $$I_k^i(u_1, u_2) := \langle d_k^i, x(T) \rangle,\ (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2,$$
$$(i = 1, \dots, n; \quad k = 1, 2).$$

Then Theorem 1. implies the existence of an equilibrium point (u_1^i, u_2^i) for Γ_i $(i=1, \dots, n)$. Further on, suppose that A, B_1 U_1 and A, B_2, U_2 satisfy the condition of general position. Then Theorem 3 implies the uniqueness of the equilibrium points (u_1^i, u_2^i).

Put $\Omega(t) := e^{A(T-t)}$, $(t \in \boldsymbol{R})$, and define the $3n \times 3n$ matrix

$$\mathbf{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

and the $3n \times 2$ matrix

$$\mathbf{D}(t):=\begin{bmatrix} B_1 & B_2 \\ \langle d_1^1, \Omega(t) B_1\rangle & 0 \\ \vdots & \vdots \\ \langle d_1^n, \Omega(t) B_1\rangle & 0 \\ 0 & \langle d_2^1, \Omega(t) B_2\rangle \\ \vdots & \vdots \\ 0 & \langle d_2^n, \Omega(t) B_2\rangle \end{bmatrix}, \quad (t\in \boldsymbol{R}).$$

THEOREM 5. *Suppose that the equilibrium points* (u_1, u_2^i), $(i=1, \ldots, n)$ *are not all the same, and let* **A**, **D** *and* $[-1, 1]$ *satisfy the condition of general position on an open interval including* $[0, T]$ ([1], § 23.). *Then*

(a) *every* L–S *solution* $(\bar{u}_1, \bar{u}_2)$ *of* Γ *consists of piece-wise constant controls;*

(b) *if, besides, both the vectors* d_1^i *and the vectors* d_2^i *are linear independent, moreover not all* u_1^i *and not all* u_2^i *coincide, then for a symmeric matrix* A *the optimal controls* $\bar{u}_1$ *and* $\bar{u}_2$ *have at most* $n-1$ *switching points;*

(c) *in the general case, if it is known that the optimal controls have at most* s *switching points, the problem of finding all extremals reduces to the solving a system of non-linear equations constructed in the proof of the theorem.*

PROOF. The proof consists of several steps.

1. *Reduction of the functional*. Denoting

$$K_k^i := \left\langle d_k^i, \int_0^T \Omega(t) B_2 u_1^i(t)\, dt\right\rangle,$$

$$L_k^i := \left\langle d_k^i, \int_0^T \Omega(t) B_2 u_2^i(t)\, dt\right\rangle,$$

for the equilibrium values of the games Γ_i we obtain

$$\boldsymbol{I}_k^i = I_k^i(u_1^i, u_2^i) = K_k^i + L_k^i \quad (k=1,2;\ i=1, \ldots, n).$$

Straightforward calculations show that the minimization of the functional

$$J(u_1, u_2) = |\varphi_1(u_1) - \boldsymbol{I}_1|^2 + |\varphi_2(u_2) - \boldsymbol{I}_2|^2,$$
$$(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$$

is equivalent to that of the functional

$$J^0(u_1, u_2) := \sum_{i=1}^n \left\{ \left[\int_0^T \langle d_1^i, \Omega(t) B_1 u_1(t)\rangle\, dt\right]^2 - \right.$$

$$-2K_1^i \int_0^T \langle d_1^i, \Omega(t) B_1 u_1(t)\rangle\, dt + \left[\int_0^T \langle d_2^i, \Omega(t) B_2 u_2(t)\rangle\, dt\right]^2 -$$

$$\left. -2L_2^i \int_0^T \langle d_2^i, \Omega(t) B_2 u_2(t)\rangle\, dt \right\}, \quad (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2 .$$

2. *Extension of the system.* The functional J^0 is neither integral, nor terminal functional for the system (7)–(8). However, it can be transformed into a terminal functional by an appropriate extension of the system, which makes us possible to apply the Pontryagin's maximum principle. Let

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}^1(t) \\ \vdots \\ \mathbf{x}^{3n}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad \mathbf{x}(t) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n .$$

Introducing the new equations

$$\dot{y}^i(t) = \langle d_1^i, \Omega(t) B_1 u_1(t) \rangle, \quad y^i(0) = 0 \quad (i = 1, \dots, n);$$

$$\dot{z}^i(t) = \langle d_2^i, \Omega(t) B_2 u_2(t) \rangle, \quad z^i(0) = 0 \quad (i = 1, \dots, n),$$

and denoting

$$w = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

we obtain the control system

(10) $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{D} w, \quad \mathbf{x}(0) = 0, \quad w \in \mathcal{U}_1 \times \mathcal{U}_2 .$$

Then the functional J^0 is equivalent to the terminal functional, generated by

(11) $$G(\mathbf{x}) := \sum_{i=1}^{n} \{[y^i]^2 - 2K_1^i y^i + [z^i]^2 - 2L_2^i z^i\},$$

i.e.

$$J^0(u_1, u_2) = J^0(w) = G(\mathbf{x}(T)),$$

where $\mathbf{x}$ is the trajectory, corresponding to w.

3. *Application of the maximum principle to the extended system.* The adjoint system of the problem (10–(11) for the function $\tilde{\Psi} = (\Psi, \eta, \zeta)$ has the form

(12) $$\dot{\Psi} = -A'\Psi, \quad \dot{\eta} = 0, \quad \dot{\zeta} = 0 .$$

On the other hand

$$G_{\mathbf{x}}(\mathbf{x}) = (0, \dots, 0, \; 2y^1 - 2K_1^1, \dots, 2y^n - 2K_1^n, \; 2z^1 - 2L_2^1, \dots, 2z^n - 2L_2^n).$$

Applying the maximum principle in the needed form ([3], Th. 5.10, Th. 5.11), for the adjoint solution $\tilde{\Psi}$ figuring in the maximum principle we have $\tilde{\Psi}(T) = -G_{\mathbf{x}}(\mathbf{x}(T))$, i.e. $\Psi_i(T) = 0$, $\eta_i(T) = 2K_1^i - 2y^i(T)$, $\zeta_i(T) = 2L_2^i - 2z^i(T)$, $(i = 1, \dots, n)$, where y^i, z^i are the components of the trajectory corresponding to the optimal control $(\bar{u}_1, \bar{u}_2)$. Thus the solution $\tilde{\Psi}$ of the system (12) satisfying the end-point condition $\tilde{\Psi}(T) = -G_{\mathbf{x}}(\mathbf{x}(T))$ is constant.

$\tilde{\Psi}$ is not a trivial solution. Namely, the triviality of $\tilde{\Psi}$ implies

$$y^i(T) = \left\langle d_1^i, \int_0^T \Omega(t) B_1 \bar{u}_1(t)\, dt \right\rangle = K_1^i;\ z^i(t) =$$

$$= \left\langle d_2^i, \int_0^T \Omega(t) B_2 \bar{u}_2(t)\, dt \right\rangle,$$

$$(i = 1, \ldots, n).$$

Moreover, by the definition of the equilibrium point, for any $u_1 \in \mathcal{U}_1$

$$\left\langle d_1^i, \int_0^T \Omega(t) B_1 u_1(t)\, dt \right\rangle + \left\langle d_1^i, \int_0^T \Omega(t) B_2 u_2^i(t)\, dt \right\rangle =$$

$$= I_1^i(u_1, u_2^i) \leq I_1^i(u_1^i, u_2^i) = K_1^i + L_1^i =$$

$$= \left\langle d_1^i, \int_0^T \Omega(t) B_1 \bar{u}_1(t)\, dt \right\rangle + L_1^i,$$

that is

$$\left\langle d_1^i, \int_0^T \Omega(t) B_1 u_1(t)\, dt \right\rangle \leq \left\langle d_1^i, \int_0^T \Omega(t) B_1 \bar{u}_1(t)\, dt \right\rangle. \tag{13}$$

This implies that

$$I_1^i(u_1, \bar{u}_2) \leq I_1^i(\bar{u}_1, \bar{u}_2), \quad (u_1 \in \mathcal{U}_1), \quad (i = 1, \ldots, n).$$

In a similar may we obtain

$$I_2^i(\bar{u}_1, u_3) \leq I_2^i(\bar{u}_1, \bar{u}_2), \quad (u_2 \in \mathcal{U}_2), \quad (i = 1, \ldots, n).$$

Thus, by the uniqueness of the equilibrium point for Γ_i, $(\bar{u}_1, \bar{u}_2) = (u_1^i, u_2^i)$ $i = 1, \ldots, n$, which contradicts to the conditions of our theorem.

According to the maximum principle, the nontrivial adjoint solution $\tilde{\Psi}$ and the optimal control $\bar{w} = (\bar{u}_1, \bar{u}_2)$ a.e. satisfy the following maximum relation.

$$\langle \tilde{\Psi}(t), \mathbf{D}(t) \bar{w}(t) \rangle = \max_{w \in U_1 \times U_2} \langle \tilde{\Psi}(t), \mathbf{D}(t) w \rangle. \tag{14}$$

Although, $\tilde{\Psi}$ depends on the control, the idea of the proof the bang-bang principle for time optimal control of nonautonom linear systems applies ([1], Th. 15.). Thus $\bar{u}_1$ and $\bar{u}_2$ are (up to a.e. equation) piece-wise constant taking only values ± 1. The statement (a) of our theorem is proved.

For the proof of the statement (b) rewright the right-hand side of (14) in the form

$$\langle \tilde{\Psi}, \mathbf{D}(t) w \rangle = \sum_{i=1}^{n} [\eta_i(T) \langle d_1^i, \Omega(t) B_1 u_1 \rangle + \zeta_i(T) \langle d_2^i, \Omega(t) B_2 u_2 \rangle] =$$

$$= \left\langle \Omega'(t) \sum_{i=1}^{n} \eta_i(T) d_1^i, B_1 u_1 \right\rangle + \left\langle \Omega'(t) \sum_{i=1}^{n} \zeta_i(T) d_2^i, B_2 u_2 \right\rangle. \tag{15}$$

Now, if in (15) e.g. $\eta_i(T)=0$ ($i=1, \ldots, n$), then, as in the proof of statement (a), we obtain (13). Thus, adding the term

$$\left\langle d_4^i, \int_0^T \Omega(t)\, B_1 u_2^i(t)\, dt \right\rangle$$

to both sides of (13), we have

$$I_1^i(u_1, u_2^i) \leq I_1^i(\bar{u}_1, u_2^i)$$

for $u_1 \in \mathcal{U}_1$ and $i=1, \ldots, n$. On the other hand,

$$I_2^i(u_2^i, u_2) \leq I_2^i(u_1^i, u_2^i) \tag{16}$$

for $u_2 \in \mathcal{U}_2$ and $i=1, \ldots, n$. Because of the separability of $I_2^i(u_1, u_2)$ in u_1 and u_2, from (16) it follows that

$$I_2^i(\bar{u}_1, u_2) \leq I_2^i(\bar{u}_1, u_2^i)$$

for $u_2 \in \mathcal{U}_2$; $i=1, \ldots, n$. Thus $(\bar{u}_1, u_2^i)$ is an equilibrium point for Γ_i, which, by the uniqueness implies $\bar{u}_1 = u_1^i$ $i=1, \ldots, n$. This contradicts to the conditions of our theorem. Therefore, because of the linear independence of the vectors d_1^i $i=1, \ldots, n$, $\Omega'(t) \sum_{i=1}^n \eta_i(T)\, d_1^i$ is a nontrivial solution of the system adjoint to (7). Consequently, as in the case of time optimal control, for a symmetric matrix A we get that the number of switching points in the control function u_1 is at most $n-1$, which is also obtained for u_2 similarly. The statement (b) is proved.

4. *Finding extremals.* The problem is not standard in the sense that $\tilde{\Psi}$ in (14) depends on the control. Using (15) the maximum principle is of the form

$$\left[B_1' \Omega'(t) \sum_{i=1}^n \eta_i(T)\right] u_1(t) + \left[B_2' \Omega'(t) \sum_{i=1}^n \zeta_i(T)\, d_2^i\right] u_2(t) =$$

$$= \max_{(u_1, u_2)\, U_1 \times U_2} \left\{\left[B_1' \Omega'(t) \sum_{i=1}^n \eta_i(T)\right] u_1 + \left[B_2' \Omega'(t) \sum_{i=1}^n \zeta_i(T)\, d_2^i\right] u_2\right\},$$

where

$$\eta_i(T) = 2K_1^i - 2y^i(T) = 2K_1^i - 2\int_0^T [B_1' \Omega'(t)\, d_1^i]\, \bar{u}_1(t)\, dt,$$

$$\zeta_i(T) = 2L_2^i - 2z^i(T) = 2L_2^i - 2\int_0^T [B_2' \Omega'(t)\, d_2^i]\, \bar{u}_2(t)\, dt.$$

Assume that both $\bar{u}_1$ and $\bar{u}_2$ have at most s switching points: $0 < \tau_1 \leq \ldots \leq \tau_s < T$ and $0 < \sigma_1 \leq \ldots \leq \sigma_s < T$ respectively. Further on, considering e.g. $\bar{u}_1$, put

$$\bar{u}_1(t) = (-1)^{\varphi + l}, \quad t \in (\tau_l, \tau_{l+1}), \quad (l = 0, 1, \ldots, s),$$

where $\tau_0 := 0$,$\tau_{s+1} := T$, and μ is a parameter taking the value 0 or 1. Then the end-point condition for the adjoint solution, as a function of the switching points, has the form

$$\eta_i(T) = 2K_1^i - 2\sum_{l=0}^{s}(-1)^{\mu+l}\int_{\tau_l}^{\tau_{l+1}} B_1' \Omega'(t)\, d_1^i\, dt =$$

$$= 2K_1^i - 2\sum_{l=0}^{s}(-1)^{\mu+l}\left(\int_{\tau_l}^{\tau_{l+1}} \Omega'(t)\, dt\right) d_1^i =: \varrho_i^\mu(\tau_1, \dots, \tau_s).$$

The functions ϱ_i^μ are known quasi-polynomials for every i and μ. Further, introduce the notations

$$M := \{\tau \in \boldsymbol{R}^s : 0 < \tau_1 \leq \dots \leq \tau_s < T\},$$

$$P_\mu^\tau(t) := B_1' \Omega'(t) \sum_{i=0}^{n} \varrho_i^\mu(\tau)\, d_1^i, \quad (t \in (0, T)),$$

$$\bar{\varrho}_i^\mu(\bar{\tau}_1, \dots, \bar{\tau}_{s-1}) := 2K_1^i - 2\sum_{l=0}^{s-1}(-1)^{\mu+l} \cdot B_1'\left(\int_{\tau_l}^{\tau_{l+1}} \Omega'(t)\, dt\right) d_1^i,$$

$$\bar{M} := \{\bar{\tau} \in \boldsymbol{R}^{s-1} : 0 < \bar{\tau}_1 \leq \dots \leq \bar{\tau}_{s-1} < T\},$$

$$P_{\bar{\tau}}^\mu(t) := B_1' \Omega'(t) \sum_{i=1}^{n} \bar{\varrho}_i^\mu(\bar{\tau})\, d_1^i, \quad (t \in (0, T)).$$

Case (α). If there exists no $\tau \in M$, such that $P_\tau^\mu(\tau_l) = 0$, $(l = 1, \dots, s)$, then an extremal control of paramerer μ can be only $u_1(t) = (-1)^\mu$, $t \in [0, T]$. The extremality of this control can be immediately verified.

Case (β). Let $\tau \in \operatorname{int} M$, $P_\tau^\mu(\tau_l) = 0$, $(l = 1, \dots, s)$. Then the function sign $P_\tau^\mu(t)$, $t \in (0, T)$ is an extremal control of parameter μ if and only if the function P_τ^μ, beginning at $(-1)^\mu$, changes his sign subsequently at the points τ_l $(l = 1, \dots, s)$.

Case (γ). Let $\tau \in M$ be a boundary point, and $P_\tau^\mu(\tau_l) = 0$, $(l = 1, \dots, s)$. Then some coordinates of the point τ are equal:

$$\tau_1 = \dots = \tau_{l_1} < \tau_{l_1+1} < \dots < \tau_{l_1+s_1} = \dots = \tau_{l_2} < \tau_{l_2+1} \dots = \tau_{l_p} = \tau_s.$$

Now the function sign P_τ^μ is an extremal control of parameter μ with l_p switching points if and only if P_τ^μ, beginning at $(-1)^\mu$ changes his sign subsequently at the points $\tau_{l_1}, \tau_{l_1+1}, \dots, \tau_{l_2}, \dots, \tau_{l_p}$.

Obviously, by means of criteria (α), (β) and (γ) we can obtain all extremals of parameter μ having a switching number $s' \leq s$, where $s - s'$ is even. Indeed, if $\tau_1 < \dots < \tau s'$ are the switching points, then the vector

$\tau := (\tau_1, \ldots, \tau_{s'}, \ldots, \tau_{s'}) \in M$ satisfies the equations $P_\tau^\mu(\tau_l) = 0$ ($l = 1, \ldots, s$). Then extremal controls of parameter μ having s' switching points, where $s - s'$ is odd, can be obtained from the solutions of the equation system $\bar{P}_{\bar{\tau}}^\mu(\bar{\tau}_l) = 0$ ($l = 1, \ldots, s$) on the set $\overline{M}$, applying considerations (α), (β) and (γ) mutatis mutandis.

Applying similar arguments to the control $\bar{u}_2$ (with a sign parameter $\nu = 0,1$) we obtain all extremal controls $\bar{w}(t)$ ($t \in [0, T]$).

If the number of extremal controls is finite (this is "expected", since both the number of equations of the type $P_\tau^\mu(\tau_l) = 0$ ($l = 1, \ldots, s$) and the number of unknowns are equal to s), then by comparing the values of the functional J^0 we get the optimal controls $\bar{w}$, i.e. the L–S solution of the considered game.

The proof of Theorem 5. is complete.

REMARK 2. For the case of N-person games with vector control functions the proof of Theorem 5 is similar, only the description of the extremals is more complicated.

REMARK 3. The assumption that the equilibrium points (u_1^i, u_2^i) are not all the same, is not strong. Indeed, in the contrary case these idendical equilibrium points give a L–S solution of the game.

REMARK 4. If the functionals I_k^i does not depend linearly on the end point of the trajectory, then in the functional J the terms depending on u_1^i and u_2^i can not be omitted. Thus, these piece-wise constant functions figure on the right-hand side of the extended system. In this case, converting the terminal functional into an integral one, the maximum principle applies on each continuity interval.

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DIE KONJUGIERTE GITTERFÖRMIGE INKONGRUENTE KREISPACKUNG DER EBENE

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Eine Punktmenge **M**′ wird konjugiert zur diskreten Punktmenge **M** genannt, wenn **M**′ aus den Eckpunkten der Dirichletschen Zelle (*D*-Zelle, s. [1]) von **M** besteht.

Über konjugierte Packung der inkongruenten Kreise sprechen wir, wenn wir um die Punkte einer Punktmenge Einheitskreise und um die Punkte der konjugierten Punktmenge Kreise mit gegebenem Radius r schreiben.

In dieser Arbeit knüpfen wir uns an unseren früheren Artikel [2] und wir bestimmen die Dichte der dichtesten inkongruenten konjugierten Kreisausfüllung der Ebene abhängig von r, wenn die Punktmenge **M** ein Punktgitter ist. Dieser Problemenkreis wurde von JENŐ HORVÁTH aufgeworfen.

In der Ebene geben es zwei Gittertype:

1. Das rechteckige Gitter, wenn die *D*-Zelle Rechteck ist. In diesem Fall bilden **M** und **M**′ ein Doppelgitter.

2. Das primitive Gitter, wenn die *D*-Zelle Sechseck ist. Der Durchschnitt eines Grundgitterparalellogramms mit den betrachteten Kreissystemen ist natürlich ein Einheitskreis und zwei Kreise vom Radius r.

SATZ 1. *Die Dichte der konjugierten rechteckgitterförmigen inkongruenten Kreisausfüllung der Ebene ist höchstens*

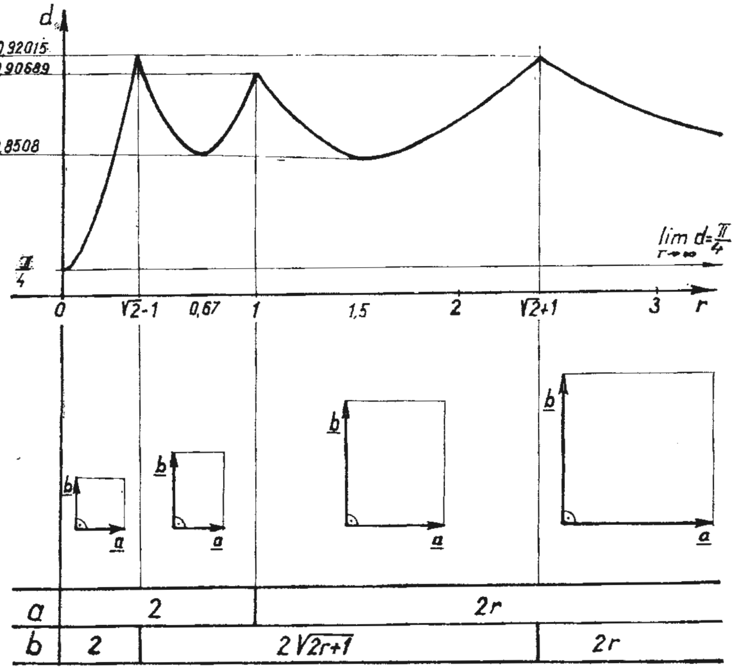
$$\pi\frac{r^2+1}{4} \qquad \textit{für} \quad 0\leq r\leq\sqrt{2}-1\,,$$

$$\pi\frac{r^2+1}{4\sqrt{r^2+2r}} \qquad \textit{für} \quad \sqrt{2}-1\leq r\leq 1\,,$$

$$\pi\frac{r^2+1}{4r\sqrt{2r+1}} \qquad \textit{für} \quad 1\leq r\leq\sqrt{2}+1\,,$$

$$\pi\frac{r^2+1}{4r^2} \qquad \textit{für} \quad r\geq\sqrt{2}+1\,.$$

Die Angabe des Gitters, womit die dichteste Ausfüllung verwirklicht werden kann, und die Funktion der maximale Dichte d(r) sind in der folgenden Tabelle auch anschaulich gegeben.



Tab. 1.

Beweis. Die Ecke der D-Zellen des rechteckigen Gitters sind die Mittelpunkte der Rechtecke, so ist die konjugierte Punktmenge zum ursprünglichen Gitter kongruent, d.h. sie bilden ein Doppelgitter.

Der Flächeninhalt des Rechtecks von der Seitenlänge ≥ 2 ist minimal, wenn es ein Quadrat von der Seitenlänge 2 ist. Die Behauptung des Satzes ist also offenbar, wenn $r \leq \sqrt{2}-1$ ist.

Ist $\sqrt{2}-1 < r \leq 1$, haben wir die Behauptung in [2] bewiesen.

Wenn $r > 1$ ist, dann ergibt sich die Behauptung aus dem Tausch der Rollen der Punktsystemen.

Satz 2. *Die Dichte der konjugierten primitivgitterförmigen inkongruenten Kreisausfüllung der Ebene ist höchstens*

$$\pi \frac{2r^2+1}{2\sqrt{3}} \qquad \textit{für} \quad 0 \leq r \leq \frac{\sqrt{12}-3}{3} \ (= 0{,}1547\ldots),$$

$$\pi \frac{(2r^2+1)(r+1)^2}{8\sqrt{r^2+2r}} \qquad \textit{für} \quad \frac{\sqrt{12}-3}{3} \leq r \leq \frac{\sqrt{17}-3}{4} \ (= 0{,}2807\ldots),$$

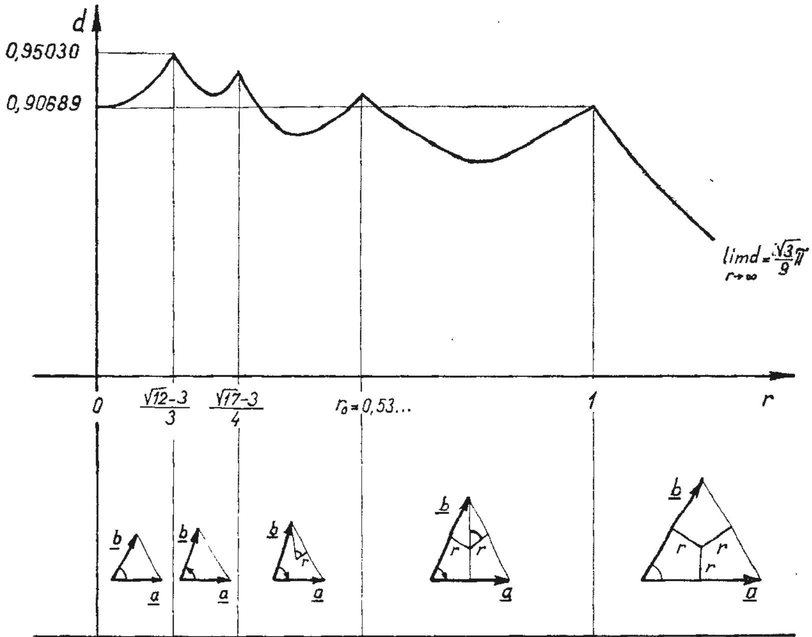
$$\pi \frac{(2r^2+1)(r+1)^2}{4\sqrt{2r+1}\,\left(r+\sqrt{2r^3+5r^2+2r}\right)} \quad \textit{für} \quad \frac{\sqrt{17}-3}{4} \le r \le r_0 (= 0{,}5334\ldots),$$

wo r_0 die positive Lösung der Gleichung $8r^3+3r^2-2r-1=0$ ist,

$$\pi \frac{(2r^2+1)(r+1)^2}{8r\sqrt{(2r+1)^3}} \quad \textit{für} \quad r_0 \le r \le 1,$$

$$\pi \frac{2r^2+1}{6\sqrt{3}\,r^2} \quad \textit{für} \quad r \ge 1.$$

Die Angabe des Gitters, womit die dichteste Ausfüllung verwiklicht werden kann, und die Funktion der maximale Dichte d(r) sind in der folgenden Tabelle auch anschaulich gegeben.



a	2			$\frac{4r\sqrt{2r+1}}{r+1}$	$2r\sqrt{3}$
b	2		$\frac{r+\sqrt{2r^3+5r^2+2r}}{0{,}5(r+1)}$	$2\sqrt{2r+1}$	$2r\sqrt{3}$
$\sphericalangle(ab)$	60°	$2\arccos\frac{1}{r+1}$	$\arccos\frac{r}{r+1}$		60°

Tab. 2

Beweis. Wir wollen eine Ausfüllung bekommen, d.h. die Kreise können keine gemeinsamen inneren Punkte haben, darum müssen die Seite des Stützdreiecks ≥ 2 (s. [1]), die Seiten der D-Zelle $\geq 2r$ und der Radius des Umkreis vom Stützdreieck $\geq r+1$ sein. Die Dichte der Ausfüllung ist maximal, wenn der Flächeninhalt des Stützdreiecks minimal ist. Diese Bedingungen können auch so abgefaßt werden: wir schreiben ein Dreieck dem Kreis **K** vom Radius $R \geq r+1$, Mittelpunkt O ein, dessen Flächeninhalt minimal und die Seite ≥ 2 sind, und dieses Dreieck enthält einen Kreis **k** vom Radius r, Mittelpunkt O.

Wenn $r \leq \frac{\sqrt{12}-3}{3}$ ist, dann gibt das reguläre Dreieck von der Seitenlänge 2 offenbar das Minimum.

Der Flächeninhalt eines Dreiecks, das einen Kreis enthält, ist minimal, wenn es ein dem Kreis umbeschriebenes reguläres Dreieck ist. Wenn $r \geq 1$ ist, verfügt dieses Dreieck auch über weitere Bedingungen, so ist die Behauptung des Satzes auch in diesem Intervall offenbar.

Im Intervall $\left(\frac{\sqrt{12}-3}{3}, 1\right)$ ist die Methode des Beweises folgendes: zuerst bestimmen wir das dem Kreis vom Radius $R \geq r+1$ einbeschriebene Dreieck mit obgenannten Bedingungen, dann beweisen wir, daß der Flächeninhalt des Dreiecks eine zunehmende Funktion von R ist, dann ist also die Dichte maximal, wenn $R = r+1$ ist.

Wenn $R \leq 2$ ist, dann ist $r \leq \frac{R}{2}$ wegen $r \leq R-1$, darum existiert das dem Kreis **K** einbeschriebene, den Kreis **k** enthaltende Dreieck ABC. Wir beweisen, daß der Flächeninhalt eines solchen Dreieck minimal ist, wenn zwei von seinen Seiten den Kreis **k** berühren. Es seien $AB \leq BC \leq CA$. Wenn AC den Kreis **k** nicht berührt, bewegen wir die Ecke A zu B dem Kreis **K** entlang (Fig. 1.). Mit dieser Bewegung nimmt der Flächeninthalt des Drei-

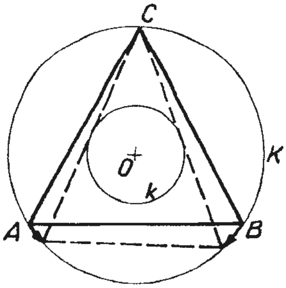


Fig. 1.

ecks ab, weil $AB \leq AC$ war. Wir können erreichen, daß AC den Kreis **k** berühren soll. Wir wiederholen die obige Bewegung mit dem Punkt B, wo AB weiter abnimmt.

Die Länge der den Kreis **k** berührenden Sehne ist

$$2\sqrt{R^2-r^2} \geq 2\sqrt{(r+1)^2-r^2} = 2\sqrt{2r+1} > 2 .$$

Die Basis des Dreiecks ist 2 Einheiten lang, wenn $R = R_0$ und $r = R_0-1$ sind, wo $R_0 > 1$ die Lösung der Gleichung $8R^3-21R^2+16R-4=0$ ist (Fig. 2.).

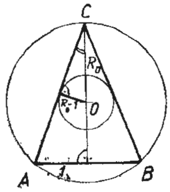


Fig. 2.

$$\sin \sphericalangle = \frac{1}{2\sqrt{R_0^2-(R_0-1)^2}} = \frac{R_0-1}{R_0},$$

$$8R_0^3-21R_0^2+16R_0-4 = 0,$$

$$R_0 = 1{,}5334\ldots$$

Es sei $R < R_0$. In diesem Fall bewegen wir den Punkt A wie oben, aber den Punkt B bis zur Lage, wo $AB = 2$ ist.

Die Länge der Schenkel des gleichschenkligen Dreiecks, dessen Basis eine den Kreis **k** berührende Sehne ist, wird 2, wenn $R = \frac{\sqrt{17}+1}{4}$ und $r = R-1$ sind (Fig. 3.).

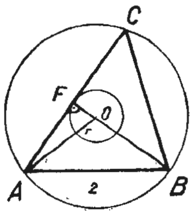


Fig. 3.

$$AO^2-OF^2+BF^2 = AB^2,$$

$$(r+1)^2-r^2+(2r+1)^2 = 4,$$

$$r = \frac{\sqrt{17}-3}{4}.$$

Ist $R < \frac{\sqrt{17}+1}{4}$, kann auch die Seite AC den Kreis **k** nicht berühren. In diesem Fall bewegen wir den Punkt A nur bis zur Lage, wo $AB = 2$ ist, dann bewegen wir die Ecke C zu B bis zur Lage $BC = 2$.

Die Flächeninhalte der obigen dreierlei Dreiecke sind:

$$f_1 = \frac{4r\sqrt{(R^2-r^2)^3}}{R^2} = 4r\sqrt{R^2-r^2}\left(1-\frac{r^2}{R^2}\right),$$

$$f_2 = 2\sqrt{R^2-r^2}\,\frac{r+\sqrt{(R^2-1)(R^2-r^2)}}{R^2} = 2\sqrt{R^2-r^2}\,f(R),$$

$$f_3 = \frac{4\sqrt{R^2-1}}{R^2} = 2\sin\left(2\arccos\frac{1}{R}\right).$$

f_1, f_3 und der erste Faktor von f_2 sind offenbar zunehmende Funktion von R. Die Ableitung des zweiten Faktors von f_2 ist umzuschreiben:

$$f'(R) = \frac{(\sqrt{R^2-r^2} - r\sqrt{R^2-1})^2}{R^3\sqrt{(R^2-1)(R^2-r^2)}}.$$

Weil $r \neq 1$ ist, so ist $f' > 0$. Das Stützdreieck des die dichteste Ausfüllung gebenden primitiven Gitters ist also dem Kreis vom kleinsten Radius $R = r+1$ einbeschrieben.

Anmerkung. Die D-Zelle ist ein reguläres Sechseck von der Seitenlänge $> 2r$, wenn $r = \frac{\sqrt{12}-3}{3}$ ist (Fig. 4.).

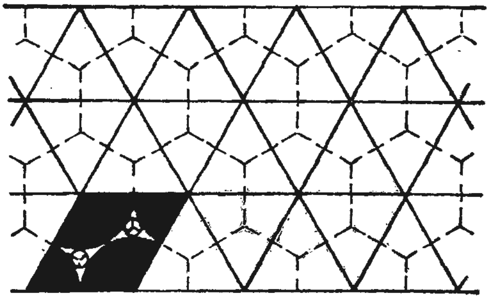


Fig. 4.

Ist $r = \frac{\sqrt{17}-3}{4}$, ist ein Seitenpaar $2r$ lang (Fig. 5.).

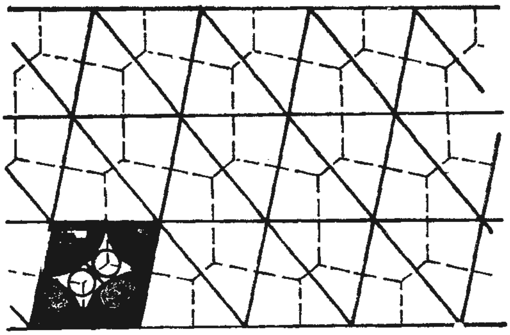


Fig. 5.

Ist $r = R_0 - 1$, sind zwei Seitenpaare $2r$ lang (Fig. 6.).

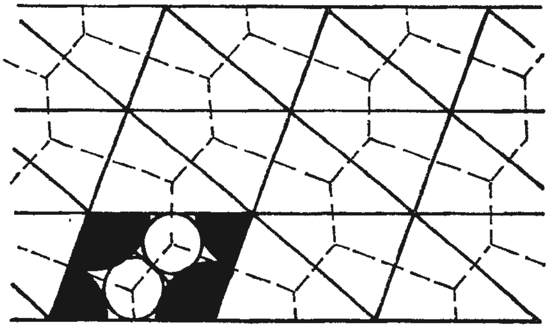


Fig. 6.

Ist $r > 1$, dann ist die *D*-Zelle ein reguläres Sechseck von der Seitenlänge $2r$ (Fig. 7.).

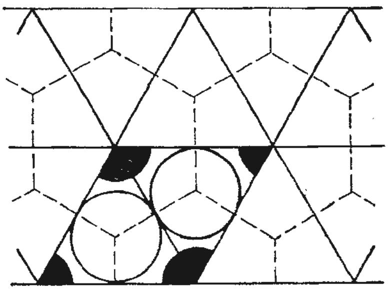


Fig. 7.

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