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ANNALES

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Пусть $a = bT_{x_1}\dots T_{x_n} \in alt(A)$, $b \in L'$. Если $T_{x_1} = L_{x_1}$, то $bL_{x_1} = x_1 b = b' \in L'$, так что по индукции элемент $a = bT_{x_2}\dots T_{x_n}$ принадлежит L. Если $T_{x_1} = R_{x_1}$, то элемент

$$a = (b \circ x_1)T_{x_2}\dots T_{x_n} - bL_{x_1}T_{x_2}\dots T_{x_n} \equiv (b \circ x_1)T_{x_2}\dots T_{x_n},$$

где $x \equiv y$ обозначает что $x - y \in L$. Предположим, что уже доказано сравнение $a \equiv ((b \circ x_1)\circ \dots \circ x_{i-1})T_{x_i}\dots T_{x_n}$, $i > 1$. Если $T_{x_i} = L_{x_i}$, то

$$\begin{aligned} a &\equiv (\dots(b \circ x_1)\circ \dots \circ x_{i-1})T_{x_i}\dots T_{x_n} = \\ &= x_i(\dots(b \circ x_1)\circ \dots \circ x_{i-1})T_{x_{i+1}}\dots T_{x_n} = \\ &= \sum b_j T_{y_1}\dots T_{y_s},\ b_j \in L',\ y_i \in A,\ s < n. \end{aligned}$$

Таким образом по индуктивному предположению

$$a \equiv \sum b_j T_{y_1}\dots T_{y_s} \equiv 0.$$

Следовательно, на основании соотношения $dx = d \circ x - xd$, если $T_{x_i} = R_{x_i}$, то $a \equiv (\dots(b \circ x_1)\circ \dots \circ x_i)T_{x_{i+1}}\dots T_{x_n}$. Следовательно, по индукции имеем $a \equiv (\dots(b \circ x_1)\circ \dots \circ x_n) \equiv 0$. Лемма доказана.

Лемма 7. *$\mathcal{D}_J(A) = \mathcal{D}(A)$, где $\mathcal{D}(A)$ — идеал порожденный ассоциаторами в алгебре A.*

Доказательство. Ясно, что $\mathcal{D}_J(A) \subseteq \mathcal{D}(A)$. Докажем включение $\mathcal{D}(A) \subseteq \mathcal{D}_J(A)$. Из тождества Теймюллера следует, что

$$x(y,z,w) + (x,y,z)w \in \mathcal{D}_J(A) \tag{5}$$

где $x,y,z,w \in A$. С другой стороны в силу включения $x(y,z,w) + (y,z,w)x \in \mathcal{D}_J(A)$, из леммы 6 и линеаризации тождества Муфанга $(x,y,z)z = (x,zy,z)$ имеем

$$x(x,y,z) \in \mathcal{D}_J(A). \tag{6}$$

Последовательно применяя линеаризации тождества Муфанга, (5), лемму 6 и (6), по модулю $\mathcal{D}_J(A)$ имеем:

$$\begin{aligned} 0 &\equiv x(y,z,w) + (y,z,w)x \equiv x(y,z,w) - (y,z,x)w \equiv x(y,z,w) + y(z,x,w) \equiv \\ &\equiv x(y,z,w) - y(x,z,w) \equiv x(y,z,w) + x(y,z,w) = 2x(y,z,w). \end{aligned}$$

Таким образом $\mathcal{D}(A) = (A,A,A) + A(A,A,A) \subseteq \mathcal{D}_J(A)$. Лемма доказана.

Теорема 2. *Пусть A — правоальтернативная алгебра над ассоциативно-коммутативным кольцом $\Phi \ni \frac{1}{2}$. Тогда в ассоциаторной алгебре $\mathcal{D}(A)$ выполняется равенство $T^3_{288}(\mathcal{D}(A)) = 0$, где степень $T^3_{288}(\mathcal{D}(A))$ рассматривается в йордановой алгебре $\mathcal{D}(A)^{(+)}$.*

Доказательство. Пусть $l(f_i)=5$, $i=1,2,3$ и

$$(7)\qquad \begin{cases} f_1,\, f_3 \in \{f \in B \mid t(f) \in T_s,\ 3(l(f)+85) \le s+3\} \\ \quad f_2 \in \{f \in B \mid t(f) \in T_s,\ 3(l(f)+92) \le s+3\}. \end{cases}$$

Согласно [3], каждый элемент из $\mathcal{D}_J(A) = \mathcal{D}(A)$ имеет вид $\sum(m \circ x)\circ y\mathcal{D}_{x_1,y_1}\dots\mathcal{D}_{x_n,y_n}$ где $m \in (A,A,A)$, $x,y,x_i,y_i, \in A^{\#}$, $n \in \mathbb{N}\cup\{0\}$.

Пусть g_i выражение, получающееся из f_i заменой символа (x,y) и $(x,y)^*$ на такие элементы d_i' из $(\mathcal{D}_J(A))'$, что $d_i = (m_i \circ x_i)\circ y_i$.

В выражении $F = g_i T_1'^6 g_2 T_1'^6 g_3$, для m_i, который стоит на нечетном месте возьмем его выражение (2), а для m_i котоый стоит на четном месте его выражение (3).

Тогда

$$F = \sum a_1'a_2'a_3'(\ )a_4'a_5'a_6'T_{267}'x_1'x_2'x_3'(\ )^*\cdot$$
$$\cdot x_4'x_5'x_6'T_1'^6 b_1'b_2'b_3'(\ )b_4'b_5'b_6'T_{288}'y_1'y_2'y_3'(\ )^* y_4'y_5'y_6'\cdot$$
$$\cdot T_1'^6 c_1'c_2'c_3'(\ )c_4'c_5'c_6'T_{267}'z_1'z_2'z_3'(\ )^* z_4'z_5'z_6',$$

где ( ) означает (x,y), а $(\ )^*$ — $(x,y)^*$, T_i' означает t_i' для $t_i \in T_i$, а из одноименных символов, например из символов $a_1',\dots,a_6'$ по крайне мере три принадлежит в $1\cdot\Phi$. Запишим его в виде

$$F = \sum a_1'a_2'a_3'h_1x_4'x_5'x_6'T_1'^6 b_1'b_2'b_3'h_2y_4'y_5'y_6'T_1'^6\cdot c_1'c_2'c_3'h_3z_4'z_5'z_6' =$$
$$= \sum a_1'a_2'a_3'H_k z_4'z_5'z_6',$$

где

$$h_1 = (\ )a_4'a_5'a_6'T_{267}'x_1'x_2'x_3'(\ )^*, \qquad h_2,\ h_3 \in \beta,$$
$$H_k = h_1x_4'x_5'x_6'T_1'^6 \dots T_1'^6 c_1'c_2'c_3'h_3.$$

Используя равенство $(x\circ y)' - x'y' = y'x'$, выражение H_k можно записать в виде

$$H_k = \sum h_1\left(\prod_{i=1}^{6} r_i't_i'h_2\prod_{i=7}^{12} r_i't_i'\right)h_3, \quad \text{где} \quad r_i \in A^{\#},\ t_i \in T_1.$$

Дальше здесь применим (I):

$$H_k = \sum h_1 \prod_{i=1}^{6} \left[(r_i \circ t_i)' + \frac{1}{2}(r_i,t_i) - \frac{1}{2}(r_i,t_i)^* \right] \cdot$$

$$\cdot h_2 \cdot \prod_{i=7}^{12} \left[(r_i \circ t_i)' + \frac{1}{2}(r_i,t_i) - \frac{1}{2}(r_i,t_i)^* \right] h_3 = \sum \overline{h_1} T_1'^k \overline{h_2} T_1'^l \overline{h_3},$$

где $\overline{h_1} = h_1 \ldots (\,)^*$ или $\overline{h_1} = h_2$, а $\overline{h_2} = (\,) \ldots h_2 \ldots (\,)^*$ или $\overline{h_2} = h_2$, $\overline{h_3} = (\,) \ldots h_3$, или $\overline{h_3} = h_3$, и $0 \le k,\ l \le 6$.

Если k или $l \ge 2$, то еще раз применяя (I) окончательно получаем, что

$$H_k = \sum \overline{\overline{h}}_1 T_1'^k \overline{\overline{h}}_2 T_1'^l \overline{\overline{h}}_3 \quad \text{где} \quad 0 \le l,\ k \le 1,$$

$$\overline{\overline{h}}_1 = \overline{\overline{h}}_1 \ldots (\,)^* \quad \text{или} \quad \overline{\overline{h}}_1 = \overline{h}_1, \ldots.$$

Если хотя бы один из чисел k или l равно нулю, то согласно лемме 3 $\overline{\overline{h}}_1 T'^k \overline{\overline{h}}_2 T'^l \overline{\overline{h_3}} = 0$, когда $\overline{\overline{h_i}} \in Q_9$. В силу (7) условие $\overline{\overline{h_i}} \in Q_9$ автоматически выполняется, так как

$$3(l(\overline{\overline{h_i}}) + 36) \le 3(l(f_i) + 15 + 36) = 3(l(f_i) + 51) \le 3(l(f_i) + 85) \le s_i + 3, \quad i = 1,\ 3;$$

$$3(l(\overline{\overline{h}}_2) + 36) \le 3(l(f_2) + 24 + 36) = 3(l(f_2) + 60) \le 3(l(f_2) + 92) \le s_2 + 3.$$

Если $k = l = 1$, то согласно лемме 5, $\overline{\overline{h}}_1 T_1' \overline{\overline{h}}_2 T_1' \overline{\overline{h}}_3 = 0$, когда $\overline{\overline{h}}_i \in Q_{18}$. В силу (7) условие $\overline{\overline{h}}_i \in Q_{18}$ тоже автоматически выполняется, так как

$$3(l(\overline{\overline{h}}_i) + 72) \le 3(l(f_i) + 13 + 72) = 3(l(f_i) + 85) \le s_i + 3,$$
$$3(5 + 85) \le s_i + 3, \quad 267 \le s_i, \quad i = 1,\ 2;$$

$$3(l(\overline{\overline{h}}_2) + 72) \le 3(l(f_2) + 20 + 72) = 3(l(f_2) + 92) \le s_2 + 3,$$
$$3(5 + 92) \le s_2 + 3,\ 288 \le s_2.$$

Таким образом при условии (7) выполяется равенство

$$F = g_1 T_1'^6 g_2 T_1'^6 g_3 = 0.$$

Если в выражении F для g_i вместо $d_i = (m_i \circ x_i) \circ y_i$ рассмотрим элемент $d_i \mathcal{D}_{a,b} \ldots \mathcal{D}_{c,d} = \sum (n_i \circ a_i) \circ b_i$, где $n_i \in (A, A, A)$, $a_i, b_i \in A$, то тоже очевидно, что выполняется равенство $F = 0$.

Если обозначим $\mathcal{D} = \mathcal{D}(A)$, $T_i = T_i(\mathcal{D}) \subseteq T_i(A)$, то из равенства $F = 0$ следует, что

$$F \supseteq F_1 = \mathcal{D}' T'_{267} \mathcal{D}' T_1'^6 \mathcal{D}' T'_{288} \mathcal{D}' T_1'^6 \mathcal{D}' T'_{267} \mathcal{D}' = 0$$

где $T_i = T_i(\mathcal{D})$, так как $T_i(\mathcal{D}) \subseteq T_i(A)$.

Из включений

$$T_1'^6 \supseteq ((T_1 \circ T_1) \circ T_1)'((T_1 \circ T_1) \circ T_1)' \supseteq T_2' T_2' \supseteq (T_2 \circ T_2)' \supseteq (U_{T_2}(T_2))' \supseteq T_3',$$

$$\mathcal{D}' T_3' \mathcal{D}' \supseteq T_3'^3 \supseteq T_4' \qquad \mathcal{D}' T'_{267} T_4' \supseteq T'^3_{267} \supseteq T'_{268}$$

следует, что

$$0 = F_1 \supseteq T'_{268} T'_{288} T'_{268} \supseteq T'^3_{288} = 0.$$

Следовательно $T^3_{288} = 0$.

Следствие. *Если A свободная правоальтернативная алгебра над $\Phi \ni 1/2$, то для любого её невырожденного радикала $R(A)$ выполняется равенство $T^3_{288}(R(A)) = 0$.*

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ON A PROBLEM OF ASAAD

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1. Introduction

Throughout this paper G denotes a finite group. Our notations are standard and follow [3]. Moreover, $\Omega_1(P)$, $\mho_1(P)$ are defined to be $\langle x \in P: x^p = 1\rangle$ and $\langle x^p : x \in P\rangle$ respectively, where P is a finite p-group.

In [1] ASAAD proved:

THEOREM (ASAAD): *Let P be a Sylow p-subgroup of G such that $|P \cap P^x| = p^{p-1}$ for all $x \in G \setminus P$. If $N_G(P) = P$, then G possesses a normal p-complement, i.e. G is p-nilpotent.*

ASAAD asked are following question:

Let P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent and that $|\Omega_1(P) \cap \Omega_1(P^x)| \leq p^{p-1}$ for all $x \in G \setminus N_G)P)$. Is G p-nilpotent?

This paper gives a positive answer for odd p and a negative one for $p = 2$. We even get a sharper rêsult:

THEOREM: *Let P be a Sylow p-subgroup of G for an odd prime p. If $|\Omega_1(P \cap P^x)| \leq p^{p-1}$ for all $x \in G \setminus N_G(P)$ and $N_G(P)$ is p-nilpotent then also G is p-nilpotent.*

From the Theorem follows a sharper version of ASAAD's Theorem in the odd case:

COROLLARY: Let P be a Sylow p-subgroup of G such that $|P \cap P^x| \leq p^{p-1}$ for all $x \in G \setminus N_G(P)$. Is $N_G(P)$ p-nilpotent then also G is p-nilpotent.

The proof of the Corollary in the case $p=2$ is similar to that of ASAAD's Theorem in [1] and is therefore omitted.

The group $GL(2,3)$ is an example which shows that our Theorem is not true for $p=2$.

2. Preliminary lemmas

We give a list of results which we use for the proof of our Theorem:

LEMMA 1: [4] (Th. III.10.2., III.10.7., III.10.13. (P. HALL), p.322–333)

a) The p-group p is regular if one of the following conditions holds:

(i) $|P| \leq p^p$

(ii) $|P/\mho_1(P)| \leq p^{p-1}$.

b) If P is a regular p-group then $|P/\mho_1(P)| = |\Omega_1(P)|$.

LEMMA 2: [4] (Th. IV.8.1. (WIELANDT), p.447)

Let P be a regular Sylow p-subgroup of G such that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.

LEMMA 3: [3] (Th. 6.3.2., p.228)

If G is p-solvable and $O_{p'}(G)=1$ then $C_G(O_p(G)) \leq O_p(G)$.

LEMMA 4: Let P be a p-group, p odd with $|\Omega_1(P)| \leq p^{p-1}$. Then $|P/\Phi(P)| \leq p^{p-1}$.

PROOF. By [2] Lemma 3 (d) either P is of maximal class or regular. In the first case we have $|P/\Phi(P)| = p^2 \leq p^{p-1}$; in the second case by Lemma 1 b)

$$|P/\Phi(P)| \leq |P/\mho_1(P)| = |\Omega_1(P)| \leq p^{p-1}.$$

LEMMA 5: [4] (Th. IV.5.8. b) (FROBENIUS), p.436)

For a finite group G are equivalent:

(i) G has a normal p-complement.

(ii) $N_G(U)$ has a normal p-complement for each p-subgroup $U \neq 1$ of G.

3. Proof of the Theorem

Let G be a counterexample of minimal order.

(i) $O_{p'}(G)=1$ is trivial.

(ii) G is p-solvable.

PROOF. By Lemma 5 we have a maximal p-subgroup U, $1\neq U<P$ such that $N_G(U)$ is not p-nilpotent. Let U_1 be a Sylow p-subgroup of $N_G(U)$. Then $U_1>U$ and $T:=N_G(U_1)$ is p-nilpotent. Clearly $\left[U_1,O_{p'}(T)\right]=1$ and $T\cap N_G(U)$ is p-nilpotent. It is easy to see, that $N_G(U)$ satisfies the assumptions of our Theorem. It follows $G=N_G(U)$. Now by the choice of U we have that $N_{G/U}(V/U)$ is p-nilpotent for all p-groups $V>U$. By Lemma 5 $G/O_p(G)$ is p-nilpotent and so G is p-solvable.

(iii) G contains no proper normal subgroup W with $\left(|G/W|,p\right)=1$.

PROOF. Otherwise W would be p-nilpotent by the choice of G. Then by (i) $O_{p'}(W)=1$. It follows $W=P$ and P is normal in G. But then G is no counterexample.

Now let M be a maximal normal subgroup of G. By (ii), (iii) G/M must be a p-group such that $|G/M|=p$. Let $P_1=P\cap M$ be a Sylow p-subgroup of M. Consider $N_M(P_1)\geq N_G(P)\cap M$. If equality holds then by the assumption of the Theorem M is p-nilpotent. But then by (i) M and also G is a p-group, a contradiction.

Hence there exists $y\in N_G(P_1)\setminus N_G(P)$. We get $P_1=P\cap P^y$. By the assumption of our Theorem $|\Omega_1(P_1)|\leq p^{p-1}$ follows. Moreover M cannot be p-nilpotent by (iii). That means $N_M(P_1)$ is not p-nilpotent by our choice of G. Thus there exists $x\in N_G(P_1)$ such that x is a p'-element and $P_1\langle x\rangle$ is not p-nilpotent. It follows $\langle P,x\rangle$ is not p-nilpotent and by the choice of G we have $G=\langle P,x\rangle$. By (iii) G has the following structure

$$G=O_{pp'p}(G)>O_{pp'}(G)>O_p(G)=P_1$$

with $|G/O_{pp'}(G)|=p$ and $|\Omega_1(P_1)|\leq p^{p-1}$. Hence $|P_1/\Phi(P_1)|\leq p^{p-1}$ by Lemma 4. Now $G/\Phi(P_1)$ has a Sylow p-subgroup of order at most p^p which is regular by Lemnma 1 a) (i). Thus $G/\Phi(P_1)$ has a normal p-complement $L/\Phi(P_1)$ by Lemma 2. Let K be a p-complement of L. Then we have $[K,P_1]\subseteq L\cap P_1=\Phi(P_1)$. Therefore K acts trivially on $P_1/\Phi(P_1)$. Hence K acts trivially even of P_1. But $P_1=O_p(G)$ and therefore $C_G(P_1)\leq P_1$ by Lemma 3. It follows the contradiction $K=1$ and the Theorem is proved.

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FLÄCHEN 2. ORDNUNG IM EINFACH ISOTROPEN RAUM $I_3^{(1)}$

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1. Einleitung

Es bezeichne P_3 den dreidimensionalen projektiven Raum, ω eine Ebene in P_3 und $A_3 := P_3 \setminus \omega$ den zugeordneten affinen Raum. Ein affiner Raum A_3 heißt ein *einfach isotroper Raum* $I_3^{(1)}$, wenn in ihm eine Metrik über die Absolutfigur $\{\omega, f_1, f_2, F\}$ induziert wird, wobei f_1, f_2 konjugiert–komplexe Geraden aus ω mit dem reellen Schnittpunkt F sind. Im folgenden bezeichnen wir mit (x,y,z) affine Koordinaten in $I_3^{(1)}$ und mit $(x_0 : x_1 : x_2 : x_3)$ die zugehörigen projektiven Koordinaten. Es ist dann üblich, die Absolutfigur $\{\omega, f_1, f_2, F\}$ durch

$$\omega \cdots x_0 = 0, \quad f_1, f_2 \cdots x_0 = x_1^2 + x_2^2 = 0, \quad F(0:0:0:1) \tag{1.1}$$

zu beschreiben. Betreibt man *einfach isotrope Bewegungsgeometrie* in $I_3^{(1)}$, so betrachtet man als Fundamentalgruppe in $I_3^{(1)}$ nicht die allgemeine projektive Automorphismengruppe von $\{\omega, f_1, f_2, F\}$ (vgl. [3,6]), sondern wählt die sechsparametrige Bewegungsgruppe $B_6^{(1)}$

$$\begin{cases} \overline{x} = c_1 + x\cos\phi - y\sin\varphi \\ \overline{y} = c_2 + x\sin\varphi + y\cos\varphi \\ \overline{z} = c_3 + c_4 x + c_5 y + z \end{cases} \tag{1.2}$$

(vgl. [3,6]) als Fundamentalgruppe.

Die Flächen 2. Ordnung $\Phi^{(2)}$ des einfach isotropen Raumes $I_3^{(1)}$, die wir unter Verwendung projektiver Koordinaten im folgenden in der Gestalt

$$\begin{aligned} &a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 \\ &\quad + 2_{23}x_2x_3 + Ax_1x_0 + Bx_2x_0 + Cx_3x_0 + Dx_0^2 = 0 \end{aligned} \tag{1.3}$$

bzw. unter Benützung affiner Koordinaten in der Form

$$\begin{aligned}(1.4)\qquad & a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz \\ & + 2a_{23}yz + Ax + By + Cz + D = 0\end{aligned}$$

ansetzen, wurden bezüglich der isotropen Bewegungsgruppe erstmals in [1] klassifiziert. Diese Klassifikation ist jedoch nicht vollständig und berücksichtigt auch nur die regulären Flächen 2. Ordnung. In [5] bestimmt K. STRUBECKER zwar die einfach isotropen Krümmungslinien auf Mittelpunktsflächen 2. Ordnung des $I_3^{(1)}$ (vgl. [5,250]), geht jedoch auf eine Klassifikation dieser Flächen nicht ein. Wir geben im folgenden eine Gesamtklassifikation der Flächen 2. Ordnung des $I_3^{(1)}$, wobei wir uns geometrischer Methoden bedienen, die z.B. auch in [4] mit Erfolg eingesetzt werden. Wir werden vor allem die *Schnittkurve* $k := \Phi^{(2)} \cap \omega$ zur Klassifikation heranziehen, und beachten weiters, daß in der Fernebene ω des projektiv erweiterten isotropen Raumes $I_3^{(1)}$ durch die Gruppe $B_6^{(1)}$ eine zur ebenen *euklidischen Geometrie duale Geometrie* induziert wird (vgl. [3,20]). Die Schnittkurve k kann sein:

A: *Ein Dual–Kegelschnitt* (i.f. bezeichnet als D–Kegelschnitt), d.h. eine irreduzible Kurve 2. Ordnung, die bezüglich der dualen euklidischen Geometrie in ω weiter klassifiziert werden muß.

B: *Ein Geradenpaar*

C: *Ein Linienelement bzw. eine Doppelgerade*

Im folgenden geben wir die detaillierte Klassifikation der Kurven 2. Ordnung in ω gemäß den Hauptfällen **A**, **B** und **C** an, wobei wir gleichzeitig die zugehörigen Flächen 2. Ordnung bestimmen, und bezüglich der Gruppe $B_6^{(1)}$ auf Normalform transformieren. Die beiden folgenden Hilfssätze werden gelegentlich nützlich sein.

HILFSSATZ 1. *Besitzt eine Fläche 2. Ordnung des* $I_3^{(1)}$ *die Darstellung*

$$(1.5)\qquad a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + Ax + By + Cz + D = 0$$

mit $a_{11} \neq 0$, *so kann man durch eine Schiebung in x–Richtung stets* $A = 0$ *erreichen. Analoges gilt sinngemäß für die Koeffizienten B und C, wenn* $a_{22} \neq 0$ *bzw.* $a_{33} \neq 0$ *ist; die Schiebungen erfolgen dann in* y*- bzw.* z*-Richtung.*

BEWEIS. Sei $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$, dann wenden wir auf (1.5) die Schiebung $\left\{x = \overline{x} - \frac{A}{2a_{11}}, y = \overline{y} - \frac{B}{2a_{22}}, z = \overline{z} - \frac{C}{2a_{33}}\right\}$ an. ∎

HILFSSATZ 2. *Besitzt eine Fläche 2. Ordnung des* $I_3^{(1)}$ *die Darstellung (1.5) mit* $a_{11}=0$, $A\neq 0$ *bzw.* $a_{22}=0$, $B\neq 0$ *bzw.* $a_{33}=0$, $C\neq 0$, *dann kann man durch eine Schiebung in* x*-bzw.* y*-bzw.* z*-Richtung stets erreichen*

$$a_{22}y^2+a_{33}z^2+Ax=0 \quad bzw. \tag{1.6a}$$

$$a_{11}x^2+a_{33}z^2+By=0 \quad bzw. \tag{1.6b}$$

$$a_{11}x^2+a_{22}y^2+Cz=0. \tag{1.6c}$$

BEWEIS. Durch Anwendung von *Hilfssatz 1* erhalten wir $a_{22}\bar{y}^2+a_{33}\bar{z}^2+$ $+A\bar{x}+\bar{D}=0$, wobei $\bar{D}$ das neue Absolutglied bezeichnet. Nun wenden wir die Schiebung $\left\{\bar{x}=x-\frac{\bar{D}}{A}, \bar{y}=y, \bar{z}=z\right\}$ an. Für (1.6 b,c) geht der Beweis analog. ∎

2. Die Fernkurve $k:=\Phi^{(2)}\cap\omega$ ist ein irreduzibler Dual-Kegelschnitt.

Die Metrik einer euklidischen Ebene E_2 wird bekanntlich durch die beiden konjugiert-komplexen absoluten Kreispunkte F_1, F_2 auf der Ferngeraden f dieser Ebene geregelt. Unterwirft man diese Absolutfigur dem Dualitätsprinzip, so erhält man die Absolutfigur $\{f_1, f_2, F\}$ der Fernebene ω des $I_3^{(1)}$. Aus den irreduziblen Kegelschnitten in E_2 entstehen durch diese Dualisierung die folgenden D-Kegelschnitte k (vgl. Abb.1.):

1. *Nullteiliger D-Kegelschnitt:* Dieser Kegelschnitt k ist dual zu einem nullteiligen Kegelschnitt in E_2.

2. *Nullteiliger Fernkreis:* Dieser Kegelschnitt k ist dual zu einem nullteiligen euklidischen Kreis in E_2. k berührt ebenfalls f_1 und f_2.

3. *Einteiliger Fernkreis:* Dieser Kegelschnitt k ist dual zu einem euklidischen Kreis, der bekanntlich F_1 und F_2 als Punkte enthält. Demnach berührt k die beiden absoluten Geraden f_1 und f_2.

4. *D-Hyperbel:* Dies ist ein Kegelschnitt in ω, der F nicht enthält und an den sich von F aus zwei reelle Tangenten legen lassen. F ist ein Außenpunkt von k.

5. *D-Ellipse:* Dies ist ein Kegelschnitt in ω, der F nicht enthält und an den sich von F aus keine reellen Tangenten legen lassen. F ist ein Innenpunkt von k.

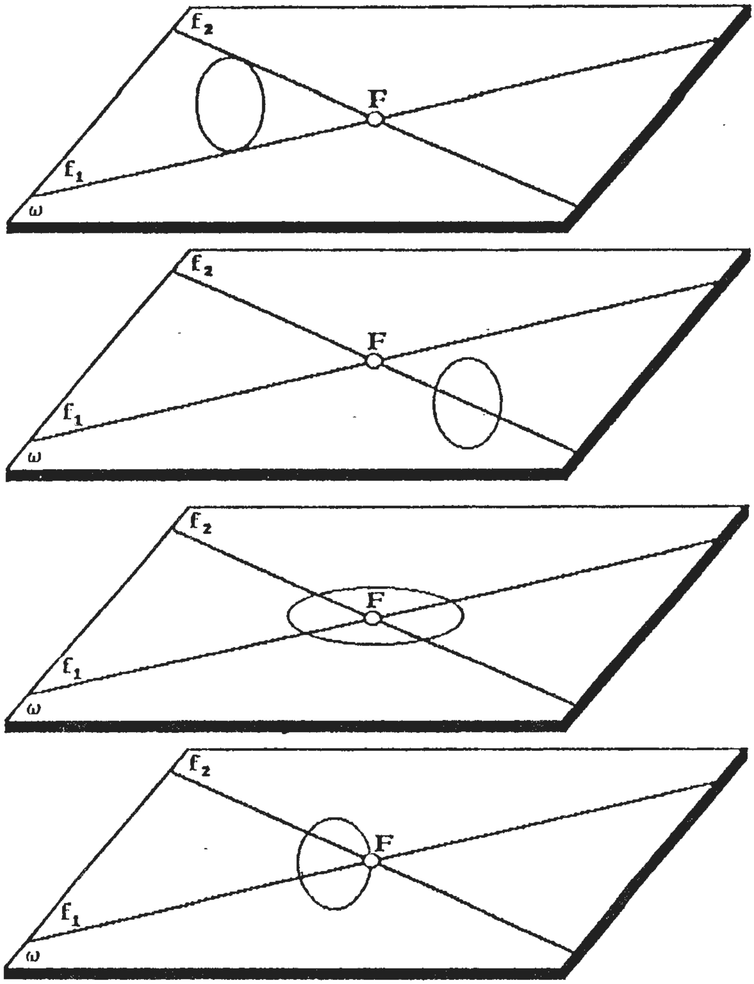

Abb. 1

6. *D-Parabel:* Dies ist ein Kegelschnitt in ω, der F enthält.

Sei zunächst $F \notin k$, d.h., wir betrachten die Fälle 1–5. Nach (1.4) lautet k

(2.1) $$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = 0$$

Wegen $F \notin k$ gilt $a_{33} \neq 0$. Wenden wir auf (2.1) die isotrope Bewegung $\left\{x = \bar{x}, y = \bar{y}, z = \bar{z} - \frac{a_{13}}{a_{33}}\bar{x} - \frac{a_{23}}{a_{33}}\bar{y}\right\}$ an, so entsteht

$$b_{11}\bar{x}^2 + b_{22}\bar{y}^2 + b_{33}\bar{z}^2 + 2b_{12}\bar{x}\,\bar{y} = 0. \tag{2.2}$$

Gilt $b_{12} \neq 0$, dann kann durch eine Drehung $\{\bar{x} = x\cos\varphi - y\sin\varphi, \bar{y} = x\sin\varphi + + y\cos\varphi, \bar{z} = z\}$ mit $\cot 2\varphi = \frac{b_{11}-b_{22}}{2b_{12}}$ die Normalform

$$c_{11}x^2 + c_{22}y^2 + c_{33}z^2 = 0 \tag{2.3}$$

von k erreicht werden. Für $b_{12} = 0$ stellt (2.2) schon die Normalform dar.

FALL 1. $\operatorname{sgn} c_{11} = \operatorname{sgn} c_{22} = \operatorname{sgn} c_{33}$.

k ist ein nullteiliger Kegelschnitt und die zugehörigen Flächen 2. Ordnung lauten

$$c_{11}x^2 + c_{22}y^2 + c_{33}z^2 + Ax + By + Cz + D = 0. \tag{2.4}$$

Nach dem *Hilfssatz 1* kann man $A = B = C = 0$ erreichen und es entsteht die Normalform

$$c_{11}x^2 + c_{22}y^2 + c_{33}z^2 + \overline{D} = 0. \tag{2.A1–3}$$

Für $\overline{D} = 0$ liegt ein *nullteiliger Kegel* $\Phi_1^{(2)}$, für $c_{11} \cdot \overline{D} > 0$ ein *nullteiliges Ellipsoid* $\Phi_2^{(2)}$ und für $c_{11} \cdot \overline{D} < 0$ ein *Ellipsoid* $\Phi_3^{(2)}$ vor.

FALL 2. $c_{11} = c_{22}$, $\operatorname{sgn} c_{11} = \operatorname{sgn} c_{33}$.

In diesem Fall ist k ein nullteiliger Fernkreis. Wie im Fall 1 erhält man als Normalform für die zugehörigen Flächen 2. Ordnung

$$x^2 + y^2 + \bar{c}_{33}z^2 + \overline{D} = 0, \tag{2.A4–6}$$

wobei $\bar{c}_{33} := \frac{c_{33}}{c_{11}} > 0$ gesetzt wurde. Für $\overline{D} = 0$ liegt ein *nullteiliger Drehkegel* $\Phi_4^{(2)}$, für $\bar{c}_{33} \cdot \overline{D} > 0$ ein *nullteiliger Drehellipsoid* $\Phi_5^{(2)}$ und für $\bar{c}_{33} \cdot \overline{D} < 0$ ein *reelles Drehellipsoid* $\Phi_6^{(2)}$ vor.

FALL 3. $c_{11} = c_{22}$, $\operatorname{sgn} c_{11} \neq \operatorname{sgn} c_{33}$.

k ist ein einteiliger Fernkreis. Wie im Fall 1 erhält man als Normalform der zugehörigen Flächen 2. Ordnung

$$x^2 + y^2 + \bar{c}_{33}z^2 + \overline{D} = 0, \tag{2.A7–9}$$

wobei $\bar{c}_{33} := \frac{c_{33}}{c_{11}} < 0$ gesetzt wurde. Für $\overline{D} = 0$ liegt ein *einteiliger Drehkegel* $\Phi_7^{(2)}$, für $\bar{c}_{33} \cdot \overline{D} > 0$ ein *einschaliges Drehhyperboloid* $\Phi_8^{(2)}$ und für $\bar{c}_{33} \cdot \overline{D} < 0$ ein *zweischaliges Drehhyperboloid* $\Phi_9^{(2)}$ vor.

FALL 4. Schreibt man (2.3) in projektiven Koordinaten $(x_1 : x_2 : x_3)$ in $\omega \cdots x_0 = 0$, so erhält man

$$c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 = 0. \tag{2.5}$$

Die Polare eines Punktes $P(p_1 : p_2 : p_3)$ bezüglich (2.5) lautet dann

$$c_{11}x_1p_1 + c_{22}x_2p_2 + c_{33}x_3p_3 = 0. \tag{2.6}$$

Speziell ergibt sich für die Polare p des absoluten Punktes $F(0 : 0 : 1)$ die Gleichung $x_3 = 0$. Im Fall einer D-Hyperbel müssen die Schnittpunkte von p mit (2.5) reell sein. Demnach muß in (2.5) $\operatorname{sgn} c_{11} \neq \operatorname{sgn} c_{22}$ gelten. Die Normalform von k lautet somit

$$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 = 0, \tag{2.7}$$

wobei $\bar{c}_{22} := \frac{c_{22}}{c_{11}} < 0$, $\bar{c}_{22} := \frac{c_{22}}{c_{11}}$ gesetzt wurde. Wie im Fall 1 gewinnt man als Normalform der entsprechenden Flächen 2. Ordnung

$$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 = 0 \tag{2.A10–11}$$

bzw.

$$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 + \overline{D} = 0 \tag{2.A12–15}$$

mit $\overline{D} \neq 0$. Für $\bar{c}_{33} > 0$ bzw. für $\bar{c}_{33} < 0$ in (2.A10–11) bezeichnen wir diesen reellen Kegel als *Kegel 1. Art* $\Phi_{10}^{(2)}$ bzw. *Kegel 2. Art* $\Phi_{11}^{(2)}$. Zur geometrischen Unterscheidung der beiden Kegeltypen bestimmen wir ihre isotropen Kreisschnitte. Mit $x^2 = -y^2$ folgt

$$z\sqrt{\bar{c}_{33}} = \pm y\sqrt{1 - \bar{c}_{22}}. \tag{2.8}$$

Wegen $\bar{c}_{22} < 0$ ist $\sqrt{1 - \bar{c}_{22}}$ stets reell. Somit stellt (2.8) für $\bar{c}_{33} > 0$ zwei reelle Ebenen dar, für $\bar{c}_{33} < 0$ sind diese Ebenen kojugiert-komplex. Daher besitzt ein Kegel 1. Art stets zwei Scharen *reeller isotroper Kreisschnitte*, während ein Kegel 2. Art nur konjugiert-komplexe Kreisschnitte besitzt. Für $\bar{c}_{33} > 0$ und $\overline{D} < 0$ bzw. für $\bar{c}_{33} < 0$ und $\overline{D} > 0$ liegt in (2.A12–15) ein *einschaliges Hyperboloid* $\Phi_{12}^{(2)}$ bzw. $\Phi_{13}^{(2)}$ mit einem Kegel 1. Art bzw. 2. Art als Asymptotenkegel vor, für $\bar{c}_{33} > 0$ und $\overline{D} > 0$ bzw. für $\bar{c}_{33} < 0$ und $\overline{D} < 0$ ergibt sich ein *zweischaliges Hyperboloid* $\Phi_{14}^{(2)}$ bzw. $\Phi_{15}^{(2)}$ mit einem Kegel 1. Art bzw. 2. Art als Asymptotenkegel.

FALL 5. Die Polare $p\{x_3 = 0\}$ schneidet die Fernkurve k jetzt nicht reell. Damit folgt aus (2.5) $\operatorname{sgn} c_{11} = \operatorname{sgn} c_{22} \neq \operatorname{sgn} c_{33}$. Wie im Fall 1 erhält man damit als Normalform einer D-Ellipse

$$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 = 0 \tag{2.9}$$

mit $\bar{c}_{22} := \frac{c_{22}}{c_{11}} > 0$ und $\bar{c}_{33} := \frac{c_{33}}{c_{11}} < 0$. Die zugehörigen Flächen 2. Ordnung lauten in Normalform

(2.A16) $$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 = 0$$

bzw.

(2.A17–18) $$x^2 + \bar{c}_{22}y^2 + \bar{c}_{33}z^2 + \overline{D} = 0, \quad \text{mit } \overline{D} \neq 0.$$

(2.A16) ist ein *reeller Kegel* $\Phi_{16}^{(2)}$, den wir als *Kegel 3. Art* bezeichnen. Die Ebenen $x = 0$ und $y = 0$ sind Symmetrieebenen von (2.A16) und besitzen somit geometrische Bedeutung, ebenso ihre Schnittgerade $g \cdots x = y = 0$, die wir als *vollisotrope Kegelachse* bezeichnen. Alle isotropen Ebenen durch g schneiden (2.A16) nach zwei stets reellen Erzeugenden, denn aus $y = k \cdot x$ folgt mittels (2.A16) $x^2(1 + \bar{c}_{22}k^2) = \bar{c}_{33}z^2$, wobei $1 + \bar{c}_{22}k^2 > 0$ und $\bar{c}_{33} > 0$ gilt. Durch diese Eigenschaft unterscheidet sich aber ein Kegel 3. Art eindeutig von einem Kegel 1. Art oder 2. Art.

Für $\overline{D} > 0$ bzw. $\overline{D} < 0$ liegt in (2.A17–18) ein *zweischaliges Hyperboloid* $\Phi_{17}^{(2)}$ bzw. ein *einschaliges Hyperboloid* $\Phi_{18}^{(2)}$ mit einem Asymptotenkegel 3. Art vor.

FALL 6. Nun sei F ein Punkt von k, d.h. k ist eine D-Parabel. Nach (1.4) gilt dann $a_{33} = 0$ und k lautet

(2.10) $$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = 0.$$

Wir wenden auf (2.10) die Drehung $\{x = \bar{x}\cos\varphi - \bar{y}\sin\varphi,\ y = \bar{x}\sin\varphi + \bar{y}\cos\varphi,\ z = \bar{z}\}$ mit $\tan\varphi = -\frac{a_{13}}{a_{23}}$ an. Dann entsteht als Gleichung von k

(2.11) $$b_{11}\bar{x}^2 + b_{22}\bar{y}^2 + 2b_{12}\bar{x}\,\bar{y} + 2b_{23}\bar{y}\,\bar{z} = 0.$$

Für $a_{23} \neq 0$ gilt sicher $b_{23} \neq 0$. Ist hingegen $a_{23} = 0$, so folgt $\varphi = \frac{\pi}{2}$ und man hat $b_{23} = -a_{13}$. Wäre $b_{23} = 0$, so würde $a_{13} = 0$ folgen und (2.10) wäre reduzibel. Wir dürfen daher i.f. stets $b_{23} \neq 0$ voraussetzen. Wenden wir nun die Transformation $\left\{\bar{x} = x, \bar{y} = y, \bar{z} = z - \frac{b_{22}}{2b_{23}}y - \frac{b_{12}}{b_{23}}x\right\}$ aus $B_6^{(1)}$ an, so entsteht als Normalform von k

(2.12) $$x^2 + 2c_{23}yz = 0$$

mit $c_{23} := \frac{b_{23}}{b_{11}}$. Die zugehörigen Flächen 2. Ordnung lauten

(2.13) $$x^2 + 2c_{23}yz + Ax + By + Cz + D = 0.$$

Mittels *Hilfssatz 1* kann $A = 0$ erreicht werden, und durch eine Schiebung in y- und z-Richtung kann noch $B = C = 0$ erzwungen werden. Damit erhält man als Normalform

(2.A19) $$x^2 + 2c_{23}yz = 0$$

bzw.

(2.A20–23) $$x^2+2c_{23}yz+\overline{D}=0, \text{ mit } \overline{D}\neq 0.$$

(2.A19) ist ein Kegel $\Phi^{(2)}_{19}$, den wir als *Kegel 4. Art* bezeichnen.

Für $c_{23}>0$ und $\overline{D}<0$ bzw. für $c_{23}<0$ und $\overline{D}<0$ liegt in (2.A20–23) ein *einschaliges Hyperboloid* $\Phi^{(2)}_{20}$ bzw. $\Phi^{(2)}_{22}$, für $c_{23}>0$ und $\overline{D}>0$ bzw. für $c_{23}<0$ und $\overline{D}>0$ ein *zweischaliges Hyperboloid* $\Phi^{(2)}_{21}$ bzw. $\Phi^{(2)}_{23}$ vor.

Wenden wir auf (2.A20–23) die Affinität $\{x=\overline{x}, y=\overline{y}+\overline{z}, z=\overline{y}-\overline{z}\}$ an, so entsteht als Flächengleichung $\overline{x}^2+2c_{23}(\overline{y}^2-\overline{z}^2)+\overline{D}=0$, woraus ersichtlich ist, daß die Flächen $\Phi^{(2)}_{20}$ und $\Phi^{(2)}_{22}$ bzw. $\Phi^{(2)}_{21}$ und $\Phi^{(2)}_{23}$ zwei Geradenscharen bzw. keine reellen Geraden tragen. $\Phi^{(2)}_{22}$ bzw. $\Phi^{(2)}_{23}$ sind durch die Spiegelung $\{x=\overline{x}, y=-\overline{y}, z=\overline{z}\}$ an der Kreisschnittebene $y=o$ auf $\Phi^{(2)}_{20}$ bzw. auf $\Phi^{(2)}_{21}$ abbildbar. Die Flächen $\Phi^{(2)}_{20}$ und $\Phi^{(2)}_{22}$ bzw. $\Phi^{(2)}_{21}$ und $\Phi^{(2)}_{23}$ sind in der Gruppe $B^{(1)}_6$ als verschieden anzusehen, da $B^{(1)}_6$ keine Spiegelung obiger Art enthält. Damit sind alle Fälle von A erledigt.

3. Die Fernkurve $k:=\Phi^{(2)}\cap\omega$ ist ein Geradenpaar

Es sei zunächst ein reelles Geradenpaar $\{g_1, g_2\}$ mit dem Schnittpunkt G in der Fernebene ω gegeben. Dieses Geradenpaar kann 3 Lagemöglichkeiten bezüglich $\{f_1, f_2, F\}$ besitzen (vgl. Abb.2.).

FALL 1. $G\neq F$ und $F\notin g_1\cup g_2$.

Durch eine isotrope Bewegung kann man erreichen, daß G die projektiven Koordinaten $G(0:1:0:0)$ erhält. Jede Gerade g durch G in ω wird dann durch $x_0=u_2x_2+u_3x_3=0$ beschrieben. Wegen $F\notin g$ gilt hierbei $u_3\neq 0$, und man kann g in affinen Koordinaten mittels $uy+z=0$ erfassen. Werden g_1 und g_2 zunächst durch $u_1y+z=0$ bzw. $u_2y+z=0$ beschrieben, so kann man durch eine isotrope Bewegung noch erreichen, daß diese Geradengleichungen $z+ay=0$ bzw. $z-ay=0$ lauten, wobei $a\neq 0$ gilt. Zum Beweis setzen wir eine isotrope Bewegung in der Gestalt $\{x=\overline{x}, y=\overline{y}, z=\overline{z}+\gamma\overline{y}\}$ an. Dann folgt aus obigen Gleichungen $\overline{y}(u_1+\gamma)+\overline{z}=0$ bzw. $\overline{y}(u_2+\gamma)+\overline{z}=0$, woraus sich $\gamma=-\frac{1}{2}(u_1+u_2)$ ergibt. Es gilt $a\neq 0$ wegen $g_1\neq g_2$. Damit lautet die Normalform von k

(3.1) $$z^2-a^2y^2=0.$$

Nach *Hilfssatz 1* gehören zu (3.1) die Flächen 2. Ordnung

(3.2) $$z^2-a^2y^2+Ax+\overline{D}=0.$$

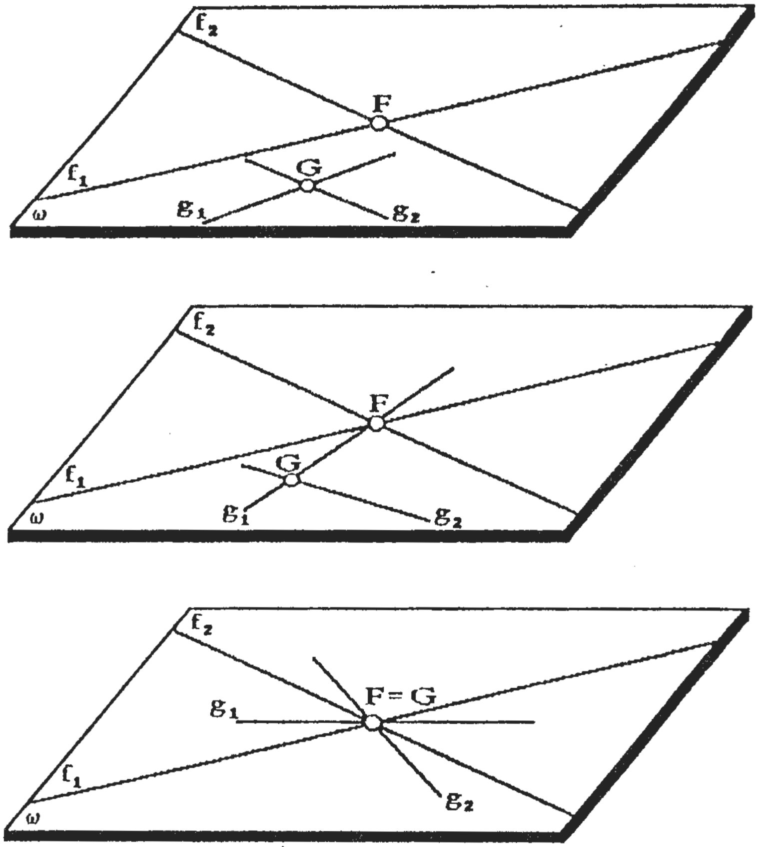

Abb. 2

Für $A \neq 0$ folgt nach *Hilfssatz* 2 die Normalform

(3.B24) $$z^2 - a^2y^2 + Ax = 0$$

eines *hyperbolischen Paraboloids* $\Phi_{24}^{(2)}$. Gilt $A = 0$, so erhalten wir

(3.B25–26) $$z^2 - a^2y^2 + \overline{D} = 0.$$

Für $\overline{D} \neq 0$ liegt ein *hyperbolischer Zylinder* $\Phi_{25}^{(2)}$, für $\overline{D} = 0$ ein ***reelles schneidendes** Ebenenpaar* $\Phi_{26}^{(2)}$ vor.

FALL 2. $G \neq F$ und $F \in g_1$.

Wie im Fall 1 erhalten wir zunächst für k

(3.3) $$y(z - ay) = 0.$$

Wenden wir auf (3.3) die isotrope Bewegung $\{x=\overline{x}, y=\overline{y}, z-ay=\overline{z}\}$ an, so gewinnen wir als Normalform von k

(3.4) $$\overline{y}\,\overline{z}=0.$$

Die zugehörigen Flächen 2. Ordnung lauten

(3.5) $$\overline{y}\,\overline{z}+A\overline{x}+B\overline{y}+C\overline{z}+D=0.$$

Durch eine Schiebung in y- und z-Richtung kann man erreichen

(3.6) $$yz+Ax+\overline{D}=0.$$

Gilt in (3.6) $A\neq 0$, dann erhält man nach *Hilfssatz 2* die Normalform

(3.B27) $$yz+Ax=0.$$

Dies ist ein *hyperbolisches Paraboloid* $\Phi_{27}^{(2)}$. Gilt $A=0$, so erhalten wir die Gleichung

(3.B28–29) $$yz+\overline{D}=0,$$

welche für $\overline{D}\neq 0$ einen *hyperbolischen Zylinder* $\Phi_{28}^{(2)}$, für $\overline{D}=0$ ein *reelles schneidendes Ebenenpaar* $\Phi_{29}^{(2)}$ beschreibt.

FALL 3. $F=G$.

Man erhält wie oben die Normalform von k zu

(3.7) $$y^2-a^2x^2=0$$

mit $a\neq 0$. Nach *Hilfssatz 1* lauten die zugehörigen Flächen 2. Ordnung

(3.8) $$y^2-a^2x^2+Cz+\overline{D}=0.$$

Für $C\neq 0$ kann man nach *Hilfssatz 2* noch $\overline{D}=0$ erreichen und man erhält

(3.B30) $$y^2-a^2x^2+Cz=0.$$

Dies ist ein *hyperbolisches Paraboloid* $\Phi_{30}^{(2)}$. Für $C=0$ findet man

(3.B31–32) $$y^2-a^2x^2+\overline{D}=0.$$

Für $\overline{D}\neq 0$ liegt ein *hyperbolischer Zylinder* $\Phi_{31}^{(2)}$, für $\overline{D}=0$ ein *reelles schneidendes isotropes Ebenenpaar* $\Phi_{32}^{(2)}$ vor.

Nun sei $\{g_1, g_2\}$ ein konjugiert-komplexes Geradenpaar mit dem reellen Schnittpunkt G. Dann gibt es 3 Lagemöglichkeiten (vgl. Abb.3.).

FALL 4. $G\neq F$.

Wie im Fall 1 kann man erreichen, daß G die projektiven Koordinaten $G(0:1:0:0)$ erhält, und die Geraden g_1 bzw. g_2 durch $z-Ly=0$ bzw.

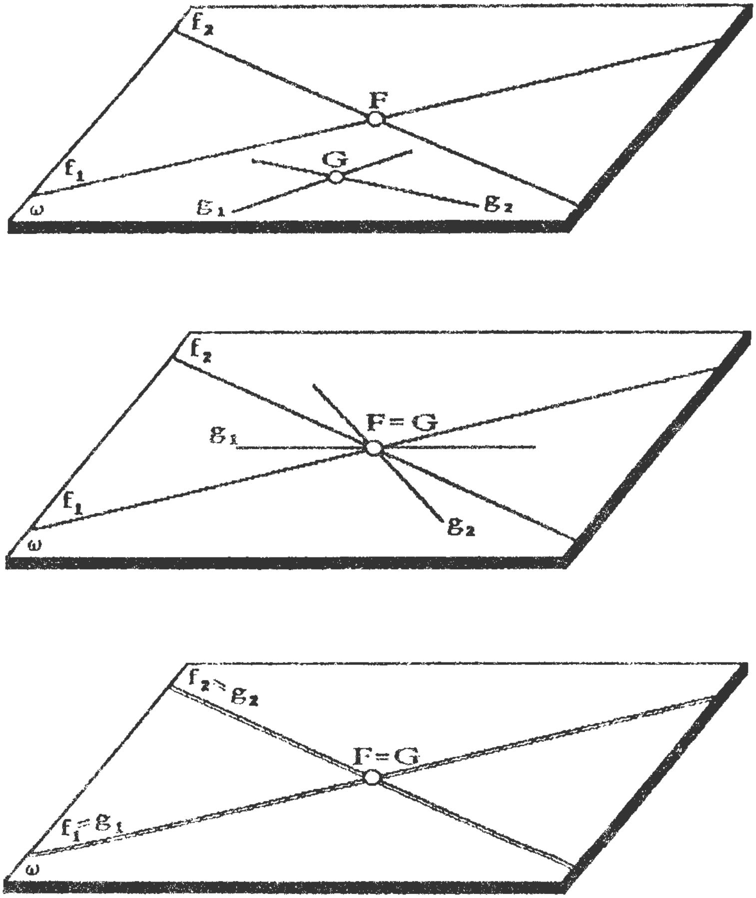

Abb. 3

$z - \overline{L}y = 0$ beschrieben werden, wobei $L = a + ib$ und $\overline{L} = a - ib$, $a,b \in \mathbb{R}$, $b \neq 0$ gilt. Wenden wir die isotrope Bewegung $\{x = \overline{x}, y = \overline{y}, z = \overline{z} + a\overline{y}\}$ an, so entsteht als Normalform von k

(3.9) $$\overline{z}^2 + b^2\overline{y}^2 = 0.$$

Die zugehörigen Flächen 2. Ordnung lauten

(3.10) $$\overline{z}^2 + b^2\overline{y}^2 + A\overline{x} + B\overline{y} + C\overline{z} + D = 0.$$

Mit *Hilfssatz 1* erhält man die Flächengleichung

(3.11) $$z^2 + b^2y^2 + Ax + \overline{D} = 0.$$

Gilt in (3.11) $A \neq 0$, dann gewinnt man nach *Hilfssatz 2*

(3.B33) $$z^2 + b^2y^2 + Ax = 0.$$

Dies ist ein *elliptisches Paraboloid* $\Phi_{33}^{(2)}$. Gilt in (3.11) $A=0$, so findet man die Gleichung

(3.B34–36) $$z^2+b^2y^2+\overline{D}=0,$$

welche für $\overline{D}>0$ einen *nullteiligen Zylinder* $\Phi_{34}^{(2)}$ bzw. für $\overline{D}<0$ einen *elliptischen Zylinder* $\Phi_{35}^{(2)}$ beschreibt. Für $\overline{D}=0$ liegt ein *konjugiert-komplexes Ebenenpaar* $\Phi_{36}^{(2)}$ vor.

FALL 5. $G=F$ und $(f_1\cup f_2)\cap(g_1\cup g_2)=F$.

Wie im Fall 4 erhält man zunächst für k

(3.12) $$(y-Lx)(y-\overline{L}x)=0$$

mit $L=a+ib$, $\overline{L}=a-ib$, $a,b\in\mathbb{R}\setminus\{0\}$. Nach Anwendung der isotropen Bewegung $\{x=\overline{x}\cos\varphi-\overline{y}\sin\varphi, y=\overline{x}\sin\varphi+\overline{y}\cos\varphi, z=\overline{z}\}$ mit $\cot 2\varphi = = \frac{1-(a^2+b^2)}{2a}$, entsteht als Normalform von k

(3.13) $$\overline{x}^2+d^2\overline{y}^2=0$$

mit

$$d^2=\frac{\sin^2\varphi+a\sin 2\varphi+(a^2+b^2)\cos^2\varphi}{\cos^2\varphi+a\sin 2\varphi+(a^2+b^2)\sin^2\varphi},$$

$d^2>0$, $d^2\neq 1$. Die zugehörige Flächengleichung lautet

(3.14) $$\overline{x}^2+d^2\overline{y}^2+A\overline{x}+B\overline{y}+C\overline{z}+D=0.$$

Nach *Hilfssatz 1* gewinnt man zunächst

(3.15) $$x^2+d^2y^2+Cz+\overline{D}=0.$$

Gilt in (3.15) $C\neq 0$, dann erhalten wir nach *Hilfssatz 2*

(3.B37) $$x^2+d^2y^2+Cz=0.$$

Dies ist ein *elliptisches Paraboloid* $\Phi_{37}^{(2)}$. Gilt in (3.15) $C=0$, so findet man die Gleichung

(3.B38–40) $$x^2+d^2y^2+\overline{D}=0,$$

welche für $\overline{D}>0$ einen *nullteiligen Zylinder* $\Phi_{38}^{(2)}$ bzw. für $\overline{D}<0$ einen *elliptischen Zylinder* $\Phi_{39}^{(2)}$ bzw. für $\overline{D}=0$ ein *konjugiert-komplexes Ebenenpaar* $\Phi_{40}^{(2)}$ beschreibt.

FALL 6. $g_1=f_1$ und $g_2=f_2$.

Die Normalform von k lautet

(3.16) $$(x - iy)(x + iy) = 0.$$

Nach *Hilfssatz 1* entsteht die zugehörige Flächengleichung

(3.17) $$x^2 + y^2 + Cz + \overline{D} = 0.$$

Gilt in (3.17) $C \neq 0$, dann erhält man nach *Hilfssatz 2*

(3.B41) $$x^2 + y^2 + Cz = 0.$$

Dies ist ein *parabolische Sphäre* $\Phi_{41}^{(2)}$ (vgl. [3,66]). Gilt in (3.17) $C = 0$, so gewinnt man

(3.B42–44) $$x^2 + y^2 + \overline{D} = 0.$$

Für $\overline{D} > 0$ liegt dann ein *nullteiliger vollisotroper Drehzylinder* $\Phi_{42}^{(2)}$, für $\overline{D} < 0$ ein *einteiliger vollisotroper Drehzylinder* $\Phi_{43}^{(2)}$ und für $\overline{D} = 0$ ein *konjugiert-komplexes Ebenenpaar* $\Phi_{44}^{(2)}$ vor. Damit sind alle Fälle von B erledigt.

4. Die Fernkurve $k := \Phi^{(2)} \cap \omega$ ist ein Linienelement bzw. eine Doppelgerade

Bezeichnet s die Doppelgerade bzw. den Träger des Linienelementes (s, S) — wobei S den inzidenten Punkt angibt — dann erhält man 3 Lagemöglichkeiten (vgl. Abb.4).

FALL 1. $F \notin s$ und $F \neq S$.

Durch eine isotrope Bewegung kann man erreichen, daß s durch $x_0 = x_3 = 0$ beschrieben wird. Im Fall eines Linienelementes kann man durch eine weitere isotrope Bewegung erreichen, daß $S(0:1:0:0)$ gilt. Damit hat man für k

(4.1) $$z^2 = 0$$

und die zugehörigen Flächen 2. Ordnung lassen sich wegen *Hilfssatz 1* in der Gestalt

(4.2) $$z^2 + Ax + By + \overline{D} = 0$$

ansetzen. Gilt $(A, B) \neq (0,0)$, dann kann in (4.2) durch eine Schiebung $\overline{D} = 0$ erreicht werden. Im Fall eines Linienelementes ist $\Phi^{(2)}$ ein *parabolischer Zylinder* $\Phi_{45}^{(2)}$, dessen Erzeugenden den Fernpunkt $S(0:1:0:0)$ enthalten. Die ebenen Schnitte von (4.2) $y = konst.$ müssen dann Geraden sein, woraus $A = 0$ folgt. Damit erhalten wir als Normalform

(4.C45) $$z^2 + By = 0.$$

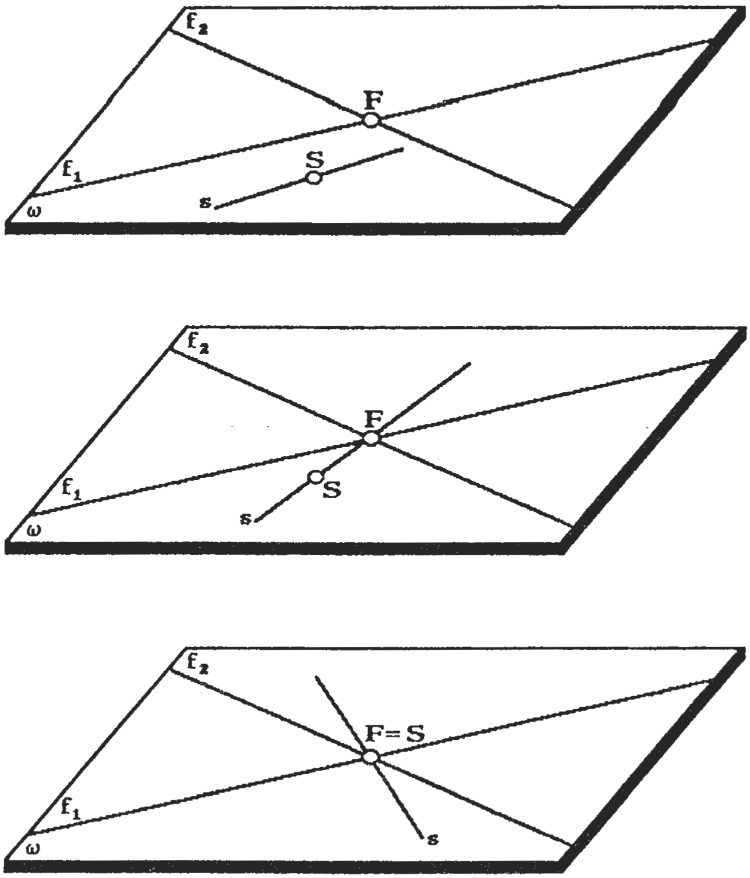

Abb. 4

Gilt $A = B = 0$, so gewinnt man aus (4.2)

$$z^2 + \overline{D} = 0 \qquad \text{(4.C46–48)}$$

und dies ist für $\overline{D} > 0$ ein *konjugiert-komplexes Ebenenpaar* $\Phi_{46}^{(2)}$, für $\overline{D} < 0$ ein *reelles, paralleles Ebenenpaar* $\Phi_{47}^{(2)}$ bzw. für $\overline{D} = 0$ eine *Doppelebene* $\Phi_{48}^{(2)}$.

FALL 2. $F \in s$ und $F \neq S$.

Wie im Fall 1 kann man erreichen, daß s durch $x_0 = x_2 = 0$ und im Fall eines Linienelementes S durch $S(0:1:0:0)$ beschrieben wird. Damit hat man für k

$$y^2 = 0 \tag{4.3}$$

und die zugehörigen Flächen 2. Ordnung lassen sich wegen *Hilfssatz 1* in der Gestalt

$$y^2 + Ax + Cz + \overline{D} = 0 \tag{4.4}$$

ansetzen. Gilt $(A, C) \neq (0,0)$, dann kann in (4.4) durch eine Schiebung noch $\overline{D} = 0$ erreicht werden. Im Fall eines Linienelementes ist $\Phi^{(2)}$ ein *parabolischer Zylinder* $\Phi^{(2)}_{49}$, dessen Erzeugenden den Fernpunkt $S(0:1:0:0)$ enthalten. Wie im Fall 1 folgt $A = 0$ und damit erhalten wir als Normalform

$$y^2 + Cz = 0. \tag{4.C49}$$

Gilt $A = C = 0$, so gewinnt man aus (4.4)

$$y^2 + \overline{D} = 0. \tag{4.C50–52}$$

Für $\overline{D} > 0$ liegt ein *konjugiert-komplexes Ebenenpaar* $\Phi^{(2)}_{50}$, für $\overline{D} < 0$ ein *reelles, paralleles Ebenenpaar* $\Phi^{(2)}_{51}$ und für $\overline{D} = 0$ eine *Doppelebene* $\Phi^{(2)}_{52}$ vor.

FALL 3. $S = F$.

Durch eine isotrope Bewegung kann man erreichen, daß s durch $x_0 = x_2 = 0$ beschrieben wird. Damit hat man für k

$$y^2 = 0, \tag{4.5}$$

und die zugehörigen Flächen 2. Ordnung lassen sich wegen *Hilfssatz 1* in der Gestalt

$$y^2 + Ax + Cz + \overline{D} = 0 \tag{4.6}$$

ansetzen. Gilt $(A, C) \neq (0,0)$, dann kann man in (4.6) durch eine Schiebung noch $\overline{D} = 0$ erreicht werden. Im Fall eines Linienelementes ist $\Phi^{(2)}$ ein *parabolischer Zylinder* $\Phi^{(2)}_{53}$, dessen Erzeugenden den Fernpunkt $F = S(0:0:0:1)$ enthalten. Wie im Fall 1 folgt $C = 0$ und die Normalform lautet

$$y^2 + Ax = 0. \tag{4.C53}$$

Gilt $A = C = 0$, so beschreibt (4.6) zwei parallele, reelle oder konjugiert-komplexe, isotrope Ebenen bzw. ene isotrope Doppelebene; diese Fälle wurden aber schon unter (4.C50–52) aufgezählt. Damit sind Fälle von C erledigt.

Wir fassen das Gesamtresultat zusammen im

SATZ 1. *Im einfach isotropen Raum existieren genau 53 Typen von Flächen 2. Ordnung, die sich bezüglich der einfach isotropen Bewegungsgruppe* $B_6^{(1)}$ *durch die Normalformen* (2.A1–23), (3.B24–44) *bzw.* (4.C45–53) *beschreiben lassen.*

Für viele Anwendungen ist die Klassifikation der Flächen 2. Ordnung im projektiv erweiterten einfach isotropen Raum notwendig.

Dann kommen noch folgende Typen hinzu: $\Phi_{54}^{(2)}$ *(Fernebene ω + nichtisotrope Ebene),* $\Phi_{55}^{(2)}$ *(Fernebene ω + isotrope Ebene),* $\Phi_{56}^{(2)}$ *(Fernebene ω als Doppelebene).* Diese lassen sich durch die Normalformen

$$x_0 x_3 = 0 \quad \text{bzw.}$$
$$(4.D54–56) \qquad x_0 x_1 = 0 \quad \text{bzw.}$$
$$x_0^2 = 0$$

beschreiben.

Wir notieren den

SATZ 2. *Im projektiv erweiterten einfach isotropen Raum existieren genau 56 Typen von Flächen 2. Ordnung.*

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AN EMBEDDING OF $E\omega$-CLIFFORD INVERSE SEMIGROUPS

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1. Introduction

An inverse semigroup in which the closure $E\omega$ of the set of idempotents is a Clifford semigroup will be called *$E\omega$-Clifford*. It is easily seen that an inverse semigroup is $E\omega$-Clifford if and only if $E\omega \subseteq E\xi$ where $E\xi$ is the centralizer of the set of idempotents. The class of $E\omega$-Clifford semigroups forms a quasivariety defined by the implication $(xy = y \Rightarrow xx^{-1} = x^{-1}x)$, and contains the quasi-variety of E-unitary inverse semigroups and the variety of Clifford semigroups. The class of $E\omega$-Clifford semigroups is one of the first classes with unknown structure of the network below (Figure 1).

O'CARROLL [1] proved that every E-unitary inverse semigroup can be embedded into a semidirect product of a semilattice by a group, and SZENDREI [4] generalized this result by proving that every E-unitary regular semigroup with regular band of idempotents can be embedded into a semidirect product of a band by a group. In the main result of this paper, we generalize O'Carroll's result in another direction. We show that each $E\omega$-Clifford inverse semigroup can be embedded into a semidirect product of a Clifford semigroup by a group.

For the undefined notions and notations the reader is referred to [2].

2. The main theorem

First we characterize $E\omega$-Clifford semigroups and give some other results.

PROPOSITION 1. *Let S be an inverse semigroup. Then the following conditions are equivalent:*

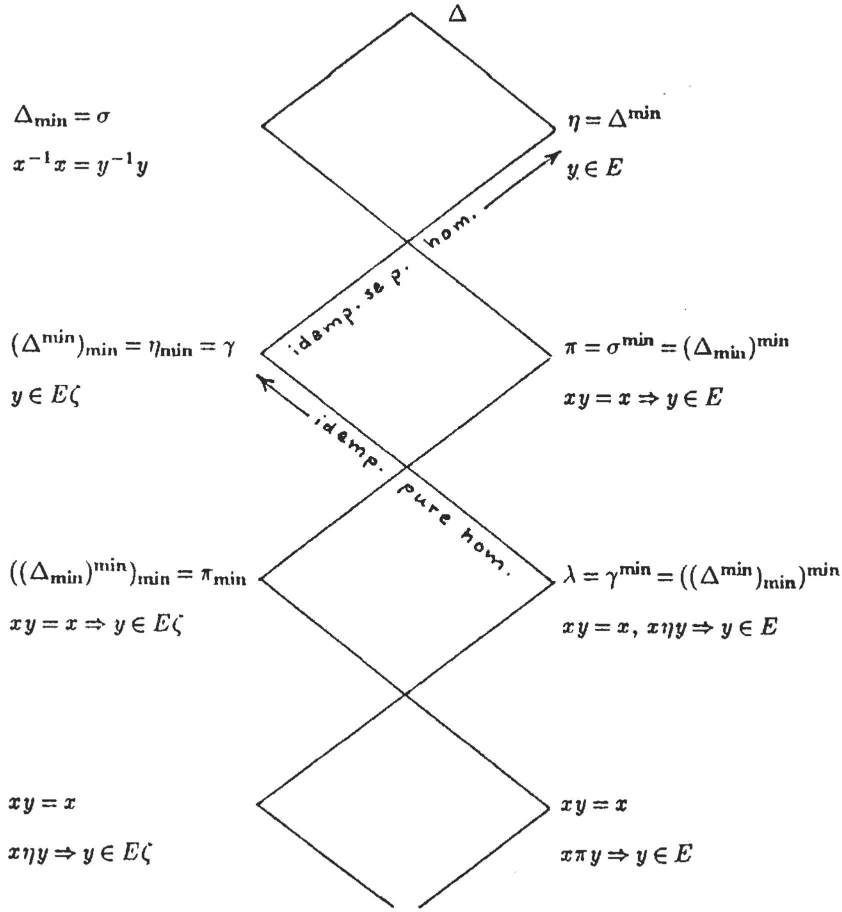

Fig. 1

(i) *S is $E\omega$-Clifford*

(ii) *$\sigma \cap \mathcal{R}$ is a congruence, where σ is the least group congruence on S.*

(iii) *π, the least E-unitary congruence on S, is idempotent separating.*

PROOF. (i) $\Rightarrow$ (ii). Since σ is a congruence and $\mathcal{R}$ is left compatible, it remains to prove that $\sigma \cap \mathcal{R}$ is right compatible. Let $(a,b) \in \sigma \cap \mathcal{R}$, and $c \in S$. Then $aa^{-1} = bb^{-1}$ and $ab^{-1} \in \ker\sigma = E\omega \subseteq E\zeta$ by (i). So we have $acc^{-1}a^{-1} = acc^{-1}a^{-1}aa^{-1} = acc^{-1}a^{-1} \cdot bb^{-1} = a \cdot a^{-1}bcc^{-1}b^{-1} =$ $= bb^{-1}bcc^{-1}b^{-1} = bcc^{-1}b^{-1}$, that is $(ac,bc) \in \mathcal{R}$. Hence $(ac,bc) \in \sigma \cap \mathcal{R}$ follows since σ is a congruence.

(ii) $\Rightarrow$ (iii). If $\sigma \cap \mathcal{R}$ is a congruence, then $\pi = (\sigma \cap \mathcal{R})^* = \sigma \cap \mathcal{R}$. Hence π is idempotent separating.

(iii) $\Rightarrow$ (i). Let π be idempotent separating. Then $\pi \subseteq \mu$, the maximal idempotent separating congruence on S. This implies $E\omega = \ker\pi \subseteq \ker\mu = E\zeta$, and so S is $E\omega$-Clifford.

COROLLARY 2. *Let S be an inverse semigroup. Then S is $E\omega$-Clifford if and only if $\varrho \cap \mathcal{R}$ is a congruence for each congruence ϱ with $\varrho \subseteq \sigma$.*

PROOF. Let S be an $E\omega$-Clifford semigroup, let ϱ be a congruence on S with $\varrho \subseteq \sigma$ and let $(a,b) \in \varrho \cap \mathcal{R}$, $c \in S$. Then $(ac,bc) \in \varrho$ and $(a,b) \in \sigma \cap \mathcal{R}$. By Proposition 1, $(ac,bc) \in \sigma \cap \mathcal{R}$. This implies $(ac,bc) \in \varrho \cap \mathcal{R}$, so $\varrho \cap \mathcal{R}$ is a congruence. The converse follows from Proposition 1.

PROPOSITION 3. *If S is an $E\omega$-Clifford semigroup, then S/τ is also $E\omega$-Clifford for every idempotent pure congruence τ on S.*

PROOF. Let $a \in E_{S/\tau}\omega$. Then there exists an idempotent in S/τ such that $ae = e$. Now let $a'\tau = a$ and $e'\tau = e$, where $a' \in S$ and $e' \in E_S$. Then we have $(a'e')\tau = a'\tau \cdot e'\tau = a \cdot e = e = e'\tau$. As τ is idempotent pure, we have $a'e' \in E_S$ whence $a' \in E_\omega$ follows which implies $a'^{-1}a' = a'a'^{-1}$. Thus, $aa^{-1} = a^{-1}a$, that is, $a \in E_{S/\tau}\zeta$. This completes the proof.

Now we recall the definition of a semidirect product.

DEFINITION 4. Let C be an inverse semigroup, and let G act on C by automorphisms on the left. Define the multiplication $(a,g)\circ(b,h) = (a \cdot gb, gh)$ on the Cartesian product $C \times G$ which thereby becomes a semigroup. This latter one is called a *semidirect product* of C by G and is denoted by $C * G$.

PROPOSITION 5. *Let $C * G$ be a semidirect product of a Clifford semigroup C by a group G. If S is an inverse subsemigroup of $C * G$, then S is $E\omega$-Clifford.*

PROOF. Let $(a,g) \in E_S\omega$. Then there exists an idempotent $(e,1) \in E_S$ such that $(a,g)(e,1) = (e,1)$. This implies $g = 1$ and $a \cdot ge = e$. Hence $(a,1)(a,1)^{-1} =$ $= (a,1)(a^{-1},1) = (aa^{-1},1) = (a^{-1}a,1) = (a,1)^{-1}(a,1)$, so $(a,g) \in E_S\zeta$.

NOTATION 6. Let $\mathcal{I}_X$ be the free semigroup with involution on the alphabet X. For a word w, denote by $c(w)$ the content of w, that is the set of letters occurring in w, and by $r(w)$ the reduced word (a word is called reduced if and only if it has no subword of the form xx^{-1} or $x^{-1}x$). By $\mathcal{E}_S$ we denote the least fully invariant semilattice congruence, and by $\mathcal{E}_C$ the least fully invariant Clifford congruence on $\mathcal{I}_X$. We have $(u,v) \in \mathcal{E}_S$ if and only if $c(u) = c(v)$ and $(u,v) \in \mathcal{E}_C$ if and only if $c(u) = c(v)$ and $r(u) = r(v)$ [2].

Now let S be an $E\omega$-Clifford semigroup. For brevity, denote S/σ by G.

Define a partial multiplication on $X = G \times S$ as follows: for s, $t \in S$ and g, $h \in G$, $(g,s) \circ (h,t)$ is defined if and only if $g \cdot s\sigma = h$, and if so, then $(g,s) \circ (h,t) = (g,st)$.

Define a map from X into X by $x = (g,s) \mapsto (g \cdot s\sigma, s^{-1}) = \bar{x}$. Observe that $\bar{\bar{x}} = x$ for every $x \in X$. Define an action of the group G on X by setting $h(g,s) = (hg,s)$, for every $h \in G$, $(g,s) \in X$. This action of G is obviously compatible with both $\circ$ and $\bar{\ }$.

Extend the action of G on X to an action on $\mathcal{I}^1_X$, the free monoid with involution on X, as follows: Let $h \in G$. For every $x \in X$, put $hx^{-1} = (hx)^{-1}$, for every word $w = x_1x_2\ldots x_n$ in $\mathcal{I}_X$ with x_1, x_2, $\ldots$, $x_n \in X \cup X^{-1}$, put $hw = hx_1 \cdot hx_2 \cdot \ldots \cdot hx_n$, and for the empty word $\Box$ put $h\Box = \Box$. The free monoid F^1_X being a submonoid of $\mathcal{I}^1_X$, this defines an action of G on F^1_X. Notice that $\bar{\ }$ can be extended to an involution on F^1_X which we will also denote by $\bar{\ }$. The action of G on F^1_X is compatible with this involution.

Let $\bar{\varrho}_C$ be the join of $\mathcal{E}_C$ and the congruence on $\mathcal{I}_X$ generated by the relation

$$\{(xy,z) : x,y,z \in X \text{ and } x \circ y = z \text{ in } X\} \bigcup \{(x^{-1},\bar{x}) : x \in X\}.$$

From the well-known description of $\mathcal{E}_C$ and that of the join of congruences we easily deduce the following lemma.

LEMMA 6. *Let u, v be words in $\mathcal{I}_X$. Then $u\bar{\varrho}_C v$ if and only if there exists a finite sequence of words $u = w_0$, w_1, $\ldots$, $w_n = v$ such that, for any $i \in \{0,1,\ldots,n-1\}$ the word w_{i+1} is obtained from w_i by one of the following steps:*

step 0: $w_i = px^{-1}q$, $w_{i+1} = p\bar{x}q$ *for some* p, $q \in \mathcal{I}^1_X$ *and* $x \in X$;

step 0': $w_i = p\bar{x}q$, $w_{i+1} = px^{-1}q$ *for some* p, $q \in \mathcal{I}^1_X$ *and* $x \in X$;

step $\bar{1}$: $w_i = pxx^{-1}q$ *or* $w_i = px^{-1}xq$, *and* $w_{i+1} = pq$ *for some* p, $q \in \mathcal{I}^1_X$ *and* $x \in X$ *such that* x *or* x^{-1} *occurs in* p *or in* q;

step $\bar{1}$': $w_i = pq$, *and* $w_{i+1} = pxx^{-1}q$ *or* $w_{i+1} = px^{-1}xq$ *for some* p, $q \in \mathcal{I}^1_X$ *and* $x \in X$ *such that* x *or* x^{-1} *occurs in* p *or in* q;

step $\bar{2}$: $w_i = pzq$, $w_{i+1} = pxyq$ *for some* p, $q \in \mathcal{I}^1_X$ *and* x, y, $z \in X$ *where* $x \circ y = z$ *in* X;

step $\bar{2}$': $w_i = pxyq$, $w_{i+1} = pzq$ *for some* p, $q \in \mathcal{I}^1_X$ *and* x, y, $z \in X$ *where* $x \circ y = z$ *in* X.

Now we show that we can restrict ourselves to words consisting of letters only from X (not from X^{-1}).

Observe that if we start with a word $u \in \mathcal{I}_X$ and apply step 0 to every letter from X^{-1} which occurs in u then a word $u' \in F_X$ is obtained. So each $u \in \mathcal{I}_X$ is $\bar{\varrho}_C$-related to an element in F_X. This implies that $\mathcal{I}_X/\bar{\varrho}_C$ is isomorphic to F_X/ϱ_C where ϱ_C denotes the restriction of $\bar{\varrho}_C$ to F_X. In particular, this shows that F_X/ϱ_C is a Clifford semigroup. It is straightforward to check by the previous Lemma that ϱ_C can be obtained as follows:

LEMMA 7. *Let u, v be words in F_X. Then $u\varrho_C v$ if and only if there exists a finite sequence of words $u = w_0, w_1, \ldots, w_n = v$ such that, for any $i \in \{0,1,\ldots,n-1\}$ this word w_{i+1} is obtained from w_i by one of the following steps:*

step 1: *$w_i = px\bar{x}q$, $w_{i+1} = pq$ for some p, $q \in F_X^1$ and $x \in X$ such that x or $\bar{x}$ occurs in p or in q;*

step 1': *$w_i = pq$, $w_{i+1} = px\bar{x}q$ for some p, $q \in F_X^1$ and $x \in X$ such that x or $\bar{x}$ occurs in p or in q;*

step 2: *$w_i = pzq$, $w_{i+1} = pxyq$ for some p, $q \in F_X^1$ and $x, y, z \in X$ where $x \circ y = z$ in X;*

step 2': *$w_i = pxyq$, $w_{i+1} = pzq$ for some p, $q \in F_X^1$ and $x, y, z \in X$ where $x \circ y = z$ in X.*

Now we define an action of G on $F_X/\varrho_C = C$. Let $u = px\bar{x}q$ and $v = pq$ for some p, $q \in F_X^1$ and $x \in X$ such that x or $\bar{x}$ occurs in p or q. Then, for any $g \in G$, we have $gu = gp \cdot gx \cdot g\bar{x} \cdot gq$ and $gv = gp \cdot gq$ where gp, $gq \in F_X^1$, $gx \in X$ and $g\bar{x} = \overline{gx}$, and so gx or $\overline{gx}$ occurs in gp or gq. If $u = pzq$ and $v = pxyq$ where p, $q \in F_X^1$ and $x, y, z \in X$ with $x \circ y = z$, then $gu = gp \cdot gz \cdot gq$ and $gv = gp \cdot gx \cdot gy \cdot gq$ where $gp, gq \in F_X^1$ and gx, gy, $gz \in X$. If $x = (h,s)$, $y = (h \cdot s\sigma, t)$ then we have $z = (h, st)$. Thus the product $gx \circ gy = (gh, s) \circ (gh \cdot s\sigma, t)$ is defined and $gx \circ gy = gz$.

Hence, by Lemma 7, the action of G is compatible with ϱ_C. Thus, the action of G on F_X determines an action of G on C in a natural way, and this defines a semidirect product $C * G$ of the Clifford semigroup C by the group G.

THEOREM 8. *If S is an $E\omega$-Clifford semigroup, then S can be embedded into the semidirect product of the Clifford semigroup C by the group G.*

We prove Theorem 8 by a series of Lemmas. Throughout S is an $E\omega$-Clifford semigroup.

LEMMA 9. *The mapping* $\Theta: S \to C * G$ *defined by* $s\Theta = ((1,s)\varrho_C, s\sigma)$ *is a homomorphism.*

PROOF. If $s, t \in S$, then

$$(st)\Theta = ((1,st)\varrho_C,(st)\sigma) = ((1,s)\varrho_C \cdot s\sigma((1,t)\varrho_C), s\sigma \cdot t\sigma) = \\ = ((1,s)\varrho_C, s\sigma)((1,t)\varrho_C, t\sigma) = s\Theta \cdot t\Theta.$$

NOTATION 10. If $x \in X$, $x = (g,t)$, then we shall write $t = L(x)$ and $g = = s(X)$. Let $w = x_1x_2\dots x_n$ be a word in F_x such that $x_1 \circ x_2 \circ \dots \circ x_n$ is defined. Suppose that $x_i = (g_i, t_i)$, $i = 1, 2, \dots, n$. Then we put $L(w) = t_1 \cdot t_2 \cdot \dots \cdot t_n$ and $S(w) = g_1$. Notice that $L(w) = L(x_1 \circ \dots \circ x_n)$ and $S(w) = (x_1 \circ \dots \circ x_n)$.

LEMMA 11. *Let* $w \in F_X$, $w = x_1x_2\dots x_n$, *such that* $x_1 \circ x_2 \circ \dots \circ x_n$ *is defined. If* $S(x_i) = S(x_1)$ *for some* $i \in \{1,2,\dots,n\}$, *then* $L(x_i\bar{x}_i w) = L(w)$.

PROOF. Let $x_i = (g_i, t_i)$, $i = 1, 2, \dots, n$. Since the product $x_1 \circ \dots \circ x_{i-1}$ is defined, we have $x_1 \circ \dots \circ x_{i-1} = (g_1, t_1) \circ \dots \circ (g_{i-1}, t_{i-1}) = (g_1, t_1, \dots, t_{i-1})$. Furthermore, also $x_1 \circ \dots \circ x_i$ is defined, so we have $g_i = g_1(t_1 \cdot \dots \cdot t_{i-1})\sigma$. By assumption, $g_i = S(x_i) = S(x_1) = g_1$, whence $(t_1 \dots t_{i-1})\sigma = 1$. So $t_1 \dots t_{i-1} \in \in \ker\sigma = E\omega \subseteq E\zeta$, and $\bar{x}_i \circ x_1$ is defined since $S(\bar{x}_i)t_i^{-1}\sigma = g_i t_i \sigma \cdot t_i^{-1}\sigma = = g_i = g_1$. Hence $L(w) = (t_1 \dots t_{i-1})t_i \dots t_n = t_i t_i^{-1} L(w) = L(x_i \bar{x}_i w)$.

LEMMA 12. *Let* w, w' *be two words in* F_X, $w = x_1 \dots x_n$, $w' = y_1 \dots y_m$ *such that* $x_1 \circ x_2 \circ \dots \circ x_n$, $y_1 \circ y_2 \circ \dots \circ y_m$ *are defined. If* $c(w) \subseteq c(w'\bar{w}')$ *and* $S(w) = S(w')$, *then* $L(w') = L(w) \cdot a$ *for some* $a \in S$.

PROOF. Without loss of generality, we can suppose $c(w) \subseteq c(w')$. Indeed, if $x_i = \bar{y}_j$, we can replace w' by $w'' = y_1 \dots y_j \bar{y}_j y_j \dots y_m$, where $S(w'') = S(w')$, $L(w'') = L(w')$. Now let $x_i = (g_i, s_i)$, $y_j = (h_j, t_j)$, $i = 1, \dots, n$, $j = 1, \dots, m$. First we show that

$$(*) \qquad L(w') = s_1 s_1^{-1} L(w').$$

Since $c(w) \subseteq c(w')$, this implies $x_1 = y_j$ for some $j \in \{1, \dots, m\}$. Thus we have also $S(y_1) = S(x_1) = S(y_j)$. Hence, by Lemma 11, we have $L(w') = L(y_j \bar{y}_j w') = = L(x_1 \bar{x}_1 w') = s_1 s_1^{-1} L(w')$.

Now we prove by induction on the length n of w that $L(w') = L(w) \cdot a$ for some $a \in S$. If w is of length 1 (i.e. $n = 1$), then $(*)$ implies that $L(w') = = L(w) \cdot ((L(w))^{-1} L(w'))$. Suppose that for any pairs of words w, w' which satisfy the conditions of the Lemma and $n = r$, the equality $L(w') = L(w) \cdot a$ holds for some $a \in S$. In order to prove it for words w, w' with $n = r + 1$, put $w^* = x_2 x_3 \dots x_n$, and $u = \bar{x}_1 w'$. It is clear that $c(w^*) \subseteq c(u)$, and the length of w^* is r. Moreover, $\bar{x}_1 \circ y_1$ is defined since $g_2 \cdot s_1^{-1}\sigma = g_1 \cdot s_1\sigma \cdot \cdot s_1^{-1}\sigma = g_1 = S(x_1) = S(y_1)$. Thus both $x_2 \circ x_3 \circ \ldots \circ x_n$ and $\bar{x}_1 \circ y_1 \circ \ldots \circ y_m$ are defined and $S(w^*) = S(u)$. Hence, by the induction hypothesis, we have $L(u) = L(w^*) \cdot a'$ for some $a' \in S$. By definition, we have $L(u) = s_1^{-1} \cdot L(w')$, whence $L(w') = s_1 s_1^{-1} L(w') = s_1 L(u) = s_1 L(w^*) \cdot a' = s_1(s_2 \ldots s_n) \cdot a' = L(w) \cdot a'$. The proof is complete.

LEMMA 13. *Let w, w' be two words in F_X, $w = x_1 \ldots x_n$, $w' = y_1 y_2 \ldots y_m$ such that $x_1 \circ x_2 \circ \ldots \circ x_n$, $y_1 \circ \ldots \circ y_m$ are defined. If $c(w\bar{w}) = c(w'\bar{w}')$, $S(w) = S(w')$ and $S(\bar{w}) = S(\bar{w}')$ (i.e. $S(x_n) \cdot L(x_n)\sigma = S(y_m) \cdot L(y_m)\sigma$), then $(L(w), L(w')) \in \pi$.*

PROOF. Since $S(w) = S(w')$ and $c(w\bar{w}) = c(w'\bar{w}')$, using Lemma 12, we shall have $L(w) = L(w') \cdot a$ and $L(w') = L(w) \cdot a'$ for some a, $a' \in S$, whence $(L(w), L(w')) \in \mathcal{R}$. Since $S(x_n) \cdot L(x_n)\sigma = S(y_m) \cdot L(y_m)\sigma$, we have $S(x_1) \cdot L(w)\sigma = S(x_1) \cdot L(x_1 x_2 \ldots x_{n-1})\sigma \cdot L(x_n)\sigma = S(x_n) \cdot L(x_n)\sigma = S(y_m) \cdot \cdot L(y_m)\sigma = S(y_1) \cdot L(y_1 \ldots y_{m-1})\sigma \cdot L(y_m)\sigma = S(y_1) \cdot L(w')\sigma = S(x_1) \cdot L(w')\sigma$, hence $L(w)\sigma = L(w')\sigma$. Thus $(L(w), L(w')) \in \sigma \cap \mathcal{R} = \pi$.

LEMMA 14. *If $s \in E\omega$ and $e \in E$ such that $(1,s)\varrho_C(1,e)$, then $s = e$.*

PROOF. If $(1,s)\varrho_C(1,e)$ then, in virtue of Lemma 7, there exist finitely many words $(1,s) = w_0$, w_1, …, $w_n = (1,e)$ in F_X such that w_{i+1} is obtained from w_i by one of the steps 1, 1', 2, 2'.

Instead of dealing with w_i, we shall define a word w_i^* such that the multiplication $\circ$ is defined for the components of w_i^* and $L(w_i^*) = s$. If $w_i = x_1 x_2 \ldots x_j x_{j+1} \ldots x_m$ then the word w_i^* will be obtained from w_i such that, for any j, $1 \leq j < m$, we insert a word $\overline{p^i(h)} \cdot p^i(g)$ between x_j and x_{j+1}, where $h = S(x_j) \cdot L(x_j)\sigma$ and $g = S(x_{j+1})$. The words $p^i(g)$ will be defined inductively on i in such a way that the multiplication $\circ$ is defined for their components and, furthermore, $c\left(p^i(g)\right) = c(w_0\bar{w}_0 w_1 \bar{w}_1 \ldots w_i \bar{w}_i)$, $S\left(p^i(g)\right) = 1$ and $L\left(p^i(g)\right)\sigma = g$ for every i, $i = 0, 1, \ldots, n$ and for every g in $\{S(x) : x \in c(w_0\bar{w}_0 w_1\bar{w}_1 \ldots w_i\bar{w}_i)\}$. For brevity, we introduce the following notations: $X_i = c(w_0\bar{w}_0 \ldots w_i\bar{w}_i)$ and $H_i = \{S(x) : x \in X_i\}$ $(i = 0, 1, \ldots, n)$.

Since $s \in E\omega$, we have $s\sigma = 1$. So $X_0 = \{(1,s)\overline{(1,s)}\}$ and $H_0 = \{1\}$. We put

$$p^0(1) = (1,s)\overline{(1,s)}.$$

Obviously, $(1,s)\circ\overline{(1,s)}$ is defined, $c\left(p^0(1)\right) = X_0$, $S\left(p^0(1)\right) = 1$ and $L\left(p^0(1)\right)\sigma = 1$.

Suppose that, for some $i = 0, 1, \ldots, n-1$, $p^i(g)$ is defined for every $g \in H_i$ such that it possesses the properties required above. Now we define $p^{i+1}(g)$ $(g \in H_{i+1})$ by means of $p^i(g)$ $(g \in H_i)$. If w_{i+1} is obtained from w_i by step 1 or 1' then $c(w_i) = c(w_{i+1})$, and so $X_{i+1} = X_i$ and $H_{i+1} = H_i$ follow. In this case, we define

$$p^{i+1}(g) = p^i(g) \qquad \text{for every } g \in H_{i+1}.$$

If w_{i+1} is obtained from w_i by step 2 then $X_{i+1} = X_i \bigcup\{x,\bar{x},y,\bar{y}\}$ and $H_{i+1} = H_i \cup \{S(y)\}$. Since $c\left(p^i(g)\right) = X_i$, we have $S(z) = S(x) \in H_i$. For brevity, put $h = S(z)$. Thus we define

$$p^{i+1}(g) = \begin{cases} p^i(h)x\bar{x}xy\bar{y} & \text{if} \quad g = S(y) \notin H_i \\ p^i(h)xy\bar{y}\bar{x}\overline{p^i(h)}p^i(g) & \text{if} \quad g \in H_i \end{cases}$$

If w_{i+1} is obtained from w_i by step 2' then $X_{i+1} = X_i \bigcup\{z,\bar{z}\}$ and $H_{i+1} = H_i$. Again, we have $S(x) = S(z) \in H_i$, and denoting $h = S(x)$, we define

$$p^{i+1}(g) = p^i(h)z\bar{z}\overline{p^i(h)}p^i(g) \qquad \text{for every } g \in H_{i+1}.$$

It is straightforward to check that $p^{i+1}(g)$ $(g \in H_{i+1})$ also possesses the properties above.

Now let

$$w_i^* = p^i(g_1)x_1\overline{p^i(g_1\cdot s_1\sigma)}p^i(g_2)x_2\overline{p^i(g_2\cdot s_2\sigma)}\ldots p^i(g_m)x_m\overline{p^i(g_m\cdot s_m\sigma)}$$

for each $i = 0, 1, \ldots, n$ where $g_j = S(x_j)$ and $s_j = L(x_j)$ for $j = 1, \ldots, m$. Here the product of consecutive components of w_i^* is defined.

First we prove that $L\left(p^i(g)\overline{p^i(g)}\right) = ss^{-1}$ for each $g \in H_i$, $i = 0, 1, \ldots, n$. For $i = 0$, $L\left(p^0(1)\overline{p^0(1)}\right) = L\left(\left((1,s)\overline{(1,s)}\right)^2\right) = L\left(\left((1,s)(1,s^{-1})\right)^2\right) = ss^{-1}$.

Suppose that $L\left(p^i(g)\overline{p^i(g)}\right) = ss^{-1}$ holds for $i = r$ $(0 \le r < n)$, and we shall prove it for $i = r+1$.

If w_{i+1} is obtained from w_i by step 1 or 1' then $p^{r+1}(g) = p^r(g)$ $(g \in H_{r+1})$ which implies $L\left(p^{r+1}(g)\overline{p^{r+1}(g)}\right) = ss^{-1}$.

If w_{i+1} is obtained from w_i by step 2 and $g = S(y) \notin H_i$, then $p^{r+1}(g) = p^r(h)x\bar{x}xy\bar{y}$,

$$L\left(p^{r+1}(g)\overline{p^{r+1}(g)}\right) = L\left(p^r(h)x\bar{x}xy\bar{y}y\bar{y}\bar{x}x\bar{x}\overline{p^r(h)}\right) =$$

$$= L(p^r(h)) \cdot L(x\bar{x}x) \cdot L(y\bar{y}y\bar{y}) \cdot L(\bar{x}x\bar{x}) \cdot L\left(\overline{p^r(h)}\right) =$$

$$= L(p^r(h)) \cdot L(x) \cdot L(y) \cdot L(\bar{y}) \cdot L(\bar{x}) \cdot L\left(\overline{p^r(h)}\right) =$$

$$= L(p^r(h)) \cdot L(x \circ y) \cdot L(\overline{x \circ y}) \cdot L\left(\overline{p^r(h)}\right) = L(p^r(h)) \cdot L(z) \cdot L(\bar{z}) \cdot L\left(\overline{p^r(h)}\right)$$

As $z \in c\left(p^r(h)\overline{p^r(h)}\right)$, the products $p^r(h)z\bar{z}\overline{p^r(h)}$ and $p^r(h)\overline{p^r(h)}$ satisfy the condition of Lemma 13, hence

$$\left(L\left(p^r(h)z\bar{z}\overline{p^r(h)}\right), L\left(p^r(h)\overline{p^r(h)}\right)\right) \in \pi.$$

Since π is idempotent separating [Proposition 1], we have

$$L\left(p^r(h)z\bar{z}\overline{p^r(h)}\right) = L\left(p^r(h)\overline{p^r(h)}\right) = ss^{-1}.$$

As for the case $g \in H_i$, we have $p^{r+1}(g) = p^r(h)xy\bar{y}\bar{x}\overline{p^r(h)}p^r(g)$. Hence

$$L\left(p^{r+1}(g)\right) = L\left(p^r(h)xy\bar{y}\bar{x}\overline{p^r(h)}\right) \cdot L(p^r(g)) = ss^{-1} \cdot L(p^r(g)),$$

and so

$$L\left(p^{r+1}(g)\overline{p^{r+1}(g)}\right) = ss^{-1} \cdot L\left(p^r(g)\overline{p^r(g)}\right) \cdot ss^{-1} = ss^{-1}.$$

If w_{i+1} is obtained from w_i by step 2', the proof is similar as in step 2.

Now we observe that $L(p^i(g)) = L(p^{i+1}(g))$ holds for every $g \in H_i$ and for every i, $i = 0, 1, \ldots, n-1$. Indeed, if w_{i+1} is obtained from w_i by step 1 or 1' then this equality trivially follows. If w_{i+1} is obtained from w_i by step 2 then we have seen formerly that $L\left(p^i(h)xy\bar{y}\bar{x}\overline{p^i(h)}\right) = L\left(p^i(g)\overline{p^i(g)}\right)$ whence $L(p^{i+1}(g)) = L\left(p^i(g)\overline{p^i(g)}\right) \cdot L(p^i(g)) = L(p^i(g))$ follows. The case when w_{i+1} is obtained from w_i by step 2' can be handled in the same way.

In particular, since $1 \in H_i$ for every i, $i = 0, 1, \ldots, n$, the equalities $ss^{-1} = L(p^0(1)) = L(p^1(1)) = \ldots = L(p^n(1))$ follows.

Now we are going to show that $L(w_i^*) = s$ for $i = 0, 1, \ldots, n$. This proof also goes by induction on i. For $i = 0$ we have $w_0^* = p^0(1)(1,s)\overline{p^0(1)}$, $L(w_0^*) = ss^{-1}sss^{-1} = s$ since $s \in E\omega$, a Clifford semigroup.

Now we suppose that $L(w_i^*) = s$, and prove that $L(w_{i+1}^*) = s$.

Assume that $w_i = x_1x_2\ldots x_m$ and put $g_k = S(x_k)$ and $s_k = L(x_k)$ for $k = 1, 2, \ldots, m$.

If we apply step 1 to obtain w_{i+1}, then

$$L(w_{i+1}^*) = L(p^{i+1}(g_1)x_1 \ldots x_j \overline{p^{i+1}(g_j \cdot s_j\sigma)} p^{i+1}(g) x \overline{p^{i+1}(g \cdot s\sigma)}$$
$$p^{i+1}(g \cdot s\sigma)\bar{x}\overline{p^{i+1}(g)}p^{i+1}(g_{j+1})x_{j+1} \ldots x_m \overline{p^{i+1}(g_m \cdot s_m\sigma)}$$

where $x \in \{x_1, \bar{x}_1, \ldots, x_m, \bar{x}_m\}$, $g = S(x)$ and $s = L(x)$. Since $p^i(g_k) = p^{i+1}(g_k)$ for $k = 1, 2, \ldots, m$, it suffices to show that

$$L\left(p^{i+1}(g)x\overline{p^{i+1}(g \cdot s\sigma)}p^{i+1}(g \cdot s\sigma)\bar{x}\overline{p^{i+1}(g)}\right) = L\left(p^{i+1}(g_{j+1})\overline{p^{i+1}(g_{j+1})}\right).$$

Lemma 13 implies that these elements are π-related. As both sides are also idempotents and π is an idempotent separating congruence, the equality follows.

If w_{i+1} is obtained from w_i by step 1' then the proof is similar.

Now consider the case when w_{i+1} is obtained from w_i by step 2. Suppose that $w_{i+1} = x_1 \ldots x_{j-1}x_j'x_j''x_{j+1} \ldots x_m$ where $x_j' \circ x_j'' = x_j$. Put $x_j' = (g_j, s_j')$ and $x_j'' = (g_j \cdot s_j'\sigma, s_j'')$. Then we have

$$L(w_{i+1}^*) = L\Big(p^{i+1}(g_1)x_1 \ldots p^{i+1}(g_j)x_j'\overline{p^{i+1}(g_j \cdot s_j'\sigma)}p^{i+1}(g_j \cdot s_j'\sigma)x_j''$$
$$\overline{p^{i+1}(g_j \cdot s_j'\sigma \cdot s_j''\sigma)} \ldots x_m \overline{p^{i+1}(g_m \cdot s_m\sigma)}\Big).$$

Let $a = L(p^{i+1}(g_1)x_1 \ldots p^{i+1}(g_j)x_j')$ and $b = L(p^{i+1}(g_j \cdot s_j'\sigma))$. Lemma 13 implies that $a\pi b$, and so $a^{-1}a = b^{-1}b$. Hence

$$L\left(p^{i+1}(g_1)x_1 \ldots p^{i+1}(g_j)x_j'\overline{p^{i+1}(g_j \cdot s_j'\sigma)}p^{i+1}(g_j \cdot s_j'\sigma)\right) =$$
$$L\left(p^{i+1}(g_1)x_1 \ldots p^{i+1}(g_j)x_j'\right)$$

follows.

Since $L(x_j'x_j'') = L(x_j)$ and $L(p^{i+1}(g_k)) = L(p^i(g_k))$ for $k = 1, \ldots, m$, we see that $L(w_{i+1}^*) = L(w_i^*)$.

If w_{i+1} is obtained from w_i by step 2', the proof is similar. Thus we have verified that $L(w_i^*) = s$ for each i, $i = 0, 1, \ldots, n$. Now $w_n^* = p^n(1)(1,e)\overline{p^n(1)}$ implies that $s = L(w_n^*) = ss^{-1}ess^{-1} = ss^{-1}e$, and this implies that $s \in E$ and $s \leq e$.

In a similar way one can prove that $e \leq s$, hence $s = e$. This completes the proof of the Lemma.

Now we return to the proof of our Theorem 8. If $(1,s)\varrho_C = (1,t)\varrho_C$ and $(s,t) \in \sigma$, then

$$(1,st^{-1})\varrho_C = ((1,s)(s\sigma,t^{-1}))\varrho_C = ((1,s)(t\sigma,t^{-1}))\varrho_C =$$
$$= ((1,t)(t\sigma,t^{-1}))\varrho_C = (1,tt^{-1}\varrho_C.$$

Hence by Lemma 14 $st^{-1} = tt^{-1}$ and so $t \leq s$. Similarly, we obtain also that $s \leq t$. Thus $s = t$, which completes the proof of the Theorem.

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While completing the manuscript to the present paper, we learned that the same result had been obtained by B. BILLHARDT [1]. His proof is much different from ours.

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MONOTONE METHODS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS NONLINEARITIES

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1. Preliminaries

Let Ω be a bounded domain in R^n, with a Lipschitz boundary $\partial\Omega$. The Sobolev space $W^{k,p}(\Omega)$, $k=0,1,\dots,1\le p\le\infty$, is the space of functions $u\in L^p(\Omega)$ with weak partial derivatives up to order k in $L^p(\Omega)$. Here, for any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ with $|\alpha|=\alpha_1+\dots+\alpha_n$, $D^\alpha=\partial^{|\alpha|}u/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, the norm on $W^{k,p}(\Omega)$ is

$$\|u\|=\left\{\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u|^p dx\right\}^{1/p}.$$

By $W_0^{k,p}(\Omega)$ will be denoted the closure in $\|\ \|_{W^{k,p}(\Omega)}$ of $C_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support contained in Ω.

We shall consider the following elliptic boundary value problem

$$\mathcal{L}u=f(x,u)\quad x\in\Omega \tag{1}$$

$$u|_{\partial\Omega}=0 \tag{2}$$

where the operator $\mathcal{L}=-\sum_{ij=1}^{n}\partial/\partial x_i\{a_{ij}(x)\partial/\partial x_j\}+\sum_{i=1}^{n}b_i(x)\partial/\partial x_i+c(x)$.

Throughout this paper, we shall assume that the coefficients of the operator $\mathcal{L}$, a_{ij}, b_i, c $(i,j=1,2,\dots,n)$ belong to $L^\infty(\Omega)$, and c is nonnegative in Ω. We shall also assume that there exists a positive constant μ such that

$$\sum a_{ij}(x)l_il_j\ge\mu|l|^2,\quad \text{for a.e.}\quad x\in\Omega, l=(l_1,\dots,l_n)\in R^n.$$

In this paper, we shall prove several existence theorems for the elliptic boundary value problem (1)–(2), where no continuity conditions are imposed on $f(x,u)$.

Elliptic problems with discontinuous nonlinearities have been investigated, for example, in [1], [2], [3], [5], [8]. The main topics of research have been existence, uniqueness and regularity of the solutions, as well as approximating methods for the numerical treatment of these problems.

AMBROSETTI A. and others (see [1], [2]) used a dual functional and Mountain Pass Theorem to deal with the discontinuous elliptic problem

$$\mathscr{L}u = h(u-a)q(u) \quad x \in \Omega$$
$$u|_{\partial\Omega} = 0$$

where $a > 0$, h denotes the Heaviside function:

$$h(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s > 0 \end{cases}.$$

If q satisfies

(a) $q \in C^0(R)$, q is nondecreasing;

(b) $q(s) \leq c_0 s + c_1$, c_0, c_1 are constants and $c_0 < \lambda_1$, the first eigenvalue of the elliptic operator $\mathscr{L}$, and

(c) $q(a)/a > 2\lambda_1 \frac{\|\Phi\|_{L^1(\Omega)}}{\|\Phi\|_{L^2(\Omega)}}$ where Φ satisfies $\mathscr{L}\phi = \lambda_1\phi$ in Ω, $\phi|_{\partial\Omega} = 0$, and $\|\phi\|_{c^0(\Omega)} = 1$, then they have proved that the problem above has two, distinct, positive solutions.

FRANC L. S. and WENDT W. D. (see [5]) considered

$$-\nabla^2 u + \lambda X(u) \ni g(x) \quad x \in \Omega$$
$$u|_{\partial\Omega} = r(x)$$

where $\lambda > 0$, X is the multivalued operator

$$X(t) = \begin{cases} h(t) \quad \text{(heaviside function)} & \text{for } t \in R \setminus \{0\} \\ [0,1] & \text{for } t = 0 \end{cases}$$

They have proved that the problem above has a solution, $u \in c^{1,\alpha}(\overline{\Omega})$, for any α with $0 < \alpha < 1$ if $r \in c^2(\partial\Omega)$, $g \in c^0(\overline{\Omega})$.

In the present paper, we shall consider a class of more general elliptic boundary value problems with discontinuous nonlinearities by means of the theory of nonlinear increasing operators in ordered Banach spaces.

In this section, we shall introduce some concepts and results in ordered Banach spaces which are needed for the other sections.

Let E be a real Banach space with norm $\| \ \|$. Let $K \subset E$ be a cone, that is, a closed convex subset of E such that $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A partial ordering $\leq$ is defined in E related to K by $v \leq u$ if and only if $u - v \in K$.

By an ordered Banach space, usually denoted by (E, K), we mean a Banach space E together with an ordering $\leq$ induced by a cone K, the positive cone of E.

The order interval $[x,y]$ is defined by $[x,y] = \{z \in E; x \leq z \leq y\}$. We shall also keep the usual terminology concerning concepts connected with $\leq$. For example, $\{x_k\}$ is said to be monotone if $\{x_k\}$ is either increasing, i.e. $x_k \leq x_{k+1}$ for all k, or decreasing, i.e. $x_k \geq x_{k+1}$ for all k. A set $M \subset E$ is said to be bounded above if M has an upper bound with respect to $\leq$, i.e. $x \leq y$ for all $x \in M$ with some $y \in E$, and SupM will denote the least upper bound of M with respect to $\leq$ if it exists. Analogously we can define InfM.

DEFINITION 1. Let E be a Banach space, $K \subset E$ a cone and $\leq$ the partial ordering defined by K. Then

(a) K is reproducing if $K - K = E$;

(b) K is called normal if there exists a constant $\delta > 0$, such that for any x_1, $x_2 \in K$ with $\|x_1\| = \|x_2\| = 1$, we have $\|x_1 + x_2\| \geq \delta$;

(c) K is regular if every increasing sequence which is bounded from above is convergent;

(d) K is minihedral if $\sup\{x,y\}$ exists for all $x, y \in E$, and strongly minihedral if every set which is bounded from above has a supremum.

PROPOSITION 1. Let $E = L^p(\Omega)$, $p \geq 1$, $0 < \text{mes}\Omega < +\infty$, and $K = \{u \in \in L^p(\Omega),\ u(x) \geq 0$ a.e. in $\Omega\}$. Then the cone K is reproducing, normal, minihedral and even strongly minihedral.

The proof of Proposition 1 can be found in [7].

DEFINITION 2. Let (E_1, K_1) and (E_2, K_2) be ordered Banach spaces, and let X be a nonempty subset of E_1. An operator $F: X \to E_2$ is called increasing if $x \leq y$ implies $F(x) \leq F(y)$.

Suppose that $E_1 = E_2$ and let Y be a nonempty subset of X. A fixed point x of an operator $F: X \to E_1$ is called a minimal (or maximal) fixed point in Y if every fixed point y of F in Y satisfies $x \leq y$ (or $y \leq x$).

THEOREM 1. *(See [7]) Let (E, K) be an ordered Banach space and the cone K be strongly minihedral. Assume that in the interval $[u_0, v_0] \subset E$, $F: [u_0, v_0] \to E$ is increasing. If the operator F satisfies the inequalities*

(3) $$u_0 \leq F(u_0), F(v_0) \leq v_0$$

then F *possesses a minimal fixed point* u_* *and a maximal fixed point* u^* *in* $[u_0, v_0]$.

2. The existence theorems

For convenience, we shall consider the elliptic boundary value problem (1)–(2) in the Hilbert space $W^{1,2}(\Omega)$.

Put $K = \{u(x) \in L^2(\Omega) | u(x) \geq 0 \text{ a.e. in } \Omega\}$. It is obvious that K is a cone in $L^2(\Omega)$. Let E be the ordered Banach space $(L^2(\Omega), K)$. The partial ordering in E is defined by $u \leq v$ if and only if $u(x) \leq v(x)$ a.e. in Ω.

The interval $[u, v]$ in E can be defined by

$$[u, v] = \{z(x) \in L^2(\Omega) | u(x) \leq z(x) \leq v(x) \text{ a.e. in} \Omega\}.$$

Because of the low regularity assumptions on the data of the elliptic problem (1)–(2), only weak solutions can be expected.

DEFINITION 3. A function $u(x) \in W_0^{1,2}(\Omega)$ is called a (weak) solution of the elliptic problem (1)–(2) if

$$(4) \quad \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} \left[\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \right] v dx = \int_{\Omega} f(x, u) v dx$$

for any function $v \in W_0^{1,2}(\Omega)$.

If a function $u_0(x) \in W^{1,2}(\Omega)$ satisfies the inequalities

$$(5) \quad \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial u_0}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} \left[\sum_{i=1}^{n} b_i \frac{\partial u_0}{\partial x_i} + cu_0 \right] v dx \leq \int_{\Omega} f(x, u_0) v dx$$

for any $v \in W_0^{1,2}(\Omega) \cap K$; and

$$(6) \quad u_0|_{\partial\Omega} \leq 0, \text{ i.e. } u_0^+ = \max\{u_0(x), 0\} \in W_0^{1,2}(\Omega)$$

we say that $u_0(x)$ is a lower solution of the elliptic problem (1)–(2). While an upper solution is defined by reversing the inequalities.

The following hypotheses will be imposed on the nonlinearity $f(x, u)$.

(H_1) There exists a constant $M \geq 0$ such that $f(x, u) + Mu$ is nondecreasing in u.

(H_2) The elliptic problem (1)–(2) has a lower solution u_0 and an upper solution v_0 such that $u_0 \leq v_0$ and $f(x, u_0(x)), f(x, v_0(x)) \in L^2(\Omega)$.

(H_3) $f(x, u(x))$ is measurable on Ω, whenever $u \in [u_0, v_0]$.

The main result of this section is the following

THEOREM 2. *Assume that conditions* (H_1)–(H_3) *are fulfilled. Then the elliptic boundary value problem* (1)–(2) *has both the least and the greatest solution in the order interval* $[u_0, v_0]$ *in* E.

Before proving Theorem 2, we shall prove

PROPOSITION 2. If the function f satisfies the conditions (H_1)–(H_3), then for any $v \in [u_0, v_0]$, the linear elliptic boundary value problem

$$\mathcal{L}u + Mu = f(x, v) + Mv \quad x \in \Omega \tag{7}$$

$$u|_{\partial\Omega} = 0 \tag{8}$$

has a unique solution $u = T(v)$ and there is a constant $c > 0$ such that

$$\|T(v)\|_{W_0^{1,2}(\Omega)} \leq c \text{ for all } v \in [u_0, v_0].$$

Moreover the operator $T : [u_0, v_0] \to E$ is an increasing operator such that

$$u_0 \leq T(u_0), T(v_0) \leq v_0 \tag{9}$$

PROOF. For any $v \in [u_0, v_0]$, i.e. $v \in L^2(\Omega)$, $u_0(x) \leq v(x) \leq v_0(x)$ a.e. in Ω, by (H_1)

$$f(x, u_0(x)) + Mu_0(x) \leq f(x, v(x)) + Mv(x) \leq f(x, v_0(x)) + Mv_0(x)$$

So

$$|f(x, v(x)) + Mv(x)| \leq |f(x, u_0(x)) + Mu_0(x)| + |f(x, v_0(x)) + Mv_0(x)| \tag{10}$$

In virtue of (H_2), (H_3) and Theorems 8.3 and 8.7 in [6], the linear elliptic problem (7)–(8) is uniquely solvable in $W_0^{1,2}(\Omega)$ and there exists a constant $c > 0$ such that

$$\|T(v)\|_{W_0^{1,2}(\Omega)} \leq c \text{ for all } v \in [u_0, v_0]$$

In order to prove inequalities (9), let $w(x) = T(u_0) - u_0$. since $T(u_0)$ is the solution of the problem

$$\mathcal{L}(T(u_0)) + M(T(u_0)) = f(x, u_0) + Mu_0 \quad x \in \Omega \tag{11}$$

$$T(u_0)|_{\partial\Omega} = 0 \tag{12}$$

and u_0 is a lower solution of the elliptic problem (1)–(2), i.e. u_0 satisfies (in weak sense) the inequalities

$$\mathcal{L}u_0 + Mu_0 \leq f(x, u_0) + Mu_0 \quad x \in \Omega \tag{13}$$

$$u_0|_{\partial\Omega} \leq 0 \tag{14}$$

thus, $w(x)$ satisfies

(15) $$\mathcal{L}w + Mw \geq 0 \quad x \in \partial\Omega$$

(16) $$w|_{\partial\Omega} \geq 0$$

Applying the maximum principle, Theorem 8.1 in [6], we get

$$w(x) \geq 0 \quad \text{a.e. in } \Omega$$

i.e. $u_0 \leq Tu_0$. Similarly can be obtained $Tv_0 \leq v_0$.

Using condition (H_1), it is easy to obtain by analogous arguments that T is an increasing operator. The proof of the Proposition is complete.

THE PROOF OF THEOREM 2: By Proposition 1, the cone $K = \{u(x) \in \in L^2(\Omega) | u(x) \geq 0$ a.e. in $\Omega\}$ is strongly minihedral. Proposition 2 tells us that in the ordered Banach space $E = (L^2(\Omega), K)$, the operator $T : [u_0, v_0] \to E$ is increasing and $u_0 \leq T(u_0)$, $T(v_0) \leq v_0$. Applying Theorem 1, the operator T possesses a minimal fixed point u_* and a maximal fixed point u^* in $[u_0, v_0]$. According to the definitions of the operator T, (see Proposition 2 for its definition). We know that u_* and u^* are least and greatest solutions of the elliptic problem (1)–(2) in the order interval $[u_0, v_0]$, respectively. So the proof of Theorem 2 is complete.

REMARK 1. In general, hyphotesis (H_1) is essential and can not be relaxed. This is seen by the following example. Let us consider the elliptic problem

(17) $$-\nabla^2 u = f(u) \quad x \in \Omega$$

(18) $$u|_{\partial\Omega} = 0$$

where the nonlinearity

$$f(s) = \begin{cases} 1 & \text{for } s \leq 0 \\ 0 & \text{for } s > 0 \end{cases}.$$

One readily verifies that $u_0 = 0$ and $v_0 = 1$ are lower and upper solutions, respectively. The nonlinearity f satisfies the assumptions of Theorem 2, but hypothesis (H_1) is violated since for any constant $M \geq 0$, the jump of $f(s) + Ms$ in $s = 0$ is downward. Thus our Theorem 2 can not be applied. Moreover, we shall show that the elliptic problem (17)–(18) has no solution.

Suppose that there would be any solution u which belongs to $W_0^{1,2}(\Omega)$. By definition, u should satisfy the relation

$$\sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f(u) v\, dx \quad \forall v \in W_0^{1,2}(\Omega)$$

Particularly, we set $v = u$,

$$\sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \int_{\Omega} f(u) u dx \tag{19}$$

By the definition of the nonlinearity f, the right hand side of (19) is nonpositive, because we can estimate as follows

$$\int_{\Omega} f(u) u dx = \int_{\{u(x) \leq 0\}} f(u) u dx + \int_{\{u(x) > 0\}} f(u) u dx \leq 0.$$

Therefore from (19) it follows that $u \equiv 0$ in Ω, which is, however, in contradiction to the assumption of u being a solution. So we have proved that elliptic problem (17)–(18) has no solution.

THEOREM 3. *Suppose that there exists a positive constant r such that $|f(x,s)| \leq c(x)|s| + r$ for all $(x,s) \in \Omega \times R^1$, where $c(x)$ is the coefficient appearing in the elliptic operator $\mathcal{L}$ (see,* (1)*). Moreover, let $f(x,u)$ be nondecreasing in u for $x \in \Omega$ and $f(x,v(x))$ be measurable on Ω for any $v \in L^2(\Omega)$. Then the nonlinear elliptic boundary value problem* (1)–(2) *has at least one solution.*

PROOF. We first consider the linear elliptic boundary value problem

$$\mathcal{L}u - c(x)u = r \quad x \in \Omega$$
$$u|_{\partial\Omega} = 0$$

i.e.

$$-\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = r \tag{20}$$

$$u|_{\partial\Omega} = 0 \tag{21}$$

By maximum principle, linear elliptic boundary value problem (20)–(21) has exactly one solution $v_0 \in W_0^{1,2}(\Omega)$ and $v_0(x) \geq 0$ in Ω.

Let $u_0(x) = -v_0(x)$. Then $u_0(x) \leq 0$ in Ω and $u_0(x)$ is the only solution of the elliptic boundary value problem

$$\mathcal{L}u - c(x)u = -r \quad x \in \Omega \tag{22}$$

$$u|_{\partial\Omega} = 0 \tag{23}$$

Therefore, from (20)–(21)

$$\mathcal{L}v_0 = r + c(x)v_0 \geq f(x, v_0) \quad x \in \Omega$$
$$v_0|_{\partial\Omega} = 0$$

here we have used the condition $|f(x,s)| \leq c(x)|s| + r$.

Analogously, from (22)–(23)

$$\mathcal{L}u_0 = -r + c(x)u_0 = -r - c(x)|u_0| \leq f(x,u_0) \quad x \in \Omega$$
$$u_0|_{\partial\Omega} = 0$$

which indicate that u_0 and v_0 are lower and upper solutions of the elliptic boundary value problem (1)–(2) and $u_0 \leq v_0$. The rest of the proof for the theorem follows from Theorem 2.

THEOREM 4. *Suppose the nonlinearity $f(x,u)$ satisfies condition* (H_1) *and is nonincreasing in u for a.e. $x \in \Omega$. If $f(x,0) \in L^q(\Omega)$ where $q > \frac{n}{2}$, then there exist a lower and an upper solution u_0, v_0 of the elliptic problem* (1)–(2) *such that $u_0 \leq v_0$ in E. Moreover, if $f(x,u_0(x))$, $f(x,v_0(x)) \in L^2(\Omega)$ and for any $v \in [u_0,v_0]$, $f(x,v(x))$ is measurable on Ω, then there exists a unique solution u of the elliptic problem* (1)–(2) *such that $u_0 \leq u \leq v_0$.*

PROOF. Since $f(x,0) \in L^q(\Omega)$, $q > \frac{n}{2}$, by Theorem 8.30 in [6], the solution $w(x)$ of the linear elliptic problem

$$\mathcal{L}w = f(x,0) \quad x \in \Omega \tag{24}$$

$$w|_{\partial\Omega} = 0 \tag{25}$$

belongs to $C^0(\overline{\Omega})$, where $C^0(\overline{\Omega})$ denotes the set of all continuous functions on Ω. Set $u_0 = -R + w(x)$, $v_0 = R + w(x)$. Choose $R > 0$ so large that $u_0(x) \leq 0 \leq v_0(x)$ in Ω. We have

$$\begin{aligned} \mathcal{L}u_0 &= -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial w}{\partial x_i} + c(x)(-R + w(x)) \\ &= \mathcal{L}w - c(x)R \\ &= f(x,0) - c(x)R \\ &\leq f(x,0) \\ &\leq f(x,u_0(x)), \quad x \in \Omega \end{aligned} \tag{26}$$

here we have used $c(x) \geq 0$, $u_0(x) \leq 0$ and the nonincreasing property of $f(x,s)$ in s.

Analogously, we have

$$\mathcal{L}v_0 \geq f(x,v_0(x)), \quad x \in \Omega. \tag{27}$$

Therefore $u_0(x)$, $v_0(x)$ are lower and upper solutions of problem (1)–(2), respectively, and $u_0 \leq v_0$. The proof of the first part of Theorem 4 is complete.

Moreover, if $f(x,u_0(x))$, $f(x,v_0(x)) \in L^2(\Omega)$ and for any $v \in [u_0,v_0]$, $f(x,v(x))$ is measurable on Ω, then $f(x,u)$ satisfies all the conditions of Theorem 2. So elliptic problem (1)–(2) has the least solution u_* and the greatest solution u^* in the order interval $[u_0,v_0]$ in E. Now we shall apply the maximum principle again to prove that $u_* = u^*$ in Ω.

Since u_* and u^* are the least and greatest solution in the order interval $[u_0,v_0]$, respectively, so

(28) $$u_*(x) \le u^*(x) \quad \text{a.e. in } \Omega.$$

Let $w_0(x) = u_*(x) - u^*(x)$. Then by the nonincreasing property of $f(x,s)$ in s,

$$Lw_0 = f(x,u_*(x)) - f(x,u^*(x)) \ge 0 \qquad x \in \Omega$$
$$w_0\Big|_{\partial\Omega} = 0$$

Using the maximum principle, we know that $w_0(x) \ge 0$ in Ω, i.e.

(29) $$u_*(x) \ge u^*(x) \qquad \text{a.e. in } \Omega$$

Combining (28) with (29),we get

$$u_*(x) = u^*(x) \quad \text{a.e. in } \Omega$$

that is, the elliptic problem (1)–(2) has a unique solution in $[u_0,v_0]$. So we have proved the assertion of the second part of Theorem 4.

REMARK 2. In Theorem 4, although we do not obviously impose any continuity conditions on $f(x,u)$, it is, in fact, continuous in u if

(a) it satisfies condition (H_1); and

(b) it is nonincreasing in u.

Now we show the assertion. By (a), there exists a constant $M \ge 0$ such that $f(x,u) + Mu$ is nondecreasing in u. So if u, $v \in R^1$, $u \le v$, then

$$f(x,u) + Mu \le f(x,v) + Mv$$

Using (b), we have

$$f(x,v) + Mu \le f(x,u) + Mu \le f(x,v) + Mv$$

i.e.

$$0 \le f(x,u) - f(x,v) \le M(v-u)$$

Therefore,

$$\lim_{v \to u+} f(x,v) = f(x,u)$$

i.e. $f(x,u)$ is right continuous in u.

Similarly, we can prove that $f(x,u)$ is left continuous in u. Hence $f(x,u)$ is continuous in u.

REMARK 3. In Theorem 4., the conditions that $f(x,u)$ satisfies condition (H_1) and it is nonincreasing in u for a.e. $x \in \Omega$ can be satisfied if for any $(x,u) \in \Omega \times R^1$,

$$-l \leq \frac{\partial f(x,u)}{\partial u} \leq 0,$$

where l is a positive constant.

3. Monotone iterative technique

Theorem 2, 3 and 4 provide only pure existence results, that is to say, they are not constructive in general. In the following, we shall impose one more condition on the nonlinearity f and show that the method of lower and upper solutions combined with monotone iteration is well suited to provide constructive existence results, where the monotone iterative technique has been used, for example, in [4] for ordinary differential equations with continuous nonlinearity.

THEOREM 5. *Suppose that* $f(x,u)$ *satisfies all the conditions in Theorem 2 and* $f(x,s)$ *is right continuous in* s (i.e. $\lim_{t \to s+} f(x,t) = f(x,s)$ a.e. in Ω). *Then we can get the greatest solution of elliptic problem* (1)–(2) *in the order interval* $[u_0, v_0]$ *by the monotone iteration scheme*

$$\mathscr{L}v_{k+1} + Mv_{k+1} = f(x, v_k) + Mv_k \quad x \in \Omega \tag{30}$$

$$v_{k+1}|_{\partial\Omega} = 0 \quad k = 0,1,2,\ldots. \tag{31}$$

PROOF. By Proposition 2, the operator $T : [u_0, v_0] \to E$ is an increasing operator, and $v_1 = Tv_0 \leq v_0$. By induction, we know that $u_0 \leq v_k \leq v_0$. $\|v_k\|_{W^{1,2}(\Omega)} \leq c$, for $k = 1,2,\ldots$, and $\{v_k\}$ is a monotone nonincreasing sequence. Using the compact imbedding $W_0^{1,2}(\Omega) \to L^2(\Omega)$ and the monotonicity of the sequence $\{v_k\}$, we get

$$v_k \to v \quad \text{strongly in } L^2(\Omega) \tag{32}$$

$$v_k \to v \quad \text{weakly in } W_0^{1,2}(\Omega) \tag{33}$$

$$v_k \to v \quad \text{a.e. in } \Omega. \tag{34}$$

Now we consider the linear elliptic boundary value problem (30)–(31) in its weak formulation and pass to the limit as $k \to \infty$. By means of (32)–

(33) and the boundedness of all coefficients of the operator $\mathscr{L}$ in (1), we immediately get

(35) $$\sum_{ij=1}^{n}\int_{\Omega} a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial u}{\partial x_i}dx+\int_{\Omega}\left[\sum_{i=1}^{n} b_i\frac{\partial v}{\partial x_i}+c(x)v\right]udx+\int_{\Omega} Mvudx$$

$$=\lim_{k\to\infty}\int_{\Omega}\left[f(x,v_k)+Mv_k\right]udx \quad \text{for all } u\in W_0^{1,2}(\Omega)$$

in order to prove that v is a solution of problem (1)–(2), it remains to show that the following equality is valid:

(36) $$\lim_{k\to\infty}\int_{\Omega}\left[f(x,v_k)+Mv_k\right]udx=\int_{\Omega}\left[f(x,v)+Mv\right]udx$$

This can be seen as follows. The function $f(x,u)$ is supposed to be right continuous in u, which obviously implies the right continuity of the function $f(x,u)+Mu$ in u. By (34) and the nonincreasing property of the sequence $\{v_k\}$, we know

(37) $$f(x,v_k)+Mv_k \underset{k\to\infty}{\longrightarrow} f(x,v)+Mv \quad \text{a.e. in } \Omega$$

Using (H_1) and $v_k\in[u_0,v_0]$,

$$f(x,u_0)+Mu_0\le f(x,v_k)+Mv_k\le f(x,v_0)+Mv_0$$

So

(38) $$|f(x,v_k)+Mv_k|\le|f(x,u_0)+Mu_0|+|f(x,v_0)+Mv_0|$$

By Lebesque's dominated convergence theorem, the identity (36) holds. Thus the limit v of the sequence $\{v_k\}$ is a solution of problem (1)–(2) and due to the inequality $u_0\le v_k\le v_0$, it is contained in $[u_0,v_0]$.

In the following we shall show that the solution constructed by the monotone iteration scheme (30)–(31) is the greatest one in the order interval $[u_0,v_0]$. This can be shown by taking any other solution $u\in[u_0,v_0]$ as a special lower solution of the problem (1)–(2). Then the iterates v_k constructed above satisfy the inequalities

$$u\le v_k\le v_0$$

and thus it follows $u\le v$. The proof is complete.

Analogously, we can prove

THEOREM 6. *Suppose that* $f(x,s)$ *satisfies all the conditions in Theorem 2 and* $f(x,s)$ *is left continuous in* s (i.e. $\lim_{t \to s-} f(x,t) = f(x,s)$ a.e. in Ω). *Then we can get the least solution of elliptic problem* (1)–(2) *in the order interval* $[u_0, v_0]$ *by the monotone iteration scheme*

$$\mathscr{L}u_{k+1} + Mu_{k+1} = f(x, u_k) + Mu_k \quad x \in \Omega \tag{39}$$

$$u_{k+1}|_{\partial\Omega} = 0. \tag{40}$$

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A C^2 RATIONAL CUBIC SPLINE WHICH HAS LINEAR DENOMINATOR AND SHAPE CONTROL

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1. Introduction

The generation of interpolating spline curves is a useful tool in Computer Graphics. Although the C^2 cubic splines have the many elegant mathematical properties discussed in [6], [7], [22], [23], the curves sometimes exhibit undesirable oscillations. Various methods have been developed to control the shape of an interpolating curve, such as those in [1]–[3], [5], [7], [10], [11], [15]–[17], [24]–[31]. Some methods are well suited for one type of shape control but not well suited for another. For this reason, a multipurpose system was developed in [18] which consists of different spline methods and uses the particular spline that is most suited for the desired type of shape control. This system uses a C^2 cubic spline to generate the initial interpolating curve, an exponential-based spline under tension ([7], [17]) and a rational spline with tension [5] are used to *flatten* or *tighten* the curve on segments between interpolation points, and piecewise cubic ν-splines ([2], [15]) are used to *sharpen* or *tighten* the corners of the curve at the interpolation points.

Thus, to avoid a multiplicity of methods, one method is desirable which is capable of generating a broad range of interpolating curves, is easy to implement, provides a shape control according to the user's wishes, visually pleasing and computationally economical. Although this kind of problem is tackled in [11] but this method is only C^1 and weights for interval tension can not be applied blindly as the curve may not achieve the interval tension. These shortcomings are removed in [1] and a C^2 method, to achieve similar shape characteristics, is developed but the shape parameters (two shape parameters per interval) are not independent and thus the user has to have again careful in application. Alternatively, in [33], another method was presented and a description and analysis of a rational geometric cubic spline, in interpolatory form, was given. This rational spline provides not only a computationally simple alternative to the exponential based spline under tension ([7], [16], [17], [26]) but also provides a GC^2 alternative to the well known existing GC^2 or C^1 methods like cubic ν-spline of NIELSON [15], γ-splines of BOEHM [13] and weighted ν-splines [11]. This method is the marriage of the *rational spline with tension* [5] and ν-spline of NIELSON ([2], [15]). The shape parameters are introduced in such a way that each interval and the control point is controlled by one parameter. They provide a variety of shape controls like point and interval tensions both locally and globally. The continuity in this scheme is GC^2.

This paper describes a parametric C^2 rational cubic spline representation which has point tension weights which can be used to control the shape of the curve, both locally and globally. Like other methods in literature, the shape parameters have maximum influence on the target area of the curve, but at the same time some neighbouring portion of the target area also has a slight effect which is not that significant. The rational spline can be considered as an alternative to the spline methods mentioned in the above paragraph. It also has the characteristic to be in the class of the rational cubics having linear denominator and thus is highly economical.

We first describe the piecewise rational cubic interpolant in the following section and the spline curve method in Section 3. An algorithm for the computation is suggested in Section 4. The effects of the shape parameters are illustrated by examples in Section 5.

1.1. Notation and Conventions

- The symbol $\mathbb{R}^N$ will be used to denote the N-dimensional real space.
- Knot partition will be assumed as:

  (1.1) $$t_0 < t_1 < \ldots < t_n.$$

- For any i transformation

  (1.2) $$\theta \equiv \theta(t) = (t - t_i)/h_i$$

  will be commonly used where

  (1.3) $$h_i = t_{i+1} - t_i.$$

- Vectors (Points) and vector valued functions are set in bold face letters.
- $\mathbf{X}_i$, $i = 0, 1, \ldots, n$, will denote the interpolatory points and $\mathbf{\Delta}_i$ will be used for the ratios of the type:

  (1.4) $$\mathbf{\Delta}_i = (\mathbf{X}_{i+1} - \mathbf{X}_i)/h_i.$$

- $\mathbf{D}_i$ will be used for the first derivative value at the knot t_i.
- Given a function such as $\mathbf{P}(t)$, we will denote the i^{th} derivative by $\mathbf{P}^{(i)}(t)$.
- $\mathbf{P} \in C^m[t_0,t_n]$ will mean that each component function of $\mathbf{P}:[t_0,t_n] \to \mathbb{R}^N$ is m-times continuously differentiable on $[t_0,t_n]$. Similarly the notation GC^m will be fixed for geometric (reparametrization) continuity.
- We will use $\|\cdot\|$ to denote the uniform norm, either on $[t_0,t_n]$ or $[t_i,t_{i+1}]$.

2. The Rational Cubic Form

The rational cubic spline curve in the following section will be constructed by using a piecewise rational cubic Hermite parametric function $\mathbf{P} \in C^1[t_0,t_n]$, with parameters α_i, β_i, $i = 0,1,\ldots,n-1$, which is defined for $t \in [t_i,t_{i+1}]$, $i = 0,1,\ldots,n-1$, by:

(2.1)
$$\mathbf{P}(t) = \mathbf{P}_i(t;\alpha_i,\beta_i) =$$
$$= \frac{(1-\theta)^3\alpha_i\mathbf{X}_i + \theta(1-\theta)^2(2\alpha_i+\beta_i)\mathbf{V}_i + \theta^2(1-\theta)(\alpha_i+2\beta_i)\mathbf{W}_i + \theta^3\mathbf{X}_{i+1}}{\alpha_i(1-\theta)+\beta_i\theta},$$

where

(2.2)
$$\mathbf{V}_i = \mathbf{X}_i + \frac{h_i\alpha_i}{2\alpha_i+\beta_i}\mathbf{D}_i, \quad \mathbf{W}_i = \mathbf{X}_{i+1} - \frac{h_i\beta_i}{\alpha_i+2\beta_i}\mathbf{D}_{i+1}$$

and $\alpha_i, \beta_i \geq 0$.

The function $\mathbf{P}(t)$ has the Hermite interpolation properties that

$$\mathbf{P}(t_i) = \mathbf{X}_i \quad \text{and} \quad \mathbf{P}^{(1)}(t_i) = \mathbf{D}_i, \quad i = 0,1,\ldots,n.$$

The denominator in (2.1) can be written as

(2.3)
$$\alpha_i(1-\theta)^3 + (2\alpha_i+\beta_i)\theta(1-\theta)^2 + (\alpha_i+2\beta_i)\theta^2(1-\theta) + \beta_i\theta^3.$$

The α_i and β_i, $i = 0,1,\ldots,n-1$, will be used as *shape* parameters to control and fine tune the shape of the curve. The case $\alpha_i = \beta_i = 1$, $i = 0,1,\ldots,i-1$, is that of cubic (polynomial case) Hermite interpolation and the restriction α_i, $\beta_i > 0$ ensures a positive denominator in (2.1).

For $\alpha_i, \beta_i \neq 0$, (2.1) can be written in the form

(2.4)
$$\mathbf{P}_i(t_i;\alpha_i,\beta_i) =$$
$$= R_0(\theta;\alpha_i,\beta_i)\mathbf{X}_i + R_1(\theta;\alpha_i,\beta_i)\mathbf{V}_i + R_2(\theta;\alpha_i,\beta_i)\mathbf{W}_i + R_3(\theta;\alpha_i,\beta_i)\mathbf{X}_{i+1},$$

where $R_j(\theta;\alpha_i,\beta_i)$, $j = 0, 1, 2, 3$, are appropriately defined rational functions with

(2.5)
$$\sum_{j=0}^{3} R_j(\theta;\alpha_i,\beta_i) = 1.$$

Moreover, these functions are rational Bernstein–Bézier weight functions which are non-negative for $\alpha_i, \beta_i > 0$. Thus in $\mathbb{R}^N$, $N > 1$ and for $\alpha_i, \beta_i > 0$ we have:

Convex hull property: The curve segment $\mathbf{P}_i$ lies in the convex hull of the control points $\{\mathbf{X}_i, \mathbf{V}_i, \mathbf{W}_i, \mathbf{X}_{i+1}\}$.

Variation diminishing property: The curve segment $\mathbf{P}_i$ crosses any (hyper) plane of dimension $N-1$ no more times than it crosses the control polygon joining $\mathbf{X}_i$, $\mathbf{V}_i$, $\mathbf{W}_i$, $\mathbf{X}_{i+1}$.

REMARK 2.1. If $P(t)$ is the interpolant for scalar data $X_i \in \mathbb{R}$, with derivatives $D_i \in \mathbb{R}$, $i \in \mathbb{Z}$, then $(t, P(t))$ can also be applied in the scalar case. We apply this to the curve segment $(t, P_i(t;\alpha_i,\beta_i)) \in \mathbb{R}^2$, $t \in (t_i, t_{i+1})$, to the data (t_i, X_i) with derivatives $(1, D_i)$, $i \in \mathbb{Z}$. This is a consequence of the property that the interpolant is capable of reproducing linear functions. Particularly, for scalar data $X_i := t$ and derivatives $D_i := 1$, $i \in \mathbb{Z}$, the interpolant reproduces the function t.

Regarding the shape control, we immediately have the following from (2.1) and (2.2):

$$(2.6)\quad \begin{cases} \lim_{\alpha_i \to \infty} \mathbf{W}_i = \mathbf{X}_{i+1} \quad \text{and} \\ \lim_{\alpha_i \to \infty} \mathbf{P}_i(t;\alpha_i,\beta_i) = (1-\theta)^2\mathbf{X}_i + \theta(1-\theta)(2\mathbf{X}_i + h_i\mathbf{D}_i) + \theta^2\mathbf{X}_{i+1}, \end{cases}$$

$$(2.7)\quad \begin{cases} \lim_{\beta_i \to \infty} \mathbf{V}_i = \mathbf{X}_i \quad \text{and} \\ \lim_{\beta_i \to \infty} \mathbf{P}_i(t;\alpha_i,\beta_i) = (1-\theta)^2\mathbf{X}_i + \theta(1-\theta)(2\mathbf{X}_{i+1} + h_i\mathbf{D}_{i+1}) + \theta^2\mathbf{X}_{i+1}, \end{cases}$$

One can note that:

Point Tension behaviour:

(i) increase in the parameter α_i, results the curve to be pulled towards the control point $\mathbf{X}_{i+1}$, (see Figure 1).

(ii) increase in the parameter β_i, results the curve to be pulled towards the control point $\mathbf{X}_i$, (see Figure 2).

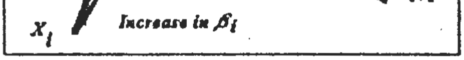

Fig. 1

No Tension behaviour:

(iii) The limits $\alpha_i = \beta_i \to \infty$ will have no effect on the shape of the curve as it reduces to the simple cubic polynomial form, (see Figure 3).

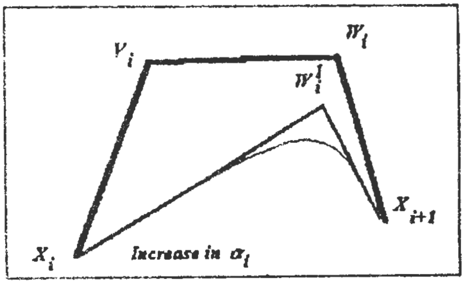

Fig. 2

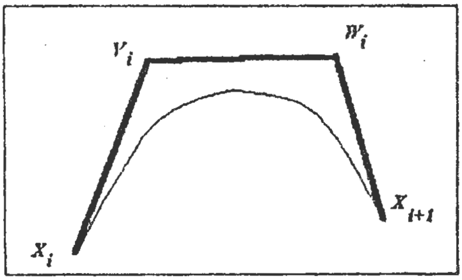

Fig. 3

3. C^2 Rational Cubic Interpolation

Now we construct a parametric C^2 rational cubic spline interpolant on $[t_0, t_n]$, using the rational cubic Hermite form of section 2. This is done by joining the piecewise rational cubics (2.1) with C^2 constraints

$$\mathbf{P}^{(2)}(t_i^+) = \mathbf{P}^{(2)}(t_i^-), \qquad i = 1, \ldots, n-1, \tag{3.1}$$

which give the tridiagonal system of *consistency equations*

$$\frac{h_i\alpha_{i-1}}{\beta_{i-1}}\mathbf{D}_{i-1} + \left(h_i\left(1+\frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\left(1+\frac{\beta_i}{\alpha_i}\right)\right)\mathbf{D}_i + \frac{h_{i-1}\beta_i}{\alpha_i}\mathbf{D}_{i+1} =$$
(3.2)
$$= h_i\left(1+\frac{2\alpha_{i-1}}{\beta_{i-1}}\right)\boldsymbol{\Delta}_{i-1} + h_{i-1}\left(1+\frac{2\beta_i}{\alpha_i}\right)\boldsymbol{\Delta}_i, \qquad i = 1, \ldots, n-1.$$

With appropriate end conditions $\mathbf{D}_0$ and $\mathbf{D}_n$, (3.2) is a tridiagonal linear system in the unknowns $\mathbf{D}_i$, $i = 1, \ldots, n-1$. Assume that

$$\alpha_i, \quad \beta_i > 0, \tag{3.3}$$

then the tridiagonal linear system is strictly diagonally dominant and hence has a unique solution which can be easily calculated by use of some appropriate algorithm. Thus a rational cubic spline interpolant can be constructed with tension parameters α_i, β_i, $i = 0, \ldots, n-1$, where the special case is such that $\alpha_i = 1 = \beta_i$, $\forall\, i$ corresponds to cubic spline interpolation.

We now prove the boundedness of the solution of the consistency equations for varying the shape parameters according to (3.3). For this we can convert the system (3.2) in unit diagonal form as:

$$a_i\mathbf{D}_{i-1} + \mathbf{D}_i + b_i\mathbf{D}_{i+1} = \mathbf{c}_i, \qquad i = 1, \ldots, n-1. \tag{3.4}$$

In matrix form, it can be written as:

$$(\mathbf{I}+\mathbf{E})\mathbf{D} = \mathbf{C}, \tag{3.5}$$

where the terms involving the end conditions have been transferred to the right hand side and **E** is a tridiagonal matrix with zero diagonal.

Let us have $\varepsilon > 0$, such that

$$1/\varepsilon \geq \alpha_i, \quad \beta_i \geq \varepsilon, \quad i = 0,\dots,n-1 \tag{3.6}$$

then we have

$$\|\mathbf{E}\| \leq 1/(1+\varepsilon) \quad \text{and} \quad \|(\mathbf{I}+\mathbf{E})^{-1}\| \leq 1 + 1/\varepsilon. \tag{3.7}$$

and $\|\mathbf{C}\|$ is bounded as

$$\|\mathbf{C}\| \leq \max\left(|\mathbf{D}_0|, |\mathbf{D}_n|\right) + 4 \max_{0 \leq i \leq n-1} |\boldsymbol{\Delta}_i| \tag{3.8}$$

Hence the boundedness property is obtained.

3.1. Shape Control Analysis

We now consider the limiting behaviour of the solution with respect to each tension parameter, where we assume that the other parameters are held constant with respect to each limit process:

(i) **Point Tension.** Let $\alpha_i \to \infty$, then the ith equation takes the form

$$\frac{h_i \alpha_{i-1}}{\beta_{i-1}} \mathbf{D}_{i-1} + \left(h_i \left(1 + \frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\right) \mathbf{D}_i = \tag{3.4}$$

$$= h_i \left(1 + \frac{2\alpha_{i-1}}{\beta_{i-1}}\right) \boldsymbol{\Delta}_{i-1} + h_{i-1} \boldsymbol{\Delta}_i,$$

(the $i+1$st equation is also affected). The equations thus become decoupled in that the system for $\mathbf{D}_{i-1},\dots,\mathbf{D}_{n-1}$ in the limit is dependent only on the data on $[t_i, t_n]$. The tension property (2.6) will now hold. Furthermore, (3.4) is consistent with setting the derivative of (2.6) at t_i equal to $\mathbf{D}_i$. Hence the rational spline is C^1 at t_i in the limit.

The case $\beta_i \to \infty$ is the dual of the above.

(ii) **Accentuated point tension.** There is no decoupling, in the case $\beta_{i-1} = \alpha_i \to \infty$ of the system (3.2) in the limit. The point tension property now holds from both left and right of t_i, where the spline interpolant becomes C^0.

(iii) **Interval Tension.** Similar as above if we let α_i, $\alpha_{i+1} \to \infty$, then the point tension at two points gives rise to the interval tension and thus the global tension can be achieved by letting $\alpha_i \to \infty$, $\forall\, i$, where all β_i's are kept fixed.

The case $\beta_i \to \infty$, for any i, where α_i's are kept fixed, is the dual of the above.

REMARK 3.1. It should be noted that by the interval tension, here, we do not mean that, in limiting case, the curve is pulled to the control polygon

$(1-\theta)X_i + \theta X_{i+1}$, $i = 0, 1, \ldots, n-1$. By the interval tension, we mean that the curve has some interaction in the sense that the convex hull region formed by the control points of the rational cubics (2.1) is reduced to that of quadratics, (see (2.6) and (2.7)).

4. Computational Method

For practical purposes, to determine and display the rational spline curve, it is proposed to adopt the LU-decomposition method for the solution of the triangular system (3.1). Once the tangent vectors are in hand, the user can obtain the points in the rational spline curve by using some efficient method for rational Bernstein–Bézier form (see [8], [32]).

5. Examples

For illustration of the tension results, consider a data set in $\mathbb{R}^2$ which defines the control points of the rational cubic spline representation (the Figure 4 displays the control polygon of this data). The curve in Figure 5 is the cubic spline whereas the curve in Figure 6 demonstrates the *accentuated point tension* effect. The interval and global tension effects, as mentioned in Remark 3.1, are displayed in the Figures 7 and 8 respectively. It should be noted that, in all these figures, where ever the shape effect is achieved it is done by taking shape parameters equivalent to 50 and otherwise they are considered equivalent to 1 everywhere.

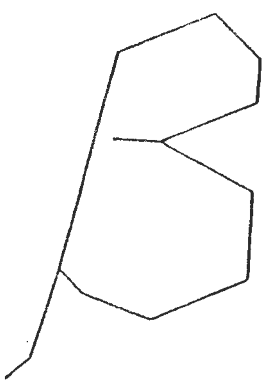

Fig. 4

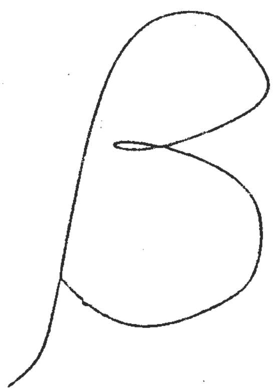

Fig. 5

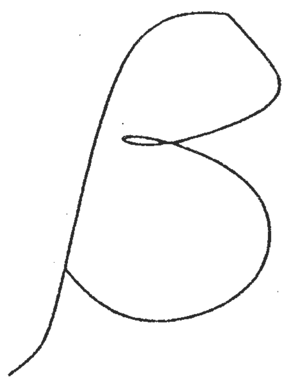

Fig. 6

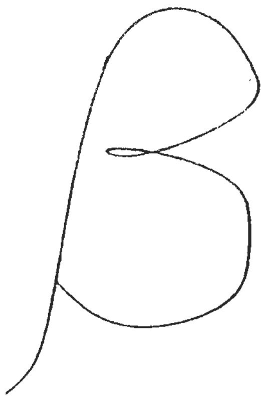

Fig. 7

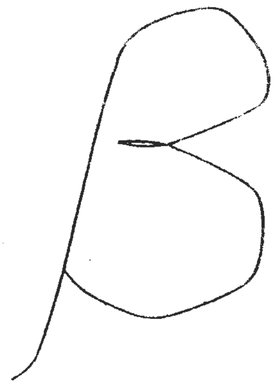

Fig. 8

6. Concluding Remarks

An analysis of a C^2 interpolatory rational cubic spline is developed with a view to its application in Computer Graphics. It is quite reasonable to construct a rational form, with cubic numerator and linear denominator, which involves shape parameters in its description and provides a variety of local and global controls like interval and point shape effects. The rational spline method can be applied to tensor product surfaces but unfortunately, in the context of interactive surface design, this tensor product surface is not that useful because any one of the tension parameters controls an entire corresponding interval strip of the surface. Thus as an application of C^2 rational spline for the surfaces, Nielson's [2] spline blended methods can be adopted. This will not only produce local shape control but also elevate the continuity as Nielson uses GC^2 ν-splines and observes only GC^1 continuity results.

The mathematics of the rational cubic spline could also be visualized via its homogeneous counter part. It could be established some equivalent constraints on the homogeneous curve regarding parametric continuity. For every degree of continuity, the rational continuity constraints probably can contain a degree of freedom not present in the corresponding continuity constraints for projected curves: relationship of these degrees of freedom with the rational cubic spline could be derived.

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INTERPOLATION BETWEEN SOME DYADIC HARDY-TYPE SPACES OF SEQUENCES AND SOME APPLICATIONS

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0. Preliminaries

0.1. Dyadic Hardy-type spaces

We begin this section with some notations. In what follows L^p will denote $L^p[0,1]$. We shall call a dyadic interval in $[0,1]$ an interval of the form $\left[k2^{-n},(k+1)2^{-n}\right)$, where $n \in \mathbb{N}$ and $0 \leq k \leq 2^n$. For any set A we shall denote the characteristic function of A by $\chi(A)$. By a dyadic step function we shall mean a finite linear combination of characteristic functions of dyadic intervals in $[0,1]$. By a Walsh polynomial we shall mean a finite linear combination of Walsh functions. We shall denote the collection of Walsh polynomials by $\mathcal{P}$. We have that the collection $\mathcal{L}$ of dyadic step functions coincides with $\mathcal{P}$, (See [3]).

For each $f \in L^1$, let

$$\varepsilon_n f := s_{2^n} f, \qquad (n \in \mathbb{N});$$

where $s_{2^n} f$ is the 2^n-th partial sum of the Walsh–Fourier series of f.

DEFINITION 1. The dyadic maximal operator for $f \in L^1$ is defined by

$$\varepsilon f := \operatorname*{Sup}_{n \in \mathbb{N}} |\varepsilon_n f|.$$

For each $f \in L^1$ and $0 < p < \infty$ set

$$\|f\|_{\mathrm{H}^p} := \|\varepsilon f\|_p.$$

On the collection of Walsh polynomials $\mathcal{P}$, the map $f \to \|f\|_{\mathrm{H}^p}$ is a norm for $1 \leq p < \infty$ and a quasi-norm for $0 < p < 1$.

DEFINITION 2. (Dyadic Hardy spaces). We define the Dyadic–Hardy spaces $\mathbf{H}^p$ to be the closure of $\mathcal{P}$ in the quasi-norm

$$\|\ldots\|_{\mathbf{H}^p} \qquad \text{for} \quad 0<p<\infty.$$

For these spaces we have

i) For $1 \leq p < \infty$, the space $\mathbf{H}^p$ is precisely the collection of functions $f \in L^1$ such that $\|f\|_{\mathbf{H}^p} < \infty$.

ii) The space $\mathbf{H}^p$ an L^p are isomorphic, with equivalent norms for $1<p<\infty$.

For a detailed study of these space see [3].

It is easy to see that

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} a_k \right) (b_k - b_{k+1}) + b_n \sum_{j=0}^{n} a_j \tag{1}$$

holds for all real sequences $(a_k, k \in \mathbb{N})$ and $(b_k, k \in \mathbb{N})$. The identity (1) will be used in the present work and will be referred to as Abel's transformation.

0.2. The Complex method of interpolation

In this section we define the interpolation spaces $\overline{A}_{[\Theta]}$ in the same way as in [1].

Given a couple $\overline{A} = (A_0, A_1)$ of Banach spaces, we shall consider the spaces $\mathcal{F}(\overline{A})$ of all functions f with values in $\sum(\overline{A})$, which are bounded and continuous in the closed strip

$$s = \{z \in \mathbb{C} \ : \ 0 \leq \operatorname{Re} z \leq 1\},$$

and analytic in the open strip

$$s_0 = \{z \in \mathbb{C} \ : \ 0 < \operatorname{Re} z < 1\},$$

moreover the functions

$$t \to f(j+it) \qquad (j=0,1)$$

are continuous functions from the real into A_j, which tend to zero as $|t| \to \infty$. $\mathcal{F}(\overline{A})$ is a vector space. We provide $\mathcal{F}(\overline{A})$ with the norm

$$\|f\|_{\mathcal{F}(\overline{A})} = \operatorname{Max}\left(\operatorname{Sup}_t \|f(it)\|_{A_0}, \operatorname{Sup}_t \|f(1+it)\|_{A_1} \right).$$

We have the following

LEMMA 1. *The space $F(\overline{A})$ is a Banach space.*

DEFINITION 3. Given a couple $\overline{A} = (A_0, A_1)$ of Banach spaces and $0 < \Theta < 1$. The space $\overline{A}_{[\Theta]}$ is defined as

$$\overline{A}_{[\Theta]} = \left\{a \in \sum(\overline{A}) \ : \ a = f(\Theta), \text{ form some } f \in \mathcal{F}(\overline{A})\right\}.$$

The space $\overline{A}_{[\Theta]}$ is a Banach space with the norm

$$\|a\|_{[\Theta]} = \inf\left\{\|f\|_{\mathcal{F}(\overline{A})} \ : \ a = f(\Theta),\ f \in \mathcal{F}(\overline{A})\right\}.$$

For $\overline{A}_{[\Theta]}$ we have the following results

PROPOSITION 1. *The space $\overline{A}_{[\Theta]}$ is an interpolation space with respect to $\overline{A}$.*

THEOREM 1. *Let $0 < \Theta < 1$. Then $\Delta(\overline{A})$ is dense in $\overline{A}_{[\Theta]}$.*

Let us denote by u_j, $j = 0, 1$ the Poisson kernels for the strip $\mathbf{S}$ i.e.

$$u_j(s+it,x) = \frac{e^{-\pi(x-t)}\sin\pi s}{\sin^2\pi s + \left(\cos\pi s - e^{ij\pi-\pi(x-t)}\right)^2}, \qquad j = 0,1.$$

For these functions we have that

$$\int_{-\infty}^{\infty} u_0(\Theta,t)dt = 1-\Theta, \qquad \int_{-\infty}^{\infty} u_1(\Theta,t)dt = \Theta, \quad 0 < \Theta < 1 \tag{2}$$

Using the above notation we also have the following result

LEMMA 2. *If $f \in \mathcal{F}(\overline{A})$ we have*

$$\|f(\Theta)\|_{[\Theta]} \le$$

$$\le \left(\frac{1}{1-\Theta}\int_{-\infty}^{\infty}\|f(it)\|_{A_0} u_0(\Theta,t)dt\right)^{1-\Theta}\left(\frac{1}{\Theta}\int_{-\infty}^{\infty}\|f(1+it)\|_{A_1} u_1(\Theta,t)dt\right)^{\Theta}.$$

For the proof of the above result see [1].

The following result, which is well known for the classical case, is proved for dyadic Hardy spaces in [2].

THEOREM 2. *Assume that* $1 \leq p_j < \infty$ *(j = 0, 1) and* $0 < \Theta < 1$. *Then*

$$\left(\mathbf{H}^{p_0}, \mathbf{H}^{p_1}\right)_{[\Theta]} = \mathbf{H}^{p}, \qquad \textit{(equivalent norm)},$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

1. Interpolation between dyadic Hardy-type spaces of sequences

In this section we get a result on interpolation between sequence space and use it to obtain some results about convergence of Walsh series.

We begin with some notation. Let B be a Banach space we consider the space $L_p(B) = L_p$, of all strongly measurable functions $f : \Omega \to B$, such that

$$\int_{\Omega} \|f(x)\|_B^p \, dU(x) < \infty,$$

where $1 \leq p < \infty$. We shall denote by $L_\infty^0(B) = L_\infty^0(\Omega, dU, B)$ the completion in the sup-norm of all simple functions. When $\Omega = \mathbb{N}$ and U is the atomic measure we denote $L_\infty^0(B) = L_\infty^0(\Omega, dU, B) = \ell_\infty(B)$. In this case

$$\left\|\{B_n\}_{n=0}^{\infty}\right\|_{\ell_\infty(B)} = \operatorname*{Sup}_{n\in\mathbb{N}} \|B_n\|_B.$$

Let now

$$\ell := \left\{a = \{a_k\}_{k=0}^{\infty} \ : \ a_k \in \mathbb{R}\right\},$$

and X a normed space that contains the simple step functions. For $a \in \ell$ and $n \in \mathbb{N}$ set

$$A_n(x) := a_k, \quad \text{if } x \in \left[k/2^n, (k+1)2^n\right), \quad k = 0, 1, \ldots, 2^n - 1.$$

We define

$$\|a\|_X := \operatorname*{Sup}_{n\in\mathbb{N}} \|A_n\|_X \quad (a \in \ell).$$

We denote by ℓ_0 the set of real sequences with finite non-zero terms. We shall denote by ℓ_X the closure of ℓ_0 in X. In what follows we shall deal with ℓ_X when X is H^p or L^p.

Given the spaces H^{p_0}, H^{p_1} we put $\overline{H} = \left(H^{p_0}, H^{p_1}\right)$, for the spaces ℓ_{H^p} we have the following result on interpolation.

THEOREM 3. *For* $0 < p_i < \infty$ $(i = 0, 1)$, $0 < \Theta < \infty$ *we have*

$$\left(\ell_{H^{p_0}}, \ell_{H^{p_1}}\right)_{[\Theta]} = \ell_{(H^{p_0}, H^{p_1})_{[\Theta]}} = \ell_{H^p},$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

PROOF. It is sufficient to prove that

$$C_1 \|a\|_{\left(\ell_{H^{p_0}}, \ell_{H^{p_1}}\right)_{[\Theta]}} \le \|a\|_{\ell_{H^p}} \le C_2 \|a\|_{\left(\ell_{H^{p_0}}, \ell_{H^{p_1}}\right)_{[\Theta]}}$$

for $a \in \ell_0$. Let us prove the inequality

$$C_1 \|a\|_{\left(\ell_{H^{p_0}}, \ell_{H^{p_1}}\right)_{[\Theta]}} \le \|a\|_{\ell_{H^p}}.$$

Let $\mathbf{a} \in \ell_0$ and $A = \{A_n\}_{n=0}^{\infty}$ the sequence of dyadic step function constructed from $\mathbf{a}$ in the way described above. Given $\lambda > 0$, there is a function $g(.,n) \in \in \mathcal{F}\left(\overline{H}\right)$ such that

$$g(.,n) \in \mathcal{F}\left(\overline{H}\right) \le (C+\lambda)\|A_n\|_{\overline{H}_{[\Theta]}}$$

and $g(\Theta, n) = A_n$. Put

$$f(z,n) = g(z,n)\left(\frac{\|A_n\|_{\overline{H}_{[\Theta]}}}{\|A\|_{\ell_\infty(\overline{H}_{[\Theta]})}}\right)^{\mathrm{Re}\,(z)-\Theta},$$

we have that $f(\Theta, n) = A_n$. Moreover

$$\|f(it,.)\|_{\ell_\infty(H^{p_0})} = \operatorname*{Sup}_{n\in\mathbb{N}} \|f(it,n)\|_{H^{p_0}} =$$

$$= \operatorname*{Sup}_{n\in\mathbb{N}} \left\| g(it,n)\left(\frac{\|A_n\|_{\overline{H}_{[\Theta]}}}{\|A\|_{\ell_\infty(\overline{H}_{[\Theta]})}}\right)^{it-\Theta} \right\|_{H^{p_0}} =$$

$$= \|A\|^{\Theta}_{\ell_\infty(H_{[\Theta]})} \operatorname*{Sup}_{n\in\mathbb{N}} \|A_n\|^{-\Theta}_{\overline{H}_{[\Theta]}} \|g(it,n)\|_{H^{p_0}} \le$$

$$\le (C+\lambda)\|A\|^{\Theta}_{\ell_\infty(H_{[\Theta]})} \operatorname*{Sup}_{n\in\mathbb{N}} \|A_n\|^{-\Theta}_{\overline{H}_{[\Theta]}} \|A_n\|_{\overline{H}_{[\Theta]}} \le$$

$$\le (C+\lambda)\|A\|_{\ell_\infty(H_{[\Theta]})} = (C+\lambda)\|a\|_{\ell(\overline{H}_{[\Theta]})}.$$

Similarly we obtain that

$$\|f(it,.)\|_{\ell_\infty(H^{p_1})} \le (C+\lambda)\|a\|_{\ell(\overline{H}_{[\Theta]})}.$$

Since $\lambda > 0$ is arbitrary the desired inequality follows from these two inequalities and Theorem 2.

Let us now prove the inequality

$$\|a\|_{\ell_{\mathrm{H}^p}} \leq C_2 \|a\|_{\ell(\mathrm{H}^{p_0},\mathrm{H}^{p_1})_{[\Theta]}}.$$

Let $f(.,n) \in \mathcal{F}\left(\overline{H}\right)$ and $f(\Theta,n) = A_n$ $(n \in \mathbb{N})$. Using lemma 2 we get that $\|A\|_{\ell_\infty(\overline{H}_{[\Theta]})} = \operatorname{Sup}_{n\in\mathbb{N}} \|A\|_{\overline{H}_{[\Theta]}} \leq$

$$\leq \operatorname{Sup}_{n\in\mathbb{N}} \left(\frac{1}{1-\Theta} \int_{-\infty}^{\infty} \|f(it,n)\|_{\mathrm{H}^{p_0}} u_0(\Theta,t)dt \right)^{1-\Theta} \cdot$$

$$\cdot \left(\frac{1}{\Theta} \int_{-\infty}^{\infty} \|f(1+it,n)\|_{\mathrm{H}^{p_1}} u_1(\Theta,t)dt \right)^{\Theta} \leq$$

$$\leq \left(\frac{1}{1-\Theta} \int_{-\infty}^{\infty} \operatorname{Sup}_{n\in\mathbb{N}} \|f(it,n)\|_{\mathrm{H}^{p_0}} u_0(\Theta,t)dt \right)^{1-\Theta} \cdot$$

$$\cdot \left(\frac{1}{\Theta} \int_{-\infty}^{\infty} \operatorname{Sup}_{n\in\mathbb{N}} \|f(1+it,n)\|_{\mathrm{H}^{p_1}} u_1(\Theta,t)dt \right)^{\Theta} \leq$$

$$\leq \left(\frac{1}{1-\Theta} \int_{-\infty}^{\infty} \|f(it)\|_{\ell_\infty(\mathrm{H}^{p_0})} u_0(\Theta,t)dt \right)^{1-\Theta} \cdot$$

$$\cdot \left(\frac{1}{\Theta} \int_{-\infty}^{\infty} \|f(1+it)\|_{\ell_\infty(\mathrm{H}^{p_1})} u_1(\Theta,t)dt \right)^{\Theta} \leq$$

$$\leq \operatorname{Sup}_t \|f(it)\|^{1-\Theta}_{\ell(\mathrm{H}^{p_0})} \operatorname{Sup}_t \|f(1+it)\|^{\Theta}_{\ell(\mathrm{H}^{p_1})} \leq \|f\|_{\mathcal{F}(\ell_\infty(\mathrm{H}^{p_0}),\ell_\infty(\mathrm{H}^{p_1}))}.$$

From this follows the desired inequality and the theorem is proved.

Let us denote by

$$U = \{u_k,\ k \in \mathbb{N} \text{ or } k \in \mathbb{Z}\}$$

an orthonormal system defined on the set I. The Dirichlet kernels with respect to the system U are denoted by

$$D_n := \sum_{|k|<n} u_k.$$

We shall denote

$$M_n(a,u) := \frac{1}{n}\int_I \left|\sum_{k=1}^{n} a_k D_k(x)\right| dx \quad (a \in \ell), \qquad M(a,U) = \operatorname*{Sup}_{n\in\mathbb{N}} M_n(a,U).$$

It was proved by F. SCHIPP that

(3) $\quad M(a,U) \le C\|a\|_{\mathcal{H}^1} \quad$ (U is the trigonometric system)

(4) $\quad M(a,U) \le C\|a\|_{\mathbf{H}^1} \quad$ (U is the Walsh system).

In case U is the Walsh system we get the following result

LEMMA 3. *For each $n \in \mathbb{N}$ and $\mathbf{a} \in \ell$ the following inequality holds*

$$\left\|2^{-n}\sum_{k=1}^{2^n-1} a_k D_k\right\|_2 \le 2\cdot 2^{n/2}\left(2^{-n}\sum_{k=1}^{2^n-1}|a_k|^2\right)^{1/2}.$$

PROOF. Let us denote s_m by $s_m = \sum_{k=1}^{m} a_k$ then by Abel transformation and Minkowski inequality we get

$$\left\|2^{-n}\sum_{k=1}^{2^n-1} a_k D_k\right\|_2 = \left\|2^{-n}\left(\sum_{k=1}^{2^n-1} s_k(-w_k) + D_{2^n-1}S_{2^n-1}\right)\right\|_2 \le$$

$$\le \left\|2^{-n}\sum_{k=1}^{2^n-1} s_k(-w_k)\right\|_2 + \left\|2^{-n} D_{2^n-1}S_{2^n-1}\right\|.$$

Now applying Parseval identity we obtain

$$\left\|2^{-n}\sum_{k=1}^{2^n-1} a_k D_k\right\|_2 \le$$

$$\le 2^{-n}\left(\sum_{k=0}^{2^n-1}(s_k)^2\right)^{1/2} + 2^{-n}|s_{2^n-1}|\left(\int_0^1 |D_{2^n-1}(x)|^2 dx\right)^{1/2} =$$

$$= 2^{-n}\left(\sum_{k=0}^{2^n-1}(s_k)^2\right)^{1/2} + (2^{-n})^{1/2}|s_{2^n-1}| =$$

$$= \left(2^{-2n}\sum_{k=0}^{2^n-1}(s_k)^2\right)^{1/2} + \left(2^{-n})\left[s_{2^n-1}\right]^2\right)^{1/2}.$$

Using the fact that

$$(s_k)^2 \le (k+1)\left(\sum_{j=0}^{k} |a_j|^2\right),$$

we get

$$\left\| 2^{-n} \sum_{k=1}^{2^n=1} a_k D_k \right\|_2 \le$$

$$\le \left(2^{-2n} \sum_{k=0}^{2^n-1} (k+1) \sum_{j=0}^{k} |a_j|^2\right)^{1/2} + \left(2^{-n}(2^n) \sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2} \le$$

$$\le \left(2^{-2n} \sum_{j=0}^{2^n-1} |a_j|^2 \sum_{k=j}^{2^n-1} (k+1)\right)^{1/2} + \left(\sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2} \le$$

$$\le \left(2^{-2n} \sum_{j=0}^{2^n-1} |a_j|^2 \sum_{k=0}^{2^n-1} (k+1)\right)^{1/2} + \left(\sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2} \le$$

$$\le \left(2^{-2n} \sum_{j=0}^{2^n-1} |a_j|^2 2^{2n}\right)^{1/2} + \left(\sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2} \le$$

$$\le 2\left(\sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2} = 2 \cdot 2^{n/2} + \left(2^{-n} \sum_{j=0}^{2^n-1} |a_j|^2\right)^{1/2}.$$

The lemma is proved.

From inequalities (3), (4), Lemma 3 and Theorem 3 we obtain the inequality

$$\text{(5)} \qquad \left\| 2^{-n} \sum_{k=0}^{2^n-1} a_k D_k \right\|_p \le C^{(2-p)/p} 2^{2/p'} \left(2^{n/2}\right)^{2/p'} \|a\|_{H^p} =$$

$$= C^{(2-p)/p} 2^{2/p'} \left(2^{n/2}\right)^{2/p'} \|a\|_{H^p}$$

for $1 \le p \le 2$.

That is

$$(5') \quad \left\| 2^{-n} \sum_{k=0}^{2^n-1} a_k D_k \right\|_p \le C^{(2-p)/p} 2^{2/p'} \left(2^{n/2}\right)^{2/p'} \left(2^{-n} \sum_{k=0}^{2^n-1} |a_k|^p\right)^{1/p},$$

$$(6) \quad \left\| \sum_{k=0}^{2^n-1} a_k D_k \right\|_p \le C^{(2-p)/p} 2^{2/p'} \left(2^{n/2}\right)^{2/p'} \left(2^{-n} \sum_{k=0}^{2^n-1} |a_k|^p\right)^{1/p}.$$

Given a sequence $\mathbf{a} = (a_k,\ k \in \mathbb{N})$ we define the first difference of $\mathbf{a}$ by

$$\Delta\mathbf{a} := (\Delta a_k,\ k \in \mathbb{N}) := (a_k - a_{k+1},\ k \in \mathbb{N}).$$

Now we introduce the norms

$$\|a\|_{\ell_*^p} := |a_0| + \sum_{n=0}^{\infty} 2^{n(1+p')/p'} \left(2^{-n} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\right)\right)^{1/p} =$$

$$= |a_0| + \sum_{n=0}^{\infty} 2^{2n/p'} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k|^p\right)^{1/p},$$

for $a \in \ell$ and $1 < p < \infty$, where p' is the index conjugated to p. For $p = 1$ we define

$$\|a\|_{\ell_*^1} := \|a\|_{\ell^1}.$$

Using this norm $\|\ldots\|_{\ell_*^p}$ we can get a result about convergence of Walsh series

THEOREM. *Let $a \in \ell$. If $\Delta a \in \ell_*^p$ $(1 \le p \le 2)$ and $2^{n/p} a_{2^n}$ tends to zero as $n \to \infty$, then series $\sum_{k=0}^{\infty} a_k w_k$ converges pointwise and in L^p on $(0,1)$; its sum $f(x) = \sum_{k=0}^{\infty} a_k w_k(x)$ belongs to L_p.*

PROOF. The pointwise convergence of the series is easy to see using the fact that $\|\Delta_a\|_{\ell^1} < \infty$, $|D_n(x)| \le \frac{2}{x}$ for all $x \in (0,1)$ and Abel's transformation.

For each $n,\ m \in \mathbb{N}$ we have

$$\left\| \sum_{k=2^n}^{2^m-1} a_k w_k \right\|_p = \left\| \sum_{k=2^n}^{2^m-1} (\Delta a_k) D_{k+1} + a_{2^m-1} D_{2^m} - a_{2^n} D_{2^n} \right\|_p \le$$

$$\leq \sum_{s=n}^{\infty} \left\| \sum_{k=2^s}^{2^{s+1}-1} (\Delta a_k) D_{k+1} \right\|_p + |a_{2^m-1}| \, \|D_{2^m}\|_p + |a_{2^n}| \, \|D_{2^n}\|_p .$$

Using inequality (6)

$$\left\| \sum_{k=2^n}^{2^m-1} a_k w_k \right\|_p \leq \sum_{s=n}^{\infty} C^{(2-p)/p} 2^{2(s+1)/p'} \left(2^{-(s+1)} \sum_{2^s}^{2^{(s+1)}-1} \Delta a_k p \right)^{1/p} + $$

$$+ |a_{2^m-1}| \, 2^{m/p'} + |a_{2^n}| \, 2^{n/p'} .$$

Thus we have $\left\| S_{2^m} - S^{2^n} \right\|_p \underset{m,n\to\infty}{\longrightarrow} 0$ i.e. $S_{2^n} = \sum_{k=0}^{2^n-1} a_k w_k$ converges in L^p-norm and consequently $f = \lim_{n\to\infty} S_{2^n}$ belongs to L^p. The theorem is proved.

Observe that is we suppose that $\{a_n\}$ tends to zero monotonously then the condition $a_{2^n} 2^{n/p'} \to 0$ is not necessary in the hypothesis of the above theorem.

I wish to express my gratitude to professor F. SCHIPP for calling my attention to the problem and his helpful comments.

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LEBESGUE FUNCTIONS FOR CONVERGENCE AND SUMMABILITY OF DOUBLE FUNCTION SERIES

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In this paper we investigate the summability of double function series. Introduce some notations. If limits of summation are not indicated, they are 0 and ∞ on each index, and the free indices assume all values $0,1,\ldots$;

$$\lim_{mn} a_{mn} = a$$

means that for every $\varepsilon > 0$ $\exists m_0$, n_0 such that $|a_{mn} - a| < \varepsilon$ if $m > m_0$ and $n > n_0$, i. e. the convergence in Pringsheim's sense.

Denote $E \subseteq \mathbb{R}^d$, μ is a positive measure on E, $\mu(E) < \infty$, $f_{mn} \in L^1_\mu(E)$. Let T be a summability method with a triangular matrix. If T is a sequence to sequence method then denote the element of the matrix with $t_{mnk\ell}$, if T is a series to sequence method then denote with $\tau_{mnk\ell}$. We assume that $\tau_{mn00} \to 1$ as $m,n \to \infty$. Let be

$$D_{mn}(u,x) := \sum_{k,\ell=0}^{m,n} f_{k\ell}(u) f_{k\ell}(x), \tag{1}$$

$$K_{mn}(u,x) := \sum_{k,\ell=0}^{m,n} \tau_{mnk\ell} f_{k\ell}(u) f_{k\ell}(x), \tag{2}$$

where $x = (x_1,\ldots,x_d)$, $u = (u_1,\ldots,u_d)$, $x,u \in E$, and K_{mn} is the kernel function of T,

$$L_{mn}(x) := \int_E |K_{mn}(u,x)| d\mu(u), \tag{3}$$

$$L'_{mn}(x) := \int_E \max_{0\le\ell\le n} |K_{m\ell}(u,x)| d\mu(u), \tag{4}$$

$$L''_{mn}(x) := \int_E \max_{0\le k\le m} |K_{kn}(u,x)|\,d\mu(u) \tag{5}$$

the Lebesgue functions of the method T.

The Lebesgue functions play a very important role in the investigation of the convergence and summability of the series (see [1]–[5], [11], [17]–[20], [22]). Let be

$$\Phi_{M,N}(u,\upsilon) := \sum_{k,\ell=0}^{M,N} \tau_{mnk\ell}\tau_{\mu\nu k\ell} f_{k\ell}(u) f_{k\ell}(\upsilon), \tag{6}$$

where $M := \min(m,\mu)$, $N := \min(n,\nu)$.

Consider the double function series

$$\sum_{m,n} c_{mn} f_{mn}(x), \tag{7}$$

where

$$\sum_{m,n} c_{mn}^2 < \infty, \tag{8}$$

and denote the T-means of (7) by σ_{mn}, i. e.

$$\sigma_{mn}(x) = \sum_{k,\ell=0}^{m,n} \tau_{mnk\ell} c_{k\ell} f_{k\ell}(x). \tag{9}$$

Let $\{\varphi_{mn}\}$ be an orthonormal system on E with respect to w, where w is a positive measure. It is known that there is a function $h \in L^2_w(E)$ such that

$$h(x) \sim \sum_{m,n} c_{mn}\varphi_{mn}(x)$$

and

$$||h||^2_{L^2_\omega(E)} = \sum_{m,n} c_{mn}^2.$$

Denote by

$$s = s(m,x), \quad t = t(n,x)$$

the least index for which

$$g_{mn}(x) := \sigma_{st}(x) = \sup_{i,j\le m,n} \sigma_{ij}(x).$$

LEMMA 1. *Assume that the conditions* (8) *and*

$$\Phi_{MN}(u,v) = O(1) \sum_{k,\ell=0}^{M,N} \xi_{k\ell}|K_{k\ell}(u,v)|, \quad \xi_{k\ell} \geq 0, \tag{10}$$

are satisfied and there exist $\xi'_k(M), \xi''_\ell(N) \geq 0$ *such that*

$$\sum_{k,\ell=0}^{M,N} \xi_{k\ell} = O(1), \quad \xi_{k\ell} = O(1)\xi'_k(M)\xi''_\ell(N), \tag{11}$$

further

$$L_{mn}(x) = O(1), \tag{12}$$

$$L'_{mn}(x) = O(1), \quad L''_{mn}(x) = O(1) \tag{13}$$

are fulfilled for $x \in \mathbf{E}$. *Then we have*

$$||g_{mn}||_{L^1_\mu(E)} \leq C||h||_{L^2_w(E)}, \quad C > 0 \text{ is a constant.}$$

PROOF. Let be

$$G_{mn}(u,x) := \sum_{k,\ell=0}^{m,n} \tau_{mnk\ell}\varphi_{k\ell}(u)f_{k\ell}(x).$$

Then, since c_{mn} are the Fourier coefficients of h, we get

$$\sigma_{mn}(x) = \int_E h(u)G_{mn}(u,x)\,dw(u),$$

consequently

$$||g_{mn}||_{L^1_\mu(E)} = \int_E h(u)\,dw(u) \int_E G_{st}(u,x)\,d\mu(x),$$

where we assumed that $c_{00} = 0$ (without loss of generality by $\tau_{mn00} \to 1$ as $m,n \to \infty$) from which $g_{mn}(x) \geq 0$. Using the Cauchy–Schwarz inequality we have

$$||g_{mn}||_{L^1_\mu(E)} \leq \sqrt{J_{mn}} \cdot ||h||_{L^2_w(E)},$$

where

$$J_{mn} := \int_E \left[\int_E G_{st}(u,x)\,d\mu(x)\right]^2 dw(u) =$$

$$= \int_E \int_E \int_E G_{\alpha\beta}(u,v)G_{\gamma\delta}(u,\eta)\,dw(u)\,d\mu(v)\,d\mu(\eta),$$

$$\alpha := s(m,v), \quad \beta := t(n,v), \quad \gamma := s(m,\eta), \quad \delta := t(n,\eta).$$

Let be $q := \min(\alpha,\gamma)$, $r := \min(\beta,\delta)$. Then

$$J_{mn} \le \int_E \int_E |\Phi_{q,r}(\upsilon,\eta)|\, d\mu(\upsilon)\, d\mu(\eta),$$

where

$$\Phi_{q,r}(\upsilon,\eta) := \int_E G_{\alpha\beta}(u,\upsilon) G_{\gamma\delta}(u,\eta)\, dw(u). \tag{14}$$

Since $\{\varphi_{mn}\}$ is an orthonormal system therefore

$$\Phi_{q,r}(\upsilon,\eta) = \sum_{k,\ell=0}^{q,r} \tau_{\alpha\beta k\ell}\tau_{\gamma\delta k\ell} f_{k\ell}(\upsilon) f_{k\ell}(\eta).$$

Divide $E \times E$ into 4 parts:

$$M_1 := \{(\upsilon,\eta) : \alpha \ge \gamma, \beta \ge \delta\}, \qquad M_2 := \{(\upsilon,\eta) : \alpha < \gamma, \beta \ge \delta\},$$
$$M_3 := \{(\upsilon,\eta) : \alpha \ge \gamma, \beta < \delta\}, \qquad M_4 := \{(\upsilon,\eta) : \alpha < \gamma, \beta < \delta\}.$$

We have to prove that

$$\int_E \int_E |\Phi_{q,r}(\upsilon,\eta)|\, d\mu(\upsilon)\, d\mu(\eta) = O(1).$$

Indeed, on M_1 we have

$$\int\int_{M_1} |\Phi_{q,r}(\upsilon,\eta)|\, d\mu(\upsilon)\, d\mu(\eta) \le \int_E \int_E |\Phi_{\gamma,\delta}(\upsilon,\eta)|\, d\mu(\upsilon)\, d\mu(\eta) =$$

$$\overset{(10)}{=} O(1) \int_E \int_E \sum_{k,\ell=0}^{\gamma,\delta} \xi_{k\ell} |K_{k\ell}(\upsilon,\eta)|\, d\mu(\upsilon)\, d\mu(\eta) =$$

$$\overset{(3)}{=} O(1) \int_E \sum_{k,\ell=0}^{\gamma,\delta} \xi_{k\ell} L_{k\ell}(\eta)\, d\mu(\eta) \overset{(12)}{=} O(1) \int_E \sum_{k,\ell=0}^{\gamma,\delta} \xi_{k\ell}\, d\mu(\eta) \overset{(11)}{=} O(1).$$

The estimate on M_4 is analogous. Now estimate on M_2:

$$\int\int_{M_2} |\Phi_{q,r}(\upsilon,\eta)|\,d\mu(\upsilon)\,d\mu(\eta) \le \int_E\int_E |\Phi_{\alpha,\delta}(\upsilon,\eta)|\,d\mu(\upsilon)\,d\mu(\eta) =$$

$$= O(1)\int_E\int_E \sum_{k,\ell=0}^{\alpha,\delta} \xi_{k,\ell}|K_{k\ell}(\upsilon,\eta)|\,d\mu(\upsilon)\,d\mu(\eta) =$$

$$= O(1)\int_E \left(\sum_{k=0}^{\alpha}\xi'_k(\alpha)\int_E\sum_{\ell=0}^{\delta}\xi''_\ell(\delta)|K_{k\ell}(\upsilon,\eta)|\,d\mu(\eta)\right)d\mu(\upsilon) =$$

$$= O(1)\int_E \left(\sum_{k=0}^{\alpha}\xi'_k(\alpha)\int_E\sum_{\ell=0}^{\delta}\xi''_\ell(\delta)\cdot\max_{0\le\ell\le\beta}|K_{k\ell}(\upsilon,\eta)|\,d\mu(\eta)\right)d\mu(\upsilon) =$$

$$= O(1)\int_E \sum_{k=0}^{\alpha}\xi'_k(\alpha)L'_{k\beta}(\upsilon)\,d\mu(\upsilon) = O(1)\int_E\sum_{k=0}^{\alpha}\xi'_k(\alpha)\,d\mu(\upsilon) = O(1).$$

The estimate on M_3 is analogous by (10), (11), (5) and (13).

Lemma 1 is proved.

REMARK. 1. We note that if $\{f_{mn}\}$ is an orthonormal system on E, then $G_{mn} = K_{mn}$ and by (6) and (14) the estimate (10) can be written in a more natural form as follows (cf. [13], p.292, [18], p.202.):

$$\left|\sum_{k,\ell}\tau_{mnk\ell}\tau_{\mu\nu k\ell}f_{k\ell}(x)f_{k\ell}(y)\right| \le \left|\int_E K_{mn}(x,t)K_{\mu\nu}(y,t)\,d\mu(t)\right| \le$$

$$\le \sum_{k,\ell=0}^{M,N}\xi_{k\ell}|K_{k\ell}(x,y)|.$$

2. The basic idea of the proof can be found in [6]–[8]. Here we used the method of [9] introducing the system $\{\varphi_{mn}\}$.

We will prove that conditions (10), (11) are fulfilled for weighted means method P of Riesz. Introduce some notations and definitions. Define the product $T := A \odot B$ of the methods A and B as follows:

(15) $$t_{mnk\ell} = a_{mk}b_{n\ell},$$

(16) $$\tau_{mnk\ell} = \alpha_{mk}\beta_{n\ell}.$$

In special case, when A and B are Riesz's weighted means method, i. e.

$$a_{mk}=\frac{a_k}{A_m},\quad A_m=\sum_{i=0}^{m}a_i,$$

$$b_{n\ell}=\frac{b_\ell}{B_n},\quad B_n=\sum_{j=0}^{n}b_j,$$

$$\alpha_{mk}=1-\frac{A_{k-1}}{A_m},\quad \beta_{n\ell}=1-\frac{B_{\ell-1}}{B_n},$$

then we have

$$t_{mnk\ell}=\frac{a_k b_\ell}{A_m B_n},\qquad \tau_{mnk\ell}=\left(1-\frac{A_{k-1}}{A_m}\right)\left(1-\frac{B_{\ell-1}}{B_n}\right).$$

For the sake of convenience denote $p_{k\ell}:=a_k b_\ell$, $P_{mn}:=A_m B_n$. In this case, obviously,

$$P_{mn}=\sum_{k,\ell=0}^{m,n}p_{k\ell}.$$

Assume that $P_{mn}\neq 0$ is fulfilled and a_k, b_ℓ are real numbers. Define

$$\Delta_k a_k:=a_k-a_{k+1},\quad \Delta_{k\ell}a_{k\ell}:=a_{k\ell}-a_{k+1,\ell}-a_{k,\ell+1}+a_{k+1,\ell+1}.$$

Obviously $\Delta_{k\ell}a_{k\ell}=\Delta_k\Delta_\ell a_{k\ell}$.

LEMMA 2. *If the method P satisfies the condition*

$$\sum_{k,\ell=0}^{m,n}|p_{k\ell}|=O(P_{mn}),\tag{17}$$

then the conditions (10), (11) *are fulfilled.*

PROOF. Obviously $t_{mnk\ell}=\Delta_{k\ell}\tau_{mnk\ell}$. From (1), (2) we obtain

$$K_{mn}(u,x)=\sum_{k,\ell=0}^{m,n}t_{mnk\ell}D_{k\ell}(u,x).\tag{18}$$

From (6) we get by Abel–Hardy transformation using (1) and the triangularity of the method T

$$\Phi_{M,N}(\upsilon,\eta)=\sum_{k,\ell=0}^{M,N}\Delta_{k\ell}(\tau_{mnk\ell}\tau_{\mu\nu k\ell})D_{k\ell}(\upsilon,\eta).\tag{19}$$

Since by the known formula for difference of products of sequences (cf. [15], p. 119) we have

$$\Delta_{k\ell}(a_{k\ell}b_{k\ell})=\sum_{\kappa\lambda=0}^{1,1}\Delta_{k\ell}^{\kappa\lambda}a_{k\ell}\cdot\Delta_{k\ell}^{1-\kappa,1-\lambda}b_{k+\kappa,\ell+\lambda}.$$

Indeed,

$$\Delta_{k\ell}(a_{k\ell}b_{k\ell})=\Delta_k\Delta_\ell(a_{k\ell}b_{k\ell})=\Delta_k(a_{k\ell}\cdot\Delta_\ell b_{k\ell}+\Delta_\ell a_{k\ell}\cdot b_{k,\ell+1})=$$
$$=a_{k\ell}\cdot\Delta_{k\ell}b_{k\ell}+\Delta_k a_{k\ell}\cdot\Delta_\ell b_{k+1,\ell}+\Delta_\ell a_{k\ell}\cdot\Delta_k b_{k,\ell+1}+\Delta_{k\ell}a_{k\ell}\cdot b_{k+1,\ell+1}.$$

Therefore by (15), (16)

$$\Delta_{k\ell}(\tau_{mnk\ell}\tau_{\mu\nu k\ell})=p_{k\ell}\Psi_{k\ell},\tag{20}$$

where

$$\Psi_{k\ell}=\left(1-\frac{A_{k-1}}{A_m}\right)\left(1-\frac{B_{\ell-1}}{B_n}\right)\frac{1}{P_{\mu\nu}}+\left(1-\frac{A_{k-1}}{A_m}\right)\left(1-\frac{B_\ell}{B_\nu}\right)\frac{1}{P_{\mu n}}+$$
$$+\left(1-\frac{A_k}{A_\mu}\right)\left(1-\frac{B_{\ell-1}}{B_n}\right)\frac{1}{P_{m\nu}}+\left(1-\frac{A_k}{A_\mu}\right)\left(1-\frac{B_\ell}{B_\nu}\right)\frac{1}{P_{mn}}.$$

From (15), (18) we have

$$P_{mn}K_{mn}(u,x)=\sum_{k,\ell=0}^{m,n}p_{k\ell}D_{k\ell}(u,x),$$

therefore from (19) and (20) by Abel–Hardy transformation

$$\Phi_{M,N}(\upsilon,\eta)=$$
$$=\sum_{k,\ell=0}^{M-1,N-1}(\Delta_{k\ell}\Psi_{k\ell})P_{k\ell}K_{k\ell}(\upsilon,\eta)+\sum_{k=0}^{M-1}(\Delta_k\Psi_{kN})P_{kN}K_{kN}(\upsilon,\eta)+$$
$$+\sum_{\ell=0}^{N-1}(\Delta_\ell\Psi_{M\ell})P_{M\ell}K_{M\ell}(\upsilon,\eta)+\Psi_{MN}P_{MN}K_{MN}(\upsilon,\eta).$$

Here

$$\Delta_{k\ell}\Psi_{k\ell}=\frac{1}{P_{\mu\nu}}\cdot\frac{a_k+a_{k+1}}{A_m}\cdot\frac{b_\ell+b_{\ell+1}}{B_n},$$

$$\Delta_k\Psi_{kN}=\frac{1}{P_{\mu\nu}}\cdot\frac{a_k+a_{k+1}}{A_m}\cdot\left(1-\frac{B_{N-1}}{B_n}\right)+\frac{1}{P_{mn}}\cdot\frac{a_k+a_{k+1}}{A_\mu}\cdot\left(1-\frac{B_N}{B_\nu}\right),$$

$$\Delta_\ell\Psi_{M\ell}=\frac{1}{P_{\mu\nu}}\cdot\frac{b_\ell+b_{\ell+1}}{B_n}\cdot\left(1-\frac{A_{M-1}}{A_m}\right)+\frac{1}{P_{mn}}\cdot\frac{b_\ell+b_{\ell+1}}{B_\nu}\cdot\left(1-\frac{A_M}{A_\mu}\right).$$

From (17) we obtain $P_{k\ell} = O(P_{mn})$ for $0 \le k,\ \ell \le m,n$, therefore

$$\Phi_{M,N}(\upsilon,\eta) =$$

$$= O(1)\left\{ |P_{MN}|^{-1} \sum_{k,\ell=0}^{M-1,N-1} |p_{k\ell}+p_{k,\ell+1}+p_{k+1,\ell}+p_{k+1,\ell+1}| \cdot |K_{k\ell}(\upsilon,\eta)| + \right.$$

$$+|A_M|^{-1} \sum_{k=0}^{M-1} (|a_k|+|a_{k+1}|)\cdot |K_{kN}(\upsilon,\eta)| +$$

$$\left. +|B_N|^{-1} \sum_{\ell=0}^{N-1} (|b_\ell|+|b_{\ell+1}|)\cdot |K_{M\ell}(\upsilon,\eta)| + |K_{MN}(\upsilon,\eta)| \right\}$$

that is

$$\xi'_k(M) = \begin{cases} |A_M|^{-1}(|a_k|+|a_{k+1}|) & \text{if } 0 \le k \le M-1, \\ 1 & \text{if } k = M, \end{cases}$$

$$\xi''_\ell(N) = \begin{cases} |B_N|^{-1}(|b_\ell|+|b_{\ell+1}|) & \text{if } 0 \le \ell \le N-1, \\ 1 & \text{if } \ell = N, \end{cases}$$

are suitable choosing.

The double sequence $\{y_{mn}\}$ is called nondecreasing if $\{y_{m,n_0}\}$ and $\{y_{m_0,n}\}$ are nondecreasing, where m_0 and n_0 are fixed.

LEMMA 3. *If $\{y_{mn}\} \subset L^1_\mu(E)$ is nondecreasing and*

$$\int_E y_{mn}(x)\,d\mu(x) = O(1),$$

then the limit functions y, y'_m and y''_n, defined by

$$y(x) := \lim_{m,n} y_{mn}(x), \qquad y'_m(x) := \lim_n y_{mn}(x), \qquad y''_n(x) := \lim_m y_{mn}(x),$$

μ a. e. on E are finite and

$$\int_E y(x)\,d\mu(x) = \lim_{m,n} \int_E y_{mn}(x)\,d\mu(x).$$

PROOF. For $\{y'_m\}$ and $\{y''_n\}$ see [11], p.18. Since $\{y_{mn}\}$ is nondecreasing then by theorem of B. Levi ([16], p.118.) we obtain

$$\int_E y'_m(x)\,d\mu(x) = \lim_n \int_E y_{mn}(x)\,d\mu(x) = O(1).$$

Since $\{y'_m\}$ is also nondecreasing we get (cf. [24], p.16)

$$\int_E y(x)d\mu(x) = \lim_m \int_E y'_m(x)d\mu(x) = \lim_{m,n} \int_E y_{mn}(x)d\mu(x),$$

moreover (cf. [16], p.144.) $y \in L^1_\mu(E)$ and y is finite μ-almost everywhere (briefly μ-a.e.) on E (cf. [16], p.136.).

In order to have latest conclusion almost everywhere in Lebesgue's sense, we set some restrictions on the measure μ for the further proceedings. Assume that μ is absolutely continuous with respect to the Lebesgue measure λ. In this case the Radon–Nikodym derivative, $d\mu/d\lambda$ (see [23], 31.§) exists and $d\mu/d\lambda$ is uniquely determined λ-almost everywhere. If $d\mu/d\lambda > 0$ on E λ-a.e. then using [23], 32.§, Theorem B, we obtain

$$\mu(D) = \int_D \frac{d\mu}{d\lambda} d\lambda$$

where $D \subset E$ is a measurable set. Hence if $\mu(D) = 0$ then we have $\lambda(D) = 0$.

We note, that μ fulfills the assumptions when

$$\mu(x) = \mu_1(x_1)\dots\mu_d(x_d),$$

where $\mu_1,\dots,\mu_d$ are positive, bounded and monotonically increasing in their domains of definition (we recall the well-known theorem which says the derivative of μ_j $(j = 1,\dots,d)$ equals zero only in sets with Lebesgue's measure zero ([16], p.45.).

THEOREM 1. *If condition* (8) *is fulfilled and the method T satisfies* (10)–(13) *then*

$$\sigma_{mn}(x) = O_x(1) \quad \text{a. e. on } E.$$

PROOF. For the sake of completeness we write it down here, since paper [12] is difficult to reach owing to a great number of references to paper [10]. However, the idea of proof is standard. Since $\{g_{mn}\}$ is nondecreasing therefore its limit function S,

$$S(x) := \lim_{m,n} g_{mn}(x)$$

because of Lemma 1 and 3, is finite a. e. on E. Writing $\{-\sigma_{ij}\}$ instead of $\{\sigma_{ij}\}$ Lemma 1 remains true. After changing we obtain a limit function S^*, which is finite a. e. on E,

$$S^*(x) := \lim_{m,n} g^*_{mn}(x), \quad g^*_{mn}(x) := \sup_{i,j \leq m,n} \{-\sigma_{ij}(x)\} \geq 0,$$

where we assumed that $c_{00} = 0$. Hence a. e. on E $|\sigma_{mn}(x)| \leq S(x) + S^*(x)$.

Theorem 1 is proved.

THEOREM 2. *If condition* (8) *is fulfilled and the method T satisfies* (10)–(13) *then there exist*

(21) $$\lim_{m,n}\sigma_{mn}(x),\quad \lim_{m}\sigma_{mn}(x),\quad \lim_{n}\sigma_{mn}(x),\quad \text{a. e. on } E.$$

PROOF. First we investigate $\lim_{m,n}\sigma_{mn}(x)$. Define the function Ω by

$$\Omega(x) := \sup_{m,n,k,\ell} |\sigma_{m+k,n+\ell}(x)-\sigma_{mn}(x)|,\quad x\in E.$$

We can assume that $c_{00}=0$, thus $S(x)\geq 0$, $S^*(x)\geq 0$ from proof of Theorem 1. Consequently $\Omega(x)\leq 2(S(x)+S^*(x))$. Using Lemma 1 and Lemma 3 and the fact that $\{g_{mn}\}$ is nondecreasing we get

$$||S||_{L^1_\mu(E)}\leq C||h||_{L^2_w(E)},\quad ||S^*||_{L^1_\mu(E)}\leq C||h||_{L^2_w(E)}.$$

Therefore

(22) $$||\Omega||_{L^1_\mu(E)}\leq 4C||h||_{L^2_w(E)}.$$

For arbitrary positive integer N we define (cf. [22], p. 254.)

$$\Omega_N(x) := \sup_{\substack{m,n\geq N\\ k,\ell\geq 0}} |\sigma_{m+k,n+\ell}(x)-\sigma_{mn}(x)|,\quad x\in E.$$

Similarly as we obtained (22), we get $||\Omega_N||_{L^1_\mu(E)}\leq 4C||h_N||_{L^2_w(E)}$, where

$$h_N(x)\sim\left(\sum_{m,n\geq N}+\sum_{m\geq N,n<N}+\sum_{m<N,n\geq N}\right)c_{mn}\varphi_{mn}(x).$$

Since using the Parseval equality

$$||h_N||_{L^2_w(E)}=\left(\sum_{m,n\geq N}+\sum_{m\geq N,n<N}+\sum_{m<N,n\geq N}\right)c^2_{mn}$$

and taking into account (8), therefore

(23) $$\lim_{N\to\infty}||\Omega_N||_{L^1_\mu(E)}=0.$$

Obviously $||-\Omega_N||_{L^1_\mu(E)}=||\Omega_N||_{L^1_\mu(E)}$. From the definition

$$-\Omega(x)=-\Omega_0(x)\leq-\Omega_1(x)\leq\ldots.$$

On the other hand (using (22))

$$\left|\int_E -\Omega_0(x)\,d\mu(x)\right| = ||\Omega||_{L^1_\mu(E)}<\infty.$$

Therefore using B. Levi's theorem

$$\lim_{N\to\infty}\int_E -\Omega_N(x)d\mu(x)=\int_E \lim_{N\to\infty}(-\Omega_N(x))d\mu(x).$$

But by (23) the left hand side is 0, thus $\lim\Omega_N(x)\to 0$ as $N\to\infty$ a. e. on E, from which we obtain (see [21], p.158, or [24], p.11.) that $\sigma_{mn}(x)$ as $m,n\to\infty$, converges a. e. on E. For the other cases the proof is similar. Theorem 2 is proved.

We denote by I the convergence method. For it $\tau_{mnk\ell}=1$.

COROLLARY 1. *If the method I satisfies* (12), (13) *and* (8) *is also satisfied, then there exist*

$$\lim_{m,n} s_{mn}(x),\quad \lim_{m} s_{mn}(x),\quad \lim_{n} s_{mn}(x),\ \textit{a. e. on } E,$$

where

$$s_{mn}(x)=\sum_{k,\ell\leq m,n} c_{k\ell}f_{k\ell}(x).$$

PROOF. Because of Theorem 2 it is enough to prove that (10), (11) are satisfied. But it is obvious, let $\xi_{k\ell}=\delta_{Mk}\cdot\delta_{N\ell}$, where δ_{mk} is the Kronecker function.

COROLLARY 2. *If condition* (8) *is fulfilled, the method P satisfies* (17) *and the Lebesgue functions of P satisfy* (12), (13) *then there exist* (21).

PROOF. From (8), (12), (13) and (17) follows immediately by Theorem 2 and Lemma 2.

REMARK. 1. In [12] and [10] there are some errors in calculation (see [12], p.219.).

2. Theorem 1 for simple series was proved in [13], pp.292–293., see also [14] and [18] for orthogonal series.

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AN EQUICONVERGENCE THEOREM FOR HERMITE–FOURIER SERIES

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Dedicated to Professor Ákos Császár on his 70th birthday

1. Introduction.

This paper consists of two parts. In the first one we prove an equiconvergence theorem concernity Hermite expansions, beyond the scope of the classical results of SZEGŐ [2] and MUCKENHOUPT [3]. The analogous statement for Laguerre expansions will be proved in SZABÓ [6]. In the second part we prove an inequality with respect to the Christoffel numbers for Hermite weight, which will be applied in a subsequent paper of the author and V. E. S. SZABÓ. Similar investigations appeared in FREUD [15], [16] and for the Laguerre case in NÉVAI [20].

2. Notations and results.

As usual, the Hermite polynomials are defined by $\sum_{n=0}^{\infty} \frac{H_n(x)r^n}{n!} = \exp(2xr - r^2)$; they are orthogonal on $(-\infty,\infty)$ with respect to the weight function $\exp(-x^2)$. Assume that $f(x)$ has an Hermite series and denote $\sigma_n(f,x)$ the n-th partial sum of that series. Then

$$\sigma_n(f,x) = \int_{-\infty}^{\infty} f(x)e^{-y^2} K_n(x,y)\,dy \tag{2.1}$$

where

$$K_n(x,y) = \frac{H_{n+1}H_n(y) - H_n(x)H_{n+1}(y)}{(x-y)\cdot 2^{n+1}\cdot n!\cdot\sqrt{\pi}}. \tag{2.2}$$

Introduce further the n-th trigonometric partial sum

$$S_n(f,x) = \frac{1}{\pi} \int_{x-\delta}^{x+\delta} \frac{\sin\sqrt{2n}(x-y)}{x-y} f(y)\,dy \tag{2.3}$$

where $\delta > 0$ can be fixed arbitrarily.

THEOREM 1. *Assume that $e^{-t^2/2} f(t)$ has bounded variation on $\mathbb{R}$. Then*

$$\sigma_n(f,x) - S_n(f,x) \to 0 \qquad (n \to \infty) \tag{2.4}$$

locally uniformly on $\mathbb{R}$ □.

Denote further $x_1 > x_2 > \ldots > x_n$ the zeros of $H_n(x)$ and let λ_k be the Christoffel numbers

$$\frac{1}{\lambda_k} = \sum_{\nu=0}^{n-1} h_\nu^2(x_k) \qquad k = 1, \ldots, n \tag{2.5}$$

where $h_\nu(x) = (\sqrt{\pi} \cdot 2^\nu \cdot h_\nu)^{-1/2} \cdot H_\nu(x)$ are the normed Hermite polynomials, i. e.

$$\int_{-\infty}^{\infty} h_n(x) k_k(x) e^{-x^2}\,dx = \delta_{n,k}. \tag{2.6}$$

THEOREM 2. *Let $m \leq \frac{n}{2}$ and denote p_m a polynomial of degree $\leq m$. Then*

$$\sum_{k=1}^{n} \lambda_k |p_m(x_k)| e^{x_k^2/2} \leq c \int_{-\infty}^{\infty} |p_m(x)| e^{-x^2}\,dx. \ \square \tag{2.7}$$

REMARK. The special case $p_m \equiv 1$ is a classical inequality of BALÁZS and TURÁN [21], see also in FREUD [15].

3. Proof of Theorem 1; equiconvergence.

Consider the partition

$$\sigma_n(f,x) = \int_{-\infty}^{\infty} K_n(x,y) e^{-y^2} f(y)\,dy = \int_{x}^{x+\gamma} + \int_{x+\gamma}^{\infty} + \int_{x-\gamma}^{x} + \int_{-\infty}^{x-\gamma}$$

$$= D_1 + D_2 + D_1' + D_2'. \tag{3.1}$$

We shall prove that given $\varepsilon > 0$ and $K > 0$ arbitrarily, we can choose $\gamma = \gamma(K,\varepsilon) > 0$ such that for $|x| \le K$

$$(3.2)\qquad \begin{cases} |D_1 - \frac{1}{\pi}\int\limits_x^{x+\gamma} \frac{\sin\sqrt{2n}(x-y)}{x-y} f(y)\,dy| < \varepsilon \\ |D_2| \le c(K,\gamma)\cdot n^{-1/4} \end{cases}$$

$$(3.3)\qquad \begin{cases} |D_1' - \frac{1}{\pi}\int\limits_{x-\gamma}^{x} \frac{\sin\sqrt{2n}(x-y)}{x-y} f(y)\,dy| < \varepsilon \\ |D_2'| \le c(K,\gamma)\cdot n^{-1/4}. \end{cases}$$

This will imply that for ε, $K > 0$ there exists $n_0 = n_0(\varepsilon, K)$ such that $n \ge n_0$ and $|x| \le K$ implies $|\sigma(f,x) - S_n(f,x)| < 4\varepsilon$ which is the statement of Theorem 1. We shall prove only the estimates concerning D_i; those concerning D_i' can be obtained similarly. Start with the estimate

$$(3.4)\qquad K_n(x,y)e^{-y^2} = \frac{1}{\pi}\frac{\sin\sqrt{2n}(x-y)}{x-y} + O(1)$$

which holds locally uniformly in x and y, see SZEGŐ [2], (9.5.24). Taking into account that f is locally bounded, we see that for $|x| \le K$ and $0 < \gamma \le 1$

$$(3.5)\qquad \left| D_1 - \frac{1}{\pi}\int\limits_x^{x+\gamma} \frac{\sin\sqrt{2n}(x-y)}{x-y} f(y)\,dy \right| < c(K)\gamma$$

so if $\gamma < \frac{\varepsilon}{c(K)}$ then the first estimate (3.2) follows.

Consider now D_2.

$$D_2 = \int\limits_{x+\gamma}^{\infty} K_n(x,y)e^{-y^2/2}e^{-y^2/2} f(y)\,dy =$$

$$= \int\limits_{x+\gamma}^{\sqrt{n}} + \int\limits_{\sqrt{n}}^{\sqrt{2n+1}-n^{-1/6}} + \int\limits_{\sqrt{2n+1}-n^{-1/6}}^{\sqrt{2n+1}+1} + \int\limits_{\sqrt{2n+1}+1}^{\infty} := T_1 + T_2 + T_3 + T_4.$$

First estimate T_1. Since $e^{-y^2/2}f(y)$ has bounded variation, we see from the second mean value theorem of integral calculus that

$$(3.7)\qquad |T_1| \le c_0 \cdot \max_{x+\gamma \le A < B \le \sqrt{n}} \left| \int\limits_A^B K_n(x,y)e^{-y^2} \right|.$$

Now let $x+\gamma \leq A < B \leq \sqrt{n}$ and compute

$$\int_A^B K_n(x,y)e^{-y^2}. \tag{3.8}$$

Using (2.2) we obtain

$$\int_A^B K_n(x,y)e^{-y^2} = \tag{3.9}$$

$$= \frac{H_{n+1}(x)}{2^{n+1}n!\sqrt{\pi}}\int_A^B \frac{e^{-y^2/2}H_n(y)}{x-y}dy - \frac{H_n(x)}{2^{n+1}n!\sqrt{\pi}}\int_A^B \frac{e^{-y^2/2}H_{n+1}(y)}{x-y}dy.$$

It is enough to estimate

$$\int_A^B \frac{e^{y^2/2}H_n(y)}{x-y}dy \tag{3.10}$$

Integrating by parts

$$\int_A^B e^{-y^2/2}\frac{1}{x-y}dy = \tag{3.11}$$

$$\left[\int_A^y e^{-t^2/2}H_n(t)dt\frac{1}{x-y}\right]_{y=A}^B - \int_A^B\int_A^y e^{-t^2/2}H_n(t)dt\frac{1}{(x-y)^2}dy =$$

$$= O(1)\sup_{A\leq y\leq B}\left|\int_A^y e^{-t^2/2}H_n(t)dt\right|\frac{1}{A-x}.$$

We know ([2], (8.22.12))

$$e^{-t^2/2}H_n(t) =$$

$$= \frac{2^{\frac{2n+1}{4}}\sqrt{n!}}{(\pi n)^{1/4}\sqrt{\sin\varphi}}\left\{\sin\left[\frac{2n+1}{4}(\sin 2\varphi - 2\varphi)+\frac{3\pi}{4}\right]+O\left(\frac{1}{n}\right)\right\},$$

where $t=\sqrt{2n+1}\cos\varphi$, $0<\varepsilon\leq\varphi\leq\pi-\varepsilon$, the error term is uniform.

Using this we obtain

$$(3.12)\qquad \int_A^y e^{-t^2/2} H_n(t)\,dt = \frac{2^{\frac{2n+1}{4}}\sqrt{n!}\sqrt{2n+1}}{(\pi n)^{1/4}}\cdot$$

$$\cdot \int_{\varphi_y}^{\varphi_A} \sqrt{\sin\varphi}\cdot\left\{\sin\left[\frac{2n+1}{4}(\sin 2\varphi - 2\varphi)+\frac{3\pi}{4}\right]+O\left(\frac{1}{n}\right)\right\}d\varphi,$$

where $\frac{A}{\sqrt{2n+1}} = \cos\varphi_A$, $\frac{y}{\sqrt{2n+1}} = \cos\varphi_y$, $0 < \varepsilon \le \varphi_A, \varphi_y < \pi - \varepsilon$.

Let

$$\varphi := \frac{\pi}{2} - \tau, \qquad \varphi_A := \frac{\pi}{2} - \tau_A, \qquad \varphi_y := \frac{\pi}{2} - \tau_y,$$

Then

$$\frac{A}{\sqrt{2n+1}} = \sin\tau_A, \qquad \frac{y}{\sqrt{2n+1}} = \sin\tau_y,$$

Hence

$$\int_{\varphi_y}^{\varphi_A} \sqrt{\sin\varphi}\,\sin\left[\frac{2n+1}{4}(\sin 2\varphi - 2\varphi)+\frac{3\pi}{4}\right]d\varphi =$$

$$(3.13)\qquad \begin{aligned} &= \int_{\tau_A}^{\tau_y} \sqrt{\cos\tau}\,\sin\left[\frac{2n+1}{4}(\sin 2\tau + 2\tau - \pi)+\frac{3\pi}{4}\right]d\tau = \\ &= \int_{\tau_A}^{\tau_y} \sqrt{\cos\tau}\,\sin\left[\frac{2n+1}{4}(\sin 2\tau + 2\tau)+\frac{1-n}{2}\pi\right]d\tau = \\ &= \pm\int_{\tau_A}^{\tau_y} \sqrt{\cos\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\tau + 2\tau)\right]d\tau. \end{aligned}$$

Integrating by parts

$$\int_{\tau_A}^{\tau_y} \sqrt{\cos\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\tau + 2\tau)\right]d\tau =$$

$$(3.14)\qquad = \left[\sqrt{\cos\tau}\int_{\tau_A}^{\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\omega+2\omega)\right]d\omega\right]_{\tau=\tau_A}^{\tau_y} +$$

$$+\frac{1}{2}\int_{\tau_A}^{\tau_y}\frac{\sin\tau}{\sqrt{\cos\tau}}\int_{\tau_A}^{\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\omega+2\omega)\right]d\omega\, d\tau.$$

We have

$$\int_{\tau_A}^{\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\omega+2\omega)\right]d\omega=\frac{1}{2}\int_{s_A}^{s_\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{2n+1}{4}s\frac{1}{1+\cos 2\omega}ds$$

where we introduced a new variable s, $\sin 2\omega+2\omega=s$, which implies $(2\cos 2\omega+2)d\omega=1\,ds$, $\omega=g(s)$, $(\sin 2\omega+2\omega)$ is strongly monotone increasing if $-\frac{\pi}{2}\le\omega\le\frac{\pi}{2}$, $\tau=g(s_\tau)$, $\tau_A=g(s_A)$, $\sin 2\tau+2\tau=s_\tau$, $\sin 2\tau_A+2\tau_A=s_A$, and since $-\frac{\pi}{2}\le\omega\le\frac{\pi}{2}$ therefore $\omega\asymp s$ (and $\operatorname{sgn}\omega=\operatorname{sgn}s$).

Integrating by parts

$$\int_{\tau_A}^{\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\omega+2\omega)\right]d\omega=\frac{1}{2}\int_{s_A}^{s_\tau}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{2n+1}{4}s\frac{1}{1+\cos 2g(s)}ds=$$

$$=\frac{2}{2n+1}\left[\frac{\left\{\begin{matrix}-\cos\\ \sin\end{matrix}\right\}\frac{2n+1}{4}s}{1+\cos 2g(s)}\right]_{s_A}^{s_\tau}-\frac{2}{2n+1}\int_{s_A}^{s_\tau}\left\{\begin{matrix}-\cos\\ \sin\end{matrix}\right\}\frac{2n+1}{4}s\frac{2\cdot\sin 2g(s)}{(1+\cos 2g(s))^2}ds=$$

$$=O\left(\frac{1}{n}\right).$$

Using this we obtain from (3.14), (3.13), (3.12)

$$\int_A^y e^{-t^2/2}H_n(t)\,dt=\frac{O\sqrt{2^n n!}}{n^{3/4}}.$$

Hence from (3.11), (3.9) we get

$$\int_A^B K_n(x,y)e^{-y^2}=O\left(\frac{1}{\sqrt{n}}\right).$$

and then by (3.7)

$$T_1=O\left(\frac{1}{\sqrt{n}}\right).$$

Now consider T_2.

$$T_2 = \frac{1}{2} \int\limits_{\sqrt{n}}^{\sqrt{2n+1}-n^{-1/6}} K_n(x,y)e^{-y^2/2}e^{-y^2/2}f(y)\,dy.$$

In this case we need estimate for

$$\int\limits_A^B K_n(x,y)e^{-y^2}.$$

where $\sqrt{n} \le A < B \le \sqrt{2n+1} - n^{-1/6}$.

Hence we have to estimate

$$\int\limits_A^B \frac{e^{y^2/2}H_n(y)}{x-y}\,dy.$$

From (3.11)

$$\int\limits_A^B \frac{e^{-y^2/2}H_n(y)}{x-y}\,dy = O(1)\frac{1}{\sqrt{n}} \sup_{A\le y\le B} \left| \int\limits_A^y e^{-t^2/2}H_n(t)\,dt \right|. \tag{3.15}$$

We know ([7], p. 22–23)

$$e^{-t^2/2}H_n(t) =$$

$$= \frac{\sqrt{2N^n}e^{-N/4}}{\sqrt{\sin\theta}} \left\{ \sin\left[\frac{2n+1}{4}(\sin 2\theta - 2\theta) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n\theta^3}\right) \right\},$$

where $t = \sqrt{2n+1}\cos\theta$, $0 < \theta \le \frac{\pi}{2}$, $N = 2n+1$, the remainder term is uniform in θ. Here

$$e^{-N/4}N^{n/2} =$$

$$= e^{(n/2)\log(2n+1) - \frac{2n+1}{4}} = e^{n\log n/2 + (n/2)(\log 2 - 1) + O(1)} \asymp \sqrt{2^n n!}\,n^{-1/4}.$$

Using this we obtain

$$\int\limits_A^y e^{-t^2/2}H_n(t)\,dt = O(1)\frac{\sqrt{2^n(2n+1)n!}}{n^{1/4}}. \tag{3.16}$$

$$\cdot \int\limits_{\theta_y}^{\theta_A} \sqrt{\sin\theta} \cdot \left\{ \sin\left[\frac{2n+1}{4}(\sin 2\theta - 2\theta) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n\theta^3}\right) \right\} d\theta$$

where $\frac{A}{\sqrt{2n+1}} = \cos\theta_A$, $\frac{y}{\sqrt{2n+1}} = \cos\theta_y$, $\frac{c}{n^{1/3}} \le \theta_B \le \theta_y \le \theta_A \le \frac{\pi}{2}$, $\frac{B}{\sqrt{2n+1}} = \cos\theta_B$. The remainder term gives

$$\int_{\theta_y}^{\theta_A} \frac{\sqrt{\theta}}{n\theta^3}\, d\theta = O(1)\frac{1}{n\theta_y^{3/2}}. \tag{3.17}$$

For the estimation of the main term it is enough to estimate

$$\int_{\theta_y}^{\theta_A} \sqrt{\sin\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\theta - 2\theta)\right] d\theta. \tag{3.18}$$

Integrating by parts

$$\int_{\theta_y}^{\theta_A} \sqrt{\sin\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\theta - 2\theta)\right] d\theta = \tag{3.19}$$

$$= -\sqrt{\sin\theta_A} \int_{\theta_y}^{\theta_A} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\omega - 2\omega)\right] d\omega +$$

$$+ \frac{1}{2}\int_{\theta_y}^{\theta_A} \frac{\cos\theta}{\sqrt{\sin\theta}} \int_{\theta}^{\theta_A} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\omega - 2\omega)\right] d\omega\, d\theta.$$

We have

$$\int_{\theta}^{\theta_A} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\omega - 2\omega)\right] d\omega = \frac{1}{2}\int_{s_\theta}^{s_A} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \frac{2n+1}{4} s \frac{1}{\cos 2\omega - 1} ds$$

where we introduced a new variable s, $\sin 2\omega - 2\omega = s$, i. e. $g(s) = \omega$, $g(s_\theta) = \theta$, $g(s_A) = \theta_A$. Since $0 \le \omega \le \frac{\pi}{2}$, hence $\omega^3 \asymp -s$.

Integrating by parts

$$\int_{\theta_A}^{\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left[\frac{2n+1}{4}(\sin 2\omega - 2\omega)\right] d\omega = \frac{1}{2}\int_{s_A}^{s_\theta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \frac{2n+1}{4} s \frac{1}{\cos 2\omega - 1} ds$$

$$= \frac{2}{2n+1}\left[\frac{\left\{\begin{matrix}-\cos\\ \sin\end{matrix}\right\} \frac{2n+1}{4}s}{\cos 2g(s)-1}\right]_{s_A}^{s_\theta} - \frac{2}{2n+1}\int_{s_A}^{s_\theta}\left\{\begin{matrix}-\cos\\ \sin\end{matrix}\right\}\frac{2n+1}{4}s\frac{\sin 2g(s)}{(\cos 2g(s)-1)^3}ds$$

$$= O(1)\frac{1}{n\sin^2\theta} + O(1)\frac{1}{n}\int_{s_A}^{s_\theta}\frac{1}{|\sin g(s)|^5}ds =$$

$$= O(1)\frac{1}{n\theta^2} + O(1)\frac{1}{n}\int_{s_A}^{s_\theta}\frac{1}{|s|^{5/3}}ds = O(1)\frac{1}{n\theta^2} + O(1)\frac{1}{n|s_\theta|^{2/3}} = O\frac{1}{n\theta^2}.$$

Using this estimate we obtain from (3.19)

$$\int_{\theta_y}^{\theta_A}\sqrt{\sin\theta}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left[\frac{2n+1}{4}(\sin 2\theta - 2\theta)\right]d\theta$$

(3.20)

$$= O(1)\frac{1}{n\theta_y^{3/2}} + O(1)\frac{1}{n}\int_{\theta_y}^{\theta_A}\frac{1}{\theta^{5/2}}d\theta = O(1)\frac{1}{n\theta_y^{3/2}}.$$

So by (3.20), (3.17) and (3.16) we get

$$\int_A^y e^{-t^2/2}H_n(t)dt = O(1)\frac{\sqrt{2^n n!}\cdot n^{-1/4}}{\sqrt{n}\cdot\theta_y^{3/2}}$$

Using this we obtain from (3.10)

$$\int_A^B K_n(x,y)e^{-y^2} = O(1)\frac{1}{n\theta_B^{3/2}}.$$

Hence

$$T_2 = O\left(\frac{1}{\sqrt{n}}\right).$$

Now consider T_3. We have to estimate

$$\sup_{A\le y\le B}\left|\int_A^y e^{-t^2/2}H_n(t)dt\right|,$$

where $\sqrt{2n+1}-n^{-1/6}\le A\le B\le\sqrt{2n+1}+1$. Using [6], p. 700. we obtain

$$\sup_{A\le y\le B}\left|\int_A^y e^{-t^2/2}H_n(t)dt\right|=O(1)\sqrt{2^n n!}.$$

Thus

$$T_3=O\left(\frac{1}{\sqrt[4]{n}}\right).$$

Using [6], p. 700. we obtain that T_4 is exponentially small. Hence from (3.6) we get $D_2=O\left(\frac{1}{\sqrt[4]{n}}\right)$ i. e. (3.2) is proved. Since (3.3) can be derived similarly; the proof of Theorem 1 is complete. ∎

4. REMARK. It can be proved that if $f(x)\cdot ln(1+|x|)\cdot e^{-x^2}\in L_\infty(\mathbb{R})$ then equiconvergence holds for f, see e. g. Muckenhoupt [3]. Here there is no need of bounded variation property. On the other hand let $f(x)=e^{(x+x_0)^2/2}$. Then (see [3]) there exists f_1 with $|f_1(x)|=|f(x)|$ $\forall\, x$ and $\sigma_n(f,y)-S_n(f,y)\not\to 0$ a. e.

Since $S_n(f_1,x)\to f_1(x)$ a. e., we get $\sigma_n(f_1,x)\not\to f_1(x)$ a. e. In what follows we prove more.

5. REMARK. We show that there exists a continuous function $f(x)$, such that its Hermite–Fourier series is divergent in every point.

Let

$$g(x):=e^{(x+x_0)^2/2},\qquad x_0\ne 0.$$

The Hermite–Fourier series of $g(x)$ is

$$\sum_{n=0}^{\infty}\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{\infty}g(z)e^{-z^2}H_n(z)dz\cdot H_n(x). \tag{5.1}$$

Denote

$$a_n:=\int_{-\infty}^{\infty}e^{(x+x_0)^2/2}\cdot e^{-x^2}\cdot H_n(x)dx.$$

Using [2], (5.5.3)

$$a_n := (-1)^n \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} \cdot \left(e^{-x^2}\right)^{(n)} dx =$$

$$= (-1)^n \left\{ \left[e^{(x+x_0)^2/2} \left(e^{-x^2}\right)^{(n-1)} \right]_{-\infty}^{\infty} - \right.$$

$$\left. - \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} (x+x_0) \left(e^{-x^2}\right)^{(n-1)} dx \right\} =$$

$$= (-1)^{n+1} \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} (x+x_0) \left(e^{-x^2}\right)^{(n-1)} dx =$$

$$= \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} (x+x_0)(-1)^{n-1} \left(e^{-x^2}\right)^{(n-1)} dx =$$

$$= \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} (x+x_0) e^{-x^2} H_{n-1}(x) dx.$$

Since ([2], (5.5.8))

$$H_n(x) + 2(n-1)H_{n-2}(x) = 2x H_{n-1}(x), \quad n = 2, 3, \ldots$$

therefore

$$a_n = \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} (x+x_0) e^{-x^2} H_{n-1}(x) dx$$

$$= x_0 \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} e^{-x^2} H_{n-1}(x) dx +$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} e^{(x+x_0)^2/2} e^{-x^2} 2x H_{n-1}(x) dx$$

$$= x_0 a_{n-1} + \frac{1}{2} a_n + (n-1) a n_{n-2}.$$

Hence

$$a_n = 2x_0 a_{n-1} + 2(n-1) a_{n-2}, \qquad n = 2, 3, \ldots$$

i. e.,

$$a_{n+2} - 2x_0 a_{n+1} - 2(n+1)a_n = 0, \quad n = 0, 1, \dots$$

where $a_0 = \sqrt{2\pi}$, $a_1 = 2\sqrt{2\pi}x_0$.

Therefore

$$a_n = \sqrt{2\pi}\, i^n H_n(-ix_0). \tag{5.2}$$

It is known ([2], 8.22.8)

(5.3)

$$\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(n+1)} e^{-x^2/2} H_n(x) = \cos\left(\sqrt{2n+1}x - \frac{n\pi}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right), \quad |x| \le c_0,$$

where the error term is uniform in x. This asymptotic formula is true if x is a complex number, but the error term $O\left(\frac{1}{\sqrt{n}}\right)$ has to be changed for $O\left(\frac{1}{\sqrt{n}}\right) e^{\sqrt{2n+1}|Im\, x|}$. It is known ([9], p. 322)

$$\Gamma(t+1) = \sqrt{2\pi t}\left(\frac{t}{e}\right)^t \left[1 + O\left(\frac{1}{t}\right)\right], \qquad t > 0.$$

Hence the n–th term of (5.1) ($n \ge 1$) is

$$\frac{1}{2^n n! \sqrt{\pi}} a_n H_n(x) = e^{x^2/2 - x_0^2/2} \cdot \frac{2}{\sqrt{\pi}} (-1)^{n/2} \frac{1}{\sqrt{n}} \left(1 + O\left(\frac{1}{n}\right)\right) \cdot$$

$$\cdot \left[\cos\left(\sqrt{2n+1}x - \frac{n\pi}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)\right] \cdot$$

$$\cdot \left[\cos\left(i\sqrt{2n+1}x_0 + \frac{n\pi}{2}\right) + O(1)\frac{e^{\sqrt{2n+1}|x_0|}}{\sqrt{n}}\right].$$

If the sum in (5.1) is convergent, then the summands tend to 0. Taking the 4k-th summand we can write

$$\frac{1}{\sqrt{4k}}\left(1 + O\left(\frac{1}{k}\right)\right)\left[\cos\sqrt{8k+1}x + O(1)\frac{1}{\sqrt{k}}\right] \cdot$$

$$\cdot \left[\cos i\sqrt{8k+1}x_0 + O(1)\frac{e^{\sqrt{8k+1}|x_0|}}{\sqrt{k}}\right] \asymp$$

$$\asymp \frac{1}{\sqrt{k}}\left[\cos\sqrt{8k+1}x + O(1)\frac{1}{\sqrt{k}}\right] \cdot e^{\sqrt{8k+1}|x_0|}$$

If we choose $k_1 < k_2 < \dots$ such that (certainly they depend on x, x is fixed) $|\cos\sqrt{8k_j+1}x| \ge \frac{1}{2}$ then we obtain that the $4k_j$-th summand is $\asymp \frac{e^{\sqrt{8k_j+1}x_0}}{\sqrt{k_j}}$, which is not tends to 0.

Hence the Hermite–Fourier series of $g(x)$ is divergent.

6. Proof of Theorem 2; Christoffel numbers.

LEMMA 1 *If* λ_i $(i = 1,\dots,n)$ *are the Christoffel–numbers on Hermite nodes, then*

$$\lambda_i \asymp e^{-x_i^2} \cdot \frac{1}{n^{1/6} \cdot i^{1/3}} \asymp e^{x_i^2} \cdot \varphi_n(x_i), \quad i = 1, \dots, \frac{n}{2} \tag{6.1}$$

where $\varphi_n(x_i) = x_i - x_{i+1}$ *(certainly* $x_1 > x_2 > \dots > x_n$*).*

The last relation follows from [18], Theorem 2, (7). (We know that $\lambda_i = \lambda_{n-i+1}$.)

PROOF. It is enough to prove for $x_i \geq 0$. We know that the Lemma is true if $|x_i| \leq a\sqrt{2n+1}$, $0 < a < 1$ arbitrary fixed number (see [15], (12), (33)). Therefore assume that $a\sqrt{2n+1} \leq x_i \leq \sqrt{2n+1}$. We know ([15], (31))

$$|H_n'(x_i)| = 2n|H_{n-1}(x_i)| = \pi^{1/4} \cdot \sqrt{2}\sqrt{2^n n!} \cdot \lambda_i^{-1/2}. \tag{6.2}$$

We have to estimate $|H_{n-1}(x_i)|$.

We distinguish 2 cases:

a) $a\sqrt{2n+1} \leq x_i \leq \sqrt{2n+1} - O(1)$

b) $\sqrt{2n+1} - O(1) \leq x_i \leq \sqrt{2n+1}$,

(where $O(1)$ means a positive, suitable absolute constant).

a) $a\sqrt{2n+1} \leq x_i \leq \sqrt{2n+1} - O(1)$.

Using [18], Theorem 2, (6) we have $i \geq d_1 \sqrt[4]{n}$. Assume that d_1 is a large enough absolute constant. Define $0 \leq \theta_{i,n-1} \leq \frac{\pi}{2}$.

$$x_{i,n} =: \sqrt{2n-1}\cos\theta_{i,n-1}. \tag{6.3}$$

We know ([18], Theorem 2)

$$x_{i,n} = \sqrt{2n+1}\cos\theta_{i,n} + O(1)\frac{1}{n^{1/6} \cdot i^{4/3}}, \tag{6.4}$$

where we can not write $o(1)$ in place of $O(1)$ and $\theta_{i,n}$ is given by

$$\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) = i\pi - \frac{3\pi}{4} \tag{6.5}$$

From (6.5) obviously $\theta_{i,n}^3 \asymp \frac{i}{n}$.

We know ([19], p. 22–23)

$$e^{-x^2} H_n(x) = (2N^n)^{1/2} \cdot \exp\left(-\frac{N}{4}\right) \cdot (\sin\theta)^{-1/2}.$$

(6.6)

$$\cdot \left\{ \sin\left[\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta - 2\theta) + \frac{3\pi}{4} \right] + O\left(\frac{1}{n\theta^3}\right) \right\},$$

where $x := \sqrt{2n+1}\cos\theta$, $0 < \theta \le \frac{\pi}{2}$, $N = 2n+1$, the remainder terms are uniform in θ.

If we want to estimate $|H_{n-1}(x_{i,n})|$ then we have to estimate

$$\sin\theta_{i,n-1} \tag{6.7}$$

and

$$\sin\left[\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) + \frac{3\pi}{4} \right]. \tag{6.8}$$

First estimate (6.7). From (6.3), (6.4) we get

$$\cos\theta_{i,n-1} = \sqrt{1 + \frac{2}{2n-1}}\cos\theta_{i,n} + O(1)\frac{1}{n^{2/3} \cdot i^{4/3}}. \tag{6.9}$$

From this

$$\cos^2\theta_{i,n-1} = \cos^2\theta_{i,n} + O\left(\frac{1}{n}\right), \quad \text{(where we used } i \ge d_1 \sqrt[4]{n}\text{), and}$$

$$\sin^2\theta_{i,n-1} = \sin^2\theta_{i,n} + O\left(\frac{1}{n}\right) = \sin^2\theta_{i,n}\left(1 + O(1)\frac{1}{n^{1/3} \cdot i^{2/3}}\right), \tag{6.10}$$

where we used $\sin^2\theta_{i,n} \asymp \theta_{i,n}^2 \asymp \frac{i^{2/3}}{n^{2/3}}$. Hence

$$\sin\theta_{i,n-1} = \sin\theta_{i,n}\left(1 + O(1)\frac{1}{n^{1/3} \cdot i^{2/3}}\right). \tag{6.11}$$

Therefore

$$\sin\theta_{i,n-1} \asymp \frac{i^{1/3}}{n^{1/3}}. \tag{6.12}$$

Now estimate (6.8). Denote $f(\theta) := \sin 2\theta - 2\theta$. Then

$$f'(\theta) = 2(\cos\theta - 1) = -4\sin^2\theta, \quad f''(\theta) = -4\sin 2\theta.$$

We have

$$f(\theta_{i,n-1}) - f(\theta_{i,n}) = (\theta_{i,n-1} - \theta_{i,n}) f'(\theta_{i,n}) + O(1) \cdot (\theta_{i,n-1} - \theta_{i,n})^2 \cdot |f''(\xi)|,$$

where $\xi \in (\theta_{i,n-1}, \theta_{i,n})$, i. e.

$$\sin 2\theta_{i,n} - 2\theta_{i,n} - (\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) = \tag{6.13}$$
$$= (\theta_{i,n} - \theta_{i,n-1}) f'(\theta_{i,n}) + O(1)(\theta_{i,n} - \theta_{i,n-1})^2 \cdot |f''(\xi)|.$$

From (6.11)

$$\theta_{i,n} - \theta_{i,n-1} \asymp \frac{\theta_{i,n}}{n^{1/3} \cdot i^{2/3}} \asymp \frac{1}{n^{2/3} \cdot i^{1/3}}. \tag{6.14}$$

Using this estimate we obtain form (6.13)

$$\sin 2\theta_{i,n} - 2\theta_{i,n} - (\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) =$$
$$= O(1)\frac{\theta_{i,n}^3}{n^{1/3} \cdot i^{2/3}} + O(1)\frac{\theta_{i,n}^3}{(n^{1/3} \cdot i^{2/3})^2} = O(1)\frac{\theta_{i,n}^3}{n^{1/3} \cdot i^{2/3}} = O(1)\frac{i^{1/3}}{n^{4/3}},$$

where we can not write $o(1)$ instead of $O(1)$. Hence

$$\sin 2\theta_{i,n-1} - 2\theta_{i,n-1} = \sin 2\theta_{i,n} - 2\theta_{i,n} + O(1)\frac{i^{1/3}}{n^{4/3}}.$$

Therefore

$$\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) + \frac{3\pi}{4} =$$
$$= \left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n} - 2\theta_{i,n}) + \frac{3\pi}{4} + O(1)\frac{i^{1/3}}{n^{1/3}} =$$
$$= \left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n} - 2\theta_{i,n}) + \frac{3\pi}{4} - \frac{1}{2}(\sin 2\theta_{i,n} - 2\theta_{i,n}) + O(1)\frac{i^{1/3}}{n^{1/3}} =$$
$$= i\pi + O(1)\theta_{i,n}^3 + O(1)\frac{i^{1/3}}{n^{1/3}} = i\pi + O(1)\frac{i^{1/3}}{n^{1/3}}.$$

Hence

$$\sin\left[\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) + \frac{3\pi}{4}\right] \asymp \frac{i^{1/3}}{n^{1/3}} \asymp \theta_{i,n-1} \asymp \theta_{i,n}. \tag{6.15}$$

Since $i \geq d_1 \cdot \sqrt[4]{n}$ therefore

$$\sin\left[\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\theta_{i,n-1} - 2\theta_{i,n-1}) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n\theta_{i,n-1}^3}\right) \asymp \frac{i^{1/3}}{n^{1/3}}. \tag{6.16}$$

Obviously

$$N^{n/2} \cdot \exp\left(-\frac{N}{4}\right) \asymp \sqrt{2^n n!} \cdot n^{-1/4},$$

therefore

$$H_{n-1}(x_i) \asymp e^{x_i^2/2} \cdot \sqrt{2n(n-1)!} \cdot n^{-1/4} \cdot \frac{n^{1/6}}{i^{1/6}} \cdot \frac{i^{1/3}}{n^{1/3}} \asymp$$
$$\asymp e^{x_i^2/2} \cdot \sqrt{2^n n!} \cdot n^{-11/12} \cdot i^{1/6}.$$

Using this

$$\lambda_i \asymp e^{-x_i^2} \cdot \frac{1}{n^{1/6} \cdot i^{1/3}} \asymp e^{-x_i^2} \cdot \varphi_n(x_i).$$

b) $\sqrt{2n+1} - O(1) \le x_i \le \sqrt{2n+1}$.

In this case $1 \le i \le d_1 \cdot \sqrt[4]{n}$. We know ([7], Theorem B, SKOVGAARD)

$$H_n(N^{1/2}x) = (2\pi)^{1/2} \cdot e^{-N/4} \cdot N^{n/2+1/6} \cdot e^{Nx^2/2} \tag{6.17}$$
$$\cdot \left(\frac{\xi}{x^2-1}\right)^{1/4} \cdot \left\{ Ai(N^{2/3}\xi) + O(1)\frac{|N^{2/3}\xi|^{-1/4}}{n} \right\},$$

where $N = 2n+1$, $0 \le x \le 1$, $\xi = -\left\{\frac{3}{4}\arccos x - \frac{3}{4}x(1-x^2)^{1/2}\right\}^{2/3}$. Here

$$e^{-N/4} \cdot N^{n/2+1/6} \asymp \sqrt{2^n n!} \cdot n^{-1/12}.$$

Using the L'Hospital–rule we get

$$\arccos x - x(1-x^2)^{1/2} \asymp (1-x^2)^{3/2}, \tag{6.18}$$

therefore

$$\left(\frac{\xi}{x^2-1}\right)^{1/4} \asymp 1. \tag{6.19}$$

Denote

$$x_n(i) := \frac{x_{i,n}}{\sqrt{2n+1}}, \qquad x_{n-1}(i) := \frac{x_{i,n}}{\sqrt{2n-1}},$$
$$\xi_{i,n} := -\left\{\frac{3}{4}\arccos x_n(i) - \frac{3}{4}x_n(i)(1-x_n^2(i))^{1/2}\right\}^{2/3}, \tag{6.20}$$
$$\xi_{i,(n-1)} := -\left\{\frac{3}{4}\arccos x_{n-1}(i) - \frac{3}{4}x_{n-1}(i)(1-x_{(n-1)}^2(i))^{1/2}\right\}^{2/3}.$$

If we want to estimate $|H_{n-1}(x_{i,n})|$ then we have to estimate

$$Ai\left((2n-1)^{2/3}\xi_{i,n-1}\right) \tag{6.21}$$

and

$$|n^{2/3} \cdot \xi_{i,n-1}| \tag{6.22}$$

We know

$$\sqrt{2n+1} - O(1) \leq x_{i,n} \leq \sqrt{2n+1} - n^{-1/6},$$

therefore

$$1 - \frac{O(1)}{\sqrt{2n+1}} \leq x_n(i) \leq 1 - \frac{1}{n^{2/3}},$$

$$\frac{1}{n^{2/3}} \leq |\xi_{i,n}| \leq \frac{1}{\sqrt{n}},$$

because of $\xi_{i,n} \asymp (1 - x_n^2(i)) \asymp (1 - x_n(i))$. If $H_n(x_{i,n}) = 0$ then

$$Ai(N^{2/3}\xi_{i,n}) = O(1)\frac{1}{n}|n^{2/3} \cdot \xi_{i,n}|^{-1/4} = O(1)\frac{1}{n}.$$

Consider the following inequality

$$Ai(\ell) = O(1)\frac{1}{n} \cdot \frac{1}{\sqrt[4]{|\ell|}}, \quad |\ell| \geq 1, \quad |\ell| \leq c \cdot n^{1/6}. \tag{6.23}$$

We look for ℓ in the form $\ell = a_i + \tau$, where $0 > a_1 > a_2 > \ldots > a_i >$ the zeros of Airy–function, and $|\tau| \leq \frac{1}{2}|a_{i-1} - a_i|, \frac{1}{2}|a_{i+1} - a_i|$. The set of τ's is connected.

We know (see e. g. [17] p. 405., (5.05)) that

$$|a_i| \asymp i^{2/3}. \tag{6.24}$$

Therefore from (6.23)

$$Ai(\ell) = O(1)\frac{1}{n} \cdot \frac{1}{i^{1/6}}. \tag{6.25}$$

Using the Taylor-expansion

$$Ai(a_j + \tau) = Ai(a_j) + \tau \cdot A'i(a_j) + O(1)\tau^2 A''i(\xi), \text{ where } \xi \in (a_j, a_j + \tau).$$

We know ([17], p. 405, (5.05)) $|A'i(a_j)| \asymp j^{1/6}$ and ([17], p. 392, (1.01)) $A''i(\xi) = \xi \cdot Ai(\xi)$. Hence

(6.26)

$$Ai(a_i + \tau) = \tau \cdot A'i(a_i) + O(1)\tau^2 \cdot i^{2/3} \cdot (i^{2/3})^{-1/4} = \tau \cdot A'i(a_i) + O(1)\tau^2 \cdot \sqrt{i}.$$

If $\tau \asymp \frac{1}{\sqrt{n}}$, then

$$O(1)\frac{1}{n} \cdot \frac{1}{i^{1/6}} = Ai(a_i + \tau) \asymp \frac{1}{\sqrt{n}} \cdot i^{1/6},$$

but it is a contradiction. Therefore $|\tau| < \frac{1}{\sqrt{n}}$, thus from (6.26)

$$|\tau| = O(1)\frac{1}{n} \cdot \frac{1}{i^{1/3}}. \tag{6.27}$$

Hence

$$\xi_{i,n} = \frac{a_i}{N^{2/3}} + O(1)\frac{1}{n^{5/3}} \cdot \frac{1}{i^{1/3}}. \tag{6.28}$$

On the other hand

$$\xi_{i,n-1} - \xi_{i,n} = (x_n(i) - x_{n-1}(i)) \cdot \left[\left\{\frac{3}{4}\arccos u - \frac{3}{4}u(1-u^2)^{1/2}\right\}^{2/3}\right]'_{u=\sigma} \asymp$$

$$\asymp (x_n(i) - x_{n-1}(i)) \cdot \{\cdot\}_{u=\sigma}^{-1/3} \cdot \{\cdot\}'_{u=\sigma} \asymp$$

$$\asymp (x_n(i) - x_{n-1}(i)) \cdot \frac{1}{\sqrt{1-\tau^2}} \cdot \sqrt{1-\tau^2} \asymp$$

$$\asymp x_{i,n}\left(\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n+1}}\right) \asymp \frac{1}{n},$$

i. e.

$$\xi_{i,n-1} = \xi_{i,n} + O(1)\frac{1}{n}, \tag{6.29}$$

where we can not write $o(1)$ instead of $O(1)$.

Using (6.28–29) we get

$$Ai\left((2n-1)^{2/3} \cdot \xi_{i,n-1}\right) = Ai\left((2n-1)^{2/3}\xi_{i,n} + O(1)\frac{1}{n^{1/3}}\right) =$$

$$= Ai\left(N^{2/3}\xi_{i,n} + [(2n-1)^{2/3} - (2n+1)^{2/3}]\xi_{i,n} + O(1)\frac{1}{n^{1/3}}\right) =$$

$$= Ai\left(a_i + O(1)\frac{1}{n^{1/3}}\right),$$

where we can not write $o(1)$ instead of $O(1)$.

But

$$Ai\left(a_i + O(1)\frac{1}{n^{1/3}}\right) =$$

$$= Ai(a_i) + O(1)\frac{1}{n^{1/3}} \cdot A_i'(a_i) + O\left(\frac{1}{n^{2/3}}\right)\sqrt{i} = O(1)\frac{i^{1/6}}{n^{1/3}},$$

where we can not write $o(1)$ instead of $O(1)$. Hence

$$|Ai((2n-1)^{2/3} \cdot \xi_{i,n-1})| \asymp \frac{i^{1/6}}{n^{1/3}}. \tag{6.30}$$

At last, from (6.17), (6.30) we obtain

$$H_{n-1}(x_i) \asymp e^{x_i^2/2}\sqrt{2^n(n-1)!}\cdot n^{-1/12}\cdot\frac{i^{1/6}}{n^{1/3}} \asymp$$

$$\asymp e^{x_i^2/2}\cdot\sqrt{2^n n!}\cdot n^{-11/12}\cdot i^{1/6}.$$

Using this

$$\lambda_i \asymp e^{-x_i^2}\cdot\frac{1}{n^{1/6}\cdot i^{1/3}} \asymp e^{-x_i^2}\cdot\varphi_n(x_i).$$

Lemma 1 is proved. ∎

LEMMA 2. *For any polynomial of degree at most* m *we have*

$$\max_{x\in\mathbb{R}}|P_m(x)e^{-x^2/2}| \asymp \max_{|x|\le 1.921623m^{1/2}}|P_m(x)e^{-x^2/2}| \tag{6.31}$$

and in the case of $|x| > 1.921623m^{1/2}$ *we have*

$$|P_m(x)e^{-x^2/2}| \le$$

$$\le c\cdot m^2\cdot\exp(-2.7\cdot 10^{-7}\cdot m)\cdot\max_{|x|\le 1.0000001m^{1/2}}|P_m(x)e^{-x^2/2}|. \tag{6.32}$$

PROOF. It is known [12], p. 111. Lemma 11 that

$$|p_m(x)|^p \le c\cdot m^2\cdot\int_{-1}^{1}|P_m(t)|^p dt \qquad (-1\le x\le 1,\quad 0<p<\infty) \tag{6.33}$$

further (see [13], p. 62, Corollary) that

$$|p_m(x)| \le M(2|x|)^m \quad (|x|\ge 1) \tag{6.34}$$

if $|p_m(x)|\le M$ for $-1\le x\le 1$.

So

$$|p_m(x)| \le c(2|x|)^m\cdot m^2\cdot\int_{-1}^{1}|P_m(t)|dt,\quad (|x|\ge 1).$$

Consequently

$$|p_m(x)| \le c2^m A^{-m}|x|^m m^{-m/2} m^{2-(1/2)}A^{-1}\int_{-A\sqrt{m}}^{A\sqrt{m}}|P_m(t)|dt,$$

$(|x| \geq Am^{1/2})$, and hence

$$|p_m(x)| \cdot e^{-x^2} \leq c2^m A^{-m-1} |x|^m m^{-(m/2)+(3/2)} \int_{-A\sqrt{m}}^{A\sqrt{m}} |P_m(t)| e^{-x^2} dt,$$

$(|x| \geq Am^{1/2})$. Here

$$\int_{-A\sqrt{m}}^{A\sqrt{m}} |P_m(t)| \cdot e^{-x^2/2} dt = \int_{-A\sqrt{m}}^{A\sqrt{m}} |P_m(t)| e^{-t^2/2} \cdot e^{(t^2-x^2)/2} dt \leq$$

$$\leq ce^{-x^2/2B} \int_{-A\sqrt{m}}^{A\sqrt{m}} |P_m(t)| e^{-t^2/2} dt,$$

where

$$B > 1, \qquad |x| \geq \sqrt{\frac{B}{B-1}} Am^{1/2}.$$

Hence

$$|p_m(x)| \cdot e^{-x^2} \leq c2^m A^{-m-1} |x|^m m^{-(m/2)+(3/2)} e^{-x^2/2B} Am^{1/2} \cdot$$

$$\cdot \max_{|t| \leq A\sqrt{m}} |P_m(t) e^{-t^2/2}|, \quad (|x| \geq \sqrt{\frac{B}{B-1}} Am^{1/2}).$$

Suppose in the following that $\sqrt{B-1} < A$. In this case maximum of the function $f(x) = x^m e^{-x^2/2B}$, $x \geq \sqrt{\frac{B}{B-1}} Am^{1/2}$, is attained at $x_0 = \sqrt{\frac{B}{B-1}} Am^{1/2}$. Hence

$$2^m A^{-m} |x|^m m^{-(m/2)+2} e^{-x^2/2B} \leq$$

$$\leq \exp\left(m \log 2 - m \log A - \frac{m}{2} \log m + 2 \log m + \right.$$

$$\left. + \frac{m}{2} \log \frac{B}{B-1} + m \log A + \frac{m}{2} \log m - \frac{A^2}{2(B-1)} m \right) =$$

$$= \exp\left(m \left(\log 2 + \frac{1}{2} \log\left(1 + \frac{1}{B-1}\right) - \frac{A^2}{2(B-1)} \right) + 2 \log m \right),$$

$$\left(|x| \geq \sqrt{\frac{B}{B-1}} Am^{1/2} \right).$$

We have to determine the minimum of the function $\sqrt{\frac{B}{B-1}}A$ under the conditions:

$$\begin{cases} B > 1 \\ B < A^2 + 1 \\ \log 2 + \frac{1}{2}\log\left(1 + \frac{1}{B-1}\right) - \frac{A^2}{2(B-1)} < 0. \end{cases}$$

Let $B = 1 + \frac{1}{\sigma}$, then $\frac{B}{B-1} = 1 + \sigma$, $\log 2 + \frac{1}{2}\log\left(1 + \frac{1}{B-1}\right) - \frac{A^2}{2(B-1)} = \log 2 +$ $+ \frac{1}{2}\log(1+\sigma) - \frac{A^2\sigma}{2}$.

Let D be such that

$$\log 2 + \frac{1}{2}\log(1+\sigma) - \frac{D^2\sigma}{2} = 0$$

i. e.

$$D = \sqrt{\frac{\log 4 + \log(1+\sigma)}{\sigma}}.$$

We are looking for the minimum of the function

$$\sqrt{\frac{B}{B-1}}D = \sqrt{1+\sigma}\sqrt{\frac{\log 4 + \log(1+\sigma)}{\sigma}} =: \sqrt{g(\sigma)}.$$

From the equation

$$g'(\sigma_0) = \frac{-\log 4 + \sigma_0 - \log(1+\sigma_0)}{\sigma_0^2} = 0$$

we get $\sigma_0 = 2.692634$. In this case

$$B = 1.371384; \qquad D = \sqrt{\frac{\log 4 + \log(1+\sigma_0)}{\sigma_0}} = 1,$$

hence we choose $A = 1.0000001 > D$ and obviously

$$\sqrt{\frac{B}{B-1}}A = \sqrt{1+\sigma_0}A = 1.921623$$

$$\log 2 + \frac{1}{2}\log(1+\sigma_0) - \frac{A^2\sigma_0}{2} = -\frac{\delta_0}{2}(A^2 - 1) \approx 2.7 \cdot 10^{-7}.$$

Lemma 2 is proved.

PROOF OF THEOREM 2. Consider the partition

$$\sum_{k=1}^{n} \lambda_{k,n} |P_m(x_{k,n})| e^{-x_{k,n}^2/2} = \sum_{\substack{k=1 \\ |x_{k,n}|<\sigma\sqrt{n}}}^{n} \lambda_{k,n} |P_m(x_{k,n})| e^{-x_{k,n}^2/2} + $$

$$+ \sum_{\substack{k=1 \\ |x_{k,n}|\geq\sigma\sqrt{n}}}^{n} \lambda_{k,n} |P_m(x_{k,n})| e^{-x_{k,n}^2/2} = S_1 + S_2,$$

where $0 < \sigma < \sqrt{2}$ is fixed.

According to [14], Lemma 5 we have

$$S_1 \leq c \left(1 + \sqrt{\frac{m}{n}}\right) \int_{-\infty}^{\infty} |p_m(x)| e^{-x^2/2} dx. \tag{6.36}$$

Using Lemma 1 we obtain

$$S_2 \leq \sum_{\substack{k=1 \\ |x_{k,n}|\geq\sigma\sqrt{n}}}^{n} \varphi_n(x_{k,n}) |P_m(x_{k,n})| e^{-x_{k,n}^2/2} \leq$$

$$\leq c \cdot \max_{|x_{k,n}|\geq\sigma\sqrt{n}} |P_m(x_{k,n})| e^{-x_{k,n}^2/2} \cdot \sum_{\substack{k=1 \\ |x_{k,n}|\geq\sigma\sqrt{n}}}^{n} \varphi_n(x_{k,n}).$$

According to $\sigma\sqrt{n} \geq 1.921623\sqrt{m}$ by Lemma 3 we get

$$S_2 \leq cm^2 \exp(-2.7 \cdot 10^{-7} m) \cdot n \cdot \max_{|x|\leq 1.0000001 m^{1/2}} |P_m(x) e^{-x^2/2}| \tag{6.37}$$

Taking into account Freud's result [16] there exists a polynomial R_m of degree $\leq c_2 m$ such that

$$R_m(x) \asymp e^{-x^2}, \quad (|x| \leq 1.0000001 m^{1/2}). \tag{6.38}$$

Using (6.33) with $p_m(x)R_m(x)$ we have

$$|p_m(x)R_m(x)| \leq cm^{2-(1/2)} \int_{-1.000000 m^{1/2}}^{1.0000001 m^{1/2}} |p_m(t)R_m(t)| dt \leq$$

(6.39)

$$\leq cm^{3/2} \int_{-\infty}^{\infty} |P_m(t) e^{-t^2/2}| dt,$$

where $|x| \leq 1.0000001 m^{1/2}$.

From (6.37–39) we obtain

$$S_2 \leq cm^2 \exp(-2.7 \cdot 10^{-7} m) \cdot n \cdot m^{3/2} \int_{-\infty}^{\infty} |P_m(t) e^{-t^2/2}| dt. \tag{6.40}$$

Here we assume that $m \geq a \cdot \log n$ (a is a suitable absolute constant), so

$$S_2 \leq c \int_{-\infty}^{\infty} e^{u^2/2} \cdot |p_m(u)| du. \tag{6.41}$$

From (6.36), (6.41) we get (2.7)

Theorem 2 is proved. ∎

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ON THE NUMBER OF q-EXPANSIONS

By

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Expansions

Let (p_i) be a sequence of positive numbers such that $P := \sum p_i < +\infty$. Given a real number $x \in [0,P]$, let us define two sequences (c_i), (d_i) of integers c_i, $d_i \in \{0,1\}$ as follows:

— Set $c_1 = c_1(x) := 1$ if $p_1 \leq x$ and $c_1 := 0$ otherwise;

— If $i > 1$ and $c_1, \ldots, c_{i-1}$ are already defined, set $c_i = c_i(x) := 1$ if $c_1p_1 + c_2p_2 + \cdots + c_{i-1}p_{i-1} \leq x - p_i$ and $c_i := 0$ otherwise;

— Set $d_1 = d_1(x) := 0$ if $\sum_{j>1} p_j \geq x$ and $d_1 := 1$ otherwise;

— If $i > 1$ and $d_1 \ldots, d_{i-1}$ are already defined, set $d_i = d_i(x) := 0$ if $\sum_{j>i} p_j \geq x - \sum_{j<i} p_j$ and $d_i = 1$ otherwise.

Clearly, we have always $\sum c_ip_i \leq x$. If we have equality: $\sum c_ip_i = x$, then we say that $\sum c_ip_i$ is the *greedy expansion* of x.

Similarly, we have always $\sum d_ip_i \geq x$. If we have equality: $\sum d_ip_i = x$, then we say that $\sum d_ip_i$ is the *lazy expansion* of x.

Obviously, if $\sum c_ip_i$ is the *greedy expansion* of x, then $\sum (1-c_i)p_i$ is the *lazy expansion* of $P - x$, and conversely.

More generally, we say that $\sum_{i=1}^{\infty} c_ip_i$ is an *expansion* of x if $c_i \in \{0,1\}$ for each i and if $\sum_{i=1}^{\infty} c_ip_i = x$. Clearly, if $x \notin [0,P]$, then x cannot have any expansion.

The following result is well–known and is easy to verify:

LEMMA 1.1 *The following three properties are equivalent:*

a) *Every $x \in [0,P]$ has a greedy expansion;*

b) *Every $x \in [0,P]$ has a lazy expansion;*

c) $p_j \le \sum_{i>j} p_i$ *for all* $j \ge 1$.

If the sequence (p_i) *is non–increasing, then a)–c) are also equivalent to*

d) *Every* $x \in [0, P]$ *has an expansion.*

Let us note that in the general case d) is strictly weaker than a)–c). Indeed, it is sufficient to observe that property d) is invariant for the permutations of the sequence (p_i) while property c) is not.

In what follows we shall only consider sequences of the form $p_i = q^{-i}$ for some $1 < q < 2$; then property c) and hence a) b) d) are satisfied. We shall often say q-expansion instead of expansion. Applying this result to $x - \sum_{i=1}^{n} c_i q^{-i}$ instead of x we obtain the

LEMMA 1.2 *Given a finite sequence* $c_1, \dots, c_n$ *in* $\{0,1\}$, *a real number* x *has a* q*-expansion beginning with* $c_1, \dots, c_n$ *if and only if*

$$\sum_{i=1}^{n} c_i q^{-i} \le x \le \frac{1}{q^n(q-1)} \sum_{i=1}^{n} c_i q^{-i}.$$

For brevity we shall use the notation

$$(c_i)(q) := \sum_{i=1}^{\infty} c_i q^{-i}.$$

It is clear that $x = 0$ and $x = 1/(q-1)$ have only one expansion i. e. their greedy and lazy expansions coincide. As it was first shown in [1], there are many values of q for which 1 has a unique q-expansion; we shall return to this question in the following section. However, in many cases x has several different expansions: for example if $1 < q < A := (1+\sqrt{5})/2$, then every $x \in (0, 1/(q-1))$ has 2^{ω} different q-expansions. This result was proved in [1] for $x = 1$ and then in [2] for the general case.

We end this section with two definitions. An expansion $x = (c_i)(q)$ is called *infinite* if $c_i = 1$ for infinitely many i; otherwise the expansion is called *finite*.

2. Greedy, lazy and unique expansions

In this section we seek necessary and/or sufficient conditions permitting to recognize whether a given expansion $x = (c_i)(q)$ is greedy, lazy or unique. The greedy expansions were studied in detail earlier by PARRY [4]; we slightly improve his results by giving a different proof. Next we give a new proof of the characterization of the unique expansions obtained earlier in [2]. Finally we give a new sufficient condition for an expansion of 1 to be lazy. We end this section by proving a simple lemma on the lazy expansion of 1. It will turn to be very useful in simplifying the discussions of the rest of this paper.

Let us introduce the lexicographic ordering between the sequences: given two sequences (c_i) and (d_i) we write $(c_i) > (d_i)$ or $(d_i) < (c_i)$ if $(c_i) \neq (d_i)$ and if $c_m > d_m$ for the first m such that $c_m \neq d_m$. One can readily verify that among all expansions of a given number $x \in [0, P]$ the greedy expansion (if exists) is the biggest one and the lazy expansion (if exists) is the smallest one.

LEMMA 2.1 *Consider an expansion* $x = (c_i)(q)$.

1. *If there is an infinite expansion* $1 = (d_i)(q)$ *of 1 such that*

$$(c_{m+i}) < (d_i) \qquad \text{whenever} \qquad c_m = 0, \tag{1}$$

then the expansion $x = (c_i)(q)$ *is greedy.*

2. *If* (1) *is satisfied for some finite expansion* $1 = (d_i)(q)$ *and if* (c_i) *does not have a periodic tail with the period* $d_1, \dots, d_{m-1}, d_m - 1$ *where* d_m *is the last non-zero term of the sequence* (d_i), *then the expansion* $x = (c_i)(q)$ *is greedy.*

PROOF. In view of Lemma 1.2 it is sufficient to prove that if $c_n = 0$ for some n, then $(c_{n+i})(q) := \sum_{i=1}^{\infty} c_{n+i} q^{-i} < 1$.

Fix an arbitrary n satisfying $c_n = 0$. We are going to construct a sequence of integers $0 = k_0 < k_1 < \dots$ such that

$$c_{n+k_j} = 0 \quad \text{and} \quad d_{k_{j+i}-k_j} = 1 \quad \text{for all} \quad j \geq 0. \tag{2}$$

Set $k_0 = 0$. If $k_0 < \dots < k_j$ are already defined for some $j \geq 0$ and $c_{n+k_j} = 0$, then let k_{j+1} be the first integer satisfying

$$k_{j+1} > k_j \qquad \text{and} \qquad c_{n+k_{j+1}} < d_{k_{j+1}-k_j}$$

(its existence follows from (1) applied with $m := n + k_j$). Then we have $c_{n+k_{j+1}} = 0$ and $d_{k_{j+1}-k_j} = 1$.

Now we have

$$(c_{n+i})(q)=\sum_{j=0}^{\infty}\sum_{i=k_j+1}^{k_{j+1}} c_{n+i}q^{-i}=\sum_{j=0}^{\infty}\left(\left(\sum_{i=1}^{k_{j+1}-k_j} d_i q^{-i-k_j}\right)-q^{-k_{j+1}}\right)$$
$$\le\sum_{j=0}^{\infty}\left(q^{-k_j}-q^{-k_{j+1}}\right)=1$$

and it remains to show that the last inequality is strict.

Assume on the contrary that the last inequality is in fact an equality. Then we have

$$\sum_{i=1}^{k_{j+1}-k_j} d_i q^{-i}=1 \quad \forall j\ge 0. \tag{3}$$

It follows from (3) that $d_i=0$ for all $j\ge 0$ and $i>k_{j+1}-k_j$. Using (2) hence we conclude that the numbers $k_{j+1}-k_j$ coincide for all $j\ge 0$. Denoting this number by m we have $d_m=1$, and $k_j=mj$, $c_{n+mj}=0$ for all $j\ge 0$. Finally, using the *definition* of the sequence (k_j) we also have $c_{n+mj+i}=d_i$, $i=1,\dots,m-1$, $j=0,1,\dots$. Thus the expansion (c_i) should have the form which was excluded in the formulation of the lemma. ∎

REMARKS. It is easy to verify that in the case which was excluded from the proceeding lemma the expansion $x=(c_i)(q)$ is *not* greedy although (1) is satisfied.

Now we give a new proof of the characterization of the greedy q-expansion of 1, obtained earlier in [2].

THEOREM 2.2 *The expansion $(a_i)(q)=1$ is greedy if and only if*

$$(a_{m+i})<(a_i) \qquad \textit{whenever} \qquad a_m=0. \tag{4}$$

A similar characterization was obtained earlier by PARRY [4] by a different proof: he assumed the condition in (4) for all m.

PROOF. Assume first that $(a_i)(q)=1$ is the greedy expansion but (4) is not satisfied for some m. Then $a_m=0$ and $(a_{m+i})\ge(a_i)$. Since $(a_i)(q)=$ $=1$ is the greedy expansion, the second relation implies that $(a_{m+i})(q)\ge 1$. Hence, applying Lemma 1.2 we conclude that there is another expansion of 1 beginning with $a_1,\dots,a_{m-1},a_m+1$. And this is impossible because $(a_i)(q)=1$ is the greedy expansion.

Now assume that (4) is satisfied and apply Lemma 2.1 with $(c_i)=(d_i):=$ $=(a_i)$. It is sufficient to show that the exceptional case of the lemma cannot occur here: then $(a_i)(q)=1$ is the greedy expansion.

If the expansion $(a_i)(q)=1$ is finite, then let a_m be its last non–zero term. Clearly we have $m=2$. Since we have $a_1=1$ by (4), it follows that the period $a_1,\dots,a_{m-1},a_m-1$ cannot be identically zero. ∎

Let (a_i) be a sequence of nonnegative integers satisfying (5). We recall from [4] that the equation $(a_i)(q)=1$ has a unique solution $q>1$. Furthermore, $a_1=1$ and $(a_i)(q)=1$ is the greedy q-expansion of 1.

Henceforth the expansion $(a_i)(q)=1$ will always denote the *greedy* expansion of 1. Now we are ready to characterize the greedy expansion of any $0\le x\le 1/(q-1)$.

THEOREM 2.3 1. *If the expansion $(a_i)(q)=1$ is infinite, then an expansion $(c_i)(q)=x$ is greedy if and only if*

$$(c_{m+i})<(a_i) \quad \textit{whenever} \quad c_m=0. \tag{6}$$

2. *If the expansion $(a_i)(q)=1$ is finite and if a_m is its last non–zero digit, then an expansion $(c_i)(q)=x$ is greedy if and only if (6) is satisfied and (c_i) has not a periodic tail with the period $a_1,\dots,a_{m-1},a_m-1$.*

A similar characterization was obtained earlier by PARRY [4] by a different proof: he assumed the condition in (6) for all m, and he also assumed (implicitly by his construction) that $x\le 1$. In fact, his result remains valid for all $x<1/(q-1)$ (but not for $x=1/(q-1)$).

PROOF. The sufficiency part follows at once from Lemma 2.1 in both cases.

The necessity of the condition (6) may be proved in the same way as the corresponding assertion of Theorem 2.2 above. Indeed, if (6) is violated for some m, then $(c_{m+i})(q)\ge 1$ because $1=(a_i)(q)$ is the greedy expansion, and then an application of Lemma 1.2 shows that x has an expansion beginning with $c_1,\dots,c_{m-1},c_m+1$. Hence the expansion $x=(c_i)(q)$ is not greedy.

In the second case the necessity of the periodicity condition is obvious. Indeed, if for some n the sequence (c_{n+i}) is periodic with the period $a_1,\dots,a_{m-1},a_m-1$, then x has another expansion $x=(c_1,\dots,c_n,a_1,\dots$ $\dots,a_m,0,0,\dots)$; since we have obviously

$$(c_1,\dots,c_n,a_1,\dots,a_m,0,0,\dots)>(c_i),$$

the expansion $x=(c_i)(q)$ is not greedy. ∎

Let us recall for the reader's convenience the following characterization of the unique expansions; it was proved earlier in a different way in [2].

THEOREM 2.4 *An expansion* $1=(c_i)(q)$ *is the unique expansion of 1 if and only if*

(8) $$(c_{m+i})<(c_i) \quad \textit{whenever} \quad c_m=0$$

and

(9) $$(1-c_{m+i})<(c_i) \quad \textit{whenever} \quad c_m=1.$$

PROOF. If $1=(c_i)(q)$ is the unique expansion of 1, then it is in particular a greedy expansion, and the dual expansion $(q-1)^{-1}-1=(1-c_i)(q)$ is also greedy. Applying for them Theorem 2.3, (8) and (9) follow.

Conversely, if (8) and (9) are satisfied, then the expansion $1=(c_i)(q)$ is greedy by Theorem 2.2 and the dual expansion $(q-1)^{-1}-1=(1-c_i)(q)$ is greedy by Theorem 2.3 (observe that condition (9) excludes the sequence (c_i) to be finite). Hence the expansion $1=(c_i)(q)$ is lazy, too. ∎

We cannot characterize the lazy expansion of 1 without using its greedy expansion. The following result is a sufficient, but not necessary condition.

PROPOSITION 2.5 *If an expansion* $1=(b_i)(q)$ *has the property*

(10) $$(1-b_{m+i})<(b_i) \quad \textit{whenever} \quad b_m=1,$$

then it is the lazy expansion of 1.

PROOF. Assume (10). Then the expansion $1=(b_i)(q)$ is infinite. Indeed, in the contrary case there would exist m such that $b_m=1$ and $b_i=0$ for all $i>m$, and (10) cannot hold for this m. The theorem now follows from the first part of Lemma 2.1. ∎

REMARK. The condition (10) is not necessary. Indeed, one can readily verify that for most $1<q<A:=(1+\sqrt{5})/2$ the lazy expansion $1=(b_i)(q)$ satisfies $b_m=1$ and $b_{m+1}=0$ for some $m\geq 1$; since $b_1=0$, (10) is not satisfied in this case. It would be interesting to find weaker sufficient (or necessary and sufficient) conditions for the lazy expansions.

In the sequel the expansion $1=(b_i)(q)$ will always denote the lazy expansion of 1.

The following result will be useful later.

LEMMA 2.6 *Consider an expansion* $x=(c_i)(q)$.

1. *If for some* m *we have* $c_m=1$ *and* $(1-c_{m+i})<(b_i)$, *then* x *has no expansion beginning with* $c_1,\dots,c_{m-1},0$.

2. *If for some* m *we have* $c_m = 0$ *and* $(c_{m+i}) < (b_i)$, *then* x *has no expansion beginning with* $c_1, \ldots, c_{m-1}, 1$.

PROOF. 1. Since the expansion $1 = (b_i)(q)$ is lazy, it follows from $(1 - c_{m+i}) < (b_i)$ that $(1 - c_{m+i})(q) < 1$. Therefore

$$\sum_{i<m} c_i q^{-i} + (c_m - 1)q^{-m} + \sum_{i>m} q^{-i} < (c_i)(q)$$

and the result follows by applying Lemma 1.2.

2. The relation $(c_{m+i}) < (b_i)$ implies that $(c_{m+i})(q) < 1$; hence

$$\sum_{i<m} c_i q^{-i} + (c_m + 1)q^{-m} > (c_i)(q)$$

and the assertion follows.

3. The number of different expansions

It was proved in [1] that there are 2^ω numbers q in $(1,2)$ for which the q-expansion of 1 is unique. This result was later generalized in [3]: for every positive integer N there are 2^ω numbers q in $(1,2)$ such that 1 has exactly N different q-expansions. Also, it was shown in [1] that there are at least ω numbers q in $(1,2)$ such that 1 has exactly ω different q-expansions. Thus the following result is not new for $N < \omega$. However, we prove this result by a new construction.

THEOREM 3.1 *For every* $1 \leq N \leq \omega$ *there are* 2^ω *numbers* $q \in (1,2)$ *for which* 1 *has exactly* N *different* q*-expansions.*

PROOF. Let us consider a sequence of the form

$$(a_i) := 1,1,1,1,1,A_1,A_2,\ldots$$

if $N = \omega$, and of the form

$$(a_i) := 1,1,1,1,1,A_1,A_2,\ldots,A_{N-1},B_1,B_2,\ldots$$

if $1 \leq N < \omega$ where each block A_i is a sequence of ten digits of the form $A_i = 0,0,0,0,0,1,a_i,0,0,1$ with arbitrary $a_i \in \{0,1\}$, and each block B_i is a sequence of ten digits of the form $B_i = 0,0,1,0,0,1,b_i,0,0,1$ with arbitrary $b_i \in \{0,1\}$. There are clearly 2^ω different sequences (a_i) of this type for every fixed $1 \leq N \leq \omega$.

Defining q by the condition $(a_i)(q) = 1$ we have $1 < q < 2$ and the expansion $(a_i)(q) = 1$ is greedy (see Remark 2 following the proof of Theorem 2.2). In particular, hence we conclude that the 2^ω different sequences (a_i) lead to 2^ω different numbers q. It remains to show that for every number q defined in this way 1 has exactly N different q-expansions.

For $N = 1$ this property follows at once by applying Theorem 2.4. We may thus assume in the sequel that $1 < N \le \omega$.

It is easy to construct new q-expansions of 1 from the given, greedy one. For every fixed integer $1 \le k < N$ we define a sequence (c_i^k) in $\{0,1\}$ by the formula

$$c_i^k := \begin{cases} a_i, & \text{if } i < 10k-5; \\ 0(=a_i-1), & \text{if } i = 10k-5 \\ a_i + a_{i-(10k-5)}, & \text{if } i > 10k-5. \end{cases}$$

It is clear that $(c_i^k)(q) = 1$ for every $1 \le k < N$. It remains to show that 1 has no other q-expansions.

For $k = 1$ the sequence (c_i^1) has the explicit form

$$(c_i^1) = 1,1,1,1,0,1,1,1,1,1,C_1,C_2,\dots$$

with some blocks $C_i = 1,*,0,0,1$ (we do not need the explicit values of the digits *) if $N = \omega$, and

$$(c_i^1) = 1,1,1,1,0,1,1,1,1,1,C_1,\dots,C_{2N-3},D_1,D_2,\dots$$

with some blocks $C_i = 1,*,0,0,1$ and $D_i = 1,*,1,0,1$ if $2 \le N < \omega$.

Applying Proposition 2.5 one can readily verify that the expansion $(c_i^1)(q) = 1$ is the lazy one; hence we shall write (b_i) instead of (c_i^1).

Next we observe that, excepting the digits $a_{10j-5} = 1$, $1 \le j < N$, for every other digit a_m we have

$$(1 - a_{m+i}) < (b_i) \quad \text{if} \quad a_m = 1 \qquad \text{and} \qquad (a_{m+i}) < (b_i) \quad \text{if} \quad a_m = 0.$$

Applying Lemma 2.6 hence we conclude that if a q-expansion $(d_i)(q) = 1$ satisfies the condition

$$d_{10j-5} = 1 \qquad \text{for all} \qquad 1 \le j < N, \tag{11}$$

then in fact $(d_i) = (a_i)$.

Now consider the expansion $(c_i^k)(q) = 1$ for some fixed $1 \le k < N$. It is easy to check that, excepting the digits $c_{10j-5}^k = 1$, $1 \le j < k$ and the digit $c_{10k-5}^k = 0$, for every other digit c_m^k we have

$$(1 - c_{m+i}^k) < (b_i) \quad \text{if} \quad c_m^k = 1 \qquad \text{and} \qquad (c_{m+i}^k) < (b_i) \quad \text{if} \quad c_m^k = 0.$$

Applying Lemma 2.6 again, it follows that if a q-expansion $(d_i)(q) = 1$ satisfies the conditions

$$d_{10j-5} = 1 \quad \text{for all} \quad 1 \le j < k \quad \text{and} \quad d_{10j-5} = 0,$$

then in fact $(d_i) = (c_i^k)$.

Since every expansion $(d_i)(q) = 1$ satisfies either (11) or (12) for some $1 \le k < N$, we conclude that 1 has exactly N different q-expansions:

$$(a_i)(q) = 1 \quad \text{and} \quad (c_i^k)(q) = 1, \quad 1 \le k < N. \qquad \blacksquare$$

Now fix a sequence (c_i) in $\{0,1\}$. For every $q \in (1,2)$ set $x := (c_i)(q)$ and denote by $N(q)$ the number of different q-expansions of x. The following result shows that $N(q)$ depends effectively on q in general.

THEOREM 3.2 *For every positive integer N there exists a sequence (c_i) in $\{0,1\}$ and N numbers $q_1,\dots,q_N \in (1,2)$ such that $N(q_j) = j, j = 1,\dots,N$.*

PROOF. Fix a positive integer k (to be chosen later) and consider the sequences (c_i) and (b_i) defined by

$$(c_i) = 1,0,0,1,A_1,\dots,A_N,B,B,\dots,$$
$$(b_i) = 1,1,1,1,0,1,0,0,C_1,\dots,C_{k-1},D,D,\dots$$

with the blocks

$$A_1 = \dots = A_N = 0,0,0,0,1,0,0,1, \quad B = 1,0,0,1,$$
$$C_1 = \dots = C_{k-1} = 1,1,0,1,0,1,0,0, \quad D = 0,1,0,0.$$

Define $1 < q < 2$ by the equation $(b_i)(q) = 1$; the application of Proposition 2.5 shows that then $(b_i)(q) = 1$ is the lazy expansion of 1.

If $k > N$, then applying Theorem 2.4 we obtain that the expansion $x = (c_i)(q)$ is in fact the unique q-expansion of 1; hence $N(q) = 1$.

Now fix $1 \le k \le N$ arbitrarily. It is sufficient to show that $N(q) = N + 2 - k$.

It is easy to verify that for every integer $1 \le n \le N + 1 - k$ we obtain a new expansion $x = (c_i^n)(q)$ of x by putting

$$c_i^n := \begin{cases} c_i, & \text{if } i < 8n-4; \\ 0(= c_i - 1), & \text{if } i = 8n-4 \\ c_i + b_{i-(8n-4)}, & \text{if } i > 8n-4. \end{cases}$$

We have, explicitly,

$$(c_i^1) = 1,0,0,0,1,1,1,1,1,1,0,1,A_1,\dots,A_{k-1},B_1,\dots,B_{N-k},C,C,\dots$$

with $A_1 = \dots = A_{k-1} = 1,1,0,1,1,1,0,1$, $B_1 = \dots = B_{N-k} = 0,1,0,0,1,1,0,1$, $C = 1,1,0,1$ if $n = 1$, and

$$(c_i^n) = 1,0,0,1,D_1,\dots,D_{n-2},0,0,0,0,1,0,0,0,1,1,1,1,1,1,0,1,$$
$$A_1,\dots,A_{k-1},B_1,\dots,B_{N-k+1-n},C,C,\dots$$

with the same blocks A_j, B_j and C as above and with $D_1 = \cdots = D_{n-2} = 0,0,0,0,1,0,0,1$ if $n \geq 2$.

It is easy to verify that, excepting the digits $c_{8j-4} = 1$, $j = 1,\ldots,N$, for every other digit c_m we have

$$(c_{m+i}) < (b_i) \quad \text{if} \quad c_m = 1 \qquad \text{and} \qquad (1 - c_{m+i}) < (b_i) \quad \text{if} \quad c_m = 0.$$

Applying Lemma 2.6 it follows that if an expansion $(d_i)(q) = x$ satisfies the conditions

$$(13) \qquad d_{8j-4} = 1, \qquad j = 1,\ldots,N,$$

then in fact $(d_i) = (c_i)$.

Similarly, one can readily verify for each fixed $1 \leq n \leq N+1-k$ that, excepting the digits $c^n_{8j-4} = 1$, $j = 1,\ldots,n-1$ and $c^n_{8n-4} = 0$, for every other digit c^n_m we have

$$(c^n_{m+i}) < (b_i) \quad \text{if} \quad c^n_m = 1 \qquad \text{and} \qquad (1 - c^n_{m+i}) < (b_i) \quad \text{if} \quad c^n_m = 0.$$

Applying again Lemma 2.6 it follows that if an expansion $(d_i)(q) = x$ satisfies for some $1 \leq n \leq N+1-k$ the conditions

$$(14) \qquad d_{8j-4} = 1 \quad \text{for} \quad j = 1,\ldots,n-1 \quad \text{and} \quad d_{8n-4} = 0,$$

then in fact $(d_i) = (c^n_i)$.

Since every expansion $(d_i)(q) = x$ satisfies either (13) or (14) for some n, the theorem follows.

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ON A MINIMAX THEOREM*

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In this paper we prove the following

THEOREM. *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *and* $\psi : [0,1] \times [0,1] \to [0,1]$ *be continuous functions satisfying*

$$(1) \qquad f(\psi(t_1,t_2),y) \geq \frac{f(t_1,y)+f(t_2,y)}{2} \qquad (t_1,t_2,y \in [0,1]).$$

Then there exists $t^* \in [0,1]$ *with* $f(t^*,t^*) \geq f(t,t^*)$ *for all* $t \in [0,1]$.

The story of the problem started with the paper [1] of HORVÁTH and SÖVEGJÁRTÓ, where the authors posed the following problem: Does any pair of functions have a Nash equilibrium point if the functions are Ky–Fan concave in the corresponding variables (i.e. if $\forall x_1,x_2,\lambda \exists x_3 : f_1(x_3,y) \geq \lambda f_1(x_1,y) + (1-\lambda) f_1(x_2,y)$ for every y and if f_2 satisfies an analogous statement in its second variable y)? In [2] a negative answer is given by constructing continuous functions f_1, f_2 Ky–Fan concave in x and y respectively and not having an equilibrium point. I. Joó investigated the case when the point x_3 depends "regularly" on x_1, x_2 and λ. In [4] he observed that if ψ is a contraction i.e. $|\psi(x_1,x_1) - \psi(x_2,x_2)| \leq \varepsilon |x_1 - x_2|$ for some fixed $0 < \varepsilon < 1$ then the conclusion of the above Theorem holds. The existence of such a point t^* (at least in n dimension) is equivalent to the existence of a Nash-equilibrium point for n functions as it is known from the literature [4]. In [3] F. FORGÓ proved that if $x_3 = \psi(x_1,x_2,\lambda)$ is continuous, then equilibrium point does exist. In this case the ideas of the proof of the Nikaido–Isoda theorem can easily be applied. If the continuity of λ is dropped, and only $\lambda = \frac{1}{2}$ is considered, we arrive at the above Theorem; the following proof is completely different from [5]; we use some ideas from [2], [3].

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The proof of the theorem is based on the approximation of f by functions $f_N(t,y)$ piecewise linear in y. Then the existence of t_N^* corresponding to f_N implies the existence of t^* by a simple continuity argument. In dealing with f_N the linearity leads to simple geometrical investigations of the corresponding curve. To be more concrete denote

$$h_i(t) := f\left(t, \frac{i}{N}\right), \qquad i = 0, 1, \dots, N; \quad t \in [0,1]$$

$$f_N(t,y) := (i+1-Ny)h_i(t) + (Ny-i)h_{i+1}(t), \quad i = 0,\dots,N;\ t \in [0,1]; \quad \frac{i}{N} \le y \le \frac{i+1}{N},$$

$$\underline{r}(t) := (h_0(t), h_1(t), \dots, h_N(t))^T.$$

Then $f_N(t,y) = \underline{\lambda}(y)\underline{r}(t)$, where

$$\underline{\lambda}(y) = (\underbrace{0}_{0}, \dots \underbrace{0}_{i-1}, \underbrace{i+1-Ny}_{i}, \underbrace{Ny-i}_{i+1}, \underbrace{0}_{i+2}, \dots \underbrace{0}_{N})$$

$$\text{for} \quad y \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \quad i = 0, 1, \dots, N.$$

Our aim is to prove the existence of $t^* \in [0,1]$ with

$$(2) \qquad f_N(t^*, t^*) \ge f_N(t, t^*) \quad \text{for all} \quad t \in [0,1]$$

and to derive from here the same statement for f. The inequality (2) means that $t \mapsto \underline{\lambda}(t^*)\underline{r}(t)$ is maximal for $t = t^*$. Denote Λ the set of all $\underline{\lambda} \in \mathbb{R}^{N+1}$, $\underline{\lambda} = (\lambda_0, \dots, \lambda_N)$ where $\lambda_i \ge 0$ for all i and $\sum_{i=0}^{N} \lambda_i = 1$; then $\underline{\lambda}(y) \in \Lambda$ for all $y \in [0,1]$. For fixed $\underline{\lambda} \in \Lambda$ denote $M_{\underline{\lambda}}$ the set of all $\underline{r}(t)$ where $\underline{\lambda}\,\underline{r}(t) = \max_{t' \in [0,1]} \underline{\lambda}\,\underline{r}(t')$. Then (2) can be reformulated as

$$(2') \qquad \text{There exists } t^* \in [0,1] \text{ with } \underline{r}(t^*) \in M_{\underline{\lambda}(t^*)}.$$

To prove this we need some lemmas.

LEMMA 1. *If $\underline{x}_n \in M_{\underline{\lambda}_n}$, $\underline{\lambda}_n \in \Lambda$, $\underline{x}_n \to \underline{x}$ and $\underline{\lambda}_n \to \underline{\lambda}$, then $\underline{x} \in M_{\underline{\lambda}}$. In particular, $M_{\underline{\lambda}}$ is closed for $\underline{\lambda} \in \Lambda$.*

PROOF. Denote $m(\underline{\lambda}) = \max\{\underline{\lambda}\,\underline{r}(t) : t \in [0,1]\}$. Then $m(\underline{\lambda}_n) = \underline{\lambda}_n \underline{x}_n \to \underline{\lambda}\,\underline{x}$. Suppose indirectly that $\underline{\lambda}\,\underline{x} < m(\underline{\lambda})$.

Since for $\underline{x}' \in M_{\underline{\lambda}}$ $\underline{\lambda}_n \underline{x}' \leq m(\underline{\lambda}_n)$ for all n, this would contradict the compactness of the curve $\{\underline{r}(t) : t \in [0,1]\}$. Consequently $\underline{\lambda}\,\underline{x} = m(\underline{\lambda})$, $\underline{x} \in M_{\underline{\lambda}}$ as we asserted.

LEMMA 2. *Let* $t_1, t_2 \in [0,1]$ *and* $\underline{\lambda} \in \Lambda$ *with* $\underline{\lambda}\,\underline{r}(t_1) = \underline{\lambda}\,\underline{r}(t_2) =: c_0$ *and let* $t_i^* := \psi(t_i, t_i)$ $(i = 1,2)$. *Then*

$$\min_{t \in [t_1^*, t_2^*]} \underline{\lambda}\,\underline{r}(t) \geq \frac{1}{2}\left(c_0 + \min_{t \in [t_1, t_2]} \underline{\lambda}\,\underline{r}(t)\right).$$

PROOF. We know from (1) that

$$\underline{r}(\psi(t', t'')) \geq \frac{\underline{r}(t') + \underline{r}(t'')}{2} \qquad t', t'' \in [0,1] \tag{3}$$

where $\geq$ means the inequality in all the $N+1$ coordinates. Hence for all $\underline{\lambda} \in \Lambda$ we have $\underline{\lambda}\,\underline{r}(\psi(t', t'')) \geq \frac{\underline{\lambda}\,\underline{r}(t') + \underline{\lambda}\,\underline{r}(t'')}{2}$. Now we have for $t \in [t_1, t_2]$

$$\underline{\lambda}\,\underline{r}(\psi(t_1, t)) \geq \frac{1}{2}(c_0 + \underline{\lambda}\,\underline{r}(t)) \geq \frac{1}{2}(c_0 + \min_{[t_1, t_2]} \underline{\lambda}\,\underline{r}(t)),$$

$$\underline{\lambda}\,\underline{r}(\psi(t, t_2)) \geq \frac{1}{2}(\underline{\lambda}\,\underline{r}(t) + c_0) \geq \frac{1}{2}(c_0 + \min_{[t_1, t_2]} \underline{\lambda}\,\underline{r}(t)).$$

If $t \in [t_1, t_2]$, $\psi(t_1, t)$ varies continuously from t_1^* to $\psi(t_1, t_2)$ and $\psi(t, t_2)$ varies from $\psi(t_1, t_2)$ to t_2^* i.e. these values run over the whole $[t_1^*, t_2^*]$ (and, may be, other values as well), so the proof is complete. □

LEMMA 3. *Let* $\underline{x}$ *be a point of the curve* $\{\underline{r}(t) : t \in [0,1]\}$ *such that there is no other curve point* $\underline{x}' \geq \underline{x}$, $\underline{x}' \neq \underline{x}$. *Then there exists* $t \in [0,1]$ *with* $\underline{r}(t) = \underline{x}$ *and* $\psi(t,t) = t$.

PROOF. Denote $T = \{t \in [0,1] : \underline{r}(t) = \underline{x}\}$. Then $t \in T$ implies $\underline{r}(\psi(t,t)) \geq$ $\geq \underline{r}(t) = x$, and hence $\underline{r}(\psi(t,t)) = \underline{x}$, $\psi(t,t) \in T$. We show first

a$_0$) $\exists t_1^*, t_2^* \in T$ such that $t_2^* = \psi(t_1^*, t_1^*)$ and $h_0(t) \geq x_0$ for all $t \in [t_1^*, t_2^*]$. Indeed, let $t_1 \in T$ be arbitrary and define inductively $t_{n+1} = \psi(t_n, t_n)$; then $t_n \in T$ for all n. Denote further $d_n := \min\{h_0(t) : t \in [t_n, t_{n+1}]\}$.

If the pairs t_n, t_{n+1} do not fulfil the conditions of a$_0$), then $d_n < x_0$ for all n. Using Lemma 2 we see that $d_{n+1} \geq \frac{d_n + x_0}{2}$ and hence $d_n \to x_0$. Let $t_1^* = \lim t_{n_k}$ for any convergent subsequence t_{n_k}, and let $t_2^* = \psi(t_1^*, t_1^*)$. Then $t_1^* \in T$ (since T is closed) and then $t_2^* \in T$; further we get by the continuity of $\underline{r}(t)$ that $\min_{t \in [t_1^*, t_2^*]} h_0(t) = \lim_{k \to \infty} d_{n_k} = x_0$. So a$_0$) is proved. The next step:

a$_1$) $\exists t_1^{**}, t_2^{**} \in T$ with $t_2^{**} = \psi(t_1^{**}, t_1^{**})$ and $h_0(t) \geq x_0$, $h_1(t) \geq x_1$ for all $t \in [t_1^{**}, t_2^{**}]$.

Indeed, let $t^*_{n+1} = \psi(t^*_n, t^*_n) \in T$. Then from Lemma 2 we see by induction that $\min_{[t^*_n, t^*_{n+1}]} h_0(t) = x_0$ for all n. If no pairs t^*_n, t^*_{n+1} fulfil a$_1$) then $e_n := \min_{[t^*_n, t^*_{n+1}]} h_1(t) < x_1$. As in a$_0$) we obtain $e_{n+1} \geq \frac{e_n + x_1}{2}$, hence $e_n \to x_1$. Let $t_1^{**} = \lim_{h\to\infty} t^*_{n_h}$ for any convergent subsequence $t^*_{n_h}$. Then $t_1^{**} \in T$, $t_2^{**} := = \psi(t_1^{**}, t_1^{**}) \in T$ and we see by continuity that $\min_{[t_1^{**}, t_2^{**}]} h_i(t) = x_i$ $(i = 0, 1)$. Thus a$_1$) is proved. Continuing this way we finally obtain:

a$_N$) $\exists t_1, t_2 \in T$ with $t_2 = \psi(t_1, t_1)$ and $\underline{r}(t) \geq \underline{x}$ for all $t \in [t_1, t_2]$. But this means by the maximality of $\underline{x}$ that $\underline{r}(t) = \underline{x}$ for all $t \in [t_1, t_2]$. Denote again $t_{n+1} = \psi(t_n, t_n)$. If the sequence t_n is monotone then $t_n \to t^* \in T$ and $\psi(t^*, t^*) = t^*$ so Lemma 3 fulfils. If t_n is not monotone, there is an index n with $t_n < t_{n+1} > t_{n+2}$. Since $\psi(t_n, t_n) > t_n$ and $\psi(t_{n+1}, t_{n+1}) < t_{n+1}$, there exists $t \in [t_n, t_{n+1}]$ with $\psi(t, t) = t$. Lemma 2 implies as above that $\underline{r}(t) \geq \underline{x}$ i.e. $\underline{r}(t) = \underline{x}$ for all $t \in [t_n, t_{n+1}]$, $n = 1, 2, \ldots$. The proof of Lemma 3 is complete. □

LEMMA 4. *The statement (2) holds for some* $t^* \in [0, 1]$.

PROOF. As we observed above, it is enough to show (2'). For $y \in [0, 1]$ denote

$$t_1(y) = \min\{t \in [0,1] : \underline{r}(t) \in M_{\underline{\lambda}(y)}, \psi(t,t) = t\},$$
$$t_2(y) = \max\{t \in [0,1] : \underline{r}(t) \in M_{\underline{\lambda}(y)}, \psi(t,t) = t\}.$$

Observe first that the set figuring here is nonempty. Indeed, the set $M_{\underline{\lambda}(y)}$ being closed, it contains a point $\underline{x}$ maximal in the lexicographical order of the $N+1$ coordinates, and then by Lemma 3, there exists t with $\underline{r}(t) = = \underline{x} \in M_{\underline{\lambda}(y)}$ and $\psi(t, t) = t$. On the other hand we have $\underline{r}(t) \in M_{\underline{\lambda}(y)}$ for all $t \in [t_1(y), t_2(y)]$. Indeed, let $d = \min\{\underline{\lambda}(y)\underline{r}(t) : t \in [t_1(y), t_2(y)]\}$. Since $\underline{\lambda}(y)\underline{r}(t_1(y)) = \underline{\lambda}(y)\underline{r}(t_2(y)) = m(\underline{\lambda}(y))$, hence by Lemma 2 $d \geq \frac{1}{2}(m(\underline{\lambda}(y)) + + d)$, i.e. $d \geq m(\underline{\lambda}(y))$, so $d = m(\underline{\lambda}(y))$ and then $\underline{r}(t) \in M_{\underline{\lambda}(y)}$ indeed for $t \in [t_1(y), t_2(y)]$. Define the set-valued mapping

$$F : [0,1] \to \mathcal{P}([0,1]), \quad F(y) := [t_1(y), t_2(y)].$$

We show that F is closed, i.e. that $y_n \to y$, $t_n \in F(y_n)$, $t_n \to t$ implies $t \in F(y)$. Indeed, taking an appropriate subsequence we can suppose $t_1(y_n) \to t_1^*$, $t_2(y_n) \to t_2^*$. Now $\psi(t_i(y_n), t_i(y_n)) = t_i(y_n)$ implies $\psi(t_i^*, t_i^*) = t_i^*$ for $i = 1, 2$, and by Lemma 1, $\underline{x}_n^i = \underline{r}(t_i(y_n)) \in M_{\underline{\lambda}(y_n)}$, $\underline{x}_n^i \to \underline{x}^i = \underline{r}(t_i^*)$ and $\underline{\lambda}(y_n) \to \underline{\lambda}(y)$ implies $\underline{x}^i = \underline{r}(t_i^*) \in M_{\underline{\lambda}(y)}$, $i = 1, 2$. Consequently $[t_1(y), t_2(y)] \supset [t_1^*, t_2^*]$. On the other hand, $t_n \in F(y_n) = [t_1(y_n), t_2(y_n)]$ implies $t \in [t_1^*, t_2^*] \subset F(y)$. So F is closed, and then the Kakutani fixed point theorem states that there exists

$y \in [0,1]$, $y \in F(y)$. As we showed above, $t \in F(y)$ implies $\underline{r}(t) \in M_{\underline{\lambda}(y)}$, so we have $\underline{r}(y) \in M_{\underline{\lambda}(y)}$. Hence (2') and then (2) fulfils. □

PROOF OF THE THEOREM. Since the function f has compact support, its continuity is uniform, hence f_N tends to f uniformly on $[0,1] \times [0,1]$. It implies that if $f_N(t_N^*, t_N^*) \geq f_N(t, t_N^*)$ for all $t \in [0,1]$ and $t_{N_k}^* \to t^*$ then $f(t^*, t^*) \geq f(t, t^*)$ for all $t \in [0,1]$ as we asserted. □

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A MINIMAX THEOREM

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M. HORVÁTH and A. SÖVEGJÁRTÓ asked in [2] whether Nikaido–Isoda theorem [13] holds for Ky Fan concave function [1] or not. The negative answer was given in [5] further it was shown in [3] that we can not obtain positive answer even at strong continuity assumption for the functions considered. Furthermore, it was remarked by the second author of the present paper that if x_3 corresponding to x_1, x_2 and λ depends "regularly" on these parameters then the statement of the Nikaido–Isoda theorem remains valid [9] even for König-convex functions. This continuity occurred in the investigations [10] naturally. Different types of generalizations of [9] were discovered independently and simultaneously in [6] and [11].

The aim of the present note is to generalize this positive result for more general class of functions. As is well known, for this it is enough to prove statement of type given in the following Theorem. Our investigations are motivated by the stimulating works of S. SIMONS, e.g. [14, 15]. We generalize the paper [11] as well where ψ does not depend on y and $\lambda = \frac{1}{2}$. The method of [11] seems not to work in our more general situation.

THEOREM. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be any continuous real function. Suppose there exists a continuous function $\psi : [0,1] \times [0,1] \times [0,1] \to [0,1]$ such that $f(\psi(x_1,x_2,y),y) \geq \lambda(f(x_1,y) \vee f(x_2,y)) + (1-\lambda)(f(x_1,y) \wedge f(x_2,y))$ $\forall\ x_1, x_2, y \in [0,1]$, where $0 < \lambda < 1$ is a fixed number. Then there exists $x_0 \in [0,1] : f(x_0,x_x) \geq f(z,x_0)\ \forall z \in [0,1]$.*

For the proof we need for following

LEMMA. *Let $M^* := \{x \in [0,1] : f(x,y^*) = m\}$ for any fixed y^*, where $m = \max_{x \in [0,1]} f(x,y^*)$. Let z_1 and z_2 be such that $z_1 < z_2$; $z_1, z_2 \in M^*$ and $(z_1,z_2) \cap M^* = \emptyset$.*

Then

a) $z_2 \leq \psi(z_2,z_2,y^*)$ *implies* $z_2 \leq \psi(z_1,z_1,y^*)$,

b) $\psi(z_1,z_1,y^*) \leq z_1$ *implies* $\psi(z_2,z_2,y^*) \leq z_1$.

PROOF. We prove only the statement a) because that of b) is similar. Suppose $z_2 \leq \psi(z_2,z_2,y^*)$ and let $u \in (z_1,z_2)$ be such that $f(u,y^*) = \min_{x\in[z_1,z_2]} f(x,y^*)$. First we show that $\psi(z_1,z_2,y^*) > z_1$. We know $m = \lambda(f(z_1,y^*) \vee f(z_2,y^*)) + (1-\lambda)(f(z_1,y^*) \wedge f(z_2,y^*)) \leq f((z_1,z_2,y^*),y^*)$, hence $\psi(z_1,z_2,y^*) \geq z_2$ because $(z_1,z_2) \cap M^* = \emptyset$. If we suppose the opposite, i.e. $\psi(z_1,z_2,y^*) \leq z_1$, then because of $z_2 \leq \psi(z_2,z_2,y^*)$ and of continuity of $\psi(.,z_2,y^*)$ there would be a $x_3 \in (z_1,z_2)$ such that $\psi(z_3,z_2,y^*) = u$ and hence

$$\lambda(f(z_3,y^*) \vee f(z_2,y^*)) + (1-\lambda)(f(z_3,y^*) \wedge f(z_3,y^*)) \leq$$
$$\leq f(\psi(z_3,z_2,y^*),y^*) = f(u,y^*) = \min_{x\in[z_1,z_2]} f(x,y^*)$$

which is a contradiction. Thus we have proved that $\psi(z_1,z_2,y^*) > z_1$. If $\phi(z_1,z_1,y^*) \leq z_1$ would fulfill, then $\psi(z_1,z_2,y^*) \geq z_2$ and by the continuity of $\psi(z_1,\ldots,y^*)$ there exists $z_3 \in (z_1,z_2)$ such that $\psi(z_1,z_3,y^*) = u$ but then

$$\lambda(f(z_3,y^*) \vee f(z_1,y^*)) + (1-\lambda)(f(z_3,y^*) \wedge (z_1,y^*)) \leq$$
$$\leq f(\psi(z_1,z_3,y^*),y^*) = \min_{x\in[z_1,z_2,]} f(x,y^*),$$

which is a contradiction. That is $\psi(z_1,z_1,y^*) > z_1$. But on the other hand $m = \lambda(f(z_1,y^*) \vee f(z_1,y^*)) + (1-\lambda)(f(z_1,y^*) \wedge f(z_1,y^*)) \leq f(\psi(z_1,z_1,y^*),y^*)$ hence $\psi(z_1,z_1,y^*) \geq z_2$ which proves the Lemma. ∎

PROOF OF THE THEOREM. Suppose indirectly that the statement of the Theorem is not fulfilled. Denote $M_y := \{(x,y) : f(x,y) = \max_{x\in[0,1]} f(x,y)\}$, $M := \bigcup_{y\in[0,1]} M_y$. It is easy to see that M is closed. According to our indirect assumption, there exists a neighbourhood S of width 2δ of the segment $[(0,0),(1,1)]$ such that $M \cap S = \emptyset$. Denote T_1 and T_2 the two connected components of $[0,1] \times [0,1] \setminus S$. Let $y_0 \in [0,1]$ be such that $M_{y_0} \cap T_1 \neq \emptyset$ and $M_{y_0} \cap T_2 \neq \emptyset$. Then $\exists\ x_1$: $x_1 = \max\{x : (x,y_0) \in M_{y_0} \cap T_1\}$, $\exists\ x_2$: $x_2 = \min\{(x,y_0) \in M_{y_0} \cap T_2\}$ because $M_{y_0} \cap T_1$ and $M_{y_0} \cap T_2$ are closed. Obviously, for any $x \in (x_1,x_2)$ we have $(x,y_0) \notin M_{y_0}$. If $\psi(x_1,x_1,y_0) > x_1$, then $\psi(x_1,x_1,y_0) \geq x_2$ because, according to $x_1 \in M_{y_0}$ we have in this case

$$\max_{x\in[0,1]} f(x,y_0) = \lambda(f(x_1,y_0) \vee f(x_1,y_0)) + (1-\lambda)(f(x_1,y_0) \wedge f(x_1,y_0)) \leq$$
$$\leq f(\psi(x_1,x_2,y_0),y_0)$$

but $((x_1,y_0),(x_2,y_0)) \cap M_{y_0} = \emptyset$. If $\psi(x_1,x_1,y_0) \le x_1$ then because of the Lemma we have $\psi(x_2,x_3,y_0) \le x_1$, i.e.

1. $\psi(x_1,x_1,y_0) \le x_1$ but then $\psi(x_2,x_2,y_0) \le x_1$ or
2. $\psi(x_1,x_1,y_0) > x_1$ but then $\psi(x_1,x_1,y_0) \ge x_2$.

We prove that in the last case $\psi(x_2,x_2,y_0) \ge x_2$. Suppose indirectly that $\psi(x_2,x_2,y_0) < x_2$, then $\psi(x_2,x_2,y_0) \le x_1$ because for $x \in (x_1,x_2)$ we have $f(x,y_0) < f(x_2,y_0)$ and

$$\max_{x\in[0,1]} f(x,y_0) = \lambda(f(x_2,y_0) \vee f(x_2,y_0)) + (1-\lambda)(f(x_2,y_0) \wedge f(x_2,y_0)) \le$$
$$\le f(\psi(x_2,x_2,y_0),y_0),$$

i.e. in this case $\psi(x_2,x_2,y_0) \le x_1$. We know for this case that $\psi(x_1,x_1,y_0) \ge \ge x_1$. One can prove similarly as above that $\psi(x_1,x_2,y_0) \notin (x_1,x_2)$. Thus $\psi(x_1,x_2,y_0) \le x_1$ or $\psi(x_1,x_2,y_0) \ge x_2$. Suppose e.g. that $\psi(x_1,x_2,y_0) \le x_1$ and denote $g(x) := \psi(x_1,x,y_0) - x$ $(x \in [x_1,x_2])$. $g(x)$ is continuous, $g(x_1) > 0$, $g(x_2) < 0$ hence $\exists\ x_3 \in (x_1,x_2)$: $0 = g(x_3) = \psi(x_1,x_3,y_0) - x_3$. On the other hand $\lambda(f(x_1,y_0) \vee f(x_3,y_0)) + (1-\lambda)(f(x_1,y_0) \wedge f(x_3,y_0)) \le f(\psi(x_1,x_3,y_0),y_0) = = f(x_3,y_0)$, i.e.

$$\lambda f(x_1,y_0) + (1-\lambda) f(x_3,y_0) \le f(x_3,y_0)$$

whence $f(x_1,y_0) \le f(x_3,y_0)$ which is a contradiction, because $x_1 \in M_{y_0}$ and $x_3 \notin M_{y_0}$. We obtain a contradiction also in the second case if we assume that $\psi(x_1,x_2,y_0) \ge x_2$. Thus we have in the second case $\psi(x_2,x_2,y_0) \ge x_2$. Thus we have the following two cases

$\alpha)$ $\psi(x_1,x_1,y_0) \le x_1$ and $\psi(x_2,x_2,y_0) \le x_2$,

$\beta)$ $\psi(x_1,x_1,y_0) \ge x_2$ and $\psi(x_2,x_2,y_0) \ge x_2$,

and it is easy to see that these cases are dual to each other. Denote

$$K := \{y \in [0,1] : \ M_y \cap T_1 \ne \emptyset \quad \text{and} \quad M_y \cap T_2 \ne \emptyset\},$$
$$K_1 := \{y \in K : \alpha) \text{ is fulfilled}\}$$
$$K_2 := \{y \in K : \beta) \text{ is fulfilled}\}.$$

Now we prove: K_1 is closed. Let $(y_n) \subset K_1$ and $y_0 \in [0,1]$ be such that $y_n \to y_0$. Let $x_n := \max\{x : (x,y_n) \in M_{y_n} \cap T_1\}$. By the Bolzano–Weierstrass theorem there exists a subsequence $(x_{n_i}) \subset (x_n)$ such that $x_{n_i} \to x_0$ for some $x_0 \in [0,1]$. M is closed and $(x_{n_i},y_{n_i}) \to (x_0,y_0)$ hence $(x_0,y_0) \in M$ hence $x_0 \in M_{y_0}$. Because $(y_{n_i}) \subset K_1$ hence $\psi(x_{n_i},x_{n_i},y_{n_i}) \le x_{n_i}$, so because of $\psi(x_{n_i},x_{n_i},y_{n_i}) \to \psi(x_0,x_0,y_0)$ and $x_{n_i} \to x_0$ we have $\psi(x_0,x_0,y_0) \le x_0$. According to the Lemma $\psi(x^*,x^*,y_0) \le x^*$, where $x^* := \max\{x : (x,y_0) \in \in M_{y_0} \cap T_1\}$, whence $y_0 \in K_1$ because $y_0 \in K$ by the Bolzano–Weierstrass theorem. Thus we have proved that K_1 is closed. One can prove similarly that K_2 is closed.

Denote

$$L_1 := \{y \in [0,1]: \ M_y \cap T_1 \neq \emptyset \quad \text{and} \quad M_y \cap T_2 = \emptyset\}$$
$$\text{and} \quad L_2 := \{y \in [0,1]: \ M_y \cap T_2 \neq \emptyset \quad \text{and} \quad M_y \cap T_1 = \emptyset\}$$

furthermore $N_1 = K_1 \cup L_1$ and $N_2 = K_2 \cup L_2$. We shall see that N_1 is closed, it can be seen in a similar way that N_2 is also closed. As $N_1 = K_1 \cup L_1$, hence according to the above arguments we have only to see that $(y_n) \subset L_1$ and $y_n \to y_0$ implies $y_0 \in N_1$.

Let $x_n := \max\{x : \ (x,y_n) \in M_{y_n} \cap T_1\}$. From Bolzano–Weierstrass theorem comes the existence of a subsequence (x_{n_i}) of (x_n), so that $x_{n_i} \to x_0$ for some $x_0 \in [0,1]$. Because of $(x_{n_i},y_{n_i}) \in T_1$, $(x_0,y_0) \in T_1$ holds true. The closedness of M and $\{x_{n_i},y_{n_i}\} \to \{x_0,y_0\}$ lead to $(x_0,y_0) \in M$, and hence $x_0 \in M_{y_0}$, and so $M_{y_0} \cap T_1 \neq \emptyset$. If $M_{y_0} \cap T_2 = \emptyset$, then $y_0 \in L_1 \subset N_1$, and we are ready. Further on suppose that $M_{y_0} \cap T_2 \neq \emptyset$. Because of $y_{n_i} \in L_1$ $M_{y_{n_i}} \cap T_2 = \emptyset$ and because of $x_{n_i} = \max\left\{x, (x,y_{n_i}) \in M_{y_{n_i}} \cap T_1\right\}$ $\psi\left(x_{n_i},x_{n_i},y_{n_i}\right) \leq x_{n_i}$; from the continuity of ψ $\psi\left(x_{n_i},x_{n_i},y_{n_i}\right) \to \psi(x_0,x_0,y_0)$ and $x_{n_i} \to x_0$ follow. So $\psi(x_0,x_0,y_0) \leq x_0$.

It is sufficient to show that $\psi(x^*,x^*,y_0) \leq x^*$ where $x^* := \max\{x : (x,y_0) \in M_{y_0} \cap T_1\}$, since in this case $y_0 \in K_1$. If it is not true, then $\psi(x^*,x^*,y_0) \geq x^{**}$ (where $x^{**} = \min\{x : \ (x,y_0) \in M_{y_0} \cap T_2\}$), and there are two cases. Let $f(u,y_0) = \min\limits_{x_0 \leq x \leq x^{**}} f(x,y)$.

a) If $\psi(x_0,x^*,y_0) \leq u$, then $\exists\, x' \in [x_0,x^*]$ so that $\psi(x',x^*,y_0) = u$, and hence $f(u,y_0) \geq \lambda f(x^*,y_0) + (1-\lambda) f(x',y_0)$.

b) If $\psi(x_0,x^*,y_0) \geq u$, then $\exists\, x'' \in [x_0,x^*]$ so that $\psi(x_0,x'',y_0) = u \Rightarrow f(u,y_0) \geq \lambda f(x_0,y_0) + (1-\lambda) f(x'',y_0)$.

We arrived at contradictions in both cases. So $\psi(x^*,x^*,y_0) \leq x^*$ holds indeed, and thus $y_0 \in K_1 \subset N_1$.

In this way we saw the closedness of N_1. Closedness of N_2 can be proved similarly. Since $K_1 \cap K_2 = \emptyset$ and $L_1 \cap L_2 = \emptyset$ thus $N_1 \cap N_2 = \emptyset$ holds. As $\forall\, y$ $M_y \neq \emptyset$, so $N_1 \cup N_2 = K_1 \cup L_1 \cup K_2 \cup L_2 = [0,1]$. In this way either $N_1 = \emptyset$ and $N_2 = [0,1]$ or $N_2 = \emptyset$ and $N_1 = [0,1]$ are true. Both cases contradict the existence of zone S, i.e. the indirect assumption.

This contradiction proves our theorem. ∎

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SOME DECOMPOSITION THEOREMS FOR QUASI-REFLEXIVE RINGS

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1. Introduction and preliminaries

Throughout this paper, R will denote a nonzero associative ring. If A and B are non-empty subsets of R, then, the *product* AB will mean the set of all finite sums of the form $\sum a_i b_i (a_i \in A, b_i \in B)$. A ring R is called *quasi-reflexive* (cf. [4]) if, and only if, $AB = 0 \Rightarrow BA = 0$ holds for all two-sided ideals, left ideals, right ideals A and B of R. A ring is called *semiprime* if no nonzero one-sided ideal of it is nilpotent. A non-empty subset X of a ring is called *nilpotent* if there exists a positive integer n such that $X^n = 0$. If $X^2 = X$ holds, then X is called *idempotent*. An additive subgroup Q of a ring R is called a *quasi-ideal* of R if, and only if, $QR \cap RQ \subseteq Q$ holds. (cf. [2]). A nonzero two-sided (left, right, quasi-) ideal Q of a ring R [semigroup with absorbing element 0] is called *minimal* [*0-minimal*] if it does not contain any nonzero two-sided (left, right, quasi) ideal. In [3] we call the sum of all idempotent minimal right ideals of a ring R the *right n-socle* of R. In this paper we denote the right n-socle of ring R by $SocR$. Dually one defines the *left n-socle* of R, and denote it by $_l SocR$. In particular, these two notions coincide in a quasi-reflexive ring [cf. [3], Prop. 3.2]. A ring R is called *n-atomic* if R is generated by its right n-socle. Likewise we call a ring R *left n-atomic* if it is generated by its left n-socle. (We define $Soc\ R$ to be equal to zero if a ring R contains no idempotent minimal right ideals. Similarly for the left case). A ring [semigroup with 0] R is called *simple* [*0-simple*] iff $R^2 \neq 0$ and $\{0\}$ is the only proper two-sided ideal of R. To simplify our terminology, throughout this paper we will write IMR (IML) ideal rather than idempotent minimal right (left) ideal.

It is interesting to know that the form of an IMR (IML) ideal of a ring R can be described in a natural way, indeed. To attain it we use the construction method of J. B. DERR and P. S. PECK (cf. [1]) which amounts to define a new ring from an arbitrary ring A with identity 1_A. We start with an arbitrary index set $\wedge$, and let λ_0 be any specified element of it. Define the ring $F_\wedge(A)$ as the set consisting of all functions $f : \wedge \to A$ for which their corresponding image $f(\wedge)$ contains at most a finite number of nonzero elements. The sum and multiplication of two elements f and g of $F_\wedge(A)$ are defined by

$$(f+g)\lambda = f(\lambda)+g(\lambda)$$
$$(f\cdot g)\lambda = f(\lambda_0)\cdot g(\lambda) \quad \text{for all} \quad \lambda \in F_\wedge(A).$$

Note the left unities of this ring are the functions u of the ring $F_\wedge(A)$ such that $u(\lambda_0) = 1_A$.

REMARK. Due to Thm. 2 of [1], any IMR ideal I of a ring R is isomorphic to a ring of the type $F_\wedge(D)$ where D denotes the division ring $I/{}^aI$. (${}^aI$ is the set of all left annihilators of I in I).

Two idempotent elements u and v of a ring are called orthogonal if $uv = vu = 0$. Following O. Steinfeld [2] we say that the quasi-ideals $Q_{\gamma\delta}$ $(\gamma \in \Gamma, \delta \in \Delta)$ of a ring R form a *complete system* K if the following three conditions hold:

1) either $Q_{\gamma\delta} = 0$ or $Q_{\gamma\delta}$ is a minimal quasi-ideal of R;

2) for every nonzero $Q_{\gamma\delta}$ there exists idempotents e_γ, f_δ in R such that $Q_{\gamma\delta} = e_\gamma Rf_\delta$;

3) for each finite subset $Q_{11}, Q_{12}, \ldots, Q_{1k}; Q_{21}, \ldots, Q_{2k}; \ldots; Q_{l1}, Q_{l2}, \ldots, Q_{lk}$ of K there exists orthogonal idempotents g_i and h_i $(i = 1,2,\ldots,r \leq k;$ $j = 1,2,\ldots,s \leq l)$ such that $\sum_{x=1}^{k}\sum_{y=1}^{l} Q_{xy} \subseteq \sum_{i=1}^{r}\sum_{j=1}^{s} g_i Rh_j$ where all the g_iRh_j are either zero or minimal quasi-ideals of R with the property $(g_iRh_j)(h_jRg_i) = g_iRg_i$. A *finite complete system* of R is defined analogously [cf.[2],p.63]. Finally, a finite complete system of quasi-ideals $Q_{11}, Q_{12}, \ldots, Q_{nn}$ of a ring R is called *homogeneous* if every $Q_{ik} (1 \leq i,j \leq n)$ is a minimal quasi-ideal of R. (cf. [2], p.67) The following fact will be used constantly: *every minimal one-sided ideal I of a ring R such that $I^2 \neq 0$ is generated by some idempotent in I* (cf.[2],p.37). Notice that much of the material in this paper is based on corresponding results for semiprime rings (cf.[2],§8]. In fact, for our purpose we extend all relevant results in that section of [2] to the quasi-reflexive case. The dual of these results can be obtained by left-right symmetry. Furthermore we emphasize that the class of semiprime rings forms a proper subclass of the class of quasi-reflexive ones. (cf.[3], Prop. 2.1).

2. Quasi-Reflexive rings generated by their one-sided n-socles.

For the sake of easy reference we list the following useful result [cf.[3], Thm. 5.8].

THEOREM 2.1. *The n-socle $SocR = \sum_{i\in\lambda} e_i R$ of a quasi-reflexive ring R is a direct sum of its n-homogeneous components. In particular, each component is a simple n-atomic ring.*

REMARK. We call the sum of all R-isomorphic IMR ideals of ring R a n-homogeneous component of $SocR$. (cf.[3]).

We begin this section by showing that any quasi-reflexive ring such that it is equal to the sum of its IML ideals has at least four characterizations. Notice that we prove the next result without making use of the generalized version of Prop. 8.5 of [2] to the quasi-reflexive case (See Prop. 3.4 below). In the semiprime case that result is of vital importance.

THEOREM 2.2. (cf.[2], Thm. 8.1). *The following statements about a ring R are equivalent.*

i) R is a quasi-reflexive ring and it is generated by its left n-socle ${}_lSoc\ R$.

ii) R is the direct sum of two-sided ideals $I_\lambda(\lambda \in \wedge)$ such that every I_λ is a simple subring of R containing at least one minimal left ideal.

iii) R is the sum of such quasi-ideals of R which form a complete system K.

iv) R is a semiprime ring and the sum of its minimal quasi-ideals.

v) R is quasi-reflexive and it is generated by its right n-socle $Soc\ R$.

PROOF. $i \Rightarrow ii)$ is a direct consequence of the dual of Thm. 2.1. For $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$ we refer to the proofs given in [2].

Assume R satisfies iv). Due to Thm. 7.2 of [2] each minimal quasi-ideal Q_i of R has the form $L_i \cap J_i$ where L_i is a minimal left ideal and J_i a minimal right ideal of R. Then $R = \sum Q_i$ implies $R = \sum(L_i \cap J_i)$. Therefore let $r \in R$, so $r = r_1 + r_2 + \ldots + r_n$, where $r_i \in J_i$ Hence $R \subseteq \sum J_i \subseteq R$ and $R = \sum J_i$. This means $R = Soc\ R$. Clearly R is quasi-reflexive. This concludes statement v). The implication $v) \Rightarrow i)$ is a direct consequence of Prop. 3.2 of [3].

COROLLARY 2.3. (cf.[2], Cor. 8.8). *The following statements about a ring R are equivalent.*

i) R is a quasi-reflexive ring and it is generated by its IML ideals which are R-isomorphic R-modules.

ii) R is a simple ring containing at least one minimal left ideal.

iii) R is the sum of such of its quasi-ideals which form a homogeneous complete system.

iv) R is a semiprime ring, it is the sum of its minimal left ideals which are R-isomorphic R-modules.

PROOF. To obtain $i) \Rightarrow ii)$ we apply $i)$ of the preceding result on the socle of R. For $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$ we refer to the proof of Cor. 8.8 of [2]. Obviously, $iv) \Rightarrow i)$.

3. Finitely generated quasi-reflexive rings

In this section we consider quasi-reflexive rings which are generated by a finite number of their idempotent minimal one-sided ideals. The next result improves Thm. 8.7 of [2].

THEOREM 3.1. *The following statements about a ring R are equivalent.*

i) R is a quasi-reflexive ring and it is generated by a finite number of its IMR ideals.

i') R is a quasi-reflexive ring and it is the sum of a finite number of its IML ideals.

ii) R is the direct sum of a finite number of its two-sided ideals $S_1, \ldots, S_r$ such that every S_j is a simple subring of R and each S_j is the sum of its minimal right ideals.

ii') Statement ii) with "right" replaced by "left".

iii) R is the sum of such quasi-ideals of R which form a finite complete system.

iv) R is semiprime and it is the sum of a finite number of its minimal quasi-ideals.

PROOF. Let $R = \sum_{i=1}^{k} I_i$ where each I_i is an IMR ideal of R. Thus the (right) n-socle $Soc\ R = \sum_{i=1}^{k} I_i \neq 0$. It is a routine exercise to check that the relation Γ, defined by $i\Gamma j$ iff I_i and I_j are R-isomorphic R-modules, is an equivalence relation on the set of IMR ideals $\{I_i | i = 1,2,\ldots,k\} = \mathcal{I}$. Let S_α denotes the sum of the IMR ideals in $\mathcal{I}$ belonging to the equivalence class K_α of K mod Γ where $K = \{1,2,\ldots,k\}$. Thus $Soc\,R = \sum_{\alpha \in \Delta} S_\alpha = R$ where

Δ indexes the set of classes K_α. The proof of $i) \Rightarrow ii)$ is now an adaptation of the arguments used in the proof of Thm. 5.8 of [3]. Here each S_α is a two-sided ideal of R and each S_α is a simple subring which is a finite sum of its minimal right ideals. In particular the sum $\sum_{\alpha\in\Delta} S_\alpha$ is direct. For $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$ we refer to the proof of Thm. 8.7 of [2]. The implication $iv) \Rightarrow i)$ is proved in the same manner we proved implication $iv) \Rightarrow v)$ of Thm. 2.2. Finally, the equivalence of $i)$, $ii)$, $iii)$ and $iv)$ implies that of $i')$, $ii')$, $iii)$ and $iv)$.

COROLLARY 3.2. *A quasi-reflexive ring R such that it is generated by a finite number of its IMR ideals is right artinian if, and only if, it is left artinian.*

PROOF. Ring R satisfies condition $i)$ and $iv)$ of the preceding theorem, iff R satisfies condition (a) of Thm. 8.7 of [2] iff R is semiprime and right artinian iff R is semiprime and left artinian (cf [2], Remark 2, p. 64). Hence the result.

We may also state the following corollary.

COROLLARY 3.3. *The following conditions on a ring R are equivalent.*

i) R is a semiprime ring, $R = Soc\ R$ where $Soc\ R$ is generated by a finite number of its minimal right ideals which are R-isomorphic R-modules.

ii) R is a quasi-reflexive ring, $R = Soc\ R$ where $Soc\ R$ is generated by a finite number of its IMR ideals which are R-isomorphic R-modules.

iii) R is a simple ring and it is generated by a finite number of its minimal left ideals.

iv) R is generated by its quasi-ideals which form a homogeneous finite complete system.

PROOF. The implication $i) \Rightarrow ii)$ is obvious. Assume $ii)$ holds.

Let $R = \sum_{i=1}^{k} I_i$ where each $I_i = e_i R (e_i^2 = e_i)$ satisfies the hypothesis of $ii)$. Consider the set $S = \bigcup_{i\in K} e_i R$ where K denotes the set $\{1,2,\ldots,k\}$. We prove firstly S is a multiplicative semigroup. Let c, $d \in S$, so $c \in e_i R$ and $d \in e_j R$ for some i, $j \in K$. By Prop. 6.12a of [2] one has $e_i R e_j \neq 0$. Consequently, $e_i R e_j R = e_i R$. Hence $cd \in e_i R$ whence $cd \in S$. Furthermore, let J be any nonzero ideal of S, and let $x \in J$, $y \in S$, Then $x = e_i x$ and $y = e_j y$ for some idempotents e_i and e_j in S. Due to Prop. 3 $ii)$ of [3], $xR = e_i R$. Moreover, $e_i R$ and $e_j R$ are right similar (cf [2], p.23). Hence by Prop. 4.9 of [2] there exists nonzero elements a, $b \in S$ such that $ab = e_i$ and $ba = e_j$. Then

$y = e_j y = (ba)^2 y = b(ab)ay = be_i ay$ belongs to J since $(e_i a)y \in J$. Thus $J = S$ proving S is 0-simple. Suppose I is any ideal of ring R. Apply the method used in the proof of Thm. 5.8 of [3] to attain $I \cap S \neq 0$. Hence $I \cap S = S$ and so $S \subseteq I$. Subsequently $R \subseteq I$ and therefore $R = I$. This proves R is simple. Finally the proofs of $iii) \Rightarrow iv)$ and $iv) \Rightarrow i)$ the similar to the corresponding proofs of $b^* \Rightarrow c^*$ and $c^* \Rightarrow a^*$ of Cor. 8.9 of [2] respectively and we therefore omit it.

We close this paper with the following improvements of Prop. 8.5 and Cor. 8.6 of [2] in the quasi-reflexive case.

PROPOSITION 3.4. *Let e and f be two nonzero idempotent elements of the quasi-reflexive ring R such that $eR \cap fR = 0$. Suppose fR is a minimal right ideal of R, then, the following assertions hold:*

i) Rf is a minimal left ideal of R; ii) $A = eR \oplus fR = eR \oplus gR$, $eg = 0$ hold for some element $0 \neq g \in R$; iii) gR is an IMR ideal of R; iv) $gR = kR$ holds for some idempotent k and such that $g + k$ is a left identity element of the right ideal A of R. Moreover, $ek = ke = 0$.

REMARK. Note that parts of the proofs of $ii)$ and $iii)$ are similar to the corresponding duals of the proof of Prop. 8.5. of [2].

PROOF. $i)$ is a direct consequence of Prop. 4.1 of [4]. To prove $ii)$ note $f \neq ef$ iff $g = f - ef \neq 0$. Hence $gf = g$ and $f = g + ef$. Let $b \in A$, so $b = ea_1 + fa_2$ where a_1, $a_2 \in R$. Then $b = ea_1 + (g + ef)a_2 = e(a_1 + fa_2) + + ga_2 \in eR + gR$. Thus $A \subseteq eR + gR$. Since $eg = 0$, $eR \oplus gR$ exists. Evidently $eR \oplus gR \subseteq A$. This proves $A = eR \oplus gR$. $iii)$. Define the nonzero mapping $\varphi : gR \to fR$ by $\varphi(ga) = fa (a \in R)$. Now $ga = ga'$ yields $ga = fa - efa = = fa' - efa'$ whence $fa - fa' = efa - efa'$. So $fa - fa' \in eR \cap fR = 0$ implies $fa = fa'$. Thus φ is well-defined. Evidently, φ is an R-homomorphism. It is easy to check φ is bijective. Hence gR is a minimal right ideal of R. We next show $(gR)^2 \neq 0$. This can be done by proving firstly $Rg \neq 0$ by contradiction. Suppose therefore $Rg = 0$, then $RgR = 0$. Consequently, $gRR = 0$ whence $gf = 0$ which is impossible since $gf = g \neq 0$. Hence $0 \neq Rg = Rgf \subseteq Rf$. The minimality of Rf (cf $i)$) ensures $Rg = Rf$. So $f = ag$ holds for some $a \in R$. Then $g = gf$ implies $g = gag = ga \cdot gf$. Hence $(gR)^2 \neq 0$. So $gR = hR$ holds for some idempotent h in gR. Let $h = gu (u \in R)$, then $eh = egu = 0$ by $ii)$ above. Put $k = h - he$. Since $kh = h - heh = h \neq 0$ this proves $kR \neq 0$. Hence $gR = hR = kR$. For the first part of the proof of $iv)$ we refer to the dual of the corresponding part of the proof of Prop. 8.5 of [2]. The second part follows easily from above considerations.

REMARK. A special case of the least result arises when one assumes the idempotent generated right ideal eR is minimal in R. We proceed to extend this special case to the case where we suppose A is generated by $r \geq 3$ different IMR ideals $I_i (i = 1,2,\ldots,r)$ of a quasi-reflexive ring. For each I_i one writes $I_i = f_i R$ for some suitable idempotent f_i in I_i. So we formulate

COROLLARY 3.5. *Let R be a quasi-reflexive ring, and let A be the right ideal of R described in above remark. Then there exist $s(< r)$ orthogonal idempotents $e_j (j = 1,\ldots,s)$ such that*

$$A = \sum_{i=1}^{r} I_i = \sum_{j=1}^{s} e_j R \quad (s < r, r \geq 3)$$

where each IMR ideal $e_j R$ of R is isomorphic to a ring of the form $F_\Lambda(D_j)$ where $D_j = {}^{e_j R}/a_{(e_j R)}$ is a division ring.

PROOF. The method of proof is that of Cor. 8.6 of [2]. For the sake of completeness we write out the proof. Consider $A = \sum_{i=1} f_i R$. For the case $r = 2$ we refer to remark above. So let $r \geq 3$. Assume the hypothesis of the corollary is true for $r - 1$ components. Then there exist orthogonal idempotents e_j in R such that $B = \sum_{i=1}^{r} f_i R = \sum_{j=1}^{t} e_j R (t \leq r - 1)$ and $g = e_1 + \ldots + e_t$ is a left identity of B. Furthermore such $e_j R$ represents a minimal right ideal of R. Since $B = gB \subseteq gR \subseteq B$ one has $B = gR$ whereas $A = B + f_r R = gR + f_r R$. If $f_r R \subseteq gR$, then $A = gR$ and the proof is completed. On the other hand assume $f_r R \nsubseteq gR$. This yields $f_r R \cap gR = 0$ and Prop. 3.4 iv) above is applicable to these right ideals. Consequently there exists an idempotent e_r in R such that $A = gR + f_r R = gR + e_r R$ where $ge_r = e_r g = 0$. Furthermore $e_j g = ge_j = e_j$ whereas $e_j e_r = e_j g e_r = 0$ and $e_r e_j = e_r g e_j = 0$. Due to part *iii*) above, $e_r R$ is a minimal right ideal of R where $e_1 + \ldots + e_t + e_r$ is a left identity of A. For the rest of the proof we refer to *Remark* of §1 above.

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PERIODIC TRAJECTORIES FOR EVOLUTION INCLUSIONS ASSOCIATED WITH TIME DEPENDENT SUBDIFFERENTIALS

By

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1. Introduction.

The purpose of this paper is to study the existence of periodic trajectories for evolution inclusions of the subdifferential type. Our work here partially extends the very recent work of VRABIE [23], who studied evolution equation driven by a time independent m-accreative operator. Of course his result concerns integral solutions, while out formulation of the problem using a subdifferential operator allows us to conclude the existence of strong periodic solutions. So our result constitutes a two-fold generalization of the work of VRABIE [23]. First we allow the vector field $F(t,x)$ to be multivalued, which is important in many applications, like obstacle problems (see CHANG [9]), optimal control (see AHMED [1] and PAPAGEORGIOU [21]), in mathematical economics (see AUBIN–CELLINA [2] and HENRY [12]) and in theoretical mechanics (see MOREAU [16]). Second, our unbounded multivalued operator operator is time dependent, as opposed to VRABIE who deals with a time invariant operator. Our approach in establishing the existence of a periodic trajectory is similar to that of VRABIE, who in turn followed the same line of reasoning as in the earlier seminar work of BECKER [5]).

The problem of existence of periodic trajectories of evolution equations was studied in the past by several authors who concentrated their efforts on semilinear systems. We refer to the works of BECKER [5], BROWDER [8] (who has a nonlinear strongly accreative perturbation term F) and PRÜSS [22] (who used a Nagumo-type tangential condition). We should also mention the important nonlinear work of YAMADA [27], who had a constant perturbation term (i.e. $F(t,x) = F(t)$), but had a periodically moving boundary. Finally, there are also the works of DEIMLING [10] and LIGHTBURNE [15], which deal with differential equations in Banach spaces. However, their formulation of the problem does not allow the presence of unbounded operators and so it precludes the applicability of their work to partial differential equations. To our knowledge, there hasn't been any work on periodic trajectories of evolution inclusions. Only differential inclusions in $\mathbb{R}^n$ were treated by AUBIN–CELLINA [2] (theorem 4, p.237) and HADDAD–LASRY [12] and differential inclusions in Banach spaces were studied by PAPAGEORGIOU [19]. All these works used flow invariance arguments on a time invariant constraint set, involving the usual Nagumo-type tangential condition and assumed that the orientor field (multivalued vector field) $F(t,x)$ is convex valued. Extensions of this approach to finite dimensional systems with time-varying constraints can be found in GUSEINOV [29], theorem 4.1. This approach is of course analogous to the one used by PRÜSS [22]. Finally we should also mention the interesting work of MACKI–NISTRI–ZECCA [30], who consider semilinear differential inclusions in $\mathbb{R}^N$ with $N =$ odd and using degree theoretical techniques, proved the existence of a periodic trajectory, when the orientor field $F(t,x)$ is measurable in t, *u.s.c.* in x and has different growth behavior for $|x|$ small and $|x|$ large. Here we consider a multivalued system monitored by a subdifferential evolution inclusion. Our approach uses a growth condition imposed on the orientor field, which eventually guarantees that the Poincarré map maps a ball into itself. Hence since it is compact it has a fixed point.

2. Preliminaries.

Let (Ω,Σ) be a measurable space and X a separable Banach space. By $P_{f(c)}(X)$ we will denote the collection of all nonempty, closed (and convex) subsets of H. A multifunction $F:\Omega \to 2^X\{\emptyset\}$ is said to be "graph measurable", if $GrF = \{(\omega,x) \in \omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the borel σ-field of X. If $F(\cdot)$ is closed valued (i.e. for all $\omega \in \Omega$, $F(\omega) \in P_f(X)$),then we say that $F(\cdot)$ is measurable is and only if $\omega \to d(x,F(x)) = inf\{||x-z|| :\in F(\omega)\}$ is measurable. For $P_f(X)$-valued multifunctions, measurability implies graph measurability and the two are equivalent if there exists a complete σ-finite measure $\mu(\cdot)$ on (Ω,Σ). For more details on the measurability of multifunctions, we refer to Wagner [24]. By S_F^p $1 \leq p \leq \infty$, we will denote the set of measurable selectors of $F(\cdot)$ that belong in the lebesgue–Bochner space $L^p(X)$; i.e. $S_F^p = \{f \in L^p(X) : f(\omega) \in F(\omega)$ μ- a.e.$\}$. This set may be empty. It is nonempty if and only if $F:\Omega \to 2^X\{\emptyset\}$ is graph measurable and $\omega \to inf\{||x|| : x \in F(\omega)\} \in L_+^p$. This follows from an easy application of Aumann's selection theorem (see Wagner [24], theorem 5.10).

Now let H be a separable Hilbert space and $\varphi : H \to \overline{\mathbb{R}} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is proper, if it is not idetically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and l.s.c. (usually this family of $\overline{\mathbb{R}}$-valued functions on H, is denoted by $\Gamma_0(H)$). By $dom\varphi(\cdot)$ we will denote the effective domain of $\varphi(\cdot)$; i.e. $dom\varphi = \{x \in H : \varphi(x) < \infty\}$. The subdifferential of $\varphi(\cdot)$ at x is the set $\partial\varphi(x) = \{x^* \in H : (x^*, y - x) \leq \varphi(y) - \varphi(x)$ for all $y \in dom\varphi\}$ (here $(\cdot,\cdot)$ denotes the inner product of H; note that by the Riesz–Frechtet theorem, we identify H with its dual (pivot space)). If $\varphi(\cdot)$ is Gateaux differentiable at x, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi(\cdot)$ is of compact type, if for every $\lambda \in \mathbb{R}$, the level set $\{x \in H : ||x||^2 + \varphi(x) \leq \lambda\}$ is compact (using the terminology of nonsmooth analysis, we can say yhat the convex function $x \to ||x||^2 + \varphi(x)$ is inf-compact.

Finally, a multifunction $G : H \to 2^H\{\phi\}$ is said to be lower semicontinuous (l.s.c), if for all $C \subseteq H$ nonempty, closed $G^+(C) = \{y \in H : G(y) \subseteq C\}$ is closed in H. In fact this is equivalent to saying that if $y_n \to n$ in H, then $G(y) \subseteq \underline{\lim} G(y_n) = \{z \in H : \lim d(z, G(y_n)) = 0\}$. Also on $P_f(H)$ we can define a generalized metric known as the Hausdorff metric, by setting $h(A, B) = \max\left[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right]$.

3. Periodic trajectories.

Let $T = [0, b]$ and H a separable Hilbert space. The problem under consideration is the following:

$$(\underline{1}) \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial(t, x(t)) + F(t, x(t)) \text{ a.e.} \\ x(0) = x(b) \end{array} \right\}$$

By a "strong solution" of ($\underline{1}$), we mean a functyion $x(\cdot) \in C(T, H)$ which is strongly absolutely continuous on $(0, b)$, $x(t) \in dom\partial\varphi(t,\cdot) = \{z \in H : \partial\varphi(t, z) \neq \emptyset\}$ a.e. and satisfies $-\dot{x} \in \partial\varphi(t, x(t)) + f(t)$ a.e. $x(0) = x(b)$, with $f \in L^2(H)$, $f(t) \in F(t, x(t))$ a.e. (i.e. $f(\cdot) \in S^2_{F(\cdot, x(\cdot))}$). Recall that since H is a Hilbert space, an absolutely continuous function from T into H is almost everywhere strongly differentiable (see for example Barbu [4] or Brezis [7]).

We will need the following hypotheses on the data of ($\underline{1}$):

$\underline{H(\varphi)}$: $\varphi : T \times H \to \underline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function s.t.

($\underline{1}$) for every $t \in T$, $\varphi(t,\cdot)$ is a proper, convex, l.s.c. (i.e. $\varphi(t,\cdot) \in \Gamma_0(H)$) and of compact type,

($\underline{2}$) for every positive integer r, there exists a constant K_r, an absolutely continuous function $g_r : T \to \mathbb{R}$ with $\dot{g}_r \in L^\beta(T)$ and a function of bounded variation $h_r : T \to \mathbb{R}$ s.t. if $t \in T$, $x \in dom\varphi(t,\cdot)$ with $||x|| \leq r$ and $s \in [t.b]$, then there exists $\hat{x} \in dom\varphi(s,\cdot)$ satisfying

$$||\hat{x} - x|| \leq |g_r(s) - g_r(t)|(\varphi(t,x) + K_r)^{\alpha}$$
$$\text{and } \varphi(s,\hat{x}) \leq \varphi(t,x) + |h_r(s) - h_r(t)| \cdot (\varphi(t,x) + K_r)$$

where $\alpha \in [0,1]$ and $\beta = 2$ if $\alpha \in [0,1/2]$, $\beta = \frac{1}{a-\alpha}$ if $\alpha \in [1/2,1]$,

(3) for every $x,y \in dom\partial\varphi(t,\cdot) = \{z \in H : \partial\varphi(t,z) \neq \emptyset\}$ and for every $x^* \in \partial\varphi(t,x)$, $y^* \in \partial\varphi(t,y)$ we have $(x^* - y^*, x - y) \geq c||x - y||^2$ with $c > 0$,

(4) $dom\varphi(b,\cdot) \subseteq dom\varphi(0,\cdot)$, $0 \in \overline{dom\varphi(0,\cdot)}$ and if $K = \{u = x(0) \in \in H :\ x(\cdot)$ is a solution of $-\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t)$ a.e., $x(0) = x_0$ with $f \in B \subseteq L^2(T,H)$ bounded and $x_0 \in dom\varphi(0,\cdot)$, $||x_0|| \leq \lambda$ for some $\lambda > 0\}$, then $\sup\left[\varphi(0,u) :\ u \in K\right] < \infty$.

REMARK. Hypotheses $H(\varphi)$ (1) and (2), which clearly impose mild restrictions on the t-dependence of $\varphi(t,x)$, were first introduced by YOTSUTANI [28]. They extend earlier ones used in the important works of WATANABE [25] and YAMADA [26].

$H(F)$: $F : T \times H \to P_f(H)$ is a multifunction s.t.

(1) $(t,x) \to F(t,x)$ is a graph measurable,

(2) $x \to F(t,x)$ is l.s.c.,

(3) $|F(t,x)| \leq \psi(t,||x||)$ a.e. with $\psi(\cdot,r) \in L^{\infty}_{+}$ and $\psi(t,\cdot)$ nondecreasing,

(4) $\overline{\lim}_{r\to\infty} \frac{1}{r}\psi(t,r) = m < c$, where $c > 0$ is as in $H(\varphi)$ (3) (here $|F(t,x)| = \sup\{||z|| : z \in F(t,x)\}$).

We will start with an auxiliary result that we will need in the sequel. So let $f_1, f_2 \in L^2(H)$ and $x_0^1, x_0^2 \in \overline{dom\varphi(0,\cdot)}$. Consider the following two Cauchy problems:

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + f_1(t) \text{ a.e., } x(0) = x_0^1$$

and

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + f_2(t) \text{ a.e., } x(0) = x_0^2.$$

Let $x_1, x_2 \in C(T,H)$ be the corresponding unique strong solutions of the above problems. Their existence and uniqueness follows from the theorem of YOTSUTANI [28].

LEMMA 3.1 *If hypotheses $H(\varphi)$ 1→3 hold, then for all $t \in T$, we have*

$$||x_1(t) - x_2(t)|| \leq e^{-ct}||x_0^1 - x_0^2|| + \int_0^t e^{-c(t-s)}||f_1(s) - f_2(s)||ds.$$

PROOF. By hypothesis we have:

$$-\dot{x}_1(t) \in \partial(t, x_1(t)) + f_1(t) \text{ a.e., } x_1(0) = x_0^1$$

and

$$-\dot{x}_2(t) \in \partial(t, x_2(t)) + f_2(t) \text{ a.e., } x_2(0) = x_0^2.$$

Subtracting the second evolution equation from the first, taking the inner product of both sides of the resulting evolution equation with $x_2(t) - x_1(t)$ and recalling that by hypothesis $H(\varphi)$ (3), our subdifferential operator is strongly monotone, we get:

$$(-\dot{x}_1(t) + \dot{x}_2(t), x_2(t) - x_1(t)) + c||x_1(t) - x_2(t)||^2 \leq$$
$$\leq (f_1(t) - f_2(t), x_2(t) - x_1(t)) \text{ a.e.}$$
$$\Rightarrow \frac{1}{2}\frac{d}{dt}||x_1(t) - x_2(t)||^2 + c||x_1(t) - x_2(t)||^2 \leq$$
$$\leq ||f_1(t) - f_2(t)|| \cdot ||x_2(t) - x_1(t)|| \text{ a.e.}$$

Temporarily let $\eta(t) = ||f_1(t) - f_2(t)|| \cdot ||x_1(t) - x_2(t)||$. Clearly $\eta(\cdot) \in L_+^2$. If $\xi(t) = ||x_1(t) - x_2(t)||^2$, we have that

$$\dot{\xi}(t) \leq -2c\xi(t) + 2\eta(t) \text{ a.e.}$$

So by a well-known differential inequality (see for example HALE [11]), we have that

$$\xi(t) \leq e^{-2ct}\xi(0) + 2\int_0^t e^{-2c(t-s)}\eta(s)ds$$
$$\Rightarrow ||x_1(t) - x_2(t)||^2 \leq e^{-2ct}||x_0^1 - x_0^2||^2 +$$
$$+2\int_0^t e^{-2c(t-s)}||f_1(s) - f_2(s)|| \cdot ||x_1(s) - x_2(s)||ds.$$

Applying lemma A.5, p. 157 of BREZIS [7], we get

$$||x_1(t) - x_2(t)|| \leq e^{-ct}||x_0^1 - x_0^2|| + \int_0^t e^{-c(t-s)}||f_1(s) - f_2(s)||ds. \quad \blacksquare$$

Now we can have our result on the existence of solutions for problem (1).

THEOREM 3.2 *If hypotheses $H(\varphi)$ and $H(F)$ hold, then problem (1) admits a strong solution.*

PROOF. Let $K=\{y\in C(T,H):y(0)\in \overline{dom\varphi(0,\cdot)}\}$. Clearly this is a closed and convex subset of $C(T,H)$. Consider the multifunction $R\colon K\to P_f(L^1(H))$ defined by

$$R(y)=S^1_{F(\cdot,y(\cdot))}.$$

From theorem 4.1 of [18], we know that $R(\cdot)$ is l.s.c., with closed and decomposable values (i.e. $f_1, f_2\in S^1_{F(\cdot,y(\cdot))}$ and $A\subseteq T$ measurable, then $\chi_A f_1+\chi_{A^c}f_2\in S^1_{F(\cdot,y(\cdot))}$). So we can apply theorem 3 of BRESSAN–COLOMBO [6] and get $r:K\to L^1(H)$ a continuous map such that $r(y)\in R(y)$ for all $y\in K$. Then consider the following evolution equation:

$$-\dot{x}(t)\in\partial\varphi(t,x(t))+r(y)(t)\ \text{a.e.}$$
$$x(0)=y(0)$$

Since $y(0)\in\overline{dom\varphi(0,\cdot)}$, from YOTSUTANI [28], we know that this problem has a unique strong solution $x(y)(\cdot)\in C(T,H)$. Let $\gamma:K\to H$ be defined by $\gamma(y)=x(y)(b)$. Recall (see the theorem of YOTSUTANI [28]), that for all $0<t\leq b$, $x(y)(t)\in dom\varphi(t,\cdot)$. So using hypothesis $H(\varphi)$ (4), we have that $\gamma(y)=x(y)(b)\in dom\varphi(b,\cdot)\subseteq dom\varphi(0,\cdot)$. Therefore $\gamma(K)\subseteq dom\varphi(0,\cdot)$. Also since $r(\cdot)$ is continuous, using lemma 3.1, we get that $\gamma:K\to H$ is continuous.

Next consider the following evolution equation:

$$-\dot{v}(t)\in\partial\varphi(t,v(t))+r(u)(t)\ \text{a.e.}$$
$$v(0)=\gamma(u)$$

Since $\gamma(u)\in dom\varphi(0,\cdot)$, again from the theorem of Yotsutani [28], we see that the above evolution equation has a unique solution $v(u)(\cdot)\in K\subseteq C(T,H)$. Consider the map $\xi:K\to K$ defined by $\xi(u)=v(u)$. Since both $r(\cdot)$ and $\gamma(\cdot)$ are continuous, using lemma 3.1, we get ξ is continuous. If we can show that $\xi(\cdot)$ has a fixed point, then this will be the desired periodic trajectory of our evolution inclusion.

To this end, first we will show that for some $\lambda>0$, $\xi(K_\lambda)\subseteq K_\lambda$, where $K_\lambda=\{y\in K:||y||_{C(T,H)}\leq\lambda\}$. Suppose not. Then there exists a sequence $\{u_n\}_{n\geq 1}\subseteq K$ such that

$$||u_n||_\infty=n\ \text{and}\ ||\xi(u_n)||\infty>n.$$

Let $v_n=\xi(u_n)$ and let $\overline{x}(\cdot)\in C(T,H)$ be the unique solution of

$$-\dot{x}(t)\in\partial\varphi(t,x(t))\ \text{a.e.}$$
$$x(0)=0$$

Recall that by hypothesis $H(\varphi)$ (4), $0 \in \overline{dom\varphi(0,\cdot)}$. So we can apply lemma 3.1 and get

$$||v_n(t) - \overline{x}(t)|| \le e^{-ct}||\gamma(u_n)|| + \int_0^t e^{-c(t-s)}||r(u_n)(s)||ds$$

$$\Rightarrow ||v_n(t)|| \le ||\overline{x}||_\infty + e^{-ct}||\gamma(u_n)|| + \int_0^t e^{-c(t-s)}||r(u_n)(s)||ds.$$

Let $t_n \in T$, $n \ge 1$ be such that $||v_n(t_n)|| = ||v_n||_\infty$. By passing to a subsequence if necessary, we may assume $t_n \to \tau$. Also note from the lemma, we have

$$||\gamma(u_n)|| \le e^{-cb}||u_n(0)|| + \int_0^b e^{-c(b-s)}||r(u_n)(s)||ds.$$

So we get

$||v_n||_\infty = ||v_n(t_n)|| \le$

$$\le ||\overline{x}||_\infty + e^{-ct_n}e^{-cb}||u_n(0)|| + e^{-ct_n}\int_0^b e^{-c(b-s)}||r(u_n)(s)||ds +$$

$$+ \int_0^{t_n} e^{-c(t-s)}||r(u_n)(s)||ds$$

$$\Rightarrow 1 < \frac{||v_n||_\infty}{n} = \frac{||v_n||_\infty}{||u_n||_\infty} \le \frac{||\overline{x}||_\infty}{||u_n||_\infty} + e^{-ct_n}e^{-cb} +$$

$$+ \frac{e^{-ct_n}}{||u_n|| - \infty}\int_0^b e^{-c(b-s)}||r(u_n)(s)||ds +$$

$$+ \frac{1}{||u_n||_\infty}\int_0^{t_n} e^{-c(t-s)}||r(u_n)(s)||ds \le$$

$$\le \frac{||\overline{x}||_\infty}{||u_n||_\infty} + e^{-ct_n}e^{-cb} + \frac{e^{-ct_n}||r(u_n)||_\infty}{||u_n||_\infty}\frac{1}{c}(1 - e^{-cb}) +$$

$$+ \frac{||r(u_n)||_\infty}{||u_n||_\infty}\frac{1}{c}(1 - e^{-ct_n})$$

(hypothesis $H(F)$ (3)).

Letting $n \to \infty$, in the limit we get using hypothesis $H(F)$ (4):

$$1 \leq e^{-c\tau} e^{-cb} + \frac{e^{-c\tau} m}{c}(1 - e^{-cb}) + \frac{m}{c}(1 - e^{-c\tau}).$$

But note that

$$\begin{aligned} 1 &\leq e^{-c\tau} e^{-cb} + \frac{e^{-c\tau} m}{c}(1 - e^{-cb}) + \frac{m}{c}(1 - e^{-c\tau}) \\ &= e^{-c\tau} e^{-cb}(1 - \frac{m}{c} + \frac{m}{c} < 1 \end{aligned}$$

a contradiction. Therefore there exists $\lambda > 0$ such that $\xi(K_\lambda) \subseteq K_\lambda$.

Next we will show that $\overline{\xi(K_\lambda)}^{C(T,H)}$ is compact in $C(T,H)$. To this let $x \in K_\lambda$ and $y = \xi(x)$. Then by definition

$$\begin{aligned} -\dot{y}(t) &\in \partial\varphi(t, y(t)) + r(x)(t) \text{ a.e.} \\ y(0) &= \gamma(x) \end{aligned}$$

Since $x \in K_\lambda$ and $\xi(K_\lambda) \subseteq K_\lambda$ as we just proved, we have that for all $t \in T$ and all $y(\cdot) \in \xi(K_\lambda)$

$$||y(t)|| \leq \lambda.$$

Also since $y(0) = \gamma(x) \in dom\varphi(b, \cdot) \subseteq dom\varphi(0, \cdot)$, from YOTSUTANI [28], we know that $\dot{y} \in L^2(H)$ and so via the Cauchy–Schwartz inequality, we get for $0 \leq t \leq t' \leq b$

$$\begin{aligned} ||y(t') - y(t)|| &\leq \int_t^{t'} ||\dot{y}(s)|| ds \leq \\ &\leq \left(\int_0^b \chi_{[t,t']}(s)^2 ds \right)^{1/2} \left(\int_0^b ||\dot{y}(s)||^2 ds \right)^{1/2} \leq (t' - t)^{1/2} M_1 \end{aligned}$$

because by lemma 6.11 of YOTSUTANI [28], since $r(K_\lambda) \subseteq L^2(H)$ is bounded (see hypothesis $H(F)$ (3)), we have that $||\dot{y}||_{L^2(H)} \leq M_1$ for some $M_1 > 0$ and all $y \in \xi(K_\lambda)$. Therefore, we get that $\xi(K_\lambda)$ is equicontinuous in $C(T,H)$.

Furthermore from inequality (7.9) of YOTSUTANI [28], we have

$$\begin{aligned} &||y(t)||^2 + \varphi(t, y(t)) \leq \\ &\leq \lambda^2 + M_2(b + ||h_\lambda||_{TV} + ||g_\lambda||_{l^\beta} + ||\psi(\cdot,\lambda)||_\infty b) + \varphi(0, y(0)) \end{aligned}$$

where $M_2 > 0$ is independent of $y \in \xi(K_\lambda)$ and $||h_\lambda||_{TV}$ is the total variation norm of $h_\lambda(\cdot)$. Also because of hypothesis $H(\varphi)$ (4), we have that $\varphi(0,y(0)) \leq M_3$ for some $M_3 > 0$ and all $y \in \xi(K_\lambda)$. Therefore finally there exists $M_4 > 0$ such that for all $t \in T$

$$||y(t)||^2 + \varphi(t,y(t)) \leq M_4$$
$$\Rightarrow \xi(K_\lambda)(t) = \{y(t) : y \in \xi(K_\lambda)\} \subseteq \{z \in H : ||z||^2 + \varphi(t,z) \leq M_4\}$$

and the latter set is compact, since by hypothesis $H(\varphi)$ (1) $\varphi(t,\cdot)$ is compact type (see Section 2). So from the Arzela–Ascoli theorem, we conclude that $\xi(K_\lambda)$ is compact in $C(T,H)$. Apply Schauder's fixed point theorem to get $x = \xi(x) \Rightarrow x(\cdot) \in C(T,H)$ is the desired solution of (1). ∎

4. Periodic optimal control.

In this section we consider the following periodic optimal control problem of Bolza type. The control space is modeled by a separable reflexive Banach space Y.

$$(2) \qquad \left\{ \begin{array}{c} J(x,u) = l(x(0)) + \int_0^b L(t,x(t),u(t))dt \to \inf = \hat{m} \\ \text{s.t.} - \dot{x}(t) \in \partial\varphi(t,x(t))u(t) \text{ a.e.} \\ x(0) = x(b), u(t) \in U(t) \text{ a.e., } u(\cdot) \text{ measurable} \end{array} \right\}.$$

We will need the following hypotheses on the data of (2):

$H(f)$: $f : T \times H \to \mathcal{L}(Y,H)$ is a map s.t.

(1) $t \to f(t,x)u$ is measurable,

(2) $x \to f(t,x)^* v$ is continuous for all $v \in H$,

(3) $||f(t,x)u|| \leq \psi(t,||x||)$ a.e. for all $u \in U(t)$, with $\psi(\cdot,r) \in L_+^\infty, \psi(t,\cdot)$ nondecreasing and $\overline{\lim_{r\to\infty}} \frac{1}{r}\psi(t,r) = m < c$, where $c > 0$ is as in $H(\varphi)$ (3).

$H(U)$: $U : T \to P_{fc}(Y)$ is a measurable multifunction with weakly compact values such that $|U(t)| = \sup\{||u|| : u \in U(t)\} \leq \alpha(t)$ a.e. $\alpha(\cdot) \in L_+^\infty$.

$H(l)$: $l : H \to \mathbb{R}_+$ is a continuous function s.t. $l(x) \to \infty$ as $||x|| \to \infty$.

$H(L)$: $L : T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an integrand s.t.

(1) $(t,x,u) \to L(t,x,u)$ is measurable,

(2) $(x,u) \to L(t,x,u)$ is l.s.c.,

(3) $u \to L(t,x,u)$ is convex,

(4) $\theta(t) - M\|u\| \leq L(t,x,u)$ with $\theta(\cdot) \in L^1$ and $M > 0$.

Since our cost integrand L is $\overline{\mathbb{R}}$-valued, we need the following feasibility hypothesis. By an admissible "state-control" pair, we main a pair $(x,u) \in$ $\in C(T,H) \times L^1(Y)$ satisfying all the constraints of problem (2). Observe that because of theorem 3.1, given $u \in S^1_U$, there exists at least one trajectory $x \in C(T,H)$ generated by that control.

H_0: There exists an admissible state-control pair (x,u) s.t. $J(x,u) < \infty$ (i.e. $\hat{m} < \infty$).

THEOREM 4.1 *If hypotheses $H(\varphi)$, $H(f)$, $H(U)$, $H(l)$, $H(L)$ and H_0 hold, then problem (2) admits a solution; i.e. there exists admissible pair (x,u) s.t. $J(x,u) = \hat{m}$.*

PROOF. Let $\{(x_n,u_n)\}_{n\geq 1} \subseteq C(T,H) \times L^1(Y)$ be a minimizing sequence of admissible state-control pairs i.e. $J(x_n,u_n) \downarrow \hat{m}$ as $n \to \infty$. Since by hypothesis H_0, $\hat{m} < \infty$ and since by hypothesis $H(l)$, $l(\cdot)$ is coercive, we get that $\{x_n(0)\}_{n\geq 1}$ is bounded in H. Knowing this and arguing as in the proof of theorem 3.2, through the Arzela–Ascoli theorem we can establish that $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T,H)$. Also from proposition 3.1 of [17], we know that S^1_U is weakly compact in $L^1(Y)$. Hence by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T,H)$ and $u_n \xrightarrow{w} u$ in $L^1(Y)$. Because of hypothesis $H(l)$, we have that $l(x_n(0)) \to l(x(0))$, Furthermore using theorem 2.1 of BALDER [3], we get that

$$\int_0^b L(t,x(t),u(t))dt \leq \underline{\lim} \int_0^b L(t,x_n(t),u_n(t))dt.$$

Thus we have:

$$l(x(0)) + \int_0^b L(t,x(t),u(t))dt \leq \hat{m}.$$

We will be done, if we can that (x,u) is an admissible state-control pair. For every $n \geq 1$ we have:

$$-\dot{x}_n(t) \in \partial\varphi(t,x_n(t)) + f(t,x_n(t))u_n(t) \text{ a.e.}$$
$$x_n(0) = x_n(b)$$

From the proof of theorem 3.2, we know that $\{\dot{x}_n\}_{n>1}$ is bounded in $L^2(H)$. So we may assume that $\dot{x}_n \xrightarrow{w} y$ in $L^2(H)$. Clearly $y = \dot{x}$. Also because of hypothesis $H(f)$ (2), we have that $f(\cdot,x_n(\cdot))u_n(\cdot)\xrightarrow{w}f(\cdot,x(\cdot))u(\cdot)$ in $L^2(H)$. Define $\Phi : L^2(H)\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ by

$$\Phi(v)=\begin{cases}\int_0^b \varphi(t,v(t))dt & \text{if } \varphi(\cdot,v(\cdot))\in L^1\\ +\infty & \text{otherwise.}\end{cases}$$

The measurability of $t\to\varphi(t,v(t))$ (see lemma 3.4 of YOTSUTANI [28]) guarantees that this is a well-defined functional belonging in $\Gamma_0(L^2(H))$. Furthermore using lemma 4.4 of YOTSUTANI [28], we have

$$[x_n,-\dot{x}_n-f(\cdot,x_n(\cdot))u_n(\cdot)]\in Gr\partial\Phi,\quad n\geq 1.$$

Recall that $Gr\partial\Phi$ is demiclosed in $L^2(H)\times L^2(H)$ (see BREZIS [7]); i.e. $Gr\partial\Phi$ is sequentially closed in $L^2(H)\times L^2(H)_w$ $L^2(H)_w$ denoting the Hilbert space $L^2(H)$ equipped with the weak topology. So in the limit as $n\to\infty$, we get

$$[x,-\dot{x}-f(\cdot,x(\cdot))u(\cdot)]\in Gr\partial\Phi.$$

Using once more lemma 4.4 of YOTSUTANI [28], we get

$$-\dot{x}(t)\in\partial\varphi(t,x(t))+f(t,x(t))u(t) \text{ a.e.}$$
$$x(0)=x(b), u(t)\in U(t)\text{a.e., } u(\cdot) \text{ measurable}$$

$\Rightarrow (x,u)$ is an admissible state-control pair and $J(x,u)=\hat{m}$. ∎

5. Examples

In this section, we illustrate the abstract theory with three examples.

Let $T=[0,b]$ and Z bounded domain in $\mathbb{R}^N$ with boundary $\partial Z=\Gamma$. First we consider the following multivalued parabolic partial differential equation. Systems of this form arise in the study of obstacle problems (see CHANG [9]). In what follows $D_k\frac{\partial}{\partial z_k}$, $k=1,2,\dots,N$.

$$(3)\qquad\left\{\begin{array}{l}\dfrac{\partial x}{\partial t}-\sum_{k=1}^{N}\int_Z D_k(a(t,z)D_k x)+\\ +x\in[f_1(t,z,x(t,z)),f_2(t,z,x(t,z))] \text{ a.e.}\\ x|_{T\times\Gamma}=0,\quad x(0,z)=x(b,z) \text{ a.e.},\quad p\geq 2\end{array}\right\}.$$

The following hypotheses will be needed:

$H(a)$: $a\in L^\infty(T\times Z)$, $|a(t,z)-a(t',z)|\leq l|t-t'|$, $l>0$ and $0<\hat{\beta}\leq a(t,z)$.

$\underline{H(f)_1}$: $f_1, f_2 : T \times Z \times \mathbb{R} \to \mathbb{R}$ are functions s.t.

($\underline{1}$) $(t,z) \to f_1(t,z,x), f_2(t,z,x)$ are measurable,

($\underline{2}$) $x \to -f_1(t,z,x), f_2(t,z,x)$ are l.s.c.,

($\underline{3}$) for every $x(\cdot) \in L^2(Z)$, $\left(\int_Z |f_1(t,z,x(z))|^2 dz\right)^{1/2} \leq \psi(t)\|x\|_2,\ t \in T$ with $\|\psi\|_\infty < \hat{\beta}$.

THEOREM 5.1 *If hypotheses $H(a)$ and $H(f)$ hold, then problem ($\underline{3}$) admits a solution $x \in C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^2(T \times Z)$.*

PROOF. Let $H = L^2(Z)$ and let $\varphi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(t,x) \begin{cases} \frac{1}{2}\sum_{k=1}^{N} \int_Z a(t,z)|D_k x|^2 dz + \frac{1}{2}\int_Z |x|^2 dz & \text{if } x \in H_0^1(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $\varphi(t,\cdot) \in \Gamma_0(H)$. Also note that $\varphi(t,x) \geq \gamma\|x\|^p_{H_0^1(Z)}$ for all $x \in H_0^1(Z)$ and some $\gamma > 0$, and $dom\varphi(t,\cdot) = H_0^1(Z)$ for all $t \in T$. In addition

$$\begin{aligned}|\varphi(t,z) - \varphi(t',x)| &\leq \sum_{k=1}^{N} \frac{1}{2}\int_Z |a(t,z) - a(t',z)| \cdot |D_k x|^2 dz \\ &\leq \gamma'|t-t'|\,\|x\|^2_{H_0^1(Z)}\end{aligned}$$

for some $\gamma' > 0$, by using hypothesis $H(a)$ and the Poincare–Friedrich's inequality. Hence we have

$$\varphi(t,x) \leq \varphi(t',x) + \gamma'|t-t'|\,\|x\|^2_{H_0^1(Z)} \leq \varphi(t',x) + \gamma'|t-t'|\frac{1}{\gamma}\varphi(t,x)$$

and so we have satisfied hypotheses $H(\varphi)$ ($\underline{1}$) and ($\underline{2}$). As in BARBU [4] (proposition 2.9, p. 63), using Green's identity, we can chech that

$$\partial\varphi(t,x) = -\sum_{k=1}^{N} D_k(a(t,z)D_k x) + x = L(t)x$$

with $D(L(t)) = \{x \in H_0^1(Z) : L(t)x \in L^2(Z)\}$.

Using this definition, we can easily check that if $x, y \in H_0^1(Z)$, then

$$\hat{\beta}\|x - y\|_2 \leq (\partial\varphi(t,x) - \partial\varphi(t,y), x - y)_{L^2(Z)}.$$

Finally note that $\varphi(0,\cdot)$ is bounded on K as in $H(\varphi)$ (4) and since by Sobolev's embedding theorem, $H_0^1(Z)$ embeds into $L^2(Z)$ compactly, we get that for every $t \in T$, $\varphi(t,\cdot)$ is of compact type. So we have satisfied hypotheses $H(\varphi)$ ($\underline{3}$) and ($\underline{4}$).

Next let $F(t,z,x) = \{u \in \mathbb{R} : f_1(t,z,x) \leq u_z \leq f_2(t,z,x)\}$. Because of hypotheses $H(f)_1$ ($\underline{1}$), $(t,z,x) \to F(t,z,x)$ is measurable, while from KLEIN-THOMPSON [14] (p. 76), we have that $x \to F(t,z,x)$ is l.s.c. Let $\hat{F} : T \times H \to P_f(H)$ be defined by

$$\hat{F}(t,z) = S^2_{F(t,\cdot,x(\cdot))}.$$

Note that

$$\begin{aligned} Gr\hat{F} = \{t,x,u) \in T \times L^2(Z) \times L^2(Z) : f_1(t,z,x(z)) \leq \\ \leq u(z) \leq f_2(t,z,x(z)) \text{ a.e. on } Z\} = \\ = \{(x,u) \in L^2(Z) \times L^2(Z) : \int_A f_1(t,z,x(z))dz \leq \\ \int_Z u(z)dz \leq \int_Z f_2(t,z,x(z))dz \text{ for all } A \in B(Z)\} \end{aligned}$$

where $B(Z)$ is the Borel σ-field of Z. Since $B(Z)$ is countably generated, we deduce that $Gr\hat{F} \in B(T) \times B(L^2(Z)) \Rightarrow \hat{F}(\cdot,\cdot)$ is graph measurable.

Furthermore from theorem 4.1 of [18], we have that $x \to \hat{F}(t,x)$ is l.s.c. Also using hypothesis $H(f_1)$ ($\underline{3}$), we can easily check that hypotheses $H(F)$ ($\underline{3}$) and ($\underline{4}$) are verified.

Next rewrite ($\underline{3}$) in the following equivalent evolution inclusion form:

$$(\underline{3})' \qquad \left\{ \begin{array}{r} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \hat{F}(t,x(t)) \text{ a.e.} \\ x(0) = x(b) \end{array} \right\}.$$

Using theorem 3.1, we get that ($\underline{3}$) has a solution $x \in C(T, L^2(Z))$ s.t. $\frac{\partial x}{\partial t} \in L^2(T \times Z)$. ∎

Next we consider the following periodic optimal control problem for a class of nonlinear systems

$$(\underline{4})\quad \left\{\begin{array}{l} \|x(0,\cdot)\|^2_{L^2(Z)} + \int_0^b \int_Z L(t,z,x(t,z),u(t,z))dz\,dt \to \inf = \hat{m} \\ \text{s.t. } \frac{\partial x}{\partial t} - \sum_{k=1}^N D_k(a(t,z)D_k x) + \beta(x) = f(t,z,x(t,z))u(t,z) \text{ a.e.} \\ x(0,z) = x(b,z) \text{ a.e.,}\quad 0 \le u(t,z) \le \gamma,\quad (t,z) \in T \times Z \end{array}\right\}.$$

We will need the following hypotheses:

$\underline{H(\beta)}$: $\beta : \mathbb{R} \to \mathbb{R}_+$ is a nondecreasing, continuous, convex function.

$\underline{H(f)_2}$: $f : T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function s.t.

($\underline{1}$) $(t,z) \to f(t,z,x)$ is measurable,

($\underline{2}$) $x \to f(t,z,x)$ is continuous,

($\underline{3}$) for all $x(\cdot) \in L^2(Z)$, $(\int_Z |f(t,z,x(z))|^2 dz)^{1/2} \le \psi(t)\|x\|_2$ a.e., with $a \le \psi(t) < \lambda_1$ and λ_1 is the first eigenvalue of $(-\Delta, H_0^1(Z))$; i.e. of $-\Delta$ with Dirichlet boundary conditions.

$\underline{H(L)_1}$: $L: T \times Z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an integrand s.t.

($\underline{1}$) $(t,z,x,u) \to L(t,z,x,u)$ is measurable,

($\underline{2}$) $(x,u) \to L(t,z,x,u)$ is l.s.c.,

($\underline{3}$) $u \to L(t,z,x,u)$ is convex,

($\underline{4}$) $\psi(t,z) - M|u| \le L(t,z,x,u)$ a.e. with $\psi(\cdot,\cdot) \in L^1(T \times Z)$ and $M \ge 0$.

THEOREM 5.2 *If hypotheses $H(a)$, $H(\beta)$, $H(f)_2$ and $H(L_1)$ hold, then problem ($\underline{4}$) admits a solution $(x,u) \in C(T, L^2(Z)) \times L^\infty(T \times Z)$ with $\frac{\partial x}{\partial t} \in L^2(T \times Z)$.*

PROOF. Let $H = L^2(Z)$ and let $\varphi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(t,x) = \begin{cases} \frac{1}{2}\int_Z a(t,z)|\nabla x|^2 dz + \int_Z \beta(x(z))dz & \text{if } x \in H_0^1(Z), \beta(x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise} \end{cases}.$$

Clearly $\varphi(t,\cdot) \in \Gamma_0(H)$ and $dom\varphi(t,\cdot)$ is independent of $t \in T$. Furthermore, since $H_0^1(Z)$ embeds into $L^2(Z)$ compactly, we can check that $\varphi(t,\cdot)$ is of compact type. As in BARBU [4], we get that $\varphi(t,x) = L(t)x$, where $L(t)x = \beta(x(\cdot)) - \sum_{k=1}^{N} D_k(a(t,z)D_k x)$ with $D(L(t)) = \{x \in H_0^1(Z) : L(t)x \in L^2(Z)\}$. Via the Poincare–Friedrich's inequality we get that

$$\hat{\beta}\lambda_1 \|x - y\|^2_{L^2(Z)} \le (\partial\varphi(t,z) - \partial\varphi(t,y), x - y)_{L^2(Z)}.$$

In addition, using $H(a)$ as in the proof of theorem 5.1, we can easily check that hypothesis $H(\varphi)$ (2) is satisfied. So we have verified hypothesis $H(\varphi)$.

Next let $U = \{u \in L^2(Z) = Y : 0 \le u(z) \le \gamma\}$ and let $\hat{f} : T \times H \to \mathcal{L}(H)$ be defined by $\hat{f}(t,x)u(\cdot) = f(t,\cdot,x(\cdot))u(\cdot)$. Using hypothesis $H(f)_2$, we can check that $H(f)$ is satisfied.

Finally let $l : H \to \mathbb{R}$ be defined by $l(x) = \|x\|_2^2$ and $\hat{L} : T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by $\hat{L}(t,x,u) = \int_Z L(t,z,x(z),u(z))dz$. Clearly then hypotheses $h(l)$ and $H(L)$ are verified.

Now rewrite problem (4), in the following equivalent abstract form:

$$(4)' \qquad \left\{ \begin{array}{c} l(x) + \displaystyle\int_0^b \hat{L}(t,x(t),u(t))dt \to \inf = \hat{m} \\ \text{s.t.} - \dot{x}(t) \in \partial\varphi(t,x(t)) + \hat{f}(t,x(t))u(t) \text{ a.e.} \\ x(0) = x(b), \quad u(t) \in U \text{ a.e.}, \quad u(\cdot) \text{ measurable} \end{array} \right\}.$$

Apply theorem 4.1 to get an optimal pair $(x,u) \in C(T, L^2(Z)) \times L^\infty(T \times Z)$ for (4), with $\frac{\partial x}{\partial t} \in L^2(T \times Z)$. ∎

Finally, our general formulation incorporates evolution inclusions of the following form:

$$(5) \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t,x(t)) \text{ a.e.} \\ x(0) = x(b) \end{array} \right\}.$$

Here $N_{K(t)}(x)$ is the normal cone to the closed, convex set $K(t)$ at x. From convex analysis we know that $N_{K(t)}(x) = \partial\beta_{K(t)}(x)$, where $\delta_{K(t)}(x) = 0$ if $x \in K(t)$ and $+\infty$ otherwise (indicator function of $K(t)$). Systems like (5) are usually called "Differential Variational Inequalities" (see AUBIN–CELLINA [2], section 5.6) and arise in mathematical economics (see AUBIN–CELLINA [2] and HENRY [13]) and theoretical mechanics (see MOREAU [16]). Other results on differential variational inequalities can be found in [20].

$\underline{H(K)}$: $K : T \to P_{fc}(\mathbb{R}^n)$ is a multifunction s.t. $h(K(t), K(t')) \leq \int_t^{t'} \theta(s)ds$, $0 \leq t \leq t' \leq b$ with $\theta(\cdot) \in L^1_+$.

Note that in this case with $\varphi(t,x) = \delta_{K(t)}(x)$, hypothesis $H(\varphi)$ is satisfied with $\dot{g}_r = \theta$, $\beta = 1$ and $h_r = 0$. So we can apply theorem 3.1 and get:

THEOREM 5.3 *If hypothesis $H(K)$ and $H(F)$ (with $H = \mathbb{R}^n$) hold then problem ($\underline{5}$) admits a solution.*

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A MINIMAX THEOREM

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§ 0. Introduction

DEFINITION 1. (SIMONS [1]) We suppose that X and Y are nonempty sets and $f : X \times Y \to \mathbb{R}$. We shall say f is upward on Y if $\forall \varepsilon > 0 \exists \delta > 0$ such that:

$$(0.1)\qquad \begin{cases} \forall y_1, y_2 \in Y \exists y_3 \in Y \text{ such that} \\ \forall x \in X \quad f(x,y_3) \le f(x,y_1) \vee f(x,y_2) \text{ and} \\ |f(x,y_1) - f(x,y_2)| \ge \varepsilon \Rightarrow f(x,y_3) \le f(x,y_1) \vee f(x,y_2) - \delta. \end{cases}$$

We shall say that f is downward on X if $\forall \varepsilon > 0 \exists \delta > 0$ such that:

$$(0.2)\qquad \begin{cases} \forall x_1, x_2 \in X \exists x_3 \in X \text{ such that} \\ \forall y \in Y f(x_3,y) \ge f(x_1,y) \wedge f(x_2,y) \text{ and} \\ |f(x_1,y) - f(x_2,y)| \ge \varepsilon \Rightarrow f(x_3,y) \ge f(x_1,y) \wedge f(x_2,y) + \delta. \end{cases}$$

S. SIMONS proved [1] the following theorem:

THEOREM 1. *Suppose that f is upward on Y, downward on X, Y is a compact topological space and $\forall\, x \in X\ f(x,.)$ is l.s.c. on Y. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in Y} f(x,y).$$

I. JOÓ and G. KASSAY proved [2] that if $f : X \times Y \to \mathbb{R}$ is downward on X and X is a finite set then (0.2) implies:

$$(0.3)\qquad \begin{cases} \forall x_1, x_2 \in X \exists x_3 \in X \text{ such that} \\ \forall y \in Y f(x_3,y) \ge f(x_1,y) \wedge f(x_2,y) \text{ and} \\ f(x_1,y) \ne f(x_2,y) \Rightarrow f(x_3,y) > f(x_1,y) \wedge f(x_2,y). \end{cases}$$

They also proved in this case $\exists x^* \in X$ such that $f(x^*,y) \geq f(x,y) \forall x \in X$, $\forall y \in Y$ and $\inf_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\inf_{y\in Y} f(x,y)$. Similarly, if $f : X \times Y \to \mathbb{R}$ is upward on Y and Y is a finite set (0.1) implies that:

$$\text{(0.4)} \qquad \begin{cases} \forall y_1,y_2 \in Y \exists y_3 \in Y \text{ such that} \\ \forall x \in X f(x,y_3) \leq f(x,y_1) \vee f(x,y_2) \text{ and} \\ f(x,y_1) \neq f(x,y_2) \Rightarrow f(x,y_3) < f(x,y_1) \vee f(x,y_2). \end{cases}$$

The main result of this paper is:

THEOREM 2. *If X and Y are compact topological spaces, $f : X \times Y \to \mathbb{R}$ is a continuous function satisfying* (0.3) *and* (0.4) *then*

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

In section 2 there are examples to prove:

a) Theorem 2 does not hold if X or Y is not compact.

b) Theorem 2 does not hold if $f(x,.)$ is only l.s.c. for some $x \in X$ and $f(.,y)$ is continuous $\forall y \in Y$.

c) If X and Y are compact topological spaces there exists continuous functions $f : X \times Y \to \mathbb{R}$ upward on Y, downward on X such that f does not satisfies (0.3) or (0.4).

§ 1.

LEMMA 1. *Suppose that X, Y are compact topological spaces, $f : X \times Y \to \mathbb{R}$ is a continuous function satisfying* (0.4). *Then:*

$$\text{(1.1)} \qquad \begin{cases} \forall \varepsilon > 0, \forall y_1,y_2 \in y \exists y_3 > 0, \exists \delta = \delta(\varepsilon,y_1,y_2,y_3) > 0 \\ \textit{such that } \forall x \in X f(x,y_3) \leq f(x,y_1) \vee f(x,y_2) \textit{ and} \\ |f(x,y_1) - f(x,y_2)| \geq \varepsilon \Rightarrow f(x,y_3) < f(x,y_1) \vee f(x,y_2) - \delta. \end{cases}$$

PROOF.

(1.2) $\quad M := \{x \in X : |f(x,y_1) - f(x,y_2)| \geq \varepsilon\}$ is a compact subset of X.

The function $g : X \to \mathbb{R}$, $g(x) := f(x,y_1) \vee f(x,y_2)$ is continuous.

(0.4) implies that $\exists y_3 \in Y$ such that:

$$\forall x \in X f(x,y_3) \leq g(x) \text{ and } |f(x,y_1) - f(x,y_2)| > 0 \Rightarrow f(x,y_3) < g(x).$$

Hence $g(x) - f(x,y_3) > 0$, $\forall x \in M$. From (1.2) $\exists \delta = \delta(\varepsilon,y_1,y_2,y_3)$ such that $g(x) - f(x,y_3) > \delta$, $\forall x \in M$.

LEMMA 2. *Let* X, Y *compact topological spaces,* $f : X \times Y \to \mathbb{R}$ *a continuous function satisfying* (0.4). *Suppose that* $\forall y \in Y$ $s(y) := \sup_{x \in X} f(x,y) \in \in (\Theta,+\infty)$. *Let* $n < \Theta, y_1, y_2 \in Y$ *and* $B(y) := \{x \in X : f(x,y) > n\}$ $\forall y \in Y$. *Then* $\exists h_1, h_2 \in Y$ *with* $B(h_i) \subseteq B(y_i)$, $\forall i \in \{1,2\}$ *and if* $h_3 \in Y$ *is given by* (1.1) *for* $\varepsilon = \Theta - n$, h_1, h_2 *then* $\exists x_1, x_2 \in X$ *such that:*

$$f(x_i,y_i) \wedge f(x_i,h_3) > n, \forall i \in \{1,2\}.$$

PROOF. Let $H_1 := \{h \in Y : B(h) \subseteq B(y_1)\} \subseteq Y$. H_1 is a closed subset of Y because, otherwise $\exists h_0 \in \overline{H}_1$ such that $h_0 \in H_1$ i.e. $\exists x_0 \in X$ such that $f(x_0,h_0) > n$ and $f(x_0,y_1) \leq n$. Since f is continuous $\exists\, V \in N(h_0)$ such that $\forall h \in V, f(x_0,h) > n$. $h_0 \in \overline{H}_1$ implies that $\exists h' \in V \cap H_1$ so that $f(x_0,h') > n$ and $f(x_0,y_1) > n$ which is a contradiction with $f(x_0,y_1) \leq n$. Thus H_1 is closed and therefore is compact. So $\exists h_1 \in H_1$ such that:

$$s(h_1) \leq s(k), \quad \forall k \in H_1. \tag{2.1}$$

Similarly $\exists h_2 \in H_2 := \{h \in Y : B(h) \subseteq B(y_2)\}$ such that

$$s(h_2) \leq s(k), \quad \forall k \in H_2. \tag{2.2}$$

From Lemma 1 with $\varepsilon = \Theta - n$, h_1, $h_2 \in Y \exists h_3 \in Y$ and $\exists \delta = \delta(\varepsilon, h_1, h_2, h_3)$ satisfying (1.1). From (2.1) and (2.2) results that $\forall i \in \{1,2\}$, $\forall k \in H_i$, $s(k) > > s(h_i) - \delta$. (1.1) implies that $B(h_3) \subseteq B(h_1) \cup B(h_2)$. Suppose that $B(h_2) \cap B(H_3) = \emptyset$. From $s(h_3) > s(h_1) - \delta$ and $s(h_3) > \Theta$ it follows that:

$$\exists x \in X \text{ such that } n < \Theta \vee (s(h_1) - \delta) < f(x,h_3). \tag{2.3}$$

$$\text{Thus: } \begin{cases} x \in B(h_3) \Rightarrow x \in B(h_2) \Rightarrow f(x,h_2) \leq n < f(x,h_1) \\ f(x,h_1) \vee f(x,h_2) - \delta = f(x,h_1) - \delta \leq s(h_1) - \delta < f(x,h_3). \end{cases}$$

From (1.1):

$$f(x,h_3) \leq f(x,h_1) \text{ and } f(x,h_1) - f(x,h_2) < \varepsilon = \Theta - n \Rightarrow$$
$$\Rightarrow f(x,h_3) \leq f(x,h_1) = f(x,h_2) + f(x,h_1) - f(x,h_2) < n \to \Theta - n = \Theta$$

which is in contradiction with (2.3). Thus $\exists x_2 \in B(h_2) \cap B(h_3)$. Similarly $\exists x_1 \in B(h_1) \cap B(h_3)$.

LEMMA 3. *Suppose that* X, Y *are compact topological spaces,* $f : X \times Y \to \mathbb{R}$ *is a continuous function satisfying* (0.3) *and* (0.4). *Let* $W \subseteq X$, u_1, $u_2 \in X$. *Then:*

$$\inf_{y \in Y} \sup_{x \in \{u_1,u_2\} \cup w} f(x,y) \leq \sup_{x' \in X} \inf_{y \in Y} \sup_{x \in \{x'\} \cup w} f(x,y).$$

PROOF. If the result fails, then $\exists \alpha_0, \alpha, \beta \in \mathbb{R}$ such that:

(3.1)
$$\forall x' \in X, \forall y \in Y \inf_{y\in Y} \sup_{x\in\{x'\}\cup w} f(x,y) \le \alpha_0 < \alpha < \beta < \sup_{x\in\{u_1,u_2\}\cup w} f(x,y).$$

Let $V := \{y \in Y : \sup_{x\in W} f(x,y) \le \alpha_0\}$. V is a compact subset of Y. From (3.1):

(3.2) $$\forall x' \in X, \forall y \in Y \inf_{y\in V} f(x',y) \le \alpha_0 < \alpha \le \beta \le f(u_1,y) \vee f(u_2,y).$$

(3.2) is equivalent with:

$$\forall x' \in X, \forall y \in Y (f(u_1,y)) \wedge (-f(u_2,y)) < -\beta < -\alpha < -\alpha_0 \le \sup_{y\in V}(-f(x',y)).$$

From Lemma 2 with $f|_{X\times V} : X \times V \to \mathbb{R}$, $\Theta = -\alpha$, $n = -\beta$, for $u_1, u_2 \in X$, $\exists x_1, x_2 \in X$ such that:

(3.3)
$$\begin{aligned}\forall y \in Y, \forall i \in \{1,2\}, f(x_i,y) > -\beta &\Rightarrow -f(u_i,y) > -\beta \Leftrightarrow \\ &\Leftrightarrow f(x_i,y) < \beta \Rightarrow f(u_i,y) < \beta \Leftrightarrow \\ &\Leftrightarrow f(u_i,y) \ge \beta \Rightarrow f(x_i,y) \ge \beta.\end{aligned}$$

Also from Lemma 2 $\exists x_3 \in X$, $\exists w_1, w_2 \in W$ such that $\forall i \in \{1,2\}$: $(-f(x_i,w_i)) \wedge (f(x_3,w_i)) > -\beta \Leftrightarrow f(x_i,w_i) \wedge f(x_3,w_i) < \beta$. We choose $n' < \beta$ such that $\forall i \in \{1,2\}$:

(3.4) $$f(x_i,w_i) \wedge f(x_3,w_i) \le n' < \beta.$$

From Lemma 2 with $X = \{x_1,x_2,x_3\}$, $Y = V$, $\Theta \in (n',\beta)$, $\varepsilon = \Theta - n'$:

(3.5)
$$\begin{cases} \text{for } w_1,w_2 \in V, \exists y_1,y_2 \in V \text{ such that:} \\ \forall i \in \{1,2\}, \forall j \in \{1,2,3\}, f(x_j,w_i) \le n' \Rightarrow f(x_j,y_i) \le n' \\ \text{and } \exists y_3 \in Y \text{ as in (1.1).}\end{cases}$$

This implies that: $\sup_{x\in W} f(x,y_3) \le \alpha_0$, thus $y_3 \in V$.

From Lemma 2 $\exists z_1, z_2 \in \{x_1,x_2,x_3\}$ such that:

(3.6) $$\forall i \in \{1,2\}, \quad n' < f(z_i,y_i) \wedge f(z_i,y_3).$$

From (3.4) and (3.5):

(3.7) $$\forall i \in \{1,2\}, \quad f(x_i,y_i) \wedge f(x_3,y_i) \le n'.$$

Hence, from (0.1):

(3.8) $$f(x_3,y_3) \le f(x_3,y_1) \wedge f(x_3,y_2) \le n'.$$

From (3.6) and (3.7): $\forall i \in \{1,2\}, f(x_i,y_i) \vee f(x_3,y_i) \le n' < f(z_i,y_i)$. hence $z_1 = x_{2'}, z_{2'} = x_1$.

From (3.6): $n < f(z_1,y_3) \wedge f(z_2,y_3) = f(x_2,y_3) \wedge f(x_1,y_3) \le f(x_3,y_3)$. This is a contradiction of (3.8).

The following Lemma is proved in [1] and the proof of Theorem 3 using Lemma 3 and Lemma 4 is like in [1].

LEMMA 4. *Let X, Y compact topological spaces, $f : X \times Y \to \mathbf{R}$ a continuous function satisfying* (0.3) *and* (0.4), *and let X_0 a finite subset of X. Then:*

$$\min_{y\in Y}\max_{x\in X_0} f(x,y) \leq \max_{x\in X_0}\min_{y\in Y} f(x,y)$$

§ 2. Examples.

a) Let $f : \left[-1,-\frac{1}{4}\right[\bigcup\left]\frac{1}{4},1\right] \times [0,1] \to \mathbf{R}, f(1,0)=0, f(1,1)=3, f\left(\frac{1}{4},1\right) = \frac{3}{4}, f\left(-\frac{1}{4},1\right) = -\frac{1}{4}, f(-1,1)=-1, f(-1,0)=0, f(-1,-1)=3, f\left(-\frac{1}{4},-1\right) = \frac{3}{4}, f\left(\frac{1}{4},-1\right) = -\frac{1}{4}$. Let $f(x,0)=0$, $\forall x \in \left[-1,-\frac{1}{4}\right[\bigcup\left]\frac{1}{4},1\right]$. Let $f(\cdot,-1)$ and $f(\cdot,1)$ be affine on intervals $\left[-1,-\frac{1}{4}\right[$ and $\left]\frac{1}{4},1\right]$, $\forall x \in \left[-1,-\frac{1}{4}\right[\bigcup\left]\frac{1}{4},1\right]$, let $f(x,\cdot)$ be affine on $[-1,0]$ and $[0,1]$. f satisfies (0.3) and (0.4) but:

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = 0 \text{ and } \sup_{x\in X}\min_{y\in Y} = -\frac{1}{4} < 0,$$

see the figure 1.

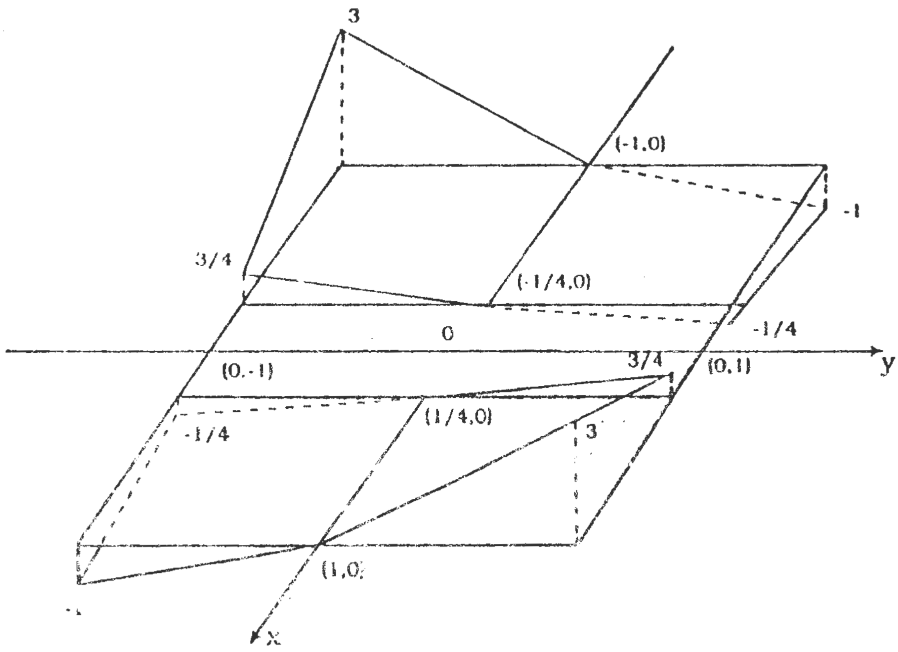

Fig. 1

b) Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$, $f(1,0) = 1$, $f(1,1) = 3$, $f(0,1) = 0$, $f(-1,1) = -1$, $f(-1,0) = -\frac{1}{2}$, $\lim\limits_{y\to 0, y<0} f(-1,y) = 2$, $f(-1,-1) = 3$, $f(0,-1) = 0$, $f(1,-1) = \frac{1}{2}$. Let $f(\cdot,-1)$, $f(\cdot,0)$ and $f(\cdot,1)$ be affine on intervals $[-1,0]$ and $[0,1]$. $\forall x \in [0,1]$ let $f(x,\cdot)$ be affine on $[-1,0]$ and $[0,1]$. $\forall x \in [-1,0[$ let $\lim\limits_{y\to 0, y<0} f(x,y) = 2x$ and let $f(x,\cdot)$ be affine on $[-1,0]$ and $[0,1]$. Thus, $f(x,\cdot)$ is l.s.c. $\forall x \in [-1,0]$, $f(x,\cdot)$ is continuous $\forall x \in [0,1]$, and $f(\cdot,y)$ is continuous $\forall y \in [-1,1]$ and: $\min\limits_{y\in Y}\max\limits_{x\in X} f(x,y) = 1 > \frac{1}{2} = \max\limits_{x\in X}\min\limits_{y\in Y} f(x,y)$ (see the figure 2.)

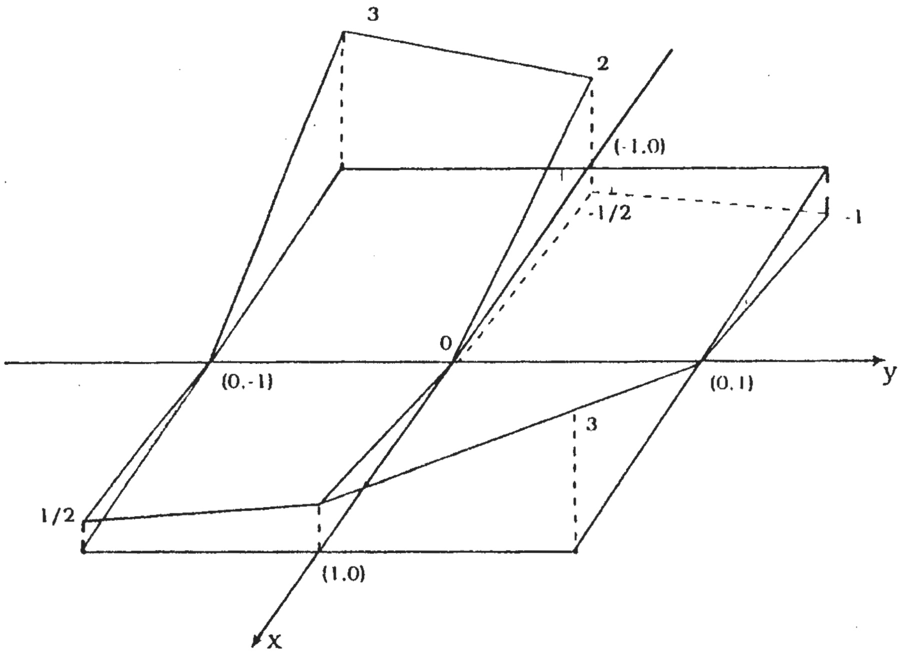

Fig. 2

c) Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$, $f(1,0) = 0$, $f(1,1) = 1$, $f(0,1) = 0$, $f(-1,1) = -1$, $f(-1,0) = 0$, $f(-1,-1) = 2$, $f(0,-1) = 1$, $f(1,-1) = 0$, $f(x,-x) = 0$ $\forall x \in [0,1]$. Let $f(\cdot,-1)$, $f(\cdot,0)$ and $f(\cdot,1)$ be affine on $[-1,1]$. $\forall x \in [-1,0]$ let $f(x,\cdot)$ be affine on $[-1,0]$ and $[0,1]$. $\forall x \in]0,1]$ let $f(x,\cdot)$ be affine on $[-1,-x]$, $[-x,0]$, $[0,1]$. f is defined to be continuous, downward on X, upward on Y. But, for $0 < x_1 < x_2 < 1$ there is no $x_3 \in [-1,1]$ to satisfy (0,4) (see the figure 3.)

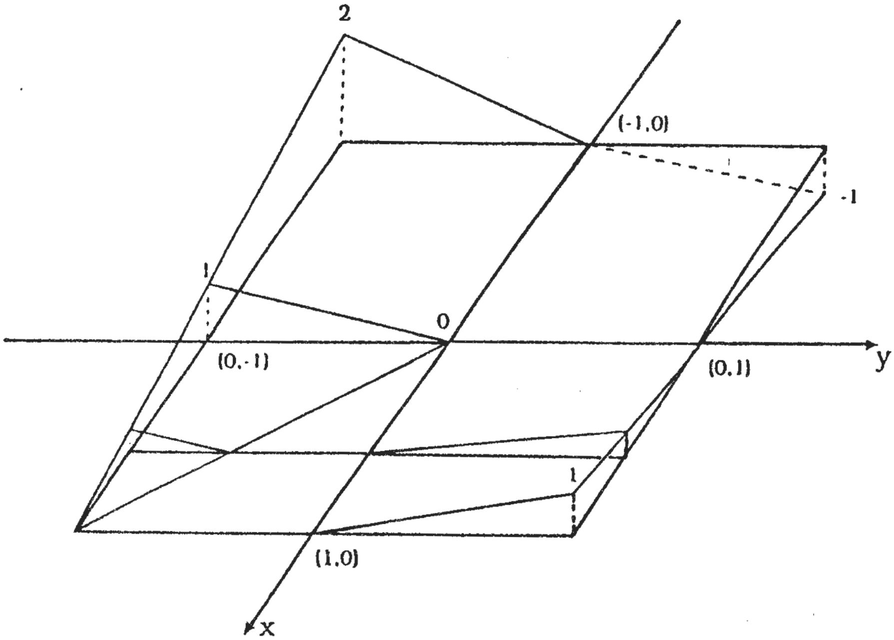

Fig. 3

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REMARK OF ADDITIVE FUNCTIONS

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Dedicated to Professor Paul Erdős on his 80^{th} birthday

In this paper we give a characterization theorem. The first characterization is apparently that of P. Erdős [1]. He proved that if a real-valued additive function f is non-decreasing or

$$f(n+1)-f(n)\to 0$$

as $n\to\infty$, then it must have the form $A\cdot\log n$ for some constant A. He had a separate argument for each case.

DEFINITION. We say that $f:\mathbb{N}\to\mathbb{R}/\mathbb{Z}$ is completely additive mod 1 if for every $n, m\in\mathbb{N}$

$$f(n\cdot m)=f(n)+f(m) \pmod 1.$$

THEOREM. *Let $a\geq 0$, b denote integers and let $(a,b)=1$. If f is completely additive* mod 1 *and*

$$f(a(n+1)+b)-f(an+b)\to 0 \pmod 1$$

as $n\to\infty$, then $o(B)\cdot f(an+B)-c_0\cdot\log(an+B))=0$ (mod 1), *where $(a,B)=$ $=1$, c_0 is a constant independent of n, $o(B)$ is the order of B* mod a.

PROOF. From the assumption of Theorem we can get easily that

$$\lim_{n\to\infty} f(an+ab+b)-f(an+b)=0 \pmod 1.$$

In this relation writing bn in place of n we get

$$\lim_{n\to\infty} f(a(n+1)b+b)-f(anb+b)=0 \pmod 1.$$

Since f is completely additive mod 1, therefore we obtain

$$\lim_{n\to\infty} f(a(n+1)+1)-f(an+1)=0 \pmod 1.$$

Let $F:\mathbb{N}\to\mathbb{R}$ such that $\{F(n)\}=\{f(n)\}$ ($\{x\}$ denotes the fractional part of x) and $K:\mathbb{R}\to\mathbb{R}/\mathbb{Z}$, $z\mapsto\{z\}+\mathbb{Z}$, further let be $F(a+1):=\{f(a+1)\}$. Choose the values of F such a way, that $F(a(n+1)+1)\in\left[F(an+1)-\frac{1}{2},F(an+1)+\frac{1}{2}\right)$. Then F is uniquely determined on the set $(an+1\,:\,n=1,2,\ldots)$. Let us use the notation

(1) $$q(n_1,n_2):=F((an_1+1)(an_2+1))-F(an_1+1)-F(an_2+1).$$

Since f is completely additive mod 1, therefore $q(n_1,n_2)\in\mathbb{Z}$. Obviously

(2) $$q(n_1,n_2+1)-q(n_1,n_2)=F((an_1+1)(an_2+a+1))- \\ -F((an_1+1)(an_2+1))-(F(an_2+a+1)+F(an_2+1)).$$

Since

$$K(F(a(n+1)+1)-F(an+1))=f(a(n+1)+1)-f(an+1), \quad \text{therefore}$$
$$\lim_{n\to\infty} K(F(a(n+1)+1)-F(an+1))=0 \pmod 1.$$

Since

$$|F(a(n+1)+1)-F(an+1)|\le\frac{1}{2}, \quad \text{therefore}$$

(3) $$\lim_{n\to\infty}(F(a(n+1)+1)-F(an+1))=0$$

Obviously

$$F((an_1+1)(an_2+a+1))-F((an_1+1)(an_2+1))= \\ =F(a[(an_1+1)n_2+an_1+n_1]+1)-F(a[(an_1+1)n_2+n_1]+1).$$

A simple consequence of (3) is that

(4) $$\lim_{n\to\infty}(F(a(n+k_1)+1)-F(a(n+k_2)+1))=0 \quad (k_1,k_2\in\mathbb{Z}).$$

The substitution $n=(an_1+1)n_2$, $k_1=an_1+n_1$, $k_2=n_1$ in (4) and (3) together yield that

(5) $$\lim_{n_2\to\infty}(q(n_1,n_2+1)-q(n_1,n_2))=0.$$

But $q(n_1,n_2+1)-q(n_1,n_2)$ is an integer, therefore

(6) $$q(n_1,n_2)=q(n_1,n_2+1)=q(n_1,n_2+2)=\ldots, \quad \text{if } n_2>N(n_1).$$

Introduce the function G,

(7) $$G(am+1):=F(am+1)+\lim_{n\to\infty}q(m,n)= \\ =\lim_{n\to\infty}(F(am+1)(an+1))-F(an+1)).$$

Now we prove, that G is completely additive on the set $(an+1 : n=1,2,\dots)$. Let be $m_1, m_2 \in \mathbb{N}$.

$$(8) \qquad G((am_1+1)(am_2+1)) =$$
$$= \lim_{n\to\infty} (F((am_1+1)(am_2+1)(an+1)) - F(an+1)) =$$
$$= \lim_{n\to\infty} (F(am_1+1)(am_2+1)(an+1)) - F((am_2+1)(an+1))+$$
$$+F((am_2+a)(an+1)) - F(an+1)).$$

Because of (7)

$$(9) \qquad \lim_{n\to\infty} (F((am_2+1)(an+1)) - F(an+1)) = G(am_2+1).$$

Further $(am_2+1)(an+1)=a(am_2n+m_2+n)+1$. Replacing n by am_2n+m_2+ $+n$ in (7) we obtain that

$$(10) \qquad \lim_{n\to\infty} (F((am_1+1)(am_2+1)(an+1)) - F((am_2+1)(an+1))) =$$
$$= G(am_1+1).$$

From (8), (9), (10) we get G is completely additive on the set $(an+1 : n =$ $= 1,2,\dots)$.

Denote

$$(11) \qquad \delta(m) := \lim_{n\to\infty} q(m,n).$$

Since G is completely additive on the set $(an+1 : n=1,2,\dots)$ therefore using its definition (7),

$$\delta(m_1)+\delta(m_2) - \delta(am_1m_2+m_1+m_2) = q(m_1,m_2).$$

Using (11) we have

$$(12) \qquad \delta(m_2) = \delta(am_1m_2+m_1+m_2) \quad \text{for all} \quad m_2 > N(m_1).$$

From (7), (11)

$$(13) \qquad G(a(m+1)+1) - G(am+1) =$$
$$= F(a(m+1)+1) - F(am+1) + \delta(m+1) - \delta(m).$$

Now we prove, that $\delta(m) = \delta(m+1) = \dots$, if $m > N_0$, where N_0 is a fixed constant. Let be $m_1 := 1$. Replacing $m_1 = 1$ in (12)

$$\delta(m_2) = \delta(am_2+1+m_2) = \delta((a+1)m_2+1), \quad \text{for all} \quad m_2 > N(m_1).$$

Iterating the argument we get

$$\delta(m_2) = \delta\left((a+1)^l m_2 + (a+1)^{l-1} + \dots + 1\right) = \delta\left((a+1)^l m_2 + \frac{(a+1)^l - 1}{a}\right),$$

where $l \in \mathbb{N}$ is arbitrary. Here

$$\delta\left((a+1)^l m_2 + \frac{(a+1)^l-1}{a}\right) = \delta\left(a\frac{(a+1)^l-1}{a}m_2 + m_2 + \frac{(a+1)^l-1}{a}\right).$$

If l is large enough, $l > L(m_2)$, then using (12) we have

$$= \delta\left((a+1)^l m_2 + \frac{(a+1)^l-1}{a}\right) = \delta\left(\frac{(a+1)^l-1}{a}\right).$$

That is

$$\delta(m_2) = \delta\left(\frac{(a+1)^l-1}{a}\right) \quad \text{it} \quad l > L(m_2).$$

Similarly

$$\delta(m_2+1) + \delta\left(\frac{(a+1)^l-1}{a}\right) \quad \text{if} \quad l > L(m_2+1).$$

This yields

(14) $$\delta(m) = \delta(m+1) = \ldots \quad \text{for all} \quad m > N_0.$$

Using (3), (14) we obtain from (13) that

(15) $$\lim_{m\to\infty}(G(a(m+1)+1) - G(am+1)) = 0.$$

Now we prove that if $h \equiv 1 \pmod a$ then

(16) $$G(h) = \alpha \cdot \log h,$$

where α is a constant independent of h. We use an idea, which can be found in [2], but we need some new ideas also.

Assume that there exist h_1 and h_2 $(h_1 \neq h_2)$ such that

$$h_1 \equiv \pmod a, \quad h_2 \equiv 1 \pmod a, \quad c_1 = \frac{G(h_1)}{\log h_1} \neq \frac{G(h_2)}{\log h_2} =: C_2$$

Let e.g. $C_2 > C_1$. Let x_0 be arbitrary but fixed number for which $C_2 > x_0 > C_1$.

Denote $G_0 := G - x_0 \cdot \log$. Then G_0 is completely additive on the set $(an+1 : n = 0, 1, \ldots)$ and

(15′) $$\lim_{n\to\infty}(G_0(a(n+1)+1) - G_0(an+1)) = 0.$$

Further

(17) $$c_2 := \frac{G_0(h_2)}{\log h_2} > \frac{G_0(h_1)}{\log h_1} =: c_1 \quad \text{and} \quad G_0(h_2) > 0 > G_0(h_1).$$

Denote $d_{h_2}(n) := G_0(an+1) - (1-\varepsilon)c_2 \cdot \log n$, where $0 < \varepsilon < 1$, we will choose later. We show that $d_{h_2}(n)$ is bounded. First we show that $d_{h_2}(n)$ is bounded above, i.e. we show that if $n > n_0(c_1,c_2,h_1,h_2,\varepsilon)$ then there exists $m < n$ for which

(18) $$G_0(an+1) - (1-\varepsilon)c_2 \cdot \log n < G_0(am+1) - (1-\varepsilon)c_2 \cdot \log m.$$

We are looking for such m for which

(19) $$h_1(am+1) > an+1.$$

Choose the smallest m for which (19) is fulfilled, i.e.

$$m > \frac{n}{h_1} + \frac{1}{ah_1} - \frac{1}{a}.$$

Hence

(20) $$m = \frac{n}{h_1} + O(1).$$

Thus $m < n$ is satisfied. Since $h_1 \equiv 1 \pmod a$ therefore $h_1(am+1) \equiv 1 \pmod 1$. Using (20) $h_1(am+1) - (an+1) = O(1)$. Thus we can use (15′). We get

$$G_0(an+1) = G_0(h_1) + G_0(am+1) + o_n(1),$$

where $o_n(1)$ means $o_n(1) \to 0$ as $n \to \infty$. From this

$$G_0(an+1) - (1-\varepsilon)c_2 \cdot \log n =$$
$$= G_0(h_1) - (1-\varepsilon)c_2 \cdot \log\frac{n}{m} + (G_0(am+1) - (1\varepsilon)c_2 \cdot \log m + o_n(1).$$

From (20) $\frac{n}{m} = h_1 + O\left(\frac{1}{m}\right)$, thus $\log\frac{n}{m} = \log h_1 + O\left(\frac{1}{m}\right)$, hence we can estimate as follows

$$G_0(an+1) - (1-\varepsilon)c_2 \cdot \log n =$$
$$G_0(h_1) - (1\varepsilon)c_2 \cdot \log h_1 + O\left(\frac{1}{m}\right) + (G_0(am+1) - (1\varepsilon)c_2 \cdot \log m) + o_n(1) =$$
$$= \log h_2(c_1 - (1-\varepsilon)\cdot c_2) + O\left(\frac{1}{m}\right) + (G_0(am+1) - (1-\varepsilon)c_2 \cdot \log m) +$$
$$+ o_n(1) < G_0(am+1) - (1-\varepsilon)c_2 \cdot \log m,$$

if we choose ε such that $0 < \varepsilon < \varepsilon_0$, $(c_1 - (1-\varepsilon)\cdot c_2) < 0$ and n is large enough. Hence $d_{h_2}(n)$ is bounded above c_1.

We can prove similarly that $d_{h_2}(n)$ is bounded below, but to this we have to modify the calculations above. Let us choose m such that we get

$$G_0(an+1) - (1-\varepsilon)c_2 \cdot \log n =$$
$$= G_0(am+1) - (1-\varepsilon)c_2 \cdot \log m + \varepsilon \cdot G_0(h_2) + O\left(\frac{1}{m}\right) + o_n(1) >$$
$$> G_0(am+1) - (1-\varepsilon)c_2 \cdot \log m,$$

using that $G_0(h_2) > 0$ and $n > n_0$. Hence $d_{h_2}(n)$ is bounded below. Consequently $d_{h_2}(n)$ is bounded.

Using the same ideas we get that $d_{h_1}(n)$ is bounded. Therefore $d_{h_2}(n) - d_{h_1}(n)$ is bounded. On the other hand

$$d_{h_2}(n) - d_{h_1}(n) = (1-\varepsilon)\cdot(c_1 - c_2)\cdot \log n$$

for every $n \in \mathbb{N}$, consequently $c_1 = c_2$ i.e. if $h \equiv 1 \pmod a$ then $G_0(h) = c \cdot \log h$. Using the definition of G_0 we obtain $G(h) = \alpha \cdot \log h$, where α is a constant. (16) is proved.

Now let us return to the proof of theorem. From (16) using (14)

$$f(an+1) = KF(an+1) = K(G-\delta)(an+1) = \\ = KG(an+1) = c_0 \cdot \log(an+1) \pmod 1.$$

For all B coprime to a we have $(an+B)^{o(B)} = am+1$, where m is a suitable positive integer. Hence

$$f\left((an+B)^{o(B)}\right) = f(am+1) = c_0 \cdot \log(am+1) = \\ = c_0 \cdot \log\left((an+B)^{o(B)}\right) \pmod 1.$$

From this

$$o(B)\cdot f(an+B) = o(B)\cdot c_0 \cdot \log(an+B) \pmod 1,$$

which implies the theorem. In the theorem $o(B)$ cannot be eliminated. E.g. take $a=3$, $b=2$, $f(p) := \log p$ if $p \equiv 1 \pmod 3$, $f(p) := \log p + \frac{1}{2}$ is $p \equiv 2 \pmod 3$, p is prime, f is completely additive mod 1 ([4]). The special case $a=1$, $b=0$ was proved in [4]. The results present paper are further generalized in [5].

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TYPEN VON FLÄCHENTRANSITIVEN WÜRFELPFLASTERUNGEN

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1. Einführung

In der Arbeit [1] haben *A. W. M. Dress, D. H. Huson* und *E. Molnár* diejenigen euklidischen Pfasterungen (T, Γ) klassifizert, wo eine Bewegungsgruppe Γ auf den Flächen der Pflasterung T transitiv wirkt. Der kombinatorische Algorithmus und das Computerprogramm, das in der Theorie des sogenannten D-Symbols beruht, ergeben auch solche Pflasterungen, die nur in verschiedenen nicht euklidischen Räumen realisierbar sind. Ein schweres Problem ist, in welchem Raum wird eine kombinatorische Pflasterung überhaupt metrisch existieren. Wie E. Molnár es bewiesen hat, kommen sehr außergewöhnliche Räume z. B. $H^2 \times R$, $S^2 \times R$ vor.

Eine Möglichkeit ist die projektiv-metrische Geometrie, wie sie in der Arbeit [2] gezeigt wurde. Wir werden im folgenden die Methoden der projektiven Metrik benutzen. In dieser Arbeit betrachten wir kombinatorische Würfelpflasterungen (T, Γ) auf den die Symmetriegruppe Γ der Pflasterung T flächentransitiv wirkt, dh. für je zwei Flächen f_1 und f_2 der Polyeder von T wenigstens eine Bewegung $\gamma \in \Gamma$ existiert, die f_1 in $f_2 = f_1^\gamma$ überführt, so daß die ganze Pflasterung T auf sich abgebildet wird.

Zwei Pflasterungen (T_1, Γ_1) und (T_2, Γ_2) sind in derselben Klasse (äquivariant), wenn es eine bijektive inzidenztreue Abbildung ϕ gibt, für die $\Gamma_2 = \phi^{-1}\Gamma_1\phi$ ist. Im Fall, wenn zwei Pflasterungen kombinatorisch isomorph sind $(T_1 \cong T_2)$ kann die Gruppe Γ_2 reichhaltiger als Γ_1 sein. In diesem Fall sagen wir daß (T_1, Γ_1) ein Symmetriebruch von (T_2, Γ_2) ist. Wir interessieren uns am besten für die Pflasterung (T, Γ) wo, die Bewegungsgruppe maximal

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ist, d. h. die Gruppe Γ ist zu der die Inzidenzstruktur erhaltenden Automorphismengruppe äquivariant ($\Gamma \cong Aut\, T$).

2. Resultate

2.1. Besonders interessant sind die Würfel deren "Keilwinkel" von der Form $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$ mit der Ungleichung $\frac{1}{p}+\frac{1}{q}>\frac{1}{2}$ gegeben sind.

Zunachst, für Einfachheit nehmen wir zusätzlich an, daß die Symmetriegruppe Γ der Pflasterung T, flächentransitiv wirkt, und die Spiegelungen an den Flächen von T zu der Gruppe Γ gehören. Wie wir es in der Sektion 4 beweisen werden, ist die Stellung der Keilwinkel des Würfels, wie sie die Abb. 1. zeigt.

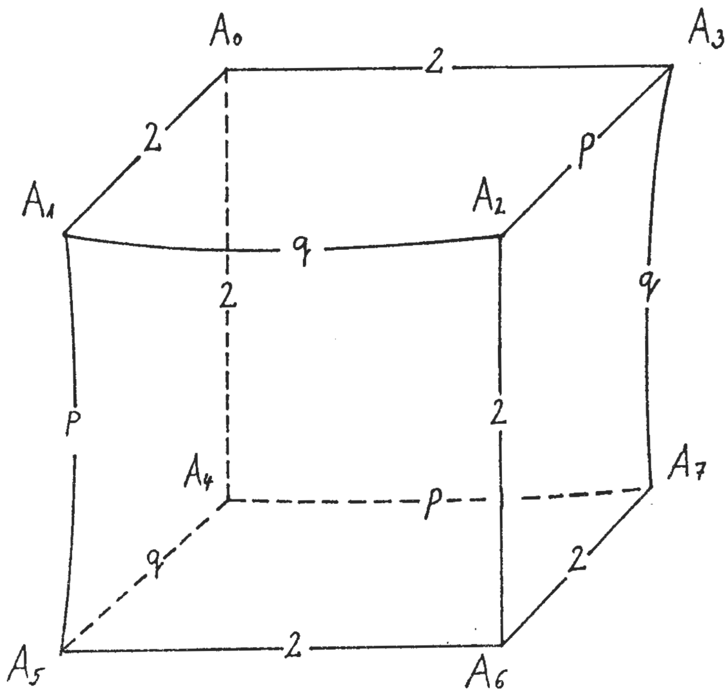


Abb. 1

2.2. In den Fällen $p > 2$; $q = 2$ bekommen wir den wohlbekannten Lambert Würfel mit 3 Keilwinkeln von $\frac{\pi}{p}$ an 3 schiefen Kanten und mit rechten Winkeln bei den anderen. Die metrische existenz eines solchen Würfels in H^3 ist wohlbekannt. Eine einfache Konstruktion beruht auf dem Vektormodell von H^3 wie E. Molnár in der Arbeit [2] gezeigt hat. (Abb. 2)

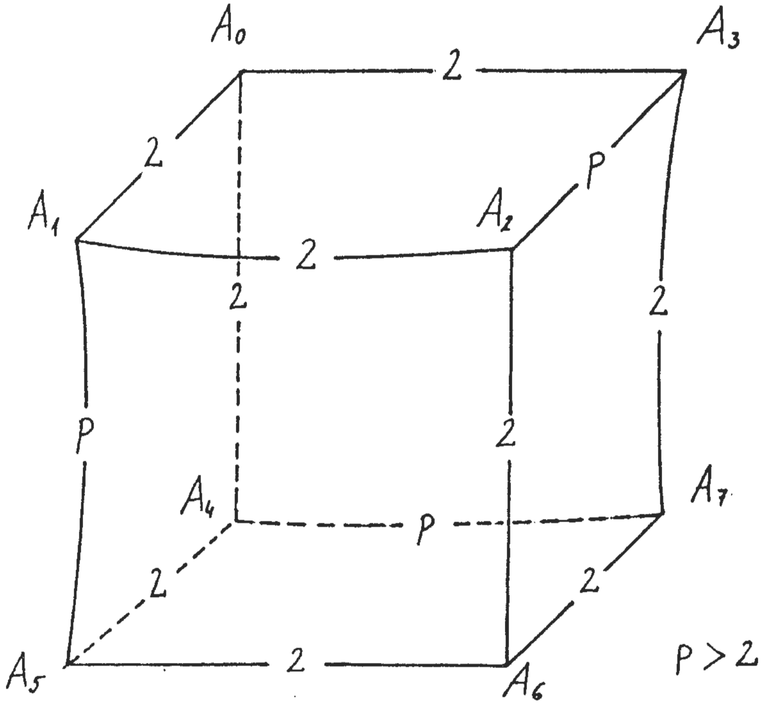


Abb. 2

Wir kennzeichen die metrische Existenz der übringen Fällen in dem folgenden Satz.

2.3. SATZ *Außer den obigen unendlich vielen Fällen,* $(p, 2)$, $p > 2$ *in* H^3 *gibt es noch 3 weitere maximalen Würfelpflasterungen* (T, Γ) *im hyperbolischen Raum* H^3 *mit* $(p, q) = (3, 3); (3, 4); (3, 5)$*, wobei jede gewünschte Bewegungsgruppe* $\Gamma \cong \mathrm{Aut}\, T$ *auf den Spiegelflächen transitiv wirkt, und jeder Würfel im* T *eigentliche Ecken besitzt.*

3. Über die D-Symbole

In der Arbeit [1] haben *A. W. M. Dress, D. H. Huson* und *E. Molnár* die Theorie des D-Symbols skizziert. In den Arbeiten [4], [5], [6], [7] befinden wir einige zwei und dreidimensionalen Anwendungen des D-Symbols.

Es sei T eine gegebene kombinatorische dreidimensionale Polyederpflasterung mit ihrer formalen baryzentrischen Unterteilung. Die baryzentrische Unterteilung von T führt zu einer Simplexzerlegung $\mathcal{C}$. Die formalen Mittelpunkte der Körper, Flächen, Kanten und die Ecken markieren wir mit 3,2,1,0 ebenso wie die entsprechenden gegenüberliegenden Simplexseiten. Die Ecken eines Simplex von der Simplexzerlegung $\mathcal{C}$ werden mit A_i, $i \in I := \{0,1,2,3\}$ bezeichnet.

Es sei $\sigma_i (i \in I)$ eine formale Spiegelung (involuntorische Operation), die die bei i-Seitenflächen benachbarten Simplexen C_1 und C_2 von $\mathcal{C}$ vertauscht ($C_2 = C_1^{\sigma_i}$). So führen wir die auf den Körpern von $\mathcal{C}$ frei wirkenden involutorischen σ_i-Operationen und die durch sie erzeugte freie Coxeter Gruppe mit der darstellung $\sum_I := \{\sigma_i : \sigma_i^2 = 1;\ i \in I\}$ ein.

Wir führen eine Matrixfunktion $M : \mathcal{C} \to N_{I \times I}\ C \to m_{ij}(C)\ i,j \in I$ ein, um die Wirkung der Gruppe $\sum_I$ und so die kombinatorische Struktur der Pflasterung T zu beschränken: $m_{ij}(C)$ ist die minimale natürliche Zahl, die die folgende Gleichung erfüllt:

$$C^{(\sigma_i\sigma_j)^{m_{ij}(c)}} = C$$

Das bedeutet, daß $2m_{ij}(C)$ benachbarte Simplicen beim Schnitt der i- und j-Seitenflächen des beliebigen Simplex von $\mathcal{C}$, sich treffen. M ist die verallgemeinerte Coxeter–Schläflische Matrixfunktion mit den folgenden zusätzlichen Eigenschaften:

$$m_{ij}(C) = m_{ji}(C) = m_{ij}(C^{\sigma_i})$$

$$m_{ii}(C) = 1; \quad m_{ij}(C) = 2 \text{ wenn } |i-j| \geq 2$$

$$m_{ij}(C) \geq 3 \text{ wenn } |i-j| = 1$$

für jedes $C \in \mathcal{C}$ und $i,j \in I$.

Nehmen wir an, daß Γ eine nicht triviale Gruppe ist, die auf T wirkt. Ferner behält jedes Element von Γ die baryzentrische Unterteilung von T und so die σ_i Operationen. Die Zerlegung $\mathcal{C}$ und auch die Matrixfunktion M wird durch die Gruppe Γ faktorisiert. Ein Element von $\mathcal{D} := \mathcal{C}/\Gamma$ ist eine Bahn (Orbit).

$$D := C^\Gamma := \{C^\gamma \in \mathcal{C}; \gamma \in \Gamma\}$$

Die induzierte Matrixfunktion ist die Folgende:

$$\mathcal{M} : \mathcal{D} \to N_{I \times I}; \quad D \to (m_{ij}(C)); \quad C \in D; \quad i,j \in I.$$

Num können wir ein sogenanntes D-Symbol, $\mathcal{D}(\sum_I, \mathcal{M})$, definieren, das zu einer Pflasterung (T, Γ) gehört. Das D-Symbol besteht aus dem D-Graph und aus der entsprechenden Matrixfunktion $\mathcal{M} = m_{ij}$.

Der D-Graph wird durch die D-Menge $\mathcal{D}$ und die Operationen $\sigma_i \in \sum_I$ $(i \in I)$ definiert. Die Bezeichnungen der Kantenlinien des D-Graphs und auch der entsprechenden σ_i-Operationen sind die Folgenden:

σ_0 : ....

σ_1 : – – – – –

σ_2 : ————

σ_3 : ~~~~~~~~

Von einem D-Symbol ausgehend, sehen wir, daß es die entsprechende Zerlegung (wenn sie existiert) bisauf äquivariante Bijektion kennzeichnet, wie es im Abschnitt 4 beschreiben werden.

4. Die Würfelpflasterungen

In der Arbeit [1] haben A. W. M. DRESS, D. H. HUSON und E. MOLNÁR mit Hilfe eines Computerprogramms, die vollständige Aufzählung der kombinatorischen Polyederpflasterungen (T, Γ) angegeben, wo die Bewegungsgruppe Γ auf den Flächen der Pflasterung transitiv wirkt, und die Polyeder von T eigentliche Ecken besitzen. Der Algorithmus des Computerprogramms beruht auf der Theorie des D-Symbols.

In diesem Arbeit interessieren wir uns für diejenigen Würfelpflasterungen, wobei die Forderungen im Punkt 2.1. erfüllt sind. Wir führen die Untersuchung derjenigen Fällen durch, wo die Bewegungsgruppe Γ maximal ist $(\Gamma \cong Aut\, T)$.

Durch D-Symbole fordert man die folgenden.

a. Der D-Graph $\mathcal{D}^2$, der von $\mathcal{D}$ durch Streichen der σ_2-Operation entsteht, hat genau eine Komponente (Flächentransitivität).

b. Die Matrixfunktion hat konstante Koeffizienten $m_{01} = 4$, $m_{12} = 3$ auf $\mathcal{D}$ (die Steine sind Würfel).

c. Das D-Symbol $\mathcal{D}$ läßt keine (eigentliche) sürjektive Abbildung auf ein anderes D-Symbol $\Psi : \mathcal{D} \to \overline{\mathcal{D}}$ mit $|\mathcal{D}| > |\overline{\mathcal{D}}|$, $\Psi(D^{\sigma_i}) = (\Psi(D))^{\overline{\sigma}_i}$, $m_{ij}(\Psi(D)) = m_{ij}(D)$ für jedes $D \in \mathcal{D}$, $i,j \in I = \{0,1,2,3\}$ (Maximalität der Gruppe $\Gamma \cong Aut\, T$).

d. Das Teilsymbol $\mathcal{D}^0(\mathcal{M}^0)$, das durch Streichen der σ_0-Operation und der 0-ten Reihen und Spalten der Matricen entsteht, hat nur solche Komponenten $\mathcal{D}_c^0$, für die die sogenannten Krümmungskonstanten

$$K_{\mathcal{D}_c^0} = \sum_{D \in \mathcal{D}_c^0} \left(\frac{1}{m_{ij}(D)} + \frac{1}{m_{23}(D)} - \frac{1}{2} \right) > 0$$

positiv sind. So haben wir ausschließlich eigentliche Ecken für die Pflasterung.

Nun bekommen wir die folgenden Fällen:

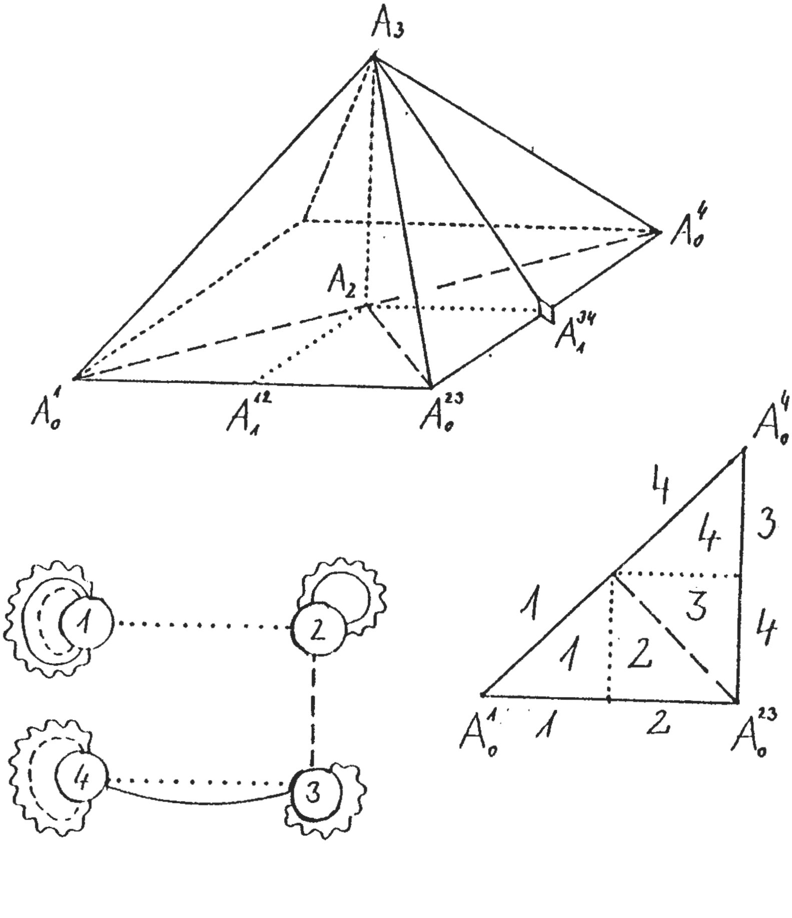


Abb. 3

4.1. In Abb. 3. sehen wir den D-Graph, der durch die folgenden involutorischen Permutationen (Transpositionen) angegeben wurden ($i \leftrightarrow D_i$).

$\sigma_0 : (1,2)(3,4)$ ....

$\sigma_1 : (1)(2,3)(4)$ – – – – –

$\sigma_2 : (1)(2)(3,4)$ ————

$\sigma_3 : (1)(2)(3)(4)$ ∿∿∿∿∿

Die Matrixfunktionen für die Würfelpflasterungen bekommen wir auch aus der Computerliste:

$$m_{01}(D) = 4, \quad m_{12}(D) = 3 \quad \text{für jedes} \quad D \in \mathcal{D},$$

$$m_{23}(D_1, D_2) = 4, \qquad m_{23}(D_3, D_4) = 6.$$

Wenn wir das Teildiagramm $\mathcal{D}^2$ nach weglassen der Kanten der σ_2 Operation aus dem D-Graph bilden, finden wir eine Komponente. Das bedeutet, daß die dem D-Symbol entsprechende Pflasterung (T, Γ) flächentransitiv ist.

In Abb. 3 sehen wir den aus 4 Simplexen zusammengeklebten Fundamentalbereich für die Gruppe Γ.

$$\mathcal{F}_\Gamma = A_3A_2A_1^{12}A_0^{1} \bigcup A_3A_2A_1^{12}A_0^{23} \bigcup A_3A_2A_1^{34}A_0^{23} \bigcup A_3A_2A_1^{34}A_0^{4}$$

Die Erzeugenden Bewegungen sind die Folgenden:

m_1 : ist eine Spiegelung an der Fläche $A_0^1A_0^4A_3$

m_2 : ist eine Spiegelung an der Fläche $A_0^1A_0^{23}A_3$

m_3 : ist eine Spiegelung an der Fläche $A_0^1A_0^{23}A_0^4$

r : ist eine Halbdrehung um die Achse $A_3A_1^{34}$

Unter diesen Erzeugenden bestehen die folgenden definierenden Relationen für die Gruppe Γ:

$$1 = m_1^2 = m_2^2 = m_3^2 = r^2 = (m_1m_2)^3 = (m_1m_3)^2 = (m_2m_3)^4 = m_2rm_1r = (m_3rm_3r)^3$$

nach dem Poincaréschen Algorithmus.

Nun können wir einen Würfel der Plasterungen aufbauen, wo die Stellung der Keilwinkel sieht so aus, wie sie Abb. 4. zeigt.

4.2. Im zweiten Fall gehen wir auch von den Ergebnissen des Computerprogramms aus. Die Abb. 5. zeigt den D-Graph, der durch die folgenden involutorischen Permutationen angegeben ist:

$\sigma_0 : (1,2)(3,4)(5,6)(7,8)$ ....

$\sigma_1 : (1,8)(2,3)(4,5)(6,7)$ – – – – –

$\sigma_2 : (1,2)(3,4)(5,8)(6,7)$ ————

$\sigma_3 : (1)(2)(3)(4)(5)(6)(7)(8)$ ∿∿∿∿∿

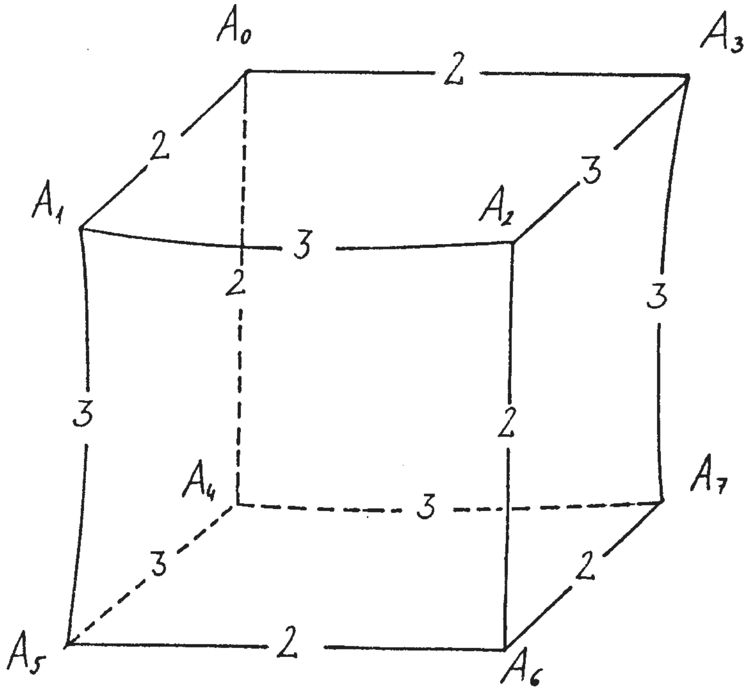


Abb. 4

Für die gewünschten Würfelpflasterungen bekommen wir die Matrixfunktion:

$$m_{01}(D) = 4; \quad m_{12}(D) = 3 \text{ für jedes } D \in \mathcal{D}$$
$$m_{23}(D_1, D_2) = 2q \qquad q = 4; 5$$
$$m_{23}(D_3, D_4) = 2p \qquad p = 3$$
$$m_{23}(D_5, D_6, D_7, D_8) = 4$$

Die erzeugenden Bewegungen sind die Folgenden:

r_1 : ist eine Halbdrehung um die Achse $A_1^{12} A_3$,

r_2 : ist eine Halbdrehung um die Achse $A_1^{34} A_3$,

r : ist eine 3-Drehung um die Achse $A_0^{67} A_3$,

m : ist eine Spiegelung an der Fläche $A_0^{18} A_0^{67} A_0^{45} A_0^{23}$.

Das Fundamentalbereich wird aus 8 Symplexen zusammengeklebt, wie Abb. 5. es darstellt.

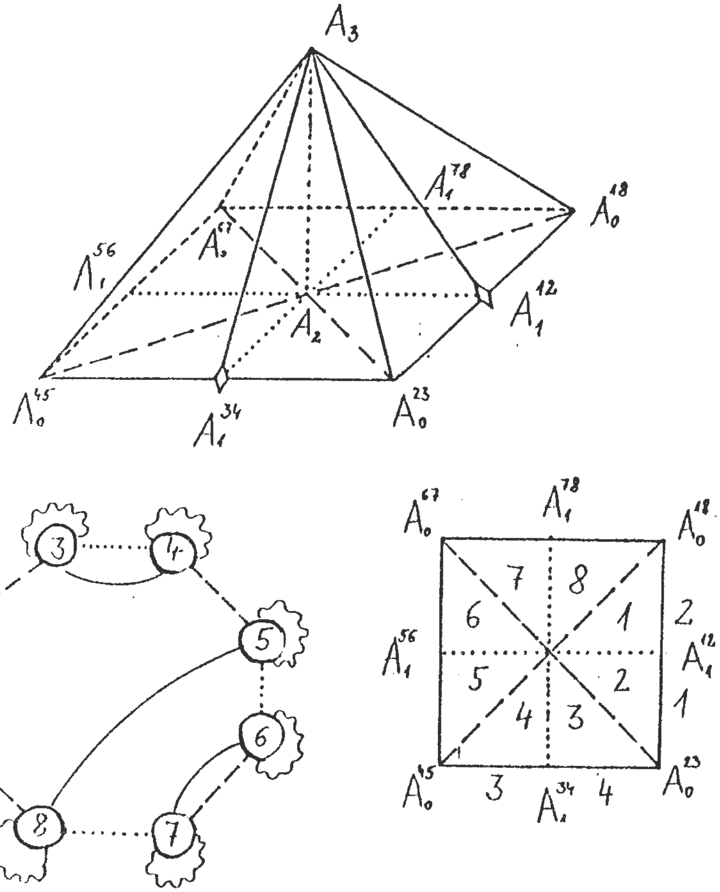


Abb. 5

Mit dem Hilfe des Poincare-Algorithmus bekommen wir die definierenden Relationen für die Gruppe Γ:

$$1 = m^2 = r^3 = r_1^2 = r_2^2 = r_1 r_2 r = (m r^{-1} m r)^2 = (r_1 m r_1 m)^p = (r_2 m r_2 m)^q .$$

Nun kriegen wir den Würfel, wobei die Stellung der Keilwinkel in der Abb. 6. gesehen werden kann.

Bemerkung: In diesem Fall bekommen wir die Lambert Würfeln (2.2) wenn $p > 2$ und $q = 2$ (Abb. 2.).

Wenn die vorigen Bedingungen unter 2.1. für die Pflasterungen erfüllt werden, dann haben wir mit Hilfe des Computerprogramms bewiesen daß die Stellung der Keilwinkel des Würfels sieht so aus wie sie Abb. 1. zeigt.

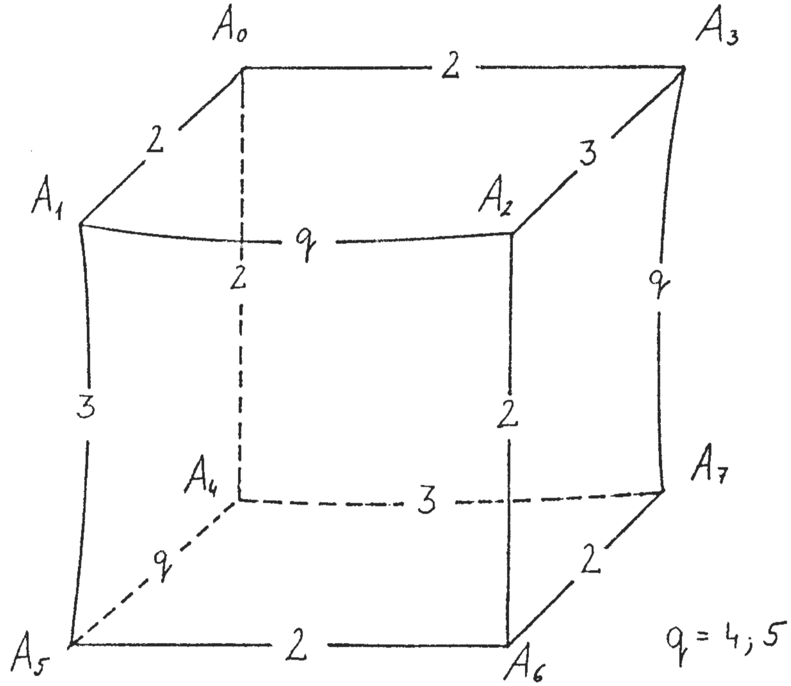


Abb. 6

5. Die metrische Existenz der Würfelpflasterungen $(p,q) = (3,3); (3,4); (3,5)$

5.1. In Abb. 1. sehen wir einen Würfel, für den die obigen Bedingungen (2.1.) erfüllt werden, wo die Spiegelungen an den Seitenflächen dieses Würfels zu Γ gehören. Wir stellen den Würfel in ein Koordinatensystem und führen zunächst die üblichen euklidischen Koordinaten: $A_0(0,0,0)$, $A_4(d,0,0)$, $A_7(x,y,0)$ ein, der Punkt $A_6(c,c,c)$ liege auf der Achse der 3-Drehung r. Die Parameter x, y, d, c seien reelle Zahlen.

Nun führen wir die Einheitskugel mit dem Zentrum A_0 ein, die das Cayley–Kleinsche Modell der hyperbolischen Raumgeometrie realisiert. Zu den euklidischen Basisvektoren $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ führen wir formal einen neuen Basisvektor $\underline{e}_0$ ein, um die homogenen Koordinaten zu gewinnen.

Die Vektoren $\underline{e}_0$, $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, bestimmen einen reellen Vektorraum V^4. Wir definieren ein Skalarprodukt mit der Formel: $\langle \underline{x}, \underline{y} \rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$ wo $\underline{x} = x^i \underline{e}_i$ und $\underline{y} = y^j \underline{e}_j$.

Über der Teilraumstruktur von V^4 gewinnen wir den projektiv metrischen Raum $P^3(V^4;\langle\langle\ ;\ \rangle\rangle)$. Der duale Raum des Vektorraums V^4 wird mit V_4 bezeichnet.

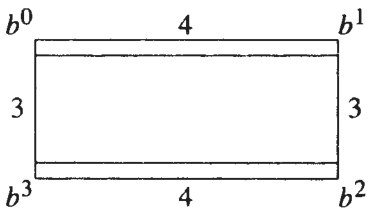
Wir nutzen aus, daß die euklidischen Bewegungen um das feste Zentrum A_0 genau die entsprechenden hyperbolischen Bewegungen des Modells darstellen, und ferner, die Geometrien der Sphären in den Räumen S^3, E^3, H^3 alle isomorph sind.

Die obigen homogenen Koordinaten der Punkte im projektiv-metrischen Raum sind:

$$\begin{array}{ll}
A_0 \sim \underline{a}_0(1;0;0;0), & A_7 \sim \underline{a}_7(1;x;y;0), \\
A_6 \sim \underline{a}_6(1;c;c;c), & A_4 \sim \underline{a}_4(1;d;0;0), \\
A_1 \sim \underline{a}_4^r(1;0;0;d), & A_3 \sim \underline{a}_4^{r^{-1}}(1;0;d;0), \\
A_2 \sim \underline{a}_7^{r^{-1}}(1;0;x;y), & A_5 \sim \underline{a}_7^r(1;y;0;x).
\end{array}$$

5.2. *Die Würfelpflasterung im Fall* $(p;q) = (3;3)$. Die Abb. 7. zeigt für den Würfel der Abb. 4. ein Spiegelungstetraeder (ein von den 9 Lannerschen Tetraedern [3]) dh. den Fundamentalbereich einer hyperbolischen Coxeterschen Raumgruppe. Das Coxeter-Diagramm und die Coxeter–Schläfli Matrix des Tetraeders sind die Folgenden:

$$(b^{ij}) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$



Hier stellen die Formen b^0, b^1, b^2, b^3 vom dualen Raum V_4 die Flächenebenen des Tetraeders dar, und die Matrixelemente $b^{ij} := -\cos(b^i, b^j)$ kennzeichnen die Keilwinkel.

Die metrische Existenz dieses Würfels $[(p;q) = (3;3)]$ folgt aus der Existenz des obigen Tetraeders. (Abb. 7.) Wir bestimmen die Parameter c, d, x, y.

Wegen Det $(b^{ij}) = -\frac{7}{16} < 0$ können wir die inverse Matrix $(a_{ij}) = (b^{ij})^{-1}$ mit $a_{ik}b^{kj} = \delta_i^j$ bilden (Einstein-Konvention).

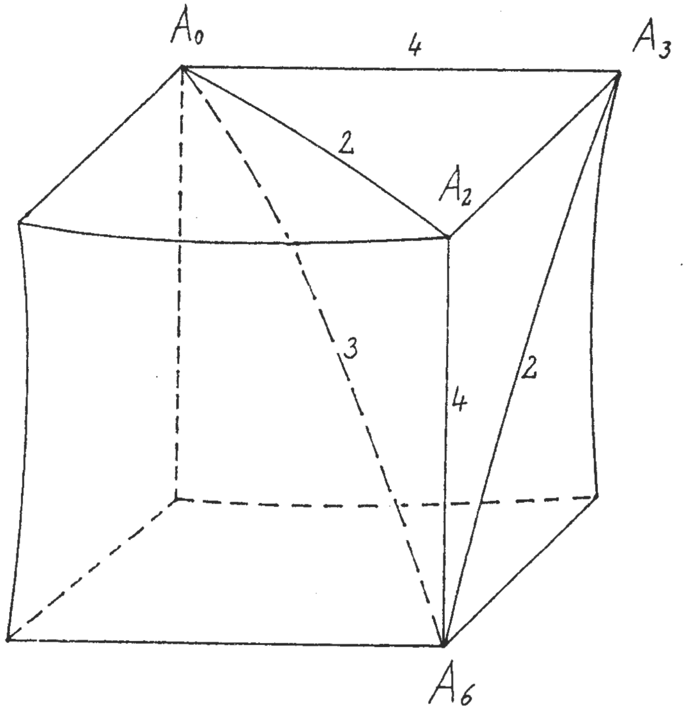


Abb. 7

Die Entfernung d_{ik} zwischen den eigentlichen Ecken $(\underline{a}_i)$, $(\underline{a}_k)$ ist:

$$chd_{ik} = \frac{-\langle \underline{a}_i ; \underline{a}_k \rangle}{\sqrt{\langle \underline{a}_i ; \underline{a}_i \rangle \langle \underline{a}_k ; \underline{a}_k \rangle}} = \frac{-a_{ik}}{\sqrt{a_{ii} a_{kk}}},$$

In diesem Fall haben wir die folgenden Resultate bekommen:

$$(p;q) = (3;3)$$

$$c \approx 0,52915026 \qquad d \approx 0,88191710$$

$$x \approx 0,66143783 \qquad y \approx 0,66143783$$

5.3. *Die Würfelpflasterungen in den Fällen* $(p;q) = (3;4), (3;5)$. Wir nutzen aus, daß die Keilwinkel des Würfels bekannt sind und, daß die Winkel α zwischen den eigentlichen Ebenen $u, v (\in V_4)$ $\cos\alpha = \frac{-\langle u,v \rangle}{\sqrt{\langle u,u \rangle \langle v,v \rangle}}$ ist. Den Würfel stellen wir in ein Koordinatensystem wie wir das im Punkt 5.1. gezeigt haben. Nun ergeben sich die Gleichungen der Seitenflächenebenen des Würfels.

Die erste Geichung gewinnen wir aus der Bedingung daß die vier Punkte A_4, A_5, A_6, A_7 in einer Ebene liegen.

Der Winkel $\frac{\pi}{3}$ zwischen den Würfelebenen $A_4A_5A_6A_7$ und $A_0A_3A_4A_7$ ergibt die zweite, der Winkel $\frac{\pi}{2}$ zwischen den Würfelebenen $A_0A_3A_4A_7$ und $A_0A_1A_4A_5$ ergibt die dritte, der Winkel $\frac{\pi}{q}$ (q = 4,5) zwischen den Würfelebenen $A_4A_5A_6A_7$ und $A_0A_1A_4A_5$ liefert die vierte Gleichung.

Nun haben wir für die vier Unbekannten c, d, x, y mit $0 < c < \sqrt{3}$, $1 - x^2 - y^2 > 0$, $0 < d < 1$, vier komplizierte Gleichungen gefunden.

Wir haben zur Lösung des Gleichungsystems ein Computerprogramm (EUREKA) benutzt.

Die exakten Lösungen des Falls $(p;q) = (3;3)$ haben wir bei der Computerlösung des Geichungsystems verwendet.

Wir haben die folgenden Resultate bekommen:

$$\begin{array}{lll} (p;q) = (3;4) & c \approx 0{,}53911695 & d \approx 0{,}94909399 \\ & x \approx 0{,}61220809 & y \approx 0{,}75625274 \\ (p;q) = (3;5) & c \approx 0{,}54261145 & d \approx 0{,}98159334 \\ & x \approx 0{,}58048682 & y \approx 0{,}80221305 \end{array}$$

Endlich gewinnen wir, daß die Würfelpflasterungen $(p;q) = (3;3)$: (3;4); (3;5) im Bolyai–Lobatschewskischen hyperbolischen Raum H^3 wirklich realisierbar sind. Der Satz ist bewiesen.

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TOPOLOGICAL SPACES THAT ADMIT BICOMPLETE QUASI-PSEUDOMETRICS

By

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1. Introduction

We shall be concerned with topological spaces that admit bicomplete quasi-pseudometrics. As a motivation for this study consider the paradigmatic example of the unit interval I with the quasi-pseudometric p defined by $p(x,y)=(y-x)\vee 0$. The decreasing sequence $\langle 1/n\rangle$ converges in the topology $T(p)$ to all points in I, however it is p^*-convergent to its "optimal" limit, namely 0.

In Section 2 we present an example of a quasi-pseudometrizable topological space which admits no compatible bicomplete quasi-pseudometric. In Section 3 we investigate conditions under which a quasi-pseudometrizable topological space admits a compatible bicomplete quasi-pseudometric. From the obtained results we deduce that the Euclidian topology on the set $\mathbb{Q}$ of rationals admits a compatible bicomplete quasi-pseudometric d such that d^{-1} generates the discrete topology on $\mathbb{Q}$. Hence a metrizable space may admit a compatible bicomplete quasi-metric without having any compatible complete metric. Finally, in Section 4 we characterize those quasi-pseudometrizable topological spaces for which every compatible quasi-pseudometric is bicomplete. In particular, it is proved that these spaces are exactly the hereditarily compact quasi-sober second countable spaces.

Our basic reference for topology is [3]. For quasi-uniform spaces it is [6] and for bitopological spaces it is [8].

If A is a subset of a set X and T is a topology on X, then $T\mathrm{cl}\,A$ denotes the closure of A in the topological space (X,T) and $T\mathrm{int}\,A$ its interior. If $A=\{x\}$, $x\in X$, we will write, simply $T\mathrm{cl}x$ and $T\mathrm{int}x$, respectively.

Recall that a quasi-pseudometric on a set X is a non-negative real valued function d on $X\times X$ such that for all x, y, $z\in X$: (i) $d(x,x)=0$ and (ii)

$d(x,y) \leq d(x,z)+d(z,y)$. If d satisfies the additional condition (iii) $d(x,y)=0 \Rightarrow x=y$, then d is called a quasi-metric on X.

A quasi-(pseudo)metric space is a pair (X,d) such that X is a nonempty set and d is a quasi-(pseudo)metric on X.

Each quasi-pseudometric d on X generates a topology $T(d)$ on X which has as a base the family of d-balls $\{B_d(x,r) : x \in X, r > 0\}$ where $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

A topological space (X,T) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X compatible with T, where d is said to be compatible with T provided that $T = T(d)$.

If d is a quasi-(pseudo)metric on X, we denote by d^{-1} the quasi-(pseudo)metric on X given by $d^{-1}(x,y)=d(y,x)$ and by d^* the (pseudo)metric $d \vee d^{-1}$.

A quasi-pseudometric d on X is called bicomplete if d^* is a complete pseudometric on X.

The quasi-pseudometric p, defined on $\mathbb{R}$ (the set of real numbers) by $p(x,y) = (y-x) \vee 0$, induces the lower topology u on $\mathbb{R}$, with basic open sets $(-\infty,a)$; the upper topology $l(= T(p^{-1}))$ has basic open sets $(b,+\infty)$. Furthermore p^* is the usual metric on $\mathbb{R}$. Hence p is a bicomplete quasi-pseudometric on $\mathbb{R}$.

Given a topological space (X,T), we denote by $T^{\#}$ the topology on X with $T\mathrm{cl}x$ as a base for the $T^{\#}$-neighbourhoods of x. The bitopological space $(X,T,T^{\#})$ is 2-separated (i.e. $(X,T \vee T^{\#})$ is a Hausdorff topological space [19]) if and only if (X,T) is T_0.

Recall that $T \vee T^{\#}$ is precisely the b-topology of (X,T), i.e. the topology $T(\mathcal{P}^*)$ where $\mathcal{P}$ denotes the Pervin quasi-uniformity of (X,T) (see [2], [20], [7], [19], [11], [12]). Since $T(\mathcal{P}^{-1}) = T^{\#}$, $(X,T,T^{\#})$ is pairwise normal and pairwise completely regular (see [19]).

A nonempty subspace A of a topological space (X,T) is called irreducible [1, Chapter 2] if each pair of nonempty A-open subsets has a nonempty intersection. A topological space (X,T) is called quasi-sober if each closed irreducible subset is of the form $T\mathrm{cl}x$ for some $x \in X$ [7, p. 154].

REMARK 1. It is known that for any quasi-pseudometric d on a set X for which $T(d) \subseteq T$, it follows that the identity map $i : (X,T,T^{\#}) \to (X,T(d),T(d^{-1}))$ is bicontinuous and, as a result, the fine quasi-uniformity of (X,T) is precisely the finest compatible quasi-uniformity of $(X,T,T^{\#})$.

2. Quasi-pseudometrizable spaces which admit no compatible bicomplete quasi-pseudometrics

There are many examples of quasi-pseudometrizable spaces that, on closer analysis, are seen to admit a bicomplete quasi-pseudometric (the Sorgenfrey line, the Kofner plane, the Pixley–Roy space on the real line and the space $(\mathbb{R},u)$ are some of these examples). The profusion of such spaces raises the question of whether every quasi-pseudometrizable space admits a compatible bicomplete quasi-pseudometric. In the absence of such criteria as Čech completeness and of a generalized Baire Category Theorem in the discussion of individual examples, it is not always clear how to proceed.

EXAMPLE 1. The rationals with the lower topology $(\mathbb{Q},u)$ admit no compatible bicomplete quasi-pseudometric.

Clearly, the basic $u^{\#}$-neighbourhoods of x in $\mathbb{Q}$ are of the form $[x,+\infty)$. The basic $u \vee u^{\#}$-neighbourhoods are of the form $[x,a)$, $x < a$. Consider any quasi-pseudometric d on $\mathbb{Q}$, with $T(d) = u$. Since the topology u determines the usual order on $\mathbb{Q}$, it is not surprising that the same is true of d. We shall establish the following facts:

(i) $a \leq b \;\Leftrightarrow\; d(b,a) = 0$.

It is clear, from $T(d) = u$, that $d(b,a) = 0 \;\Leftrightarrow\; b \in (u\mathrm{cl}\, a) \;\Leftrightarrow\; a \leq b$.

(ii) $l \subseteq T(d^{-1}) \subseteq u^{\#}$.

Let F be an l-closed set and $x \in \mathbb{Q} \setminus F$. Then $y < x$ for all $y \in F$. Since $x \notin F$, there is $r \in \mathbb{Q}$ such that for all $y \in F$, $y < r < x$. By (i) above we can write $d(r,x) = 2s > 0$. Then $B_d(r,s)$ and $B_{d^{-1}}(x,s)$ are disjoint. Also, $y \in F$ gives $d(r,y) = 0$ by (i), so that $F \subseteq B_d(r,s)$. Hence $B_{d^{-1}}(x,s) \subseteq \mathbb{Q} \setminus F$. Finally the inclusion $T(d^{-1}) \subseteq u^{\#}$ is known (see Remark 1). Consequently, $u \vee l \subseteq T(d^*) \subseteq u \vee u^{\#}$. Hence, for all $x \in \mathbb{Q}$,

$$T(d^*)\mathrm{int}(T(d^*)\mathrm{cl}\, x) \subseteq (u \vee u^{\#})\mathrm{int}((u \vee l)\mathrm{cl}\, x) = (u \vee u^{\#})\mathrm{int}\, x = \emptyset.$$

Thus, the metrizable space $(X, T(d^*))$ is a countable union of nowhere dense sets and, consequently, d^* cannot be complete.

3. Topological spaces which admit at least one compatible bicomplete quasi-pseudometric

If not all quasi-pseudometrizable spaces (X,T) admit a compatible bicomplete quasi-pseudometric, which ones do? Not much is known about the solution of this seemingly difficult problem; see also [18].

Before we give criteria akin to the characterization of Čech completeness for Tychonoff spaces, we shall consider an example of a metrizable space which admits no compatible complete metric, that is, it is not Čech complete and, nevertheless, admits a bicomplete quasi-metric. It seems appropriate to point out here that Fletcher and Lindgren showed in [5] that a metrizable space is completely metrizable if and only if it has compatible complete quasi-metric (in the sense of [5] and [6]) and that this result has been generalized in [16] to metrizable left K-sequentially complete quasi-metric spaces. Therefore, the next example shows that the situation is quite different in the bicomplete case.

EXAMPLE 2. The rationals with the Sorgenfrey topology S.

Basic open sets of S are of the form $[x,a)$, $x<a$ in $\mathbb{Q}$. Then $(\mathbb{Q},S)$ is a regular Hausdorff space with a countable base, hence metrizable. (In fact it is homeomorphic to $\mathbb{Q}$ equipped with its usual Euclidian topology.) Since it is a countable union of nowhere dense sets (singletons) it is not Čech complete. However, consider the quasi-metric d on $\mathbb{Q}$ given by $d(x,y)=1$ if $x>y$ and $d(x,y)=\min\{1,y-x\}$ if $x\leq y$. Then $T(d)=S$ and $d^*(x,y)=1$ if $x\neq y$. Thus d is a compatible bicomplete quasi-metric on $(\mathbb{Q},S)$.

For second countable spaces the bitopological theorem [18, Theorem 2.1] yields a purely topological condition that ensures the existence of a compatible bicomplete quasi-pseudometric.

For a base $\mathcal{B}_0$ of a topology T, let $\mathcal{B}$ consist of all finite unions of finite intersections of members of $\mathcal{B}_0$. Then $\mathcal{B}$ is a base for T and is itself closed under finite unions and finite intersections. Moreover, if $\mathcal{B}_0$ is countable, so is $\mathcal{B}$. Now $\mathcal{B}$ determines a companion topology S, turned towards the other side, as it were, with base $\mathcal{B}^c=\{X\setminus B\ :\ B\in\mathcal{B}\}$. Then $T\vee S=:T(\mathcal{B}^*)$ where $\mathcal{B}^*$ is the smallest Boolean algebra in 2^X containing $\mathcal{B}$ and $\mathcal{B}^c$. A base for $T\vee S$ consists of all sets $B_1\setminus B_2$ where B_1, $B_2\in\mathcal{B}$.

THEOREM 1. *Let $\mathcal{B}_0$ be a countable base for a topology T on a set X. If $T(\mathcal{B}^*)$ is Čech complete, then (X,T) admits a compatible bicomplete quasi-pseudometric.*

PROOF. If $\mathcal{B}_0$ is countable, so is $\mathcal{B}^c$. Then (X,T,S) is pairwise regular and both T and S are second countable, so the space (X,T,S) admits a compatible quasi-pseudometric [8]. If the pseudometrizable space $(X,T\vee S)$ is Čech complete, then it admits a compatible complete metric. By [18, Theorem 2.1] it follows that (X,T,S) admits a compatible bicomplete quasi-pseudometric, hence so does (X,T).

A slight modification of the proof given in Theorem 1 provides the following

PROPOSITION 1. *Let (X,T) be a countable quasi-pseudometrizable space. If $(X,T \vee T^{\#})$ is a Čech complete space then $(X,T,T^{\#})$ admits a compatible bicomplete quasi-pseudometric.*

EXAMPLE 3. Let m denote the usual topology on the set $\mathbb{Q}$ of rationals. Then $m^{\#}$ is the discrete topology on $\mathbb{Q}$. Hence $(\mathbb{Q},m \vee m^{\#})$ is Čech complete and, thus, $(\mathbb{Q},m,m^{\#})$ admits a compatible bicomplete quasi-metric.

4. Quasi-pseudometrizable topological spaces for which every compatible quasi-pseudometric is bicomplete

In Theorem 2 we shall give a characterization of such spaces in terms of disjoint-absorbing sequences.

DEFINITION 1. Let (X,T) be a topological space. A sequence $\langle x_n \rangle$ in X is said to be *disjoint-absorbing*, a *T-DA*, or simply, a *DA-sequence*, if it converges to all its cluster points. (DA-sequences were called primitive sequences by J. M. G. FELL in his paper "A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space", *Proc. Amer. Math. Soc.* **13** (1962), 472–476.)

Equivalently, $\langle x_n \rangle$ is a DA-sequence if and only if given $x \in X$ there is a T-open set V containing x such that $\langle x_n \rangle$ is eventually off V, or else, $\langle x_n \rangle$ is eventually in every neighbourhood of x.

DEFINITION 2. A quasi-pseudometrizable space (X,T) will be called *fully complete* if every compatible quasi-pseudometric is bicomplete.

THEOREM 2. *For a quasi-pseudometrizable space (X,T) the following conditions are equivalent:*

(1) *(X,T) is fully complete.*

(2) *Each DA-sequence in X has a $T(d^*)$-cluster point for any quasi-pseudometric d on X compatible with T.*

(3) *Each DA-sequence in X has a b-cluster point.*

(4) *Each DA-sequence in X is b-convergent.*

(5) *(X,T) is hereditarily compact and quasi-sober.*

(6) *The b-topology of (X,T) is compact.*

PROOF. (1) $\Rightarrow$ (2). Let $\langle x_n \rangle$ be a DA-sequence in X and d a quasi-pseudometric on X compatible with T. For each pair $n,k \in \mathbb{N}$ put

$$V_n = \{(x,y) : d(x,y) < 2^{-n}\}, \quad A_k = \{x_n : n \geq k\} \quad \text{and}$$

$$U_{nk} = V_n \cup \left[V_n^{-1}(A_k) \times V_n(A_k) \right].$$

As in the proof of [6, Lemma 7.35] or [17, Proposition on page 226], $\{U_{nk} : n, k \in \mathbb{N}\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X. Hence there is a quasi-pseudometric e on X that generates $\mathcal{U}$ and such that $\langle x_n \rangle$ is an e^*-Cauchy sequence. We want to show that e is compatible with T. Clearly, $T(e) \subseteq T$ because $V_n(x) \subseteq U_{nk}(x)$ for all $x \in X$ and $n, k \in \mathbb{N}$. In order to prove the reverse inclusion, take $x \in X$ and suppose, firstly, that x is a T-limit point of $\langle x_n \rangle$. Then, given $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $A_k \subseteq B_d(x, 2^{-(n+1)})$. Let $y \in U_{(n+1)k}(x)$. Then $d(x,y) < 2^{-(n+1)}$, or there is $j \geq k$ such that $d(x_j, y) < 2^{-(n+1)}$. Since $d(x,x_j) < 2^{-(n+1)}$ it follows that $d(x,y) < 2^{-n}$. Therefore $U_{(n+1)k}(x) \subseteq B_d\left(x, 2^{-n}\right)$. If x is not a T-limit point of $\langle x_n \rangle$, then there are n and k with $B_d\left(x, 2^{-n}\right) \cap A_k = \emptyset$. So, $U_{nk}(x) = B_d\left(x, 2^{-n}\right)$. Thus, e is compatible with T. Hence there is $y \in X$ such that $e^*(y, x_n) \to 0$ and, hence, $\langle x_n \rangle$ is eventually in each $U_{nk}^{-1}(y)$. Then there exists a subsequence $\langle x_{n(m)} \rangle$ of $\langle x_n \rangle$ such that $d\left(x_{n(m)}, y\right) \to 0$. Since $d(y, x_n) \to 0$, we conclude that y is a $T(d^*)$-cluster point of $\langle x_n \rangle$.

(2) $\Rightarrow$ (3). Let $\langle x_n \rangle$ be a DA-sequence in X. Fix a quasi-pseudometric p on X compatible with T. Then there exists a point $z \in X$ and a subsequence $\langle x_{n(m)} \rangle$ of $\langle x_n \rangle$ such that $p^*\left(z, x_{n(m)}\right) \to 0$. Let $d(x,y) = 1$ if $x \notin T\mathrm{cl}\, z$ and $y \in T\mathrm{cl}\, z$, and $d(x,y) = 0$ otherwise. Put $q = d \vee p$. Then q is a quasi-pseudometric on X compatible with T. Since $\langle x_{n(m)} \rangle$ is a DA-sequence, there are an $a \in X$ and a subsequence $\left\langle x_{n(m_j)} \right\rangle$ of $\langle x_{n(m)} \rangle$ such that $q^*(a, x_{n(m_j)}) \to$ $\to 0$. Hence, $p^*(a, x_{n(m_j)}) \to 0$ and, thus, $p^*(a,z) = 0$. Since p and q and compatible with T, $q^*(a,z) = 0$. Hence $x_{n(m_j)} \in T\mathrm{cl}\, z$ eventually and thus z is a b-cluster point of $\langle x_n \rangle$.

(3) $\Rightarrow$ (4). We first show that for each quasi-pseudometric d on X compatible with T, $T(d^*) = T \vee T^{\#}$. In fact, let d be any such quasi-pseudometric. Since $T(d^{-1}) \subseteq T^{\#}$ we have $T(d^*) \subseteq T \vee T^{\#}$. Now let $d^*(x, x_n) \to 0$ for some $x \in X$ and some sequence $\langle x_n \rangle$ in X. Then $\langle x_n \rangle$ is a DA-sequence. Hence, it has a b-cluster point $y \in X$. So there is a subsequence $\langle x_{n(m)} \rangle$ of $\langle x_n \rangle$ such that $d\left(y, x_{n(m)}\right) \to 0$ and $d\left(x_{n(m)}, y\right) = 0$ for all $m \in \mathbb{N}$. By the triangle inequality $d(y,x) = 0$ and $d\left(x_{n(m)}, x\right) = 0$ for all $m \in \mathbb{N}$. Therefore x is a b-cluster point of $\langle x_n \rangle$ and, thus $T \vee T^{\#} \subseteq T(d^*)$.

Now let $\langle x_n \rangle$ be a DA-sequence in X. As in the proof of (1) $\Rightarrow$ (2), there is a quasi-pseudometric e on X compatible with T such that $\langle x_n \rangle$ is an e^*-Cauchy sequence. Since $T\left(e^*\right) = T \vee T^{\#}$ and, by hypothesis, $\langle x_n \rangle$ has a b-cluster point $x \in X$, it follows that $e^*(x, x_n) \to 0$. So $\langle x_n \rangle$ is b-convergent to x.

(4) $\Rightarrow$ (5). Assume that each DA-sequence in the quasi-pseudometrizable space (X,T) is b-convergent. Consider the set $\mathcal{H}=\{G\subseteq X\ :\ G$ is open and not compact in $(X,T)\}$ ordered by the set-theoretic inclusion. We want to show that $\mathcal{H}$ is empty. Of course, this will mean that (X,T) is hereditarily compact. Assume that $\mathcal{H}\neq\emptyset$. Since for any nonempty chain $\mathcal{K}\subseteq\mathcal{H}$ we have $\cup\mathcal{K}\in\mathcal{H}$, we conclude by Zorn's Lemma that there exists a maximal element $Y\in\mathcal{H}$. We consider two cases:

Case (a): Suppose that $L=\{T\mathrm{cl}x\ :\ x\in X\setminus Y\}$ is countable. For each $(T\mathrm{cl}y)\in L$ choose exactly one $x\in(T\mathrm{cl}y)$ such that $T\mathrm{cl}x=T\mathrm{cl}y$. Enumerate the chosen elements by $\{x_n\ :\ n\in K\}$ where K is an initial segment of $\mathbb{N}$. We are going to show that Y is a sequentially compact subspace of (X,T). Assume the contrary. Let $\left\langle y_n^0\right\rangle$ be a sequence in Y without cluster point in the subspace Y of (X,T). Inductively define for each $s\in K$ a sequence $\langle y_n^s\rangle$ such that $\langle y_n^s\rangle$ is a subsequence of $\left\langle y_n^{s-1}\right\rangle$ as follows: Suppose that $j\in K$ and that $\langle y_n^s\rangle$ is defined for all $s<j$. If x_j is a cluster point of $\left\langle y_n^{j-1}\right\rangle$ in (X,T), then choose a subsequence $\left\langle y_n^j\right\rangle$ of $\left\langle y_n^{j-1}\right\rangle$ that converges to x_j in (X,T). (We always assume that the sequence $\langle n(k)\rangle$ of indices of a subsequence $\langle z_{n(k)}\rangle$ of $\langle z_n\rangle$ is strictly increasing.) Otherwise let $y_n^j=y_n^{j-1}$ whenever $n\in\mathbb{N}$. Finally for each $n\in\mathbb{N}$, set $z_n=y_n^n$ if $K=\mathbb{N}$, and set $z_n=y_n^{\max(K\cup\{0\})}$ if $\max(K\cup\{0\})$ exists.

We next verify that $\langle z_n\rangle$ is a DA-sequence in (X,T): Let a be a cluster point of $\langle z_n\rangle$ in (X,T). Then $T\mathrm{cl}a=T\mathrm{cl}x_s$ for some $s\in K$, because $\langle z_n\rangle$ is a subsequence of $\left\langle y_n^0\right\rangle$. Thus by definition, $\langle y_n^s\rangle$ converges to x_s in (X,T), since $\langle z_n\rangle$ is finally a subsequence of $\left\langle y_n^{s-1}\right\rangle$ and x_s is a cluster point of $\langle z_n\rangle$. Consequently $\langle z_n\rangle$ which is finally a subsequence of $\langle y_n^s\rangle$ converges to a in (X,T). We have shown that $\langle z_n\rangle$ is a DA-sequence in (X,T).

Thus it converges to some $x\in X$ with respect to the b-topology on (X,T). Consequently $\langle z_n\rangle$ converges to x in (X,T). Since Y is b-closed and $z_n\in Y$ whenever $n\in\mathbb{N}$, we have $x\in Y$. Since $\langle z_n\rangle$ is a subsequence of $\left\langle y_n^0\right\rangle$, $\left\langle y_n^0\right\rangle$ has the cluster point x in the subspace Y of (X,T) — a contradiction. We conclude that Y is a sequentially compact subspace of (X,T).

We next show that the following condition $(*)$ is satisfied:

For each $y\in Y$ there is $m_y\in(T\mathrm{cl}y)\cap Y$ such that $a\in(T\mathrm{cl}\,m_y)\cap Y$ implies that $(T\mathrm{cl}\,a)\cap Y=(T\mathrm{cl}\,m_y)\cap Y$.

Indeed, otherwise we define inductively a sequence $\langle x_n\rangle$ in Y with $x_1 = y$ such that $(T\mathrm{cl}x_{n+1})\cap Y \subsetneq (T\mathrm{cl}x_n)\cap Y$ whenever $n\in\mathbb{N}$. Now take a quasi-pseudometric d on X compatible with T. Then $d(x_{n+1},x_n)=0$. Obviously $\langle x_n\rangle$ is a DA-sequence in (X,T). Therefore $\langle x_n\rangle$ converges to some $x\in X$ with respect to the b-topology on (X,T). Thus $x\in\bigcap\{T\mathrm{cl}x_n : n\in\mathbb{N}\}$ since $\bigcap\{T\mathrm{cl}x_n : n\in\mathbb{N}\}$ is b-closed and $x_{n+1}\in(T\mathrm{cl}x_n)$ whenever $n\in\mathbb{N}$. On the other hand, $X\setminus\bigcap\{T\mathrm{cl}x_n : n\in\mathbb{N}\}$ is b-closed and $x_n\in X\setminus(T\mathrm{cl}x_{n+1})$ whenever $n\in\mathbb{N}$. Therefore $x\in X\setminus\bigcap\{T\mathrm{cl}x_n : n\in\mathbb{N}\}$ — a contradiction. We conclude that there is no such sequence $\langle x_n\rangle$ and the condition $(*)$ holds.

Set $A=\{x\in Y : (T\mathrm{cl}z)\cap Y=(T\mathrm{cl}x)\cap Y$ for all $z\in(T\mathrm{cl}x)\cap Y\}$. Note that for each $y\in Y$ the elements m_y defined above belong to A. We choose exactly one element y in every set of the form $(T\mathrm{cl}x)\cap Y$ with $x\in A$. (Note that these sets are pairwise disjoint). Let M be the set of all the chosen elements. Then M is a compact quasi-metric subspace of (X,d): If $x,y\in M$ and $d(x,y)=0$ then $x\in(T\mathrm{cl}x)\cap Y$. It follows that $(T\mathrm{cl}x)\cap Y=(T\mathrm{cl}y)\cap Y$, i.e. $x=y$ according to the definition of M. Therefore d is a quasi-metric on M. If $\langle x_n\rangle$ is a sequence in M, there are a subsequence $\langle x_{n(k)}\rangle$ of $\langle x_n\rangle$ and $x\in Y$ such that $d(x,x_{n(k)})\to 0$ because Y is a sequentially compact subspace of (X,T). By $(*)$ there is $z\in A\cap T\mathrm{cl}x$. Thus we have $d(y,z)=0$ and $d(z,x)=0$ for the chosen $y\in(T\mathrm{cl}z)\cap M$. There fore $\langle x_{n(k)}\rangle$ converges to y in the subspace M of (X,T). We conclude that M is a sequentially compact subspace of (X,T). Since a sequentially compact quasi-metric space is compact [14], we see that M is a compact subspace of (X,T).

Let $\{G_i : i\in I\}$ be an open cover of the set Y in (X,T) and let $\{G_i : i=1,\dots,n\}$ be a finite subcover for M. By $(*)$ for each $x\in Y$ there is $z\in M\cap(T\mathrm{cl}x)\cap Y$; in particular $d(z,x)=0$. The element $z\in M$ is covered by some G_m with $1\le m\le n$. Therefore $x\in G_m$. We have shown that $\{G_i : i=1,\dots,n\}$ covers Y. Hence Y is a compact subspace of (X,T) — a contradiction. We conclude that L cannot be countable.

Case (b): We suppose that L is uncountable. Take a quasi-pseudometric d on X compatible with T. If $\langle G_n\rangle$ is a sequence of open sets in (X,T) such that $\langle G_n\setminus Y\rangle$ is increasing and $G_1\setminus Y\neq\emptyset$, then for some $m\in\mathbb{N}$, $G_m\setminus Y=$ $=G_{m+1}\setminus Y$, because $\bigcup\{G_n\cup Y : n\in\mathbb{N}\}$ is compact in (X,T) by the definition of Y. Thus each strictly increasing sequence of open subsets in the subspace $X\setminus Y$ of (X,T) is finite and thus $X\setminus Y$ is hereditarily compact. Therefore (compare e.g. [10]) it admits a unique totally bounded quasi-uniformity and the Pervin quasi-uniformity of the subspace $X\setminus Y$ of (X,T) is coarser than the quasi-uniformity induced by d on $X\setminus Y$. We conclude that d^* induces the b-topology on $X\setminus Y$. We next show that $X\setminus Y$ is a quasi-sober subspace of (X,T). Let $\langle x_n\rangle$ be a sequence in $X\setminus Y$. By

[13, Theorem 3] $\langle x_n \rangle$ has a left K-Cauchy subsequence $\langle x_{n(k)} \rangle$ in (X,d), since the subspace $X \setminus Y$ of (X,T) is hereditarily compact. Since $\langle x_{n(k)} \rangle$ is a DA-sequence in (X,T) [15, Theorem 1], it converges to some $x \in X$ with respect to the b-topology of (X,T). Because $X \setminus Y$ is b-closed and $x_n \in X \setminus Y$ whenever $n \in \mathbb{N}$, we have $x \in X \setminus Y$. Thus $X \setminus Y$ is a (sequentially) compact subspace of the pseudometric space (X,d^*). By [7, Corollary 3.2] the subspace $X \setminus Y$ of (X,T) is hereditarily compact and quasi-sober. Since $X \setminus Y$ is a subspace of (X,T) that according to [11, Proposition 3] admits a unique quasi-uniformity, d induces the Pervin quasi-uniformity on $X \setminus Y$. Since this Pervin quasi-uniformity has a countable base, the topology of the subspace $X \setminus Y$ of (X,T) is countable [4, Proposition 1]. It follows that L is countable — a contradiction.

We finally conclude that $\mathcal{H} = \emptyset$ and that (X,T) is hereditarily compact. The argument presented in the last part of the proof (with Y set equal to $\emptyset$) shows that (X,T) is quasi-sober.

(5) $\Leftrightarrow$ (6). Recall that a space is hereditarily compact and quasi-sober if and only if its b-topology is compact (see [7], [19], [11], [12]). It follows that (5) $\Rightarrow$ (1) , since each quasi-pseudometric d on X compatible with T is bicomplete (compare [11, Proposition 3]).

REMARK 2. In the proof of implication (4) $\Rightarrow$ (5) of the above theorem we have used methods from [9, Proposition] and [10, Proposition 2.4].

COROLLARY. *A quasi-metrizable space (X,T) is fully complete if and only if X is a finite set.*

COROLLARY. *A topological space is a fully complete quasi-pseudometrizable space if and only if it is a hereditarily compact quasi-sober second countable space.*

PROOF. The necessary condition follows from Theorem 2(5), [11, Proposition 3] and [4, Proposition 1]. The converse is a consequence of [11, Proposition 3] and [6, Chapter 7].

In our final result we shall characterize the fully complete quasi-pseudometrizable spaces having a linear specialization order.

PROPOSITION 2. *Let (X,T) be a nonempty fully complete quasi-pseudometrizable space with linear specialization order $x \leq y \Leftrightarrow x \in T\mathrm{cl}\, y$. Then (X,T) is with respect to order and topology isomorphic to $(\alpha+1, L, \leq)$ where α is a countable ordinal, L is the upper topology on $\alpha+1$ and $\leq$ denotes the usual order on $\alpha+1$.*

PROOF. Let d be a quasi-pseudometric on X compatible with T. Then (X,T) cannot have strictly decreasing infinite sequences, since they are DA-sequences without b-cluster point. Thus $\leq$ is a well-order on X. Since open sets are increasing, (X,T) is irreducible. Hence X is a point-closure, because (X,T) is quasi-sober. Thus X has a largest element α. Since $T(d^*)$ is the b-topology on X it is clear that $(a,b]=(T\text{cl}\, b)\setminus(T\text{cl}\, a)$ is $T(d^*)$-open whenever $a<b$, i.e. the usual compact T_2-order topology on $\alpha+1=X$ is coarser than $T(d^*)$. Since $T(d^*)$ is a compact Hausdorff topology, $T(d^*)$ is equal to the order topology on $\alpha+1$. Then the quasi-uniformity $\mathcal{U}_d$, induced by d, is the unique quasi-uniformity that determines the compact ordered space $(X,T(d^*),\leq)$. In particular, $T(\mathcal{U}_d)=T$ is the upper topology of $(X,T(d^*),\leq)$ (see e.g. [11, p. 240]). Since T is first countable, α is a countable ordinal. On the other hand, obviously each space $(\alpha+1,L,\leq)$ where α is a countable ordinal is hereditarily compact (quasi-)sober and second countable.

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A FIXED POINT THEOREM

By

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Dedicated to Professor Á. Császár on the occasion of his 70^{th} birthday

In this paper we prove a fixed point theorem for set-valued functions. Denote $K[0,1]$ the set of closed connected subsets of $[0,1]$ and let X be a connected topological space (see e.g. [1]).

DEFINITION. We say that a function $F: X \to K[0,1]$ is continuous if the $F(x)$ is $[F_0(x), F_1(x)]$ where F_0, F_1 are continuous.

DEFINITION. If A, $B \subseteq \mathbb{R}$ then $A - B := \{x : x \in \mathbb{R}, x = a - b, a \in A, b \in$ $\in B\}$.

DEFINITION. If A, $B \subseteq \mathbb{R}$ then $A < B$ means that $a \in A$, $b \in B$ imply $a < b$.

THEOREM. *Let F, $G: X \to K[0,1]$ be continuous functions and assume that*

$$\bigcup_{x \in X} F(x) = [0,1].$$

Then there exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.

PROOF. The proof is indirect. Assume that $F(x_0) \cap G(x_0) = \emptyset$ for all $x_0 \in X$. It is clear that $F - G: X \to K[-1,1]$ is continuous. (Here $(F - G)(x) = F(x) - G(x)$.) By the assumptions $0 \notin (F - G)(x)$ for all $x \in X$. Since $(F - G)(x)$ is a connected interval therefore $(F - G)(x) \subset [-1,0)$ or $(F - G)(x) \subset (0,1]$ for all $x \in X$. Denote $X_1 := \{x \in X : (F - G)(x) \subset [-1,0)\}$ and $X_2 := \{x \in X : (F - G)(x) \subset (0,1]\}$. By the continuity of $F - G$ we

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get X_1, X_2 are open in X. Obviously $X_1 \cap X_2 = \emptyset$. Since X is connected therefore $X_1 = X$ or $X_2 = X$. If $X_1 = X$ then $F(x) < G(x)$ for all $x \in X$ but in this case there exists $x_1 \in X$ such that $1 \in F(x_1) < G(x_1)$ which is a contradiction since $G(x_1) \subset [0,1]$. If $X_2 = X$ then $G(x) < F(x)$ for all $x \in X$ but in this case there exists $x_2 \in X$ such that $0 \in F(x_2)$, thus $G(x_2) < 0$ which is a contradiction since $G(x_2) \subset [0,1]$. The theorem is proved.

COROLLARY. If $f, g : [0,1] \to [0,1]$ are number-valued continuous functions and range$f = [0,1]$, then there exists a $x_1 \in [0,1]$ such that $f(x_1) = g(x_1)$.

Indeed, choose $X := [0,1]$, $F(x) := \{f(x)\}$, $G(x) := \{g(x)\}$.

REMARK. If we want to generalize this theorem for arbitrary topological space Y, then we cannot do it without restriction on Y. Indeed, let $X := = [0,2\pi]$ and $Y := \{x \in \mathbb{R}^2 : \|x\|_e \leq 1\}$. Define $F(0)$ and $G(0)$ in the following way. Let $F(0)$, $G(0)$ be closed disjoint subsets of Y, such that $F(0)$ contains a half-disc. Let $F(x)$, $G(x)$ be obtained from $F(0)$, $G(0)$ by a rotation with angle x around the centre of Y. Then F and G are continuous functions and

$$\bigcup_{x \in X} F(x) = Y.$$

But it is clear that $F(x_0) \cap G(x_0) = \emptyset$ for all $x \in X$.

PROBLEM. The Corollary and Remark motivate the next question. How can we generalize the Theorem for other spaces instead of $[0,1]$?

REMARK (in August 9, 1994). We can answer partially this problem in [2].

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ON MIXED QUASIMONOTONE PARABOLIC SYSTEMS

By

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In [5] HEIKKILE and LAKSHMIKANTHAM considered the following system of periodic boundary value problems (PBVS)

$$(*)\qquad \begin{aligned} &-u_i''(t)=f_i(t,u_i(t),[u]_{p_i}(t),[u]_{q_i}(t),u_i'(t)) \qquad \text{for a.e. } t\in J\\ &u_i(0)=u_i(T),\quad u_i'(0)=u_i'(T), \qquad i=1,\dots,n \end{aligned}$$

with $f_i\colon J\times\mathbb{R}^{n+1}\to E$, $i=1,\dots,n$ where E is an ordered Banach space with regular order cone K and $J=[0,T]$, $T>0$. In the notation $u=(u_i,[u]_{p_i},[u]_{q_i})$ for $u=(u_1,\dots,u_n)\in E^n$ the term $[u]_{p_i}$ is formed by p_i coordinates of u, different from u_i and $[u]_{q_i}$ contains the remaining ones, so that $p_i+q_i=n-1$. They have proved that the PBVS $(*)$ has coupled extremal quasisolutions if for each $i=1,\dots,n$ there is a Lebesque integrable function $h:J\to\mathbb{R}^+$ such that the function

$$f_i(t,u_i,[u]_{p_i},[u]_{q_i},v)-h_i(t)v$$

is increasing with respect to $[u]_{q_i}$ and decreasing with respect to u_i, $[u]_{p_i}$ and to v, adding some other assumptions.

We shall consider the following parabolic system

$$(1)\qquad \frac{\partial u_k}{\partial t}+L_k u_k=f_k(x,t,u_k,[u]_{p_k},[u]_{q_k}) \qquad \text{in } Q$$

$$(2)\qquad u_k=0 \qquad \text{on } S$$

$$(3)\qquad u_k(x,0)=g_k(x) \qquad \text{in } \Omega$$

$k = 1,\dots,n$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0,T)$, $S = \partial\Omega \times (0,T)$, $T > 0$ and

$$L_k u_k = -\sum_{ij=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial u_k}{\partial x_j}\right) + \sum_{i=1}^{N} b_i^k \frac{\partial u_k}{\partial x_i} + c^k u_k \quad k = 1,\dots,n.$$

There is no continuity condition imposed on f_k $(k = 1,\dots,n)$.

An existence result for a parabolic system with a discontinuous nonlinearity has been given by VON WAHL in [8]. In his paper, the discontinuous nonlinearity is approximated by continuous ones and it is shown that there exists a subsequence of solutions to the approximating nonlinear continuous problems converging to a solution of the original discontinuous problem.

Recently, GIANNI and MANNUCCI in [3] has considered

$$\frac{\partial u}{\partial t} - \Delta u = H(u-1) \qquad \text{in} \quad Q$$

where H is the Heaviside function. By using approximating methods they proved that the problem above with initial-boundary value conditions is solvable.

In the present paper we shall prove that the problem (1)–(3) has coupled extremal quasisolutions if f_k $(k = 1,\dots,n)$ satisfy conditions (f1)–(f3) stated later. As consequences we also get some existence theorems on solutions of parabolic systems and equations with discontinuous nonlinearities.

2. Hypotheses and auxiliary results

Assume now that the functions a_{ij}^k, b_i^k, c^k are bounded, measurable on Q, c^k are nonnegative in Q, and that $g_k \in L^2(\Omega)$, $i,j = 1, \dots, N$; $k = 1, \dots, n$. We shall also assume that there exist two positive constants $\mu_1 < \mu_2$ such that

$$\mu_1|\xi|^2 \le \sum_{ij=1}^{N} a_{ij}^k(x,t)\xi_i\xi_j \le \mu_2|\xi|^2 \quad \text{a.e. in } Q$$

for all $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^N$.

$W_2^{1,0}(Q)$ is the Hilbert space with scalar product

$$(u,v)_{W_2^{1,0}(Q)} = \int_0^T\int_\Omega \left(uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right) dx\,dt$$

and $W_2^{1,1}(Q)$ is the Hilbert space with scalar product

$$(u,v)_{W_2^{1,1}(Q)} = \int_0^T \int_\Omega \left(uv + \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right) dxdt.$$

$V_2(Q)$ is the Banach space consisting of all elements of $W_2^{1,0}(Q)$ having a finite norm cf. [6]

$$|u|_Q = \operatorname{vrai}\max_{0\le t\le T} \|u(x,t)\|_{2,\Omega} + \|\nabla u\|_{2,Q}$$

where

$$\|u(x,t)\|_{2,\Omega}^2 = \int_\Omega (u(x,t))^2 dx$$

$$\|\nabla u\|_{2,Q}^2 = \int_\Omega |\nabla u|^2 dxdt \equiv \int_Q \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2 dxdt.$$

$V_2^{1,0}(Q)$ is the Banach space consisting of all elements of $V_2(Q)$ that are continuous in t in the norm of $L^2(\Omega)$, with the norm

$$|u|_\Omega = \operatorname{vrai}\max_{0\le t\le T} \|u(x,t)\|_{2,\Omega} + \|\nabla u\|_{2,Q}.$$

The space $V_2^{1,0}(\Omega)$ can be obtained by completing the set $W_2^{1,1}(Q)$ in the norm of $V_2(Q)$ (cf. [6] for details).

A zero over $W_2^{1,0}(Q)$, $W_2^{1,1}(Q)$, $V_2(Q)$, $V_2^{1,0}(Q)$ means that only those elements of these spaces are taken which vanish on S.

For convenience, denote $\left[V_2^{1,0}(Q)\right]^n$ by V, $\left[\mathring{V}_2^{1,0}(Q)\right]^n$ by $\mathring{V}$ and let

$$I^k(t,v_k,z_k) \equiv \int_\Omega v_k(x,t)z_k(x,t)dx - \int_0^t \int_\Omega v_k \frac{\partial z_k}{\partial t} dxdt +$$
$$+ \int_0^t \int_\Omega \left[\sum_{ij=1}^N a_{ij}^k \frac{\partial v_k}{\partial x_i}\frac{\partial z_k}{\partial x_j} + \sum_{i=1}^N b_i^k \frac{\partial v_k}{\partial x_i} z_k + c^k v_k z_k \right] dxdt.$$

We say that the function $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ are coupled quasisolutions of the problem (1)–(3) if $v, w \in V$, and

(4) $$\frac{\partial v_k}{\partial t} + L_k v_k = f_k(x,t,v_k,[v]_{p_k},[w]_{q_k}) \qquad \text{in } Q$$

(5) $v_k = 0$ on S

(6) $v_k(x,0) = g_k(x)$ in Ω

(7) $\dfrac{\partial w_k}{\partial t} + L_k w_k = f_k(x,t,w_k,[w]_{p_k},[v]_{q_k})$ in Q

(8) $w_k = 0$ on S

(9) $w_k(x,0) = g_k(x)$ in Ω

$k = 1,\dots,n$ in weak sense. It means that the functions $v,w \in \overset{\circ}{V}$, and they satisfy the following equalities:

$$I^k(t,v_k,z_k) = \int_0^t \int_\Omega f_k(x,t,v_k,[v]_{p_k},[w]_{q_k}) z_k \, dx\,dt + \int_\Omega g_k(x) z_k(x,0)\,dx \tag{10}$$

$$I^k(t,w_k,z_k) = \int_0^t \int_\Omega f_k(x,t,w_k,[w]_{p_k},[v]_{q_k}) z_k \, dx\,dt + \int_\Omega g_k(x) z_k(x,0)\,dx \tag{11}$$

$k = 1, \dots, n$ for all $z = (z_1,\dots,z_n)$ from $\left[\overset{\circ}{W}{}_2^{1,1}(Q)\right]^n$

The functions $\hat{v} = (\hat{v}_1,\dots,\hat{v}_n)$, $\hat{w} = (\hat{w}_1,\dots,\hat{w}_n) \in V$ are said to be coupled upper and lower quasisolutions of the problem (1)–(3), if $\hat{v}$, $\hat{w}$ satisfy the following inequalities in weak sense

(12) $\dfrac{\partial \hat{v}_k}{\partial t} + L + k\hat{v}_k \le f_k\left(x,t,\hat{v}_k,[\hat{v}]_{p_k},[\hat{w}]_{q_k}\right)$ in Q

(13) $\hat{v}_k \le 0$ on S

(14) $\hat{v}_k(x,0) \le g_k(x)$ in Ω

(15) $\dfrac{\partial \hat{w}_k}{\partial t} + L + k\hat{w}_k \ge f_k\left(x,t,\hat{w}_k,[\hat{w}]_{p_k},[\hat{v}]_{q_k}\right)$ on Q

(16) $\hat{w}_k \ge 0$ on S

(17) $\hat{w}_k(x,0) \ge g_k(x)$ in Ω

$k = 1, \dots, n$. It means that $\hat{v}$, $\hat{w} \in V$ and they satisfy the following inequalities

$$(18)\qquad I^k(t,\hat{v}_k,z_k) \le \int_0^t \int_\Omega f_k\left(x,t,\hat{v}_k,[\hat{v}]_{p_k},[\hat{w}]_{q_k}\right) z_k\,dx\,dt + \int_\Omega g_k(z) z_k(x,0)\,dx$$

$$(19)\qquad I^k(t,\hat{w}_k,z_k) \ge \int_0^t \int_\Omega f_k\left(x,t,\hat{w}_k,[\hat{w}]_{p_k},[\hat{v}]_{q_k}\right) z_k\,dx\,dt + \int_\Omega g_k(z) z_k(x,0)\,dx$$

for all $z=(z_1,\ldots,z_n)$ from $\left[\mathring{W}_2^{1,1}(Q)\right]^n \cap \left[L^2_+(Q)\right]^n$, where $L^2_+(Q)$ consists of all nonnegative elements of $L^2(Q)$. At the same time, $\max\{\hat{v}_k,0\} \in \mathring{V}_2(Q)$ $\max\{-\hat{w}_k,0\} \in \mathring{V}_2(Q)$, $k=1,\ldots,n$.

Put

(20) $P=\{u=(u_1,\ldots,u_n) \mid u_k \in L^2(Q),\ u_k(x,t)\ge 0 \text{ a.e. in } Q\ k=1,\ldots,n\}$.

It is easy to show that P is a cone in the Banach space $\left[L^2(Q)\right]^n$. Let E be the Banach space $\left[L^2(Q)\right]^n$, which is partially ordered by the cone P. The partial ordering in E is defined by

$$v \le w \text{ if and only if } v_k(x,t) \le w_k(x,t) \text{ a.e. in } Q\ k=1,\ldots,n.$$

If $v \le w$, the order interval $[v,w]$ in E can be defined by

$$[v,w] = \{u=(u_1,\ldots,u_n) \in E \mid v_k \le u_k \le w_k \text{ a.e. in } Q\ k=1,\ldots,n\}.$$

Now we shall impose the following hypotheses on the functions f_k $(k=1,\ldots,n)$.

(f1) there exists coupled upper and lower quasisolutions $\hat{v}$, $\hat{w}$ of the problem (1)–(3) such that $\hat{v} \le \hat{w}$ and

$$f_k\left(x,t,\hat{v}_k,[\hat{v}]_{p_k},[\hat{w}]_{q_k}\right),\ f_k\left(x,t,\hat{w}_k,[\hat{w}]_{p_k},[\hat{v}]_{q_k}\right) \in L^2(Q).$$

(f2) $f_k(x,t,u(x,t))$ are measurable on Q, whenever $u \in [\hat{v},\hat{w}]$.

(f3) there is a positive constant M such that the functions

$$f_k\left(x,t,u_k,[u]_{p_k},[u]_{q_k}\right) + Mu_k : Q \times R^n \to R$$

are increasing with respect to u_k on

$$\left[\hat{v}_k(x,t),\hat{w}_k(x,t)\right]$$

to $[u]_{p_k}$ on

$$\left[[\hat{v}]_{p_k}(x,t),[\hat{w}]_{p_k}(x,t)\right]$$

and decreasing with respect to $[u]_{q_k}$ on

$$\left[[\hat{v}]_{q_k}(x,t),[\hat{w}]_{q_k}(x,t)\right]$$

for a.e. $(x,t) \in Q$ and for each $k = 1, \ldots, n$.

Note that in the assumption (f3) the intervals are real intervals, for example, $\left[\hat{u}_k(x,t),\hat{w}_k(x,t)\right] \{s \in R \mid \hat{u}_k(x,t) \le s \le \hat{w}_k(x,t)\}$ and if $[u]_{p_k}$ is formed by the p_k coordinates $\left(a_{j_1},\ldots,u_{j_{p_k}}\right)$ of $u = u(u_1,\ldots,u_n)$, then

$$\left[[\hat{u}]_{p_k}(x,t),[\hat{w}]_{p_k}(x,t)\right] =$$
$$= \left[\hat{v}_{j_1}(x,t),\hat{w}_{j_1}(x,t)\right] \times \left[\hat{v}_{j_2}(x,t),\hat{w}_{j_2}(x,t)\right] \times \ldots \times \left[\hat{v}_{j_{p_k}}(x,t),\hat{w}_{j_{p_k}}(x,t)\right].$$

LEMMA 1. Let f_k, $k = 1, \ldots, n$ satisfy conditions (f1)–(f3). Define the operator $G : [\hat{v},\hat{w}]^2 \to E$ by $G(v,w) = (G_1(v,w),\ldots,G_n(v,w))$, where $G_k(v,w) = f_k\left(x,t,v_k,[v]_{p_k},[w]_{q_k}\right) + Mv_k$. Then the operator G is bounded from $[\hat{v},\hat{w}]^2$ to E, furthermore, $G(\cdot,z)$ is increasing and $G(z,\cdot)$ is decreasing for each $z \in [\hat{v},\hat{w}]$.

PROOF. Assume $v \le w$ in E, so $v_k \le w_k$ a.e. in Q. By (f3),

$$G_k(v,z) - G_k(w,z) =$$
$$= f_k\left(x,t,v_k,[v]_{p_k},[z]_{q_k}\right) + Mv_k - \left[f_k\left(x,t,w_k,[w]_{p_k},[z]_{q_k}\right) + Mw_k\right] \le 0$$
$$G_k(z,v) - G_k(z,w) =$$
$$= f_k\left(x,t,z_k,[z]_{p_k},[v]_{q_k}\right) + Mz_k - \left[f_k\left(x,t,z_k,[z]_{p_k},[w]_{q_k}\right) + Mz_k\right] \ge 0$$

i.e.

(21) $$G_k(v,z) \le G_k(w,z), \qquad G_k(z,v) \ge G_k(z,w) \quad \text{a.e. in } Q$$

which imply that $G(\cdot,z)$ is increasing and $G(z,\cdot)$ is decreasing. Especially we get

$$G_k(\hat{z},\hat{w}) \le G_k(v,w) \le G_k(\hat{w},\hat{v}) \qquad \text{for any } v,w \in [\hat{v},\hat{w}].$$

So we have

(22) $$|G_k(v,w)| \le |G_k(\hat{v},\hat{w})| + |G_k(\hat{w},\hat{v})| \qquad \text{for any } v,w \in [\hat{v},\hat{w}]$$

$k = 1, \ldots, n$.

By use of (f2) and the inequalities (22), it follows that G is bounded from $[\hat{v},\hat{w}]^2$ to E by applying standard theorems on the integrability of measurable functions bounded by integrable ones. The proof of Lemma 1 is complete.

LEMMA 2. Let f_k, $k = 1, \ldots, n$ satisfy conditions (f1)–(f3). Then for any $v, w \in [\hat{v},\hat{w}]$, the linear parabolic problem

$$\frac{\partial u_k}{\partial t} + L_k u_k + M u_k = G_k(v,w) \quad \text{in } Q \tag{23}$$

$$u_k = 0 \quad \text{on } S \tag{24}$$

$$u_k(x,0) = g_k(x) \quad \text{in } \Omega \tag{25}$$

$k = 1, \ldots, n$ has a unique solution u_k denoted by $A_k(v,w)$ and $u = (u_1,\ldots \ldots,u_n) \in \mathring{V}$.

PROOF. From Lemma 1, the right-hand sides of equations (23) belong to $L^2(Q)$. The linear parabolic problems (23)–(25) are uniquely solvable and the solutions $u_k \equiv A_k(v,w) \in \mathring{V}_2^{1,0}(Q)$ cf. [6], which implies the assertion of Lemma 2.

By Lemma 2, we have defined an operator $A : [\hat{v},\hat{w}]^2 \to \mathring{V} \subset E$ by $A(v,w) = (A_1(v,w),\ldots,A_n(v,w))$. In the following, we shall study the operator equations

$$A(v,w) = v, \qquad A(w,v) = w. \tag{26}$$

LEMMA 3. Let f_k, $k = 1, \ldots, n$ satisfy conditions (f1)–(f3). Then the operator A defined by Lemma 2 maps $[\hat{v},\hat{w}]^2$ into $[\hat{v},\hat{w}]$ such that $A(\cdot,z)$ is increasing and $A(z,\cdot)$ is decreasing for each $z \in [\hat{v},\hat{w}]$.

PROOF. First we shall prove that

$$\hat{v} \le A(\hat{v},\hat{w}), \qquad A(\hat{w},\hat{v}) \le \hat{w}. \tag{27}$$

Let $u = A(\hat{v},\hat{w})$. Then u is the solution of the problems

$$\frac{\partial u_k}{\partial t} + L_k u_k + M u_k = G_k(\hat{v},\hat{w}) \quad \text{in } Q$$

$$u_k = 0 \quad \text{on } S$$

$$u_k(x,0) = g_k(x) \quad \text{in } \Omega$$

$k = 1, \ldots, n$. Denoting $z = \hat{v} - u$, condition (f1) implies that z satisfies the following inequalities

$$\frac{\partial z_k}{\partial t} + L_k z_k + M z_k \le 0 \quad \text{in } Q$$

$$z_k \le 0 \quad \text{on } S$$

$$z_k(x,0) \le 0 \quad \text{in } \Omega$$

$k = 1, \ldots, n$.

Applying the maximum principle (see Chapt. 3 in [6]) we get $z_k \le 0$ a.e. in Q, thus $\hat{v} \le A(\hat{v},\hat{w})$. In the same way we obtain that $A(\hat{v},\hat{w}) \le \hat{w}$.

Now assume that v, $w \in [\hat{v},\hat{w}]$ and $v \le w$. Let $u = A(v,z) - A(w,z)$. By the definition of the operator A, u is the solution of the following problem

$$\begin{aligned} &\frac{\partial u_k}{\partial t} + L_k u_k + M u_k = G_k(v,z) - G_k(w,z) \le 0 && \text{in } Q \\ &u_k = 0 && \text{on } S \\ &u_k(x,0) = 0 && \text{in } \Omega \end{aligned}$$

$k = 1, \ldots, n$. Here we have applied the results in Lemma 1. Hence, $u_k \le 0$, $k = 1, \ldots, n$ by the maximum principle, i.e. $A(v,z) \le A(w,z)$. Similarly, we have $A(z,v) \ge A(z,w)$. So we have shown the monotonicity of A. In virtue of this property and (27), we obtain

$$\hat{v} \le A(\hat{v},\hat{w}) \le A(v,w) \le A(\hat{w},\hat{v}) \le \hat{w} \qquad \text{for any } v,w \in [\hat{v},\hat{w}],$$

which indicate that A maps $[\hat{v},\hat{w}]^2$ into $[\hat{v},\hat{w}]$. The proof of Lemma 3 is complete.

LEMMA 4. Let f_k, $k = 1, \ldots, n$ satisfy conditions (f1)–(f3). Then the operator equations (26) have solutions $\bar{v}$, $\bar{w}$ satisfying $\hat{v} \le \bar{v} \le \bar{w} \le \hat{w}$ such that v, $w \in [\bar{v},\bar{w}]$ whenever v, $w \in [\hat{v},\hat{w}]$ are solutions of (26).

PROOF. First we define a new space $E_2 = E \times E$. Put $P_2 = \{(v,w) \mid v \in \in P, -w \in P\}$ where P is the positive cone in E defined by (20). Then P_2 is a cone in E_2 and E_2 is an ordered Banach space by the cone P_2. The partial ordering in E_2 is defined by

$$(v,w) \le_2 (u,z) \quad \text{in } E_2 \text{ if and only if } v \le u,\ w \ge z \text{ in } E,$$

where "$\le_2$" denotes the partial ordering sign in E_2 and "$\le$" denotes the partial ordering sign in E. In this way, we can define the order interval $[(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$ in E_2.

Since the cone $K = \{k(x) \in L^2(Q) \mid k(x) \ge 0 \text{ a.e. in } Q\}$ in $L^2(Q)$ is strongly minihedral cf. [2], [4], [7], so it easily follows that the cone P in E is strongly minihedral. Therefore the cone P_2 in E_2 is strongly minihedral, too.

Now let $A_2 : [(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2 \to E_2$ defined by

$$A_2(v,w) = (A(v,w), A(w,v)) \quad \text{for any } (v,w) \in [(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2.$$

We shall prove

(a) A_2 is increasing in $[(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$;

(b) $(\hat{v},\hat{w}) \le_2 A_2(\hat{v},\hat{w})$, $A_2(\hat{w},\hat{v}) \le_2 (\hat{w},\hat{v})$.

To prove (a), let (v,w), $(u,z) \in [(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$ and $(v,w) \leq_2 (u,z)$ i.e. $v \leq u$, $w \geq z$ in E. In virtue of the monotonicity property in Lemma 3, we have

$$A(v,w) \leq A(u,w) \leq A(u,z), \qquad A(w,v) \geq A(z,v) \geq A(z,u)$$

which imply that

$$A_2(v,w) = (A(v,w), A(w,v)) \leq_2 (A(u,z), A(z,u)) = A_2(u,z)$$

that is, A_2 is increasing in E_2.

Applying (27) and (a), it is obvious that

$$(\hat{v},\hat{w}) \leq_2 A_2(\hat{v},\hat{w}) \leq_2 A_2(\hat{w},\hat{v}) \leq_2 (\hat{w},\hat{v}).$$

Using (a), (b) and the strongly minihedral property of P_2, as an application of Theorem 2.3 in [4] or Theorem 19.1 in [2], we obtain a maximal fixed point $(\underline{v},\underline{w})$ and a minimal fixed point $(\bar{v},\bar{w})$ of A_2 in $[(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$.

According to the extremal property of $(\bar{v},\bar{w})$ and $(\underline{v},\underline{w})$, we know that for any fixed point $(v,w) \in [(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$ of A_2

$$(\bar{v},\bar{w}) \leq_2 (v,w) \leq_2 (\underline{v},\underline{w}) \text{ in } E_2 \tag{28}$$

i.e. $\bar{v} \leq v \leq \underline{v}$ and $\bar{w} \geq w \geq \underline{w}$ in E.

According to the definition of the operator A_2, we know that if (v,w) is a fixed point of A_2, so is (w,v). Therefore $(\bar{w},\bar{v})$, $(\underline{w},\underline{v})$ are fixed points of A_2 too. Substituting (v,w) by $(\underline{w},\underline{v})$ and $(\bar{w},\bar{v})$ in (28), respectively, we have

$$\bar{v} \leq \bar{w} \leq \underline{v}, \qquad \bar{w} \geq \bar{v} \geq \underline{w} \text{ in } E.$$
$$\bar{v} \leq \underline{w} \leq \underline{v}, \qquad \bar{w} \geq \underline{v} \geq \underline{w} \text{ in } E.$$

which imply that

$$\bar{w} = \underline{v}, \qquad \underline{w} = \bar{v} \quad \text{and} \quad \bar{v} \leq \bar{w}$$

It means that there exists a fixed point $(\bar{v},\bar{w})$ of A_2 in $[(\hat{v},\hat{w}),(\hat{w},\hat{v})]_2$ satisfying $\hat{v} \leq \bar{v} \leq \bar{w} \leq \hat{w}$ such that v, $w \in [\bar{v},\bar{w}]$ whenever v, $w \in [\hat{v},\hat{w}]$ are solutions of (26). The proof of Lemma 4 is complete.

3. Main results

In virtue of Lemma 4 and by the definition of the operator A, we obtain the following existence theorem.

THEOREM 1. *Let f_k, $k = 1, \dots, n$ satisfy conditions (f1)–(f3). Then the problem (1)–(3) has coupled quasisolutions $\bar{v}$, $\bar{w}$ satisfying $\hat{v} \le \bar{v} \le \bar{w} \le \hat{w}$ such that v, $w \in [\bar{v}, \bar{w}]$ whenever v, $w \in [\hat{v}, \hat{w}]$ are coupled quasisolutions of the problem (1)–(3).*

PROOF. By the definition of the operator A (see, Lemma 2), we know that if v, $w \in [\hat{v}, \hat{w}]$ are solutions of (26), v, w satisfy the following equations:

$$\begin{aligned}
&\frac{\partial v_k}{\partial t} + L_k v_k + M v_k = G_k(v,w) && \text{in } Q\\
&v_k = 0 && \text{on } S\\
&v_k(x,0) = g_k(x) && \text{in } \Omega\\
&\frac{\partial w_k}{\partial t} + L_k w_k + M w_k = G_k(w,v) && \text{in } Q\\
&w_k = 0 && \text{on } S\\
&w_k(x,0) = g_k(x) && \text{in } \Omega
\end{aligned}$$

$k = 1, \dots, n$. By the definition of $G(v,w)$, the equations above are equivalent to the following equations

$$\begin{aligned}
&\frac{\partial v_k}{\partial t} + L_k v_k = f_k\left(x,t,v_k[v]_{p_k},[w]_{q_k}\right) && \text{in } Q\\
&v_k = 0 && \text{on } S\\
&v_k(x,0) = g_k(x) && \text{in } \Omega\\
&\frac{\partial w_k}{\partial t} + L_k w_k = f_k\left(x,t,w_k,[w]_{p_k},[v]_{q_k}\right) && \text{in } Q\\
&w_k = 0 && \text{on } S\\
&w_k(x,0) = g_k(x) && \text{in } \Omega
\end{aligned}$$

$k = 1, \dots, n$ which mean that $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ are coupled quasisolutions of problem (1)–(3).

From Lemma 4, it easily follows that the conclusions of Theorem 1 are true.

THEOREM 2. *Suppose that there exists n nonnegative functions $d^1, \dots, d^n \in L^2(Q)$ such that*

$$\left|f_k\left(x,t,u_k,[u]_{p_k},[u]_{q_k}\right)\right| \le c^k(x,t)\,|\,|u_k| + d^k(x,t)$$

for all $(x,t,u_k,[u]_{p_k},[u]_{q_k}) \in Q \times R^n$, *where* $c^k(x,t)$, $k = 1, \ldots, n$ *are the coefficients appearing in the operators* L_k *(see (1)). Moreover, there is a positive constant* M *such that the functions* $f_k(x,t,u_k,[u]_{p_k},[u]_{q_k}) + Mu_k$: $Q \times R^n \to R$ *are increasing with respect to* u_k, *to* $[u]_{p_k}$, *and decreasing with respect to* $[u]_{q_k}$ *for a.e.* $(x,t) \in Q$. *Then if* $f_k(x,t,u(x,t))$ *are measurable on* Q *for any* $u \in \left[L^2(Q)\right]^n$, *and* $g_k(x) \geq 0$ *a.e. in* Ω *for each* $k = 1, \ldots, n$, *the nonlinear parabolic system (1)–(3) has coupled quasisolutions.*

PROOF. We first consider the linear parabolic problems

$$\begin{aligned} &\frac{\partial u_k}{\partial t} + L_k u_k - c^k u_k = d^k && \text{in } Q \\ &u_k = 0 && \text{on } S \\ &u_k(x,0) = g_k(x) && \text{in } \Omega \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{\partial u_k}{\partial t} - \sum_{ij=1}^{N} \frac{\partial u_k}{\partial x_i}\left(a_{ij}^k \frac{\partial u_k}{\partial x_j}\right) + \sum_{i=1}^{N} b_i^k \frac{\partial u_k}{\partial x_i} = d^k && \text{in } Q \\ &u_k = 0 && \text{on } S \\ &u_k(x,0) = g_k(x) && \text{in } \Omega. \end{aligned}$$

Applying the results about linear parabolic equations in [6], we obtain that the linear parabolic problems above have exactly one solution $\hat{w}_k \in \mathring{V}_2^{1,0}(Q)$ and $\hat{w}_k(x,t) \geq 0$ a.e. in Q, $k = 1, \ldots, n$. Therefore, $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n) \geq 0$ in the ordered Banach space E. Let $\hat{v} = -\hat{w}$. Then $\hat{v} \leq 0 \leq \hat{w}$ in E and $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ satisfies the following equations

$$\begin{aligned} &\frac{\partial v_k}{\partial t} + L_k \hat{v}_k - c^k \hat{v}_k = -d^k && \text{in } Q \\ &\hat{v}_k = 0 && \text{on } S \\ &\hat{v}_k(x,0) = -g_k(x) && \text{in } \Omega \end{aligned}$$

Hence, we can get

$$\frac{\partial \hat{v}_k}{\partial t} + L_k \hat{v}_k = c^k \hat{v}_k - d^k = -c^k |\hat{v}_k| - d^k \leq f_k\left(x,t,\hat{v}_k,[\hat{v}]_{p_k},[\hat{w}]_{q_k}\right) \quad \text{in } Q$$

$$\frac{\partial \hat{w}_k}{\partial t} + L_k \hat{w}_k = c^k \hat{w}_k + d^k = c^k |\hat{w}_k| + d^k \geq f_k\left(x,t,\hat{w}_k,[\hat{w}]_{p_k},[\hat{v}]_{q_k}\right) \quad \text{in } Q$$

$$\begin{aligned} &\hat{v}_k = \hat{w}_k = 0 && \text{on } S \\ &\hat{v}_k(x,0) = -g_k(z) \leq g_k(x), \quad \hat{w}_k(x,0) = g_k(x) && \text{in } \Omega \end{aligned}$$

which indicates that $\hat{v} = (\hat{v}_1,\dots,\hat{v}_n)$ and $\hat{w} = (\hat{w}_1,\dots,\hat{w}_n)$ are coupled upper and lower quasisolutions of the problem (1)–(3). The rest of the proof for Theorem 2 follows from Theorem 1.

In the case when $q_k = 0$ for each $k = 1, \dots, n$ we shall rewrite the system (1)–(3) in the form

$$\frac{\partial u_k}{\partial t} + L_k u_k = f_k(x,t,u_1,u_2,\dots,u_n) \quad \text{in } Q \tag{29}$$

$$u_k = 0 \quad \text{on } S \tag{30}$$

$$u_k(x,0) = g_k(x) \quad \text{in } \Omega \tag{31}$$

$k = 1, \dots, n$.

We say that $u \in V$ is an upper solutions (resp. a lower solution) of (29)–(31) if u satisfies (29)–(31) with all the equalities replaced by $\geq$ (resp. $\leq$). As a consequence of Theorem 1 we obtain

THEOREM 3. *Assume that the system (29)–(31) has a lower solution $\hat{v}$ and an upper solution $\hat{w}$ such that $\hat{v} \leq \hat{w}$ in E, $f_k(x,t,\hat{v})$, $f_k(x,t,\hat{w}) \in$ $\in L^2(Q)$, and for each $k = 1, \dots, n$ there is a positive constant M such that $f_k(x,t,u_1,u_2,\dots,u_n) + Mu_k$ is increasing with respect to all u_i ($i = 1, \dots, n$) on $[\hat{v}_i(x,t), \hat{w}_i(x,t)]$ for a.e. $(x,t) \in Q$. If condition (f2) also holds, then the system (29)–(31) has extremal solutions between $\hat{v}$ and $\hat{w}$.*

In [3], GIANNI and MANNUCCI considered

$$\text{(I)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} - \nabla^2 u = H(u-1) && \text{in } Q \\ &u(x,t) = k(x,t) && \text{on } S \\ &u(x,0) = h(x) && \text{in } \Omega \end{aligned}$$

where H is the Heaviside function. They have proved existence and continuous dependence theorems if the functions $k(x,t)$, $h(x)$ satisfy some additional conditions.

Let $\hat{v}(x,t)$, $\hat{w}(x,t)$ be the solutions of the following linear parabolic problems respectively,

$$\text{(II)} \qquad \begin{aligned} &\frac{\partial \hat{v}}{\partial t} - \nabla^2 \hat{v} = 0 && \text{in } Q \\ &\hat{v}(x,t) = k(x,t) && \text{on } S \\ &\hat{v}(x,0) = h(x) && \text{in } \Omega \end{aligned}$$

$$\text{(III)} \qquad \begin{aligned} &\frac{\partial \hat{w}}{\partial t} - \nabla^2 \hat{w} = 1 && \text{in } Q \\ &\hat{w}(x,t) = k(x,t) && \text{on } S \\ &\hat{w}(x,0) = h(x) && \text{in } \Omega. \end{aligned}$$

Then it is obvious that $\hat{v} \leq \hat{w}$, and $\hat{v}$ and $\hat{w}$ are lower and upper solutions of (I). By Theorem 3 ($n = 1$), we also get that the problem (I) has a solution.

Using our Theorem 2 ($n = 1$), we also can get the results of CARL in [1].

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A NOTE ON RARE BASES OF ORDER h

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Dedicated to my beloved father, Professor E. Fried on his 65^{th} birthday

1. Definition: We call a set $B_h(N)$ a basis of order h (or an h-basis) for the (integers of the) interval $[1,N]$, if

$$\{b_1+b_2+\ldots+b_h \mid b_i \in B_h(N),\ 1 \le i \le h\} \supseteq [1,N].$$

That is, the integer elements of the interval $[1,N]$ can be represented as the sum of (at most) h elements of $B_h(N)$.

We are looking for the least possible number of elements of such a set $B_h(N)$. Let us denote this number by $A_h(N)$.

For the case $h=2$ it is easy to see that $2\sqrt{N} \ge A_2(N) \ge \sqrt{2}\cdot\sqrt{N}$.

It was shown in [3] that $A_2(N) \le \sqrt{3.6}\cdot\sqrt{N}$. In [2] I improved this result to $\sqrt{3.5}\cdot\sqrt{N}$ by a simple construction.

In this paper I shall give upper estimates for $A_h(N)$ for any h.

2. We can clearly represent any number $n \le N$ as the sum of at most h numbers if we consider the number system of base $\sqrt[h]{N}$ (or more precisely

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$\lceil \sqrt[h]{N} \rceil$). In this way we have an h-basis of the elements:

$$\begin{array}{rl} \text{First subsequence:} & 1,\ 2,\ ,\ldots,\ \sqrt[h]{N} \\ \text{Second subsequence:} & \sqrt[h]{N},\ 2\sqrt[h]{N},\ \ldots,\ \sqrt[h]{N}^2 \\ \text{Third subsequence:} & \sqrt[h]{N}^2,\ 2\sqrt[h]{N}^2,\ \ldots,\ \sqrt[h]{N}^3 \\ & \vdots \\ h\text{-th subsequence:} & \sqrt[h]{N}^{h-1},\ 2\sqrt[h]{N}^{h-1},\ \ldots,\ N. \end{array}$$

This basis consists of approximately $h\sqrt[h]{N}$ elements. That is,

$$\frac{A_h(N)}{\sqrt[h]{N}} \le h.$$

3. In this part we will use the rare basis given in [2]. For technical reasons we will need the constant $C = \sqrt{3.5}/2$. Notice, that $C < 1$. Let $\sqrt[h]{N}$ be denoted by M. Let us consider first a rare 2-order basis $B_2(M^2)$ of the interval $\left[1, M^2\right]$ having $\sqrt{3.5}\sqrt{M^2} = 2CM$ elements. We multiply the elements of this basis by M^2 (to obtain the set $M^2 \cdot B_2(M^2)$) then by M^4 (to obtain the set $M^4 \cdot B_2(M^2)$), etc.

We replace the first two subsequences of the basis in Section 2 by $B_2(M^2)$; the next two subsequences by $M^2 \cdot B_2(M^2)$, etc.

If h is an odd number, we leave the last subsequence untouched.

It is easy to see, that we still have a basis of order h.

If h is even, then this basis consists of $h/2$ blocks, each containing $2CM$ elements, i.e. our basis has hCM elements altogether. It means that

$$\frac{A_h(N)}{\sqrt[h]{N}} \le hC < h,$$

since $C < 1$.

If h is odd, then this basis consists of $(h-1)/2$ blocks, each containing $2CM$ elements, plus one block containing M elements, i.e. the basis has $\left[(h-1)/2\right] \cdot 2CM + M$ elements in total. This gives

$$\frac{A_h(N)}{\sqrt[h]{N}} \le C(h-1)+1,$$

a slightly weaker result than for even values of h.

4. One can feel that the above model is not well balanced for odd values of h; the last block is relatively too "large" compared to the others. Therefore we keep just the first CM elements of the last block, i.e. we omit the last $(1-C)M$ elements. This way we obtain a basis for the interval $N' = CMM^{h-1}$ (instead of $N = M^h$) and this basis contains hCM elements in total. This yields

$$\frac{A_h(N')}{\sqrt[h]{N'}} \leq \frac{hC}{\sqrt[h]{C}}.$$

Since $C < 1$, this is a "better" (that is smaller) constant than the one we got for odd orders in Section 3, but it is still weaker than the one we have for even values of h.

We can summarize our results in the following

THEOREM.

$$\frac{A_h(N)}{\sqrt[h]{N}} \leq \begin{cases} hC, & \text{if } h \text{ is even,} \\ \frac{hC}{\sqrt[h]{C}}, & \text{if } h \text{ is odd.} \end{cases}$$

where $C = \sqrt{3.5}/2$.

5. Unfortunately, I do not see possibilities of further improvement by "assembling" the h-basis from 2-basis blocks more cleverly.

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ON PÁL INTERPOLATION

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Dedicated to Professor László Pál on his 65th birthday

L. G. PÁL introduced the following modification of the Hermite–Fejér interpolation. Let

$$-\infty < x_{n,n} < \ldots < x_{1,n} < +\infty$$

be a finite system of distinct nodal points for $n = 1, 2, \ldots$ and

$$w_n(x) := \prod_{i=1}^{n}(x - x_{i,n}), \qquad w_n'(x) := n \cdot \prod_{i=1}^{n}(x - x_{i,n}^*).$$

It is obvious that

$$x_{n,n} < x_{n-1,n} < \ldots < x_{1,n}^* < x_{1,n}.$$

Determine a polynomial R_n of lowest possible degree satisfying the conditions

$$R_n(x_{i,n}) = y_{i,n} \quad (i = 1, \ldots, n), \qquad R_n'(x_{i,n}^*) = y_{i,n}' \quad (i = 1, \ldots, n-1),$$

where $y_{i,n}$ and arbitrarily real numbers.

PÁL [7] proved that if $a \neq x_{i,n}$ $(i = 1, \ldots, n;\ n = 1, 2, \ldots)$, then there exists a unique polynomial R_n of degree $\leq 2n - 1$ satisfying the requirements and $R_n(a) = 0$. He has also given the explicit form of this polynomial. In [1] SZILI proved:

If the interpolated function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $f(0) = 0$,

$$\lim_{|x| \to \infty} x^{2r} e^{-x^2/2} f(x) = 0 \ (r = 0, 1, \ldots) \quad \text{and} \quad \lim_{|x| \to \infty} f'(x) e^{-x^2/2} = 0,$$

furthermore

$$y_{i,n}=f(x_{i,n})\quad (i=1,2,\dots,n),\qquad y'_{i,n}=f'(x^*_{i,n})\quad (i=1,2\dots,n-1),$$

where $x_{i,n}$ are the roots of the Hermite polynomials, then the sequence of the interpolation polynomials R_n ($n=2,4,6,\dots$) satisfy the following estimate:

$$e^{-\gamma x^2}|f(x)-R_n(x)|=O\left(w\left(f',\frac{1}{\sqrt{n}}\right)\log n\right),$$

which holds on the whole real line, where $\gamma>1$ and O does not depend on x; w is the so called Freud modulus of continuity, which is defined for a function $g:\mathbb{R}\to\mathbb{R}$ by

$$w(g,\delta):=\sup_{0\le t\le\delta}\left\|e^{-(x+t)^2/2}g(x+t)-e^{-x^2/2}g(x)\right\|+\left\|\tau(\delta x)e^{x^2/2}g(x)\right\|.$$

The aim of this paper is to prove a sharper estimate.

The polynomials R_n we can give in the explicit form (see: [1] Theorem 1)

$$R_n(x)=\sum_{i=1}^{n}y_{i,n}A_{i,n}(x)+\sum_{i=1}^{n-1}y'_{i,n}B_{i,n}(x)$$

and

$$R_n(0)=-2\sum_{i=1}^{n}y_{i,n}\left[\frac{H_n(0)}{H'_n(x_{i,n})}\right]^2,$$

where

$$A_{i,n}(x)=\frac{H'_n(x)}{H'_n(x_{i,n})}\cdot l_{i,n}(x)+2n\cdot\frac{H_n(x)}{H'_n(x_{i,n})}\cdot\int_0^z l_{i,n}(t)dt-2\left[\frac{H_n(x)}{H'_n(x_{i,n})}\right]^2,$$

$$l_{i,n}(x)=\frac{H_n(x)}{H'_n(x_{i,n})(x-x_{i,n})}\qquad (i=1,2,\dots,n)$$

(the Lagrenge fundamental polynomials corresponding to the nodal points $x_{i,n}$),

$$B_{i,n}(x)=\frac{H_n(x)}{H_n(x^*_{i,n})}\cdot\int_0^z l^*_{i,n}(t)dt,\qquad (i=1,2,\dots,n-1),$$

$$l^*_{i,n}(x)=\frac{H'_n(x)}{H_n(x^*_{i,n})''(x-x^*_{i,n})},\qquad (i=1,2,\dots,n-1),$$

(the Lagrenge fundamental polynomials corresponding to the nodal points $x^*_{i,n}$).

THEOREM. *If the interpolated function* $f:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable*

(1) $$\lim_{|x|\to+\infty} x^{2r}\cdot e^{-x^2/2}f(x)=0 \quad (r=0,1,2\ldots) \quad \textit{and} \quad \lim_{|x|\to\infty} f'(x)e^{-x^2/2}=0$$

furthermore, $f(0)=0$,

(2) $$y_{i,n}=f(x_{i,n}) \quad (i=1,2,\ldots,n), \qquad y'_{i,n}=f'(x^*_{i,n}) \quad (i=1,2,\ldots,n-1),$$

then the sequence of the above interpolation polynomials R_n $(n=2,4,6,\ldots)$ *satisfy the following estimate*

(3) $$e^{-x^2}|f(x)-R_n(x)|=O(1)w\left(f',\frac{1}{\sqrt{n}}\right)+O(1)\frac{1}{\sqrt{n}},$$

which holds on the whole real line and $O(1)$ *is independent on* x *and* n.

For the proof we need some lemmas.

LEMMA 1. *Let* n *be even, then*

(4) $$\sum_{i=1}^{n} e^{x_{i,n}^2/2}\cdot|A_{i,n}(x)|=O(1)e^{x^2}\sqrt{n}.$$

PROOF. Taking into account [1], [4], it is enough to estimate

(5) $$|H_n'(x)\cdot\sum_{i=1}^{n}\frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})|}\cdot|l_{i,n}(x)|,$$

(6) $$n|H_n(x)|\cdot\sum_{i=1}^{n}\frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})|}\cdot\left|\int_0^x l_{i,n}(t)dt\right|,$$

and

(7) $$H_n^2(x)\sum_{i=1}^{n}\frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})|^2}.$$

First estimate (5). We know ([6])

(8) $$\sum_{i=1}^{n}\frac{a^{\delta x_{i,n}^2}}{|H_n'(x_{i,n})|^2}=O(1)\frac{1}{2^n n!}, \qquad (0\le\delta<1).$$

Using this we obtain

(9) $$\sum_{i=1}^{n}\frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})|}\cdot|l_{i,n}(x)|\le$$

$$\leq \left(\sum_{i=1}^{n} \frac{1}{|H_n'(x_{i,n}|^2} \right)^{1/2} \cdot \left(\sum_{i=1}^{n} e^{x_{i,n}^2} \cdot l_{i,n}^2(x) \right)^{1/2} = O(1) \frac{1}{\sqrt{2^n n!}} e^{x^2/2},$$

where we used (see e.g. [3], [6])

$$\sum_{i=1}^{n} e^{x_{i,n}^2} \cdot l_{i,n}^2(x) = O(1) e^{x^2} \qquad (x \in \mathbb{R}). \tag{10}$$

Using [5], p.700 table, we obtain from (5)

$$|H_n'(x)| \sum_{i=1}^{n} \frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})|} \cdot |l_{i,n}(x)| = \tag{11}$$

$$= O(1) \frac{e^{x^2/2}}{\sqrt{2^n n!}} \cdot n \cdot |H_{n-1}(x)| = O(1) e^{x^2} \frac{\sqrt{n}}{n^{1/12}}.$$

Now estimate (6). We know (see [2], 15.3.6 and [9], Lemma 1)

$$|H_n'(x_{i,n}| \asymp e^{x_{i,n}^2/2} \cdot \sqrt{2^n n!} \cdot \varphi_n^{-1/2}(x_{i,n}).$$

Using this we obtain

$$\sum_{i=1}^{n} \frac{e^{x_{i,n}^2/2} \cdot |H_n(x)|}{|H_n'(x_{i,n})|} \cdot \left| \int_0^x l_{i,n}(t) dt \right| = O(1) \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{i=1}^{n} \varphi_n^{1/2}(x_{i,n}) \cdot \left| \int_0^x l_{i,n}(t) dt \right|.$$

In Lemma 3 we will estimate a similar expression and we obtain

$$= O(1) \frac{|H_n(x)|}{\sqrt{2^n n!}} \sum_{i=1}^{n} \varphi_n^{1/2}(x_{i,n}) \cdot \left| \int_0^x l_{i,n}(t) dt \right| = O(1) \frac{e^{x^2}}{\sqrt{n}}.$$

Therefore from (6)

$$n \cdot |H_n(x)| \sum_{i=1}^{n} \frac{e^{x_{i,n}^2/2}}{|H_n'(x_{i,n})} \cdot \left| \int_0^x l_{i,n}(t) dt \right| = O(1) e^{x^2} \cdot \sqrt{n}. \tag{12}$$

At last using (8) we obtain from (7)

$$H_n^2(x) \cdot \sum_{i=1}^{n} \frac{e^{x_{ii,n}^2/2}}{(H_n'(x_{i,n}))^2} = O(1) \left(\frac{H_n(x)}{\sqrt{2^n n!}} \right)^2 = O(1) e^{x^2} \cdot \frac{1}{n^{1/6}}. \tag{13}$$

Hence (4) is proved.

LEMMA 2. *Let n be an arbitrary nonnegative integer and $\varphi_n(x_{i,n}):=x_{i,n}-$ $-x_{i+1,n}$ (certainly $x_{1,n}>x_{2,n}>\ldots>x_{n,n}$). Then the following estimate holds*

$$|H_n(x_{i,n}^*)| \asymp e^{x_{i,n}^2/2}\cdot\sqrt{2^{n-1}(x-1)!}\cdot\varphi_{n-1}^{-1/2}(x_{i,n}^*) \qquad (i=1,2,\ldots,n-1),$$

where $a_n\asymp b_n$ means that $|a_n|=O(b_n)$ and $|b_n|=O(a_n)$.

PROOF. The $x_{i,n}^*$ are the roots of H_{n-1} since $H_n'(x)=2n\cdot H_{n-1}(x)$, therefore $x_{i,n}^*=x_{i,n-1}$. Using [2], (5.5.8) we obtain

$$|H_n(x_{i,n}^*)|=|H_n(x_{i,n-1})|=2(n-1)\cdot|H_{n-2}(x_{i,n-1})|.$$

Hence

$$|H_n(x_{i,n}^*)|=|H_{n-1}'(x_{i,n-1})|.$$

But we know (see [2], 15.3.6 and [9], Lemma 1)

$$|H_n'(x_{i,n}|\asymp e^{x_{i,n}^2/2}\cdot\sqrt{2^n n!}\cdot\varphi_n^{-1/2}(x_{i,n}) \qquad (i=1,\ldots,n).$$

Lemma 2 is proved.

LEMMA 3. *If n is even then*

$$\sum_{i=1}^{n} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|=O(1)e^{x^2} \qquad (x\in\mathbb{R}), \tag{14}$$

where $O(1)$ independent on x and n.

PROOF. Without loss of generality we may assume that $x\ge 0$. Using Lemma 2 we obtain

$$\sum_{i=1}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)=|H_n(x)|\cdot\sum_{i=1}^{n-1}\frac{e^{x_{i,n}^{*2}/2}}{|H_n(x_{i,n}^*)|}\cdot\left|\int_0^x l_{i,n}^*(t)dt\right|=$$

$$=O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\sum_{i=1}^{n-1}\varphi_{n-1}^{1/2}(x_{i,n-1})\cdot\left|\int_0^x l_{i,n-1}(t)dt\right|.$$

Here

$$\sum_{\substack{i=1\\|x_{i,n-1}|\ge 2\sqrt{\log(n-1)}}}^{n-1}\varphi_{n-1}^{1/2}(x_{i,n-1})\cdot\left|\int_0^x l_{i,n-1}(t)dt\right|=$$

$$=\sum_{\substack{i=1\\|x_{i,n-1}|\ge 2\sqrt{\log(n-1)}}}^{n-1} e^{-x_{i,n-1}^2/2}\cdot\varphi_{n-1}^{1/2}(x_{i,n-1})\cdot\left|\int_0^x e^{x_{i,n-1}^2/2}\cdot l_{i,n-1}(t)dt\right|=$$

$$= O(1)\frac{n^{-1/12}}{n^2}\cdot\int_0^x\sum_{i=1}^{n-1}\left|e^{x_{i,n-1}^2/2}\cdot l_{i,n-1}(t)\right|dt =$$

$$= O(1)\frac{n^{-1/12}}{n^2}\cdot\int_0^x\left(\sum_{i=1}^{n-1}1\right)^{1/2}\cdot\left(\sum_{i=1}^{n-1}e^{x_{i,n-1}^2}\cdot l_{i,n-1}^2(t)dt\right)^{1/2}dt =$$

$$= O(1)\frac{1}{n^{19/12}}\cdot\int_0^x e^{t^2/2}dt = O(1)\frac{1}{n^{19/12}}\cdot\frac{e^{x^2/2}}{1+x},$$

where we used $\varphi_{n-1}^{1/2}(x_{i,n-1}) = O(1)n^{-1/12}$ $(i = 1,\dots,n-1)$ and (10).

Hence

$$(15)\qquad \sum_{\substack{i=1\\ |x_{i,n-1}|\geq 2\sqrt{\log(n-1)}}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| =$$

$$= O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{19/12}}\cdot\frac{e^{x^2/2}}{1+x} = O(1)e^{x^2}\cdot\frac{1}{n^{4/3}}.$$

where we used $e^{-x^2/2}\cdot x^{-1}|H_n(x)| = O(1)\sqrt{2^n n!}\cdot n^{-1/4}$, $(|x|\geq 1)$ (see [2], (8.91.8)).

Furthermore

$$\sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}}}^{n-1} \varphi_{n-1}^{1/2}(x_{i,n-1})\cdot\left|\int_0^x l_{i,n-1}(t)dt\right| =$$

$$= O(1)\frac{1}{n^{1/4}}\sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}}}^{n-1}\left|\int_0^x l_{i,n-1}(t)dt\right|.$$

Using

$$\sum_{i=1}^{n-1}|l_{i,n-1}(x)| = O(1)e^{x^2/2}\qquad (|x|\geq n^{1/4})$$

(see [4], (19)) we obtain

$$\sum_{\substack{i=1\\|x_{i,n-1}|<2\sqrt{\log(n-1)}}}^{n-1} \left|\int_0^x l_{i,n-1}(t)dt\right| \leq \int_0^x \sum_{i=0}^{n-1} |l_{i,n-1}(t)dt| = O(1)\frac{e^{x^2/2}}{1+x}, \; x \geq \sqrt{n}.$$

Hence

(16)

$$\sum_{\substack{i=1\\|x_{i,n-1}|<2\sqrt{\log(n-1)}}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| = O(1)\frac{|H_n(x)}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\cdot\frac{e^{x^2/2}}{1+x} = O(1)e^{x^2}.$$

for all $x \geq \sqrt{n}$.

Using

$$\sum_{|x-x_{i,n-1}|\geq 1} |l_{i,n-1}(x)| = O(1)(e^{x^2/2}) \qquad |x| \leq \sqrt{n}$$

(see [4], (17)) we obtain

$$\sum_{|x-x_{i,n-1}|\geq 1} \left|\int_0^x l_{i,n-1}(t)dt\right| = O(1)\frac{e^{x^2/2}}{1+x}.$$

Hence

(17)

$$\sum_{\substack{i=1\\|x_{i,n-1}|<2\sqrt{\log(n-1)}\\|x-x_{i,n-1}|\geq 1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| = O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\cdot\frac{e^{x^2/2}}{1+x} = O(1)e^{x^2}$$

for all $0 \leq x \leq \sqrt{n}$.

We need an estimate for

$$\sum_{\substack{i=1\\|x_{i,n-1}|<2\sqrt{\log(n-1)}\\|x-x_{i,n-1}|\geq 1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|, \tag{18}$$

Obviously

$$\sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|\geq 1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| \asymp \tag{19}$$

$$\asymp \frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\cdot \sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|\geq 1}}^{n-1} \left|\int_0^x l_{i,n-1}(t)dt\right|.$$

Here

$$\int_0^x l_{i,n-1}(t)dt = \frac{1}{H'_{n-1}(x_{i,n-1})}\int_0^x \frac{H_{n-1}(t)}{t-x_{i,n-1}}dt \asymp$$

$$\asymp \frac{1}{e^{x_{i,n-1}^2/2}}\cdot\frac{1}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\cdot\int_0^x \frac{H_{n-1}(t)}{t-x_{i,n-1}}dt.$$

Thus

$$\sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| \asymp \tag{20}$$

$$\asymp \frac{|H_n(x)|}{\sqrt{2^n n!}}\cdot \sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|<1}}^{n-1} \frac{e^{-x_{i,n-1}^2/2}}{\sqrt{2^n(n-1)!}}\cdot\left|\int_0^x \frac{H_{n-1}(t)}{t-x_{i,n-1}}dt\right|.$$

We investigate the cases:

1. $-1 \leq x_{i,n-1} < 0$,
2. $x_{i,n-1} = 0$,
3. $0 < x_{i,n-1} \leq 1$,
4. $1 < x_{i,n-1} < 2\sqrt{\log(n-1)}$.

1. $-1 \le x_{i,n-1} < 0$. Integrating by parts we obtain

$$(21) \qquad \int_0^x \frac{H_{n-1}(t)}{t - x_{i,n-1}} dt = \frac{1}{2n} \cdot H_n(x) \cdot \frac{1}{x - x_{i,n-1}} - $$

$$- \frac{1}{2n} \cdot H_n(0) \cdot \frac{1}{0 - x_{i,n-1}} + \frac{1}{2n} \cdot \int_0^x H_n(t) \cdot \frac{1}{(t - x_{i,n-1})^2} dt.$$

From this we obtain

$$\int_0^x H_{n-1}(t) \cdot \frac{1}{t - x_{i,n-1}} dt =$$

$$= O(1)\frac{1}{n} \cdot \frac{e^{x^2/2} \cdot \sqrt{2^n n!}}{n^{1/4}} \cdot \sqrt{n} + O(1)\frac{1}{n} \cdot \frac{e^{2^n/2} \cdot \sqrt{2^n n!}}{n^{1/4}} \int_0^x \frac{1}{(t - x_{i,n-1})^2} dt =$$

$$= O(1)\frac{e^{x^2/2} \cdot \sqrt{2^n n!}}{n^{3/4}}.$$

Therefore

$$\sum_{\substack{i=1 \\ -1 \le x_{i,n-1} < 0 \\ |x - x_{i,n-1}| < 1}}^{n-1} \frac{e^{-x_{i,n-1}^2/2}}{\sqrt{2^n (n-1)!}} \cdot \left| \int_0^x \frac{H_{n-1}(t)}{t - x_{i,n-1}} dt \right| =$$

$$= O(1)\frac{e^{x^2/2}}{n^{1/4}} \cdot \sum_{\substack{i=1 \\ -1 \le x_{i,n-1} < 0 \\ |x - x_{i,n-1}| < 1}}^{n-1} e^{-x_{i,n-1}^2/2} = O(1)e^{x^2/2} \cdot n^{1/4}.$$

Hence

$$(22) \qquad \sum_{\substack{i=1 \\ -1 \le x_{i,n-1} < 0 \\ |x - x_{i,n-1}| < 1}}^{n-1} e^{x_{i,n}^{*2}/2} \cdot |B_{i,n}(x)| = O(1)\frac{|H_n(x)|}{\sqrt{2^n n!}} \cdot e^{x^2/2} \cdot n^{1/4} = O(1)e^{x^2}.$$

2. $x_{i,n-1}=0$. In this case

$$(23)\quad \sum_{\substack{i=1\\ x_{i,n-1}=0\\ |x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|\asymp$$

$$\asymp\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\sum_{\substack{i=1\\ x_{i,n-1}=0\\ |x-x_{i,n-1}|<1}}^{n-1}\left|\int_0^x l_{i,n-1}(t)dt\right|=$$

$$=O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\int_0^x\sum_{\substack{i=1\\ x_{i,n-1}=0\\ |x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n-1}^2/2}\cdot|l_{i,n-1}(t)|dt=$$

$$=O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\frac{1}{n^{1/4}}\int_0^x\left(\sum_{\substack{i=1\\ x_{i,n-1}=0\\ |x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n-1}^2}\cdot l_{i,n-1}^2(t)\right)^{1/2}dt=$$

$$=O(1)\frac{|H_n(x)|}{\sqrt{2^n(n-1)!}}\cdot\frac{1}{n^{1/4}}\cdot e^{x^2/2}=O(1)e^{x^2}.$$

3. $0<x_{i,n-1}\le 1$. We distinguish 3 cases: a) $x\le x_{i,n-1}-\frac{c_1}{\sqrt{n}}$, b) $x_{i,n-1}-\frac{c_1}{\sqrt{n}}\le x\le x_{i,n-1}+\frac{c_1}{\sqrt{n}}$, c) $x\ge x_{i,n-1}+\frac{c_1}{\sqrt{n}}$, where $c_1>0$ is a small absolute constant such that $x_{n/2,n-1}-\frac{c_1}{\sqrt{n}}\ge 0$.

a) $x\le x_{i,n-1}-\frac{c_1}{\sqrt{n}}\ge 0$. In this case from (21)

$$\int_0^x H_n(t)\cdot\frac{1}{t-x_{i,n-1}}dt=O(1)\frac{1}{n}\cdot\frac{e^{x^2/2}\cdot\sqrt{2^n n!}}{n^{1/4}}\cdot\sqrt{n}=O(1)\frac{e^{x^2/2}\cdot\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore

$$\sum_{\substack{i=1\\ 0<x_{i,n-1}\le 1\\ |x-x_{i,n-1}|<1\\ x\le x_{i,n-1}-\frac{c_1}{\sqrt{n}}}}^{n-1} \frac{e^{-x_{i,n-1}^2/2}}{\sqrt{2^n(n-1)!}}\cdot\left|\int_0^x \frac{H_{n-1}(t)}{t-x_{i,n-1}}dt\right| =$$

$$= O(1)\frac{e^{x^2/2}}{n^{1/4}} \sum_{\substack{i=1\\ 0<x_{i,n-1}\le 1\\ |x-x_{i,n-1}|<1\\ x\le x_{i,n-1}-\frac{c_1}{\sqrt{n}}}}^{n-1} e^{-x_{i,n-1}^2/2} = O(1)e^{x^2/2}\cdot n^{1/4}.$$

Hence

$$\sum_{\substack{i=1\\ 0<x_{i,n-1}\le 1\\ |x-x_{i,n-1}|<1\\ x\le x_{i,n-1}-\frac{c_1}{\sqrt{n}}}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| = O(1)\frac{|H_n(x)|}{\sqrt{2^n n!}}\cdot e^{x^2/2}\cdot n^{1/4} = O(1)e^{x^2}. \tag{24}$$

b) $x_{i,n-1}-\frac{c_1}{\sqrt{n}}\le x\le x_{i,n-1}+\frac{c_1}{\sqrt{n}}$. In this case

$$\int_0^x H_{n-1}(t)\cdot\frac{1}{t-x_{i,n-1}}dt = \int_0^{x_{i,n-1}-\frac{c_1}{\sqrt{n}}} + \int_{x_{i,n-1}-\frac{c_1}{\sqrt{n}}} =: I_1+I_2.$$

Here

$$I_1 = O(1)e^{\left(x_{i,n-1}-\frac{c_1}{\sqrt{n}}\right)^2/2}\cdot\frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Now we estimate I_2. Obviously

$$I_2 = \int_{x_{i,n-1}-\frac{c_1}{\sqrt{n}}}^{x} H_{n-1}(t)\cdot\frac{1}{t-x_{i,n-1}}dt = \int_{x_{i,n-1}-\frac{c_1}{\sqrt{n}}}^{x} \frac{H_{n-1}(t)-H_{n-1}(x_{i,n-1})}{t-x_{i,n-1}}dt.$$

Since $H_n'(x) = 2n \cdot H_{n-1}(x)$, therefore using the mean-value theorem we obtain

$$I_2 = O(1)e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Hence

$$\int_0^x H_{n-1}(t)\frac{1}{t - x_{i,n-1}}dt = O(1)e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore

(25)

$$\sum_{\substack{i=1 \\ 0<x_{i,n-1}\le 1 \\ |x-x_{i,n-1}|<1 \\ x_{i,n-1}-\frac{c_1}{\sqrt{n}}\le x\le x_{i,n-1}+\frac{c_1}{\sqrt{n}}}}^{n-1} e^{x_{i,n}^{*2}/2} \cdot |B_{i,n}(x)| = O(1)\frac{|H_n(x)|}{\sqrt{2^n n!}} \cdot \frac{e^{x^2/2}}{n^{1/4}} = O(1)\frac{e^{x^2}}{\sqrt{n}}.$$

c) $c \ge x_{i,n-1} + \frac{c_1}{\sqrt{n}}$. In this case

$$\int_0^x H_{n-1}(t) \cdot \frac{1}{t - x_{i,n-1}}dt = \int_0^{x_{i,n-1}-\frac{c_1}{\sqrt{n}}} + \int_{x_{i,n-1}-\frac{c_1}{\sqrt{n}}}^{x} =: I_1 + I_2.$$

Here

$$I_1 = O(1)e^{\left(x_{i,n-1}-\frac{c_1}{\sqrt{n}}\right)^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}},$$

$$I_2 = O(1)e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore (similarly as in case a))

(26)
$$\sum_{\substack{i=1 \\ 0<x_{i,n-1}\le 1 \\ |x-x_{i,n-1}|<1 \\ x\ge x_{i,n-1}+\frac{c_1}{\sqrt{n}}}}^{n-1} e^{x_{i,n}^{*2}/2} \cdot |B_{i,n}(x)| = O(1)e^{x^2}.$$

Hence from (24)–26)

$$\sum_{\substack{i=1\\0<x_{i,n-1}\le 1\\|x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| = O(1)e^{x^2}. \tag{27}$$

4. $1 < x_{i,n-1} < 2\sqrt{\log(x-1)}$.

We distinguish 3 cases: a) $x_{i,n-1} - 1 < x \le x_{i,n-1} - \frac{1}{\sqrt{n}}$, b) $x_{i,n-1} - \frac{1}{\sqrt{n}} \le x \le x_{i,n-1} + \frac{1}{\sqrt{n}}$, c) $x_{i,n-1} - \frac{1}{\sqrt{n}} \le x < x_{i,n-1} + 1$.

a) $x_{i,n-1} - 1 < x \le x_{i,n-1} - \frac{1}{\sqrt{n}}$. In this case

$$\int_0^x H_{n-1}(t)\frac{1}{t-x_{i,n-1}}dt = \int_0^{x_{i,n-1}-1} + \int_{x_{i,n-1}-1}^{x} =: I_1 + I_2,$$

Here

$$I_1 = O(1)e^{(x_{i,n-1}-1)^2/2}\,.\,\frac{\sqrt{2^n n!}}{n},$$
$$I_2 = O(1)e^{x^2/2}\,.\,\frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Hence

$$\int_0^x H_{n-1}(t)\frac{1}{t-x_{i,n-1}}dt = O(1)e^{x^2/2}\,.\,\frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore (similarly as in 3.a))

$$\sum_{\substack{i=1\\1<x_{i,n-1}<2\sqrt{\log(n-1)}\\|x-x_{i,n-1}|<1\\x_{i,n-1}-1<x\le x_{i,n-1}-\frac{1}{\sqrt{n}}}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)| = O(1)e^{x^2}. \tag{28}$$

b) $x_{i,n-1} - \frac{1}{\sqrt{n}} \le x \le x_{i,n-1} + \frac{1}{\sqrt{n}}$. In this case

$$\int_0^x H_{n-1}(t) \frac{1}{t - x_{i,n-1}} dt = \int_0^{x_{i,n-1} - \frac{1}{\sqrt{n}}} + \int_{x_{i,n-1} - \frac{1}{\sqrt{n}}}^{x} =: I_1 + I_2.$$

Here

$$I_1 = O(1) e^{\left(x_{i,n-1} - \frac{1}{\sqrt{n}}\right)^2/2} \cdot \frac{\sqrt{s^n n!}}{n^{3/4}},$$

$$I_2 = O(1) e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Hence

$$\int_0^x H_{n-1}(t) \frac{1}{t - x_{i,n-1}} dt = O(1) e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore

$$(29) \qquad \sum_{\substack{i=1 \\ 1 < x_{i,n-1} < 2\sqrt{\log(n-1)} \\ |x - x_{i,n-1}| < 1 \\ x_{i,n-1} - \frac{1}{\sqrt{n}} \le x \le x_{i,n-1} + \frac{1}{\sqrt{n}}}}^{n-1} e^{x_{i,n}^{*2}/2} \cdot |B_{i,n}(x)| = O(1) \frac{e^{x^2}}{\sqrt{n}}.$$

c) $x_{i,n-1} + \frac{1}{\sqrt{n}} \le x < x_{i,n-1} + 1$. In this case

$$\int_0^x H_{n-1}(t) \frac{1}{t - x_{i,n-1}} dt = \int_0^{x_{i,n-1} + \frac{1}{\sqrt{n}}} + \int_{x_{i,n-1} + \frac{1}{\sqrt{n}}}^{x} =: I_1 + I_2.$$

Here

$$I_1 = O(1) e^{\left(x_{i,n-1} + \frac{1}{\sqrt{n}}\right)^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}},$$

$$I_2 = O(1) e^{x^2/2} \cdot \frac{\sqrt{2^n n!}}{n^{3/4}}.$$

Therefore

$$\sum_{\substack{i=1\\ 1<x_{i,n-1}<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|<1\\ x_{i,n-1}-\frac{1}{\sqrt{n}}\le x\le x_{i,n-1}+\frac{1}{\sqrt{n}}}} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|=O(1)e^{x^2}. \tag{30}$$

Hence from (28)–(29)

$$\sum_{\substack{i=1\\ 1<x_{i,n-1}<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|<1}} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|=O(1)e^{x^2}. \tag{31}$$

From (22), (23), (27), (31) we obtain an estimate of (18),

$$\sum_{\substack{i=1\\ |x_{i,n-1}|<2\sqrt{\log(n-1)}\\ |x-x_{i,n-1}|<1}}^{n-1} e^{x_{i,n}^{*2}/2}\cdot|B_{i,n}(x)|=O(1)e^{x^2}. \tag{32}$$

From (15), (32) follows the lemma.

PROOF OF THE THEOREM. From lemmas we get (see [1], (22), (23))

$$\begin{aligned}
&e^{-x^2}\cdot|f(x)-R_n(x)|=\\
&=O(1)w\left(f'\frac{1}{\sqrt{n}}\right)+O(1)e^{-x^2}\cdot\left|\frac{H_n(x)}{H_n(0)}\right|\cdot\left|p_n(0)+2\sum_{i=1}^{n}p_n(x_{i,n})\frac{H_n^2(0)}{H_n'^2(x_{i,n})}\right|=\\
&=O(1)w\left(f'\frac{1}{\sqrt{n}}\right)+O(1)e^{-x^2}\cdot\left|\frac{H_n(x)}{H_n(0)}\right|\cdot|p_n(0)|+O(1)\frac{1}{\sqrt{n}}=\\
&=O(1)w\left(f'\frac{1}{\sqrt{n}}\right)+O(1)\frac{1}{\sqrt{n}}.
\end{aligned}$$

REMARK. If $e^{-x^2}\cdot|f(x)-R_n(x)|=\sigma_n(1)$, then for $x:=0$, we obtain $|f(0)-R_n(0)|=\sigma_n(1)$, but $R_n(0)=O\left(\frac{1}{\sqrt{n}}\right)$, therefore $f(0)=0$.

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NOTE ON ADDITIVE FUNCTIONS SATISFYING SOME CONGRUENCE PROPERTIES III.

By

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Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all integer-valued additive and completely additive functions, respectively.

K. KOVÁCS [5] proved the following

THEOREM A. *If $f \in \mathcal{A}^*$ and for some integers $a \geq 1$, b, c the congruence*

$$f(an+b) \equiv c \pmod{n}$$

holds for all $n \in \mathbb{N}$, then

$$f(n) = 0 \qquad \text{for all} \quad (n,a) = 1.$$

In [1] we achieved the same result for $f \in \mathcal{A}$ in the case $a = 1$. After this in [2] we proved the following generalization of the above result:

THEOREM B. *Let $A > 0$, B and C be integers. If $f \in \mathcal{A}$ satisfies the condition*

$$f(An+B) \equiv C \pmod{n} \quad \text{for all} \quad n > \max\left(0, -\frac{B}{A}\right),$$

then $f(n) = 0$ for all $n \in \mathbb{N}$ which are coprime to A.

Using the results of [3] and [4] we have:

THEOREM C. *Let $A > 0$, $B > 0$ and C be integers. If F and $G \in \mathcal{A}^*$ such that*

$$F(An+B) \equiv G(n) + C \pmod{n}$$

holds for all $n \in \mathbb{N}$, then $G \equiv 0$ and $F(n) = 0$ for all $n \in \mathbb{N}$ which are coprime to A.

In the present paper we improve the above results by proving the following:

THEOREM. *Let* $A > 0$, B *and* C *be integers. If* $F \in \mathcal{A}$ *and* $G \in \mathcal{A}^*$ *satisfy the condition*

(1) $$F(An + B) \equiv G(n) + C \pmod{n}$$

for all $n > \max\left(0, -\frac{B}{A}\right)$, *then:*

For $B \neq 0$, $G \equiv 0$ *and* $F(n) = 0$ *for all* $n \in \mathbb{N}$ *which are coprime to* A.

For $B = 0$, $F(A) = C$ *and* $F(n) = G(n)$ *for all* $n \in \mathbb{N}$ *which are coprime to* A.

The proof of our theorem is based on the following result:

LEMMA. If $g \in \mathcal{A}^*$ satisfies the congruence

(2) $$g(n+1) \equiv g(n) \pmod{n} \qquad \text{for all} \quad n \in \mathbb{N},$$

then $g \equiv 0$.

PROOF OF THE LEMMA. The lemma follows from Theorem C, but here we give another proof. First we show that from (2) it follows that for all n, $k \in \mathbb{N}$,

(3) $$g(n+k) \equiv g(n) \pmod{n}.$$

For $k = 1$, it is the condition of the lemma. We proceed by induction. If (3) holds for k, then

$$g(n+k) + g(n+1) = g(n(n+k+1)+k)$$

implies that

$$g(n) + g(n) \equiv g(n(n+k+1)) \pmod{n}$$

i.e.

$$g(n+k+1) \equiv g(n) \pmod{n}.$$

Thus, (3) is proved.

Let p be any fixed positive integer. Applying (3) with $k = (p-1)n$ we have

$$g(pn) \equiv g(n) \pmod{n}$$

which implies

$$g(p) \equiv 0 \pmod{n} \qquad \text{for all} \quad n \in \mathbb{N}.$$

Thus, we have $g(p) = 0$ which completes the proof of the lemma.

PROOF OF THE THEOREM. First we consider the case $B > 0$. Assume that (1) holds for all $n > n_0$ where $n_0 = \max\left(0, -\frac{B}{A}\right)$. Then replacing n by $3B^2k$, we have

(4) $$F(3ABk+1) \equiv G(k) + C_1 \pmod{k},$$

where $C_1 = G(3) + 2G(B) - F(B)$. Since

$$\left(3ABm+1,(3ABm)^2 - 3ABm+1\right) = $$
$$= (3ABm+1,(3ABm+1)(3ABm-2)+3) = 1$$

holds for all m, replacing k by $3^2A^2B^2m^3$ in (4) yields

$$(5)\quad F(3ABm+1)+F\left(3^2A^2B^2m^2 - 3ABm+1\right) \equiv 3G(m)+C_2 \pmod{m},$$

where $C_2 = G\left(3^2A^2B^2\right) + C_1$.

On the other hand, the substitution $k = 3ABm^2 - m$ in (4) provides

$$F\left(3^2A^2B^2m^2 - 3ABm+1\right) = F\left(3AB(3ABm^2-m)+1\right) \equiv$$
$$(6)\qquad \equiv G(3ABm-1)+G(m)+C_1 \pmod{m}.$$

Thus, the congruences (4), (5) and (6) yield

$$(7)\qquad G(3ABm-1) \equiv G(m)+C_3 \pmod{m},$$

where $C_3 = C_2 - 2C_1$. Replacing m by $3ABm^3$ in (7) we have

$$G\left(3^2A^2B^2m^2 - 1\right) \equiv G(3ABm^2)+C_3 \pmod{m}.$$

This combined with (7) implies

$$(8)\qquad G(3ABm+1) \equiv G(m)+G(3AB) \pmod{m}.$$

We shall deduce from (8) that

$$(9)\qquad G(3ABm+i) \equiv G(m)+G(3AB) \pmod{m}$$

holds for all $m, i \in \mathbb{N}$.

The proof of (9) is similar to the proof of the Lemma. Using that

$$(3ABm+1)(3ABm+k) = 3ABm(3ABm+k+1)+k$$

and the previous steps of the induction, we have

$$2G(m)+2G(3AB) \equiv G(3ABm+1)+G(3ABm+k) =$$
$$= G(3ABm(3ABm+k+1)+k) \equiv$$
$$\equiv G(m)+G(3ABm+k+1)+G(3AB) \pmod{m}.$$

The choice $i = 3AB$ in (9) yields

$$G(3ABm+2AB) \equiv G(m)+G(3AB) \pmod{m}.$$

Thus, $G(m+1) \equiv G(m) \pmod{m}$. By the lemma then $G \equiv 0$. Therefore, by (1) we have

$$F(An+B) \equiv C \pmod{n}.$$

We can apply Theorem B to get $F(n)=0$ for all $(n,A)=1$, if $B>0$.

For $B<0$, we can prove the Theorem in the same way as above using $|B|$ instead of B whenever we want to multiply by B. This proves the claim for $B\neq 0$.

Finally, we consider the case when $B=0$. Assume that F, $G\in\mathcal{A}$ and integers $A>0$, C satisfy

$$F(An)\equiv G(n)+C \pmod n \tag{10}$$

for all $n\in\mathbb{N}$.

Let k be a positive integer which is coprime to A. Applying (10) with $n=km$, we have

$$F(Akm)\equiv G(km)+C \pmod m,$$

which implies that

$$F(Am)+F(k)\equiv G(m)+G(k)+C \pmod m \tag{11}$$

for all $m\in\mathbb{N}$, $(m,k)=1$. It follows from (10) and (11) that

$$F(k)\equiv G(k) \pmod m \quad \text{for all} \quad m\in\mathbb{N},\quad (m,k)=1.$$

This implies that

$$F(k)=G(k) \quad \text{for all} \quad k\in\mathbb{N},\quad (k,A)=1. \tag{12}$$

Using (10) and (12) we obtain that

$$F(A)\equiv C \pmod n$$

for all $n\in\mathbb{N}$, $(n,A)=1$. Consequently $F(A)=C$, which completes the proof of our Theorem.

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SOME RESULTS CONCERNING THE CONTROL OF STRINGS AND MEMBRANES

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Introduction

This paper contains some results concerning earlier work of the author*. In point 1° we extend the result [7] on the independent oscillations of the points of a vibrating membrane to the case of string. We prove that the independence holds only for short time intervals (contrarily to the case of membranes). In 2° we investigate the independence of the motion of points in a circular membrane, if the origin is one of these points. The answer is similar: independence holds only for limited time intervals, namely if and only if $T < 2$. Point 3° contains a description of those situations when the set of all movement states reachable in time T is strictly increasing in T. In Point 4° we disprove the possibility of independent motions of a vibrating beam. Finally 5° extends the controllability properties of a circular membrane in the spaces $\mathcal{H}_r = W_{r+1} \oplus W_r$, $W_r = D\left((I-\Delta)^{r/2}\right)$, $r < -\frac{1}{2}$ (proved in [8]) to the case $r = -\frac{1}{2}$.

1° Consider the string equation

$$(1) \qquad \varrho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y(x,t)}{\partial x}\right), \qquad 0 < x < 1, \quad 0 < t < T$$

with density $\varrho(x)$ and modulus of elasticity $p(x)$, $p, \varrho \in C^2(0,1)$.

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* These are solutions of problems and conjectures posed by professors J.-L. LIONS, L. SIMON and R. KERSNER, the opponents of my dissertation.

Let $y(0,t)=y(1,t)=0$ (or we can prescribe any strongly regular boundary conditions). Using an appropriate substitution in variable x in y we get a new function $u(x,t)$ satisfying

$$(2)\quad u_{tt}=u_{xx}+q(x)\cdot u, \qquad 0<x<l,\quad 0<t<T,\quad l=\int_0^1\sqrt{\frac{\varrho}{p}},\quad q\in C[0,l]$$

(see HORVÁTH [1]).

THEOREM 1. *Let* $x_1,\dots,x_N\in(0,l)$ *be different numbers and*

$$Au(t)=(u(x_1,t),\dots,u(x_N,t)).$$

Then there exists $T>0$ *depending only on* N *such that the functions* Au *run over a dense subset of* $C^\infty(0,T;\mathbb{C}^N)$, *where* u *runs over the* C^2*-solutions of* (2).

Remark that since the dependence in t remains unchanged, the same Theorem holds also for system (1), too.

PROOF. Consider the eigenvalue problem.

$$(*)\qquad v_n''+qv_n+\lambda_n v_n=0,\qquad v_n(0)=v_n(l)=0$$

(see [1] for more general boundary conditions). It is known from Neumark's monography [2] that there exists a complete orthogonal system (v_n) satisfying $(*)$ and

$$(3)\qquad v_n(x)=\sin\frac{n\pi}{l}x+O\left(\frac{1}{n}\right),\quad \sqrt{\lambda_n}=\frac{n\pi}{l}+O\left(\frac{1}{n}\right).$$

Define $\Lambda=\{\lambda_n\ :\ n=1,2,\dots,\}$. Clearly every finite sum

$$(4)\qquad u(x,t)=\sum v_n(x)\left(\alpha_n e^{i\sqrt{\lambda_n t}}+\beta_n e^{-i\sqrt{\lambda_n t}}\right)$$

satisfies (2) and $u(0,t)=u(l,t)=0$. So we consider the system

$$e(\Lambda)=\left\{e^{\pm i\sqrt{\lambda_n t}}\begin{pmatrix}v_n(x_1)\\ \vdots\\ v_n(x_N)\end{pmatrix}\ :\ n=1,2,\dots\right\}.$$

It is enough to show that $e(\Lambda)$ contains a subsystem, complete in $C^\infty(0,T;\mathbb{C}^N)$ if $T>0$ is sufficiently small.

LEMMA 1. *The vectors*

$$\begin{pmatrix}\sin\frac{n\pi}{l}x_1\\ \vdots\\ \sin\frac{n\pi}{l}x_N\end{pmatrix},\qquad 1\le n\le 3N-2,\quad \operatorname{span}\mathbb{C}^N$$

PROOF. Let $\beta_1 \sin\frac{n\pi}{l}x_1 + \ldots + \beta_N \sin\frac{n\pi}{l}x_N = 0$, $1 \le n \le 3N-2$. We have to show that $\beta_i = 0$ for all i. The equalities imply that $\beta_1 T(x_1) + \ldots$ $\ldots + \beta_N T(x_N) = 0$ for every impair trigonometric polynomial of period $2l$ and of degree $\le 3N-2$. If we set $T(x) = \sin\frac{\pi x}{l} \prod_{j \ne i} \left(\sin^2\frac{\pi x}{l} - \sin^2\frac{\pi x_j}{l}\right)$, we get $\beta_i = 0$ unless there exists j with $x_j = l - x_i$. In that case let $T(x) =$ $= \sin\frac{\pi x}{l} \prod_{j \ne i} \left(\sin^2\frac{\pi x}{l}\cos\frac{\pi x}{l} - \sin^2\frac{\pi x_j}{l}\cos\frac{\pi x}{l}\right)$ which implies again $\beta_i = 0$. ∎

Fix a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ out of the vectors of Lemma 1. Then we have

LEMMA 2. *For every $\varepsilon > 0$ and for every sufficiently large $M > 0$ the system*

$$e_0(\Lambda) = \left\{ e^{\pm i\sqrt{\lambda_n t}} \begin{pmatrix} \sin\frac{n\pi}{l}x_1 \\ \vdots \\ \sin\frac{n\pi}{l}x_N \end{pmatrix} \; : \; n \ge 1 \right\}$$

contains a subsystem of its elements indexed by $n = n_{k,j}$ such that

$$\left.\begin{array}{ll} \text{a)} & n_{k,j} = M \cdot k + O(1) \\ \text{b)} & \left\| \begin{pmatrix} \sin\frac{n_{k,j}\pi}{l}x_1 \\ \vdots \\ \sin\frac{n_{k,j}\pi}{l}x_N \end{pmatrix} - e_j \right\| < \varepsilon \end{array}\right\} \quad j = 1, \ldots, N; \quad k = 1, \ldots$$

where the implicit constant in $O(1)$ depend only on ε.

PROOF. It is based on the Kronecker theorem on simultaneous diophanite approximation (see e.g. Cassels [3]). Let $e_j = \left(\sin\frac{n_j\pi}{l}x_1, \ldots, \sin\frac{n_j\pi}{l}x_N\right)^T$, then for $n = n_{k,j}$ we require $n = M \cdot k + O(1)$, $|||nx_i - n_j x_i||| < \varepsilon$ $(i = 1, \ldots, N)$. Such an n exists by [3]. ∎

LEMMA 3. *The system $e(\Lambda)$ contains a Riesz basis in $L_2\left(0, \frac{2l}{M}; \mathbb{C}^N\right)$.*

PROOF. The system

$$e^*(\Lambda) = \{\mathbf{e}_j \cdot e^{\pm i\frac{k\pi}{l}t} \; : \; k \in \mathbb{N}, \; 1 \le j \le N\}$$

is Riesz basis in $L_2(0, 2l; \mathbb{C}^N)$. If the vectors $\mathbf{e}_j$ are moved at a distance $< \varepsilon$ with sufficiently small $\varepsilon > 0$ then their basis properly remains unchanged, hence

$$e_1(\Lambda) = \left\{ \begin{pmatrix} v_{n_{k,j}}(x_1) \\ \vdots \\ v_{n_{k,j}}(x_N) \end{pmatrix} e^{\pm i\frac{k\pi}{l}t} \; : \; k \in \mathbb{N}, \; 1 \le j \le N \right\}$$

is also a Riesz basis. Since $k = \frac{n_{k,j}}{M} + O\left(\frac{1}{M}\right)$ by Lemma 2 a), we can use a stability theorem like that of Duffin and Eachus stating that any sufficiently small shift of the exponents (within an error bound ε) preserves Riesz basis property, see HORVÁTH [4] Lemma 3 or JOÓ [5]

$$e_2(\Lambda) = \left\{ \begin{pmatrix} v_{n_{k,j}}(x_1) \\ \vdots \\ v_{n_{k,j}}(x_N) \end{pmatrix} e^{\pm i \frac{n_{k,j}\pi}{lM} t} \; : \; k \in \mathbb{N},\ 1 \le j \le N \right\}$$

is Riesz basis in $L_2(0,2l;\mathbb{C}^N)$. By the same argument

$$e_3(\Lambda) = \left\{ \begin{pmatrix} v_{n_{k,j}}(x_1) \\ \vdots \\ v_{n_{k,j}}(x_N) \end{pmatrix} e^{\pm i \sqrt{\frac{\lambda_{n_{k,j}}}{M}} t} \; : \; k \in \mathbb{N},\ 1 \le j \le N \right\}$$

is also Riesz basis over $(0,2l)$ i.e. $e(\Lambda)$ contains a Riesz basis in $L_2\left(0, \frac{2l}{m}; \mathbb{C}^N\right)$. Lemma 3 is proved. ∎

LEMMA 4. *Let $s \in \mathbb{N}$ be arbitrary. If M is sufficiently large, the system $e(\Lambda)$ contains a Riesz basis in the Soboleff space $H^s\left(0, \frac{2l}{M}; \mathbb{C}^N\right)$. Here M is independent of s*

PROOF. Consider the system $e_s(\Lambda)$ constructed in Lemma 3. We can construct a system $H \subset e(\Lambda) \setminus e_3(\Lambda)$,

$$H = \{v_j e^{i\mu_j t} \; : \; j = 1, \dots, sN\}$$

with the properties

a) H is linearly independent.

b) If $\lim H$ resp. $\lim H_k$ denotes the linear hull of H resp.

$$H_k = \left\{ \begin{pmatrix} v_{n_{k,j}}(x_1) \\ \vdots \\ v_{n_{k,j}}(x_N) \end{pmatrix} e^{\pm i \frac{\sqrt{\lambda_{n_{k,j}}}}{M} t} \; : \; 1 \le j \le N \right\},$$

then

$$\lim H \cap \lim H_k = \{0\}.$$

c) The vectors $v_1, \dots, v_{sN}$ form s bases in $\mathbb{C}^N$. This can be done: if s sequences of N members $n_{k,j}$, $1 \le j \le N$, $k_0 \le k \le k_0 + s - 1$ are constructed as in Lemma 2 (with $M + \frac{1}{2}$ instead of M) then c) follows from Lemma 2 b), a) follows from Lemma 2 a) and b); finally b) is a consequence of the fact that, by construction, the exponents of H and H_k the different. Now we refer to Proposition 2 of Joó [11] which states that the above assumptions a), b), c) imply that the unified system $H \cup \bigcup_k H_k$ is Riesz basis in $H^s\left(0, \frac{2l}{M}; \mathbb{C}^N\right)$ as we asserted. ∎

PROOF OF THEOREM 1. By Lemma 4, $e(\Lambda)$ is complete in $H^s\left(0, \frac{2l}{M}; \mathbb{C}^N\right)$ for all $s \in \mathbb{N}$ if M is sufficiently large. This implies also the completeness in $C^\infty\left(0, \frac{2l}{M}; \mathbb{C}^N\right)$, which completes the proof. ∎

REMARK 1. In the above Theorem only the values $T \le 2l$ can occur. Indeed, if for some T the Theorem holds then $\{e^{\pm i\sqrt{\lambda_n}} \ : \ n = 1,2,\dots\}$ is complete in $L_2(0,T)$. On the other hand $\sqrt{\lambda_n} = \frac{n\pi}{T} + O\left(\frac{1}{n}\right)$ implies by the above mentioned Lemma 3 of [4] that $\{e^{\pm i\frac{n\pi}{T}t} \ : n < n_0\} \cup \{e^{\pm i\sqrt{\lambda_n t}} \ : n > n_0\}$ is Riesz basis in $L_2(0,2l)$. From Lemma 7 of Joó [11] we can shift the remaining finitely many exponents from $\frac{n\pi}{T}$ to $\sqrt{\lambda_n}$, hence $\{1, e^{\pm i\sqrt{\lambda_n t}} \ : n = 1,2,\dots\}$ is Riesz basis in $L_2(0,2l)$. Consequently in $L_2(0,T)$, $T > 2l$ this system has infinite-dimensional deficiency: it cannot be complete.

2° REMARK 2. In the statement of the theorem of Joó [7] on the independent motions of different points of a circular membrane the origin is excluded. That theorem states that if Ω is the unit disk $0 \ne P_1,\dots,P_N \in \Omega$ then taking the set U of all solutions $u(t,x,y) \in C^\infty(\mathbb{R}\times\Omega)\cap C(\mathbb{R}\times\overline{\Omega})$ of the problem $u_{tt} = \Delta_u$ on $\mathbb{R}\times\Omega$, $u = 0$ on $\mathbb{R}\times\partial\Omega$, the mapping $A : U \to C^\infty(\mathbb{R},\mathbb{C}^N)$

$$Au = (u(\cdot,P_1),\dots,u(\cdot,P_N))$$

has dense range in $C^\infty(\mathbb{R},\mathbb{C}^N)$. If $P_1 = (0,0)$ then the mapping $A : U \to C^\infty\left(\left(-\frac{T}{2},\frac{T}{2}\right)\mathbb{C}^N\right)$,

$$Au = (u(\cdot,P_1),\dots,u(\cdot,P_N))$$

has dense range in C^∞ if $T < 2$ (and for no larger T). Indeed, since $r_1 = 0$, we have $J_m\left(\lambda_n^{(m)} r_1\right) = 0$ unless $m = 0$. Hence

$$u(t,0,0) = \sum \left(a_k e^{i\lambda_k^{(0)}t} + b_k e^{-i\lambda_k^{(0)}t}\right).$$

On the other hand, the distribution of the zeros $\lambda_k^{(0)}$ of J_0 is linear: $\lambda_k^{(0)} = k\pi - \frac{\pi}{4} + O\left(\frac{1}{k}\right)$. Consequently the system $\left\{e^{\pm i\lambda_k^{(0)}t} \ : \ k = 1,2,\dots\right\}$ is complete in $L_2\left(-\frac{T}{2},\frac{T}{2}\right)$ only for $T<2$. This means that for $T\geq 2$ the range set of A is not dense even in L_2. For $T<2$ we can argue as follows. In the set

$$e(\Lambda)=\left\{e_{m,k}e^{\pm i\lambda_k^{(m)}t},f_{m,k}e^{\pm\lambda_k^{(m)}t} \;:\; m=0,1,\ldots;\; k=1,2,\ldots\right\}$$

(investigated in [7]) we use the members of index $m=0$ to approximate the first coordinate function f_1 of $\begin{pmatrix} f_1\\ \vdots\\ f_N\end{pmatrix}$; this gives an approximation of $\begin{pmatrix} f_1\\ g_2\\ \vdots\\ g_N\end{pmatrix}$. The members of indices $m\geq 1$ of $e(\Lambda)$ are then used to approximate $\begin{pmatrix} 0\\ f_1-g_2\\ \vdots\\ f_N-g_N\end{pmatrix}$, as described in [7]. So A has dense range indeed for $T<2$. ∎

REMARK 3. In the above Remark and in JOÓ [7] only the density of the range of A is proved. In fact, A is onto in the following sense. Introduce the function spaces $\mathcal{W}_r$ and $\mathcal{H}_r$ as in JOÓ [8]. Then we have the following

THEOREM 2. *Let $r\geq 0$ be an integer. Suppose that the origin doesn't occur, $P_i\neq 0\ \forall i$. Then for every function $F\in H^r(0,T;\mathbb{C}^N)$, $T>0$ there exists an initial state $(u_0,u_1)\in\mathcal{H}_r$ $(u_0(x)=u(0,x),\ u_1(x)=u_t'((0,x))$ such that*

$$Au=F \qquad on \quad (0,T).$$

Here $\mathcal{H}^r=\mathcal{W}_2^r$ is the usual Soboleff space.

A similar statement has been proved in AVDONIN–IVANOV–JOÓ [9] for rectangular membranes and in JOÓ [19] for the Dirichlet condition in the square membrane. In case of Newmann condition the proof of [19] can be copied step by step so we give here only an outline of the proof. The idea is that the system $e(\Lambda)$ contains a Riesz basis in $H^r(0,T;\mathbb{C}^N)$ for every natural number r. In case $r=0$ this can be shown as follows. It is proved in [7] that there exists a system $\hat{\Phi}=\hat{e}_0(\lambda)\cup\Phi$ the $\hat{e}_0(\Lambda)\subset e_0(\Lambda)$ and Φ is finite such that $\hat{\Phi}$ is Riesz basis in $L_2(0,T,\mathbb{C}^N)$, so $\hat{e}_0(\Lambda)$ has finite codimension. Since $e(\Lambda)$ is complete by [7], we can join (step by step) appropriate elements of $e(\Lambda)$ to $\hat{e}_0(\Lambda)$ to obtain bases with smaller and smaller codimension. In finitely many steps we get a Riesz basis whose elements belong to $e(\Lambda)$. This proves the case $r=0$ because the coefficients of the sums defining u_0 and u_1 belong to l_2.

In case $r > 0$ we join finitely many elements of $e(\Lambda)$ to the above constructed basis to obtain a Riesz basis in $H^r(0,T;\mathbb{C}^N)$. This can be done by Proposition 2 of [8], already mentioned in the proof of Lemma 4 above. The same procedure works also for the case of string. If the origin is allowed i.e. $P_1 = 0$ then using the ideas of Remark 2 above we see that the statement of Theorem 2 remains true only for $T < 2$.

REMARK 4. It remained open the explicit determination pf the optimal time $T < 2$ for which N points of the string can obtain arbitrary movements states and how this time depend on the number N of points considered.

3° REMARK 5. In several papers the authors investigated the growth properties of the reachability set associated to a hyperbolic equation, see e.g. [8], [1], [9]. A formal setting of the problem is the following one. Denote $(\varphi_n) \subset L_2(0,T;\mathbb{C}^N)$ a system of vector exponentials and let

$$R(T) = \left\{ \left(\langle f, \varphi_n \rangle_{L_2} \right)_n \; : f \in L_2(0,T;\mathbb{C}^N) \right\}$$

be the reachability set (more precisely, the moment space associated to it). Suppose that the functions φ_n are named such that $R(T) \subset l_2$ and that for every T there exists a subsystem of (φ_n) forming a Riesz basis in $L_2(0,T;\mathbb{C}^N)$. Then

PROPOSITION. *The statements below are equivalent*

a) $R(T)$ *increases strictly as* $T \to \infty$

b) $R(T)$ *is not dense in* $\overline{\bigcup_{T>0} R(T)}$ *for every* $T > 0$ *(the line denotes closure in* l_2*).*

The proof requires

LEMMA 5. $R(T)$ *is closed in* l_2 *for every* $T > 0$.

PROOF. Let $f_k \in L_2(0,T;\mathbb{C}^N)$ and suppose that

$$\left(\langle f_k, \varphi_n \rangle_{L_2} \right)_n \xrightarrow{l_2} (c_n)_n .$$

Since a subsystem of (φ_n) is Riesz basis on $(0,T)$ hence the mapping $f \mapsto \left(\langle f, \varphi_{n_i} \rangle \right)_i$ is an isomorphism of $L_2(0,T;\mathbb{C}^N)$ onto l_2. Consequently (f_k) is a Cauchy-sequence in L_2 hence $\exists\, f \in L_2 \,:\, f_k \xrightarrow{L_2} f$. This implies $\langle f_k, \varphi_n \rangle \to \langle f, \varphi_n \rangle$ hence $\langle f, \varphi_n \rangle = c_n$ for every n. ∎

LEMMA 6. *If* $R(T_1) = R(T_2)$ *for some* $T_1 < T_2$ *then*

$$R(T_1) = R(t) \qquad \textit{for all} \quad T > T_1 .$$

PROOF. $R(T_1) = R(T_2)$ implies that for all functions $f \in L_2$, supp$f \subset [T_1, T_2]$ there exists $g \in L_2$, supp$g \subset [0, T_1]$ with the same moment sequence (we use below the notations of [5])

$$\int_{T_1}^{T_2} e^{iw_{m_k}\tau} \langle f(\tau), e_{mn}^{\pm} \rangle \, d\tau = \int_0^{T_1} e^{iw_{m_k}\tau} \langle g(\tau), e_{mn}^{\pm} \rangle \, d\tau \Rightarrow$$

$$\Rightarrow \int_{T_1+h}^{T_2+h} e^{iw_{m_k}t} \langle f(t-h), e_{mn}^{\pm} \rangle \, dt = \int_h^{T_1+h} e^{iw_{m_k}t} \langle g(t-h), e_{mn}^{\pm} \rangle \, dt.$$

In other words: any function on $[T_1+h, T_2+h]$ can be substituted by an appropriate function with support $[h, T_1+h]$ (in the sense that they produce the same moment sequences). Let now supp$f \subset [0, T_1 + 2(T_2 - T_1)]$, then it can be substituted by some g_1, supp$g_1 \subset [T_2 - T_1, T_2]$ $(h = T_2 - T_1)$, hence f can be substituted by $f\big|_{[0,T_2]} + g_1 = g_2$. Then $g_2\big|_{[T_1, T_2 - T_1]}$ can be substituted by some g_3, supp$g_3 \subset [0, T_1]$, finally f can be substituted by $f\big|_{[0,T_1]} + g_1\big|_{[0,T_1]} + g_3$. This shows that $R(T_1 + 2(T_2 - T_1)) = R(T_1)$. Repeating this argument we see $R(T_1 + k(T_2 - T_1)) = R(T_1)$ which implies by the monotonicity of $R(T)$ that

$$R(T_1) = R(T) \qquad \text{for} \quad T > T_1. \qquad \blacksquare$$

PROOF OF THE PROPOSITION.

a) $\Rightarrow$ b) Indirectly suppose the a) holds and b) does not. Then for some T_0, $R(T_0)$ is dense in $\overline{\cup R(T)}$ i.e. by Lemma 5, $R(T_0) = \overline{\bigcup_{T>0} R(T)}$. But then $R(T) \subset R(T_0)$ for any value of T and hence $R(T) = R(T_0)$ for $T > T_0$, in contradiction with a).

b) $\Rightarrow$ a) Suppose that $R(T_1) = R(T_2)$ for some $T_1 < T_2$, i.e. that a) fails. Then by Lemma 6, $R(T) = R(T_1)$ for $T > T_1$ and hence $\bigcup_{T>0} R(T) = R(T_1)$, so b) can not fulfil.

Proposition is proved. $\blacksquare$

1° REMARK 6. For the vibrating beam statement of Theorem 1 can not hold for any $T > 0$. In this case the system equation is $-\ddot{u} = u^{(4)}$ where solution can be developed into a series

$$u(x,t) = \sum \left(\alpha_u e^{in^2 t} + \beta_u e^{-in^2 t} \right) v_u(x)$$

where

$$v_n^{(4)} = n^4 v_n \qquad v_n''(0) = v_n''(\pi)$$
$$v_n(0) = v_n(\pi) \qquad v_n'''(0) = v_n'''(\pi)$$
$$v_n'(0) = v_n'(\pi)$$

is a complete system in $L_2(0,\pi)$: $v_n = \cos nx$. Then

$$Au(t) = \sum \left(\alpha_u e^{in^2 t} + \beta_u e^{-in^2 t}\right) \begin{pmatrix} v_n(x_1) \\ \vdots \\ v_n(x_N) \end{pmatrix}.$$

Such a sum can not be dense over any $(0,T)$, because in that case $\left(e^{\pm in^2 t}\right)$ would be dense in $L_2(0,T)$. But this is not true. For example we can use the famous stability theorem of Avdonin [10] which shows that $\left(e^{\pm in^2 t}\right)$ can be completed for every T to a Riesz basis in $L_2(0,T)$ (consequently it is not complete). On the other hand it would be reasonable to describe the subspace of L_2 spanned by $\left(e^{\pm in^2 t}\right)$.

5° The result of Joó [8] holds also for $r = -\frac{1}{2}$, too. First we recall some notations. Let Ω be the unit circle, $P_1, \ldots, P_N \in \Omega$, $s_{N+1}, \ldots, s_M \in \partial\Omega$ and consider the membrane

$$u_{tt} = \Delta u + \sum_{j=1}^{N} \delta((x,y) - p_j)v_j \qquad \text{on} \quad (0,T) \times \Omega$$

$$\frac{\partial u}{\partial r} = \sum_{j=N+1}^{M} \delta(s - s_j)v_j \qquad \text{on} \quad (0,T) \times \partial\Omega$$

controlled by $v_j \in L_2(0,T)$. Denote

$$W_r = D\left((I - \Delta)^{r/2}\right)$$

the domain of the $\frac{r}{2}$-th power of $I - \Delta$ and let

$$\mathcal{H}_r = W_{r+1} \oplus W_r.$$

THEOREM. [8] *For $r < -\frac{1}{2}$ the movement state (u, u_t) belongs to $C\left([0,T], \mathcal{H}_r\right)$ if the controls v_j are in $L_2(0,T)$.*

Denote J_m the Bessel function of m-th order and let $\lambda_n^{(m)}$, $n = 1, 2, \ldots$ be the positive zeros the derivative J_m'. We prove first

LEMMA 1.

a) $$\sum_{\substack{n,m \\ k\le\lambda_n^{(m)}\le k+1 \\ n\le cm}} \frac{1}{\sqrt{m(n-n^*+1)}} = O(1)$$

where n^* *is the smallest natural number* n *(depending on* m *and on a constant* $0<r<1$*) for which* $m+\frac{1}{2}m^{\frac{1}{3}} \le \lambda_n^{(m)}\cdot r$.

b) $$\sum_{\substack{n,m \\ k\le\lambda_n^{(m)}\le k+1 \\ n\le cm}} m^{-\frac{1}{3}}n^{-\frac{2}{3}} = O(1).$$

PROOF OF b). Denote Ai the Airy function [2]. It is well-known that $\lambda_1^{(m)} > m+\frac{1}{2}m^{1/3}$, $m_0\ge 1$, m_0 is an absolute constant. We know ([2], 345. p.)

$$\varphi\left(\frac{\lambda_n^{(m)}}{m}\right) = -\frac{a_n'}{m^{2/3}} + O(1)\frac{1}{m^{4/3}}, \tag{1}$$

where a_n' is the n-th negative zero of Ai', $\frac{2}{3}\left[\varphi(x)\right]^{3/2} = \int_1^x \left(1-\frac{1}{t^2}\right)^{1/2} dt$, $1\le x<\infty$. It is known ([3], 10.4.94.–10.4.105)

$$a_n' = -\left\{\frac{3}{8}\pi(4n-3)\right\}^{2/3} + O\left(\frac{n^{2/3}}{n^2}\right) = -\left(\frac{3\pi}{2}\right)^{2/3} n^{2/3} + O\left(\frac{1}{n^{1/3}}\right). \tag{2}$$

Obviously φ is strictly monotone increasing. If $k\le\lambda_n^{(m)}\le k+1$, then $k\ge m+1$ if $m\ge m_0$. Hence

$$\sum_{\substack{n,m \\ k\le\lambda_n^{(m)}\le k+1 \\ n\le cm \\ m\ge m_0}} m^{-\frac{1}{3}}n^{-\frac{2}{3}} =$$

$$= \sum_{\substack{n,m \\ \varphi\left(\frac{k}{m}\right)\le\varphi\left(\frac{\lambda_n^{(m)}}{m}\right)\le\varphi\left(\frac{k+1}{m}\right) \\ n\le cm \\ m\ge m_0}} \le c \sum_{\substack{n,m \\ \varphi\left(\frac{k}{m}\right)\le -\frac{a_n'}{m^{2/3}}+O(1)\frac{1}{n^{1/3}m}\le\varphi\left(\frac{k+1}{m}\right) \\ n\le cm \\ m\ge m_0}} \le$$

$$\leq c \sum_{\substack{n,m \\ \varphi\left(\frac{k}{m}\right) \leq \left(\frac{3\pi}{2}\right)^{2/3} \cdot \frac{n^{2/3}}{m^{2/3}} + O(1)\frac{1}{n^{1/3}m^{2/3}} \leq \varphi\left(\frac{k+1}{m}\right) \\ n \leq cm \\ m \geq m_0}} \leq$$

$$\leq c \sum_{\substack{\left(\frac{2}{3\pi}\right)^{2/3} m^{2/3}\varphi\left(\frac{k}{m}\right) \leq n^{2/3}\left(1+O\left(\frac{1}{n}\right)\right) \leq \left(\frac{2}{3\pi}\right)^{2/3} m^{2/3}\varphi\left(\frac{k+1}{m}\right) \\ n \leq cm \\ m \geq m_0}} \leq$$

$$\leq c \sum_{\substack{n,m \\ \frac{m}{p}\cdot\frac{2}{3}\varphi^{3/2}\left(\frac{k}{m}\right)+O(1) \leq n \leq \frac{m}{\pi}\cdot\frac{2}{3}\varphi^{3/2}\left(\frac{k+1}{m}\right)+O(1) \\ n \leq cm \\ m \geq m_0}} =: A.$$

If this sum is nonempty then $\frac{m}{\pi} \cdot \frac{2}{3}\varphi^{3/2}\left(\frac{k}{m}\right) + O(1) \leq cm$ which implies $\varphi^{3/2}\left(\frac{k}{m}\right) = O(1)$, that is, $\frac{k}{m} = O(1)$. Hence $k \leq cm \leq m + cm^{5/3}$, from this $\frac{k}{m} - 1 \leq cm^{2/3}$, which gives $\frac{c}{m} \leq \frac{1}{\left(\frac{k}{m}-1\right)^{3/2}} \leq c\varphi^{3/2}\left(\frac{k}{m}\right)$. Furthermore in A

$$\frac{m}{\pi} \cdot \frac{2}{3}\varphi^{3/2}\left(\frac{k+1}{m}\right) + O(1) - \frac{m}{\pi} \cdot \frac{2}{3}\varphi^{3/2}\left(\frac{k}{m}\right) - O(1) =$$

$$= O(1) + \frac{m}{\pi} \int_{\frac{k}{m}}^{\frac{k+1}{m}} \left(1 - \frac{1}{t^2}\right)^{1/2} dt = O(1).$$

Using these we can write

$$A \leq c \sum_{\substack{m \\ ck \leq m \leq k-1 \\ m \geq m_0}} m^{-\frac{1}{3}} \left(m\varphi^{3/2}\left(\frac{k}{m}\right)\right)^{-\frac{2}{3}} \leq c \sum_{\substack{m \\ ck \leq m \leq k-1 \\ m \geq m_0}} \frac{1}{m} \cdot \frac{1}{\varphi\left(\frac{k}{m}\right)} \leq$$

$$\leq c \sum_{ck \leq m \leq k-1} \frac{}{m} = \frac{1}{m}\left(\frac{k}{m} - 1\right) = O(1).$$

Hence

$$\sum_{\substack{n,m \\ k \leq \lambda_n^{(m)} \leq k+1 \\ n \leq cm}} m^{-\frac{1}{3}} n^{-\frac{2}{3}} = O(1).$$

$$\sum_{\substack{n,m \\ k \leq \lambda_n^{(m)} \leq k+1 \\ m \geq m_0 \\ n^* \leq n \leq cm \\ m+\frac{1}{2}m^{1/3} \leq \lambda_{n^*}^{(m)} \cdot r_i \text{ and } n^* \text{ is minimal}}} \frac{1}{\sqrt{m(n-n^*+1)}} \leq c \sum_{\substack{n,m \\ k \leq \lambda_n^{(m)} \leq k+1 \\ m \geq m_0 \\ n^* \leq n \leq cm \\ m \leq \lambda_{n^*}^{(m)} \cdot r_i \text{ and } n^* \text{ is minimal}}} \leq$$

$$\leq c \sum_{\substack{n,m \\ \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{2/3}\left(\frac{k}{m}\right) + O(1) \leq n \leq \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k+1}{m}\right) + O(1) \\ n^* \leq n \leq cm \\ m \geq m_0 \\ \varphi\left(\frac{1}{r_i}\right) \leq \varphi\left(\frac{\lambda_{n^*}^{(m)}}{m}\right)}} \leq$$

$$\leq c \sum_{\substack{n,m \\ \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k}{m}\right) + O(1) \leq n \leq \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k+1}{m}\right) + O(1) \\ \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{1}{r_i}\right) + O(1) \leq n \leq cm \\ m \geq m_0}} .$$

If this sum is nonempty then $\frac{k}{m} = O(1)$. Furthermore

$$\frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k+1}{m}\right) + O(1) - \frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k}{m}\right) - O(1) = O(1).$$

$$\sum_{\substack{n,m \\ \cdots}} = \sum_{\substack{n,m \\ \cdots \\ \frac{1}{r_i} \le \frac{k}{m} - \frac{c}{m}}} + \sum_{\substack{n,m \\ \cdots \\ \frac{k}{m} - \frac{c}{m} \le \frac{1}{r_i} \le \frac{k}{m} + \frac{c}{m}}} + \sum_{\substack{n,m \\ \cdots \\ \frac{k}{m} + \frac{c}{m} \le \frac{1}{r_i} \le \frac{k+1}{m} + \frac{c}{m}}} =: S_1 + S_2 + S_3.$$

Since φ is strictly monotone increasing therefore

$$\frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{k}{m}\right) + O(1) - \left(\frac{m}{\pi} \cdot \frac{2}{3} \varphi^{3/2}\left(\frac{1}{r_i}\right) + O(1)\right) \ge c > 0.$$

Thus

$$S_1 \le c \sum_{\substack{m \\ ck \le m \le k-1 \\ \frac{1}{r_i} \le \frac{k}{m} - \frac{c}{m} \\ m \ge m_0}} \frac{1}{\sqrt{m\left(m \cdot \frac{2}{3}\varphi^{3/2}\left(\frac{k}{m}\right) - m \cdot \frac{2}{3}\varphi^{3/2}\left(\frac{1}{r_i}\right)\right)}} \le$$

$$\le c \sum_{\substack{m \\ ck \le m \le k-1 \\ \frac{1}{r_i} \le \frac{k}{m} - \frac{c}{m} \\ m \ge m_0}} \frac{1}{m\sqrt{\frac{k}{m} - r_i}} \le c \sum_{\substack{m \\ c_k \le m \le k-1 \\ c \le k - r_i m}} \frac{1}{\sqrt{m}\sqrt{k - r_i m}} \le$$

$$\le c \sum_{\substack{t \\ c \le t \le ck}} \frac{1}{\sqrt{k}} \frac{1}{\sqrt{t}} = O(1).$$

Now we estimate S_2.

$$S_2 \le c \sum_{\substack{m \\ \frac{k}{m} - \frac{c}{m} \le \frac{1}{r_i} \le \frac{k}{m} + \frac{c}{m}}} \frac{1}{\sqrt{m}} = c \sum_{\substack{m \\ r_i k - c \le m \le r_i k + c}} \frac{1}{\sqrt{m}} = O(1).$$

Now estimate S_3.

Similarly as in the estimate of S_2 we obtain $S_3 = O(1)$. Hence

$$\sum_{\substack{n,m \\ k \le \lambda_n^{(m)} \le k+1 \\ n^* \le n \le cm \\ m + \frac{1}{2} m^{1/3} \le \lambda_{n^*}^{(m)} \cdot r_i \text{ and } n^* \text{ is minimal}}} \frac{1}{\sqrt{m(n - n^* + 1)}} = O(1).$$

Using the above Lemma, we can show that Theorem 1 in [5] remains true for $r = -\frac{1}{2}$, too, namely

THEOREM 1'. *For* $r=-\frac{1}{2}$ *and for any control* $v \in L^2(0,T;\mathbb{C}^N)$ $(u,u_t) \in$ $\in C\left([0,T],\mathcal{H}_{-\frac{1}{2}}\right)$.

PROOF. As it was seen in the proof of the case $r<-\frac{1}{2}$, we have to verify that

$$(22)\qquad \left(\frac{\left(\lambda_n^{(m)}\right)^{1/2}}{\left[\left(\lambda_n^{(m)}\right)^2-m^2\right]^{1/4}}\int\limits_0^t e^{iw_{mk}(t-\tau)}\langle v(\tau),e_{mn}^{\pm}\rangle\,d\tau\right)\in l_2.$$

(Here we use the formula numbering of [5].) For the pairs m, k with $M\geq m$ (M fixed), the original proof applies. For the case $m\geq M$, $n\geq cm$ (where c is a large constant) we apply the following device: denote

$$F_i(z)=\int\limits_0^t v_i(\tau)e^{-iz\tau}\,d\tau$$

then

$$|F_i(z_0)|^2=O\left(\int\limits_{|z-z_0|\leq 1}|F_i(z)|^2dz\right)$$

and

$$\int\limits_{|\mathrm{Im}\,z|\leq 1}|F_i(z)|^2dz\leq c\int\limits_0^t|v_i(\tau)|^2d\tau$$

see Young [4]. Hence from $\lambda_n^{(m)}\asymp n$, $\left(\lambda_n^{(m)}\right)^2-m^2\asymp n^2$ and from

$$(23)\qquad e_{mn}^{\pm}=O\left(\frac{1}{\sqrt{n}}\right)$$

we get

$$\sum_{m\geq M}\sum_{n\geq cm}\frac{1}{n}\left|\int\limits_0^t e^{iw_{mk}(t-\tau)}v_i(\tau)d\tau\right|^2\leq c\sum_{m\geq M}\sum_{n\geq cm}\frac{1}{n}\int\limits_{\left|z-\lambda_n^{(m)}\right|\leq 1}|F_i(z)|^2dz=$$

$$= c \sum_n \frac{1}{n} \sum_{M \le m \le \frac{n}{c}} \int_{|z-\lambda_n^{(m)}| \le 1} |F_i|^2 \le \int_{|\mathrm{Im}\, z| \le 1} |F_i|^2 \le c \int_0^t |v_i|^2.$$

In case $n \le cm$, $m \ge M$, $r_i = 1$ we have to estimate

$$\sum_{m \ge M} \sum_{n \le cm} m^{-\frac{1}{3}} n^{-\frac{2}{3}} \left| \int_0^t e^{-iw_{mk}\tau} v_i(\tau) dz \right|^2 \le$$

$$\le c \sum_{k=1}^{\infty} \sum_{\substack{n,m \\ k \le \lambda_n^{(m)} \le k+1 \\ n \le cm}} m^{-\frac{1}{3}} n^{-\frac{2}{3}} \int_{|z-\lambda_n^{(m)}| \le 1} |F_i(z)|^2 dz \le$$

$$\le c \sum_{k=1}^{\infty} \int_{\substack{|\mathrm{Im}\, z \le 1 \\ k-1 \le \mathrm{Re}\, z \le k+1}} |F_i(z)|^2 dz \cdot \sum_{\substack{n,m \\ k \le \lambda_n^{(m)} \le k+1 \\ n \le cm}} m^{-\frac{1}{3}} n^{-\frac{2}{3}} \le$$

$$\le c \int_{|\mathrm{Im}\, z| \le 1} |F_i(z)|^2 dz \le c \int_0^t |v_i(\tau)|^2 d\tau$$

(we used Lemma 1 b) above). Similarly, if $n \le cm$, $m \ge M$, $r_i < 1$ then we estimate as follows

$$\sum_{m \ge M} \sum_{n \le cm} \frac{1}{\sqrt{m(n - n^* + 1)}} \left| \int_0^t e^{-iw_{mk}\tau} v_i(\tau) d\tau \right|^2 \le$$

$$\le c \sum_{k=1}^{\infty} \int_{\substack{|\mathrm{Im}\, z| \le 1 \\ k \le 1 \le \mathrm{Re}\, z \le k+1}} |F_i(z)|^2 dz \cdot \sum_{\substack{n,m \\ k \le \lambda_n^{(m)} \le k+1 \\ n \le cm}} \frac{1}{\sqrt{m(n - n^* + 1)}} \le$$

$$\le c \int_{|\mathrm{Im}\, z| \le 1} |F_i(z)|^2 fz \le c \int_0^t |v_i(\tau)|^2 d\tau.$$

The estimate for the sum where $\lambda_n^{(m)} r_i \leq m + \frac{1}{2} m^{\frac{1}{3}}$, remains the same as for $r < -\frac{1}{2}$. The proof is complete. ∎

THEOREM 2'. *The system is not approximately controllable for $r = -\frac{1}{2}$ and for any $T > 0$.*

The original proof works because (see p.249 in [5]) we used only that

$$\left(n^r \int_0^T e^{iw_{mk}(t-\tau)} \left\langle v(\tau), e_{mn}^{\pm} \sqrt{n} \right\rangle d\tau \right) \in l_2$$

in the coordinates $m \geq M$, $n \geq cm$. But this is proved above in Theorem 1'. ∎

THEOREM 5'. *Let $r = -\frac{1}{2}$, then*

a) $\bigcup_{T>0} R(T) \subsetneq \mathcal{H}_{-\frac{1}{2}}$

b) *If $e^{\pm} \neq 0 \ \forall \ m, n$ and e_{mn}^{+}, e_{mn}^{-} are not parallel vectors, then*

$$\overline{\bigcup_{T>R} R(T)} = \mathcal{H}_{-\frac{1}{2}}.$$

PROOF. We have to show (using the notations of the original proof) that a) $\tilde{R}(\infty) \subsetneq l_2$, b) $\overline{\tilde{R}(\infty)} = l_2$.

a) The system $e^{-i(w_{0,k} - i)\tau}$ is Riesz basis in $L_2(0,\infty)$ in its closed linear hull. From $\frac{|w_{0,k}|^r}{\sqrt{\gamma_{0,n}}} \asymp n^{r+\frac{1}{2}} = 1$ it follows that the factor of $e^{-i(w_{0,k} - i)\tau}$ is $\asymp |e_{0,n}^{+} = o(1)$, hence the moment sequence is a nontrivial subspace of l_2, $\tilde{R}(\infty) \subsetneq l_2$.

b) The original proof works. ∎

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ORTHOGONAL TYPE RELATIONS FOR THE BESSEL POLYNOMIALS

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1. Introduction

The Bessel polynomials constitute an important class of polynomials with many applications [4], pp. 131–149. GROSSWALD [4], pp. 25–33 discussed only the orthogonality of the Bessel polynomials on the unit circle and the corresponding moments. EXTON gave the orthogonality property [3], p. 215, (14):

$$(1.1)\qquad \int_0^\infty x^{a-2}e^{-1/x}y_m(x;a,1)y_n(x;a,1)dx = \\ = \frac{(-1)^n n!(n+a-2)\pi}{\Gamma(a+n)(2n+a-1)\sin(\pi a)}\delta_{m,n}, \quad \mathrm{Re}a < 1-m-n.$$

SRIVASTAVA [6] noted that the orthogonality property of the Bessel polynomials:

$$(1.2)\qquad \int_0^\infty x^{1-a}e^{-x}y_m(1;a,x)y_n(1;a,x)dx = n!\Gamma(2-a-n)\delta_{m,n}$$

obtained by HAMZA [5] is incorrect.

The author [2], in view of the remark of SRIVASTAVA, using (1.1) derived the following orthogonality property:

(1.3)

$$\int_0^\infty x^{-a}e^{-x}y_m(1;a,x)y_n(1;a,x)dx = \frac{n!\Gamma(2-a-n)}{(1-a-2n)}\delta_{m,n}, \quad \mathrm{Re}a < 1-m-n.$$

The Bessel polynomials are defined by the relation [4], p. 38, (1):

$$y_n(x;a,b) = {}_2F_0\left(-n, n+a-1; -; -\frac{x}{b}\right). \tag{1.4}$$

In view of the relation [1], p. 325, (5):

$$ {}_2F_0(-n;a;-x) = (a)_n(-1)^n x^n {}_1F_1\left(-n; 1-a-n; -\frac{1}{x}\right) \tag{1.5}$$

the Bessel polynomials can be expressed as:

$$y_n(x;a,b) = (a+n-1)_n \left(\frac{x}{b}\right)^n {}_1F_1\left(-n; 2-a-2n; \frac{b}{x}\right), \tag{1.6}$$

$$y(b;a,x) = (a+n-1)_n \left(\frac{b}{x}\right)^n {}_1F_1\left(-n; 2-a-2n; \frac{x}{b}\right). \tag{1.7}$$

In view of the interest shown in the Bessel polynomials [4], it appears worthwhile to investigate further, the matter of the orthogonality of the Bessel polynomials and the functions related to them.

In this paper, we establish an orthogonal type relation for the Bessel polynomials over the interval $(0,\infty)$ with respect to multiplier function $x^{m-n-a+2}e^{-x/b}$.

The following identity which follows from (1.4) may be found useful in the study of the Bessel polynomials:

$$y_n(x;a,b) = y_n(1/b;a,1/x). \tag{1.8}$$

2. Orthogonal type relation

The orthogonal type relation to be established is

$$\int_0^\infty x^{m-n-a+2}e^{-x/b} y_m(b;a,x) y_n(b;a,x)\,dx =$$

$$= 0, \quad \text{if} \quad m < n-1 \tag{2.1}$$

$$= b^{2-a} n!\Gamma(2-a-n), \quad \text{if} \quad m = n-1 \tag{2.2}$$

$$= b^{3-a} n!(2-a)\Gamma(2-a-n), \quad \text{if} \quad m = n \tag{2.3}$$

$$= 3b^{4-a}(n+1)!(2-a)\Gamma(2-a-n), \quad \text{if} \quad m = n+1 \tag{2.4}$$

$$= \frac{(-1)^n(a+n+k-1)_{n+k}(a+n-1)_n\Gamma(3-a-2n)}{b^{a-k-3}\Gamma(2-n)(2-a-2n)_n} \times$$

$$\times {}_3F_2\left[\begin{matrix} -n-k, 2, 3-a-2n; 1 \\ 2-a-2n-2k, 2-n \end{matrix}\right] \tag{2.5}$$

if $m = n + k \quad (k = -1, 0, 1, 2, \dots)$ where $2n + \mathrm{Re}\,a < 3$.

PROOF. In view of (1.7), the integral (2.1) can be written as

$$(2.6)\quad (a+m-1)_m(a+n-1)_n(b)^{m+n}\cdot \int_0^\infty x^{2-2n-a}e^{-x/b}\,{}_1F_1\left[\begin{matrix}-m;\ \frac{x}{b}\\ 2-a-2m\end{matrix}\right]{}_1F_1\left[\begin{matrix}-n;\ \frac{x}{b}\\ 2-a-2n\end{matrix}\right]dx.$$

We next express the hypergeometric functions as infinite series [1], p.322, (10.1), interchange the order of integration and summation, which, incidentally is justified due to absolute convergence of the integral and summations involved, and write (2.6) as

$$(2.7)\quad (a+m-1)_m(a+n-1)_n b^{m+n}\cdot \sum_{r=0}^{m}\frac{(-m)_r b^{-r}}{(2-a-2m)_r r!}\sum_{u=0}^{n}\frac{(-n)_u b^{-u}}{(2-a-2n)_u u!}\int_0^\infty x^{2-a-2n+r+u}e^{-x/b}dx.$$

Now, the integral in (2.7) can be evaluated with the help of the definition of the gamma function:

$$\int_0^\infty x^n e^{-x/b}dx = \Gamma(n+1)b^{n+1}, \quad \mathrm{Re}\,n > -1.$$

Thus (2.7) is equivalent to

$$(2.8)\quad \frac{(a+m-1)_m(a+n-1)}{b^{a+2n-3}}b^{m+n}\cdot \sum_{r=0}^{m}\frac{(-m)_r}{(2-a-2m)_r r!}\sum_{u=0}^{n}\frac{(-n)_u\Gamma(3-a-2n+r+u)}{(2-a-2n)_u u!}.$$

On simplifying, (2.8) reduces to

$$(2.9)\quad \frac{(a+m-1)_m(a+n-1)_n}{b^{a+2n-3}}b^{m+n}\cdot \sum_{r=0}^{m}\frac{(-m)_r\Gamma(3-a-2n+r)}{(2-a-2m)_r r!}\,{}_2F_1\left[\begin{matrix}-n, 3-a-2n+r; 1\\ 2-a-2n\end{matrix}\right].$$

Applying Vandermonde's theorem

$${}_2F_1(-n;b;c;1) = \frac{(c-b)_n}{(c)_n}, \qquad n = 0,\ 1,\ 2,\ \dots$$

the relation (2.9) reduces to the form:
(2.10)

$$\frac{(a+m-1)_m(a+n-1)_n}{b^{a+2n-3}}b^{m+n}\sum_{r=0}^{m}\frac{(-m)_r(-r-1)_n\Gamma(3-a-2n+r)}{(2-a-2m)_r(2-a-2n)_n r!}.$$

If $r < n-1$, the numerator of (2.10) vanishes, and since r runs from 0 to m, it follows that (2.10) also vanishes, when $m < n-1$. Now, it is clear that for $m < n-1$ all terms of (2.10) vanish, which proves (2.1).

When $m = n-1$, using the standard result $(-n)_n = (-1)^n n!$, and $\Gamma(1-a-n) = \frac{(-1)^n\Gamma(1-a)}{(a)_n}$ we have

$$\text{(2.11)}\qquad \int_0^\infty x^{1-a}e^{-x/b}y_{n-1}(b;a,x)y_n(b;a,x)dx = b^{2-a}n!\Gamma(2-a-n),$$

which proves (2.2).

For $m=n$, we employ the standard results like $(-n)_{n-1} = (-1)^{n-1}n!$ and $(-n-1)_n = (-1)^n(n+1)!$ and add the resulting two terms ($r = n-1, n$), with the help of $\Gamma(1-a-n) = \frac{(-1)^n\Gamma(1-a)}{(a)_n}$ to obtain

$$\text{(2.12)}\qquad \int_0^\infty x^{2-a}e^{-x/b}\{y_n(b;a,x)\}^2dx = b^{3-a}n!(2-a)\Gamma(2-a-n),$$

which proves the relation (2.3).

For $m = n+1$, we use the standard result [1], p. 274, (8.3):

$$(-k)_n = \begin{cases} \frac{(-1)^n k!}{(k-n)!}, & 0 \le n \le k \\ 0, & n > k. \end{cases}$$

and add the resulting three terms ($r = n-1, n, n+1$), then simplify to obtain
(2.13)

$$\int_0^\infty x^{3-a}e^{-x/b}y_{n+1}(b;a,x)y_n(b;a,x)dx = 3b^{4-a}(a-2)(n+1)!\Gamma(2-a-n),$$

which proves (2.4).

NOTE. The formulae for $m = n+2, n+3, n+4, \ldots$, can be obtained as above.

Since, we see that there is no symmetry in (2.11), (2.12) and (2.13), therefore it is not possible to find a general formula in a compact form.

However on setting $m = n + k$ $(k = -1, 0, 1, 2, \ldots)$ in (2.10), we obtain the general formula in following form:

$$(2.14) \qquad \int_0^\infty x^{k-a+2} e^{-x/a} y_{n+k}(b;a,x) y_n(b;a,x) dx =$$

$$= \frac{(-1)^n (a+n+k-1)_{n+k} (a+n-1)_n \Gamma(3-a-2n)}{b^{a-k-3} \Gamma(2-n)(2-2a-2n)_n} \times$$

$$\times {}_3F_2 \left[\begin{matrix} -n-k, 2, 3-a-2n; 1 \\ 2-a-2n-2k, 2-n \end{matrix} \right].$$

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ON UNIFORM CONVERGENCE OF POLYNOMIALS CONVERGING IN L^p NORM*

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In [1] it is shown by L. CZÁCH that exponential convergence of a sequence of polynomials in L^2 implies uniform convergence. In this paper this result is generalized with a weaker condition on the speed of convergence in L^p spaces ($p \geq 1$). A possible benefit in applications is transition to uniform convergence when a Banach space method yields an approximating sequence of polynomials converging in L^p.

First a theorem with a general convergence assumption is proved, then corollaries are given for power order and exponential convergence. Finally some consequences are proved concerning the smoothness of the limit function and the convergence of the derivatives.

THEOREM 1. *Let $S \subset \mathbb{R}^N$ be a closed ball around the origin, $(p_n) : \mathbb{R}^N \to \mathbb{R}$ a sequence of polynomials with $gr(p_n) = n$, $p \geq 1$ and $f \in L^p(S)$. Further, let $(\varepsilon_n) \subset \mathbb{R}$ such that $\varepsilon_n \to 0$ decreasingly.*

If $\|f - p_n\|_{L^p(S)} \leq \varepsilon_n$ for all $n \in \mathbb{N} := \{1,2,\ldots,\}$ then the following inequalities hold:

(1) *if $\sum n^{2N}\varepsilon_n < \infty$ and $\eta_n := \sum_{j=n}^{\infty} j^{2M}\varepsilon_j$ then $\exists\, k_1 > 0$ such that*

$$\|f - p_n\|_{C(S)} \leq k_1\eta_n \qquad (n \in \mathbb{N});$$

(2) *if $\sum n^{N}\varepsilon_1 < \infty$ and $\mu_n := \sum_{j=n}^{\infty} j^{N}\varepsilon_j$ then for every closed set $F \subset \operatorname{int} S$ $\exists\, k_2 > 0$ such that*

$$\|f - p_n\|_{C(F)} \leq k_2\mu_n \qquad (n \in \mathbb{N}).$$

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Before the proof of Theorem 1 we prove an analogous assertion for $p=1$ and with an interval instead of S.

PROPOSITION. *Let I denote $\bigtimes_{i=1}^{N} [a_1,b_1] \subset \mathbb{R}^N$ and let $(p_n), f$ and ε_n be as in the theorem. If $\|f-p_n\|_{L^1(I)} \le \varepsilon_n$ $(n \in \mathbb{N})$ then the following inequalities hold:*

(i) *if $\sum n^{2N}\varepsilon_n < \infty$ and η_n is as in the theorem then $\exists\, k_1 > 0$ such that*

$$\|f-p_n\|_{C(I)} \le k_1\eta_n \qquad (n \in \mathbb{N});$$

(ii) *if $\sum n^N \varepsilon_1 < \infty$ and μ_n is as in the theorem then for every closed set $F \subset \operatorname{int} I$ $\exists\, k_2 > 0$ such that*

$$\|f-p_n\|_{C(F)} \le k_2\mu_n \qquad (n \in \mathbb{N}).$$

PROOF. Let $q_n := p_n - p_{n-1}$ $(n = 2, 3, \ldots)$. The assumption $\|f-p_n\|_{L^1(I)} \le \varepsilon_n$ $(n \in \mathbb{N})$ implies

(3) $$\|q_n\|_{L^1(I)} \le \varepsilon_n + \varepsilon_{n-1} \le 2\varepsilon_{n-1} \qquad (n = 2,3,\ldots).$$

Let $Q_n(x_1,x_2,\ldots,x_N) := \int_{a_1}^{x_1}\int_{a_2}^{x_2}\ldots\int_{a_N}^{x_N} q_n$ $\quad((x_1,x_2,\ldots,x_N) \in I)$. It is clear that

(4) $$\partial_1\partial_2\ldots\partial_N Q_n = q_n \qquad \text{and}$$

(5) $$\|Q_n\|_{C(I)} \le \|q_n\|_{L^1(I)}.$$

Thus, considering (3), we have

(6) $$\|Q_n\|_{C(I)} \le 2\varepsilon_{n-1}.$$

First we prove (i). The iterated applications of Markov's inequality (i.e. for any polynomial P on I with $\operatorname{gr} P = n$ and $l = 1,2,\ldots,N$ $\|\partial_l P\|_{C(I)} \le \le Kn^2\|P\|_{C(I)}$ K, depending on I only; see [3]) lead to

$$\|\partial_1\partial_2\ldots\partial_N Q_n\|_{C(I)} \le$$

$$\le K^N \left[\prod_{k=0}^{N}(n+k)\right]^2 \|Q_n\|_{C(I)} \le K^N \left[n^N(N+1)!\right]^2 \|Q_n\|_{C(I)},$$

i.e.

$$\|\partial_1\partial_2\ldots\partial_N Q_n\|_{C(I)} \le K_1 n^{2N}\|Q_n\|_{C(I)}$$

$(K_1 := K^N\left((N+1)!^2\right)$, which, by (4) and (6) and with $M := 2K_1$ means

$$\|q_n\|_{C(I)} \leq M n^{2N} \varepsilon_{n-1}. \tag{7}$$

The assumption on ε_n implies that $\sum \|q_n\|_{C(I)} < \infty$, thus $\sum q_n$ converges uniformly, which just means that (p_n) converges uniformly and clearly the limits is f itself.

Finally,

$$\|f - p_n\|_{C(I)} \leq \sum_{j=n+1}^{\infty} \|q_j\|_{C(I)} \leq M \sum_{j=n+1}^{\infty} j^{2N} \varepsilon_{j-1} \leq$$
$$\leq k_1 \sum_{j=n+1}^{\infty} (j-1)^{2N} \varepsilon_{j-1} = k_1 \eta_n,$$

thus (i) is proved. (k_1 can be chosen $2^{2N}M$.)

The proof of (ii) is similar, using Markov's second inequality (i.e. for any closed set $F \subset \operatorname{int} I$ $\exists$ $L > 0$, depending on F and I, such that for any polynomial P on I with $\operatorname{gr} P = n$ and $l = 1, 2, \ldots, N$ $\|\partial_l P\|_{C(F)} \leq Ln\|P\|_{C(I)}$ holds; see [3]). Thus, with $M' := 2L_1$ (where $L_1 := L^N(N+1)!$), we have

$$\|q_n\|_{C(F)} \leq M' n^N \varepsilon_{n-1}, \tag{7'}$$

which leads to

$$\|f - p_n\|_{C(F)} \leq M' \sum_{j=n+1}^{\infty} (j-1)^N \varepsilon_{j-1} \leq k_2 \sum_{j=n+1}^{\infty} \varepsilon_{j-1} = k_2 \mu_2$$

(k_2 can be chosen $2^N M'$). Hence (ii) is also proved. ∎

PROOF OF THE THEOREM is node is three steps.

(a) Let $I := [-a, a]^N$ $(a > 0)$ and let $T_\varphi : \mathbb{R}^N \to \mathbb{R}^N$ denote rotation by $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{N-1}) \in \left[0, \frac{\pi}{2}\right)^{N-1}$. Then the proposition proved above holds also on $T_\varphi(I)$ with the same constants k_1 and k_2.

This is straightforward if we apply the proposition to the sequence $(p_n \circ T_\varphi)$ which also consists of polynomials of degree n and in (ii) to the set $T_\varphi^{-1}(F)$.

(b) Now the proof on S follows briefly for $p = 1$: denoting by r the radius of S and by I the cube $\left[-\frac{r}{\sqrt{N}}, \frac{r}{\sqrt{N}},\right]^N$ (the straight cube with vertices on the sphere), in case (1) for any $x \in S$ we can find $\varphi \in \left[0, \frac{\pi}{2}\right)^{N-1}$ such that

$x \in I_\varphi := T_\varphi(I)$. Hence $|f(x)-p_n(x)| \le \|f-p_n\|_{C(I_\varphi)} \le k_1\eta_n$, where the latter inequality follows from $\|f-p_n\|_{L^1(I_\varphi)} \le \|f-p_n\|_{L^1(S)} \le \varepsilon_n$ and part (i) of the proposition. Since $x \in S$ was arbitrary, we have $\|f-p_n\|_{C(S)} \le k_1\eta_1$.

To prove (2), we use the fact that in Markov's second inequality the exact value of the constant $L = L(F,I)$ is $\frac{1}{\operatorname{dist}(F,\partial I)}$. In the proposition we have $k_2 = 2^{N+1}L^N(N+1)!$. Now for any $x \in F$ we can find $\varphi \in \left[0,\frac{\pi}{2}\right)^{N-1}$ such that $x \in I_\varphi := T_\varphi(I)$ and $\operatorname{dist}(x,\partial I) \ge \varrho := \frac{\operatorname{dist}(F,\partial S)}{\sqrt{N}}$. Then $F_x := = F \cap \overline{S}\left(x,\frac{\varrho}{2}\right)$ is a closed subset of $\operatorname{int} I$ with $\operatorname{dist}(F_x,\partial I) \ge \frac{\varrho}{2}$. Thus for any $x \in F$, part (ii) of the proposition is valid for F_x and I_φ with $k_2 := = 2^{2N+1}\varrho^{-N}(N+1)!$, a constant independent of x. Now we can proceed as above: $|f(x)-p_n(x)| \le \|f-p_n\|_{C(F_x)} \le k_2\mu_2$, where the latter inequality follows from $\|f-p_n\|_{L^1(I_\varphi)} \le \|f-p_n\|_{L^1(S)} \le \varepsilon_n$ and part (ii) of the proposition with k_2 given above. Since $x \in F$ was arbitrary, we have $\|f-p_n\|_{C(F)} \le k_2\mu_n$.

(c) For $p > 1$ the proof follows from the case with $p = 1$ and the fact that for any $p > 1$ we have $L^p(S) \subset L^1(S)$ and $\|g\|_{L^1(S)} \le \text{const} \cdot \|g\|_{L^p(S)}$ for any $g \in L^p(S)$. ∎

In two of the most important cases, namely, for $\varepsilon_n := \frac{c}{n^\gamma}$ $(c > 0, \gamma > 1)$ and $\varepsilon_n = cq^n$ $(c > 0, 0 < q < 1)$ the application of Theorem 1 can be completed by a simpler estimate for η_n and μ_n as follows:

COROLLARY 1. *Let $S \subset \mathbb{R}^N$ be the unit ball, $(p_n) : \mathbb{R}^N \to \mathbb{R}$ a sequence of polynomials with $\operatorname{gr}(p_n) = n$, $p \ge 1$ and $f \in L^p(S)$. Assume that $\exists\, c > 0$ and $\gamma > 1$ such that $\|f-p_n\|_{L^p(S)} \le \frac{c}{n^\gamma}$ for all $n \in \mathbb{N}$. Then the following inequalities hold:*

a) *if $\gamma > k_N := 2N+1$ then $\exists\, K_1 > 0$ such that*

$$\|f-p_n\|_{C(S)} \le \frac{K_1}{n^{\gamma-k_N}} \qquad (n \in \mathbb{N}); \tag{8}$$

b) *if $\gamma > m_N := N+1$ and F is a closed subset of $\operatorname{int} S$ then $\exists\, K_2 > 0$ such that*

$$\|f-p_n\|_{C(F)} \le \frac{K_2}{n^{\gamma-m_N}} \qquad (n \in \mathbb{N}). \tag{9}$$

PROOF follows from the theorem by an elementary calculation:

For any $\beta > 1$ and $n \in \mathbb{N}$, $\sum_{j=n}^{\infty} \frac{1}{j^\beta} \leq \frac{M_\beta}{n^{\beta-1}}$ holds (with $M_\beta := \sum_{j=1}^{\infty} \frac{1}{j^\beta}$):

$$\sum_{j=n}^{\infty} \frac{1}{j^\beta} = \frac{1}{n^\beta} \sum_{j=0}^{\infty} \left(\frac{n}{n+j}\right)^\beta = \frac{1}{n^\beta} \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \left(\frac{n}{jn+k}\right)^\beta \leq \frac{1}{n^\beta} \sum_{j=1}^{\infty} \frac{n}{j^\beta} = \frac{1}{n^{\beta-1}} M_\beta .$$

Hence

a)

$$\eta_n := \sum_{j=n}^{\infty} j^{2N} \varepsilon_j = c \sum_{j=n}^{\infty} \frac{1}{j^{\gamma-2N}} \leq \frac{c M_{\gamma-2N}}{n^{\gamma-2N-1}} = \frac{c_1}{n^{\gamma-k_N}} \qquad (c_1 := c M_{\gamma-2N})$$

if $\gamma > k_N$; $K_1 := c_1 k_1$;

b)

$$\mu_n := \sum_{j=n}^{\infty} j^{N} \varepsilon_j = c \sum_{j=n}^{\infty} \frac{1}{j^{\gamma-N}} \leq \frac{c M_{\gamma-N}}{n^{\gamma-N-1}} = \frac{c_2}{n^{\gamma-m_N}} \qquad (c_2 := c M_{\gamma-N})$$

if $\gamma > m_N$; $K_2 := c_2 k_2$. ∎

COROLLARY 2. *Let $S \subset \mathbb{R}^N$ be the unit ball, $(p_n) : \mathbb{R}^N \to \mathbb{R}$ a sequence of polynomials with* $\operatorname{gr}(p_n) = n$, $p \geq 1$ *and* $f \in L^p(S)$. *Assume that* $\exists\, c > 0$ *and* $0 < q < 1$ *such that* $\|f - p_n\|_{L^p(S)} \leq c q^n$ *for all* $n \in \mathbb{N}$.

Then $\exists\, c_1 > 0$ *and* $q < q_1 < 1$ *such that*

(10) $$\|f - p_n\|_{C(S)} \leq c_1 q_1^n \qquad (n \in \mathbb{N}.$$

PROOF. We can find $r > 1$ and $q < q_1 < 1$ such that $n^{2N} q^n \leq r q_1^n$ $(n \in \mathbb{N})$. Hence $\eta_n := \sum_{j=n}^{\infty} j^{2N} \varepsilon_j = c \sum_{j=n}^{\infty} j^{2N} q^j \leq cr \sum_{j=n}^{\infty} q_1^j = \frac{cr}{1-q_1} q_1^n$, which yields the desired estimate with $c_1 := \frac{k_1 cr}{1-q_1}$.

REMARKS. 1. Obviously, we obtain a constant smaller than c_1 if we only need estimate (10) in the case $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

2. We have more rapid convergence in this case as well if we consider a closed subset F of $\operatorname{int} S$; the same proof as above leads to $\|f - p_n\|_{C(F)} \leq c_2 q_2^n$ $(n \in \mathbb{N})$ where $q_2 < q_1$ since now $n^N q^n \leq r q_1^n$ $(n \in \mathbb{N})$ is required.

Finally we discuss the facts that the proof of Theorem 1 yields about the smoothness of f and the convergence of the derivatives of (p_n).

THEOREM 2. *Under the assumptions of Theorem 1, let $m \in \mathbb{N}$ be such that $\sum n^{N+m}\varepsilon_n < \infty$ and let $\tau_n^{(k)} := \sum_{j=k}^{\infty} j^{N+k}\varepsilon_j$ for each $k = 1,2,\ldots,m$.*

Then $f \in C^m(\operatorname{int} S)$ and for any $k = 1,2,\ldots,m$ and for each closed set $F \subset \operatorname{int} S$ $\exists$ $d_k > 0$ such that

$$\|f^{k)} - p_n^{(k)}\|_{C(F)} \leq d_k \tau_n^{(k)} \qquad (n \in \mathbb{N}).$$

If $\sum n^{2N+2m}\varepsilon_n < \infty$, $k = 1,2,\ldots,m$ and $\vartheta_n^{(k)} := \sum_{j=n}^{\infty} j^{2N+2k}\varepsilon_j$ then $\exists$ $r_k > 0$ such that

$$\|f^{(k)} - p_n^{(k)}\|_{C(S)} \leq r\vartheta_n^{(k)} \qquad (n \in \mathbb{N}).$$

PROOF. We only prove the theorem for an interval I instead of a ball and for $p = 1$. Then the rest can be done just as in Theorem 1.

Let F be a closed subset of $\operatorname{int} I$. We can find closed intervals I_k ($k = 0,1,\ldots,m-1$) such that $\operatorname{int} I \supset I_0$, $\operatorname{int} I_0 \supset I_1$, $\ldots$, $\operatorname{int} I_{m-2} \supset I_{m-1}$, $\operatorname{int} I_{m-1} \supset F$, and for all $k = 1,\ldots,m-1$ $\operatorname{dist}\left(I_{k-1}, I_k\right) = \operatorname{dist}(I, I_0) = \operatorname{dist}\left(I_{m-1}, F\right)$.

Part (ii) of the proposition after Theorem 1 can be applied with I_0 in the role of the closed subset of $\operatorname{int} I$. In (7)′ during the proof we obtain the estimate $\|q_n\|_{C(I_0)} \leq M' n^N \varepsilon_{n-1}$ ($n \in \mathbb{N}$) implying the uniform convergence of $\sum q_n$ on I_0, i.e. that $p_n \to f$ uniformly on I_0. Then by Markov's inequality applied to I_1 as a closed subset of $\operatorname{int} I_0$ we have

$$\|q_n'\|_{C(I_1)} := \max\left\{\|\partial_l q_n\|_{C(I_1)} \;:\; l = 1,2,\ldots,n\right\} \leq LM' n^{N+1}\varepsilon_{n-1}.$$

Thus $\sum \|q_n'\|_{C(I)} < \infty$, hence $\sum q_n'$ converges uniformly on I_1, i.e. (p_n') converges uniformly on I_1. This implies that $f \in C^1(I_1)$ and $f' = \lim p_n'$.

If $m > 1$ then we can repeat this argument for f' on I_2 as a closed subset of $\operatorname{int} I_1$. Owing to Markov we now have $\|q_n''\|_{C(I_2)} \leq L^2 M' n^{N+2}\varepsilon_{n-1}$ ($n \in \mathbb{N}$), which finally yields that $f \in C^2(I_2)$ and $f'' = \lim p_n''$. This can be continued by induction as long as $\sum n^{N+k}\varepsilon_n < \infty$, i.e. at least up to m. Eventually we obtain that $f \in C^m(F)$. Since this holds for any closed $F \subset \operatorname{int} I$, we have $f \in C^m(\operatorname{int} I)$.

The estimate for the convergence of the derivatives is now proved similarly to that in the proposition, using the estimate

$$\|q_n^{(m)}\|_{C(F)} \leq L^m M' n^{M+n}\varepsilon_{n-1}$$

received by the above induction:

$$\|f^{(m)} - p_n^{(m)}\|_{C(F)} \leq \sum_{j=n+1}^{\infty} \|q_j^{(m)}\|_{C(F)} \leq L^m M' \sum_{j=n+1}^{\infty} j^{N+m} \varepsilon_{j-1} \leq$$

$$\leq L^m M'' \sum_{j=n+1}^{\infty} (j-1)^{N+m} \varepsilon_{j-1} = d_1 \tau_n$$

(with $M'' := M' 2^{N+m}$, $d_1 := L^m M''$).

For the second estimate we start from (7) and apply Markov's first inequality on I m times to receive the estimate

$$\|q_n^{(m)}\|_{C(I)} \leq K^m M n^{2N+2m} \varepsilon_{n-1}.$$

With this the proof is the same as above. ∎

We may apply this theorem as well for $\varepsilon_n := \frac{c}{n^\gamma}$ $(c > 0, \gamma > 1)$ and $\varepsilon_n := cq^n$ $(c > 0,\ 0 < q < 1)$ using the estimates in Corollaries 1 and 2.

COROLLARY 3. *Let $S \subset \mathbb{R}^N$ be the unit ball, $(p_n) : \mathbb{R}^N \to \mathbb{R}$ a sequence of polynomials with* $\mathrm{gr}(p_n) = n$, $p \geq 1$ *and* $f \in L^p(S)$. *Assume that* $\exists\, c > 0$, $l \in \mathbb{N}$ *and* $0 < \alpha \leq 1$ *such that* $\|f - p_n\|_{L^p(S)} \leq \frac{c}{n^{l+\alpha}}$ *for all* $n \in \mathbb{N}$.

Then $f \in C^{l-N-1}(\mathrm{int}\, S)$ *and for any* $k = 1, 2, \ldots, l - N + 1$ *and for each closed set* $F \subset \mathrm{int}\, S$ $\exists\, a_k > 0$ *such that*

$$\left\|f^{(k)} - p_n^{(k)}\right\|_{C(F)} \leq \frac{a_k}{n^{l+\alpha-(N+k+1)}} \qquad (n \in \mathbb{N}).$$

For $k = 1, 2, \ldots, \left\lfloor \frac{l-1}{2} \right\rfloor - N$ *we have* $b_k > 0$ *such that*

$$\left\|f^{(k)} - p_n^{(k)}\right\|_{C(S)} \leq \frac{b_k}{n^{l+\alpha-(2N+2k+1)}} \qquad (n \in \mathbb{N}).$$

REMARK. The result on the differentiability of f is analogous to the classical inverse theorems of Bernstein ([2]) concluding the smoothness of f from the speed of uniform approximation by polynomials.

COROLLARY 4. *Let $S \subset \mathbb{R}^N$ be the unit ball, $(p_n) : \mathbb{R}^N \to \mathbb{R}$ a sequence of polynomials with* $\mathrm{gr}(p_n) = n$, $p \geq 1$ *and* $f \in L^p(S)$. *Assume that* $\exists\, c > 0$ *and* $0 < q < 1$ *such that* $\|f - p_n\|_{L^p(S)} \leq cq^n$ *for all* $n \in \mathbb{N}$. *Then* $f \in C^\infty(\mathrm{int}\, S)$ *and for any* $k \in \mathbb{N}$ $\exists\, c_k > 0$ *and* $0 < q_n < 1$ *such that*

$$\left\|f^{(k)} - p_n^{(k)}\right\|_{C(S)} \leq c_k q_k^n \qquad (n \in \mathbb{N}).$$

REMARK 1. Applying Corollary 2 and Bernstein's theorem on exponential approximation by polynomials, we obtain the analiticity of f.

REMARK 2. The assertion of this paper admit natural generalizations, this follows directly from the way they are proved.

1. Instead of S we may consider any domain that can be represented as the union of balls or cubes with diameter bounded from below, e.g. a convex domain having a smooth boundary with bounded curvature.

2. The assumption $\mathrm{gr}\, p_n = n$ can be omitted for $\sum(\mathrm{gr}\, p_n)^{2N}\varepsilon_n < \infty$ and $\sum(\mathrm{gr}\, p_n)^{N}\varepsilon_n < \infty$, respectively.

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