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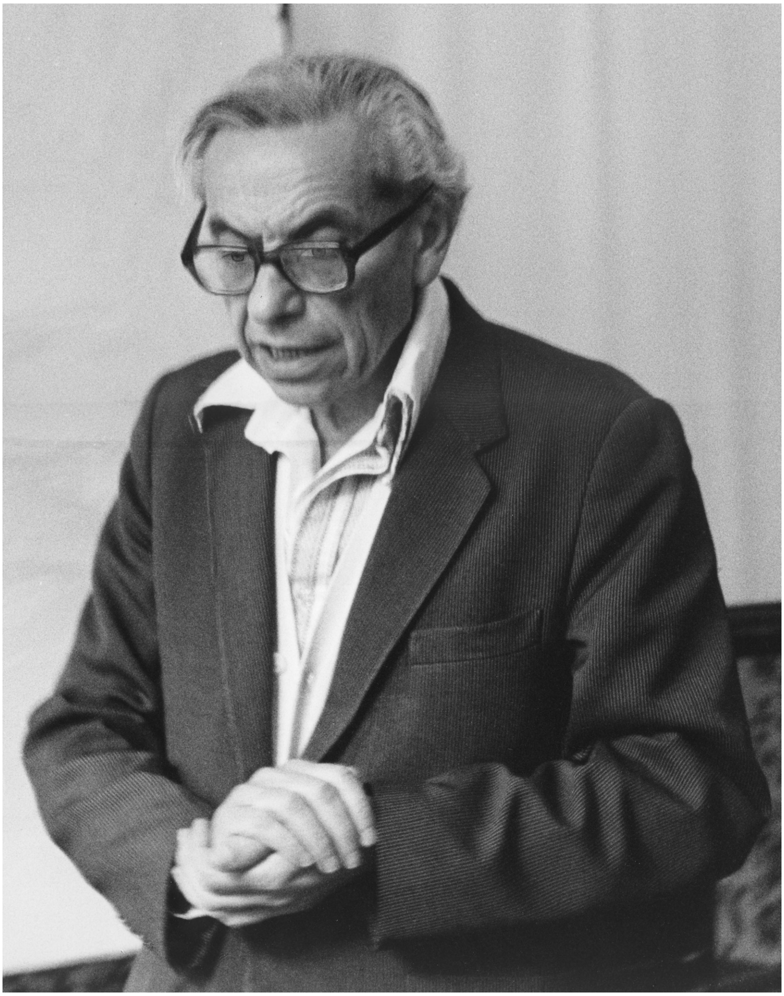

PÁL (PAUL) ERDŐS (1913–1996)

honorary doctor of the L. Eötvös University has been a master of all of us and of a great number of mathematicians in the whole world

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A SIMPLE CLASS OF CUBIC SYSTEMS WITHOUT CYCLES

By

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1. Introduction

A general technique to exclude the existence of closed orbits of second order dynamic systems is given by Bendixson criterion [1], [3]. This criterion was generalized by H. DULAC [1], [4] as follows:

Let

$$\dot{x} = P(x,y) \qquad \dot{y} = Q(x,y) \tag{1}$$

be an analytic dynamic system, G a simply connected subregion of the domain of definition of (1). If there exists a continuously differentiable function $B(x,y)$ defined in G such that the function

$$\frac{\partial}{\partial x}\left[B(x,y)P(x,y)\right] + \frac{\partial}{\partial y}\left[B(x,y)Q(x,y)\right]$$

does not change sign in G, then there are no closed orbits in G.

The procedure of construction of Dulac Functions for (1) concerns a solution to the partial differential equation

$$\text{div}\left[B(x,y)(P(x,y), Q(x,y))\right] = \phi(x,y)$$

where $\phi(x,y)$ is some suitable function positive or negative definite in G.

A different approach to obtain the function $B(x,y)$ is based on the search for integrating factors as we show in the following.

THEOREM 1. *Let* (1) *be an analytical system on the plane, and let G be a simply connected subregion of the domain of definition of the system. If there exists an analytic system on the plane.*

$$\dot{x} = P_0(x,y) \qquad \dot{y} = Q_0(x,y) \tag{2}$$

such that the system

$$\dot{x} = P(x,y) - P_0(x,y) \qquad \dot{y} = Q(x,y) - Q_0(x,y) \tag{3}$$

has and integrating factor $\mu(x,y) \in C^1(G)$ *and* $\operatorname{div}(\mu(x,y)(P_0(x,y), Q_0(x,y)))$ *does not change sign in G, then system* (1) *does not admit any closed orbit lying entirely in G.*

PROOF. If $\mu(x,y)$ is an integrating factor for (3), then

$$\operatorname{div}\,(\mu(x,y)(P(x,y) - P_0(x,y), Q(x,y) - Q_0(x,y))) \equiv 0$$

and this implies that

$$\operatorname{div}\mu(x,y)(P(x,y), Q(x,y)))$$

does not change sign in G. Then $B(x,y)=\mu(x,y)$ is a Dulac Function for (1) in G. ∎

2. Main result

Using the above procedure we have the following theorem.

THEOREM 2. *Let us consider the cubic system.*

$$\begin{aligned} \dot{x} &= x(a_{30}x^2 + a_{21}xy + a_{12}y^2 + a_{20}x + a_{11}y + a_{10}) \\ \dot{y} &= y(b_{21}x^2 + b_{12}xy + b_{03}y^2 + b_{11}x + b_{02}y + b_{01}) \end{aligned} \tag{4}$$

where $a_{ij}, b_{ij} \in \mathbb{R}$.

If $\delta = a_{20}b_{02} - a_{11}b_{11} \neq 0$.

$$(a_{30}(k+2) + hb_{21})(ka_{10} + hb_{01}) > 0$$

and

$$((k+1)a_{21} + (h+1)b_{12})^2 - 4(a_{30}(k+2) + hb_{21})(ka_{12} + (h+2)b_{03}) < 0$$

where

$$k = \frac{b_{02}(b_{11} - a_{20})}{\delta}, \qquad h = \frac{a_{20}(a_{11} - b_{02})}{\delta}$$

then system (4) *has no closed orbit in the whole plane.*

PROOF. Since the coordinate axes are invariant they cannot be intersected by any closed orbit. Therefore, every closed orbit must lie in some quadrant.

As $\delta \neq 0$, we can choose

$$\begin{aligned} P_0(x,y) &= x(a_{30}x^2 + a_{21}xy + a_{12}y^2 + a_{10}) \\ Q_0(x,y) &= y(b_{21}x^2 + b_{12}xy + b_{03}y^2 + b_{01}) \end{aligned}$$

then $\mu(x,y)=x^{k-1}y^{h-1}$ is an integrating factor for the system

$$\dot{x} = x(a_{20}x + a_{11}y) \qquad \dot{y} = y(b_{11}x + b_{02}y).$$

Furthermore

$$\operatorname{div}\left(x^{k-1}h^{h-1}(P_0(x,y), Q_0(x,y))\right) = x^{k-1}y^{h-1}\Big(x^2(a_{30}(k+2) + hb_{21})+$$
$$+xy\,((k+1)a_{21} + (h+1)b_{12}) + y^2\,(ka_{12} + (h+2)b_{03}) + ka_{10} + hb_{01}\Big)$$

does not change sign in any quadrant provided that

$$(a_{30}(k+2) + hb_{21})(ka_{10} + hb_{01}) > 0$$

and

$$((k+1)a_{21} + (h+1)b_{12})^2 - 4(a_{30}(k+2) + hb_{21})(a_{12} + (h+2)b_{03}) < 0$$

and this proves our theorem. ∎

REMARK 1. By the technique given in Theorem 1, we can found the same Dulac Function obtained by Bautin [2] for the quadratic system.

$$\dot{x} = x(a_{20} + a_{11}y + a_{10}) \qquad \dot{y} = y(b_{11}x + b_{02}y + b_{01}).$$

In fact, a periodic solution must contain at least one singular point in its interior, which then must be the intersection point of the two lines $a_{20}x + a_{11}y + a_{10} = 0$ and $b_{11}x + b_{02}y + b_{01} = 0$.

Given $\delta = a_{20}b_{02} - a_{11}b_{11} \neq 0$, we can choose $P_0(x,y) = a_{10}x$ and $Q_0(x,y) = b_{01}y$. Then $\mu(x,y) = x^{k-1}y^{h-1}$ is an integrating factor for the system

$$\dot{x} = x(a_{20}x + a_{11}y) \qquad \dot{y} = y(b_{11}x + b_{02}y)$$

where

$$k = \frac{b_{02}(b_{11} - a_{20})}{\delta}, \qquad h = \frac{a_{20}(a_{11} - b_{02})}{\delta}.$$

Furthermore

$$\operatorname{div}\left(x^{k-1}y^{h-1}(a_{10}x, b_{01}y)\right) = x^{k-1}y^{h-1}(a_{10}k + b_{01}h)$$

does not change sign in any quadrant, provided that $a_{10}k + b_{01}h \neq 0$, and $B(x,y)=\mu(x,y)$ is the Dulac Function in the whole plane.

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ABOUT MONOMIALITY OF π-CHARACTERS OF CERTAIN π-SOLVABLE GROUPS

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1. Introduction

The main result of this paper is the following:

THEOREM. *Let G be a π-solvable group, $\sigma = \pi \cup \{2\}$, $2 \notin \pi$. Let us suppose that for every π-element $x \in G^{\#}$, its centralizer $C_G(x)$ satisfies the following conditions:*

a) *it is solvable,*

b) *the Sylow 2-subgroup of its Hall σ-subgroup H is normal in H,*

c) *for every $p \in \pi$ and $q \in \pi(G)$ such that $o(p) \bmod q \equiv 0$ (2), $(p,q) \not\leq C_G(x)$.*

Then the irreducible π-characters of every subgroup of G are monomial.

As we shall see from Lemma 4, condition c) in the Theorem can be replaced by the following condition: c)' for every $p \in \pi$ having a pair $q \in \pi(G)$ such that $o(p) \bmod q \equiv 0$ (2), the p-chief factors of every subgroup of $C_G(x)$ are odd dimensional vector spaces over $GF(p)$.

The reason why we formulated our Theorem with condition c) is that it is easier to proof, that it is inherited by homomorphic images. One can prove that the two conditions are in fact equivalent, but as we shall need only that c)' implies c), we shall prove only this in Lemma 4.

Chief sections are often considered when proving monomiality. For example in [8] PRICE proved that every irreducible character of every subgroup of a solvable group is monomial if and only if the group does not have any ramified chief sections. This condition is satisfied for example if the solvable group does not have any chief section of square order, which means that for every prime divisor p of the order of the group, every p-chief factor of every subgroup is an odd dimensional vector space over $GF(p)$.

Similarly as in Theorem 6.22 [6] on can prove easily:

PROPOSITION 1. *Let G be a π-solvable group, $\pi \subseteq \pi(G)$. If for every $p \in \pi$ every p-chief factor of every subgroup of G is an odd dimensional vector space over $GF(p)$, then every irreducible π-character of every subgroup of G is monomial.*

For $|G|$ odd, our Theorem gives that the conditions of Proposition 1 can be weakened. We need restrictions only for those p-chief sections where $p \in \pi$ and $o(p) \bmod q \equiv 0\ (2)$ for some $q \in \pi(G)$. If π consist of "good" primes then monomiality of irreducible π-characters can be derived without any assumption on chief sections, namely the following is true.

PROPOSITION 2. *Let G be a solvable group of odd order, $\pi \subseteq \pi(G)$. Let us suppose that for every $p \in \pi$ and $q \in \pi(G)$, $o(p) \bmod q \equiv 1(2)$. Then every irreducible π-character of every subgroup of G is monomial.*

Using results on minimal nonmonomial groups of Price [8] and van der Waall [9], one can prove the following propositions:

PROPOSITION 3. *Let G be a solvable group of odd order. Let us suppose that for every $p \in \pi(G)$ such that $o(p) \bmod q \equiv 0\ (2)$ for some $q \in \pi(G)$, then the p-chief factors of every subgroup of G are odd dimensional vector spaces over $GF(p)$. Then every irreducible character of every subgroup of G is monomial.*

PROPOSITION 4. *Let G be a solvable group of odd order such that $(p,q) \not\leq G$ for such primes p, $q \in \pi(G)$, for which $o(p) \bmod q \equiv 0\ (2)$. Then every irreducible character of every subgroup of G is monomial.*

Our Theorem also can be considered as a generalization of Proposition 3 and Proposition 4.

Throughout the paper all groups are finite. (p,q)-group means a minimal non-p-nilpotent group G, with $\pi(G) = \{p,q\}$ and $G' \in \mathrm{Syl}_p(G)$. $(p,q) \not\leq G$ means that G does not contain such a subgroup. For further results on (p,q)-groups the reader is referred to [2].

2. The result

First we shall prove that the conditions of the Theorem are inherited to homomorphic images.

LEMMA 1. *Let G be a π-solvable group such that for every π-element $x \in G^{\#}$, its centralizer, $C_G(x)$ is solvable. This property is inherited to homomorphic images.*

PROOF. As in the proof of Theorem B in [2], we can take G as a minimal counterexample with $H \blacktriangleleft G$ a minimal normal subgroup, $\overline{x} \in Z(G/H)$, $|\overline{x}| = p \in \pi$ a prime, $|x| = p^k$ for some k, $H \leq \Phi(G)$. H and also the group $B = H < x >$ can be supposed to be an elementary abelian p-group. As G is π-solvable, there exists a Hall π'-subgroup L in G. By the theorem of Maschke, $B = H \oplus Y$, where Y is an L-invariant complement to H in B. Then $L \leq C_G(Y)$, so it is solvable, but then G is also solvable and so is $C_{\overline{G}}(\overline{x}) = G/H$.

LEMMA 2. *Let G be a π-solvable group. $\sigma = \pi \cup \{2\}$, $2 \notin \pi$. Then property b) and c) are inherited by homomorphic images.*

PROOF. By Lemma 2 in [2], property b) is equivalent to the fact that $(u,2) \nleq C_G(x)$ for every prime $u \in \pi$ and every π-element $x \in G^{\#}$. So the statement follows from Theorem B in [2]. Property c) is also inherited by every homomorphic image, by the same theorem.

To the proof of the Theorem we shall also need the following result on symplectic modules.

LEMMA 3. *Let L be a solvable group of odd order acting faithfully and irreducibly on a finite dimensional symplectic space W over $GF(p)$ with G-invariant symplectic form $(. , .)$. If every irreducible subspace of every p'-element of L is odd dimensional, then W is isotropic.*

PROOF. We use induction on $\dim W + |L|$. Let L and W be a counterexample, where $\dim W + |L|$ is minimal. Let M be a minimal normal subgroup of L. We state that then W_M is a homogeneous M-module. Otherwise

$$W_M = \oplus W_i,$$

where the W_i, $i = 1, \dots t$ are homogeneous components, which are irreducible $\mathrm{Stab}(W_i)$-modules by Clifford theorem. By induction, the W_i-s are isotropic $\mathrm{Stab}\,(W_i)/\mathrm{Ker}\,(\mathrm{Stab}\,(W_i))$-modules. As W is not isotropic, by its irreducibility it is nonsingular. Now we can follow a similar argument to that of Theorem 1.2 in [1]. Set $W_i^{\perp} = \{w \in W \mid (w, w_i) = 0 \quad \text{for all} \quad w_i \in W_i\}$. For

$w \in W$ we consider the map $f_w \in W_i^* := \mathrm{Hom}_{GF(p)}(W_i, GF(p))$, defined by $f_w(w_i) = (w, w_i)$, $w_i \in W_i$. Then $w \to f_w$ $w \in W$, induces a G-isomorphism between $W/W_i^\perp$ and the dual space W_i^*. Since W_M is completely reducible, there exists an M-module U_i such that $W_M = W_i \oplus U_i$. Thus $U_i \simeq W_i^*$ is homogeneous and consequently $U_i = W_{\pi(i)}$ for a permutation $\pi \in S_t$. As $W_i \subseteq W_i^\perp$, $\pi(i) \neq i$ for all $i = 1, \ldots, t$. This gives a partition of the set of indices into pairs, which is a contradiction as $t = |L : \mathrm{Stab}\,(W_i)|$ is odd. So W_M is homogeneous.

Let $W_M = \oplus V_i$, where the V_i $i = 1, \ldots s$ are isomorphic irreducible modules over M. Now each V_i is faithful, otherwise the element acting trivially on V_i, would act trivially on the whole W. As M is elementary abelian, and also it has a faithful irreducible module over $GF(p)$, M is a cyclic group of prime order $q \neq p$. M cannot act irreducibly over W, as then it should be odd dimensional, but as it is nonsingular, this cannot be the case. So $\dim V_i < \dim W$. Let $\langle a \rangle = M$, then the minimal polynomial of a on W has degree less than or equal to $\dim V_i$, and it has the same minimal polynomial on every irreducible component. Let us take the cyclic subspace of a vector $u \in W : \langle u, au, a^2u, \ldots \rangle$. As it is an M-subspace of W of dimension less than or equal to $\dim V_i$, it is irreducible and it is isomorphic to V_i. By the inductive hypothesis it is isotropic. Let u, $v \in W$, then by the previous argument $0 = (u+v, (u+v)a) = (u, ua) + (v, va) + (u, va) + (v, ua) = (u, va) - (ua, v)$. So $(u, va) = (u, va^{-1})$ and we have that for every $w \in W$, $(u, wa^2) = (u, w)$. In particular, $\langle u \rangle^\perp$ is a^2-invariant, and as $|L|$ is odd, it is also a-invariant. As W is nonsingular, $\dim \langle u \rangle^\perp = \dim W - 1$, so there is an i such that $V_i \nleq \langle u \rangle^\perp$. Then $V_i \cap \langle u \rangle^\perp = 0$, and so $\dim V_i = 1$. Then the element a acts on W in such a way, that vectors in each component are multiplied by the same number λ. As $(ua, va) = \lambda^2(u, v)$, we have that $\lambda^2 = 1$, so either $\lambda = 1$ or $\lambda = -1$. As W is faithful, $\lambda \neq 1$ and as $|L|$ is odd, a^2 cannot act identically. So we have a contradiction.

REMARK. Instead of the second part of the above proof, we could have also referred to Lemma 5.2 in [7], which says that if a cyclic group M acts on a nonsingular symplectic space W over $GF(p)$ faithfully and completely reducibly, and also W is a homogeneous M-module, then if every irreducible subspace of M is isotropic, then $|M| \leq 2$.

PROOF OF THE THEOREM. Let G be a counterexample of minimal order. As the conditions are inherited to every subgroup of G, by induction every irreducible π-character of every proper subgroup of G is monomial. Let $\chi \in \mathrm{Irr}\,(G)$ be a nonmonomial π-character. As by Lemma 1 and Lemma 2 the conditions are inherited to homomorphic images, so we may assume that $\mathrm{Ker}_\chi = 1$. Also we may assume that χ is primitive.

We shall present the proof in several steps:

1. $O_{\pi'}(G) \le Z(G)$:

As the irreducible constituents of $\chi\big|_{O_{\pi'}(G)}$ are linear, we get that $O_{\pi'}(G)' \le \mathrm{Ker}_\chi = 1$. As χ is primitive, $O_{\pi'}(G) \le Z(G)$.

2. G is solvable, moreover, there is a prime $p \in \pi$ and a p-element $x \in G^{\#}$, such that $C_G(x) = G$:

Take a minimal normal subgroup in $G/Z(G)$, let N be its inverse image in G. We state that $N/Z(G)$ is a π-group. Otherwise let R be the π-part of $Z(G)$ and let $H \in \mathrm{Hall}_{\pi'}(N)$. Then $N = HR$ and so $H \blacktriangleleft N$. As H is also characteristic in N, $H \blacktriangleleft G$, which is a contradiction as $H \not\le O_{\pi'}(G)$. So $N/Z(G)$ is a p-group for some $p \in \pi$, and so N is nilpotent. Let $P \in \mathrm{Syl}_p(N)$, then $P \blacktriangleleft G$ and $1 \ne Z(P) \le Z(G)$. So if $1 \ne x \in Z(P)$ then $C_G(x) = G$, and by condition a), G is solvable.

3. Let N and P as in the previous paragraph. Let $\overline{P} = P/Z(P)$ and $\overline{G} = G/F(G)$. Then $\overline{P}$ is a nonsingular, irreducible, symplectic $\overline{G}$-module, $\overline{G}$ is of odd order and every irreducible subspace of every p'-element of $\overline{G}/C_{\overline{G}}(\overline{P})$ is odd dimensional over $GF(p)$:

By the previous step, and by the indirect assumption, $C_G(F(G)) \le F(G) \ne = G$. Let $F(G) = F_\pi \times O_{\pi'}(G)$, then $F_\pi \not< Z(G)$. But $F_\pi \blacktriangleleft R \in \mathrm{Hall}_\sigma(G)$ and by condition b) $S \in \mathrm{Syl}_2(R)$ is normal in R. It follows that $S \le C_G(F_\pi) \le F(G)$ and so $G/F(G)$ is of odd order. As $P' \ne 1$, by Theorem 32.6 in [3] $P = Z(P)E$, where E is an extraspecial p-group. By Theorem 34.6 in [3] $\overline{P}$ is a nonsingular symplectic G-module. As $\overline{P} \simeq N/Z(G)$, $\overline{P}$ is also an irreducible module. Let z be an arbitrary p'-element in $\overline{G}/C_{\overline{G}}(\overline{P})$. Let $z = \prod z_i$, where the z_i are pairwise commuting and $o(z_i) = q_i^{m_i}$, for some primes q_i. By 3.10 Satz in p. 165 of [4], the dimension of an irreducible z-module in $\overline{P}$ is $o(p)$ mod $\prod q_i^{m_i}$. This number is the least common multiple of the numbers $o(p)$ mod $q_i^{m_i}$. Using condition c) and the fact that $G = C_G(x)$ for some π-element $x \ne 1$, we get that $(p,q) \not\lesssim G$, where $o(p)$ mod q is even. By Theorem A in [2] this property is inherited to $\overline{G}/C_{\overline{G}}(\overline{P})$. But $\overline{P}\langle z_i \rangle > (p, q_i)$, so $o(p)$ mod $q_i \equiv 1(2)$, and hence $o(p)$ mod $q_i^{m_i} \equiv 1(2)$. So we are done.

4. End of the proof:

Apply Lemma 3 to $L = \overline{G}/C_{\overline{G}}(\overline{P})$ and $W = \overline{P}$. We get that $\overline{P}$ is isotropic, which is a contradiction.

Corollaries

LEMMA 4. *Let G be a π-solvable group. If for any $p \in \pi$ such that* $o(p)$ mod $q \equiv 0$ (2) *for some $q \in \pi(G)$, the p-chief factors of every subgroup of G are odd dimensional vector spaces over $GF(p)$, then $(p,q) \not\leq G$ for such primes p and q.*

PROOF. If by contradiction there would be such a (p,q)-subgroup U in G, then $U = PQ$, where $P \blacktriangleleft U$, $Q = \langle x \rangle$ and $x^q \in Z(U)$, P is a special p-group and, $\langle x \rangle / \langle x^q \rangle$ acts faithfully and irreducibly on P/P'. So by 3.10 Satz of p. 165 in [4] $\dim_{GF(p)} P/P' = o(p)$ mod q, which is even by assumption. This contradicts to the assumption on dimensions of p-chief factors for subgroups of G.

COROLLARY 1. *Let G be a π-solvable group, $\sigma = \pi \cup \{2\}$, $2 \notin \pi$. Let us suppose that for every π-element $x \in G^{\#}$, $C_G(x)$ satisfies the following conditions*

a) it is solvable

b) the Sylow 2-subgroup of its Hall σ-subgroup H is normal in H

c') for every $p \in \pi$ having a pair $q \in \pi(G)$ such that $o(p)$ mod q *is even, the p-chief factors of every subgroup of $C_G(x)$ are odd dimensional vector spaces over $GF(p)$.*

Then the irreducible π-characters of every subgroup of G are monomial.

PROOF. By Lemma 4, condition c') implies condition c) in the Theorem.

COROLLARY 2. *In Proposition 2 the conditions of Corollary 1 are satisfied, so we get a proof for that, too.*

COROLLARY 3. *In Proposition 3 the conditions of Corollary 1 are satisfied, with $\pi = \pi(G)$, so we get a proof of that, too.*

COROLLARY 4. *In Proposition 4 the conditions of the Theorem are satisfied, with $\pi = \pi(G)$, so we get a proof of that, too.*

COROLLARY 5. *Let G be a π-solvable group of odd order. Let us suppose that for every π-element $x \in G^{\#}$, $C_G(x)$ is solvable and for every $p \in \pi$ it is p-supersolvable. Then every irreducible π-character of G is monomial.*

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ON NORMAL ANTISIMPLE RADICAL OF Γ_N-RINGS

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0. Introduction

The notion of antisimple radical of a ring was first defined by ANDRUNAKIEVICH [2]. Recently, this notion has been extended to Γ-rings in [5] and the relationships between the antisimple radicals of Γ-ring M and the corresponding radicals of $\Gamma_{n,m}$-ring $M_{m,n}$ and the right operator ring R of Γ-ring M are established. However, F. A. SZÁSZ [12] proposed the following problem: Problem 55. Let $\mathcal{K}$ be the class of all subdirectly irreducible rings, whose Jacobson radical is zero. Examine the upper radical determined by the class $\mathcal{K}$, we called this radical the antisimple primitive radical. The purpose of this paper is to extend the concepts of antisimple radical and antisimple primitive radical to the theory of normal antisimple radical in Γ-rings. Our results will encompass those of Booth's and it in particular answers positively the Szász's problem. Let $(R, {}_RV_S, {}_SW_R, S)$ be a Morita context (the reader is referred to [1], [11] for the definition), we call $(R, {}_RV_S, {}_SW_R, S)$ S-faithful if $S \neq 0$ and $VsW = 0$ implies $s = 0$ for $s \in S$. Then a normal class $\mathcal{P}$ is defined to be a class of prime rings such that if $(R, {}_RV_S, {}_SW_R, S)$ is an S-faithful context with $R \in \mathcal{P}$ then necessarily $S \in \mathcal{P}$. These normal classes enjoy many pleasant properties and the reader is referred to [1], [11] for details. We note that many well known classes of rings are normal classes: the class of prime rings, the class of primitive rings, the class of prime Levitzki semisimple rings, the class of prime subdirect irreducible rings, the class of primitive rings with nonzero socle, the class of weak primitive rings, the class of k-primitive rings, the class of prime Johnson rings, and the class of nonsingular prime rings. For a normal class of rings ϱ, we say M is a $\varrho - \Gamma$-ring or $\varrho(\Gamma)$-ring if the right operator ring R of M belongs to ϱ and $M\Gamma x = 0$ implies $x = 0$. We will use $\varrho(\Gamma)$ to denote the class of all $\varrho - \Gamma$-rings. We give examples to show that normal antisimple radical is different from antisimpled radical and antisimple primitive radical. For any Γ_N-ring M, there are five related rings: the Γ-ring M, the right (left) operator ring $R(L)$ of Γ-ring M, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ and the ring M_2. In this paper, the relationships between the normal antisimple radicals of Γ-ring M and the corresponding radicals of $\Gamma_{n,m}$-ring $M_{m,n}$, the right operator ring R of Γ-ring M, M-ring Γ and the ring M_2 are established.

Throughout this paper, ϱ will always denote a normal class of ring. "Γ-ring" means a Γ-ring in the sense of BARNES [3], the notion "$I \trianglelefteq M$" will means "I is an ideal of M", M will denote an arbitrary Γ-ring, and $R(L)$ its right(left) operator rings. For the definition of weak Γ_N-rings, Γ_N-rings and the opertator rings of Γ-rings, we refer [4] and [10].

Let (M,Γ) be a weak Γ_N-ring and $A \subseteq M$, $P \subseteq R$, $Q \subseteq L$ and $\Phi \subseteq \Gamma$, then we define

$P^* = \{x \in M : [\beta, x] \in P \text{ for all } \beta \in \Gamma\}$, $Q^+ = \{x \in M : [x, \mu] \in Q \text{ for all } \mu \in \Gamma\}$,

$A^{*'} = \{r \in R : Mr \subseteq A\}$, $A^{+'} = \{y \in L : yM \subseteq A\}$,

$\Gamma(A) = \{\mu \in \Gamma : M\mu M \subseteq A\}$ and $M(\Phi) = \{x \in M : \Gamma x \Gamma \subseteq \Psi\}$.

We have that if $A \trianglelefteq M$, $P \trianglelefteq R$, $Q \trianglelefteq L$ and $\Psi \trianglelefteq \Gamma$, then $A^{*'} \trianglelefteq R$, $A^{+'} \trianglelefteq L$, $P^* \trianglelefteq M$, $Q^+ \trianglelefteq M$, $\Gamma(A) \trianglelefteq \Gamma$ and $M(\Psi) \trianglelefteq M$. Moreover, $A \subseteq M(\Gamma(A))$ and $\Psi \subseteq \Gamma(M(\Psi))$.

Let (M,Γ) be Γ_N-ring. Let R and L denote respectively, the right and left opertator rings of Γ-ring M. The set

$$M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} = \{\begin{pmatrix} r & \gamma \\ m & l \end{pmatrix} \mid r \in R, \gamma \in \Gamma, m \in M, l \in L\}$$

is a ring with respect to the obvious operations of matrix multiplication and addition. For details, see [4,10]. Moreover, if $I \trianglelefteq M$, then it is easily verified that

$$I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix} \trianglelefteq M_2 \qquad \text{and} \qquad \begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \trianglelefteq M_2.$$

For further details of Γ-rings, matrix Γ-ring and their opertator rings, we refer to [8] and [9].

1. Normal antisimple radicals of Γ-rings

Following S. KYUNO, a Γ-ring M is said to be subdirectly irreducible (abbreviated as sdi) if the intersection of all nonzero ideals of M is not zero. The uniquely determined minimal ideal of a sdi Γ-ring M is called the heart of M and denoted by $H(M)$.

DEFINITION 1.1. We called Γ-ring M is a (right) $\varrho-\Gamma$-ring of $\varrho(\Gamma)$-ring, if the right opertator ring R of M belongs to ϱ and $M\Gamma x = 0$ implies $x = 0$. Left $\varrho-\Gamma$-ring or $\varrho(\Gamma)$-ring can be defined similarly. An Γ-ring M is said to be ϱ-normal subdirectly irreducible (abbreviated as $\varrho - NSDI$) if M is a $\varrho-\Gamma$-ring and sdi Γ-ring. An ideal I of M is called a ϱ-$NSDI$ ideal of M if M/I is a ϱ-$NSDI$ Γ-ring. It is easy to prove that $\varrho(\Gamma)$-rings are prime Γ-rings.

DEFINITION 1.2. Let $\mathcal{J}$ be the class of all $\varrho(\Gamma)$-rings which are sdi, $\mathcal{A}_\varrho$ be the upper radical determined by the class $\mathcal{J}$ and called it ϱ-normal antisimple radical of Γ-rings. Γ-ring M is called ϱ-normal antisimple if $\mathcal{A}_\varrho(M) = M$.

Radical classes of Γ-rings, special radical and the upper radical $\mu\mathcal{M}$ determined by a class $\mathcal{M}$ of Γ-rings are defined exactly as for rings. See for example, [6] and [7]. For any normal class ϱ of rings, by [7], Theorem 2.6, there exists a unique ϱ-normal antisimple radical of Γ-rings.

LEMMA 1.3. *Let $\mathcal{R}$ be a hereditary radical of Γ-rings, M is a sdi Γ-ring with heart H. Then M is $\mathcal{R}$-semisimple if and only if H is $\mathcal{R}$-semisimple. Moreover, H is a simple Γ-ring or zero.*

LEMMA 1.4. *Let M be a semiprime Γ-ring and $A \trianglelefteq M$. If $\langle \Gamma, A\rangle$ denotes the right operator ring of the Γ-ring A, then $\langle \Gamma, A\rangle \cong [\Gamma, A]$ (see* [7], *Lemma* 2.3*).*

PROPOSITION 1.5. *Let $\mathcal{K}$ be the class of all sdi Γ-rings, whose Jacobson radical is* 0. *Then for any Γ-ring M, $M \in \mathcal{K}$ if and only if M is a primitive sdi Γ-ring.*

PROOF. It is clear that M is primitive and sdi imply $M \in \mathcal{K}$. Conversely, if $M \in \mathcal{K}$, we have $J(M) = 0$ and M is a sdi Γ-ring with heart H. By Lemma 1.3, H is J-semisimple and then $H\Gamma H \neq 0$ and H is simple Γ-ring. Hence M is a prime Γ-ring and H is a primitive and sdi Γ-ring. Since $0 \neq [\Gamma, H] \trianglelefteq R = [\Gamma, M]$ and $[\Gamma, H]$ is a primitive ring by Proposition 1.4, by [11], p. 11, Corollary 2, R is a primitive ring, thus M is a primitive and sdi Γ-ring.

Following HEYMAN and ROOS [7], p. 202, we called a class $\mathcal{K}$ of Γ-ring is special if $\mathcal{K}$ satisfies:

(a) each $M \in \mathcal{K}$ is a prime Γ-ring.

(b) for every $M \in \mathcal{K}$, if $I \trianglelefteq M$ then $I \in \mathcal{K}$.

(c) if $0 \neq I \trianglelefteq \circ M$ (essential ideal of M) and $I \in \mathcal{K}$. Then $M \in \mathcal{K}$.

The upper radical determined by a special class of Γ-rings is called a special radical of Γ-rings.

THEOREM 1.6. *The class $\mathcal{M}_\varrho$ of all ϱ-$NSDI$ Γ-rings is a special class of Γ-rings.*

PROOF. Clearly, $\mathcal{M}_\varrho$ consists of prime Γ-rings. Suppose that $M \in \mathcal{M}_\varrho$ with heart $H(M)$ and $A \trianglelefteq M$. If $A = 0$, then $A \in \mathcal{M}_\varrho$. Suppose $A \neq 0$, we will prove that $A \in \mathcal{M}_\varrho$ with heart $H(M)$. Suppose that $0 \neq I \trianglelefteq A$. Then $H(M) \subseteq I$, whence $H(M) = H(M)\Gamma H(M)\Gamma H(M) \subseteq I^*\Gamma I^*\Gamma I^* \subseteq I$, where $I^* = I + I\Gamma M + M\Gamma I + M\Gamma I\Gamma M$. Hence A is a sdi Γ-ring with heart $H(M)$. On the other hand, since $R = [\Gamma, M] \in \varrho$ and $0 \neq [\Gamma, A] \trianglelefteq R$, it follows that $[\Gamma, M] \in \varrho$ by [11], p. 11, Corollary 2. Thus $A \in \mathcal{M}_\varrho$.

Finally, suppose that M is a Γ-ring, $0 \neq I \trianglelefteq \circ M$ and $I \in \mathcal{M}_\varrho$ with heart $H(I)$. We first prove that M is a prime Γ-ring. Since, if P, $Q \trianglelefteq M$ such that $P\Gamma Q = 0$ then $(P \cap I)\Gamma(Q \cap I) = 0$, by primeness of I, $P \cap I = 0$ or $Q \cap I = 0$, i.e. $P = 0$ or $Q = 0$ by $I \trianglelefteq \circ M$. Hence M is a prime Γ-ring. Next, we prove that M is a sdi Γ-ring. If $0 \neq A \trianglelefteq M$, then $A \cap I \neq 0$. Hence $H(I) \subseteq A$ so M is sdi. Finally, since $[\Gamma I] \in \varrho$ and $0 \neq [\Gamma, A] \trianglelefteq R = [\Gamma, M]$ is prime, it follows that $R \in \varrho$ by [11], p. 11, Corollary 2. Thus $M \in \mathcal{M}_\varrho$ as required.

From [6], Proposition 2.7 and Theorem 2.8, we have

THEOREM 1.7. *For any Γ-ring M, $\mathcal{A}_\varrho(M) = \cap\{I \trianglelefteq M \mid M/I$ is a ϱ-$NSDI$ Γ-ring$\}$.*

THEOREM 1.8. *Let M be a Γ-ring and I an ideal of M, then $\mathcal{A}_\varrho(I) = I \cap \cap \mathcal{A}_\varrho(M)$.*

THEOREM 1.9. *Γ-ring M is a subdirect sum of ϱ-$NSDI$ Γ-rings if and only if $\mathcal{A}_\varrho(M) = 0$.*

PROOF. From Theorem 1.7 and [9], Lemma 2 it is clear.

The next result gives characterization of normal antisimple radical Γ-rings. Its proof is similar to that of the corresponding result for the case of antisimple rings or antisimple Γ-rings (see [12], Proposition 12.4 or [5], Proposition 2.8), and will be omitted.

PROPOSITION 1.10 *The following are equivalent for a* Γ*-ring* M*:*

(a) $\mathscr{A}_\varrho(M)=M$;

(b) *Every homomorphic images of* M *is a subdirect sum of sdi* Γ*-rings* $M_i : i \in I$ *such that* $\mathscr{A}_\varrho(H(M_i))=H(M_i)$ *for each* $i \in I$;

(c) M *does not contain any* $\varrho(\Gamma)$*-ideal* P *such that* M/P *has a minimal ideal;*

(d) *No ideal of* M *can be mapped homomorphically onto a nonzero simple* $\varrho(\Gamma)$*-ring.*

The following examples show that in general the normal antisimple radical is different from the antisimple primitive radical and antisimple radical.

EXAMPLE 1.11. Let R be the ring of all power series in non-commutating indeterminates x and y over a field F. Let I be the ideal of all power series with constant term 0. Then I is a radical ring in the sense of Jacobson and $x \notin (x-yx^2y)_I$ for the ideal, generated by $x-yx^2y$ in I. By Zorn's lemma, there exists an ideal A of I maximal with respect to the condition $A \supseteq (x-yx^2y)_I$ and $x \notin A$. Write $S=I/A$, $\overline{x}=x+A$ and $\overline{y}=y+A$. Let T be the ideal generated by $\overline{x}$ in S. By [12], example 32.7, we have T is a simple radical ring in the sense of Jacobson and $T^2=T$. Hence T is a prime sdi ring but not primitive sdi ring. In fact, T is an ansimple primitive ring.

EXAMPLE 1.12. Let K be a field of characteristic zero, and α an automorphism of infinite order of K. Let R be the set of all polynomials $a_0+za_1+\ldots+z^nz_n$ in an indeterminate z over K with coefficients a_i from the field K. Let equality and addition of these polynomials be defined as usual. Let the multiplication be given by $kz = zk^\alpha$ for every $k \in K$. Let $T = xR$, it is an ideal of R. Then, by [12], example 32.1, T is a right primitive ring and not a simple ring. Furthermore, T is a radical ring for the upper radical determined by the class of all right primitive simple rings. We can prove that T has no ideal which can be homomorphism onto simple rings. By Proposition 1.10(d), T is an antisimple primitive ring.

2. Normal antisimple radical of operator rings

In this section, the relationships between normal antisimple radical of Γ-ring M and it right operator rings are established. Analogous results for the left operator ring can be proved similarly.

PROPOSITION 2.1. *Let M be a Γ-ring, and let R be the right operator ring of M. Then M is a ϱ-$NSDI$ Γ-ring if and only if R is a ϱ-$NSDI$ ring and $M\Gamma x=0$ implies $x=0$. Furthermore, $H(R)=[\Gamma,H(M)]$ and $H(M)=MH(R)$.*

PROOF. Suppose that M is a ϱ-$NSDI$ Γ-ring. Then $R\in\varrho$. For every nonzero ideal I of R, $MI\supseteq H(M)$ and $[\Gamma,H(M)]\subseteq\Gamma,H(M)I]\subseteq I$. By the primeness of M, we have that $M\Gamma x=0$ implies $x=0$ and hence $[\Gamma,H(M)]\neq$ $=0$. Thus R is a ϱ-$NSDI$ ring.

Conversely, suppose now that R is a ϱ-$NSDI$ ring, and let P be a nonzero ideal of Γ-ring M. By the primeness of R, M is a prime Γ-ring and hence $M\Gamma x=0$ implies $x=0$. Hence $0\neq[\Gamma,P]\trianglelefteq R$ and $0\neq MH(R)\subseteq$ $\subseteq M[\Gamma,P]\subseteq P$. Thus M is a ϱ-$NSDI$ Γ-ring. By the above proof, it is clear that $H(R)=[\Gamma,H(M)]$ and $H(M)=MH(R)$.

The next lemmas help to establish the relationship between ϱ-$NSDI$ ideals of Γ-rings and that of the right opertator rings.

LEMMA 2.2. *If A is an ideal of the Γ-ring M, R and $[\Gamma,M/A]$ are the right operator rings of Γ-ring M and Γ-ring M/A, respectively, then we have $[\Gamma,M/A]\cong R/A^{*'}$ under the mapping*

$$\sum_i[\gamma_i,x_i+A]\to\sum_i[\gamma_i,x_i]+A^{*'}.$$

LEMMA 2.3. *Let M be a Γ-ring with right operator ring R. P an ideal of R and $[\Gamma,M/P^*]$ be the right operator rings of Γ-ring M/P^*. If M has right unity or P is a prime ideal of R, then we have $[\Gamma,M/P^*]\cong R/P$ under the mapping*

$$\sum_i[\gamma_i,x_i+P^8]\to\sum_i[\gamma_i,x_i]+P.$$

The proof of Lemma 2.2 and 2.3 may easily be verified by direct computation.

THEOREM 2.4. *Let M be a Γ-ring with right opertator rings R. Then the mapping $P\to P^*$ defines a one-to-one correspondence between the sets of ϱ-$NSDI$ ideals of R and that of Γ-ring M. Moreover, $(P^*)^{*'}=P$.*

PROOF. This follows immediately from Proposition 2.1 and Lemma 2.2.

THEOREM 2.5. *Let M be a Γ-ring with right operator ring R. Then $\mathscr{A}_\varrho(R)=[\mathscr{A}_\varrho(M)]^{*'}$.*

PROOF. By Theorem 1.6 and 2.4, we have that

$$\mathscr{A}_\varrho(R) = \cap\{I^{*'} \mid I \text{ is a } \varrho - NSDI \text{ ideal of } M\}$$
$$= (\cap\{I \mid I \text{ is a } \varrho - NSDI \text{ ideal of } M\})^{*'} = [\mathscr{A}_\varrho(M)]^{*'}$$

3. Normal antisimple radical of matrix Γ-rings

For the definition of matrix Γ-rings, we refer to [8]. We now prove the next theorem which indicate one way to construct new ϱ-$NSDI\,\Gamma$-rings from given ones.

THEOREM 3.1. *M is a ϱ-$NSDI\,\Gamma$-ring if and only if $M_{m,n}$ is a ϱ-$NSDI\,\Gamma_{n,m}$-ring. Furthermore, $H(M_{m,n})=(H(M))_{m,n}$.*

PROOF. Suppose that M is ϱ-$NSDI\,\Gamma$-ring. Then, by Proposition 2.1, $R=[\Gamma, M]\in\varrho$ and R is a sdi ring and satisfies $M\Gamma x=0$ implies $x=0$. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$, recall that $[\Gamma_{n,m}, M_{m,n}]\cong$ $\cong R_n$ (see [9], p. 376). By the fact ϱ-$NSDI$ is a Morita invariant property (see [11], p. 12), we have that $[\Gamma_{n,m}, M_{m,n}]$ is a ϱ-$NSDI$ ring. Also, if $M_{m,n}\Gamma_{n,m}(x_{i,j})=0$, $(x_{i,j})\in M_{m,n}$, then for all $m\in M$, $\gamma\in\Gamma$, we have that

$$0 = (me_{ik})(\gamma e_{kj})(x_{st}) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ m\gamma x_{j1} & \dots & m\gamma x_{jn} \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix} (i).$$

Therefore, $M\Gamma x_{ij}=0$ for all $1\le i\le m$, $1\le j\le n$ and consequently $x_{ij}=0$ and $(x_{ij})=0$. Hence $M_{m,n}$ is a ϱ-$NSDI\,\Gamma_{n,m}$-ring by Proposition 2.1.

Conversely, suppose that $M_{m,n}$ is a ϱ-$NSDI\,\Gamma_{n,m}$-ring. Then $[\Gamma_{n,m}, M_{m,n}]\cong R_n$ is a ϱ-$NSDI$ ring. Hence R is a ϱ-$NSDI$ ring. Also, if $M\Gamma x=0$ $x\in M$, then $M_{m,n}\Gamma_{n,m}xe_{11}=0$ and consequently $xe_{11}=0$, i.e. $x=0$. Hence M is a ϱ-$NSDI\,\Gamma$-ring.

LEMMA 3.2. *If $I\trianglelefteq M$, then the matrix $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$ (see [8] Lemma 4).*

LEMMA 3.3. *Let M be an arbitrary Γ-ring. Then the prime ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a prime ideal of the Γ-ring M (see [9] Theorem 2).*

As a concesquence of Lemma 3.3 and Theorem 3.1, we have

THEOREM 3.4. *Let M be an arbitrary Γ-ring. Then the ϱ-$NSDI$ ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a ϱ-$NSDI$ ideal of the Γ-ring M.*

THEOREM 3.5. *If M is a Γ-ring, then $\mathcal{A}_\varrho(M_{m,n})=(\mathcal{A}_\varrho(M))_{m,n}$.*

PROOF. By Theorem 1.6 and 3.4, we have that

$$\mathcal{A}_\varrho(M_{m,n}) = \cap\{(I)_{m,n} \mid I \text{ is a } \varrho - NSDI \text{ ideal of } M\} =$$
$$= (\cap\{I \mid I \text{ is a } \varrho - NSDI \text{ ideal of } M\})_{m,n} = [\mathcal{A}_\varrho(M)]_{m,n}.$$

4. Normal antisimple radicals of M-ring Γ and the ring M_2

In this section, let (M,Γ) be Γ_N-ring. Let R and L denote respectively, the right and left operator rings of Γ-ring M. We shall establish the relationships between ϱ-$NSDI$ ideals and the normal antisimple radicals of Γ-ring M, M-ring Γ and the ring M_2.

The proof of the following Lemma may easily be verified by direct computation.

LEMMA 4.1 *Let (M,Γ) be a Γ_N-ring. Then the left operator ring L' of the M-ring Γ is isomorphic to $[\Gamma,M]/K$, where $K=\{x\in[\Gamma,M] \mid x\Gamma=0\}$.*

THEOREM 4.2. *Let (M,Γ) be a Γ_N-ring. If $\Gamma x\Gamma=0$ implies $x=0$ or Γ-ring M has left and right unities. Then the Γ-ring M is ϱ-$NSDI$ if and only if the M-ring Γ is ϱ-$NSDI$.*

PROOF. Suppose that Γ-ring M is ϱ-$NSDI$. By the primeness of M and Lemma 4.1, the left operator ring L' of the M-ring Γ is isomorphic to $R=[\Gamma,M]\in\varrho$ and hence $L'\in\varrho$, if $\gamma M\Gamma=0$, $\gamma\in\Gamma$, then $(M\gamma M)\Gamma M=0$ and hence $M\gamma M=0$ whence $\gamma=0$. By Proposition 2.1, Γ is a (left) $\varrho(M)$-ring. Let Φ be any nonzero ideal of M-ring Γ, then $0\neq M\Phi M\trianglelefteq M$ and hence $M\Phi M\supseteq H(M)$. Therefore, $\Phi\supseteq\Gamma M\Phi M\Gamma\supseteq\Gamma H(M)\Gamma\neq 0$. This proves that M-ring Γ is ϱ-$NSDI$. The proof of the converse is similar. This completes the proof.

The following lemma will be useful to characterize to ϱ-$NSDI$ property of M_2 in the sequel.

LEMMA 4.3. *Let (M,Γ) be a Γ_N-ring and A be an ideal of M_2. Then $\{x\in M\colon$ there exist $r\in R$, $\gamma\in\Gamma$, $s\in L$ such that $\begin{pmatrix} r & \gamma \\ x & s\end{pmatrix}\in A\}=0$ implies $A=0$.*

PROOF. Suppose that $\begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} \in A$. It is sufficient to show that $\begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} = 0$. By assumption, $x = 0$. Since $A \trianglelefteq M_2$, for any $m, n \in M$,

$$\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ mr & [m,\gamma] \end{pmatrix} \in A,$$

$$\begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} [\gamma, n] & 0 \\ sn & 0 \end{pmatrix} \in A$$

and

$$\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ m\gamma n & 0 \end{pmatrix} \in A.$$

By assumption, we have that $mr = 0$, $sn = 0$ and $m\gamma n = 0$, for every $m, n \in M$, whence $r = 0$, $s = 0$ and $\gamma = 0$. This completes the proof.

THEOREM 4.4 *Let* (M, Γ) *be a* Γ_N*-ring with right unity. Then the ring* $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$ *is a* ϱ*-NSDI ring if and only if* M *is a* ϱ*-NSDI* Γ*-ring. Furthermore,*

$$H(M_2) = \begin{pmatrix} [\Gamma, H(M)] & \Gamma(H(M)) \\ H(M) & [H(M), \Gamma] \end{pmatrix}.$$

PROOF. Suppose that M_2 is a ϱ-$NSDI$ ring and let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, by [11] p. 11, Corollary 2 and the fact $eM_2e \cong R$, we get $R \in \varrho$, whence M is a $\varrho(\Gamma)$-ring by Proposition 2.1 and [4], Theorem 3.5. On the other hand, for any $0 \neq I \trianglelefteq M$, we have that

$$M_2 \trianglerighteq \begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \supseteq H(M_2) \neq 0.$$

Thus $\cap\{I \mid 0 \neq I \trianglelefteq M\} \neq 0$, otherwise, we have

$$\cap \left\{ \begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \mid 0 \neq I \trianglelefteq M \right\} = 0,$$

a contradiction. From this, we have that M is a ϱ-$NSDI\,\Gamma$-ring.

Conversely, suppose that M is a ϱ-$NSDI\,\Gamma$-ring. Then $R \in \varrho$ and M is a prime Γ-ring. By [4], Theorem 3.5, M_2 is a prime ring. Again, by [11] p. 11, Corollary 2 and the fact $eM_2e \cong R$, we get that $M_2 \in \varrho$. For any nonzero ideal A of M_2, let

$$I = \{x \in M : \text{ there exist } r \in R,\ \gamma \in \Gamma, s \in L \text{ such that } \begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} \in A\}.$$

It is easily verified that $I \trianglelefteq M$. Then, by Lemma 4.3, $I \neq 0$ and $I \supseteq H(M)$. Thus $\cap\{A : 0 \neq A \trianglelefteq M_2\} \neq 0$ and M_2 is a ϱ-$NSDI$ ring.

Finally, the equality $H(M_2) = \begin{pmatrix} [\Gamma, H(M)] & \Gamma(H(M)) \\ H(M) & [H(M), \Gamma] \end{pmatrix}$ is easily proved by the above proof.

LEMMA 4.5. *Let* (M, Γ) *be a* Γ_N*-ring. If* A *an ideal of the* Γ*-ring* M, R *and* $[\Gamma/\Gamma(A), M/A]$ *are the right operator rings of* Γ*-ring* M *and* $\Gamma/\Gamma(A)$*-ring* M/A, *respectively then we have* $[\Gamma/\Gamma(A), M/A] \cong R/A^{*'}$ *under the mapping*

$$\sum_i [\gamma_i + \Gamma(A), x_i + A] \to \sum_i [\gamma_i, x_i] + A^{*'}.$$

COROLLARY 4.6. *The notations as Lemma* 4.5, *if* P *is a prime ideal of* R. *Then we have* $[\Gamma/\Gamma(P^*), M/P^*] \cong R/P$ *under the mapping*

$$\sum_i [\gamma_i + \Gamma(P^*), x_i + P^*] \to \sum_i [\gamma_i, x_i] + P.$$

The proof of Lemma 4.5 and Corollary 4.6 may easily be verified by direct computation.

NOTE. Analogous results for the left operator ring can be proved similarly.

If (M, Γ) is a Γ_N-ring. By Lemma 4.5, it is easily verified that an ideal P of M is a ϱ-$NSDI$ ideal if and only if M/P is a ϱ-$NSDI$ $\Gamma/\Gamma(M)$-ring.

THEOREM 4.7. *Let* (M, Γ) *be a weak* Γ_N*-ring. Then the mapping* $P \to \Gamma(P)$ *defines a one-to-one correspondence between the sets of* ϱ-$NSDI$ *ideals of the* Γ*-ring* M *and that of the* M*-ring* Γ.

PROOF. It is immediate from Lemmas 4.1, 4.5 and [4, Theorem 3.3] and the fact if $I \trianglelefteq M$, then $(\Gamma/\Gamma(I), M/I)$ is a Γ_N-ring.

As an immediate consequence of Theorem 4.7 we have:

COROLLARY 4.8. *Let* (M, Γ) *be a weak* Γ_N*-ring. Then* $\mathcal{A}_\varrho(\Gamma) = \Gamma(\mathcal{A}_\varrho(M))$.

COROLLARY 4.9. *Let* (M, Γ) *be a weak* Γ_N*-ring. Then* Γ*-ring* M *is* ϱ*-normal antisimple if and only if* M*-ring is* ϱ*-normal antisimple.*

THEOREM 4.10. *Let* (M, Γ) *be a* Γ_N*-ring has right unity, and let* R *and* L *denote, respectively, the right and left operator rings of* Γ*-ring* M. *Then a subset* P_2 *of* M_2 *is a* ϱ-$NSDI$ *ideal of* M_2 *if and only if*

$$P_2 = \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}, \quad \textit{where } P \textit{ is a } \varrho - NSDI \textit{ ideal of } M.$$

PROOF. Suppose firstly that P is a ϱ-$NSDI$ ideal of Γ-ring M. Then M/P is a ϱ-$NSDI$ $\Gamma/\Gamma(P)$-ring, whence

$$(M/P)_2 = \begin{pmatrix} [\Gamma/\Gamma(P), M/P] & \Gamma/\Gamma(P) \\ M/P & [M/P, \Gamma(P)] \end{pmatrix} \cong \begin{pmatrix} R/P^{*'} & \Gamma/\Gamma(P) \\ M/P & L/P^{+'} \end{pmatrix}$$

is a ϱ-$NSDI$ ring by Theorem 4.4. Hence, $M_2/\begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}$ is a ϱ-$NSDI$ ring, i.e. P_2 is a ϱ-$NSDI$ ideal of M_2.

Conversely, suppose that the subset P_2 is a ϱ-$NSDI$ ideal of M_2. Then, by [4], Theorem 3.6, $P_2 = \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}$, where P is a prime ideal of M. But $M_2/P_2 \cong \begin{pmatrix} R/P^{*'} & \Gamma/\Gamma(P) \\ M/P & L/P^{+'} \end{pmatrix}$ is a ϱ-$NSDI$ ring. Hence, by Theorem 4.4, P is a ϱ-$NSDI$ ideal of M. This concludes the proof.

The following result encompasses and generalizes the corresponding result of KYUNO [10].

COROLLARY 4.11. *Let (M,Γ) be a Γ_N-ring has right unity. Then we have*

$$\mathcal{A}_\varrho(M_2) = \begin{pmatrix} \mathcal{A}_\varrho(R) & \mathcal{A}_\varrho(\Gamma) \\ \mathcal{A}_\varrho(M) & \mathcal{A}_\varrho(L) \end{pmatrix}$$

PROOF. This follows immediately from Theorems 1.7, 2.5, 4.10 and Corollary 4.8.

COROLLARY 4.12. *Let (M,Γ) be a Γ_N-ring has right unity. Then M_2 is a ϱ-normal antisimple ring if and only if M is a ϱ-normal antisimple Γ-ring.*

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A MINIMAX THEOREM FOR TWO FUNCTIONS

By

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A famous two-function minimax theorem is Nikaido-Isoda one, connected with classical convexity. We recommend the interested reader investigation of the references. The aim of this paper is to prove Theorem 3 [12] in a special case, but without using Zorn's Lemma. The proof of Theorem 3 [12] is similar to proof of Theorem 2 [12], but it is much more complicated, [7].

Let $\varrho_i : \mathbb{R}^2 \Rightarrow \mathbb{R}$ ($\mathbb{R}$ denotes the real line, $i = 1, 2$) be any continuous functions such that

(1) $$x < y \quad \Rightarrow \quad x < \varrho_i(x,y) < y, \quad x < \varrho_i(y,x) < y$$

(2) $$x = y \quad \Rightarrow \quad x = y = \varrho_i(x,y)$$

THEOREM. *Let X, Y be any non-empty sets (without any topology), $f, g : X \times Y \to \mathbb{R}$ any functions defined on $X \times Y$ such that*

(3) $f \leq g$ *on* $X \times Y$, f *is bounded*

(4) $\forall\, y_1, y_2 \in Y \quad \exists\, y_3 \in Y : g(x, y_3) \leq \varrho_2(f(x,y_1), g(x,y_2)) \quad \forall\, x \in X,$

(5) $\forall\, x_1, x_2 \in X \quad \exists\, x_3 \in X : f(x_3, y) \geq \varrho_1(f(x_1,y), g(x_2,y)) \quad \forall\, y \in Y.$

Then for any finite set $H = \{x_1, x_2\} \subset X$ we have

(6) $$\inf_Y \max\{f(x_1,y), g(x_2,y)\} \leq \sup_X \inf_Y g(x,y).$$

REMARK. In [12] the H was any finite set and was stated that

$$\inf_Y \sup_H f(x,y) \leq \sup_X \inf_Y g(x,y).$$

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PROOF. Suppose indirectly that for some pair x_1, x_2

$$\inf_Y \max\{f(x_1,y), g(x_2,y)\} = \alpha > \beta = \sup_X \inf_Y g(x,y). \tag{7}$$

For our convenience denote the pair (x_1,x_2) by (x_1,y_1) and build inductively the sequence (x_n,y_n) as follows. Suppose that x_n and y_n have already been constructed and

$$\inf_Y \max\{f(x_n,y), g(y_n,y)\} \geq \alpha. \tag{8}$$

Let x^* be defined by

$$f(x^*,y) \geq \varrho_1(f(x_n,y), g(y_n,y)). \tag{9}$$

By (5) such x^* exists. Let γ be an arbitrary number such that

$$\alpha > \gamma > \beta. \tag{10}$$

Introduce the notation

$$N_\gamma f(x) := \{y \in Y : f(x,y) \leq \gamma\}. \tag{11}$$

In the first step we show that

$$N_\gamma g(x^*) \subset N_\gamma f(x_n) \quad \text{or} \quad N_\gamma g(x^*) \subset N_\gamma g(y_n). \tag{12}$$

Since $\gamma > \beta$ we have $N_\gamma g(x) \neq \emptyset$ for all $x \in X$, furthermore $g \geq f$ implies $N_\gamma f(x) \neq \emptyset$ for all $x \in X$. From $\alpha > \gamma$ it follows that

$$N_\gamma f(x_n) \cap N_\gamma g(y_n) = \emptyset. \tag{13}$$

From $g(x^*,y) \geq f(x^*,y) \geq \varrho_1(f(x_n,y), g(y_n,y))$ we see that

$$N_\gamma g(x^*) \subset N_\gamma f(x^*) \subset N_\gamma f(x_n) \cup N_\gamma g(y_n). \tag{14}$$

Suppose indirectly that (12) does not hold; then there exist z_1, z_2 such that

$$z_1 \in N_\gamma g(x^*) \cap N_\gamma f(x_n), \qquad z_2 \in N_\gamma g(x^*) \cap N_\gamma g(y_n). \tag{15}$$

From $m \geq 3$ we define z_m by induction as follows.

Denote i_m (resp. j_m) the largest index $< m$ satisfying $z_{i_m} \in N_\gamma f(x_n)$ (resp. $z_{j_m} \in N_\gamma g(y_n)$). Define z_m by (4), that is

$$g(x, z_m) \leq \varrho_2(f(x, z_{i_m}), g(x, z_{j_m})). \tag{16}$$

From $z_1, z_2 \in N_\gamma g(x^*)$ we see by induction (using the monotonity of ϱ_2) that $z_m \in N_\gamma g(x^*)$ for all m, i.e.,

$$z_m \in N_\gamma f(x_n) \quad \text{or} \quad z_m \in N_\gamma g(y_n). \tag{17}$$

Suppose first that there are infinitely many m with $z_m \in N_\gamma f(x_n)$; denote (z_{m_k}) the subsequence of these z_m. From $f(x_n, z_{m_{k+1}}) \leq \gamma < \alpha$ it follows that

$$\alpha \leq g(y_n, z_{m_{k+1}}) \leq \varrho_2(f(y_n, z_{m_k}), g(y_n, z_{j_{m_{k+1}}})). \tag{18}$$

Here $g(y_n,z_{jm_{k+1}})\leq\gamma$ implies that

$$f(y_n,z_{n_k}) \geq g(y_n,z_{m_{k+1}}) \geq f(y_n,z_{m_{k+1}}),$$

i.e., $f(y_n,z_{m_k}) \geq \alpha$ is monotone decreasing when $k \to \infty$. Denote $\lim_{k\to\infty} f(y_n,z_{m_k})=A\geq\alpha$. The inequality

$$f(y_n,z_{m_{k+1}})\leq g(y_n,z_{m_{k+1}})\leq\varrho_2\left(f(y_n,z_{m_k}),g(y_n,z_{jm_{k+1}})\right)\leq\varrho_2(f(y_n,z_{m_k}),\gamma)$$

gives for $k\to\infty$ that $A\leq\varrho_2(A,\gamma)$, which is contradiction since $A\geq\alpha>\gamma$ and ϱ_2 satisfies (1). This shows that there are no infinitely many $z_m\in N_\gamma f(x_n)$, hence for all $m\geq N$ we have $z_m\in N_\gamma g(y_n)$ and

$$g(x,z_{m+1}) \leq \varrho_2(f(x,z_{m_0}),g(x,z_m)) \qquad (m\geq N).$$

Now $g(y_n,z_m)\leq\gamma$ implies $f(x_n,z_m)\geq\alpha$, i.e.,

$$\alpha \leq f(x_n,z_{m+1}) \leq g(x_n,y_{m+1}) \leq \varrho_2(f(x_n,z_{m_0}),g(x_n,z_m)) \qquad (m\geq N).$$

Here $f(x_n,z_{m_0}) \leq \gamma$, hence $g(x_n,z_m)\geq\alpha$ is decreasing when $m\to\infty$ and a contradiction can be obtained just like some lines above. This proves that (12) holds indeed.

In the second step we show that using (12) we have

(19) $$N_\gamma f(x^*) \subset N_\gamma f(x_n) \qquad \text{or} \qquad N_\gamma g(x^*) \subset N_\gamma g(y_n).$$

For this it is enough to prove that $N_\gamma g(x^*) \subset N_\gamma f(x_n)$ implies $N_\gamma f(x^*) \subset$ $\subset N_\gamma f(x_n)$. Since $N_\gamma f(x^*) \subset N_\gamma f(x_n)\cup N_\gamma g(y_n)$, the indirect assumption $N_\gamma f(x^*) \not\subseteq N_\gamma f(x_n)$ implies the existence of $z_2\in N_\gamma f(x^*)\cap N_\gamma g(y_n)$. Take any element $z_1\in N_\gamma g(x^*)\subset N_\gamma f(x_n)$ and define the sequence z_m by induction using (4) as follows

$$g(x,z_{m+1}) \leq \varrho_2(f(x,z_2),g(x,z_m)) \qquad (m\geq 3),$$
$$g(x,x_3) \leq \varrho_2(f(x,z_2),g(x,z_1)).$$

Since $f(x^*,z_2)\leq\gamma$ and $g(x^*,z_1)\leq\gamma$ we see $g(x^*,z_3)\leq\gamma$ and we get by induction on m that

$$g(x^*,z_{m+1}) \leq \varrho_2(f(x^*,z_2),g(x^*,z_m)) \leq \gamma \qquad (m\geq 3).$$

That is, $z_{m+1}\in N_\gamma g(x^*)\subset N_\gamma f(x_n)$ and then $g(y_n,z_{m+1})\geq\alpha$ and

(20) $$\alpha \leq g(y_n,z_{m+1}) \leq \varrho_2(f(y_n,z_2),g(y_n,z_m)) \qquad (m\geq 3).$$

Here $f(y_n,z_2)\leq g(y_n,z_2)\leq\gamma$ (since $z_2\in N_\gamma g(y_n)$). From (20) we see that $g(y_n,z_{m+1})$ decreases for $m\geq 4$. Let $\alpha\leq B:=\lim_{m\to\infty} g(y_n,z_{m+1})$, then (20) implies $B\leq\varrho_2(\gamma,B)$ in contradiction with (1). Hence (19) is true.

If $N_\gamma f(x^*) \subset N_\gamma f(x_n)$ then let $(x_{n+1}, y_{n+1}) := (x^*, y_n)$,

if $N_\gamma g(x^*) \subset N_\gamma g(y_n)$ then let $(x_{n+1}, y_{n+1}) := (x_n, x^*)$.

In the first case $y \in N_\gamma f(x^*)$ implies $y \in N_\gamma f(x_n)$ and from (8) follows that $g(y_n, y) \geq \alpha$. Hence

$$\inf_Y \max\{f(x^*, y), g(y_n, y)\} \geq \gamma$$

which implies (8) with $n+1$ instead of n. In the second case we argue similarly: $y \in N_\gamma g(x^*)$ implies $y \in N_\gamma f(x^*) \cap N_\gamma g(y_n)$ and then $f(x_n, y) \geq \alpha$ so (8) holds also in this case with $n+1$. The construction of the sequences x_n, y_n implies that

$$N_\gamma f(x_{n+1}) \subset N_\gamma f(x_n), \qquad N_\gamma g(y_{n+1}) \subset N_\gamma g(y_n) \qquad (n \geq 1).$$

If $N_\gamma f(x^*) \subset N_\gamma f(x_n)$ then $f(x_{n+1}, y) \geq \varrho_1(f(x_n, y), g(y_n, y))$.

Since $y \in N_\gamma f(x_{n+1}) \Rightarrow y \in N_\gamma f(x_n) \Rightarrow g(y_n, y) \geq \alpha$ therefore

$$f(x_{n+1}, y) \geq \varrho_1(f(x_n, y), \alpha) \qquad \text{for} \qquad y \in N_\gamma f(x_{n+1}),$$

i.e.,

$$\inf_Y f(x_{n+1}, y) \geq \varrho_1\left(\inf_Y f(x_n, y), \alpha\right). \tag{21}$$

If $N_\gamma g(x^*) \subset N_\gamma g(y_n)$ then we obtain similarly

$$\inf_Y g(y_{n+1}, y) \geq \varrho_1\left(\alpha, \inf_Y g(y_n, y)\right). \tag{22}$$

Then for the sequences $c_n := \inf_Y f(x_n, y)$, $d_n := \inf_Y g(y_n, y)$ we have

$$c_n \leq \beta < \alpha, \qquad c_{n+1} \geq \varrho_1(c_n, \alpha) > c_n \qquad (n \geq 1)$$

and

$$d_n \leq \beta < \alpha, \qquad d_{n+1} \geq \varrho_1(\alpha, d_n) > d_n \qquad (n \geq 1).$$

Taking limit we obtain

$$c^* = \lim c_{n+1} \geq \varrho_1(\lim c_n, \alpha) > \lim c_n = c^*$$

and

$$d^* = \lim d_{n+1} \geq \varrho_1(\alpha, \lim d_n) = d^*$$

which is contradiction. Hence (7) is false. The Theorem is proved. ∎

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ON RETARDED EVOLUTION INCLUSIONS OF THE SUBDIFFERENTIAL TYPE

By

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1. Introduction

The purpose of this paper is to consider retarded, time varying subdifferential evolution inclusions, establish the existence of extremal trajectories and show that these trajectories are in fact dense in the set of trajectories of the convex problem for the norm topology of the Banach space $C(T,H)$ ("strong relaxation theorem"). Then we show that this density result allows us to establish a nonlinear "bang-bang principle" for a large class of infinite dimensional control systems. An example of a concrete parabolic disturbed parameter control system is worked out in detail.

Our work here extends those of N. S. PAPAGEORGIOU [9], [10], where evolution inclusions with no delay were considered.

2. Mathematical preliminaries

Let $\hat{T} = [-r,b]$ (with $r > 0$ being the delay), $T = [0,b]$, $T_0 = [-r,0]$ and H a separable Hilbert space. The retarded evolution inclusion under consideration is the following:

$$\tag{1} \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + F(t,x_t), & \text{a.e. on } T, \\ x(\tau) = v(\tau), & \tau \in T_0. \end{cases}$$

Here $x_t(\,.\,) \in C(T_0,H)$ is the function defined by $x_t(\tau)=x(t+\tau)$, for all $\tau \in T_0$. So $x_t(.)$ describes the past evolution of the state, from time $t-r$ until the present time t. By a "strong solution" of (1), we understand a function $x \in C(\hat{T},H)$ with the property $x(.)\in W^{1,2}(T,H)$ and such that

1. $x(t)\in \operatorname{dom}\varphi(t,.)=\{z\in H:\varphi(t,z)<\infty\}$, a.e. on T;

2. $\exists\, f \in L^2(T,H)$ such that $f(t) \in F(t,x_t)$ and $-\dot{x}(t) \in \partial\varphi(t,x(t))+f(t)$, a.e. on T;
3. $x(\tau)=v(\tau)$, $\forall\, \tau \in T_0$.

Recall (see for example [1], Theorem 2.2, p. 19) that $W^{1,2}(T,H)$ can be identified with the space $AC^{1,2}(T,H)$ of all absolutely continuous functions $x : T \to H$ such that $\dot{x}(.) \in L^2(T,H)$. Note for $x(.)$ being absolutely continuous into H, is almost everywhere strongly differentiable; see [1], Theorem 2.1, p. 16.

In conjunction with (1), we also consider the following multivalued Cauchy problem:

$$(2) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \operatorname{ext} F(t,x_t), & \text{a.e. on } T, \\ x(\tau) = v(\tau), & \tau \in T_0. \end{cases}$$

Here by ext $F(t,y)$ we denote the extreme points of the orientor field $F(t,y)$. A strong solution of (2) is defined analogously as for (1), with $f \in L^2(T,H)$ and $f(t) \in \operatorname{ext} F(t,x_t)$ a.e. on T. We will call the solutions of (2) extremal solutions (or extremal trajectories). In that follows, by $S(v) \subset C(\hat{T},H)$ we will denote the solution set of (1) and by $S_e(v)$ the solutions set of (2). Clearly $S_e(v) \subset S(v)$.

The following hypothesis on the function φ was first introduced by S. Yotsutani [15]:

$H(\varphi)$: $\quad \varphi : T \times H \to \mathbb{R} \cup \{+\infty\}$ is a function such that

i) $\forall\, t \in T$, $x \mapsto \varphi(t,x)$ is proper, convex, l.s.c. and of compact type (i.e. for all $\lambda \in \mathbb{R}$, the level set $\{z \in H : \varphi(t,z)+\|z\|^2 \le \lambda\}$ is compact in H);

ii) for each integer $r > 0$, there exists $K_r > 0$, an absolutely continuous function $g_r : T \to \mathbb{R}$ with $\dot{g}_r \in L^\beta(T,\mathbb{R})$ and a function of bounded variation $h_r : T \to \mathbb{R}$ such that if $t \in T$, $x \in \operatorname{dom}\varphi(t,\,.\,)$ with $\|x\| \le r$ and $s \in [t,b]$, there exists $\hat{x} \in \operatorname{dom}\varphi(s,.)$ satisfying

$$\|x - \hat{x}\| \le |g_r(s) - g_r(t)|\,(\varphi(t,x) + K_r)^\alpha$$

and

$$\varphi(s,\hat{x}) \le \varphi(t,x) + |h_r(s) - h_r(t)|\,(\varphi(t,x) + K_r)$$

where $\alpha \in [0,1]$ and $\beta = 2$ if $\alpha \in [0,1/2]$ or $\beta = 1/(1-\alpha)$ if $\alpha \in [1/2,1]$.

This is a general hypothesis and incorporates earlier ones introduced in the important works of Watanabe [13] and Yamada [14].

Let X be a separable Banach space. We will be using the following notations:

$$P_{f(c)}(x) = \{A \subset X : A \text{ nonempty, closed and (convex)}\},$$

$$P_{(w)k(c)}(x) = \{A \subset X : A \text{ nonempty, (weakly-)compact convex}\}.$$

If (Ω,Σ,μ) is a finite measure space, a multifunction $F:\Omega\to P_f(X)$ is said to be measurable, if for all $x\in X$, the function $\omega\mapsto d(x,F(\omega)) = \inf\{\|x-z\|:z\in F(\omega)\}$ is measurable. If $F(.)$ is measurable, then $\operatorname{Gr}F = \{(\omega,x)\in\Omega\times X : x\in F(\omega)\}\in\Sigma\times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability), while the converse is true if Σ is μ-complete. By S_F^p $(1\le p\le\infty)$ we will denote the set of all measurable selectors of $F(\,.\,)$ that belong in the Lebesgue–Bochner space $L^p(\Omega,X)$; i.e. $S_F^p=\{f\in L^p(\Omega,X):f(\omega)\in F(\omega)\ \mu\text{-a.e.}\}$. In general this set may be empty. It is easy to check using Aumann's selection Theorem (cf. [12], theorem 5.10), that for a graph measurable multifunction $F:\Omega\to 2^X\setminus\{\emptyset\}$, S_F^p is nonempty if and only if the function $\omega\mapsto\inf\{\|z\|:z\in F(\omega)\}$ belongs to $L^p(\Omega,\mathbb{R}^+)$. Recall that a subset K of $L^p(\Omega,X)$ is decomposable if for every triple $(f,g,A)\in K\times K\times\Sigma$, we have $f\chi_A+g\chi_{A^c}\in K$, where χ_A denotes the characteristic function of the set A. Clearly S_F^p is decomposable.

Recall that on $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric", by setting, for $A,\ B\in P_f(X)$,

$$h(A,B)=\max\{\sup\{d(a,B):a\in A\},\ \sup\{d(b,A):b\in B\}\}$$

(where $d(a,B)=\inf\{\|a-b\|:b\in B\}$; similarly for $d(b,A)$). It is well known (cf. [7]), that the metric space $(P_f(X),h)$ is complete, while $(P_{fc}(X),h)$ is a closed subspace of it. Also $(P_k(X),h)$ is a separable, complete metric space (i.e. a Polish space). A multifunction $F:T\to P_f(X)$ is said to be Hausdorff continuous (H-continuous) if it is continuous from T into the metric space $(P_f(X),h)$.

3. Existence of extremal solutions

In this section we establish the nonemptiness of the solution set $S_e(v)$. For this we will need the following hypothesis on the orientor field F:

$H(F)_1$: $F:T\times C(T_0,H)\to P_{wkc}(H)$ is a multifunction such that

j) $\forall\ y\in C(T_0,H)$, $t\mapsto F(t,y)$ is measurable;

jj) for a.e. $t\in T$, $y\mapsto F(t,y)$ is H-continuous;

jjj) $\exists\ a,\ c\in L^2(T,\mathbb{R}^+)$:

$$\|F(t,y)\|=\sup\{\|z\|:z\in F(t,y)\}\le a(t)+c(t)\|y\|_\infty,$$

a.e. in T, $\forall\ y\in C(T_0,H)$.

From Krein–Milman Theorem, we know that for all $(t,y)\in T\times C(T_0,H)$, $\operatorname{ext}F(t,y)\neq\emptyset$. Also hypotheses $H(F)$ j) and jj) and theorem 3.3 of [8] imply that $(t,y)\mapsto F(t,y)$ is jointly measurable.

THEOREM 3.1. *If hypotheses* $H(\varphi)$, $H(F)_1$ *hold,* $v \in C(T_0, H)$ *and* $v(0) \in \in \mathrm{dom}\,\varphi(0,.)$, *then* $S_e(v) \neq \emptyset$.

PROOF. First we will establish an a priori bound for the elements of $S(v)$. To this end let $x \in S(v)$. By definition we have that there exists $f \in L^2(T,H)$ such that $f(t) \in F(t, x_t)$, a.e. on T, and

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t), & \text{a.e. on } T; \\ x(\tau) = v(\tau), & \forall\, \tau \in T_0. \end{cases}$$

Let $y \in C(T,H)$ be the unique solution of the Cauchy problem

$$\begin{cases} -\dot{y}(t) \in \partial\varphi(t, y(t)), & \text{a.e. on } T; \\ x(0) = v(0). \end{cases}$$

Its existence is guaranteed by the existence theorem of S. YOTSUTANI [15]. Because of the monotonicity of the subdifferential operator, we have

$$(-\dot{x}(t) + \dot{y}(t), y(t) - x(t)) \leq (f(t), y(t) - x(t)), \quad \text{a.e. on } T,$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|y(t) - x(t)\|^2 \leq (f(t), y(t) - x(t)) \quad \text{a.e. on } T,$$

$$\Rightarrow \frac{1}{2}\|y(t) - x(t)\|^2 \leq \int_0^t (f(s), y(s) - x(s))\, ds \leq$$

$$\leq \int_0^t \|f(s)\| \cdot \|y(s) - x(s)\| ds, \quad \forall\, t \in T.$$

Invoking Lemma A.5, p. 157, of [4], we get

$$\|y(t) - x(t)\| \leq \int_0^t \|f(s)\| ds, \quad \forall\, t \in T,$$

$$\Rightarrow \|x(t)\| \leq \|\hat{y}\|_\infty + \int_0^t \left(a(s) + c(s)\|x_s\|_\infty\right) ds, \quad \forall\, t \in T,$$

where $\hat{y}(t) = y(t)$ for $t \in T$ and $\hat{y}(\tau) = v(\tau)$ for $\tau \in T_0$. Then we have

$$\|x_t\|_\infty \leq \|\hat{y}\|_\infty + \|a\|_1 + \int_0^t c(s)\|x_s\|_\infty ds, \forall\, t \in T.$$

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ such that, for all $t \in T$ and all solutions $x \in S(v)$, we have $\|x_t\|_\infty \leq M_1$. Hence without any loss of generality, put $\psi(t)=a(t)+c(t)M_1$, $\psi\in L^2(T,\mathbb{R}^+)$, and assume that

$$\|F(t,x)\| = \sup\{\|z\| : z\in F(t,x)\le\psi(t), \quad \text{a.e. in } T,\ \forall\, x\in C(T_0,H).$$

Let $V=\left\{h\in L^2(T,H): \|h(t)\|\le\psi(t) \text{ a.e. in } T\right\}$ and let $\eta : L^2(T,H)\to$ $\to C(T,H)$ be the map which assigns to each $h\in L^2(T,H)$ the unique solution of the Cauchy problem

$$\begin{cases} -\dot z(t)\in\partial\varphi(t,z(t))+h(t), & \text{a.e. on } T;\\ z(0)=v(0), \end{cases}$$

(cf. [15]). Let $K=\overline{\text{co}}\,\eta(V)$. From the proof of theorem 3.1, p. 486, of [9], we have that K is a compact and convex subset of $C(T,H)$. Extend the functions of K on all of $\hat T=[-r,b]$, by letting them be equal to $v(\,.\,)$ on T_0. Call the resulting compact and convex subset of $C(\hat T,H)$, $\hat K$.

Next let $\Gamma: K\to P_{wkc}(L^2(T,H))$ by

$$\Gamma(x)=\left\{h\in L^2(T,H) : h(t)\in F(t,x_t), \text{ a.e. on } T\right\}.$$

Apply Theorem 1.1 of [11], to get a continuous function $\gamma:\hat K\to L^1_w(T,H)$ such that $\gamma(x)\in\text{ext}\,\Gamma(x)$, $\forall\, x\in\hat K$ (recall that $L^1_w(T,H)$ denotes the space of all equivalence classes of Bochner integrable functions $x : T\to H$ equipped with the (weak) norm $\|f\|_w=\sup\left\{\left\|\int_t^{t'} f(s)ds\right\| : 0\le t\le t'\le b\right\}$).

Since, for every $x\in\hat K$,

$$\text{ext}\,\Gamma(x)=\left\{h\in L^2(T,H) : h(t)\in \text{ext}\,F(t,x_t), \text{ a.e. in } T\right\}$$

(cf. [2]), it follows that $\gamma(x)(t)\in\text{ext}F(t,x_t)$, a.e. in T. Also let $\hat\eta : L^2(T,H)\to$ $\to C(\hat T,H)$ be defined by $\hat\eta(h)(\,.\,)=\eta(h)(\,.\,)$ on T and $\hat\eta(h)(\,.\,)=v(\,.\,)$ on T_0. Then let $\theta=\hat\eta\circ\gamma:\hat K\to\hat K$. From Proposition 3.1 of [10], we have that $\theta(\,.\,)$ is continuous. So, by Schauder's fixed point theorem, we have that there exists $x\in\hat K$ such that $x=\theta(x)$. Clearly then $x\in S_e(v)$.

4. Strong relaxation

A natural question that arises, with important implications in applied areas (like control systems), is whether or not we can approximate with arbitrary degree of accuracy, trajectories in $S(v)$ by extremal trajectories. Such a result leads to "bang-bang" principles for large classes of nonlinear infinite dimensional control systems. Results of this type are known as "relaxation theorems". Relaxation theorems for subdifferential evolution inclusions with no delay (i.e. $r=0$), were proved in [9] and [10]. Note that $\operatorname{ext} F(x,y)$ is not to be continuous (or even l.s.c.), even of $F(t,.)$ is h-continuous or even better h-Lipschitz.

We will need the following stronger hypothesis on the orientor field:

$H(F)_2$: $F: T \times C(T_0,H) \to P_{wkc}(H)$ is a multifunction such that

j) $\forall\, y \in C(T_0,H)$, $t \mapsto F(t,y)$ is measurable;

jj)' $\exists\, k \in L^1(T,\mathbb{R}^+) : h(F(t,y),F(t,y')) \leq k(t)\|y-y'\|_\infty$, a.e. in T, $\forall\, y$, $y' \in C(T_0,H)$;

jjj) $\exists\, a,\, c \in L^2(T,\mathbb{R}^+) : \|F(t,y)\| = \sup\{\|z\| : z \in F(t,y)\} \leq a(t)+c(t)\|y\|_\infty$, a.e. in T, $\forall\, y \in C(T_0,H)$.

THEOREM 4.1. *If hypothesis $H(\varphi)$, $H(F)_2$ hold, $v \in C(T_0,H)$ and $v(0) \in \operatorname{dom}\varphi(0,.)$, then $S_e(v)$ is dense in $S(v)$ for the $C(\hat{T},H)$-topology.*

PROOF. Let $x \in S(v)$. Then by definition we have

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t))+f(t), & \text{a.e. on } T; \\ x(\tau) = v(\tau), & \forall\, \tau \in T_0. \end{cases}$$

with $f \in L^2(T,H)$, $f(t) \in F(t,x_t)$, a.e. in T. Let $\hat{K}$ be the compact subset of $C(\hat{T},H)$ as in the proof of theorem 3.1. Given $y \in \hat{K}$ and $\varepsilon > 0$, let $\Gamma_\varepsilon :\to 2^H \setminus \{\emptyset\}$ be defined by

$$\Gamma_\varepsilon(t) = \{u \in H : \|f(t)-u\| < \varepsilon + d(f(t),F(t,y_t)),\ u \in F(t,y_t)\}\,.$$

Note that because of hypotheses $H(F)_2$ j) and jj)' and Theorem 3.3 of [8] we have that $(t,y) \mapsto F(t,y_t)$ is measurable, which implies that $t \mapsto F(t,y_t)$ is measurable and so $\operatorname{Gr} F(\,.\,,y_.) \in B(T)\times B(H)$ with $B(T)$ (resp. $B(H)$) being the Borel σ-filed of T (resp. of H). Therefore $t \mapsto d(y(t),F(t,y_t))$ is measurable. Thus

$$\operatorname{Gr}\Gamma_\varepsilon = \{(t,u)\in \operatorname{Gr} F(\,.\,,y_.) : \|f(t)-u\| < \varepsilon + d(f(t),F(t,y_t))\} \in B(T)\times B(H).$$

Apply Aumann's selection Theorem (cf. [12], theorem 5.10) to get a measurable map $h_\varepsilon : T \to H$ such that $h_\varepsilon(t) \in \Gamma_\varepsilon(t)$ a.e. in T. Then let $Z_\varepsilon : \hat{K} \to 2^{L^1(T,H)}$ be defined by

$$Z_\varepsilon(y) = \Big\{h \in L^1(T,H) : h(t) \in F(t,y_t) \text{ and } \|f(t)-h(t)\| < \varepsilon + d(f(t),F(t,y_t)) \text{ a.e. in } T\Big\}.$$

We have just seen that, for all $y \in \hat{K}$ and all $\varepsilon > 0$, $Z_\varepsilon(y) \neq \emptyset$. Furthermore, from Proposition 4 of [3], we know that $y \mapsto Z_\varepsilon(y)$ is l.s.c. with decomposable values, hence $y \mapsto \overline{Z_\varepsilon(y)}$ is l.s.c. and it has nonempty, closed and decomposable values. Apply Theorem 3 of [3] to get a continuous map $u_\varepsilon : \hat{K} \to L^1(T,H)$ such that $u_\varepsilon(y) \in \overline{Z_\varepsilon(y)}$, $\forall\, y \in \hat{K}$. Hence we have:

$$\|f(t) - u_\varepsilon(y)(t)\| \le \varepsilon + d(f(t), F(t,y_t)) \le \varepsilon + k(t)\|x_t - y_t\|_\infty, \quad \text{a.e. in } T.$$

Also from Theorem 1.1 of [11] we know that we can find a continuous map $w_\varepsilon : \hat{K} \to L^1_w(T,H)$ such that

$$w_\varepsilon(y) \in \left\{h \in L^1(T,H) : h(t) \in \operatorname{ext} F(t,y_t), \text{ a.e. in } T\right\} \qquad \text{and}$$

$$\|u_\varepsilon(y) - w_\varepsilon(y)\|_w < \varepsilon, \qquad \forall\, y \in \hat{K}.$$

Let $\varepsilon = 1/n$, $u_{1/n} = u_n$ and $w_{1/n} = w_n$. Let $\hat{\eta} : L^2(T,H) \to C(\hat{T},H)$ be as in the proof of theorem 3.1. Using Proposition 3.1 of [10], we can check that $(\hat{\eta} \circ w_n)(\,.\,)$ is continuous and so by Schauder's fixed point theorem, we can find $x_n \in \hat{K}$ such what $x_n = \hat{\eta}(w_n,(x_n))$. Clearly $x_n \in S_e(v)$. Also since $\hat{K}$ is a compact subset of $C(\hat{T},H)$, by passing to a subsequence if necessary, we may assume that $x_n \to \hat{x}$ in $C(\hat{T},H)$. As before, exploiting the monotonicity of the subdifferential, we get

$$(-\dot{x}_n(t)+\dot{x}(t), x(t)-x_n(t)) \le (w_n(x_n)(t)-f(t), x(t)-x_n(t)), \text{ a.e. on } T, \forall\, n \in \mathbb{N},$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|x_n(t)-x(t)\|^2 \le (w_n(x_n)(t)-f(t), x(t)-x_n(t)), \quad \text{a.e. on } T, \forall\, n \in \mathbb{N}.$$

$$\Rightarrow \frac{1}{2}\|x_n(t) - x(t)\|^2 \le \int_0^t (w_n(x_n)(s) - f(s), x(s) - x_n(s))ds \le$$

$$\le \int_0^t (w_n(x_n)(s) - u_n(x_n)(s), x(s) - x_n(s))ds + \int_0^t \|u_n(x_n)(s) - f(s)\|\,\|x(s) - x_n(s)\|ds \le$$

$$\le \int_0^t (w_n(x_n)(s) - u_n(x_n)(s), x(s) - x_n(s))ds +$$

$$+ \int_0^t \left(\frac{1}{n} + k(s)\|x_s - (x_n)_s\|_\infty\right)\|x(s) - x_n(s)\|ds.$$

Note that, from Proposition 3.1 of [10], we have $w_n(x_n) - u_n(x_n) \to 0$ weakly in $L^2(T,H)$. So we obtain

$$\lim_{n\to+\infty} \int_0^t (w_n(x_n)(s) - u_n(x_n)(s), x(s) - x_n(s))ds = 0. \tag{4.1}$$

On the other hand

$$\int_0^t \left(\frac{1}{n} + k(s)\|x_s - (x_n)_s\|_\infty\right) \|x(s) - x_n(s)\| ds \le$$

$$(4.2) \qquad \le \frac{2M_1 b}{n} + \int_0^t k(s)\|x_s - (x_n)_s\|_\infty^2 ds \to \int_0^t k(s)\|x_s - \hat{x}_s\|_\infty^2 ds,$$

as $n \to \infty$, $\forall\, t \in T$.

Hence, by (4.1) and (4.2), it follows

$$\|x_t - \hat{x}_t\|_\infty^2 \le 2\int_0^t k(s)\|x_s - \hat{x}_s\|_\infty^2 ds, \quad \forall\, t \in T.$$

Applying Gronwall's inequality we get $x = \hat{x}$. Since $x_n \in S_e(v)$ and $x_n \to x$ in $C(\hat{T}, H)$, we have that $S(v)$ is included in the closure of $S_e(v)$ in $C(\hat{T}, H)$. But $S(v)$ is compact on $C(\hat{T}, H)$ (cf. [9], theorem 4.1). So finally we have that $S_e(v)$ is dense in $S(v)$ for the $C(\hat{T}, H)$-topology.

5. Control systems

In this section we use theorem 4.1 to prove a “bang-bang” principle for infinite dimensional, nonlinear control systems with delay and a priori feedback.

Let Y be a separable Banach space, modelling the control space. We will be dealing with the following nonlinear, feedback control system:

$$(3) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + b(t, x_t)u(t), & \text{a.e. on } T, \\ x(\tau) = v(\tau), & \tau \in T_0, \\ u(t) \in U(t, x_t), & \text{a.e. on } T. \end{cases}$$

In conjunction with (3) we also consider the control system in which the admissible controls are the “bang-bang” controls of (3); i.e.

$$(4) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + b(t, x_t)u(t), & \text{a.e. on } T, \\ x(\tau) = v(\tau), & \tau \in T_0, \\ u(t) \in \operatorname{ext} U(t, x_t), & \text{a.e. on } T. \end{cases}$$

First we establish the existence of admissible trajectories for (4). For this purpose, we will need the following hypotheses on the data:

$H(b)_1$: $b: T\times C(T_0,H)\to\mathscr{L}(Y,H)$ is a map such that

1. $\forall\ (y,u)\in C(T_0,H)\times Y$, $t\mapsto b(t,y)u$ is measurable;
2. $\forall\ t\in T$, $y\mapsto b(t,y)$ is continuous from $C(T_0,H)$ into $\mathscr{L}(Y,H)$ with the operator norm topology;
3. $\exists\ a,\ c\in L^2(T,\mathbb{R}^+)$: $\|b(t,y)\|_{\mathscr{L}}\le a(t)+c(t)\|y_t\|_\infty$, a.e. in T, $\forall\ y\in$ $\in C(T_0,H)$.

$H(U)_1$: $U: T\times C(T_0,H)\to P_{wkc}(Y)$ is a multifunction such that

1. $(t,y)\mapsto U(t,y)$ is H-continuous;
2. $\exists\ M>0: \|U(t,y)\|=\sup\{\|u\|: u\in U(t,y)\}\le M$, $\forall\ (t,y)\in T\times C(T_0,H)$.

THEOREM 5.1. *If hypotheses* $H(\varphi)$, $H(b)_1$, $H(U)_1$ *hold,* $v\in C(T_0,H)$ *and* $v(0)\in$ dom $\varphi(0,\ .\ )$*, then control system* (4) *has a nonempty set of admissible trajectories.*

PROOF. Let $F: T\times C(T_0,H)\to P_{wkc}(H)$, defined by

$$F(t,y)=b(t,y)U(t,y),\qquad \forall\ (t,y)\in T\times C(T_0,H).$$

We claim that F satisfies hypotheses $H(F)_1$.

Indeed note that $\operatorname{Gr}F(\ .\ ,y)=\{(t,z)\in T\times H: z\in F(t,y)\}=\{(t,z)\in T\times$ $\times H: (z,h)\le\sigma(h,F(t,y))$, for all $h\in H\}$, where $\sigma(\ .\ ,F(t,y))$ is the support function of the set $F(t,y)$; i.e. $\sigma(h,F(t,y))=\sup\{(h,z): z\in F(t,y)\}$. Let $u_n: T\times H\to Y$ be measurable functions such that $U(t,y)=\operatorname{cl}\{u_n(t,y)\}_{n\in\mathbb{N}}$. Their existence is guaranteed by hypothesis $H(U)_1$ and Theorem 4.2 of [12]. Then $\sigma(h,F(t,y))=\sup\{(h,b(t,y)u_n(t,y)): n\in\mathbb{N}\}$, which implies that $t\mapsto$ $\sigma(h,F(t,y))$ is measurable. Let $\{h_m\}_{m\in\mathbb{N}}$ be dense in H. So

$$\operatorname{Gr}F(\ .\ ,y)=\bigcap_{m\in\mathbb{N}}\{(t,z)\in T\times H: (z,h_m)\le\sigma(h_m,F(t,y))\}.$$

Since $\sigma(.,F(t,y))$ is continuous, we have that $\operatorname{Gr}F(.,y)\in\mathscr{L}(T)\times B(H)$, with $\mathscr{L}(T)$ being the Lebesgue σ-field of T (i.e. $F(\ .\ ,y))$ is graph measurable); therefore $t\mapsto F(t,y)$ is Lebesgue measurable on T.

Also if $y_m\to y$ in $C(T_0,H)$ we have

$$\begin{aligned} &h(F(t,y_m),F(t,y))\le\\ &\quad\le h(b(t,y_m)U(t,y_m),b(t,y)U(t,y_m))+h(b(t,y)U(t,y_m),b(t,y)U(t,y))\le\\ &\qquad\le M\|b(t,y_m)-b(t,y)\|_{\mathscr{L}}+\|b(t,y)\|_{\mathscr{L}}h(U(t,y_m),U(t,y)).\end{aligned}$$

Because of hypotheses $H(b)_1$ and $H(U)_1$ we have that $\|b(t,y_m)-$ $-b(t,y)\|_{\mathscr{L}}\to 0$ and $\|b(t,y)\|_{\mathscr{L}}h(U(t,y_m),U(t,y))\to 0$ as $m\to\infty$. Hence $F(t,.)$ is H-continuous.

Using hypothesis $H(b)_1(3)$ and $H(U)_1(2)$, we get

$$\|F(t,y)\| \leq \hat{a}(t) + \hat{c}(t)\|y\|, \qquad \text{a.e. in } T, \forall\, y \in C(T_0,H),$$

where $\hat{a}(\,.\,) = Ma(\,.\,)$, $\hat{b}(\,.\,)Mb(\,.\,) \in L^2(T,\mathbb{R}^+)$. Thus we have satisfied hypothesis $H(F)_1$. Then via Aumann's selection theorem, we verify that the problem (3) is equivalent to the following retarded subdifferential evolution inclusion (deparametrized or control free system):

$$\text{(5)} \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + F(t,x_t), & \text{a.e. on } T, \\ x(\tau) = v(\tau), & \tau \in T_0. \end{cases}$$

Note that ext $F(t,x_t) \subset b(t,x_t)$ext $U(t,x_t)$ (cf. [5], p.94). Since $t \mapsto \text{ext}\, F(t,x_t)$ is graph measurable (cf., for example [2]), using Aumann's selection theorem, we can see that an extremal trajectory of (5) is also a trajectory of (4). So an application of theorem 3.1, completes the proof.

In similar fashion, we can also treat systems, in which the control space Y is the dual of a separable Banach space; i.e. $Y = V^*$ with V being a separable Banach space.

$H(U)_1'$: $U : T \times C(T_0,H) \to 2^Y \setminus \{\emptyset\}$ is a multifunction with w^*-compact, convex values such that

1. $(t,y) \mapsto U(t,y)$ is H-continuous;
2. $\exists\, M > 0$: $\|U(t,y)\| = \sup\{\|u\| : u \in U(t,y)\} \leq M$, $\forall\, (t,y) \in T \times C(T_0,H)$.

THEOREM 5.2. *If hypotheses $H(\varphi)$, $H(b)_1$, $H(U)_1'$ hold, $v \in C(T_0,H)$ and $v(0) \in$ dom $\varphi(0,\,.\,)$, then control system* (4) *has a nonempty set of admissible trajectories.*

If we strengthen out hypotheses, we can say more about systems (3) and (4).

$H(b)_2$: $b : T \times C(T_0,H) \to \mathscr{L}(Y,H)$ is a map such that

1. $\forall\, (y,u) \in C(T_0,H) \times Y$, $t \mapsto b(t,y)u$ is measurable;
2. $\exists\, k_1 \in L^1(T,\mathbb{R}^+) : \|b(t,y) - b(t,y')\|_{\mathscr{L}} \leq k_1(t)\|y - y'\|_\infty$, a.e. in T, $\forall\, y$, $y' \in C(T_0,H)$;
3. $\exists\, a \in L^2(T,\mathbb{R}^+) : \|b(t,y)\|_{\mathscr{L}} \leq a(t)$, a.e. in T, $\forall\, y \in C(T_0,H)$.

$H(U)_2$: $U : T \times C(T_0,H) \to P_{wkc}(Y)$ is a multifunction such that

1. $(t,y) \mapsto U(t,y)$ is H-continuous;
2. $\exists\, k_2 \in L^2(T,\mathbb{R}^+) : h(U(t,y), U(t,y')) \leq k_2(y)\|y - y'\|_\infty$, a.e. in T, $\forall\, y$, $y' \in C(T_0,H)$;
3. $\exists\, M > 0$: $\|U(t,y)\| = \sup\{\|u\| : u \in U(t,y)\} \leq M$, $\forall\, (t,y) \in T \times C(T_0,H)$.

Or if $Y = V^*$, then we assume

$H(U)'_2$: $U : T \times C(T_0, H) \to 2^Y \setminus \{\emptyset\}$ is a multifunction with w^*-compact, convex values such that

1. $(t,y) \mapsto U(t,y)$ is H-continuous;
2. $\exists\ k_2 \in L^2(T, \mathbb{R}^+)$: $h(U(t,y), U(t,y')) \leq k_2(t)\|y - y'\|_\infty$, a.e. in T, $\forall\ y$, $y' \in C(T_0, H)$;
3. $\exists\ M > 0$: $\|U(t,y)\| = \sup\{\|u\| : u \in U(t,y)\} \leq M$, $\forall\ (t,y) \in T \times C(Y_0, H)$.

Let $S'(v) \subset C(\hat{T}, H)$ be the set of admissible trajectories of (3) and $S'_e(v) \subset C(\hat{T}, H)$ the admissible trajectories of (4).

THEOREM 5.3. *If hypotheses $H(\varphi)$, $H(b)_2$, $H(U)_2$ (or $H(U)'_2$) hold, $v \in C(T_0, H)$ and $v(0) \in \operatorname{dom} \varphi(0, .)$, then $S'_e(v)$ is dense in $S'(v)$ for the $C(\hat{T}, H)$-topology.*

In particular then, if $\tilde{\eta} : C(\hat{T}, H) \to \mathbb{R}$ is a continuous cost functional and, we put $m = \inf\{\tilde{\eta}(x) : x \in S'(v)\}$, we consider the minimization problem

(P) does there exist $x^* \in S'(v)$ such that $m = \tilde{\eta}(x^*)$?

We have the following existence and approximation result.

THEOREM 5.4. *If hypotheses $H(\varphi)$ holds, $v \in C(T_0, H)$ and $v(0) \in$ $\in \operatorname{dom}\varphi(0, .)$, then*

i) *if $H(b)_1$, $H(U)_1$ (or $H(U)'_1$) hold, (P) admits a solution;*

ii) *if $H(b)_2$, $H(U)_2$ (or $H(U)'_2$) hold, given $\varepsilon > 0$ we can find an extremal trajectory $x \in C(\hat{T}, H)$ (i.e. $x \in S'_e(v)$) such that $\tilde{\eta}(x) - m < \varepsilon$ (i.e. x is ε-optimal).*

6. An example

In this section we present an example of a nonlinear parabolic distributed parameter system with delay.

So let Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. The system under consideration is the following

$$(6)\quad \begin{cases} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial z_k}\left(a(t,z)\left|\frac{\partial x}{\partial z_k}\right|^{p-2}\frac{\partial x}{\partial z_k}\right) + \beta x|x|^{p-2} = \\ \qquad = B(t,z,x(t-r,z))u(t,z), & \text{a.e. in } T \times Z, \\ x\Big|_{T\times\Gamma} = 0, \quad x(\tau,z) = v(\tau,z), & \text{a.e. on } Z, \text{ for all } \tau \in T_0, \\ |u(t,z)| \leq g(t, x(t-r, .)), & \text{a.e. on } T \times Z,\ p \geq 2,\ \beta > 0. \end{cases}$$

We will need the following hypotheses on the data of (6):

$H(a)$: $a : T\times Z \to \mathbb{R}$ is a function such that

i) $(t,z)\mapsto a(t,z)$ is measurable;

ii) there exits m_1, $m_2\in\mathbb{R}^+$ such that $0<m_1\leq a(t,z)\leq m_2$, $\forall\ (t,z)\in T\times Z$;

iii) there exists $\gamma : Z\to\mathbb{R}^+$ such that $\gamma\in L^\infty(Z,\mathbb{R})$ and $\left|a(t',z)-a(t,z)\right|\leq$ $\leq\gamma(z)|t'-t|$, $\forall\ t,\ t'\in T$, a.e. on Z.

$H(B)$: $B : T\times Z\times\mathbb{R}\to\mathbb{R}$ is a function such that

i) $\forall\ x\in\mathbb{R}$, $(t,z)\mapsto B(t,z,x)$ is measurable;

ii) $\exists\ k_1\in L^\infty(T\times Z,\mathbb{R}^+)$ such that $\left|B(t,z,x)-B(t,z,x')\right|\leq k_1(t,z)\left|x-x'\right|$, a.e. in $T\times Z$ and $\forall\ x,\ x'\in\mathbb{R}$;

iii) there exist $\alpha\in L^2(T\times Z,\mathbb{R}^+)$ such that $|B(t,z,x)|\leq\alpha(t,z)$, a.e. in $T\times Z$, $\forall\ x\in\mathbb{R}$.

$H(g)$: $g : T\times L^2(Z,\mathbb{R})\to\mathbb{R}^+$ is a function such that

i) $(t,w)\mapsto g(t,w)$ is continuous;

ii) $\exists\ k_2\in L^\infty(T,\mathbb{R}^+)$ such that $|g(t,w)-g(t',w')|\leq k_2(t)\|w-w'\|_2$, a.e. in T and $\forall\ w,\ w'\in L^2(Z,\mathbb{R})$;

iii) there exist $M>0$ such that $g(t,w)\leq M$, $\forall\ (t,w)\in T\times L^2(Z,\mathbb{R})$.

THEOREM 6.1. *If hypotheses $H(a), H(B), H(g)$ hold, $v\in C(T_0, L^2(Z,\mathbb{R}))$, $v(0,\ .\ )\in W^{1,p}(Z,\mathbb{R})$ and $x\in C(\hat{T}, L^2(Z,\mathbb{R}))$ is a trajectory of (6), then for every $\varepsilon>0$ there exists $\hat{x}\in C(\hat{T}, L^2(Z,\mathbb{R}))$ another trajectory of (6), generated by a control $\hat{u}\in L^\infty(T,\times Z,\mathbb{R})$ such that*

$$\lambda\left\{t\in T : |\hat{u}(t,z)|\neq g\left(t,\hat{x}(t-r,\ .\ )\right),\ \text{a.e. on } Z\right\}=0$$

and

$$\sup\left\{\|x(t,\ .\ )-\hat{x}(t,\ .\ )\|_2 : t\in T\right\}<\varepsilon.$$

(here $\lambda(.)$ stands for the Lebesgue measure on T).

PROOF. Let $H=L^2(Z,\mathbb{R})$ and define $\varphi : T\times H\to\mathbb{R}\cup\{+\infty\}$ by

$$\varphi(t,x)=\begin{cases}\frac{1}{p}\sum\limits_{k=1}^{n}\int\limits_{Z}a(t,z)\left|\frac{\partial x}{\partial z}\right|^p dz+\frac{\beta}{p}\int\limits_{Z}|x(z)|^p dz, & \text{if } x\in W_0^{1,p}(Z,\mathbb{R}),\\ +\infty, & \text{otherwise.}\end{cases}$$

Clearly $x\mapsto\varphi(t,x)$ is proper, l.s.c. and convex and note that for every $\lambda\geq 0$ the level set

$$L_\lambda=\left\{x\in H : \|x\|_2^2+\varphi(t,x)\leq\lambda\right\}$$

is bounded in $W_0^{1,p}(Z,\mathbb{R})$. Since $W_0^{1,p}(Z,\mathbb{R})$ embeds compactly in $L^2(Z,\mathbb{R})$ (the Sobolev embedding theorem), we get that L_λ is compact in $L^2(Z,\mathbb{R})$; i.e. $\varphi(t,.)$ is of compact type. Using $H(a)$, we can easily check that there exist two positive constant $\hat{c}$ and $\hat{k}$ such that

$$\hat{c}\|x\|^p_{W_0^{1,p}} \leq \varphi(t,x), \quad \forall\, t \in T \quad \text{and} \quad \forall\, x \in W_0^{1,p}(Z,\mathbb{R}),$$

and for all $t,\, t' \in T$ and $x \in W_0^{1,p}(Z,\mathbb{R})$, we have

$$\left|\varphi(t,x) - \varphi(t',x)\right| \leq \hat{k}|t-t'|\|x\|^p_{W_0^{1,p}} \leq \frac{\hat{k}}{\hat{c}}|t-t'|\varphi(t',x).$$

So we have satisfied hypothesis $H(\varphi)$. Furthermore as in [1] using Green's theorem we can check that

$$\partial\varphi(t,x) = -\sum_{k=1} \frac{\partial}{\partial z_k}\left(a(t,z)\left|\frac{\partial x}{\partial z_k}\right|^{p-2}\frac{\partial x}{\partial z_k}\right) + \beta x|x|^{p-2} = L_p^\beta(x),$$

with $x \in D_p = \left\{y \in W_0^{1,p}(Z,\mathbb{R}) : L_p^\beta(y) \in L^2(Z,\mathbb{R})\right\}$.

Next let $Y = L^\infty(Z,\mathbb{R})$ (the controls space) and define $b : T \times C(T_0,H) \to \to \mathcal{L}(Y,H)$ by

$$(b(t,y)u)(\,.\,) = B(t,\,.\,,y(-r,\,.\,))u(\,.\,), \quad \text{for all} \quad u \in L^\infty(Z,\mathbb{R}).$$

Using hypothesis $H(B)$, we can check that,

$$\left\|\left(b(t,y) - b(t,y')\right)u\right\|_2^2 \leq \int_Z |B(t,z,y(-r,z)) - B(t,z,y'(-r,z))|^2\|u\|_\infty^2 dz,$$

which implies that

$$\|b(t,y) - b(t,y')\|_{\mathcal{L}} \leq \tilde{k}_1(t)\|y-y'\|_\infty, \quad \text{a.e. in } T,\ \forall\, y,\, y' \in C(T_0,H),$$

where $\tilde{k}_1(t) = \|k_1(t,.)\|_\infty$.

Moreover we have that $\|b(t,y)\|_{\mathcal{L}} \leq \|\alpha(t,.)\|_2$, a.e. in T, $\forall\, y \in C(T_0,H)$.

So we have satisfied hypothesis $H(b)_2$.

Finally let

$$U(t,y) = \{u \in L^\infty(Z,\mathbb{R}) = Y : \|u\|_\infty \leq g(t,y(-r,\,.\,))\},$$

for all $(t,y) \in T \times C(T_0,H)$. Using hypothesis $H(g)$, we can check that $H(U)_2'$ is valid.

So the problem (6) can be equivalently rewritten as the problem (3). Applying theorem 5.3, we get that for every trajectory of (6), $x \in C(\hat{T}, L^2(Z,\mathbb{R}))$

and for every $\varepsilon > 0$ there exists another trajectory of (6) $\hat{x} \in C(\hat{T}, L^2(Z,\mathbb{R}))$, generated by a control $\hat{u} \in L^\infty(T \times Z, \mathbb{R})$ such that $\hat{u}(t) \in \operatorname{ext} U(t, \hat{x}_t)$, a.e. on T and $\sup\{\|x(t,\,.\,) - \hat{x}(t,\,.\,)\|_2 : t \in T\} < \varepsilon$. From [6], example 1, p. 79, we know that

$$\operatorname{ext} U(t,y) = \{u \in L^\infty(Z,\mathbb{R}) : |u(t,z)| = g(t, y(t-r,\,.\,)), \quad \text{a.e. on } Z\},$$

then

$$\lambda\,\{t \in T : |\hat{u}(t,z)| \neq g\,(t, \hat{x}(t-r,\,.\,)), \quad \text{a.e. on } Z.\} = 0.$$

So if we have to minimize the terminal cost function

$$\overline{\eta} : C(\hat{T}, L^2(Z,\mathbb{R})) \to \mathbb{R}, \qquad \overline{\eta}(x) = \int_Z \eta(z, x(b,z))dz,$$

over the set of solutions of the problem (6), with $\eta : Z \times \mathbb{R} \to \mathbb{R}$ being a continuous, bounded integrand, then we can always find a trajectory generated by a “bang-bang” control, which is ε-optimal.

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PROPERTIES OF SEPARABILITY ON SPACES OF SUBSETS

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1. Terminology and notations

For a topological space $(X,\mathcal{T})$ we denote

$$\mathcal{P}_0(X)=\{A\subset X : A\neq\emptyset\};$$
$$\mathcal{F}_n(X)=\{A\in\mathcal{P}_0(X) : \text{card } A\leq n\},\ n\in\mathbb{N}^*;$$
$$\mathcal{F}(X)=\bigcup_{n\in\mathbb{N}^*}\mathcal{F}_n(X).$$

For a family $\{G_i\}_{i\in I}$ of subsets of X we denote

$$\left[\{G_i\}_{i\in I}\right]=\{A\in\mathcal{P}_0(X) : A\cap G_i\neq\emptyset,\ i\in I\}$$

and

$$\langle\{G_i\}_{i\in I}\rangle=\left\{A\in\mathcal{P}_0(X) : A\subset\bigcup_{i\in I}G_i,\ A\cap G_i\neq\emptyset,\ i\in I\right\}.$$

DEFINITION 1. Let $(X,\mathcal{T})$ be a topological space.

a) The topology $\mathcal{T}_l^{\uparrow}$ generated by the subbasis $\mathcal{S}_l=\{[\{G\}]:G\in\mathcal{T}\}$ is called the lower semifinite topology on $\mathcal{P}_0(X)$;

b) The topology $\mathcal{T}_u^{\uparrow}$ generated by the basis $\mathcal{B}_u=\{\langle\{G\}\rangle:G\in\mathcal{T}\}$ is called the upper semifinite topology on $\mathcal{P}_0(X)$;

c) The topology $\mathcal{T}^{\uparrow}$ generated by the subbasis $\mathcal{S}_l\cup\mathcal{B}_u$ is called the finite topology on $\mathcal{P}_0(X)$.

2. Relationships between the density properties of X and $\mathcal{P}_0(X)$

In the following we say that a set A is dense with respect to another set B in a topological space $(X,\mathcal{T})$ if $\overline{A} \supset B$, where $\overline{A}$ is the closure of A.

THEOREM 1. *Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{A}$ and $\mathcal{B}$ be two families of nonempty subsets of X. If $\mathcal{B}$ is dense with respect to $\mathcal{A}$ in the topology $\mathcal{T}_l^{\uparrow}$ on $\mathcal{P}_0(X)$, then $\bigcup_{B\in\mathcal{B}} B$ is dense with respect to $\bigcup_{A\in\mathcal{A}} A$.*

PROOF. Let $x \in \bigcup_{A\in\mathcal{A}} A$. Then there exists an $A \in \mathcal{A}$ such that $x \in A$. Let $G \in \mathcal{T}$ with $x \in G$. It results that $A \in [\{G\}]$, whence $[\{G\}] \cap \mathcal{B} \neq \emptyset$. Therefore there exists $B \in \mathcal{B}$ such that $B \in [\{G\}]$. From here it follows $B \cap G \neq \emptyset$, hence $x \in \overline{\bigcup_{B\in\mathcal{B}} B}$. Since x is arbitrarily chosen in $\bigcup_{A\in\mathcal{A}} A$ it results that $\bigcup_{A\in\mathcal{A}} A \subset \overline{\bigcup_{B\in\mathcal{B}} B}$, which is what we wanted to prove.

COROLLARY. *Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{A}$ be a family of countable subsets of X. If $\mathcal{A}$ endowed with the relativized topology of $\mathcal{T}_l^{\uparrow}$ is separable, then $\bigcup_{A\in\mathcal{A}} A$ is separable in the relativized topology of $\mathcal{T}$.*

PROOF. By hypothesis there exists a countable set $\mathcal{B} \subset \mathcal{A}$ such that $\overline{\mathcal{B}} = \mathcal{A}$ in the topology $\mathcal{T}_l^{\uparrow}/\mathcal{A}$. Let $D = \bigcup_{B\in\mathcal{B}} B$. This subset of X is countable as a countable union of countable sets. By the previous theorem it results $\overline{D} = \bigcup_{A\in\mathcal{A}} A$ is separable in the relativized topology of $\mathcal{T}$.

THEOREM 2. *Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{A}$ be a family of separable subsets of X. If $\mathcal{A}$ is separable in the relativized topology of $\mathcal{T}_l^{\uparrow}$, then $\bigcup_{A\in\mathcal{A}} A$ is separable in the relativized topology of $\mathcal{T}$.*

PROOF. Let $E = \bigcup_{A\in\mathcal{A}} A$. Since $\mathcal{A}$ is separable in the relativized topology of $\mathcal{T}_l^{\uparrow}$ it results that there exists a countable family $\mathcal{B} \subset \mathcal{A}$ such that $\overline{\mathcal{B}} = \mathcal{A}$ in the relativized topology of $\mathcal{T}_l^{\uparrow}$. Taking into account Theorem 1 it results $\overline{\bigcup_{B\in\mathcal{B}} B} = E$ in the relativized topology of $\mathcal{T}$. Since $\mathcal{B} \subset \mathcal{A}$, it follows that $B \in \mathcal{B}$ is separable in the relativized topology of $\mathcal{T}$. Hence there exists a countable set $C_B \subset B$ such that $\overline{C_B} = B$ in the relativized topology of $\mathcal{T}$. We will show that $E = \overline{\bigcup_{B\in\mathcal{B}} C_B}$. Let $x \in E$. Then, for every $G \in \mathcal{T}$ with $x \in G$, the relation $G \cap \left(\bigcup_{B\in\mathcal{B}} B \right) \neq \emptyset$ holds. Hence there is $B \in \mathcal{B}$ such that $G \cap B \neq \emptyset$. Since $\overline{C_B} = B$ it results that $G \cap C_B \neq \emptyset$, that is $G \cap \left(\bigcup_{B\in\mathcal{B}} C_B \right) \neq \emptyset$. Therefore $x \in \overline{\bigcup_{B\in\mathcal{B}} C_B}$.

Since x is arbitrarily chosen from E it follows that $E \subset \overline{\bigcup_{B\in\mathcal{B}} C_B} \subset \overline{\bigcup_{B\in\mathcal{B}} B}$. From here it results $E = \overline{\bigcup_{B\in\mathcal{B}} C_B}$ and the theorem follows.

REMARK. Theorems 1 and 2 remain valid if on $\mathcal{P}_0(X)$ we consider the finite topology $\mathcal{T}^{\uparrow}$.

THEOREM 3. *Let $(X,\mathcal{T})$ be a topological space.*

a) *If $(X,\mathcal{T})$ is separable, then any family $\mathcal{Q}(X)$ of nonempty subsets of X with the property $\mathcal{F}(X) \subset \mathcal{Q}(X)$ is separable in the relativized topology of $\mathcal{T}^{\uparrow}$;*

b) *If $\mathcal{Q}(X) \subset \mathcal{P}_0(X)$ is separable in the relativized topology of $\mathcal{T}_u^{\uparrow}$ and $\mathcal{F}_1(X) \subset \mathcal{Q}(X)$, then the space $(X,\mathcal{T})$ is separable.*

PROOF. a) By hypothesis there exists a countable set $B = \{x_1, x_2, \ldots, x_n, \ldots$ $\ldots\} \subset X$ such that $\overline{B} = X$. Since $\mathcal{F}(X) \subset \mathcal{Q}(X)$ it results that $\mathcal{F}(B) \subset \mathcal{Q}(X)$. We will show that $\overline{\mathcal{F}(B)} = \mathcal{Q}(X)$ in the topology $\mathcal{T}^{\uparrow}$ relativized to $\mathcal{Q}(X)$. Let $A \in \mathcal{Q}(X)$ and let $\mathcal{G}$ be an open neighbourhood of A in the topology $\mathcal{T}^{\uparrow}$ relativized to $\mathcal{Q}(X)$. Then, there exist $G_1, G_2, \ldots, G_k \in \mathcal{T}$ such that $A \in \left\langle \{G_i\}_{i\in\overline{1,k}} \right\rangle \cap \mathcal{Q}(X) \subset \mathcal{G}$.

Since $\overline{B} = X$ it results that $B \cap G_i \neq \emptyset$ for every $i \in \overline{1,k}$. Let $y_i \in B \cap G_i$, $i \in \overline{1,k}$. Then the set $B_k = \{y_1, y_2, \ldots, y_k\} \in \mathcal{F}(B)$ and $B_k \subset \bigcup_{i=1}^{k} G_i$, $B_k \cap G_i \neq \emptyset$, $\forall i \in \overline{i,k}$. Therefore $\mathcal{G} \cap \mathcal{F} \neq \emptyset$, that is $\mathcal{F}(B)$ is dense in $\mathcal{Q}(X)$ endowed with the topology $\mathcal{T}^{\uparrow}$ relativized to $\mathcal{Q}(X)$.

b) By hypothesis there exists a countable family $\mathcal{A} = \{A_1, A_2, \ldots, A_n, \ldots$ $\ldots\} \subset \mathcal{Q}(X)$ such that $\overline{\mathcal{A}} = \mathcal{Q}(X)$ in the topology $\mathcal{T}_u^{\uparrow}$ relativized to $\mathcal{Q}(X)$. Let $B = \{x_1, x_2, \ldots, x_n, \ldots\}$ such that $x_i \in A_i$, $i \in \mathbb{N}^*$. We will show that B is dense in X. Let $x \in X$ and $G \in \mathcal{T}$ with $x \in G$. Then $\{x\} \in \langle\{G\}\rangle \cap \mathcal{Q}(X)$ and $\langle\{G\}\rangle \cap \mathcal{Q}(X) \cap \mathcal{A} \neq \emptyset$. Hence there exists $j \in \mathbb{N}^*$ such that $A_j \subset G$. Then it follows that $x_j \in G$, whence $G \cap B \neq \emptyset$. Therefore $x \in \overline{B}$. Since x is arbitrarily chosen from X it results $X = \overline{B}$.

3. Applications

In this section we will utilise the above results to prove the invariance of density properties by lower semicontinuous multifunctions [2], [3].

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces and let $F : X \multimap Y$ be a multifunction onto.

DEFINITION 2. The multifunction F is said to have the pointwise property "P" if $F(x)$ has the property "P" for all $x \in X$.

For a multifunction $F : X \multimap Y$ by $F(A)$, $A \subset X$ we mean the set $\bigcup_{x \in A} F(x)$.

To a multifunction $F : X \multimap Y$ we attach the function $\mathbb{F} : X \to \mathcal{P}_0(Y)$, defined by $\mathbb{F}(x) = F(x)$, $x \in X$.

It is known [1] that the multifunction $F : (X, \mathcal{T}) \multimap (Y, \mathcal{U})$ is lower semicontinuous (*l.s.c.*) iff the function $\mathbb{F} : (X, \mathcal{T}) \to (\mathcal{P}_0(X), \mathcal{U}_l^{\uparrow})$ is continuous.

PROPOSITION 1. *If $A \subset X$ is dense in $(X, \mathcal{T})$ and if $F : (X, \mathcal{T}) \multimap (Y, \mathcal{U})$ is a surjective lower semicontinuous multifunction, then the set $F(A)$ is dense in $(Y, \mathcal{U})$.*

PROOF. If F is l.s.c., then the function $\mathbb{F} : (X, \mathcal{T}) \to (\mathcal{P}_0(Y), \mathcal{U}_l^{\uparrow})$ is continuous. Since $\overline{A} = X$ it results that $\mathbb{F}(A)$ is dense with respect to $\mathbb{F}(X) \subset \mathcal{P}_0(Y)$. Taking $\mathcal{A} = \mathbb{F}(X)$ and $\mathcal{B} = \mathbb{F}(A)$, by Theorem 1, it follows $\overline{\bigcup_{x \in A} F(x)} = \bigcup_{x \in X} F(x) = Y$.

COROLLARY. *If the topological space $(X, \mathcal{T})$ is separable and the surjective multifunction $F : (X, \mathcal{T}) \multimap (Y, \mathcal{U})$ is l.s.c. and pointwise countable, then the space $(Y, \mathcal{U})$ is separable.*

PROPOSITION 2. *If the topological space $(X, \mathcal{T})$ is separable and $F : (X, \mathcal{T}) \multimap (Y, \mathcal{U})$ is a surjective l.s.c. multifunction and pointwise separable, then the space $(Y, \mathcal{U})$ is separable.*

PROOF. Since F is l.s.c. it results $\mathbb{F}:(X,\mathcal{J})\to(\mathcal{P}_0(Y),\mathcal{U}_l^{\uparrow})$ is continuous. Then, X being separable, it follows $\mathbb{F}(X)$ is separable in the topology $\mathcal{U}_l^{\uparrow}$ relativized to $\mathbb{F}(X)$. Now, applying Theorem 2, it results that $\bigcup_{x\in X} F(x)=Y$ is separable in the topology $\mathcal{U}$.

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ON THE LINDELÖF PROPERTY OF SPACES OF SUBSETS

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1. Preliminaries

Let $(X,\mathcal{T})$ be a topological space. $\mathcal{P}_0(X)$ will denote the family of non-empty subsets of X, and $\mathcal{K}(X)$ will denote the family of non-empty compact subsets of X. By $\mathcal{F}_1(X)$ we will denote the family of singleton subsets of X.

We also use for the definitions of the subbase (base) of the lower semifinite (upper semifinite, finite) topology on $\mathcal{P}_0(X)$ the following notations:

$$\left[\{G_i\}_{i\in I}\right] = \{A \in \mathcal{P}_0(X) : A \cap G_i \neq \emptyset,\ i \in I\}.$$

$$\langle\{G_i\}_{i\in I}\rangle = \left\{A \in \mathcal{P}_0(X) : A \subset \bigcup_{i\in I} G_i,\ A \cap G_i \neq \emptyset,\ i \in I\right\}.$$

DEFINITION 1. ([1], [4]) If $(X,\mathcal{T})$ is a topological space, then:

a) The topology $\mathcal{T}_l^{\uparrow}$ generated by the subbasis

$$\mathcal{S}_l = \{[\{G\}] : G \in \mathcal{T}\}$$

is called the lower semifinite topology on $\mathcal{P}_0(X)$;

b) The topology $\mathcal{T}_u^{\uparrow}$ generated by the basis

$$\mathcal{B}_u = \{\langle\{G\}\rangle : G \in \mathcal{T}\}$$

is called the upper semifinite topology on $\mathcal{P}_0(X)$;

c) The topology $\mathcal{T}^{\uparrow}$ generated by the subbasis $\mathcal{S}_l \cup \mathcal{B}_u$ is called the finite topology on $\mathcal{P}_0(X)$.

2. On the Lindelöf property

THEOREM 1. *Let* $(X,\mathcal{T})$ *be a topological space and let* $\mathcal{A}$ *be a family of non-empty Lindelöf subsets of* X. *If* $\mathcal{A}$ *has the Lindelöf property on the topological space* $(\mathcal{P}_0(X),\mathcal{T}_u^{\uparrow})$, *then the union of the members of the family* $\mathcal{A}$ *has the Lindelöf property in* $(X,\mathcal{T})$.

PROOF. Let $B=\bigcup\limits_{A\in\mathcal{A}} A$ and let $\{G_i\}_{i=I}$ be an open cover of B. Then, for any member $A\in\mathcal{A}$ we have $A\subset\bigcup\limits_{i\in I} G_i$.

Since A has the Lindelöf property, it results that there exists a countable subcover $\left\{G_{i_j}^A\right\}_{j\in\mathbb{N}}$ of the cover $\{G_i\}_{i\in I}$ which covers A. Then $\mathcal{U}_A =$ $= \left\langle\left\{\bigcup\limits_{j\in\mathbb{N}} G_{i_j}^A\right\}\right\rangle$ is an open neighbourhood of A in $(\mathcal{P}_0(X),\mathcal{T}_u^{\uparrow})$. Hence the family $\{\mathcal{U}_A\}_{A\in\mathcal{A}}$ is an open cover of $\mathcal{A}$ in $(\mathcal{P}_0(X),\mathcal{T}_u^{\uparrow})$. Since $\mathcal{A}$ has the Lindelöf property, it results that there exists a countable subcover $\left\{\mathcal{U}_{A_k}\right\}_{k\in\mathbb{N}}$ of the cover $\{\mathcal{U}_A\}_{A\in\mathcal{A}}$ which covers $\mathcal{A}$.

Then the family $\left\{G_{i_j}^{A_k} : j\in\mathbb{N},\ k\in\mathbb{N}\right\}$ is a countable subcover of the given cover of B. Hence B has the Lindelöf property in $(X,\mathcal{T})$.

CONSEQUENCE 1. *If* $\mathcal{A}$ *is a Lindelöf subspace of the topological space* $(\mathcal{P}_0(X),\mathcal{T}_u^{\uparrow})$ *whose elements are open hereditarily Lindelöf subsets of* X, *then the union of the members of* $\mathcal{A}$ *is a hereditarily Lindelöf subspace of* $(X,\mathcal{T})$.

PROOF. Let $B=\bigcup\limits_{A\in\mathcal{A}} A$. By the previous theorem it results that B is a Lindelöf set. Hence, there exists a countable subcover of B, $\{A_i\}_{i\in\mathbb{N}}$, $A_i\in\mathcal{A}$. Let D be an arbitrary subset of B, and let $D_i = D\cap A_i$, $i\in\mathbb{N}$. Since A_i is a hereditarily Lindelöf set for any $i\in\mathbb{N}$, it results that D_i is a Lindelöf subset of B. Hence D is a Lindelöf set as countable union of the Lindelöf sets D_i, $i\in\mathbb{N}$.

REMARK 1. The previous statements hold if instead of the topology $\mathcal{T}_u^{\uparrow}$ we take a finite topology $\mathcal{T}^{\uparrow}$.

THEOREM 2. *Let* $(X,\mathcal{T})$ *be a topological space and let* $\mathcal{F}_1(X)\subset Q(X)\subset$ $\subset\mathcal{P}_0(X)$.

The space $(X,\mathcal{T})$ *is Lindelöf if and only if* $Q(X)$ *is a Lindelöf space in the relativized topology of* $\mathcal{T}_l^{\uparrow}$.

PROOF. Suppose that $(X,\mathcal{T})$ is a Lindelöf topological space and let $\Omega = \{\mathcal{G}_i\}_{i\in I}$ be an open cover of $Q(X)$ in the relativized topology of $\mathcal{T}_l^{\uparrow}$. Then, for any $x \in X$ there exists $\mathcal{G}_{i_x} \in \Omega$ such that $\{x\} \in \mathcal{G}_{i_x} \in \mathcal{T}_l^{\uparrow}\big|_{Q(X)}$. Hence there exists $G_1^x, G_2^x, \ldots, G_k^x \in \mathcal{T}$ such that

$$\{x\} \in \left[\{G_1^x\}\right] \cap \left[\{G_2^x\}\right] \cap \ldots \cap \left[\{G_k^x\}\right] \cap Q(X) \subset \mathcal{G}_{i_x}.$$

Then it results that $x \in G_i^x$ for every $i \in \overline{1,k}$, that is $x \in \bigcap_{i=1}^{k} G_i^x = G^x$.

It is evident that $\{x\} \in \left[\{G^x\}\right] \cap Q(x) \subset \mathcal{G}_{i_x}$, thus $\{G^x\}_{x\in X}$ is a cover of X. Since $(X,\mathcal{T})$ is a Lindelöf space, it results that there exists a countable subcover of the cover $\{G^x\}_{x\in X}$ which covers X. Hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $X = \bigcup_{n\in\mathbb{N}} G^{x_n}$. We will show that $\left\{\mathcal{G}_{i_{x_n}}\right\}_{n\in\mathbb{N}} \subset \{\mathcal{G}_i\}_{i\in I}$ is a countable cover of $Q(X)$.

Let $A \in Q(X)$. Then $A \subset \bigcup_{n\in\mathbb{N}} G^{x_n}$. Taking $J = \{i \in \mathbb{N} : A \cap G^{x_i} \neq \emptyset\}$, we have $A \subset \bigcup_{j\in J} G^{x_j}$, whence $A \in \left[\left\{\bigcup_{j\in I} G^{x_j}\right\}\right] = \bigcup_{j\in J} \left[\{G^{x_j}\}\right]$. Therefore there is $j \in J$ such that $A \in \left[\left\{G^{x_j}\right\}\right] \cap Q(X) \subset \mathcal{G}_{i_{x_j}}$. From here it results that $\left\{\mathcal{G}_{i_{x_j}}\right\}_{j\in\mathbb{N}}$ is a cover of $Q(X)$, that is $Q(X)$ is a Lindelöf topological space with the relativized topology of $\mathcal{T}_l^{\uparrow}$.

CONVERSE. Suppose that $Q(X)$ is a Lindelöf topological space with the relativized topology of $\mathcal{T}_l^{\uparrow}$. Let $\{G_i\}_{i\in I}$ be an arbitrary open cover of X in the topology $\mathcal{T}$, that is $X = \bigcup_{i\in I} G_i$. Then $Q(X) \subset \left[\left\{\bigcup_{i\in I} G_i\right\}\right] = \bigcup_{i\in I} \left[\{G_i\}\right]$. Since $Q(X)$ is a Lindelöf space in the relativized topology of $\mathcal{T}_l^{\uparrow}$, it results that there exists $(i_n)_{n\in\mathbb{N}} \subset I$ such that $Q(X) \subset \left[\left\{\bigcup_{n\in\mathbb{N}} G_{i_n}\right\}\right]$. From $\mathcal{F}_1(X) \subset Q(X)$ it results that $\mathcal{F}_1(X) \subset \left[\left\{\bigcup_{n\in\mathbb{N}} G_{i_n}\right\}\right] = \bigcup_{n\in\mathbb{N}} \left[\{G_{i_n}\}\right]$. Therefore for every $x \in X$, there exists $k \in \mathbb{N}$ such that $\{x\} \subset G_{i_k}$, that is $X \subset \bigcup_{n\in\mathbb{N}} G_{i_n}$, what means that $(X,\mathcal{T})$ is a Lindelöf space.

THEOREM 3. *The topological space* $(X,\mathcal{T})$ *is a hereditarily Lindelöf space if and only if every subspace* $Q(X)$ *of* $(\mathcal{P}_0(X),\mathcal{T}_l^{\uparrow})$ *with the property*

$$\mathcal{F}_1\left(\bigcup_{A\in Q(X)} A\right) \subset Q(X)$$

is Lindelöf.

PROOF. Suppose $(X,\mathcal{T})$ is a hereditarily Lindelöf space. It results that $B = \bigcup_{A\in Q(X)} A \subset X$, is a Lindelöf subspace of X.

Since $\mathcal{F}_1(B) \subset Q(X) \subset \mathcal{P}_0(B)$, by Theorem 2 it results $Q(X)$ is a Lindelöf subspace of $(\mathcal{P}_0(X),\mathcal{T}_l^{\uparrow})$.

CONVERSE. Suppose that with subspace $Q(X)$ of $(\mathcal{P}_0(X),\mathcal{T}_l^{\uparrow})$, with the property $\mathcal{F}_1\left(\bigcup_{A\in Q(X)} A\right) \subset Q(X)$, is Lindelöf. Let $B \subset X$, and $Q(X) = \mathcal{F}_1(B)$. Then $\mathcal{F}_1(B)$ is a Lindelöf subspace of $(\mathcal{P}_0(X),\mathcal{T}_l^{\uparrow})$, whence B is a Lindelöf subspace of $(X,\mathcal{T})$. Therefore $(X,\mathcal{T})$ is a hereditarily Lindelöf space.

3. Applications

In the sequel we will prove that the Lindelöf and hereditarily Lindelöf properties are preserved, under certain conditions, by an upper semicontinuous (continuous) multifunction. To facilitate our discussion we review the notations and the definitions from multifunction theory [4], which will be used in the following.

A multifunction on X into Y will be denoted by $F : X \multimap Y$.

The image of $A \subset X$ by F is the subset $F(A) \subset Y$, defined as $F(A) = \bigcup_{x\in A} F(x)$.

To a multifunction $F : X \multimap Y$, the corresponding function $\mathbb{F} : X \to P_0(Y)$ is defined by $\mathbb{F}(x) = F(x)$, $x \in X$.

DEFINITION 2. Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces. The multifunction $F: X \multimap Y$ is said to be pointwise Lindelöf (hereditarily Lindelöf) iff for all $x \in X$, $F(x)$ is a Lindelöf (hereditarily Lindelöf) subspace of $(Y,\mathcal{U})$.

It is well known [1] that a multifunction $F:(X,\mathcal{T}) \multimap (Y,\mathcal{U})$ is upper semicontinuous (u.s.c.) iff the function $\mathbb{F}:(X,\mathcal{T}) \to (\mathcal{P}_0(Y),\mathcal{U}_u^{\uparrow})$ is continuous.

PROPOSITION 1. *Let* $F:(X,\mathcal{T}) \multimap (Y,\mathcal{U})$ *be an upper semicontinuous and pointwise Lindelöf multifunction. If* $A \subset X$ *is a Lindelöf subspace of* $(X,\mathcal{T})$, *then* $F(A)$ *is a Lindelöf subspace of* $(Y,\mathcal{U})$.

PROOF. Since F is u.s.c., it results that $\mathbb{F} : (X,\mathcal{T}) \to (\mathcal{P}_0(Y),\mathcal{U}_u^{\uparrow})$ is continuous.

By hypothesis A being Lindelöf it follows $\mathbb{F}(A)$ is Lindelöf in $(\mathcal{P}_0(Y),\mathcal{U}_u^{\uparrow})$. Then, taking into account Theorem 1 and the fact that $F(x)$ is Lindelöf for all $x \in A$, it follows that $F(A) = \bigcup_{F(x)\in\mathbb{F}(A)} F(x)$ is a Lindelöf subspace of $(Y,\mathcal{U})$.

PROPOSITION 2. *If* $F:(X,\mathcal{T}) \multimap (Y,\mathcal{U})$ *is an upper semicontinuous and pointwise open hereditarily Lindelöf multifunction, and* $A \subset X$ *is a Lindelöf subspace of* $(X,\mathcal{T})$, *then* $F(A)$ *is a hereditarily Lindelöf subspace of* $(Y,\mathcal{U})$.

PROOF. F is u.s.c. iff $\mathbb{F} : (X,\mathcal{T}) \to (\mathcal{P}_0(Y),\mathcal{U}_u^{\uparrow})$ is continuous. Since $F(A) = \bigcup_{F(x)\in\mathbb{F}(A)} F(x)$, where $F(x)$ is hereditarily Lindelöf in $(Y,\mathcal{U})$, and $\mathbb{F}(A)$ is Lindelöf in $(\mathcal{P}_0(Y),\mathcal{U}_u^{\uparrow})$, by Consequence 1, it results that $F(A)$ is a hereditarily Lindelöf subspace of $(Y,\mathcal{U})$.

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A SUP-INF CONDITION FOR THE EXISTENCE OF ZEROS OF CERTAIN NONLINEAR OPERATORS[1]

By

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1. Introduction

In this paper we deal with the following problem: given a compact topological space X, a real normed space E with topological dual E^* and an operator $A : X \to E^*$, find $x^* \in X$ such that $A(x^*) = 0_{E^*}$, where 0_{E^*} denotes the origin of the space E^*. By using a new, unconventional approach, we derive a necessary and sufficient condition for the existence of solutions. As a consequence, we also obtain some sufficient conditions under which the above problem admits a solution.

2. Results

Before giving our results, we briefly recall some definitions. If S, V are topological spaces, a multifunction $F : S \to 2^V$ is said to be upper semicontinuous if the set $\{s \in S : F(s) \cap \Omega \neq \emptyset\}$ is closed in S for each closed set $\Omega \subseteq V$. When $F(s) = \{\varphi(s)\}$, where $\varphi : S \to V$ is a single-valued map, the above definition reduces to the ordinary continuity for the function φ. If F is upper semicontinuous with nonempty closed values, then the graph of F (namely, the set $\{(s,v) \in S \times V : v \in F(s)\}$) is closed. When $\psi : S \to \mathbb{R}$ is a single-valued function, we say that ψ is lower (resp. upper) semicontinuous on S if for each $r \in \mathbb{R}$ the set $\{s \in S : \psi(s) \leq r\}$ (resp. $\{s \in S : \psi(s) \geq r\}$) is closed.

Our main result is the following.

[1] This research was presented during the workshop "Equilibrium problems with side constraints. Lagrangian theory and duality", held in Catania, Italy, on December 9–10, 1994.

THEOREM 1. *Let X be a topological space, E a real normed space, with* $\dim(E) \geq 2$, $r > 0$, $Y = \{y \in E : \|y\| = r\}$, $A : X \to E^*$ *a weakly-star continuous operator.*

Then, one has

$$\sup_{(u,\alpha,\beta) \in C^0([0,1],X) \times \Lambda} \inf_{y \in Y} \inf_{t \in [\alpha(y),\beta(y)]} \langle A(u(t)), y \rangle \leq 0,$$

where $\Lambda = \{(\alpha,\beta)/\alpha : Y \to [0,1]$ *is l.s.c.,* $\beta : Y \to [0,1]$ *is u.s.c. and* $\alpha(y) \leq \beta(y)\ \forall y \in Y\}$. *When X is compact,* $A^{-1}(0_{E^*})$ *is nonempty if (and only if) one has*

$$\sup_{(u,\alpha,\beta) \in C^0([0,1],X) \times \Lambda} \inf_{y \in Y} \inf_{t \in [\alpha(y),\beta(y)]} \langle A(u(t)), y \rangle = 0.$$

PROOF. Arguing by contradiction, assume that

$$\sup_{(u,\alpha,\beta) \in C^0([0,1],X) \times \Lambda} \inf_{y \in Y} \inf_{t \in [\alpha(y),\beta(y)]} \langle A(u(t)), y \rangle > 0.$$

Hence, there are $(u,\alpha,\beta) \in C^0([0,1],X) \times \Lambda$ and $\eta > 0$ such that

$$\inf_{t \in [\alpha(y),\beta(y)]} \langle A(u(t)), y \rangle \geq \eta$$

for all $y \in Y$. Now, put

$$S = \{(y,t) \in Y \times [0,1] : \langle A(u(t)), y \rangle < \eta\}.$$

Observe what follows:

(a) the projection of S on $[0,1]$ is equal to $[0,1]$;

(b) for each $y \in Y$, the set $\{t \in [0,1] : \langle A(u(t)), y \rangle < \eta\}$ is open in $[0,1]$ since the function $t \to \langle A(u(t)), y \rangle$ is continuous;

(c) since $\dim(E) \geq 2$, for each $t \in [0,1]$ the set $\{y \in Y : \langle A(u(t)), y \rangle < \eta\}$ is connected (see [1]).

Now, let $\Psi : Y \to 2^{[0,1]}$ be the multifunction defined by setting $\Psi(y) = [\alpha(y), \beta(y)]$. It is not difficult to realize that Ψ is upper semicontinuous with nonempty closed values, hence its graph is closed. Then, by Theorem 2.5-(δ_2) of [2], the graph of Ψ must intersect S, that is to say

$$\langle A(u(\hat{t})), \hat{y} \rangle < \eta$$

for some $\hat{y} \in Y$ and $\hat{t} \in \Psi(\hat{y})$, a contradiction. So, the first part of our thesis is proved. Now, assume that X is compact and that

$$\sup_{(u,\alpha,\beta) \in C^0([0,1],X) \times \Lambda} \inf_{y \in Y} \inf_{t \in [\alpha(y),\beta(y)]} \langle A(u(t)), y \rangle = 0.$$

Again arguing by contradiction, let $A(x) \neq 0_{E^*}$ for all $x \in X$. Since the function $T \to \|T\|_{E^*}$ is weakly-star lower semicontinuous, the function $x \to \|A(x)\|_{E^*}$ is lower semicontinuous in X. By assumption, this latter function is everywhere positive in X which is compact. So, if $\gamma = \inf_{x\in X} \|A(x)\|_{E^*}$, we have $\gamma > 0$. Fix $\epsilon \in]0,\gamma[$. Furthermore, choose $(u,\alpha,\beta) \in C^0([0,1],X)\times \Lambda$ in such a way that

$$\inf_{t\in[\alpha(y),\beta(y)]} \langle A(u(t)), y\rangle \geq -r\epsilon$$

for all $y \in Y$. Now, we get a contradiction proceeding exactly as in the first part of the proof, the role of η being assumed by $-r\epsilon$. ∎

REMARK. We point out that the compactness assumption in the second part of Theorem 1 is essential. To see this, let $g:\mathbb{R}\to\mathbb{R}$ be defined by

$$g(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq 1 \\ \frac{1}{\lambda} & \text{if } |\lambda| > 1, \end{cases}$$

and let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $r = 1$, $A : \mathbb{R} \to \mathbb{R}^2$ the operator defined by setting $A(\lambda) = (g(\lambda), 0)$ for each $\lambda \in \mathbb{R}$. Let $\epsilon > 0$ be fixed, and let $\tilde{\lambda} \in \mathbb{R}$ be such that $|g(\tilde{\lambda})| < \epsilon$. Let $\tilde{u} \in C^0([0,1],\mathbb{R})$ be defined by putting $\tilde{u}(t) = \tilde{\lambda}$ for each $t \in [0,1]$ and let $(\alpha,\beta) \in \Lambda$ be arbitrary. We get

$$\inf_{y\in Y} \inf_{t\in[\alpha(y),\beta(y)]} \langle A(\tilde{u}(t)), y\rangle = \inf_{y\in Y} \langle A(\tilde{\lambda}), y\rangle = -|g(\tilde{\lambda})| > -\epsilon,$$

hence

$$\sup_{(u,\alpha,\beta)\in C^0([0,1],\mathbb{R})\times\Lambda} \inf_{y\in Y} \inf_{t\in[\alpha(y),\beta(y)]} \langle A(u(t)), y\rangle \geq 0.$$

By the first part of Theorem 1 we get

$$\sup_{(u,\alpha,\beta)\in C^0([0,1],\mathbb{R})\times\Lambda} \inf_{y\in Y} \inf_{t\in[\alpha(y),\beta(y)]} \langle A(u(t)), y\rangle = 0.$$

However, the operator A has no zeros.

The next result is an immediate corollary of Theorem 1.

THEOREM 2. *Let X be a topological space, E a real normed space, with* $\dim(E) \geq 2$, $r > 0$, $Y = \{y \in E : \|y\| = r\}$, $A : X \to E^*$ *a weakly-star continuous operator.*

Then, one has

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle \leq 0.$$

When X is compact, $A^{-1}(0_{E^*})$ is nonempty if (and only if) one has

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle = 0.$$

By Theorem 1 we can also obtain the following result.

THEOREM 3. *Let X be a compact topological space, E a real normed space, with $\dim(E)\geq 2$, $r>0$, $Y=\{y\in E: \|y\|=r\}$, $A: X\to E^*$ a weakly-star continuous operator. Assume that for each $\sigma<0$ there exist a continuous function $u_\sigma:[0,1]\to X$ and two sets A_σ, $B_\sigma\subseteq Y$, with A_σ open in the relative topology of Y, B_σ closed, and $A_\sigma\subseteq B_\sigma$, such that*

$$\text{(1)}\qquad \begin{cases} \langle A(u_\sigma(0)), y\rangle \geq \sigma & \text{for all } y\in Y\setminus B_\sigma \\ \langle A(u_\sigma(1)), y\rangle \geq \sigma & \text{for all } y\in A_\sigma \\ \langle A(u_\sigma(t)), y\rangle \geq \sigma & \text{for all } y\in B_\sigma\setminus A_\sigma,\ t\in[0,1]. \end{cases}$$

Then there exists $\hat{x}\in X$ such that $A(\hat{x})=0_{E^*}$.

PROOF. For each fixed $\sigma<0$, let $\alpha_\sigma, \beta_\sigma: Y\to[0,1]$ be the characteristic functions of the sets A_σ and B_σ, respectively. Of course, α_σ is lower semicontinuous, β_σ is upper semicontinuous, and $\alpha_\sigma(y)\leq\beta_\sigma(y)$ for all $y\in Y$. By (1), it is easy to realize that

$$\inf_{y\in Y}\ \inf_{t\in[\alpha_\sigma(y),\beta_\sigma(y)]} \langle A(u_\sigma(t)), y\rangle \geq \sigma,$$

hence we have

$$\sup_{(u,\alpha,\beta)\in C^0([0,1],X)\times\Lambda}\ \inf_{y\in Y}\ \inf_{t\in[\alpha(y),\beta(y)]} \langle A(u(t)), y\rangle \geq \sigma,$$

where Λ is defined as in the statement of Theorem 1. By the arbitrariness of $\sigma<0$ and Theorem 1 our claim follows. ∎

Now we present a consequence of Theorem 2, whose proof is based on a classical extension result for Lipschitzian maps.

THEOREM 4. *Let $A:[0,1]\to\mathbb{R}^n$ $(n\geq 2)$ be a continuous operator, $Y=\{y\in\mathbb{R}^n: \|y\|=1\}$. Assume that for each $\sigma<0$ there exists $L_\sigma>0$ such that, for each finite set $\{y_1,\ldots,y_k\}\subseteq Y$, there exists a set $\{t_1,\ldots,t_k\}\subseteq[0,1]$ such that*

$$\text{(2)}\qquad \begin{cases} \langle A(t_i), y_i\rangle \geq \sigma & \text{for each } i=1,\ldots,k \\ |t_i-t_j|\leq L_\sigma\|y_i-y_j\| & \text{for all } i,\ j=1,\ldots,k. \end{cases}$$

Then there exists $t^*\in[0,1]$ such that $A(t^*)=0_{\mathbb{R}^n}$.

PROOF. Fix $\sigma<0$. Let Σ be the space of all functions $\varphi: Y\to[0,1]$ that are Lipschitzian with constant L_σ, endowed with uniform convergence topology. By the Ascoli-Arzelà theorem the space Σ is compact. For each $y \in Y$, let $F_y = \{\varphi \in \Sigma : \langle A(\varphi(y)), y \rangle \geq \sigma\}$. The continuity of A implies that each set F_y is closed. Now, let $D = \{y_1, \dots, y_k\}$ be any finite subset of Y, and let t_1, ..., $t_k \in [0,1]$ satisfying (2). Let $g : D \to [0,1]$ be defined by setting $g(y_i) = t_i$ for each $i \in \{1, \dots, k\}$. By (2) we have

$$|g(y_i) - g(y_j)| \leq L_\sigma \|y_i - y_j\| \quad \text{for all} \qquad i,\, j \in \{1, \dots, k\},$$

hence g is Lipschitzian on D with constant L_σ. By a classical extension result (see [3]), there exists a function $\psi : Y \to [0,1]$ which is Lipschitzian on Y with the same constant L_σ such that $\psi\,|_D = g$. Therefore, $\psi \in \bigcap_{i=1}^{k} F_{y_i}$ and the family $\{F_y\}_{y \in Y}$ has the finite intersection property. Since Σ is compact, there exists $\tilde{\varphi} \in \bigcap_{y \in Y} F_y$. That is, $\tilde{\varphi} \in C^0(Y, [0,1])$ and

$$\inf_{y \in Y} \langle A(\tilde{\varphi}(y)), y \rangle \geq \sigma.$$

By Theorem 2 our claim follows. ∎

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A GENERALIZATION OF EULER'S φ-FUNCTION WITH RESPECT TO A SET OF POLYNOMIALS

By

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1. Introduction

The Euler function $\varphi(n)$ is defined as the number of integers $x \pmod n$ such that $(x,n)=1$. The well-known Jordan function $J_u(n)$ is a generalized Euler function defined as the number of ordered u-tuples $\langle x_1,x_2,\ldots,x_u\rangle \pmod n$ such that $((x_j),n)=1$, where $(x_j)=(x_1,x_2,\ldots,x_u)$, the gcd of $x_1,x_2,\ldots x_u$. E. COHEN [5], [6] defines the generalized Euler function $\varphi_k(n)$ as the number of integers $x \pmod{n^k}$ such that $(x,n^k)_k=1$, where $(a,b)_k$ denotes the greatest common k-th power divisor of a and b. For an arbitrary set S of positive integers E. COHEN [7] defines the generalized Euler function $\varphi_S(n)$ as the number of integers $x \pmod n$ such that $(x,n)\in S$. P. J. MCCARTHY [12] involves NARKIEWICZ'S [14] convolution in generalized Euler functions.

In [9] the second author of the present paper combines the above generalizations of Euler's function.

The following generalization of $\varphi(n)$ is due to P. KESAVA MENON [11], see also H. STEVENS [17] and J. CHIDAMBARASWAMY [2], [3]. For a polynomial f with integral coefficients let $\varphi_f(n)$ be the number of integers $x \pmod n$ such that $(f(x),n)=1$. P. G. GARCIA and S. LIGH [8] introduce another generalization of $\varphi(n)$, namely for an arithmetic progression $D(s,d,n)=\{s,s+ +d,s+2d,\ldots s+(n-1)d\}$, where $(s,d)=1$, let $\varphi(s,d,n)$ denote the number of elements x in $D(s,d,n)$ such that $(x,n)=1$. Observe that $\varphi(s,d,n)$ is a special case of the function $\varphi_f(n)$ for $f(x)=s+(x-1)d$.

In this paper we combine the generalization of [9] and the function $\varphi_f(n)$ as follows.

Let A be a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $A(n)$ consists entirely of divisors of n. The A-convolution of arithmetical functions f and g is defined as

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g(n/d).$$

W. Narkiewicz [14] defined an A-convolution to be regular if

(a) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function E, defined by $E(n) = 1$ for all $n \in \mathbb{N}$, ha an inverse μ_A with respect to the A-convolution and $\mu(p^a) \in \{-1, 0\}$ for every prime power p^a.

In this paper we consider regular A-convolutions, see also [13], Chapter 4, [16]. For example, the Dirichlet convolution D, where $D(n) = \{d \in \mathbb{N} : d \mid n\}$, and the unitary convolution U, where $U(n) = \{d \in \mathbb{N} : d \mid n, (d, n/d) = 1\}$, are regular.

For $k \in \mathbb{N}$, let $A_k(n) = \{d \in \mathbb{N} : d^k \in A(n^k)\}$. It is well-known that the A_k-convolution is regular whenever the A-convolution is regular. The symbol $(a,b)_{A,k}$ denotes the largest k-th power divisor of a which belongs to $A(b)$. Note that $(a,b)_{D,k} = (a,b)_k$.

Now let $\mathbf{n} = \langle n_1, n_2, \ldots, n_u \rangle$ be an ordered u-tuple of positive integers and let k be a positive integer. We say that the u-tuples $\langle x_1, x_2, \ldots, x_u \rangle$ and $\langle y_1, y_2, \ldots y_u \rangle$ are congruent (mod $\mathbf{n}^k$) if $x_i \equiv y_i \pmod{n_i^k}$ for every $i = 1$, 2, …, u. Let $F = \{f_1, f_2, \ldots, f_u\}$ be a set of polynomials with integral coefficients, S be an arbitrary set of positive integers, A be a regular convolution and $n \mid (n_1, n_2, \ldots n_u)$. We define the generalized Euler function $\varphi_{F,S,A,k}(\mathbf{n}, n)$ as the number of incongruent u-tuples $\langle x_1, x_2, \ldots, x_u \rangle$ (mod $\mathbf{n}^k$) such that $(((f_j(x_j)), n^k)_{A,k})^{1/k} \in S$. We give an arithmetical evaluation and an asymptotic formula for our new generalization of Euler's function. In the asymptotic formula we confine ourselves to a special case of Narkiewicz's regular convolution, called *cross-convolution*, including the Dirichlet convolution and the unitary convolution. The method we use here is elementary, it is described in detail and applied for various types of arithmetical functions in [21] and [22].

For special cases of our results we refer to the papers given in the bibliography and to the book of P. J. McCarthy [13].

2. Preliminaries

If A is a regular convolution, then for every prime power p^a $(a \geq 1)$ there exists a positive integer $t = t_A(p^a)$, called the type of p^a with respect to A, such that $A(p^a) = \{1, p^t, p^{2t}, \ldots, p^{st}\}$, $st = a$ and $p^t \in A(p^{2t})$, $p^{2t} \in A(p^{3t})$, $\ldots$, $p^{(s-1)t} \in A(p^a)$.

A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. It follows that the Möbius-type function μ_A is multiplicative and for all prime powers p^a $(a \geq 1)$,

$$\mu_A(p^a) = \begin{cases} -1, & \text{if } p^a \text{ is } A\text{-primitive (i.e. } t_A(p^a) = a), \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let S be a subset of $\mathbb{N}$ and let ϱ_S denote the characteristic function of the set S, that is $\varrho_S(n) = 1$ if $n \in S$, and $\varrho_S(n) = 0$ if $n \notin S$. The generalized Möbius function $\mu_{S,A}$ is defined by

$$\mu_{S,A} *_A E = \varrho_S, \tag{2}$$

where $E(n) = 1$ for all $n \in \mathbb{N}$. If $S = \{1\}$, then $\mu_{S,A} = \mu_A$, and if $A = D$, then $\mu_A = \mu$, the classical Möbius function. For further special cases of $\mu_{S,A}$ we refer to [9].

We say that S is multiplicative if its characteristic function ϱ_S is multiplicative, i.e. $1 \in S$ and $mn \in S$ if and only if $m \in S$, $n \in S$ for every m, $n \in \mathbb{N}$ with $(m,n) = 1$.

LEMMA 1. *The function $\mu_{S,A}$ is multiplicative if and only if S is multiplicative, and in this case*

$$\mu_{S,A}(n) = \prod_{p^a \| n} (\varrho_S(p^a) - \varrho_S(p^{a-t})),$$

for every $n \in \mathbb{N}$, where $t = t_A(p^a)$ is the type of p^a with respect to A and $p^a \| n$ means $p^a \mid n$ and $p^{a+1} \nmid n$. If S is multiplicative, then $\mu_{S,A}(n) \in \{-1, 0, 1\}$ for every $n \in \mathbb{N}$.

PROOF. This is an immediately consequence of (2) and (1).

REMARK 1. In particular, if S is multiplicative, then

$$\mu_{S,D}(n) = \prod_{p^a \| n} (\varrho_S(p^a) - \varrho_S(p^{a-1})),$$

$$\mu_{S,U}(n) = \prod_{p^a \| n} (\varrho_S(p^a) - 1),$$

for every $n \in \mathbb{N}$.

LEMMA 2. *For every subset* S *and for every regular convolution* A *we have* $|\mu_{S,A}(n)| \le \tau(n)$ *for every* $n \in \mathbb{N}$*, where* $\tau(n)$ *stands for the number of divisors of* n*, and* $\mu_{S,A}(n) = O(n^{\varepsilon})$ *for every* $\varepsilon > 0$*.*

PROOF. By (2) and by Möbius inversion

$$|\mu_{S,A}(n)| = \left| \sum_{d \in A(n)} \varrho_S(d) \mu_A(n/d) \right| \le$$
$$\le \sum_{d \in A(n)} |\mu_A(n/d)| = 2^{\omega(n)} \le \tau(n) = O(n^{\varepsilon})$$

for every $\varepsilon > 0$, where $\omega(n)$ denotes the number of distinct prime factors of n.

3. Arithmetical Evaluations

For a polynomial f with integral coefficients let $N_f(n)$ denote the number of incongruent solutions (mod n) of the congruence $f(x) \equiv 0$ (mod n). It is well-known that the function N_f is multiplicative. Define the function N_F by $N_F(n) = N_{f_1}(n) N_{f_2}(n) \dots N_{f_u}(n)$ for each $n \in \mathbb{N}$. It follows that the function N_F is multiplicative.

THEOREM 1. *For every* F, S, A, k, $\mathbf{n}$ *and* n *with* $n \mid (n_1, n_2, \dots n_u)$ *we have*

$$\varphi_{F,S,A,k}(\mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \sum_{e \in A_k(n)} \mu_{S,A_k}(e) e^{-ku} N_F(e^k).$$

PROOF. By the definition of $\varphi_{F,S,A,k}(\mathbf{n}, n)$, by (2) and using that $d^k \in A((a,b)_{A,k})$ if and only if $d^k \mid a$ and $d^k \in A(b)$, see [16], Theorem 4.2, we get

$$\varphi_{F,S,A,k}(\mathbf{n}, n) = \sum_{x_1 (\mathrm{mod}\ n_1^k)} \sum_{x_2 (\mathrm{mod}\ n_2^k)} \dots \sum_{x_u (\mathrm{mod}\ n_u^k)} \varrho_S(((f_j(x_j), n^k)_{A,k})^{1/k}) =$$
$$\sum_{x_1 (\mathrm{mod}\ n_1^k)} \sum_{x_2 (\mathrm{mod}\ n_2^k)} \dots \sum_{x_u (\mathrm{mod}\ n_u^k)} \sum_{\substack{e \in A_k(n) \\ e^k \mid f_j(x_j) \\ j = 1,2,\dots,u}} \mu_{S,A_k}(e) =$$
$$\sum_{e \in A_k(n)} \mu_{S,A_k}(e) \sum_{\substack{x_1 (\mathrm{mod}\ n_1^k) \\ e^k \mid f_1(x_1)}} \sum_{\substack{x_2 (\mathrm{mod}\ n_2^k) \\ e^k \mid f_2(x_2)}} \dots \sum_{\substack{x_u (\mathrm{mod}\ n_u^k) \\ e^k \mid f_u(x_u)}} 1.$$

Here for each $i=1, 2, \ldots, u$ the number of incongruent solutions (mod n_i^k) of the congruence $f_i(x)\equiv 0$ (mod e^k) is $N_{f_i}(e^k)(n_i/e)^k$. Thus

$$\varphi_{F,S,A,k}(\mathbf{n},n) = \\ = \sum_{e\in A_k(n)} \mu_{S,A_k}(e) N_{f_1}(e^k)(n_1/e)^k N_{f_2}(e^k)(n_2/e)^k \ldots N_{f_u}(e^k)(n_u/e)^k,$$

which completes the proof.

THEOREM 2. *If S is multiplicative, then*

$$\varphi_{F,S,A,k}(\mathbf{n},n) = \\ = (n_1 n_2 \ldots n_u)^k \prod_{p^a \| n} (1 + \sum_{i=1}^{a/t} (\varrho_S(p^{it}) - \varrho_S(p^{(i-1)t}))p^{-itku} N_F(p^{itk})),$$

where $t=t_{A_k}(p^a)$ is the type of p^a with respect to A_k.

PROOF. Theorem 2 is a direct consequence of Theorem 1 and Lemma 1.

COROLLARY 1. *If $S=\{1\}$, then*

$$\varphi_{F,S,A,k}(\mathbf{n},n) = (n_1 n_2 \ldots n_u)^k \prod_{p^a \| n} (1 - N_F(p^{tk}) p^{-tku}),$$

where $t=t_{A_k}(p^a)$.

If $f_i(x) = s_i + (x-1)d_i^k$, $i = 1, 2, \ldots, u$, then let $\varphi_{F,S,A,k}(\mathbf{n},n) = =\varphi_{S,A,k}(\mathbf{s},\mathbf{d},\mathbf{n},n)$, which is the number of ordered u-tuples $\langle x_1, x_2, \ldots, x_u\rangle$ in $D_k(\mathbf{s},\mathbf{d},\mathbf{n})$ such that $(((x_j), n^k)_{A,k})^{1/k} \in S$, where $D_k(\mathbf{s},\mathbf{d},\mathbf{n}) = D_k(s_1,d_1,n_1)\times \times D_k(s_2,d_2,n_2)\times\ldots\times D_k(s_u,d_u,n_u)$ and $D_k(s_i,d_i,n_i)=\{s_i, s_i+d_i^k, s_i+2d_i^k, \ldots, s_i+ +(n_i^k-1)d_i^k\}$.

This function is a direct generalization of the function $\varphi(s,d,n)$ of P. G. GARCIA and S. LIGH [8] and of the functions investigated by the first author of the present paper in [20].

Taking into account that in this case $N_{f_i}(n) = (d_i^k, n)$ if $(d_i^k, n) \mid s_i$ and $N_{f_i}(n)=0$ otherwise, from Theorem 1 we get the following

COROLLARY 2. *For every S, A, k, $\mathbf{s}$, $\mathbf{d}$, $\mathbf{n}$ and n with $n \mid (n_1, n_2, \ldots, n_u)$ we have*

$$\varphi_{S,A,k}(\mathbf{s},\mathbf{d},\mathbf{n},n)= \\ =(n_1n_2\dots n_u)^k \sum_{\substack{e\in A_k(n)\\ (e,d_i)^k\mid s_i\\ i=1,2,\dots,u}} \mu_{S,A_k}(e)e^{-ku}(e,d_1)^k(e,d_2)^k\dots(e,d_u)^k.$$

COROLLARY 3. *If* $(s_i,d_i^k)_k=1$, $i=1,2,\dots,u$, *then*

$$\varphi_{S,A,k}(\mathbf{s},\mathbf{d},\mathbf{n},n)=(n_1n_2\dots n_u)^k \sum_{\substack{e\in A_k(n)\\ (e,d_i)=1\\ i=1,2,\dots u}} \mu_{S,A_k}(e)e^{-ku}.$$

PROOF. Since $(s_i,d_i^k)_k=1$, we have $(e,d_i)^k\mid s_i$ if and only if $(e,d_i)=1$.

COROLLARY 4. *If* $S=\{1\}$ *and* $(s_i,d_i^k)_k=1$, $i=1,2,\dots,u$, *then*

$$\varphi_{S,A,k}(\mathbf{s},\mathbf{d},\mathbf{n},n)=(n_1n_2\dots n_u)^k \prod_{\substack{p^t\in A_k(n)\\ p\nmid d}} (1-p^{-tku}),$$

where $d=d_1d_2\dots d_u$ *and the product is over the* A_k*-primitive prime powers* p^t *such that* $p^t\in A_k(n)$ *and* $p\nmid d$.

REMARK 2. It should be noted that we do not need the assumption $(s_i,d_i^k)_k=1$, $i=1,2,\dots,u$, in defining the function $\varphi_{S,A,k}(\mathbf{s},\mathbf{d},\mathbf{n},n)$. This kind of assumption is made in the previous arithmetical evaluations of generalized Euler functions in arithmetic progressions, see [8], [20].

Now for an arbitrary set F of polynomials and for $n_1=n_2=\dots=n_u=n$, let $\varphi_{F,S,A,k}(\mathbf{n},n)=\varphi_{F,S,A,k}(n)$.

THEOREM 3. *If* S *is multiplicative, then the arithmetical function* $\varphi_{F,S,A,k}(n)$ *is multiplicative.*

PROOF. It has been noted that $N_F(n)$ is multiplicative. If S is multiplicative, then μ_{S,A_k} is also multiplicative by Lemma 1. Hence, according to Theorem 1, $\varphi_{F,S,A,k}(n)$ is the A_k-convolution of two multiplicative functions and it is multiplicative too.

4. Asymptotic Formulae

We need the following well-known estimates.

LEMMA 3.

$$\sum_{n\le x} n^{-s} = O(x^{1-s}), \qquad 0 < s < 1, \tag{3}$$

$$\sum_{n\le x} n^{-s} = \begin{cases} O(1), & s > 1, \\ O(\log x), & s = 1, \end{cases} \tag{4}$$

$$\sum_{n> x} n^{-s} = O(x^{1-s}), \qquad s > 1. \tag{5}$$

LEMMA 4. (see [19], Lemma 5 and Lemma 8).

$$\sum_{n\le x} \frac{\tau(n)}{n^s} = \begin{cases} O(x^{1-s}\log x), & 0 < s < 1, \\ O(\log^2 x), & s = 1, \\ O(1), & s > 1, \end{cases} \tag{6}$$

$$\sum_{n\le x} \frac{\tau^2(n)}{n^s} = \begin{cases} O(x^{1-s}\log^3 x), & 0 < s < 1, \\ O(\log^4 x), & s = 1, \\ O(1), & s > 1, \end{cases} \tag{7}$$

$$\sum_{n> x} \frac{\tau(n)}{n^s} = O(x^{1-s}\log x), \qquad s > 1. \tag{8}$$

LEMMA 5. (see [1], Lemma 2.3). *If* $s \ge 0$ *and* $a \in \mathbb{N}$, *then*

$$\sum_{\substack{n\le x\\(n,a)=1}} n^s = \frac{\varphi(a)x^{s+1}}{a(s+1)} + O(x^s\tau(a)).$$

Let f be a nonconstant polynomial with integral coefficients. If its decomposition into irreducible factors is $f = cg_1^{r_1}g_2^{r_2}\dots g_m^{r_m}$, then define $h(f) = \max_{1\le j\le m} r_j$.

LEMMA 6. *For every set* F *of nonconstant polynomials and for every* $\varepsilon > 0$ *we have*

$$N_F(n) = O(n^{u-h+\varepsilon}),$$

where $h=1/h(f_1)+1/h(f_2)+\ldots+1/h(f_u)$.

PROOF. According to a result of R. SITARAMACHANDRARAO and P. V. KRISHNAIAH [15], Lemma 3, $N_{f_i}(n)\le C_i^{\omega(n)}n^{1-1/d_i}$ for every $n\in\mathbb{N}$, where d_i is the degree of f_i and $C_i>0$ are constants. Observe that this result remains valid if we have $h(f_i)$ instead of d_i. Now using the familiar relation $C_i^{\omega(n)}=O(n^{\varepsilon})$ for each $\varepsilon>0$ and the definition of $N_F(n)$ we get the desired result.

LEMMA 7. *For every set* F *of nonconstant polynomials, every subset* S *of* N, *every regular convolution* A *and every* $k,u\in\mathbb{N}$ *we have*

$$\sum_{e\le x}|\mu_{S,A}(e)|N_F(e^k)e^{-ku+\varepsilon/2}=$$

$$=\begin{cases} O(1), & \text{if } kh>1 \text{ and } 0<\varepsilon<kh-1,\\ O(x^{1-kh+\varepsilon}), & \text{if } kh\le 1 \text{ and } 0<\varepsilon<kh,\end{cases}$$

$$\sum_{e>x}|\mu_{S,A}(e)|N_F(e^k)e^{-ku-1}=O(x^{\varepsilon-kh}),\qquad \text{if}\qquad \varepsilon<kh.$$

PROOF. By Lemmas 2 and 6 we have for every $\varepsilon>0$

$$|\mu_{S,A}(e)|N_F(e^k)=O(e^{ku-kh+\varepsilon/2}),\tag{9}$$

$$\sum_{e\le x}|\mu_{S,A}(e)|N_F(e^k)e^{-ku+\varepsilon/2}=O(\sum_{e\le x}e^{-kh+\varepsilon}).$$

For $kh>1$ and $\varepsilon<kh-1$ we get $-kh+\varepsilon<-1$ and we use (4). For $kh\le 1$ and $0<\varepsilon<kh$ we have $-1<-kh+\varepsilon<0$ and we use (3).

By (9) we have with ε instead of $\varepsilon/2$

$$\sum_{e>x}|\mu_{S,A}(e)|N_F(e^k)e^{-ku-1}=O\left(\sum_{e>x}e^{-kh-1+\varepsilon}\right),$$

where $-kh-1+\varepsilon<-1$ and applying (5) we get the second estimate.

Let A be a regular convolution such that $A=A_k$ for every $k\in\mathbb{N}$. Then for every prime p we have either $A(p^a)=\{1,p,p^2,\ldots p^a\}=D(p^a)$ or $A(p^a)=\{1,p^a\}=U(p^a)$ for every $a\in\mathbb{N}$, see [16], Theorem 3.3. We say that A is a *cross-convolution*. Let P and Q be the set of the primes of the first and second kind of above, respectively. For $Q=\emptyset$ we have the Dirichlet convolution D and for $P=\emptyset$ we obtain the unitary convolution U.

Furthermore let $(P)=\{1\}\cup\{n\in\mathbb{N}:\text{each prime factor of } n \text{ belongs to } P\}$, $(Q)=\{1\}\cup\{n\in\mathbb{N}: \text{ each prime factor of } n \text{ belongs to } Q\}$. It is clear that every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P)$, $n_Q \in (Q)$ and $(n_P, n_Q) = 1$. In this case $A(n) = \{d : d \mid n, (d, n/d) \in (P)\}$ for every $n \in \mathbb{N}$.

LEMMA 8. *If $s \geq 0$ and $a \in \mathbb{N}$, then*

$$\sum_{\substack{n \leq x \\ (n,a) \in (P)}} n^s = \frac{\varphi(a_Q) x^{s+1}}{a_Q (s+1)} + O(V(x)),$$

where $V(x) = x^s$ or $x^s \tau(a)$, according as Q is finite or Q is infinite, respectively.

PROOF. Observe that $(n,a) \in (P)$ if and only if $(n, \gamma(a_Q)) = 1$, where $\gamma(m)$ denotes the product of distinct prime factors of m. Hence

$$\sum_{\substack{n \leq x \\ (n,a) \in (P)}} n^s = \sum_{\substack{n \leq x \\ (n,\gamma(a_Q))=1}} n^s = \frac{\varphi(\gamma(a_Q)) x^{s+1}}{\gamma(a_Q)(s+1)} + O(x^s \tau(\gamma(a_Q))),$$

by Lemma 5. Here $\varphi(\gamma(a_Q))/\gamma(a_Q) = \varphi(a_Q)/a_Q$ and if Q is finite, then $\tau(\gamma(a_Q)) \leq \tau(\prod_{p \in Q} p) = C$, a constant, which completes the proof.

REMARK 3. We have $V(x) = O(x^s a^{\varepsilon})$ for every Q and for every $\varepsilon > 0$.

LEMMA 9. *For every set F of nonconstant polynomials, every regular A, every S and every k, $u \in \mathbb{N}$ the series*

$$\sum_{n=1}^{\infty} \frac{\mu_{S,A}(n) N_F(n^k) \varphi(n_Q)}{n^{ku+1} n_Q}$$

is absolutely convergent. Let $\alpha_{F,S,A,k}$ denote its sum. If A is a cross-convolution and S is multiplicative, then

$$\alpha_{F,S,A,k} = \prod_{p \in P} \left(1 + \sum_{l=1}^{\infty} \frac{(\varrho_S(p^l) - \varrho_S(p^{l-1})) N_F(p^{lk})}{p^{l(ku+1)}} \right) \cdot$$
$$\cdot \prod_{p \in Q} \left(1 + (1 - \frac{1}{p}) \sum_{l=1}^{\infty} \frac{(\varrho_S(p^l) - 1) N_F(p^{lk})}{p^{l(ku+1)}} \right).$$

PROOF. The absolute convergence of the series follows at once by Lemmas 2 and 6: the general term is

$$O(n^{ku-kh+\varepsilon}/n^{ku+1}) = O(1/n^{1+kh-\varepsilon}),$$

and we choose $\varepsilon < kh$.

If S is multiplicative, then the general term is multiplicative and the series can be expanded into an infinite product of Euler-type:

$$\alpha_{F,S,A,k} = \prod_p \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,A}(p^l)N_F(p^{lk})\varphi((p^l)_Q)}{p^{l(ku+1)}(p^l)_Q}\right) =$$

$$= \prod_{p\in P} \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,D}(p^l)N_F(p^{lk})}{p^{l(ku+1)}}\right) \prod_{p\in Q} \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,U}(p^l)N_F(p^{lk})\varphi(p^l)}{p^{l(ku+1)}p^l}\right) =$$

$$= \prod_{p\in P} \left(1 + \sum_{l=1}^{\infty} \frac{\left(\varrho_S(p^l) - \varrho_S\left(p^{l-1}\right)\right) N_F(p^{lk})}{p^{l(ku+1)}}\right) \cdot$$

$$\cdot \prod_{p\in Q} \left(1 + \sum_{l=1}^{\infty} \frac{(\varrho_S(p^l) - 1)N_F(p^{lk})}{p^{l(ku+1)}} \left(1 - \frac{1}{p}\right)\right),$$

by Remark 1.

REMARK 4. If $S=\{1\}$ and A is a cross-convolution, then

$$\alpha_{F,S,A,k} = \prod_{p\in P} \left(1 - \frac{N_F(p^k)}{p^{ku+1}}\right) \prod_{p\in Q} \left(1 - \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{N_F(p^{lk})}{p^{l(ku+1)}}\right).$$

THEOREM 4. *For every set F of nonconstant polynomials, every cross-convolution A, every S and every $k \in \mathbb{N}$ we have*

$$\sum_{n\le x} \varphi_{F,S,A,k}(n) = \frac{\alpha_{F,S,A,k}}{ku+1} x^{ku+1} + O(R(x)),$$

where $\alpha_{F,S,A,k}$ is defined in Lemma 9 and $R(x)=x^{ku}$ or $x^{ku-kh+\varepsilon+1}$, according as $kh > 1$ or $kh \le 1$, respectively, for every $0<\varepsilon<kh$.

PROOF. Using Theorem 1 with $n_1=n_2=\ldots=n_u=n$, Lemma 8 and Remark 3 we get for every $\varepsilon > 0$

$$\sum_{n\le x} \varphi_{F,S,A,k}(n) =$$

$$= \sum_{n\le x} \sum_{\substack{er=n \\ (e,r)\in(P)}} \mu_{S,A}(e)N_F(e^k)r^{ku} = \sum_{e\le x} \mu_{S,A}(e)N_F(e^k) \sum_{\substack{r\le x/e \\ (r,e)\in(P)}} r^{ku} =$$

$$= \sum_{e\le x} \mu_{S,A}(e)N_F(e^k) \left(\frac{\varphi(e_Q)(x/e)^{ku+1}}{e_Q(ku+1)} + O((x/e)^{ku}e^{\varepsilon/2})\right) =$$

$$= \frac{x^{ku+1}}{ku+1} \sum_{e \le x} \frac{\mu_{S,A}(e) N_F(e^k) \varphi(e_Q)}{e_Q e^{ku+1}} + O\left(x^{ku} \sum_{e \le x} \frac{|\mu_{S,A}(e)| N_F(e^k)}{e^{ku-\varepsilon/2}} \right) =$$

$$= \frac{\alpha_{F,S,A,k}}{ku+1} x^{ku+1} + O\left(x^{ku+1} \sum_{e > x} \frac{|\mu_{S,A}(e)| N_F(e^k)}{e^{ku+1}} \right) +$$

$$+ O\left(x^{ku} \sum_{e \le x} \frac{|\mu_{S,A}(e)| N_F(e^k)}{e^{ku-\varepsilon/2}} \right).$$

Here the first O-term is $O(x^{ku+1} x^{\varepsilon - kh}) = O(x^{ku-kh+\varepsilon+1})$ for $\varepsilon < kh$ and the second O-term is $O(x^{ku})$ for $kh > 1$ and $\varepsilon < kh-1$ and it is $O\left(x^{ku} x^{1-kh+\varepsilon}\right) =$ $= O(x^{ku-kh+\varepsilon+1})$ for $kh \le 1$ and $\varepsilon < kh$, by Lemma 7, which proves the theorem.

COROLLARY 5. *For every set F of nonconstant polynomials, every cross-convolution A, every S and every k the average order of the function $\varphi_{F,S,A,k}(n)$ is $\alpha_{F,S,A,k} n^{ku}$.*

For $n_1 = n_2 = \ldots = n_u = n$ let $\varphi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n) = \varphi_{S,A,k}(\mathbf{s}, \mathbf{d}, n)$.

COROLLARY 6. *If $(s_i, d_i^k)_k = 1$ for $i = 1, 2, \ldots, u$ and A is a cross-convolution, then the average order of the function $\varphi_{S,A,k}(\mathbf{s}, \mathbf{d}, n)$ is $\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) n^{ku}$, where*

$$\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) = \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} \frac{\mu_{S,A}(n) \varphi(n_Q)}{n^{ku+1} n_Q},$$

$d = d_1 d_2 \ldots d_u$. If in addition S is multiplicative, then

$$\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) =$$
$$= \prod_{\substack{p \in P \\ p \nmid d}} \left(1 + \sum_{l=1}^{\infty} \frac{\varrho_S(p^l) - \varrho_S(p^{l-1})}{p^{l(ku+1)}} \right) \prod_{\substack{p \in Q \\ p \nmid d}} \left(1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{\varrho_S(p^l) - 1}{p^{l(ku+1)}} \right).$$

PROOF. In this case $N_F(n^k) = \begin{cases} 1, & \text{if } (n, d_i) = 1 \text{ for } i = 1, 2, \ldots, u \\ 0, & \text{otherwise} \end{cases}$ and we use Lemma 9.

REMARK 5. If S is multiplicative and $A=D$, then

$$\alpha_{S,D,k}(\mathbf{s},\mathbf{d}) = \frac{\zeta_S(ku+1)}{\zeta(ku+1)M_{S,d}(ku+1)},$$

where ζ is the Riemann zeta-function, $\zeta_S(z)=\sum_{n=1}^{\infty}\varrho_S(n)n^{-z}$ (see [7]) and $M_{S,d}(z)=\prod_{p|d}(1+\sum_{l=1}^{\infty}\mu_S(p^l)p^{-lz})$.

If $S=\{1\}$ and $A=D$, then

$$\alpha_{\{1\},D,k}(\mathbf{s},\mathbf{d}) = \frac{d^{ku+1}}{\zeta(ku+1)\varphi_{ku+1}(d)}.$$

If $S=\{1\}$ and $A=U$, then

$$\alpha_{\{1\},U,k}(\mathbf{s},\mathbf{d}) = \prod_{p\nmid d}\left(1-\frac{p-1}{p(p^{ku+1}-1)}\right),$$

where the product is over all primes with $p\nmid d$.

REMARK 6. The remainder term of the above asymptotic formula can be improved if we have more information on F, S and A, see [4], [10], [18], [20], [23].

As an example we prove

THEOREM 5. *If* $(s_i,d_i^k)=1$ *and* A *is a cross-convolution, then*

$$\sum_{n\le x}\varphi_{S,A,k}(\mathbf{s},\mathbf{d},n) = \frac{\alpha_{S,A,k}(\mathbf{s},\mathbf{d})}{ku+1}x^{ku+1} + O(T(x)),$$

where $T(x)=x^{ku}$ $(ku>1)$, $x\log^2 x$ *(*$ku=1$, Q *is finite),* $x\log^4 x$ *(*$ku=1$, Q *is infinite). If* $ku=1$ *and* S *is multiplicative, then the error term can be improved into* $T(x)=x\log x$ *(*Q *is finite),* $x\log^2 x$ *(*Q *is infinite).*

PROOF. By Corollary 3 and Lemma 8 we get

$$\sum_{n\le x}\varphi_{S,A,k}(\mathbf{s},\mathbf{d},n) = \sum_{\substack{e\le x\\(e,d)=1}}\mu_{S,A}(e)\sum_{\substack{r\le x/e\\(r,e)\in(P)}} r^{ku} =$$

$$= \sum_{\substack{e\le x\\(e,d)=1}}\mu_{S,A}(e)\left(\frac{\varphi(e_Q)(x/e)^{ku+1}}{e_Q(ku+1)} + O(V(x/e))\right) =$$

$$= \frac{x^{ku+1}}{ku+1} \sum_{\substack{e \le x \\ (e,d)=1}} \frac{\mu_{S,A}(e)\varphi(e_Q)}{e_Q e^{ku+1}} + O\left(\sum_{e\le x} |\mu_{S,A}(e)|\, V(x/e)\right) =$$

$$= \frac{x^{ku+1}}{ku+1}\alpha_{S,A,k}(\mathbf{s},\mathbf{d}) + O\left(x^{ku+1}\sum_{e>x}\frac{|\mu_{S,A}(e)|}{e^{ku+1}}\right) + O\left(\sum_{e\le x}|\mu_{S,A}(e)|\, V(x/e)\right),$$

where the first O-term is

$$O\left(x^{ku+1}\sum_{e>x}\tau(e)e^{-ku-1}\right) = O(x^{ku+1}x^{-ku}\log x) = O(x\log x),$$

by Lemma 2 and (8). If S is multiplicative, then it is $O(x)$, by Lemma 1 and (5). If Q is finite, the second O-term is

$$O\left(\sum_{e\le x}|\mu_{S,A}(e)|(x/e)^{ku}\right) = O\left(x^{ku}\sum_{e\le x}|\mu_{S,A}(e)|e^{-ku}\right) =$$

$$= \begin{cases} O(x^{ku}\sum_{e\le x}\tau(e)e^{-ku}) = O(x^{ku}) & \text{for } ku > 1, \\ O(x\sum_{e\le x}\frac{\tau(e)}{e}) = O(x\log^2 x) & \text{for } ku = 1, \\ O(x\sum_{e\le x}\frac{1}{e}) = O(x\log x) & \text{for } ku = 1 \text{ and } S \text{ multiplicative,} \end{cases}$$

by Lemmas 1 and 2, by (4) and (6).

If Q is infinite the second O-term is

$$O\left(\sum_{e\le x}|\mu_{S,A}(e)|(x/e)^{ku}\tau(e)\right) = O\left(x^{ku}\sum_{e\le x}\frac{|\mu_{S,A}(e)|\tau(e)}{e^{ku}}\right) =$$

$$= \begin{cases} O(x^{ku}\sum_{e\le x}\frac{\tau^2(e)}{e^{ku}}) = O(x^{ku}) & \text{for } ku > 1, \\ O(x\sum_{e\le x}\frac{\tau^2(e)}{e}) = O(x\log^4 x) & \text{for } ku = 1, \\ O(x\sum_{e\le x}\frac{\tau(e)}{e}) = O(x\log^2 x) & \text{for } ku = 1 \text{ and } S \text{ multiplicative,} \end{cases}$$

by Lemmas 1 and 2, by (6) and (7).

REMARK 7. For $k=1$, $S=\{1\}$, $A=D$ and $k=1$, $S=\{1\}$, $A=U$ this result was found by the first author [20], Theorem 2.4, Theorem 4.4.

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SOME RESONANCE THEOREMS

By

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In this paper we investigate the norm divergence phenomenon of continuous linear operators defined on some Banach spaces. To this first introduce the concept of inverse limit [1]. Let A be a directed set, i.e. a partially ordered set such that

$$\forall \alpha, \beta \in A \ \exists \gamma : \alpha, \beta \geq \gamma .$$

Let D and $B_\alpha, \alpha \in A$ be Banach spaces and

$$F_{\alpha,\beta} : B_\alpha \to B_\beta (\alpha \geq \beta; \alpha, \beta \in A)$$

be continuous linear mapping satisfying

$$F_{\alpha,\alpha} = Id_{B_\alpha}$$

$$F_{\beta,\gamma} \circ F_{\alpha,\beta} = F_{\alpha,\gamma} (\alpha \geq \beta \geq \gamma)$$

The inverse limit of the system $\{F_{\alpha,\beta}, B_\alpha\}$ is defined to be the set

$$\underset{\leftarrow}{B}_\infty := \underset{\leftarrow}{\lim}\{F_{\alpha,\beta}, B_\alpha\} := \{(l_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} B_\alpha : l_\beta = F_{\alpha,\beta} l_\alpha \ \forall \alpha \geq \beta\}$$

The system $\{F_{\alpha,\beta}, B_\alpha\}$ is called inverse system. Denote by p_{α_0} the α_0-th projection

$$p_{\alpha_0} : \prod_{\alpha \in A} B_\alpha \to B_{\alpha_0},$$

thus

$$F_{\alpha,\beta} \circ p_\alpha = p_\beta : \underset{\leftarrow}{B}_\infty \to B_\beta (\alpha \geq \beta).$$

Define in

$$\underset{\leftarrow}{B}_\infty$$

the projective limit topology, it is a locally convex vectortopology, see [1]. If we are given continuous linear operators

$$G_\alpha : D \to B_\alpha$$

such that

$$F_{\alpha,\beta} \circ G_\alpha = G_\beta (\alpha \geq \beta)$$

then there exists a mapping

$$G : D \to \underset{\leftarrow}{B}_\infty$$

defined by

$$p_\alpha \circ G = G_\alpha (\alpha \in A).$$

In this case G is also continuous [1].

THEOREM *Let*

$$D, B_\alpha (\alpha \in A)$$

be Banach spaces and suppose that the directed set A is (downward) cofinal with its countable part

$$A' \subseteq A$$

with the above properties. Let further

$$L_n : D \to \underset{\leftarrow}{B}_\infty$$

be continuous linear operators and suppose that

$$\sup_n \|p_\alpha \circ L_n\| = \infty \quad (\alpha \in A).$$

Then there exists $f \in D$ satisfying

$$\sup_n \{\|p_\alpha (L_n (f)\|_\alpha\} = \infty \quad (\alpha \in A).$$

COROLLARY. *Let D be a Banach space and*

$$\{B_\alpha, F_{\alpha,\beta}\}$$

be an inverse system. Suppose that A is (downward) cofinal with its countable part $A' \subset A$ and that

$$\sup\{\|F_{\alpha,\beta}\| : \alpha \geq \beta\} < \infty$$

and let

$$L_n : D \to \underset{\leftarrow}{B}_\infty$$

be continuous linear operator and suppose that

$$\forall \alpha \in A \; \exists x_\alpha \in D : \sup_n \|p_\alpha (L_n x_\alpha \|_\alpha = \infty.$$

Then

$$\exists x \in D: \ \forall \alpha \in A, \quad \sup_n\{\|p_\alpha(L_n x)\|_\alpha\} = \infty.$$

PROOF OF THE THEOREM. The Banach-Steinhaus theorem states for any fixed α that

$$\sup_n\{\|p_\alpha \circ L_n\|_\alpha\} = \infty$$

implies

$$\sup_n\{\|p_\alpha(L_n f)\|_\alpha\} = \infty$$

for any f not belonging to a set of first category in D. This set can be meant universal for all $\alpha' \in A'$ since $|A'| = \aleph_0$. Let now $\alpha \in A$, then there exists $\alpha' \in A'$ with $\alpha' \leq \alpha$ and then

$$\|F_{\alpha,\alpha'}\| \cdot \|p_\alpha(L_n f)\| \geq \|p_{\alpha'}(L_n f)\|$$

implies that

$$\sup_n\{\|p_\alpha(L_n f)\|\} = \infty,$$

for all f except for a set of first category. The Theorem is proved.

The corollary follows immediately from the Theorem.

We can formulate two other corollaries of the Theorem.

a) Let D and B_p be Banach spaces for $p_0 < p \leq \infty$; suppose $B_p \supseteq B_q$ and

$$\| \ \|_p \leq \| \ \|_p \quad (p_0 < p \leq q).$$

Let further

$$L_n : D \to B_\infty$$

be continuous, linear and

$$\lim_{n\to\infty} \sup_{\|f\|_{D=1}} \|L_n f\|_p = \infty \quad (p_0 < p \leq \infty).$$

Then there exists $f \in D$ such that

$$\sup_n \|L_n f\|_p = \infty \quad (p_0 < p \leq \infty).$$

b) Let

$$L_n : B \to L^1(0,1)$$

be continuous and linear, B be a Banach space. Suppose that for all compact interval $I \subset (0,1)$ there exists $x_I \in B$, $\|x_I\| = 1$ such that

$$\sup_n \|L_n x_I\|_{L^1(I)} = \infty.$$

Then there exists $x \in B$ such that for all compact intervals $I \subset (0,1)$ we have

$$\sup_n \|L_n x\|_{L^1(I)} = \infty.$$

Remark that in case a) the mappings $F_{\alpha,\beta}$ are the inclusions $B_q \to B_p$, $q \geq p > p_0$ and in case b) the restriction operators $L^1(I) \to L^1(J)$, $f \vdash f\mid_J$ for $I \supseteq J$.

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A NEW APPROXIMATION FOR THE FOURIER TRANSFORM II

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In [3] a new approach was given to approximation to the Fourier transform of a given function by expressions using values of this function in finite number of points. This work continued investigations in this area fulfilled in [1] and [2]. In [3] several open problems were posed. The main one was to get an estimate of the error of approximation uniform on the whole real line instead of that uniform on compact intervals given there. Also smoothness conditions on a given function were superfluous in [3].

Thus our goal is to give a uniform estimate of the error of approximation to the Fourier transform under sharp conditions on a given function.

Let a function f be defined on the real line $\mathbb{R}=(-\infty,\infty)$ and satisfy the following conditions for some positive integer r:

The function f and its derivates $f^{(p)}$, for $p = 1, \dots, 2r$, are locally absolutely continuous functions on $\mathbb{R}$; and $f^{(2r+1)}$ is continuous on $\mathbb{R}$;

$$\lim_{|t|\to\infty} t^{2(r+1)}f(t)=0, \qquad \lim_{|t|\to\infty} t^{2r}f'(t)=0, \qquad \dots, \qquad \lim_{|t|\to\infty} t^{2}f^{(r)}(t)=0,$$

$$\lim_{|t|\to\infty} t^{2(r+1)}f^{(r+1)}(t)=0, \quad \lim_{|t|\to\infty} t^{2r}f^{(r+2)}(t)=0, \quad \dots, \quad \lim_{|t|\to\infty} t^{2}f^{(2r+1)}(t)=0.$$

Let

$$\hat{f}(x)=\int_{\mathbb{R}} f(t)e^{-ixt}\,dt$$

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be the Fourier transform of the function f. We will approximate to it by the following expression:

$$f_n(x) = \\ = \frac{\pi}{n}\sum_{k=1}^{n}\cos^{-2}\left(\frac{2k-1-n}{2n}\pi\right) f_x\left(\tan\frac{2k-1-n}{2n}\pi\right)\exp\left(-ix\tan\frac{2k-1-n}{2n}\pi\right)$$

where

$$f_x(t) = \begin{cases} f(t), & |x| < 1, \\ (-ix)^{-(r+1)} f^{(r+1)}(t), & |x| \geq 1. \end{cases}$$

Denote

$$\Delta_n(f,x) = \hat{f}(x) - f_n(x).$$

THEOREM. *For f satisfying the above conditions with $r \geq 2$ we have*

$$\Delta_n(f,x) = o(n^{-r}) \qquad as \qquad n \to \infty$$

uniformly with respect to $x \in \mathbb{R}$.

PROOF. For $|x| \geq 1$ let us integrate by parts $r+1$ times and obtain by assumptions on behavior of f at infinity:

$$\hat{f}(x) = (-ix)^{-r-1}\int_{\mathbb{R}} f^{(r+1)}(t)e^{-ixt}dt = \int_{\mathbb{R}} f_x(t)e^{-ixt}dt.$$

Now we will apply a procedure used in [3], to the Fourier transform of the function f_x. After change of variables $t \to \tan s$ we get

$$\hat{f}(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_x(\tan s)\cos^{-2} s\, e^{-ix\tan s} ds.$$

It is obvious that the function under the sign of integral

$$F_x(s) = f_x(\tan s)\cos^{-2} s\, e^{-ix\tan s}$$

satisfies the same smoothness conditions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as the given function satisfies on $\mathbb{R}$. Therefore

$$\hat{f}(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_x(s)ds = \pi\hat{F}_x(0)$$

where

$$\hat{F}_x(m) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_x(s) e^{-i2ms} ds$$

is the m-th Fourier coefficient of the π-periodic continuation of the function F_x. Let us estimate the rate of decay of these Fourier coefficients. Integrating by parts r times and applying the boundary conditions for F_x at the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (for f_x at $\mp\infty$, respectively), we obtain

$$\hat{F}_x(m) = \frac{1}{\pi}(-2im)^{-r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_x^{(r)}(s) e^{-i2ms} ds.$$

It is important to mention that x may appear as a factor in $F_x^{(r)}$ at most in the r-th power. Let us now apply to the integral the Riemann–Lebesgue theorem in the form given e.g. in [4], Ch. II, Sec. 4:

$$|\hat{F}(m)| \leq \frac{1}{2}\omega_1 \left(F; \frac{\pi}{2|m|} \right)$$

where $m \neq 0$ and

$$\omega_1(F; h) = \sup_{0 \leq t \leq h} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |F(u+t) - F(u)| du$$

is the modulus of continuity of a given function F in L^1-metrics. It tends to zero as $h \to 0$, and this is the best possible rate of decay of Fourier coefficients of F. Hence we have for $m \neq 0$

$$|\hat{F}_x(m)| \leq \frac{1}{2\pi}(2|m|)^{-r} \omega_1 \left(F_x^{(r)}; \frac{\pi}{2|m|} \right).$$

We must know how the right hand side depends on x. We have already mentioned $|x|^r$ as the largest power in which x appears before use of the Riemann–Lebesgue theorem. The function $F_x^{(r)}$ is of the form

$$F_x^{(r)}(s) = e^{-ix \tan s} \Phi_x(s) = e^{-ix \tan s} \sum_{p=0}^{r} x^p g_p(s)$$

where g_p are continuous functions independent of x. Therefore for some absolute positive constant C we have

$$\omega_1\left(F_x^{(r)};\frac{\pi}{2|m|}\right)\leq C\left[\omega_1\left(\Phi_x;\frac{\pi}{2|m|}\right)+\frac{|x|}{|m|}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\Phi_x(s)|ds\right]$$

where the last term is a simple exercise on calculation the modulus of continuity of the function $e^{-ix\tan s}$. Using the well-known properties of moduli of continuity we can write for $m\neq 0$

$$|\hat{F}_x(m)|\leq C\omega_1\left(\Phi_x;\frac{\pi}{2|m|}\right).$$

Consider now

$$\Delta_n(f,x)=\pi\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}F_x(s)ds-\frac{\pi}{n}\sum_{k=1}^{n}F_x\left(\frac{2k-1-n}{2n}\pi\right)=$$

$$=\pi\hat{F}_x(0)-\frac{\pi}{2}\sum_{k=1}^{n}\lim_{N\to\infty}\sum_{m=-N}^{N}\hat{F}_x(m)\exp\left(\frac{i2m\pi(2k-1-n)}{2n}\right)=$$

$$=-\frac{\pi}{2}\sum_{k=1}^{n}\lim_{N\to\infty}\sum_{1\leq|m|\leq N}\hat{F}_x(m)\exp\left(\frac{im\pi(2k-1-n)}{n}\right)=$$

$$=-\frac{\pi}{n}\lim_{N\to\infty}\sum_{1\leq|m|\leq N}\hat{F}_x(m)\sum_{k=1}^{n}\exp\left(\frac{i2m\pi(2k-1-n)}{2n}\right).$$

All these operations are legal since F_x is at least one time differentiable function, so its Fourier series converges uniformly (see e.g. [4], Ch. II, Sec. 10). Let us use now simple property that the sum

$$\sum_{k=1}^{n}\exp\left(\frac{im\pi(2k-1-n)}{n}\right)=\exp\left(-\frac{im\pi(n+1)}{n}\right)\sum_{k=1}^{n}\exp\left(\frac{i2m\pi k}{n}\right)$$

vanishes for $m\neq nq$, where q is integer, while for $m=nq$ it is equal to

$$n\exp(-iq\pi(n+1))=n(-1)^{q(n+1)}.$$

Hence we obtain that

$$|\Delta_n(f,x)| = \left| -\pi \lim_{N\to\infty} \sum_{1\le|q|\le N} (-1)^{q(n+1)} \hat{F}_x(qn) \right| \le$$

$$\le C \sum_{q\neq 0} (|q|n)^{-r} \omega_1 \left(\Phi_x; \frac{\pi}{2|q|n}\right).$$

The last value does not exceed

$$C_n^{-r} \begin{cases} \int\limits_n^\infty t^{-1}\omega_1\left(\Phi_x; \frac{\pi}{2t}\right) dt, & r = 1, \\ \omega_1\left(\Phi_x; \frac{\pi}{2n}\right), & r > 1. \end{cases}$$

Now the estimate

$$\Delta_n(f,x) = o(n^{-r})$$

follows from the fact that the modulus of continuity tends to zero as $n\to\infty$. Also this estimate is uniform with respect to x because of the definition of the function f_x and the fact that for $|x| < 1$ the factor x appears only in positive powers an for $|x| \ge 1$ at most in the power $r+1$.

The proof is complete. ∎

COROLLARY. *The theorem holds to be true also for* $r=1$ *under additional assumption*

$$\int\limits_1^\infty t^{-1}\omega_1\left(\Phi_x; \frac{\pi}{2t}\right) dt < \infty.$$

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ON PISOT NUMBERS

By

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The algebraic integer $1 < q < 2$ is called a Pisot number if $|q_i| < 1$ for all of its conjugates. Introduce further, as in [2] the set

$$Y = \left\{ \sum_0^n \varepsilon_i q^i : n \geq 0, \quad \varepsilon_i \in \mathbb{Z}, \quad 0 \leq \varepsilon_i \leq 2 \right\}$$

and let

$$l_2(q) = \inf\{|y_1 - y_2| : y_1, y_2 \in Y, \ y_1 \neq y_2\}.$$

Analogously the quantities $l_k(q)$ can be introduced, where in Y, $|\varepsilon_i| \leq k$ stands instead of $|\varepsilon_i| \leq 2$. BUGEAUD proved on [2] that for $1 < q < 2$

$$q \quad \text{is Pisot} \ \Leftrightarrow \ l_k(q) > 0 \quad \text{for all} \quad k \geq 1.$$

A former result of [8] states that if q is Pisot then $l_1(q) > 0$. The same proof shows that q is Pisot $\Rightarrow l_k(q) > 0$ for all k. In this note we strengthen this result stating that

THEOREM. *Let* $1 < q < A = \frac{1+\sqrt{5}}{2}$. *Then*

$$q \quad \textit{is Pisot} \ \Leftrightarrow \ l_2(q) > 0.$$

REMARK. The proof improves some ideas from [7].

For the proof we need a series of lemmas.

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LEMMA 1. $l_2(q)>0$ *implies that the number of distances* $\leq \frac{1}{q-1}$ *between numbers of form* $y=\sum^{n} \varepsilon_i q^i$, $\varepsilon_i = \begin{cases} 0 \\ 1 \end{cases}$ *is finite.*

PROOF. Otherwise there would be almost equal distances: $y_1-y_2 \approx y_1'-y_2'$, hence $(y_1+y_2')-(y_2+y_1')$ is as small as we want and y_1+y_2', $y_2+y_1' \in Y$, contradiction. ∎

LEMMA 2. $l_2(q)>0 \Rightarrow q$ *is algebraic integer.*

PROOF. Let $1=\sum_1^{\infty} \frac{\varepsilon_n}{q^n}$, $\varepsilon_i = \begin{cases} 0 \\ 1 \end{cases}$ be an expansion. Then $0 \leq q^n - \sum_1^n \varepsilon_i q^i \leq$ $\leq \frac{1}{q}+\frac{1}{q^2}+\ldots=\frac{1}{q-1}$. By Lemma 1 the number of these values are finite, hence there are two identical

$$q^m - \sum_1^m \varepsilon_i q^i = q^n - \sum_1^n \varepsilon_i q^i$$

proving that q is algebraic integer. ∎

LEMMA 3. $l_2(q)>0 \Rightarrow$ *for all conjugates* $|q_i|>1$ *we have*

$$\sum_0^{\infty} s_n q^{-n} = 0, \qquad s_n = 0, \ \pm 1 \Rightarrow \sum_0^{\infty} s_n q_i^{-n} = 0.$$

The proof is identical with that of [5], Corollary 3.2. ∎

LEMMA 4. *Let* $l_2(q)>0$, $q \leq A$. *Take angles* α, β, $\alpha \neq 0$ *with* $|\alpha|, |\beta| \leq \pi$ *and suppose that* $n\alpha$ *is never parallel to* β. *Denote* $\mathbb{N}=\mathbb{N}_1 \cup \mathbb{N}_2$ *the partition of natural numbers by the property whether the ray of angle* $n\alpha$ *is in the one or other half-plane bounded by the line of angle* β. *Then there exists a non trivial expansion*

$$(*) \qquad \sum_{\mathbb{N}_1} \frac{\varepsilon_n}{q^n} = \sum_{\mathbb{N}_2} \frac{\varepsilon_n}{q^n} \qquad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases} .$$

REMARK. $(*)$ can not hold for q_i, $|q_i|>1$ if α is the angle of q_i. Indeed, the left resp. right sum is in the left resp. right half-space bounded by the line of angle β. If $\alpha>0$ i.e. $q_i>1$ then Lemma 3 can not be true. Indeed, let $1=\sum_1^{\infty} \frac{\varepsilon_n}{q^n}$, $\varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}$. Then $q_i>q \Rightarrow 1>\sum \frac{\varepsilon_n}{q_i^n}$; $q_i<q \Rightarrow 1<\sum \frac{\varepsilon_n}{q_i^n}$. Anyway this contradiction proves that $q_i>1$ is impossible.

PROOF OF LEMMA 4. Distinguish two cases

a) For all $n \geq 1$, at least one neighbour of n belongs to the same set ($\mathbb{N}_1$ or $\mathbb{N}_2$) as $\mathbb{N}$. For example, 0, 1, …, $k \in \mathbb{N}_1$, $k+1 \in \mathbb{N}_2$. Then $k+2 \in \mathbb{N}_2$. Let

$$\varepsilon_0 = \ldots = \varepsilon_{k-1} = 0, \qquad \varepsilon_k = 1.$$

Then the left sum is larger in $(*)$ but using $k+1$, $k+2 \in \mathbb{N}_2$ we can make the right sum larger. Suppose in general that the left sum is larger and it became larger at $n \in \mathbb{N}_1$.

a$_1$) If $n+1 \in \mathbb{N}_2$ then $n+2 \in \mathbb{N}_2$. By $q \leq A$ we have

$$\frac{1}{q^n} \leq \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}}.$$

Since $\sum\limits_{\substack{k \leq n \\ k \in \mathbb{N}_1}} - \sum\limits_{\substack{k \leq n \\ k \in \mathbb{N}_2}} \leq \frac{1}{q^n}$, hence using $\frac{1}{q^{n+1}}$ and $\frac{1}{q^{n+2}}$, the right sum can be made larger.

a$_2$) If n, $n+1 \in \mathbb{N}_1$, $n+2$, $n+3 \in \mathbb{N}_2$ and if $\frac{1}{q^n}$ can be changed by $\frac{1}{q^{n+1}}$ to keep the property that the left sum be larger then we argue as in a$_1$). If not then $\sum\limits_{\substack{k \leq n \\ k \in \mathbb{N}_1}} - \sum\limits_{\substack{k \leq n \\ k \in \mathbb{N}_2}} \leq \frac{1}{q^n} - \frac{1}{q^{n+1}} \leq \frac{1}{q^{n+2}}$ so again the right sum is larger.

a$_3$) If n, $n+1$, $n+2$, …, $n+i \in \mathbb{N}_1$, $n+i+1 \in \mathbb{N}_2$ then $\frac{1}{q^n}$ will be changed in the left of $(*)$ to $\frac{1}{q^{n+1}}$ or to $\frac{1}{q^{n+1}} + \frac{1}{q^{n+2}}$ if $\frac{1}{q^{n+1}}$ is not enough. Repeating this step we finally obtain a$_1$) or a$_2$).

b) There exists $n > 0$ with no neighbour in the same set. Then we must have $|\alpha| > \frac{\pi}{2}$.

b$_1$) $|\alpha| = \pi$. Then $\mathbb{N}_1$ is the set of odd, $\mathbb{N}_2$ that of even numbers. Let $\varepsilon_0 = 1$. Since for $q \leq A$

$$1 \leq \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \ldots$$

hence if the left sum is larger, the right one can be made larger and vica versa.

b$_2$) $\frac{\pi}{2} < |\alpha| < \pi$, $\frac{\alpha}{\pi}$ is rational. Then the rays $n\alpha$ go through the nodes of a regular n-gon. The line joining the n-th and $(n+1)$-th nodes takes any notated position hence in $\mathbb{N}_1$ or in $\mathbb{N}_2$ there are infinitely many neighbours. E.g. let $n-1 \in \mathbb{N}_2$, $n+1 \in \mathbb{N}_1$. Set $\varepsilon_0 = \varepsilon_{n-2} = 0$, $\varepsilon_{n-1} = 1$; then the left sum is larger but the right one can be made larger using $\frac{1}{q^n}$ and $\frac{1}{q^{n+1}}$. Suppose that in constructing the digits the left sum became larger at $\frac{1}{q^n}$, $n \in \mathbb{N}_1$. If there exists $m > n$, $m \in \mathbb{N}_1$ such that $\frac{1}{q^n}$ can be changed to $\frac{1}{q^m}$ and the left sum dominates, do it. If not, then in case $n+1$, $n+2\in\mathbb{N}_2$ the right sum can be made larger. If $n+1\in\mathbb{N}_1$, $n+2\in\mathbb{N}_2$ then

$$\frac{1}{q^n}-\frac{1}{q^{n+1}}<\frac{1}{q^{n+2}}$$

gives the same conclusion. We analogously argue if $n+1\in\mathbb{N}_2$, $n+2\in\mathbb{N}_1$. Finally $n+1$, $n+2\in\mathbb{N}_1$ is impossible since $n\in\mathbb{N}_1$ and $|\alpha|>\frac{\pi}{2}$.

b$_3$) $\frac{\pi}{2}<|\alpha|<\pi$, $\frac{\alpha}{\pi}$ is irrational. Then in $\mathbb{N}_1$ and $\mathbb{N}_2$ there are infinitely many consecutive pairs, and the construction of b$_2$) works. ∎

REMARK. For $q>A$, Lemma 4 is not true; e.g. we can not construct

$$\sum\frac{\varepsilon_{2n+1}}{q^{2n+1}}=\sum\frac{\varepsilon_{2n}}{q^{2n}}$$

since $1>\frac{1}{q}+\frac{1}{q^3}+\dots$ and then if the first digit 1 occurs on the left then the left sum dominates.

LEMMA 5. *Let $l_2(q)>0$, $q\le A$. There is no conjugate q_i of q with $|q_i|=1$.*

PROOF. Since ± 1 can not be a conjugate of $q>1$, q_i must be complex. Checking the proof of [5], Corollary 3.2 we see at once that

(∗∗) $\displaystyle\sum_{|s_n|\le 1}^{\infty}\frac{s_n}{q^n}=0\Rightarrow$ the partial sums $\displaystyle\sum_0^N\frac{S_n}{q_i^n}$ are in one of finitely many circles with center at the origin.

Now take the expansion

$$\sum_{\mathbb{N}_1}\frac{\varepsilon_n}{q^n}-\sum_{\mathbb{N}_2}\frac{\varepsilon_n}{q^n}=0 \qquad (***)$$

constructed in Lemma 4. It must contain infinitely many members if α is the angle of q_i; otherwise q is the zero of a polynomial and $\sum_{\mathbb{N}_1}\frac{\varepsilon_n}{q_i^n}-\sum_{\mathbb{N}_2}\frac{\varepsilon_n}{q_i^n}=0$ which is impossible. From (∗∗) we know that the partial sum of (∗∗∗) are bounded with q_i instead of q. Since there are infinitely many summand of modulus 1 and each of them has a negative component, orthogonal to β, it is possible only when the angle of the summands tend to $\beta(\pm\pi)$. Consequently there is a situation where the partial sum goes unity in almost β direction, then unity in almost $\beta+\pi$ direction. Hence it must arrive to the same circle. If the angles are close enough to $\beta(\pm\pi)$ this can occur only when the returning point is identical to the starting point. But this is impossible since every summand has a negative contribution, orthogonal to β. ∎

PROOF OF THE THEOREM. The above Lemmas show that $l_2(q) > 0$ implies that q is Pisot. Conversely if q is Pisot then all of $l_k(q)$ are positive, as it is shown in [2] (his proof is identical to that of [3] for the special case $k=l$). ∎

OPEN PROBLEMS.

1. Is the Theorem true for $A < q < 2$?
2. Prove or disprove: q is Pisot $\Leftrightarrow$ $l_1(q) > 0$.

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ANNALES UNIV. SCI. BUDAPEST., **39** (1996), 101–105

CORRECTION OF MY PAPER 'ON DOUBLE-LATTICELIKE SPHEREPACKING'[1]

By

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The main result of the abovementioned paper [1] is based on the following statement:

STATEMENT. Let L be a 3-dimensional lattice and denote $\{\underline{e}_1,\underline{e}_2,\underline{e}_3\}$ the successive minima of L. Then the parallelepiped $P[\underline{e}_1,\underline{e}_2,\underline{e}_3]$ (which is spanned by the system $\{\underline{e}_1,\underline{e}_2,\underline{e}_3\}$) can be decomposed to L-polyhedra.

This statement in this form is not true, but a simple investigation shows that the other results of this paper follow from the undermentioned weaker statement:

STATEMENT*. Let L be a 3-dimensional lattice. Then there exists such a system of successive minima $\{\underline{e}_1,\underline{e}_2,\underline{e}_3\}$ of L for which the parallelepiped $P[\underline{e}_1,\underline{e}_2,\underline{e}_3]$ can be decomposed by L-polyhedra.

The first part (1) of the proof of the *statement* (i.e. when for $i \neq j$ $\left\langle \underline{e}_i,\underline{e}_j \right\rangle > 0$) is correct so we have to discuss only the second part (2) of this proof. We shall distinguish four cases. Note that we can choose $\epsilon_i \in \{-1,1\}$ such that either all of $\left\langle \epsilon_i\underline{e}_i,\epsilon_j\underline{e}_j \right\rangle$, $(i \neq j)$ are non-negative, or all of them are negative while also $\{\epsilon_i\underline{e}_i\}$ forms a successive minimum system. Therefore we will suppose that either all of $\left\langle \underline{e}_i,\underline{e}_j \right\rangle$ $(i \neq j)$ are non-negative or all of them are negative.

1. If all three vectors are orthogonal to each other then the lattice is a 'brick-lattice' and the spanned parallelepiped is an L-polyhedron.

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2. If two vectors are orthogonal to the third (e.g. $\underline{e}_1$ and $\underline{e}_2$ are orthogonal to $\underline{e}_3$, while $\langle \underline{e}_1, \underline{e}_2 \rangle > 0$) then in the plane of these vectors the triangles $H(0, \underline{e}_1, \underline{e}_2)$ and $H(\underline{e}_1, \underline{e}_2, \underline{e}_1 + \underline{e}_2)$ are L-simplices prisms above these triangles are the L-polyhedra.

3. If $\underline{e}_1$ is orthogonal to $\underline{e}_2$ but the angles of these vectors with $\underline{e}_3$ are acute we decompose the examined parallelepiped to the union of two pyramids (with a rectangle-face) and two simplices. At this time it can be proved easily from the 'diagonal condition' that the examined simplices are L-simplices. This means that the union of two such pyramids which have a common rectangular face can be decomposed to L-polyhedra. Since those lattice-simplices which contain the main diagonal of the double-pyramid (which is not in the plane of the rectangular faces) as edge are not L-simplices we have only two cases for the decomposition. If there is no circumscribed-ball of the double pyramid then the original parallelepiped can be decomposed to L-polyhedra, namely to two simplices and two rectangular pyramids. In the other case when the double-pyramid is an L-polyhedron then there hold the equalities:

$$|\underline{e}_1| = |\underline{e}_1 - \underline{e}_3| \qquad |\underline{e}_2| = |\underline{e}_2 - \underline{e}_3|.$$

This means that for example the vector system $\{-\underline{e}_1, \underline{e}_2 - \underline{e}_3, \underline{e}_3\}$ is such a system of successive minima for which the parallelepiped spanned by it can be decomposed to L-polyhedra.

4. Finally let, for $i \neq j$, $\left\langle \underline{e}_i, \underline{e}_j \right\rangle < 0$. Then we can decompose each face of $P[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ by its diagonal containing 0 or $\underline{e}_1 + \underline{e}_2 + \underline{e}_3$ into acute triangles in virtue of the diagonal condition. Then $P[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ can be decomposed to six tetrahedra, each being the convex hull of the diagonal $0, \underline{e}_1 + \underline{e}_2 + \underline{e}_3$, and an edge of $P[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ not containing any of 0 and $\underline{e}_1 + \underline{e}_2 + \underline{e}_3$. These have volume $\frac{1}{6} V(P[\underline{e}_1, \underline{e}_2, \underline{e}_3])$, and therefore if each of their faces is an acute triangle, then they are L-simplices. Their faces lying on the boundary of $P[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ are acute triangles. Let us consider a face in $P[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ e.g. $0, \underline{e}_1, \underline{e}_1 + \underline{e}_2 + \underline{e}_3$ (because of symmetry each other face is treated similarly). By the diagonal condition we have $|\underline{e}_1| \leq |\underline{e}_1 + \underline{e}_2 + \underline{e}_3|$, so we need to consider the angles at the vertices $\underline{e}_1$, 0 only. The first of these is acute by $\langle -\underline{e}_1, \underline{e}_2 + \underline{e}_3 \rangle > 0$, the second one is non-obtuse by $\langle \underline{e}_1, \underline{e}_1 + \underline{e}_2 + \underline{e}_3 \rangle = \frac{1}{2}(|\underline{e}_1 + \underline{e}_2|^2 - |\underline{e}_2|^2) + + \frac{1}{2}(|\underline{e}_1 + \underline{e}_3|^2 - |\underline{e}_3|^2) \geq 0$ using the diagonal condition. If however in this last inequality equality holds, then $|\underline{e}_1 + \underline{e}_2| = |\underline{e}_2|$ and thus $\{\underline{e}_1, \underline{e}_2 + \underline{e}_1, -\underline{e}_3\}$ is a successive minimum system, for which the enclosed angles are all acute, by $|\underline{e}_1 + \underline{e}_2| = |\underline{e}_2|$. Thus for this successive minimum system there holds the assumption of part (1) of the proof.

This completes the proof.

REMARK 1. In the third case if the successive minima are minima then the double-pyramid is regular and the lattice is the so-called regular simplex one.

REMARK 2. In the proof of Lemma 3, we replace the text beginning with ‘It can be seen’ in line 13 of p.55, till the end of proof, by the following: However it is well-known, (KORKIN-ZOLOTAREV, cf. GRUBER-LEKKERKERKER, §39.5 p.409) that in E^3 the only extremal lattice (i.e. a lattice having locally maximal packing density) is the lattice generated by three edge-vectors of a regular tetrahedron, having a common end-point. Now we finish the proof of Lemma 3 like in the last two sentences of part a, in the proof Lemma 3 in [1].

REMARK 3. The third sentence in the proof of the Theorem is replaced by the following: From this statement already follows the statement of the theorem, because, in the case of $\underline{e}_3$ perpendicular to the hyperplane $[\underline{e}_1, \underline{e}_2]$ the density is maximal if and only if the volume of the basic parallelepiped is minimal. Now the L-cells are prisms with generatrices $\underline{e}_3$, over the L-cells of the 2-dimensional lattice L' spanned by $\underline{e}_1$, $\underline{e}_2$, that are either rectangles or acute triangles, of circumradius, say, r. Then $r^2 + \left(\frac{|\underline{e}_3|}{2}\right)^2 = R^2 = 4$, and by fixed r, $|\underline{e}_3|$, the area of the basic parallelogram of L' is minimal if either it is a $\sqrt{4r^2-4} \times 2$ rectangle, or if the L-cells of L' are isosceles triangles of equal sides 2. Using still that the edges of the L-decomposition of L' are not less than two and $|\underline{e}_3| \geq 2$ we have $2 \leq r^2 \leq 3$, resp. $1 \leq 2\sin\frac{\gamma}{2} \leq \sqrt{2}$ where γ is the angle of the equal sides of the triangular L-cells. In the two cases we have

$$V = 8\sqrt{(r^2-1)(4-r^2)}, \quad \text{resp.} \quad V = 8\sqrt{(2sin\frac{\gamma}{2})^2[3-(2\sin\frac{\gamma}{2})^2]},$$

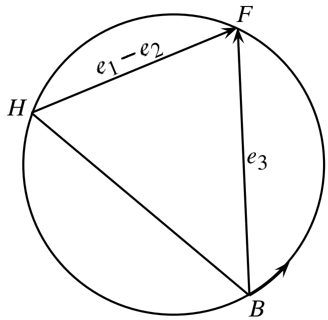
and these attain their minima at the end-points of their domains, and these minima are equal to $8\sqrt{2}$. The quadratic forms of the basic lattices L of the optimal systems are the following:

$$f_1(x_1,x_2,x_3) = 4(x_1^2 + x_2^2 + 2x_3^2) \qquad f_1(x_1,x_2,x_3) = 4\left(x_1^2 + x_2^2 + x_1x_2 + \frac{8}{3}x_3^2\right)$$

1. First we investigate the case $\left\langle \underline{e}_i, \underline{e}_j \right\rangle \geq 0$. The cases 1., 2., 3., of the proof of the *statement** and the original proof of the *statement* show that if one of the shortest diagonals of the octahedron $EBCHFD$ is BH, that implies $\langle \underline{e}_2, \underline{e}_1 - \underline{e}_3 \rangle \geq 0$, $\langle \underline{e}_3, \underline{e}_1 - \underline{e}_2 \rangle \geq 0$, then the tetrahedra $ABDE$, $EBFH$ and $FBHC$ are parts of L-polyhedra, respectively.

The inequality $|\underline{e}_3| \leq |\underline{e}_3 - \underline{e}_i|$ in the third line of (1) in p.59 is replaced by:‘$|\underline{e}_3| \leq |\underline{e}_3 - n_1\underline{e}_1 - n_2\underline{e}_2|$, n_1, n_2 integers’.

The third line of (2) in p.60 is replaced by: 'Then $\langle \underline{e}_1 - \underline{e}_2, \underline{e}_3 \rangle < 0$' and correspondingly *Fig. 4/b* is replaced by:



2. After (3) in p.60 add following text: Second we investigate the case $\left\langle \underline{e}_i, \underline{e}_j \right\rangle$,0. In 4. in the proof of the *statement** we distinguished two cases. In the first subcase we have to examine the following three case:

(1) the circumscribed ball of the tetrahedron $ABFG$ is the support ball with the greatest radius;

(2) the circumscribed ball of the tetrahedron $ABCG$ is the support ball with the greatest radius;

(3) the circumscribed ball of the tetrahedron $AEFG$ is the support ball with the greatest radius.

These tetrahedra are parts of *L*-polyhedra, too. The suitable motions can be constructed easily as in the above cases of the proof of the Theorem, for example in case (1) we define the corresponding motions with *Fig. 6*.

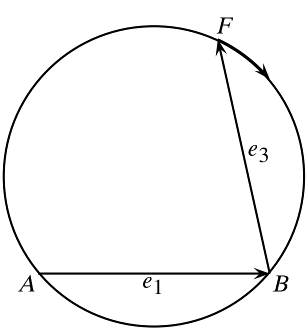


Fig. 6.

When the second subcase holds we have to change the basic parallelepiped P with the new one $\Pi[\underline{e}_1, \underline{e}_2 + \underline{e}_1, -\underline{e}_3]$. For this basic parallelepiped the enclosed angles are all non-obtuse and we can use the first part of the proof of this Theorem.

In case of (2) and (3), in the first subcase we use the motions shown in *Figures 7* and 8, respectively, while in the second subcase we proceed as in case (1).

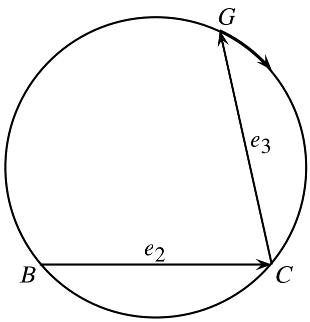


Fig. 7.

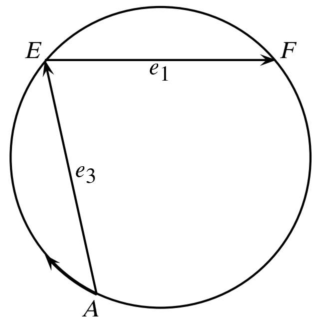


Fig. 8.

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THE RIGIDITY OF SPECIAL d CUBE GRIDS

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1. Introduction

One of the simplest structures in statics are the frameworks.

DEFINITION 1. A framework consists of rigid rods connected by rotatable joints.

DEFINITION 2. A framework is rigid if any continuous motion of the joints that keeps the length of every rod fixed, also keeps fixed the distance between every pair of vertices in the framework.

Let us consider an $n_1 \times n_2 \times \ldots \times n_d$ cube grid in the d space. The corresponding rod and joint framework is a mechanism in the d space. Let the length of the rods be unit. This cube grid framework consists of $(n_1 + {} + 1) \ldots (n_{i-1} + 1) n_i (n_{i+1} + 1) \ldots (n_d + 1)$ pieces of parallel V_i rods, V_i denotes the i-th axe of the d dimensional coordinate system, for $1 \leq i \leq d$. There are $(n_1 + 1)(n_2 + 1) \ldots (n_d + 1)$ pieces of rotatable joints in the grid.

We suppose that each cube is a rhomboid during any motion of the vertices (thus we disregard those motions of the cube where the vertices of any 'square' face do not remain coplanar). (Throughout, quotation marks will refer to the original situation). Thus the 2^{d-1} pieces of 'parallel' edges of the unit cubes are parallel during the motion of the vertices. If the distance of the two middle points of the opposite edges of each two dimensional square face are unit then the special assumption is satisfied obviously. For instance, this special assumption has to be realized technically by medianrods joining with joints the middle points of the two opposite rods of each two dimensional square face (*Fig. 1.*). This construction allows infinitesimal motion, but it is rigid according to the Definition 2. Naturally, there is not necessary to use the

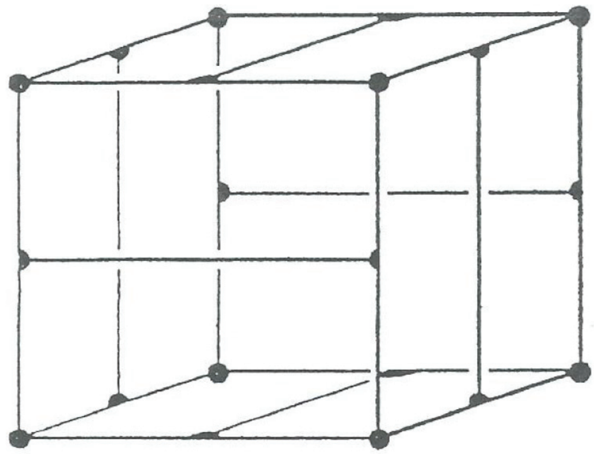

Fig. 1

medianrods in every two dimensional square face, although its exact number and places are interesting and open problem.

BOLKER and CRAPO solved the case $d=2$, when the special assumption is not needed. We want to make the special cube grid rigid using some diagonals along the square faces of the unit cubes as diagonal bracing. The consequence of our special assumption is that the rods between opposite hyperfaces of a cube are parallel to each other. Thus every rod between two 'hyperplanes' are parallel. Naturally these 'parallel hyperplanes' are 'perpendicular' to the V_i. Consider those rods that are 'parallel' to V_i. There are $(n_1+1)\dots(n_{i-1}+ +1)(n_{i+1}+1)\dots(n_d+1)$ pieces of parallel rods between the j-th and $(j+1)$-th neighbour hyperplanes, $1 \le j \le n_i$. These rods are characterized by the vector v_{ij}, $1 \le j \le n_i$, (*Fig. 2.*).

Every square of the grid is characterized by two pieces of former vectors v_{ij}, v_{kl} $(1 \le k \le d,\ 1 \le l \le n_k,\ i \neq k)$. If there is a diagonal brace in one of the characterized squares then the two vectors are perpendicular during any motion of the vertices. Thus applying a diagonal brace for any of the characterized square will fix the others as well.

The special cube grid is rigid if and only if every 'square' remains square during any motion of the joints. Let us define the bracing graph of the special cube grid framework. We have d point classes: V_i, $1 \le i \le d$, and there are n_i points in the point class V_i. The points are also denoted by v_{ij}, because they correspond to the vectors v_{ij}. Let two vertices v_{ij}, v_{kl} $(1 \le k \le d)$, $1 \le l \le n_k$, $i \neq k)$ be adjacent if and only if there is a diagonal brace in the two dimensional square characterized by vectors v_{ij}, v_{kl}.

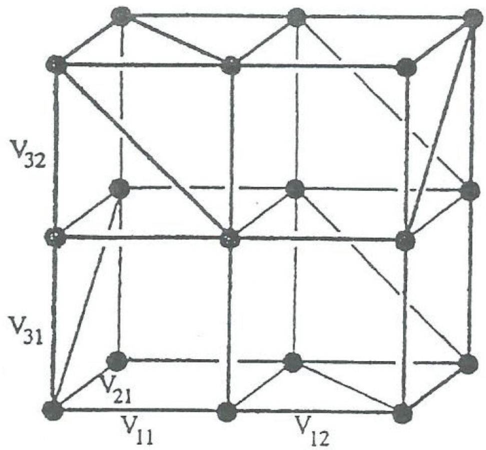

Fig. 2

Thus we have a d-partite graph D. We indicate the diagonal braces as edges in the d-partite graph. If D is complete, that is, if there exists an edge between every pair of points if they are in different point classes, then the cube grid is rigid, because each rhomboid is a cube in the grid.

However, less diagonal bracing may also be sufficient. Consider the $\binom{d}{2}$ pieces of bipartite subgraphs of the d-partite bracing graph D.

2. Necessity

THEOREM 1. *If the special d cube grid bracing with diagonal braces is rigid then the bipartite subgraphs of the d-partite bracing graph are connected.*

PROOF. If the bipartite subgraph V_i V_k is disconnected then there exists a motion which can be composed from elementary motions parallel to the coordinate axes V_i and V_k, as a consequence of the following theorem of Bolker and Crapo: An $n \times m$ square grid bracing with diagonal braces or without diagonal brace is rigid if and only if its bracing graph is connected.

The necessary condition is also sufficient:

THEOREM 2. *The special d cube grid as a bar and joint framework bracing with diagonal braces or without diagonal brace is rigid if and only if the bipartite subgraphs of the d-partite bracing graph are connected.*

Before the proof of the sufficiency we introduce a special framework on the d dimensional sphere. V_{ij} are the end points of the respective grid vectors on the unit sphere. The special cube grid is rigid if and only if every two vectors v_{ij} and v_{kl} are perpendicular if $i \neq k$, during any motion. If there is a diagonal brace between any two rods characterized by vectors v_{ij}, v_{kl} then we denoted it by a rod on the unit sphere between the points V_{ij}, V_{kl}. Thus a framework s arises on the sphere (*Fig. 3.*) If this framework s is rigid on the sphere then the special cube grid is rigid in the space.

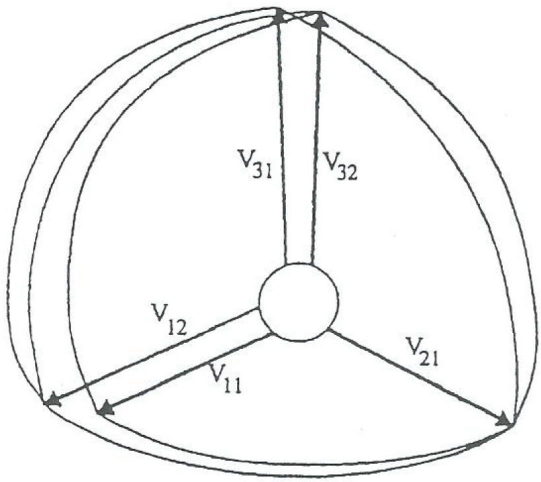

Fig. 3

Define the graph c of framework F as follows: the vertices of the graph c correspond to the joints of F and there is an edge between two points of c if there is a rod between the corresponding two joints of the framework.

We have a framework on the sphere and its graph c is d-partite and its bipartite subgraphs are connected. It is clear from the definitions of the graph c of framework s and the bracing graph D that the graph c of s and the d-partite bracing graph D are isomorphic. D denotes the bracing graph of the original framework and c denotes the graph of the fictitious framework on the unit sphere.

3. Sufficiency

If we can prove that the framework s is rigid on the sphere the Theorem 2 is true. In this case we need the coordinates of the points V_{ij}. Let us introduce a new system of coordinates V_i', where the origin of the coordinate system is still at the centre of unit sphere and let the hyperplane determined by the points V_{ij} be perpendicular to one of the new axes, for example V'_d. The rank of the rigidity matrix [7] depends on the new V'_i coordinates of the points V_{ij} and on the graph c only, since these points form a regular $d-1$ dimension simplex which is parallel to the new hyperplane V'_i, $1 \le i \le d-1$. The d-th coordinates of the V_{ij} are equal to each other. Thus we can simplify the original problem.

Consider a special cube grid bracing with some diagonal braces along square faces; it is rigid if and only if framework p is rigid in the former hyperplane. The joints of p are the same as those of framework s and there is a rod between two joints if there is rod in framework s. Since joints of s are in a hyperplane at the beginning of the motion, it suffices to consider the rigidity of the framework p in the $d-1$ dimensional hyperplane.

Let us introduce stil another system of coordinates V''_i, where the origin of the coordinate system is at V'_{dj} and let the hyperplane determined by the points V_{ij}, $1 \le i \le d-1$ be perpendicular to one of the new axes, for example V''_{d-1}. The rank of the rigidity matrix depends on the new V''_i coordinates of the points V_{ij} and on the graph c only, since these points form a regular $d-2$ dimensional simplex which is parallel to the new hyperplane V''_i, $1 \le i \le d-2$. The $d-1$-th coordinate of the V_{ij}'s are equal to each other. Thus the rank of the rigidity matrix decreases at least by $n_1+n_2+\ldots+n_{d-1}+(d-1)(n_d-1)$ if we delete the joints V_{dj} and the incident rods of framework p.

Repeating the former idea several times we get to the joints V_{1j}, V_{2j} and their framework which is rigid in dimension 1. This completes the proof of the theorem.

In the proof we gave a new framework (framework p) in a hyperplane that is rigid in its plane if and only if the special cube grid is rigid in the space and the graph c of the framework p is isomorphic to the bracing graph of the special cube grid.

Thus we need $(d-1)(n_1+n_2+\ldots+n_d)-\binom{d}{2}$ face diagonal braces for the rigidity of an $n_1 \times n_2 \times \ldots \times n_d$ special cube grid.

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STABILIZATION OF DIRAC EXPANSIONS BY RIESZ AND OTHER MEANS

By

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In the study of eigenfunction expansions the following stabilization problem arises. In the partial sums or means of spectral expansions we use the Fourier coefficients of the function expanded. If the coefficients of the function are known only with a small but unknown error, how does it influence the convergence? How much the error bound has to be diminished? This question has been posed and answered for the Laplace expansions in the interesting paper of KRUKOVSKII [1]. In this paper we aim to extend Krukovskii's result to Dirac expansions.

Consider first the case of one-dimensional Dirac operator. The corresponding eigenvalue problem is

$$\begin{aligned} u_2' + (V(x) + m)u_1 &= \lambda u_1 \\ -u_1' + (V(x) - m)u_2 &= \lambda u_2. \end{aligned} \tag{1}$$

Here $m > 0$ is the rest mass of the particle, $V \in L_1^{\text{loc}}(G)$ is the potential, $x \in G \subset \mathcal{R}$ is a finite or infinite open interval, $\lambda \in \mathcal{C}$ is arbitrary.

In HORVÁTH [2] a square sum estimate of eigenfunctions of (1) is proved in the following form. Consider a system

$$u_n = \begin{pmatrix} u_{n1} \\ u_{n2} \end{pmatrix} \qquad u_n \in L_2(G, \mathcal{C}^2), \quad n \in \mathcal{I}$$

of eigenfunctions of (1) with eigenvalues

$$\lambda_n \in \mathcal{C}, \qquad \lambda_n = \varrho_n + i\nu_n.$$

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Suppose that there exists a biorthogonal system.

$$V_n \in L_2(G, \mathscr{C}^2), \qquad (u_n, v_k) = \delta_{nk}.$$

Suppose further that $(u_n)_{n\in\mathscr{I}}$ is a Bessel-system i.e. there exists $c > 0$ such that

$$\sum_{n\in\mathscr{I}} |(f, u_n)|^2 \le c\|f\|^2_{L_2}, \qquad f \in L_2(G, \mathscr{C}^2). \tag{2}$$

Now if $V \in L_2^{\mathrm{loc}}(G)$ and $K \subset G$ is a compact subinterval then there exists $R_0 = R_0(K, m, V)$ such that for every $\mu \ge 1$

$$\sum_{|\mu - |\varrho_n|| \le 1} \|u_n\|^2_{L_\infty(K, \mathscr{C}^2)} e^{2|\nu_n|R_0} \le c \tag{3}$$

where $c = c(K, V, m, R_0)$ is independent of μ.

First we prove the following converse of (3) in the stronger form.

THEOREM 1. *Suppose that $V \in L_2^{\mathrm{loc}}$ and that $(u_n)_{n\in\mathscr{I}}$ is a Riesz–Fischer system i.e. (2) holds and with reversed inequality (2) holds also with another constant. Then there exist $M > 0$ and $0 < R < R_0$ such that*

$$\sum_{|\mu - |\varrho_n| \le M} |u_n(x)|^2 e^{2|\nu_n|R} \ge c, \qquad x \in G \tag{4}$$

locally uniformly in G.

PROOF. Recall the mean value formulae ([2] or [4])

$$\begin{aligned} u_{n,1}(x+t) &= \cos\lambda_n t \cdot \mu_{n,1}(x) - \sin\lambda_n t \cdot \mu_{n,2}(x) - \\ &- \int_x^{x+t} \big[-(m + V(\xi)) \sin\lambda_n (x+t-\xi) n_{n,1}(\xi) + \\ &\qquad +(m - V(\xi)) \cos\lambda_n (x+t-\xi) n_{n,2}(\xi)\big]\, d\xi. \end{aligned} \tag{5}$$

$$\begin{aligned} u_{n,2}(x+t) &= \sin\lambda_n t \cdot u_{n,1}(x) - \cos\lambda_n t \cdot \mu_{n,2}(x) - \\ &- \int_x^{x+t} \big[(m + V(\xi)) \cos\lambda_n (x+t-\xi) \mu_{n,1}(\xi) + \\ &\qquad +(m - V(\xi)) \sin\lambda_n (x+t-\xi) \mu_{n,2}(\xi)\big]\, d\xi. \end{aligned} \tag{6}$$

Denote I_1 and I_2 the integrals contained in (5) and (6), respectively.

Take an $R_0 > 0$ with the property

$$\sum_{|\mu - |\varrho_n|| \le 1} \|u_n\|^2_{L_\infty(K, \mathscr{C}^2)} e^{2|\nu_n|R_0} \le c, \qquad 0 < R \le R_0. \tag{3}$$

We apply the reversed inequality (2) to the function

$$f(y) = \begin{pmatrix} \cos\mu(y-x) \\ 0 \end{pmatrix} \mathcal{X}_{(x,x+R)}(y)$$

and suppose that

$$x \in K_1, \qquad R \le \frac{1}{2}|K|$$

where K_1 denotes the left half segment of K. We get from (5)

$$c \le \sum_{n\in\mathcal{I}} |(u_n, f)|^2 = \sum_{n\in\mathcal{I}} \left| a_n u_{n,1}(x) - b_n u_{n,2}(x) - \int_0^R \cos\mu t \cdot I_1 dt \right|^2$$

where

$$a_n = \int_0^R \cos\mu t \cos\lambda_n t dt, \qquad b_n = \int_0^R \cos\mu t \sin\lambda_n t dt.$$

We put the absolute value into the integral to obtain

$$\left| \int_0^R \cos\mu t \cdot I_1 dt \right| \le$$
$$\le \int_0^R e^{|\nu_n|t} \left[\int_x^{x+R} |m + V(\xi)||u_{n,1}(\xi)|d\xi + \right.$$
$$\left. + \int_x^{x+R} |m - V(\xi)||u_{n,2}(\xi)|d\xi \right] dt \le$$
$$\le \int_0^R e^{|\nu_n} t ||u_n||_{L_\infty(K,\mathcal{C}^2)} \cdot 2 \int_x^{x+R} (m + |V(\xi)|) d\xi dr =$$
$$= \overline{\overline{o}}_R(1) ||u_n||_{L_\infty(K,\mathcal{C}^2)} \cdot \begin{cases} R & \text{if } |\nu_n| \le B \\ \frac{e^{|\nu_n|R}}{|\nu_n|} & \text{if } |\nu_n| \ge B \end{cases}$$

where $B > 0$ is an arbitrary fixed constant. Further

$$|a_n| = \left| \int_0^R \frac{\cos(\mu - \lambda_n)t + \cos(\mu + \lambda_n)t}{2} dt \right| =$$
$$= \left| \frac{\sin(\mu - \lambda_n)R}{2(\mu - \lambda_n)} + \frac{\sin(\mu + \lambda_n)R}{2(\mu + \lambda_n)} \right| \le \left\{ \begin{matrix} R \\ \frac{e^{|\nu_n|R}}{|\nu_n} \end{matrix} \right\}$$

and the same estimate holds for b_n. Consequently

$$c \le \sum_{|\mu - |\varrho_n|| \ge M} \left| a_n u_{n,1}(x) - b_n u_{n,2}(x) - \int_0^R \cos\mu t \cdot I_1 dt \right|^2 +$$

$$+\sum_{\mu-|\varrho_n||^2\le M}|a_n u_{n,1}(x)-b_n u_{n,2}(x)|^2+$$

$$+\sum_{|\mu-|\varrho_n||\le M}2|a_n u_{n,1}(x)-b_n u_{n,2}(x)|\left|\int_0^R\cos\mu t\cdot I_1\,dt\right|+$$

$$+\sum_{|\mu-|\varrho_n||\le M}\left|\int_0^R\cos\mu t\cdot I_1\,dt\right|^2=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4. \tag{7}$$

Here we have

$$\Sigma_2\le 2\sum_{|\mu-|\varrho_n||\le M}|u_n(x)|^2\left\{\frac{R^2}{\frac{e^{2|\nu_n|R}}{|\nu_n|^2}}\right\}, \tag{8}$$

$$\Sigma_3,\Sigma_4\le \overline{\overline{o}}_R(1)\sum_{|\mu-|\varrho_n||\le M}||u_n||^2_{L_\infty(K,\mathcal{C}^2)}\left\{\frac{R^2}{\frac{e^{2|\nu_n|R}}{|\nu_n|^2}}\right\}\le cM\overline{\overline{o}}_R(1) \tag{9}$$

since the $\overline{\overline{o}}$-term is uniform in $x\in K_1$ and its bound depends only on V, m and K. To estimate Σ_1 we need that

$$|a_n|\le\left|\frac{\sin(\mu+\lambda_n)R}{2(\mu+\lambda_n)}+\frac{\sin(\mu-\lambda_n)R}{2(\mu-\lambda_n)}\right|\le\frac{e^{|\nu_n|R}}{|\nu_n-|\varrho_n||}$$

and analogously

$$|b_n|\le\frac{e^{\nu_n R}}{|\nu_n-|\varrho_n||}.$$

Finally, for $0<a<R$

$$\left|\int_a^R\cos\mu t\sin\lambda_n(t-a)dt\right|=\left|\cos\mu a\int_a^R\cos\mu(t-a)\sin\lambda_n(t-a)dt-\right.$$
$$\left.-\sin\mu a\int_a^R\sin\mu(t-a)\sin\lambda_n(t-a)dt\right|\le 2\frac{e^{|\nu_n|R}}{|\nu_n-|\varrho_n||}$$

and analogously

$$\left|\int_a^R\cos\mu t\cos\lambda_n(t-a)dt\right|\le 2\frac{e^{|\nu_n|R}}{|\nu_n-|\varrho_n||}$$

hence the integral in Σ_1 can be estimated as follows

$$\left|\int_a^R \cos\mu t \int_x^{x+t} \left[-(m+V(\xi))\sin\lambda_n(x+t-\xi)u_{n,1}(\xi)+\right.\right.$$
$$\left.\left.+(m-V(\xi))\cos\lambda_n(x+t-\xi)u_{n,2}(\xi)\right] d\xi dt\right| =$$
$$=\left|\int_x^{x+R}\left[(m+V(\xi))u_{n,1}(\xi)\int_{\xi-x}^R \cos\mu t\sin\lambda_n(t-(\xi-x))dt+\right.\right.$$
$$\left.\left.+(m-V(\xi))u_{n,2}(\xi)\int_{\xi-x}^R \cos\mu t\cos\lambda_n(t-(\xi-x))dt\right]d\xi\right| \leq$$
$$\leq 2\frac{e^{|\nu_n|R}}{|\nu_n-|\varrho_n||}||u_n||_{L_\infty(K,\mathscr{C}^2)}\int_x^{x+R}\left[|m+V(\xi)|+|m-|V(\xi)|\right]d\xi \leq$$
$$\leq c2\frac{e^{|\nu_n|R}}{|\nu_n-|\varrho_n||}||u_n||_{L_\infty(K,\mathscr{C}^2)}$$

with $c=c(K,m,V)$. Hence

$$\Sigma_1 \leq c\sum_{|\mu-|\varrho_n||\geq M}\frac{||u_n||_\infty^2 e^{2|\nu_n|R}}{|\mu-|\varrho_n||^2}\leq$$
$$\leq c\sum_{k=M}^\infty\frac{1}{k^2}\sum_{k\leq|\mu-|\varrho_n||\leq k+1}||u_n||_\infty^2 e^{2|\nu_n|R}\leq c\sum_{k=M}^\infty\frac{1}{k^2}\leq\frac{c}{M}$$

if $M\geq 1$. Putting (8), (9) and (10) into (7) yields

$$c\leq\frac{c_1}{M}+2\sum_{|\mu-|\varrho_n||\leq M}|u_n(x)|^2\left\{\begin{matrix}R^2\\ \frac{e^{2|\nu_n|R}}{|\nu_n|^2}\end{matrix}\right\}+c_2M\overline{\overline{o}}_R(1).$$

Now we choose M large enough to ensure $c_1/M\leq c/4$ and for such an M fixed we choose $0<R<\min(R_0,|K|/2)$ such that $c_2M\overline{\overline{o}}_R(1)\leq c/4$. Then

$$\frac{c}{2}\leq 2\sum_{|\mu-|\varrho_n||\leq M}|u_n(x)|^2\left\{\begin{matrix}R^2\\ \frac{e^{2|\nu_n|R}}{|\nu_n|^2}\end{matrix}\right\}\leq c_3\sum_{|\mu-|\varrho_n||\leq M}|u_n(x)|^2e^{2|\nu_n|R},$$
$$x\in K_1.$$

By symmetry the same estimate holds if $x\in K_2$ i.e. if x is in the right half interval of K. Theorem 1 is proved. ∎

In [1] I. M. Krukovskii introduced the notion of (φ,λ)-means by a monotone decreasing function

$$\varphi[0,\infty]\to[0,1],\qquad \varphi(0)=1,\qquad \lim_\infty\varphi=0.$$

The corresponding means are

$$\sigma_\lambda(f,x) = \sum_{n\in\mathcal{I}} (f,v_n)\varphi\left(\frac{|\varrho_n|}{\lambda}\right) u_n(x).$$

Here (u_n) is supposed to be a Riesz basis in $L_2(G,\mathscr{C}^2)$.

Remark that for $\varphi = \mathcal{X}_{[0,1)}$ we get back the partial sums

$$\sigma_\lambda^0(f,x) = \sum_{|\varrho_n|<\lambda} (f,v_n)u_n(x)$$

and for $\varphi(t) = (1-t)^s \mathcal{X}_{[0,1)}(t)$ the Riesz means

$$R_\lambda^s(f,x) = \sum_{|\varrho_n|<\lambda} \left(1 - \frac{|\varrho_n|}{\lambda}\right)^s (f,v_n)u_n(x)$$

of index $s > 0$. Concerning the stability problem mentioned in the introduction we can state

THEOREM 2. *Let $\lambda(\delta)$ be a decreasing function of $\delta > 0$, $\lim_{\delta\to 0}\lambda(\delta) = \infty$, $\lim_{\delta\to 0+}\delta\sqrt{\lambda(\delta)} = 0$. Suppose that*

$$\int_0^\infty \varphi^2 < \infty$$

and that for a given $f \in L_2(G,\mathscr{C}^2)$ and $x \in G$ we have

$$\sigma_\lambda(f,x) \to f(x), \qquad \lambda \to \infty.$$

Then taking any $f_\delta \in L_2(G,\mathscr{C}^2)$, $\|f - f_\delta\|_2 \le \delta$ we have

$$\sigma_\lambda(f_\delta,x) \to f(x), \qquad \lambda = \lambda(\delta), \quad \delta \to 0+.$$

If the convergence $\sigma_\lambda f \to f$ is uniform in some compact subinterval $K \subset G$, then $\sigma_\lambda f_\delta \to f$ is also uniform in K.

PROOF. We know that

$$\sigma_\lambda(f_\delta,x) - f(x) = \sigma_\lambda(f,x) - f(x) + \sigma_\lambda(f_\delta - f,x).$$

It is enough to show that

$$\sigma_\lambda(f_\delta - f,x) \to 0, \qquad \lambda = \lambda(\delta), \quad \delta \to 0+ \tag{11}$$

locally uniformly in G. We count

$$|\sigma_\lambda(f_\delta - f,x)| \le \sum_{-\infty}^{\infty} |(f_\delta - f,v_n)|\varphi\left(\frac{|\varrho_n|}{\lambda}\right)|u_n(x)| \le$$

$$\leq \left(\sum |(f_\delta - f, v_n)|^2\right)^{\frac{1}{2}} \left(\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right)|u_n(x)|^2\right)^{\frac{1}{2}} \leq$$

$$\leq c\|f_\delta - f\|_2 \left(\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right)|u_n(x)|^2\right)^{\frac{1}{2}}. \tag{12}$$

We use the monotonity of φ and the square sum estimate (3) to get

$$\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right)|u_n(x)|^2 \leq \sum_{m=1}^{\infty} \varphi^2\left(\frac{m-1}{\lambda}\right) \sum_{m-1\leq|\varrho_n|\leq m} |u_n(x)|^2 \leq$$

$$\leq c(K)\sum_{m=1}^{\infty} \varphi^2\left(\frac{m-1}{\lambda}\right) \leq c(K)\left(1+\sum_{m=2}^{\infty}\lambda\int_{\frac{m-2}{\lambda}}^{\frac{m-1}{\lambda}} \varphi^2(t)dt\right) =$$

$$= c(K)\left(1+\lambda\int_0^\infty \varphi^2\right) \leq c(K)\lambda.$$

Substituting this into (12) we obtain

$$|\sigma_\lambda(f_\delta - f, x)| \leq c(K)\sqrt{\lambda}\|f_\delta - f\|_2 \leq c(K)\delta\sqrt{\lambda} \tag{13}$$

which finishes the proof. ∎

REMARK. Denote

$$w(y) = \begin{cases} \frac{\sin\mu(y-x)}{\pi(y-x)}, & \text{if } |y-x| < R \\ 0, & \text{if } |y-x| \geq R \end{cases}$$

and

$$S_n(f,x) = \begin{pmatrix} (f_1,\omega)_{L_2(G)} \\ (f_2,\omega)_{L_2(G)} \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

be the μ-th trigonometric partial sum of $f \in L_2(G,\mathscr{C}^2)$. As it is proved in Horváth [2], if $V \in L_p^{\mathrm{loc}}(G)$ for some $p > 2$ then

$$S_\lambda(f,x) - \sigma_\lambda^0(f,x) \to 0 \qquad (\lambda\to\infty)$$

locally uniformly in G. Taking into account Theorem 2 and in particular (13) we get the following

COROLLARY. *If $V \in L_p^{\mathrm{loc}}(G)$ for some $p > 2$ and if $\lambda(\delta)$ satisfy the properties from Theorem 2 then*

$$S_\lambda(f,x) - \sigma_\lambda^0(f_\delta,x) \to 0 \qquad (\lambda=\lambda(\delta), \delta\to 0+)$$

locally uniformly in x, where $f_\delta \in L_2$ can be arbitrarily chosen with the assumption $\|f - f_\delta\|_2 \leq \delta$. ∎

This corollary implies another statements by the known properties of trigonometric expansions, e.g. that

$$\sigma_\lambda^0(f_\delta, x) \to f(x) \qquad \text{a.e., } \delta \to 0+, f \in L_2$$

and

$$\sigma_\lambda^0(f_\delta, x) \to f(x) \qquad \text{locally uniformly, } \delta \to 0, f \in C^1$$

and so on.

We shall see that the above used condition $\lambda = \overline{\overline{o}}(\delta^{-2})$ is the best possible in the following sense.

THEOREM 3. *If* $\lim_{\delta \to 0+} \lambda(\delta) = \infty$ *but* $\lim_{\delta \to 0+} \delta \sqrt{\lambda(\delta)} \not\to 0$ *then for every* $f \in L_2(G, \mathscr{C}^2)$*, for every* x_0 *satisfying* $\sigma_\lambda(f, x_0) \to f(x_0)$ *and for every* $\delta > 0$ *there exists* $f_\delta \in L_2(G, \mathscr{C}^2)$ *such that* $\|f - f_\delta\|_2 = \delta$ *but*

$$\sigma_\lambda(f_\delta, x_0) \not\to f(x_0), \qquad \lambda = \lambda(\delta), \quad \delta \to 0+.$$

PROOF. We have to show the exactness of the estimate (13). Define f_δ by the coefficients

$$(f_\delta, v_n) = (f, v_n) + c_1 \cdot \delta \cdot \frac{|u_n(x_0)| \varphi\left(\frac{|\varrho_n|}{\lambda}\right) \cdot \epsilon_n}{\left(\sum_{-\infty}^{\infty} |u_k(x_0)|^2 \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right)\right)^{1/2}}, \qquad |\epsilon_n| = 1.$$

Then

$$\sum |(f_\delta - f, v_n)|^2 = c_1^2 \delta^2 \asymp \|f_\delta - f\|_2^2$$

and we can choose c_1 such that $\|f_\delta - f\|_2 = \delta$. Clearly, c_1 varies with δ but it has positive lower and upper bounds by the Riesz basis property of (u_n). The ϵ_n depends also on δ. Now

$$\sigma_\lambda(f_\delta - f, x_0) =$$

$$= \frac{c_1 \delta}{\left(\sum_{-\infty}^{\infty} |u_k(x_0)|^2 \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right)\right)^{1/2}} \cdot \sum_{-\infty}^{\infty} \epsilon_n \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right) |u_n(x_0)| u_n(x_0).$$

If we set $\epsilon_n = |u_{n,1}(x_0)| / u_{n,1}(x_0)$ then

$$|\sigma_\lambda(f_{\delta,1} - f, x_0)| \geq c_1 \delta \frac{\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right) |u_{n,1}(x_0)|^2}{\left(\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right) |u_n(x_0)|^2\right) 1/2}$$

and if $\epsilon_n = |u_{n,2}(x_0)| / u_{n,2}(x_0)$ then

$$|\sigma_\lambda(f_{\delta,2} - f, x_0)| \geq c_1 \delta \frac{\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right) |u_{n,2}(x_0)|^2}{\sum \varphi^2\left(\frac{|\varrho_n|}{\lambda}\right) |u_n(x_0)|^2}.$$

Hence taking $f_\delta - f_{\delta,1}$ or $f_\delta = f_{\delta,2}$ we can obtain

$$|\sigma_\lambda(f_\delta - f, x_0)| \geq \frac{1}{2} c_1 \delta \left(\sum \varphi^2 \left(\frac{|\varrho_n|}{\lambda} \right) |u_n(x_0)|^2 \right)^{1/2} \geq$$

$$\geq \frac{c_1}{2} \delta \left(\sum_{m=1}^{\infty} \varphi^2 \left(\frac{mM}{\lambda} \right) \sum_{(m-1)M \leq |\varrho_n| \leq mM} |u_n(x_0)|^2 \right)^{1/2} \geq$$

$$\geq c\delta \left(\sum_{m=1}^{\infty} \left(\frac{mM}{\lambda} \right) \right) \geq c\delta \left(\sum_{m=1}^{\infty} \frac{\lambda}{M} \int_{\frac{mM}{\lambda}}^{\frac{(m+1)M}{\lambda}} \varphi^2(t) dt \right)^{1/2} =$$

$$= c\delta \left(\frac{\lambda}{M} \int_{\frac{M}{\lambda}}^{\infty} \varphi^2 \right) \geq c\delta \sqrt{\lambda} \not\to 0 \qquad (\delta \to 0+, \lambda = \lambda(\delta))$$

which proves Theorem 3. ∎

Next we consider the corresponding problem for the Dirac operator in the space $\mathcal{R}^3$ without any potential. The operator has the form

$$H = hc \sum_{k=1}^{3} \gamma_4 \gamma_k \partial x_k + mc^2 \gamma_4$$

Where m is the rest mass, h is the Planck constant, c is the velocity of light[1] and the 4×4 matrices γ_i are given by

$$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Consider the eigenvalue problem

$$Hu_n = \lambda_n u_n, \quad \lambda_n \in \mathcal{C}_1,$$
$$u_n = (u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4})^T \quad u_{n,i} = u_{n,i}(x_1, x_2, x_3).$$

[1] It can easily seen that the above formula for operator H might be expressed by the *fine structure constant* (α) as well, because $hc = e^2/\alpha$, where e is the elementary charge of electron. We remark that for the analytic determination of fine structure constant the formula $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$ was proposed by [6] and the numeric value of $4\pi^3 + \pi^2 + \pi \sim 137.03630\ldots$ fits to the most recent experimentally measured values of α. The possible hypothesized precise value of fine structure constant could ensure another “stabilization effect”.

It is known that the $u_{n,i}$ are eigenfunctions of the Laplace operator:

$$-\Delta u_{n,i} = \frac{\lambda_n^2 - \mu^2 c^4}{c^2 h^2} u_{n,i}.$$

If the system (u_n) is Bessel-system in $L_2(\Omega, \mathscr{C}^4)$ with a domain $\Omega \subset \subset \mathscr{R}^3$ then $(u_{n,i})$ is also Bessel-system; it can be seen if the property is written to the functions $f = (f_1, 0, 0, 0)$, $f = (0, f_2, 0, 0)$, $f = (0, 0, f_3, 0)$ or $f = = (0, 0, 0, f_4)$. Consequently the systems $(u_{n,1})$, $(u_{n,2})$, $(u_{n,3})$, $(u_{n,4})$ satisfy square sum estimates of type (3) as it is proved in HORVÁTH [5], i.e.

$$\sum_{|\mu - |\varrho_n|| \le 1} |u_n(x)|^2 e^{2|\nu_n|R} \le c\mu^2, \qquad x = (x_1, x_2, x_3).$$

Analogously, if (u_n) is Riesz–Fischer system in $L_2(\Omega, \mathscr{C}^4)$ then $(u_{n,1})$ is Riesz–Fischer system, consequently [5] there exist $M > 0$ such that

$$\sum_{|\mu - |\varrho_n|| \le M} |u_n(x)|^2 e^{2|\nu_n|R} \ge c\mu^2.$$

These estimates are locally uniform. Using this we can prove

THEOREM 4. *Let* $\lambda(\delta)$ *be decreasing*, $\lim_{0+} \lambda(\delta) = \infty$, $\lim_{0+} \delta\lambda^3(\delta) = 0$ *and* $\int_0^\infty \varphi^2 < \infty$ *and that for a given* $f \in L_2(\Omega, \mathscr{C}^4)$ *and* $x \in \Omega$ *we have*

$$\sigma_\lambda(f, x) \to f(x), \qquad \lambda \to \infty.$$

Then given any f_δ, $\|f - f_\delta\| \le \delta$ *we have*

$$\sigma_\lambda(f_\delta, x) \to f(x), \qquad \delta \to 0+, \quad \lambda = \lambda(\delta).$$

If $\sigma_\lambda f \to f$ *locally uniformly then* $\sigma_\lambda f_\delta \to f$ *locally uniformly.*

THEOREM 5. *If* $\lambda(\delta) \to \infty$ *but* $\delta\lambda^3 \not\to 0$ *then for every* $f \in L_2(\Omega, \mathscr{C}^4)$, *for every* $x_0 \in \Omega$ *satisfying* $\sigma_\lambda(f, x_0) \to f(x_0)$ *and for every* $\delta > 0$ *there exists* $f_\delta \in L_2(\Omega, \mathscr{C}^4)$ *such that* $\|f - f_\delta\|_2 = \delta$ *and*

$$\sigma_\lambda(f_\delta, x_0) \not\to f(x_0), \qquad \delta \to 0+, \quad \lambda = \lambda(\delta).$$

The proofs are very similar to the one-variable case, so we omit them. ∎

Using the convergence theorems proved in HORVÁTH [3] we get the following consequences:

COROLLARY 1. Let $f \in \mathring{H}_p^\alpha(\Omega, \mathscr{C}^4)$, $\operatorname{supp} f \subset \Omega$, $\alpha p > 3$, $1 < p < \infty$ and $0 \le s < 1/2$, $\alpha + s \ge 3/2$. Then $\lambda(\delta) \to \infty$, $\delta\lambda(\delta)^3 \to 0$ implies that

$$R_\lambda^s(f_\delta, x) \to f(x) \qquad \delta \to 0+, \quad \lambda = \lambda(\delta)$$

locally uniformly in $x \in \Omega$, whenever $||f - f_\delta||_2 \leq \delta$. ∎

COROLLARY 2. Let $f \in \mathring{H}_2^\alpha(\Omega, \mathscr{C}^4)$, $\operatorname{supp} f \subset \Omega$, $\alpha > 0$, $0 \leq s < 1/2$, $\alpha + s \geq 3/2$. Then $\lambda(\delta) \to \infty$, $\delta\lambda(\delta)^3 \to 0$ implies that

$$R_\lambda^s(f_\delta, x) \to 0 \qquad \delta \to 0+, \quad \lambda = \lambda(\delta)$$

locally uniformly in $x \in \Omega \setminus \operatorname{supp} f$, whenever $||f - f_\delta||_2 \leq \delta$. ∎

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CERTAIN COMPLEMENTARY SPACES AND MULTIPLIERS FOR DOUBLE WALSH SERIES

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1. Dyadic martingales

We shall denote the set of non-negative integers by **N**, the set of positive integers by **P**, the set of real numbers by **R**, and the set of dyadic rationals in the unit interval $[0,1]$ by **Q**. In particular, each element of **Q** has the form $k/2^n$ for some $k,n\in\mathbf{N}$ with $0\le k\le 2^n$. Furhermore, let $\mathbf{I}:=[0,1)$ be the unit interval.

By a dyadic interval in **I** we mean intervals of the form $[m/2^n,(m+1)/2^n)$ for some $m,n\in\mathbf{N}$ with $0\le m<2^n$. Given $n\in\mathbf{N}$ and $x\in\mathbf{I}$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. We denote the collection of dyadic intervals by $\mathcal{J}$.

For any set $X\ne\emptyset$ we denote by X^2 the cartesian product $X\times X$. Thus $\mathbf{N}^2$ is the collection of integral latice points in the first quadrant, and $\mathbf{I}^2$ is the unit square. A sequence and a double sequence will be represented in the form $(z_n,n\in\mathbf{N})$ and $(z_{mn};m,n\in\mathbf{N})$, respectively. Furhermore, if the limits of a $\sum$ or a $\bigcup$ are not indicated, then it is taken for each index in **N**.

Let $\mathcal{J}^2$ denote the collection of two dimensional dyadic intervals in $\mathbf{I}^2$, i.e. the sets of the form $I=I_1\times I_2$, where $I_1,I_2\in\mathcal{J}$. Then for given $(x,y)\in\mathbf{I}^2$, the dyadic intervals in $\mathcal{J}^2$ containing (x,y) are of the form

$$I_{mn}(x,y):=I_m(x)\times I_n(y), \tag{1.1}$$

where $(m,n)\in\mathbf{N}^2$.

For each $n\in\mathbf{N}$ let $\mathcal{A}^n$ represent the σ-algebra generated by the dyadic intervals $I\in\mathcal{J}$ of length 2^{-n}. Thus every element of $\mathcal{A}^n$ is a finite union of intervals of the form $[k/2^n,(k+1)/2^n)$, where $k\in\mathbf{N}$ and $k<2^n$. Denote $\mathcal{A}^{mn}$ the σ-algebra generated by the two dimensional intervals of the form (1.1), where $(x,y) \in \mathbf{I}^2$ and $(m,n) \in \mathbf{N}^2$. The collection of $\mathcal{A}^{mn}$-measurable real functions on $\mathbf{I}^2$ will be denoted by $L(\mathcal{A}^{mn})$ and the set of double dyadic step functions by $\mathcal{P}$, i.e.

$$\mathcal{P} := \bigcup_{m,n} L(\mathcal{A}^{mn})$$

(cf. [23], p. 75). The set of *W-continuous* functions, i.e. the closure of the double dyadic step functions in the supremum norm

$$\|f\| = \sup_{(s,t)\in\mathbf{I}^2} |f(s,t)|$$

is denoted by $C_W := C_W(\mathbf{I}^2)$. (For detailes and another characterization of C_W see [23], p. 9 and p. 50.) For $0 < p \leq \infty$ the $L^p := L^p(\mathbf{I}^2)$ quasi-norm or norm of any function $f \in L^p$ will be denoted by $\|f\|_p$. We denote by $|Y|$ the Lebesgue measure of the measurable subset Y in $\mathbf{I}$.

The conditional expectation of the function $f \in L^1$ with respect to $\mathcal{A}^{mn}$ is denoted by $E_{mn}f$ and can be given in the form

$$(E_{mn}f)(x,y) = \frac{1}{|I_{mn}(x,y)|} \int_{I_{mn}(x,y)} f(s,t)\,ds\,dt \qquad (1.2)$$

$$(x,y \in \mathbf{I}; m,n \in \mathbf{N}).$$

A double sequence $F=(F_{mn}; m,n \in \mathbf{N})$ of functions $F_{mn} : \mathbf{I}^2 \to \mathbf{R}$ is called a *two-parameter dyadic martingale* (in the sequel dyadic martingale) if each F_{mn} belongs to $L(\mathcal{A}^{mn})$ and

$$E_{mn}F_{MN} = F_{mn} \quad \text{for all} \quad m \leq M, n \leq N \quad \text{and} \quad m,M,n,N \in \mathbf{N}. \qquad (1.3)$$

For (1.2) and (1.3) cf. [23], p. 29, 75-76 and 318, [28], pp. 2-4. The *dyadic martingale maximal function* is defined by

$$F^*(s,t) := \sup_{m,n} |F_{mn}(s,t)| \quad (s,t \in \mathbf{I}). \qquad (1.4)$$

The set of dyadic martingales will be denoted by $\mathcal{M}$. For $0 < p \leq \infty$ and $F_{mn} \in L^p$ $(m,n \in \mathbf{N})$ we define the L^p-norm of the martingale F by

$$\|F\|_p := \sup_{m,n\in\mathbf{N}} \|F_{mn}\|_p.$$

If $\|F\|_p < \infty$, then the martingale F is called *L^p-bounded* (see [28], p. 2).

For $f \in L^1$ we define a special double sequence $F=(F_{mn}; m,n \in \mathbf{N})$ by

$$F_{mn} := E_{mn}f \quad (m,n \in \mathbf{N}). \qquad (1.5)$$

The sequence F is a martingale. Martingales of this type are called *regular* (cf. [23], p. 75). In this case we set

$$f^* := F^*.$$

Moreover, if $f \in L^p$ for some $1 \le p < \infty$, then F is L^p-bounded and

$$\lim_{m,n} \|F_{m,n} - f\|_p = 0, \tag{1.6}$$

where the limit (1.6) is taken in Pringsheim's sense.

If $1 < p < \infty$ then the martingale F can be written in the form (1.5) with a function $f \in L^p$ if and only if F is L^p-bounded (see [19], p. 68, and [28], p. 3). Thus $f \mapsto F$ is a norm-preserving map from L^p onto the space of L^p-bounded martingales if $1 < p < \infty$ and consequently the two spaces can be identified.

To define dyadic Hardy spaces H^p we shall use the martingale maximal function (1.4). For $0 < p < \infty$ denote by H^p the set of martingales $F = (F_{mn}; m, n \in \mathbf{N})$ for which

$$\|F\|_{H^p} := \|f^*\|_p < \infty. \tag{1.7}$$

Since

$$|f(x,y)| = |\lim_{n\to\infty}(E_{nn}f)(x,y)| \le |f^*(x,y)| \quad (\text{a.e.}\,(x,y) \in \mathbf{I}^2),$$

therefore (1.7) implies that H^p can be identified by a subspace of L^1 if $p \ge 1$.

The L^p-norm of F and F^* are equivalent (in notation $\|F\|_p \sim \|F\|_{H^p}$), i.e. for $1 < p < \infty$ we have (see [23], p. 81, [6], pp. 1-27)

$$\|f\|_p \le \|F^*\|_p \le q\|f\|_p \quad (f \in L^p), \tag{1.8}$$

where q is the conjugate index of p, i.e. $1/p + 1/q = 1$. Thus L^p and H^p can be identified if $1 < p < \infty$.

For any subspace $Y \subseteq L^1$ we denote by Y_0 the set of elements in Y whose integral is zero, i.e.

$$Y_0 := \left\{ f \in Y : \int_{\mathbf{I}^2} f(s,t)\, ds\,dt = 0 \right\}.$$

It is well known that for $1 \le p < \infty$ the dual space of L^p is L^q (see [15], p. 174). In one dimensional case the dual of H^1 is the space of functions with *bounded mean oscillation*, i.e. the space BMO (for the definition of BMO-norm see [23], p. 107). The space VMO is the closure of the set of one dimensional dyadic step functions in BMO-norm and the dual of VMO is H_0^1 (see [23], p. 114, Theorem 10, [28], p. 110).

In the case of two variable the dual space of H_0^1 is the two dimensional BMO space. For detailes and for the BMO-norm see [27], [22] and [5]. Moreover, if VMO is defined as the closure of $\mathcal{P}$ in BMO-norm, then the dual space of VMO is the Hardy space H_0^1 (see [28], p. 110 and [27]).

We shall use also the hibrid Hardy space $H^\#$ based on the maximal function

$$F^\#(x,y) := \sup_{k\in\mathbf{N}} \left| \int_{I_k(x)} f(s,y)\,ds \right| \quad (x,y \in \mathbf{I}, f \in L^1).$$

Thus $H^\#$ will represent the collection of functions $f \in L^1$ satisfying

$$\|F\|_\# := \|F^\#\|_1 < \infty.$$

We see that $H^1 \subset H^\#$, and if f is non-negative then f belongs to $H^\#$ if and only if $f \in L\log^+ L$, i.e.

$$\int_{\mathbf{I}^2} |f(s,t)| \log^+ |f(s,t)|\,ds\,dt < \infty.$$

In the sequel we shall use Orlicz spaces too. To fix the notations let Φ and Ψ a pair of absolutely continuous complementary Young functions (for detailles see [16], p. 134, or [29], p. 77). We shall suppose that the function Φ satisfies the $\triangle_2$-condition

$$\Phi(2t) = O(1)\,\Phi(t) \quad (t \geq t_0)$$

for some $t_0 \geq 0$. We denote by L_Φ the Orlicz space generated by the Young function Φ, i.e. L_Φ is the set of all measurable functions f, for which the norm

$$\|f\|_\Phi = \sup\left\{ \int_{\mathbf{I}^2} |f(s,t)\,g(s,t)|\,ds\,dt : \quad g \in M_\Psi \right\}$$

is finite, where

$$M_\Psi = \left\{ g : \int_{\mathbf{I}^2} \Psi(|g(s,t)|)\,ds\,dt \leq 1 \right\}$$

(see [29], p. 79, or [16], p. 145). It is known (see [29], p. 81, [2], p. 203) that $\triangle_2$-condition implies

$$L_\Phi = \left\{ f : \int_{\mathbf{I}^2} \Phi(|f(s,t)|)\, ds\, dt < \infty \right\}. \tag{1.9}$$

The Orlicz space L_Ψ can be defined in a similar way. Since $\triangle_2$-condition is not required for Ψ, the analogue of (1.9) for L_Ψ is not necessary true.

Dyadic martingales are closely connected to quasi-measures and double Walsh series.

2. Quasi-measures and homogeneous Banach spaces

The *dyadic addition* of x and y is defined (see [23], p. 10) by

$$x \dot{+} y = \sum |x_k - y_k|\, 2^{-(k+1)}.$$

Using the dyadic addition we introduce the *dyadic translation operators* $\tau_{x,y}$ for any $(x,y) \in \mathbf{I}^2$, defined (cf. [23], p. 13) by

$$(\tau_{x,y} f)(s,t) = f(s \dot{+} x, t \dot{+} y) \quad (x,y,s,t \in \mathbf{I}),$$

where $f : \mathbf{I}^2 \to \mathbf{R}$ is an arbitrary function. Dyadic translations are norm preserving, i.e. for all $f \in L^1$ and $x,y \in \mathbf{I}$ we have $\tau_{x,y} f \in L^1$ and $\|\tau_{x,y} f\|_1 = \|f\|_1$.

The *dyadic convolution* of $f, g \in L^1$ is defined by

$$(f * g)(x,y) := \int_{\mathbf{I}^2} (\tau_{x,y} f)(s,t)\, g(s,t)\, ds\, dt \qquad (x,y \in \mathbf{I}). \tag{2.1}$$

The Banach subspaces of L^1 mentioned before are homogeneous with respect to the dyadic translation. A Banach space $X \subseteq L^1$ with the norm $\|\cdot\|_X$ is called a *homogeneous Banach space* if the set $\mathcal{P}$ of double dyadic step functions is dense in X,

$$\|f\|_1 \le \|f\|_X \quad \forall f \in X \tag{2.2}$$

and its norm is translation invariant, i.e.

$$\text{if } f \in X \text{ and } x,y \in \mathbf{I} \quad \text{then} \quad \tau_{x,y} f \in X \quad \text{and} \quad \|\tau_{x,y} f\|_X = \|f\|_X.$$

LEMMA 2.1. *For a homogeneous Banach space X the following inequality holds: if $f\in L^1$ and $g\in X$ then*

$$\|f*g\|_X\le\|f\|_1\|g\|_X. \tag{2.3}$$

PROOF. First we prove (2.3) for $g\in\mathcal{P}$. In this case there exists an $n\in\mathbf{N}$ such that g is constant on every two dimensional interval

$$I^n_{k\ell}:=[k/2^n,(k+1)/2^n)\times[\ell/2^n,(\ell+1)/2^n)\quad(0\le k,\ell<2^n,n\in\mathbf{N}).$$

By the definition (2.1) of convolution we have

$$(f*g)(x,y)=\sum_{k,\ell=0}^{2^n-1}\int_{I^n_{k\ell}}f(u,v)\,g(x\dot{+}u,y\dot{+}v)\,du\,dv,\quad(x,y)\in\mathbf{I}^2.$$

Since $g(x\dot{+}u,y\dot{+}v)=g(x\dot{+}k/2^n,y\dot{+}\ell/2^n)$ for almost all $(u,v)\in I^n_{k\ell}$, therefore

$$(f*g)(x,y)=\sum_{k,\ell=0}^{2^n-1}g(x\dot{+}k/2^n,y\dot{+}\ell/2^n)\int_{I^n_{k\ell}}f(u,v)\,du\,dv\quad(x,y\in\mathbf{I}),$$

and consequently

$$f*g=\sum_{k,\ell=0}^{2^n-1}\tau_{k/2^n,\ell/2^n}g\int_{I^n_{k\ell}}f(u,v)\,du\,dv.$$

By translation invariance of the norm we obtain

$$\|f*g\|_X\le\sum_{k,\ell=0}^{2^n-1}\|\tau_{k/2^n,\ell/2^n}g\|_X\left|\int_{I^n_{k\ell}}f(u,v)\,du\,dv\right|\le$$

$$\le\|g\|_X\sum_{k,\ell=0}^{2^n-1}\int_{I^n_{k\ell}}|f(u,v)|\,du\,dv=\|g\|_X\|f\|_1$$

and (2.3) is proved for $g\in\mathcal{P}$.

Suppose now that $g\in X$. By the definition of homogeneous Banach space we can choose a sequence $(P_n,n\in\mathbf{N})$ of double Walsh polynomials, such that $\|g-P_n\|_X\to0$ as $n\to\infty$. It follows from the case already considered that for all $m,n\in\mathbf{N}$

$$\|f*P_m-f*P_n\|_X=\|f*(P_m-P_n)\|_X\le\|f\|_1\|P_m-P_n\|_X.$$

Hence $(f * P_n, n \in \mathbf{N})$ is a Cauchy sequence in X. It remains to prove that the limit is $f * g$. In view of (2.2) from the inequality $\|h\|_1 \leq \|h\|_X$ $(h \in X)$ it follows that the sequence in question converges in L^1-norm too. From (2.1) and Fubini's theorem we have

$$\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$$

and by this and (2.2)

$$\|f * P_n - f * g\|_1 \leq \|f\|_1 \|P_n - g\|_1 \leq \|f\|_1 \|P_n - g\|_X,$$

for all $n \in \mathbf{N}$, so $f * P_n \to f * g$ in L^1-norm and consequently in X-norm. Taking the limit in

$$\|f * P_n\|_X \leq \|f\|_1 \|P_n\|_X \quad (n \in \mathbf{N})$$

as $n \to \infty$, we get (2.3) in general case.

By duality argument a similar inequality can be proved for the dual space X' of any homogeneous Banach space X.

LEMMA 2.2. *Let X be a homogeneous Banach space. If $f \in L^1$ and $g \in X' \bigcap L^1$, then*

$$\|f * g\|_{X'} \leq \|f\|_1 \|g\|_{X'}. \tag{2.4}$$

PROOF. Indeed by Fubini's theorem we get for any $h \in X$ that

$$\int_{\mathbf{I}^2} h(x,y)\,(f * g)(x,y)\,dx\,dy = \int_{\mathbf{I}^2} g(s,t)\,(f * h)(s,t)\,ds\,dt.$$

Hence, applying that $X \subseteq L^1$ and (2.3) it follows that (see [15], p. 168)

$$\left| \int_{\mathbf{I}^2} h(x,y)\,(f * g)(x,y)\,dx\,dy \right| \leq \|g\|_{X'} \, \|f * h\|_X \leq \|g\|_{X'} \, \|f\|_1 \|h\|_X.$$

Taking the supremum with respect to $\|h\|_X \leq 1$ we get the required inequality (2.4).

From the translation invariance it follows that for $1 \leq p < \infty$ the L^p and H^p spaces and also the hibrid Hardy spaces $H^{\#}$ are homogeneous Banach spaces (see [23], p. 155). By definition the same hold for the spaces C_W and VMO.

The Orlicz spaces L_Φ with $\Psi(1) \leq 1$ are also homogeneous Banach spaces. Indeed, since Φ satisfies $\triangle_2$-condition, therefore $\mathcal{P}$ is dense in L_Φ (see [29], p. 86). Inequality $\|f\|_1 \leq \|f\|_\Phi$ follows from the definitions of these norms. At the end translation invariance of the integral yields that L_Φ is a homogeneous Banach space.

Furthermore the spaces L^∞, L_Ψ and BMO are duals of homogeneous Banach spaces (see [29], p. 150, and [28], p. 33, 100).

3. Double Walsh series, quasi-measures and multipliers

Let r be the function defined on $\mathbf{I}$ by

$$r(x) := \begin{cases} +1, & x \in [0, 1/2) \\ -1, & x \in (1/2, 1) \end{cases}$$

extended to $\mathbf{R}$ by periodicity of period 1. The Rademacher system $r := (r_n, n \in \in \mathbf{N})$ is defined (see [23], p. 1, cf. [1], p. 51) by

$$r_n(x) := r(2^n x) \qquad (x \in \mathbf{R}, n \in \mathbf{N}).$$

Denote $w = (w_n, n \in \mathbf{N})$ *the Walsh system in Paley's ordering*. If $m \in \mathbf{N}$ has the binary coefficients $(m_k, k \in \mathbf{N})$, then

$$w_m(x) := \prod_{k=0}^{\infty} r_k^{m_k}(x).$$

The values of the Walsh functions at $x \in \mathbf{I}$ can be expressed by the bits of x in the binary expansion $x = \sum x_k 2^{-(k+1)}$, where each $x_k = 0$ or 1. If $x \in \mathbf{I} \setminus \mathbf{Q}$, then this expansion is uniquely determined. By the dyadic expansion of $x \in \mathbf{Q}$, we shall always mean the one which terminates in 0's. This implies that the Walsh functions behave almost like characters with respect to dyadic addition, namely for almost every $x, y \in \mathbf{I}$

$$w_n(x \dotplus y) = w_n(x)\, w_n(y) \quad (n \in \mathbf{N}). \tag{3.1}$$

The double Walsh system $(w_{mn}; m, n \in \mathbf{N})$ is the Kronecker product system generated by the Walsh system, i.e.

$$w_{mn}(x, y) := (w_m \times w_n)(x, y) := w_m(x)\, w_n(y) \qquad ((m, n) \in \mathbf{N}^2, (x, y) \in \mathbf{I}^2)). \tag{3.2}$$

There is a direct connection between dyadic expansions and double Walsh functions, namely

$$w_{mn}(x, y) = (-1)^{[m,x]+[n,y]},$$

where

$$[m, x] := \sum m_k x_k \pmod 2 \quad (m \in \mathbf{N}, x \in \mathbf{I}),$$

and $m_k \in \{0,1\}$ are the binary coefficients of $m \in \mathbf{N}$, i.e. $m = \sum m_k 2^k$.

From (3.1) and (3.2) we obtain that for a.e. $x, y, s, t \in \mathbf{I}$

$$w_{mn}(x,y)\, w_{mn}(s,t) = w_{mn}(x \dotplus s, y \dotplus t). \tag{3.3}$$

Denote by $\Re$ the algebra of sets generated by the dyadic two dimensional intervals in $\mathbf{I}^2$. By a quasi-measure we shall mean a real-valued set-function which is finitely additive on $\Re$. The restriction of every finite Borel-measure on $\mathbf{I}^2$ to $\Re$ is a quasi-measure, but not conversely (cf. [23], p. 30).

We shall denote the collection of quasi-measures on $\Re$ by QM. Let VM be the set of quasi-measures with finite bounded variation, let BM be the set of finite Borel-measures on $\mathbf{I}^2$, and denote by AM the set of absolutely continuous measures in BM. We recall (cf. [23], p. 266) that the map $f \mapsto \nu^f$ defined by

$$\nu^f(J) := \int_J f(s,t)\, ds dt \qquad (J \in \Re) \tag{3.4}$$

is a 1-1 transformation from L^1 onto AM. Moreover, if $\|\nu\|$ denotes the total variation of $\nu \in VM$ then $\|\nu^f\| = \|f\|_1$, i.e. the map in (3.4) is isometric.

For any $\nu \in QM$ and $f \in \mathcal{P}$ the map

$$f \to \int_{\mathbf{I}^2} f(s,t)\, d\nu(s,t)$$

is a linear functional on $\mathcal{P}$. Moreover every linear functional on $\mathcal{P}$ is of this form .

There is a natural map between QM and $\mathcal{M}$, which can be given by the Walsh transform, defined below. If $\nu \in QM$ then the *Walsh-Fourier-Stieltjes coefficients* (shortly *Walsh coefficients*) of ν are defined by

$$\hat{\nu}(m,n) := \int_{\mathbf{I}^2} w_{mn}(s,t)\, d\nu(s,t) \qquad (m,n \in \mathbf{N}). \tag{3.5}$$

Since each Walsh function is constant on sufficiently small dyadic intervals, this definition makes sense.

The map $\nu \mapsto \hat{\nu}$ is a 1-1 function from QM onto the space of double sequences

$$\mathbf{s} := \{x : x = (x_{mn}; m,n \in \mathbf{N}),\ x_{mn} \in \mathbf{R}\}$$

is called the *Walsh transform* (cf. [23], p. 30). For $f \in L^1$ we shall denote by

$$\hat{f}(m,n) := \int_{\mathbf{I}^2} f(s,t)\, w_{mn}(s,t)\, ds dt \qquad (m,n \in \mathbf{N}) \tag{3.6}$$

the (m,n)-th Walsh-Fourier coefficient of f. From (3.4), (3.5) and (3.6) it follows (see [8], p. 180, Corollary 6) that

$$\widehat{\nu^f} = \hat{f} \qquad (f \in L^1). \tag{3.7}$$

The dyadic convolution (2.1) of $f,g \in L^1$ satisfies (cf. [23], p. 25)

$$(\widehat{f * g})(m,n) = \hat{f}(m,n) \cdot \hat{g}(m,n) \quad (m,n \in \mathbf{N}).$$

The convolution (2.1) can be extended for $\nu \in QM$ and $f \in \mathcal{P}$ by

$$(f * \nu)(x,y) := \int_{\mathbf{I}^2} f(x \dot{+} s, y \dot{+} t)\, d\nu(s,t) \quad (x,y \in \mathbf{I}). \tag{3.8}$$

The double Walsh series of $\nu \in QM$ is defined by

$$S\nu := \sum_{m,n} \hat{\nu}(m,n)\, w_{mn}, \tag{3.9}$$

and the set of double Walsh series of quasi-measures will be denoted by $\mathcal{W}$. The double Walsh series of $f \in L^1$ is defined by

$$Sf := S\nu^f \tag{3.10}$$

and by (3.7) and (3.9) we have

$$Sf = \sum_{m,n} \hat{f}(m,n)\, w_{mn}.$$

The rectangular partial sums of $S\nu$ are defined by

$$S_{00}\nu = S_{0n}\nu = S_{m0}\nu = 0, \quad S_{mn}\nu = \sum_{k,\ell=0}^{m-1,n-1} \hat{\nu}(k,\ell)\, w_{k\ell} \qquad (m,n \in \mathbf{P}). \tag{3.11}$$

The sequence of $(2^m,2^n)$-th partial sums of any double Walsh series is a dyadic martingale and conversely, every dyadic martingale can be obtained in this way (cf. [23], p. 75). Thus the investigation of $(2^m,2^n)$-th partial sums of double Walsh series leads to a study of dyadic martingales. Here we remark that also the map $\nu \mapsto (S_{2^m,2^n}\nu; m,n \in \mathbf{N})$ is an isomorphism from QM onto the collection $\mathcal{M}$ of dyadic martingales . The inverse of this map is of the form $(F_{mn}; m,n \in \mathbf{N}) \mapsto \nu$ from $\mathcal{M}$ to QM, and can be given (cf. [23], p. 30) by

$$\nu(J) := \lim_{m,n} \int_J F_{mn}(s,t)\, ds\, dt \quad (J \in \Re). \tag{3.12}$$

By (3.12) we have four pairwise isomorphic linear spaces $QM, \mathcal{M}$, $\mathbf{s}$ and $\mathcal{W}$, and the isomorphism can be given by the double Walsh system as follows:

$$\nu \mapsto (S_{2^m,2^n}\nu;\, m,n \in \mathbf{N}) \quad \text{from} \quad QM \quad \text{onto} \quad \mathcal{M},$$

$$\nu \mapsto \hat{\nu} \quad \text{from} \quad QM \quad \text{onto} \quad \mathbf{s},$$

$$\nu \mapsto S\hat{\nu} \quad \text{from} \quad QM \quad \text{onto} \quad \mathcal{W}.$$

4. Characterizations of function spaces by double Walsh series

The connection between quasi-measures and dyadic martingales can be used to characterize certain measure and function spaces in term of martingales, i.e. by the $(2^m, 2^n)$-th partial sums of the double Walsh-Fourier-Stieltjes series of the measure.

We give a characterization of these spaces by using certain summability methods. Let $A=(\alpha_{mnk\ell})$ be a triangular matrix summability method which maps series into sequences. The A-means of the double Walsh series $S\nu$ are denoted by $\sigma_{mn}\nu$, i.e.

$$\sigma_{mn}\nu := \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\,\hat{\nu}(k,\ell)\,w_{k\ell}. \tag{4.1}$$

By the definition (3.5) of Walsh coefficients and by (3.3) we have

$$(\sigma_{mn}\nu)(x,y) = \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\,w_{k\ell}(x,y) \int_{\mathbf{I}^2} w_{k\ell}(s,t)\,d\nu(s,t) =$$

$$= \int_{\mathbf{I}^2} \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\,w_{k\ell}(x,y)\,w_{k\ell}(s,t)\,d\nu(s,t) =$$

$$= \int_{\mathbf{I}^2} \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\,w_{k\ell}(x\dotplus s, y\dotplus t)\,d\nu(s,t).$$

Denote K_{mn} the kernels of the summability method A. In this case the kernels can be expressed by a two variable function, i.e.

$$K_{mn} := \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\,w_{k\ell}.$$

Thus σ_{mn} can be written in the form

$$(\sigma_{mn}\nu)(x,y) = \int_{\mathbf{I}^2} K_{mn}(x\dot{+}s, y\dot{+}t)\, d\nu(s,t).$$

and by (3.8)

$$\sigma_{mn}\nu = K_{mn} * \nu \quad (m,n \in \mathbf{N}). \tag{4.2}$$

If $\nu \in AM$, i.e. if $\nu = \nu^f$ for some $f \in L^1$ then analogously to (3.10) we have

$$\sigma_{mn}f := \sigma_{mn}\nu^f$$

and by (3.7), (3.6) and (2.1) we obtain that

$$\sigma_{mn}f = K_{mn} * f \quad (m,n \in \mathbf{N}). \tag{4.3}$$

The numbers

$$L_{mn} := \|K_{mn}\|_1 \quad (m,n \in \mathbf{N})$$

are called the *Lebesgue constants* of the summability method A.

In the sequel we suppose that A satisfies the conditions

$$\sup_{k,\ell,m,n} |\alpha_{mnk\ell}| < \infty, \quad \lim_{m,n} \alpha_{mnk\ell} = 1, \tag{4.4}$$

$$L_{mn} = O(1). \tag{4.5}$$

For the Cesàro method (C,α,β), i.e. if

$$\alpha_{mnk\ell} = (A^\alpha_{m-k}/A^\alpha_m)\cdot(A^\beta_{n-\ell}/A^\beta_n) \quad \text{with} \quad A^\alpha_n = \binom{n+\alpha}{n}$$

the condition (4.4) is satisfied. Since A is a factorizable method, therefore (4.5) follows from the calculations for simple series. Hence the claim (4.5) is known for $\alpha,\beta > 0$ (see [4], p. 297), but is not satisfied if $\alpha = 0$ or $\beta = 0$ (see [18], p. 104, [23], p. 34).

In the next theorem we characterize the elements of homogeneous Banach spaces and the elements of its duals by A-means (4.1).

THEOREM 4.1. *Let X be a homogeneous Banach space. Suppose the the matrix A satisfies* (4.4) *and* (4.5). *Then the double sequence*

$$(\sigma_{mn}\nu;\ m,n \in \mathbf{N}) \tag{4.6}$$

boundedly converges in X if and only if

$$\exists f \in X : \nu = \nu^f. \tag{4.7}$$

PROOF. *Necessity*. If the double sequence (4.6) boundedly converges in X to $f \in X$, then the inequality (2.2) implies weak convergence to the same limit. Thus for any $k, \ell \in \mathbf{N}$

$$\lim_{m,n} \int_{\mathbf{I}^2} (\sigma_{mn}\nu)(s,t)\, w_{k\ell}(s,t)\, ds dt = \int_{\mathbf{I}^2} f(s,t)\, w_{k\ell}(s,t)\, ds dt = \hat{f}(k,\ell).$$

Since for $k, \ell \le m, n$ we have

$$\int_{\mathbf{I}^2} (\sigma_{mn}\nu)(s,t)\, w_{k\ell}(s,t)\, ds\, dt = \alpha_{mnk\ell}\, \hat{\nu}(k,\ell),$$

therefore from (4.4) it follows that

$$\hat{\nu}(k,\ell) = \hat{f}(k,\ell) \quad (k,\ell \in \mathbf{N}), \tag{4.8}$$

i.e. (4.7) is true by (3.7).

Sufficiency. From assumption (4.7) and (3.7) it follows, that $\sigma_{mn}\nu^f = = \sigma_{mn}f$. We showe that for any $f \in X$ the double sequence $(\sigma_{mn}f; m,n \in \mathbf{N})$ boundedly converges to f in X-norm. Indeed by (4.3) and Lemma 2.1 from (2.3) we have

$$\|\sigma_{mn}f\|_X = \|K_{mn} * f\|_X \le \|K_{mn}\|_1 \|f\|_X \quad (m,n \in \mathbf{N}).$$

Consequently by condition (4.5) for the norms of the linear operators σ_{mn} : $X \to X$ we have

$$\|\sigma_{mn}\| = O(1). \tag{4.9}$$

The double sequence of operators σ_{mn} boundedly converges on the double Walsh system. The limit in this sense is denoted in sequels by $b-\lim$. Indeed by (4.4)

$$\text{b-}\lim_{m,n} \|\sigma_{mn}w_{k\ell} - w_{k\ell}\|_X = \text{b-}\lim_{m,n} |\alpha_{mnk\ell} - 1| \|w_{k\ell}\|_X = 0.$$

Since the linear hull of the double Walsh system coincides with the set $\mathcal{P}$ of double dyadic step functions and $\mathcal{P}$ is dense in X, then our claim in view of (4.9) is a consequence of the Banach-Steinhaus theorem (see [13], p. 41, [15], p. 204, [17], p. 12, Theorem II, [8], p. 55, Theorem 18).

THEOREM 4.2. *Let X be a homogeneous Banach space and X' its dual. Suppose that the matrix A satisfies* (4.4) *and* (4.5). *Then the double sequence* (4.6) *is bounded in X' if and only if*

$$\exists\, f \in X' : \nu = \nu^f. \tag{4.10}$$

PROOF. *Necessity*. The linear hull of the double Walsh system with rational coefficients is dense in X, consequently X is separable. Thus the unite ball in X' is weakly compact (see [13], p. 37, Theorem 2.10.1, or [29], p. 159, Theorem 8). Therefore every double sequence (4.6) satisfying the condition

$$\|\sigma_{mn}\nu\|_{X'} = O(1)$$

has a weakly convergent subsequence $(\sigma_{m_r,n_r}\nu,\ r \in \mathbf{N})$. Using this fact, our claim follows in a similar way as in Theorem 4.1.

Sufficiency. If (4.10) is true, then by (4.3) and (2.4) and Lemma 2.2 by condition (4.5) we have

$$\|\sigma_{mn}\nu\|_{X'} = \|K_{mn} * f\|_{X'} \leq \|K_{mn}\|_1 \|f\|_{X'} = O(1)\|f\|_{X'}$$

and our theorem is proved.

5. Complementary spaces and multipliers

We shall investigate linear subspaces $U, V, \ldots$ of the space of quasi-measures QM and denote by $\hat{U}, \hat{V}, \ldots$ the corresponding space of the Walsh coefficients, i.e.

$$\hat{U} := \{\hat{\nu} : \nu \in U\}.$$

Since L^1 can be identified by the subspace AM of QM, subspaces of L^1 also can be obtained in this way.

A double sequence $\lambda = (\lambda_{mn}; m,n \in \mathbf{N})$ is called a *multiplier of class* (U, V) if for every $\nu \in U$ we have

$$(\lambda_{mn}\hat{\nu}(m,n);\ m,n \in \mathbf{N}) \in \hat{V}.$$

The aim of this paper is to find effective conditions for multipliers with respect important subspaces of QM using the notions of complementary spaces and summability factors. We generalize our results from [4] to double Walsh series.

Let X be a vector subspace of QM and denote $A=(\alpha_{mnk\ell})$ a triangular matrix summability method. The collection of all quasi-measures $\mu \in QM$ for which the double series

$$\sum_{k,\ell} \hat{\nu}(k,\ell)\hat{\mu}(k,\ell) \tag{5.1}$$

is boundedly A-summable for all $\nu \in X$, is a vector subspace of QM. This space we call the *A-complementary space* of X and will be denoted by $(X \to \to A)$. Thus $\mu \in (X \to A)$ if and only if the double sequence $(F_{mn}\nu; m,n \in \mathbf{N})$, defined by

$$F_{mn}\nu := \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell}\hat{\nu}(k,\ell)\hat{\mu}(k,\ell) \quad (m,n \in \mathbf{N}) \tag{5.2}$$

boundedly converges for all $\nu \in X$. If $\nu = \nu^f$ for some $f \in L^1$, then denote

$$F_{mn}f := F_{mn}\nu^f$$

and the elements of the the double sequence $(F_{mn}\nu\,; m,n \in \mathbf{N})$ can by (3.7), (3.5), (4.1) and (4.3) be written in the form

$$F_{mn}f = \int_{\mathbf{I}^2} (K_{mn} * f)(s,t)\, d\mu(s,t) \quad (m,n \in \mathbf{N}). \tag{5.3}$$

Hence from (2.1), Fubini's theorem and (3.8), by the definition of $K_{mn} * \mu$ we get

$$F_{mn}f = \int_{\mathbf{I}^2} f(s,t)\,(K_{mn} * \mu)(s,t)\, ds dt \quad (m,n \in \mathbf{N}). \tag{5.4}$$

We shall prove that the complementary space of any homogeneous Banach space X coincides with the dual space X' of X, provided that X and X' belong to L^1.

In the following we use the next *remark*. Let $F : X \to \mathbf{R}$ be a continuous linear functional. The restriction of F to $\mathcal{P}$ can be generated by a finitely additive measure μ in the form

$$Ff = \int_{\mathbf{I}^2} f(s,t)\, d\mu(s,t).$$

The X'–norm of μ is defined by the following:

$$\|\mu\|_{X'} := \sup\{|Ff| : \|f\|_X \le 1\} = \|F\|_{X'}.$$

Thus we have for $f \in \mathcal{P} \subset X \subset L^1$ that

$$|Ff| \le \|f\|_X \|\mu\|_{X'}. \tag{5.5}$$

THEOREM 5.1. *Let $X \subseteq L^1$ be a homogeneous Banach space and X' its dual. If A satisfies* (4.4) *and* (4.5) *then*

$$(X \to A) = X',$$

PROOF. To prove the inclusion $X' \subseteq (X \to A)$ we fix $\mu \in X'$. By (5.3), (5.5) and Lemma 2.1 we obtain for $f \in X$, since $K_{mn} * f \in \mathcal{P}$, that

$$|F_{mn}f| \le \|K_{mn} * f\|_X \|\mu\|_{X'} \le \|f\|_X \|K_{mn}\|_1 \|\mu\|_{X'},$$

and consequently by (4.5) the norms satisfies

$$\|F_{mn}\| = O(1). \tag{5.6}$$

If $f = w_{k\ell}$ for some $k, \ell \in \mathbf{N}$ then by (5.2), (3.7), (3.6), (4.4) and (5.6) we have

$$F_{mn} w_{k\ell} = \alpha_{mnk\ell} \hat{\mu}(k,\ell) \to \hat{\mu}(k,\ell) \quad \text{as} \quad m,n \to \infty$$

for all $k, \ell \in \mathbf{N}$, i.e. the double sequence $(F_{mn}; m,n \in \mathbf{N})$ of bounded linear functionals boundedly converges on the dense linear subspace $\mathcal{P} \subseteq X$. Thus by (5.6) and by Banach-Steinhaus theorem this double sequence boundedly converges on X and consequently we obtain that $\mu \in (X \to A)$.

To see the opposite inclusion $(X \to A) \subseteq X'$ suppose that for some $\mu \in QM$ the double sequence $(F_{mn}f; m,n \in \mathbf{N})$ defined by (5.3) boundedly converges for all $f \in X$. Then by Banach-Steinhaus theorem condition (5.6) holds. By (5.4) in view of the definition and estimates of the norm in dual space (see [21], p. 78, [23], p. 114, [27], [28], p. 33, 100, and [29], p. 138) and by (4.2) we have

$$\|F_{mn}\| \sim \|K_{mn} * \mu\|_{X'} = \|\sigma_{mn}\mu\|_{X'}.$$

Thus by Theorem 4.2 we have (4.10) for μ instead of ν and for some $f \in X'$. Since the map $f \to \mu^f$ is an 1-1 map from $X' \subseteq L^1$ into AM, therefore $\mu \in X'$.

THEOREM 5.2. *Let $X \subseteq L^1$ be a homogeneous Banach space and X' its dual. If A satisfies* (4.4) *and* (4.5) *then*

$$(X' \to A) = X.$$

PROOF. To see that $X \subseteq (X' \to A)$ let $f \in X$. If $\nu = \nu^f$, then (3.7) implies that $\sigma_{mn}\nu^f = \sigma_{mn}f$ and by Theorem 4.1 the double sequence $(\sigma_{mn}f; m,n \in \mathbf{N})$ boundedly converges in X. Denote

$$\tilde{F}_{mn}\mu := \sum_{k,\ell=0}^{m,n} \alpha_{mnk\ell} \hat{\mu}(k,\ell) \hat{\nu}(k,\ell) \quad (m,n \in \mathbf{N}).$$

Then by (3.5), (4.1) and (5.5) we obtain

$$|\tilde{F}_{m+k,n+\ell}\mu - \tilde{F}_{mn}\mu| = \left| \int_{\mathbf{I}^2} (\sigma_{m+k,n+\ell}f - \sigma_{mn}f)(s,t)\, d\mu(s,t) \right| \leq$$

$$\leq \|\sigma_{m+k,n+\ell}f - \sigma_{mn}f\|_X \|\mu\|_{X'},$$

therefore $(\tilde{F}_{mn}\mu; m,n \in \mathbf{N})$ is a double Cauchy sequence and (see [24], p. 158, or [20], p. 255) boundedly converges for every $\mu \in X'$, i.e. $f \in (X' \to A)$.

For the opposite inclusion we can be prove in similar way as in Theorem 5.1 that $\|\sigma_{mn}\mu\|_{X''} = O(1)$. Since $X \subseteq X''$ (see [13], p. 214), therefore

$$\|\sigma_{mn}\mu\|_X = O(1). \tag{5.7}$$

Thus by the principle of uniform boundedness ([29], p. 135, Theorem 1) it follows that there exists a constant $\gamma > 0$, such that

$$\|\sigma_{mn}\mu\|_X \le \gamma\|\mu\|_X. \tag{5.8}$$

Since the set of double Walsh polynomials is dense in X, therefore there exists a double sequence $(P_{mn}; m,n \in \mathbf{N})$ of double Walsh polynomials such that

$$\mathrm{b} - \lim_{m,n} \|\mu - P_{mn}\|_X = 0. \tag{5.9}$$

From (5.8) for any fix $M,N \in \mathbf{N}$ we get

$$\|\sigma_{mn}\mu - \sigma_{m+k,n+\ell}\mu\|_X \le \|\sigma_{mn}\mu - \sigma_{mn}P_{MN}\|_X +$$
$$+\|\sigma_{mn}P_{MN} - \sigma_{m+k,n+\ell}P_{MN}\|_X + \|\sigma_{m+k,n+\ell}P_{MN} - \sigma_{m+k,n+\ell}\mu\|_X \le$$
$$\le 2\gamma\|\mu - P_{MN}\|_X + \|\sigma_{mn}P_{MN} - \sigma_{m+k,n+\ell}P_{MN}\|_X.$$

From (4.4) it follows that

$$\lim_{m,n} \sup_{k,\ell\in\mathbf{N}} \|\sigma_{mn}P_{MN} - \sigma_{m+k,n+\ell}P_{MN}\|_X = 0,$$

consequently

$$\mathrm{b} - \lim_{m,n} \sup_{k,\ell\in\mathbf{N}} \|\sigma_{mn}\mu - \sigma_{m+k,n+\ell}\mu\|_X \le 2\gamma\|\mu - P_{MN}\|_X$$

for any $M,N \in \mathbf{N}$ and by (5.9) the left hand side is zero. Thus $(\sigma_{mn}\mu; m,n \in \in \mathbf{N})$ is a double Cauchy sequence in X and therefore by (5.7) it boundedly converges in X (see [7], p. 11, Theorem 5) and by Theorem 4.1 we have $\mu = \mu^f$ for some $f \in X$, and consequently $\mu \in X$.

For single trigonometric Fourier series the notion of complementary space was introduced by Goes ([9], p. 348; [10], p. 373; [11], p. 151, [12],p. 135) for the Cesàro method $A = (C,\alpha)$ with $\alpha > 0$ and by Tõnnov ([16], p. 75) for any regular A. For double trigonometric Fourier series in the case of triangular method A see [3], p. 208.

Let $A = (\alpha_{mnk\ell})$ and $B = (\beta_{mnk\ell})$ two triangular matrix summability methods which map double series into double sequences. The double sequence $\lambda = (\lambda_{mn}; m,n \in \mathbf{N})$ is called a *summability factor of type* (A_b, B_b) if for each boundedly A-summable double series

$$\sum_{k,\ell} u_{k\ell}$$

the double series

$$\sum_{k,\ell} \lambda_{k\ell} u_{k\ell}$$

is boundedly B-summable. In this case we shall write $\lambda \in (A_b, B_b)$.

We prove (cf. [12], p. 143, [26], p. 94, [3], p. 211)

THEOREM 5.3. *If the double sequence λ is a summability factor of type (A_b, B_b) then λ is a multiplier of the class $((X \to A), (X \to B))$ for any space X and any summability methods A and B.*

PROOF. Let $\mu \in (X \to A)$ be arbitrary. Then the double series (5.1) is boundedly A-summable for any $\nu \in X$. Define the map $T: QM \to QM$ by

$$(\widehat{T\mu})(k,\ell) := \lambda_{k\ell}\hat{\mu}(k,\ell) \quad (k,\ell \in \mathbf{N}).$$

If $\lambda \in (A_b, B_b)$, then

$$\sum_{k,\ell} \hat{\nu}(k,\ell)\,(\widehat{T\mu})(k,\ell)$$

is boundedly B-summable for any $\nu \in X$, and therefore $T\mu \in (X \to B)$ by the definition of $(X \to B)$. Thus $\lambda \in ((X \to A), (X \to B))$.

If we apply Theorem 5.3 for a homogeneous Banach space X or for its dual X' and put $B = A$, then we get

COROLLARY 5.4. *Let X be a homogeneous Banach space. If λ is a summability factor of type (A_b, A_b) then λ is a multiplier of the class (X, X) and of the class (X', X').*

Using the notations

$$\Delta_m^{\gamma}\lambda_{mn} := \sum_{k=m}^{\infty} A_{k-m}^{-\gamma-1}\lambda_{kn},$$

$$\Delta_{mn}^{\gamma,\delta}\lambda_{mn} := \sum_{k,\ell=m,n}^{\infty} A_{k-m}^{-\gamma-1}A_{\ell-n}^{-\delta-1}\lambda_{k\ell},$$

and applying the known theorem of summability factors of type (A_b, A_b) for the method $A = (C,\alpha,\beta)$ with $\alpha,\beta > 0$ (see [2], Theorem 2, and [14], Supplement to Theorem 6), we obtain the following result.

COROLLARY 5.5. *Let X be a homogeneous Banach space. If for some $\alpha,\beta > 0$ the conditions*

$$\lim_n \Delta_m^{\alpha+1}\lambda_{mn} = \lim_m \Delta_n^{\beta+1}\lambda_{mn} = 0, \tag{5.10}$$

$$\lambda_{mn} = O(1), \quad \sum_{m,n}(m+1)^{\alpha}(n+1)^{\beta}|\Delta_{mn}^{\alpha+1,\beta+1}\lambda_{mn}| < \infty \tag{5.11}$$

are satisfied, then λ is a multiplier of the class (X, X) and of the class (X', X').

It is known (see [14]) if (5.11) is satisfied for some $\alpha,\beta > 0$, then it is satisfied for any $\gamma \in (0,\alpha)$ and $\delta \in (0,\beta)$.

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METRISCHE REALISIERUNGEN VON ZWEI FAMILIEN DER DREIDIMENSIONALEN KÖRPERTRANSITIVEN SYMPLEXPFLASTERUNGEN*

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1. Einführung

Wir betrachten diejenigen kombinatorischen dreidimensionalen Polyederpflasterungen, die wir aus der Arbeit [10] bekommen. In diesem Artikel haben E. Molnár und I. Prok die körpertransitiven kombinatorischen dreidimensionalen Symplexpflasterungen (T,Γ) klassifiziert, und sie haben — mit Hilfe eines kombinatorischen Algorithmus und eines Computerprogramms, die auf der Theorie der sogenannten D-Symbole beruhen — ihre vollständige Aufzählung angegeben. Eine Polyederpflasterung T ist körpertransitiv, wenn es zu zwei beliebigen Polyedern T_1 und T_2 der Pflasterung T ein Element $\gamma \in \Gamma$ gibt, das das Polyeder T_1 in $T_2 = T_1^\gamma$ überführt; so daß die ganze Pflasterung T auf sich abgebildet wird.

Zwei Pflasterungen (T_1,Γ_1) und (T_2,Γ_2) sind in derselben Klasse (äquivariant), wenn es eine bijektive inzidenztreue Abbildung $\Phi : T_1 \to T_2$ gibt, für die $\Gamma_2 = \Phi^{-1}\Gamma_1\Phi$ ist. Falls zwei Pflasterungen kombinatorisch isomorph sind ($T_1 \cong T_2$) kann die Gruppe Γ_2, reichhaltiger als Γ_1, sein. In diesem Fall sagen wir, daß (T_1,Γ_1) ein Symmetriebruch von (T_2,Γ_2) ist. Wir interessieren uns dann für die Pflasterung (T,Γ) mit maximaler Bewegungsgruppe, d.h., daß die Wirkung der Gruppe Γ auf T zur Wirkung der die Inzidenzstruktur erhaltenden Automorphismengruppe äquivariant ist ($\Gamma \cong \mathrm{Aut}\, T$).

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Besonders interessant sind diejenigen Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, die zur Familie 24 bzw. zur Familie 31 gehören und die durch die Abbildungen 1 und 2 dargestellt werden. Die zukünftigen erzeugenden Bewegungen r_0, r_1, r_2, r_3 sind für die Gruppe $\Gamma^{24}_{(a,b)} := \Gamma_{18}(4a, 8b)$ Halbdrehungen. Unter diesen Erzeugenden bestehen die folgenden definierenden Relationen in der Gruppe:

$$\Gamma_{18}(4a, 8b) \quad a \geq 1,\ b \geq 1,$$
$$r_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_0 r_1 r_3 r_1)^a = (r_1 r_2 r_3 r_0 r_2 r_0 r_3 r_2)^b = 1.$$

Wir bezeichnen die zukünftigen erzeugenden Bewegungen für die Gruppe $\Gamma^{31}_{(a,b)} := \Gamma_{43}(4a, 8b)$ mit r_0, r_1, z. Unter diesen Erzeugenden bestehen die folgenden definierenden Relationen in der Gruppe:

$$\Gamma_{43}(4a, 8b), \quad 1 \leq a,\ 1 \leq b,$$
$$r_0,\ r_1,\ z:\ A_0 A_3 A_1 \to A_0 A_1 A_2,$$
$$r_0^2 = r_1^2 = (z r_0 z r_0)^a = (r_0 r_1 z^{-2} r_1 z^2 r_1)^b = 1.$$

Wenn $a \neq 2b$, dann sind diese Pflasterungen maximal. Sonst, im Falle $a = 2b$, wäre Aut T eine obere Gruppe von Γ_{18} (bzw. von Γ_{43}), denn wir können mit regulären Tetraedern von äußeren Ecken eine Pflasterung im hyperbolischen Raum $\mathbf{H}^3$ mit größerer Symmetrie realisieren. Im Fall $(a, b) = (1, 1)$ würden die Eckpunkte des Polyeders der Polyederpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$ auf dem Absolutgebilde eines Raumes liegen und die Stabilisatorgruppen würden planare euklidische Gruppen sein, wenn es diese Pflasterungen überhaupt gibt. Wir konnten aber in der Arbeit [12] und im Abschnitt 4 die metrische Realisierbarkeit in diesen Fällen ($a = 1$, $b \geq 1$) widerlegen. Die Frage wird in einer kommenden Arbeit von E. MOLNÁR und J. WEEKS weitgehend diskutiert werden.

In den Fällen $a > 1$, $b \geq 1$ und $a = 1$, $b > 1$ sind die Eckpunkte der Pflastersteine außerhalb des Absolutgebildes, und die Stabilisatorgruppen der Eckpunkte sind planare hyperbolische Gruppen (Abb. 1, Abb. 2).

Die Frage nach der metrischen Realisierbarkeit einer kombinatorischen Symplexpflasterung ist im allgemeinen schwierig (siehe noch [4], [11], [12], [13], [14], [15]). Wie man aus den Arbeiten [8], [10], [12] sehen kann, kommen sehr außergewöhnliche Räume z.B. $\mathbf{H}^2 \times \mathbf{R}$, $\mathbf{S}^2 \times \mathbf{R}$, **Nil** und noch weitere Phänomene vor

E. MOLNÁR hat in den Arbeiten [9], [12] eine Methode für die Kennzeichnung der metrischen Realisierbarkeit der kombinatorischen Polyederpflasterungen entwickelt, deren Grundbegriffe ich in dem Abschnitt 2 darlegen werde

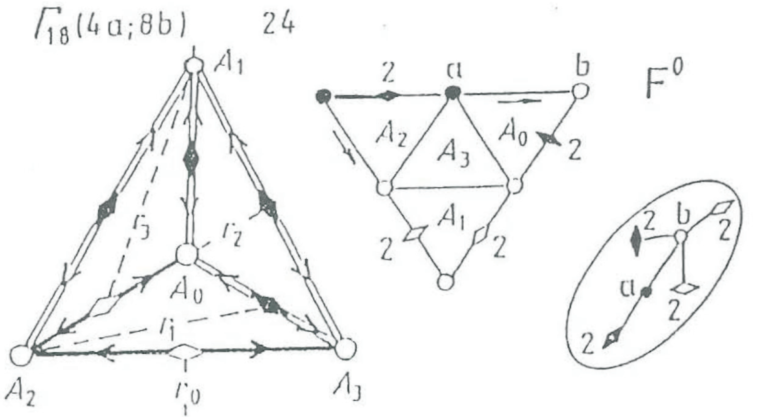

Abb. 1.

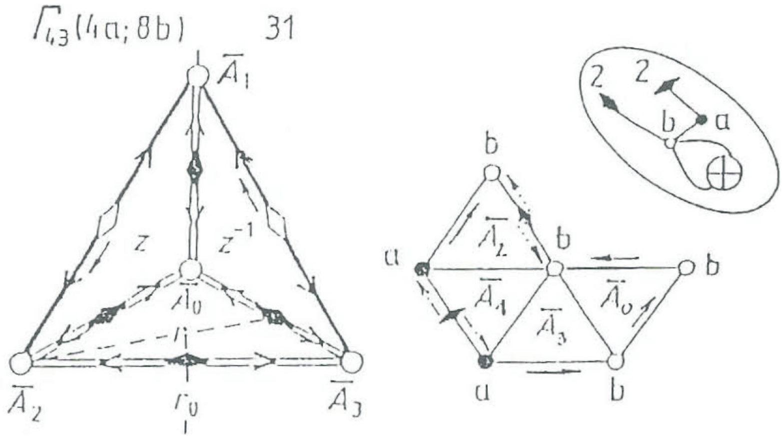

Abb. 2.

Mit Hilfe dieser Methode, die auf der Theorie der projektiv-metrischen Geometrie beruht, beweisen wir im Satz 3.1, daß die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$ $(a \geq 2,\ b \geq 1)$ im hyperbolischen Raum $\mathbf{H}^3$ metrisch realisierbar sind. Dann ist Γ also eine diskrete Bewegungsgruppe des projektiv erweiterten hyperbolischen Raumes $\mathbf{H}^3$, die auf den Pflastersteinen transitiv wirkt.

Im vierten Abschnitt werden wir im Satz 4.1 beweisen, daß die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a=1,\ b\geq 2)$ metrisch nicht realisierbar sind, d.h., daß man keine Geometrie unter den 8 maximalen, 3-dimensionalen homogenen Geometrien (von THURSTON, [9]), fiden kann, in deren Raum die obigen Pflasterungen metrisch existieren.

2. Über die projektiv–metrischen Räumen

2.1. Es sei $\mathbf{V}^4$ ein 4-Vektorraum über dem reellen Körper $\mathbf{R}$. Der duale Raum von $\mathbf{V}^4$ wird mit V_4 bezeichnet.

Die Vektoren $\mathbf{x} \in \mathbf{V}^4$ bzw. die Formen $\boldsymbol{x} \in V_4$ werden mit stehenden fetten bzw. mit kursiven fetten Buchstaben bezeichnet. Betrachten wir eine Basis von $\mathbf{V}^4$: $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{e}_i\}$.

$$\mathbf{x} = x^0\mathbf{e}_0 + x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = (x^0, x^1, x^2, x^3)\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} =: x^i\mathbf{e}_i. \tag{2.1}$$

wobei wir die Einstein–Schouten Konventionen benutzen. Die duale Vektoren oder linearen Formen $\boldsymbol{e}^j \in V_4$, $j \in \{1,2,3,4\}$ bilden die duale Basis zu $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{e}_i\}$ mit der Gleichung $\mathbf{e}_i\boldsymbol{e}^i = \delta_i^j$ (das Kronecker Symbol). Ferner sei

$$\boldsymbol{u} = \boldsymbol{e}^0u_0 + \boldsymbol{e}^1u_1 + \boldsymbol{e}^2u_2 + \boldsymbol{e}^3u_3 = (\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3)\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix} =: \boldsymbol{e}^k u_k.$$

Dann ist das formale innere Produkt $\mathbf{x}\boldsymbol{u}$ nach der Formel (2.1) das folgende:

$$\boldsymbol{xu} = (x^i\boldsymbol{e}_i)(\boldsymbol{e}^k u_k) = x^i(\boldsymbol{e}_i\boldsymbol{e}^k)u_k = x^i\delta_i^k u_k = x^i u_i. \tag{2.2}$$

Wir können eine lineare Polarität wie üblich definieren:

$$\begin{aligned} &(_*) : V_4 \to \mathbf{V}^4 : \boldsymbol{x} \mapsto \boldsymbol{x}_* =: \mathbf{x}, \\ &\text{die linear:} \quad (\boldsymbol{x}a + \boldsymbol{y}b)_* = a\boldsymbol{x}_* + b\boldsymbol{y}_* = a\mathbf{x} + b\mathbf{y}, \\ &\text{und symmetrisch ist:} \quad \boldsymbol{x}_*\boldsymbol{y} = \mathbf{x}\boldsymbol{y} = \mathbf{y}\boldsymbol{x} = \boldsymbol{y}_*\boldsymbol{x}. \end{aligned} \tag{2.3}$$

Wenn die Basen $\{\mathbf{e}_i\}$, $\{\boldsymbol{e}^j\}$ festgelegt sind, dann können wir die Matrix (b^{ij}) der linearen Polarität aufschreiben :

$$\boldsymbol{e}^i_* = \mathbf{e}^i = b^{ij}\mathbf{e}_j. \tag{2.4}$$

Wenn wir nicht entartete Polarität haben, dann können wir die inverse Polarität definieren

$$({}^*) : \mathbf{V}^4 \to \boldsymbol{V}_4, \quad \mathbf{x} \mapsto \mathbf{x}^* =: \boldsymbol{x}.$$

Die Matrix (a_{jk}) dieser inversen Polarität ist durch die folgende Formel anzugeben:

$$\mathbf{e}_j^* = \boldsymbol{e}_j = \boldsymbol{e}^k a_{jk}. \tag{2.5}$$

Nach den folgenden Gleichungen ist es zu sehen, daß die Matrix (b^{ij}) die inverse Matrix von (a_{jk}) ist:

$$\mathbf{e}_i = (\mathbf{e}_i^*)_* = (\boldsymbol{e}^j a_{ji})_* = a_{ij}(\boldsymbol{e}^j)_* = a_{ij} b^{jk} \mathbf{e}_k \;\Rightarrow\; a_{ij} b^{jk} = \delta_i^k. \tag{2.6}$$

Daraus folgt die Zusammenhang zwischen den Koordinaten x^i und x_j wobei $\mathbf{x}{=}x^i \mathbf{e}_i$ und $\boldsymbol{x}{=}\boldsymbol{e}^j x_j$:

$$\mathbf{x} = \boldsymbol{x}_* = x_j \boldsymbol{e}_*^j = x_j b^{jk} \mathbf{e}_k \;\Rightarrow\; x^k = x_j b^{jk},$$
$$\boldsymbol{x} = \mathbf{x}^* = \mathbf{e}_i^* x^i = \boldsymbol{e}^j a_{ji} x^i \;\Rightarrow\; x_j = a_{ji} x^i.$$

Wir können mittels $({}_*)$, d.h. mittels (b^{ij}) ein symmetrisches Skalarprodukt

$$\langle\ ,\ \rangle : \boldsymbol{V}_4 \times \boldsymbol{V}_4 \to \mathbf{R}, \quad \langle \boldsymbol{u}, \boldsymbol{v} \rangle := \left\langle \boldsymbol{e}^i u_i, \boldsymbol{e}^j v_j \right\rangle = u_i b^{ij} v_j \tag{2.7}$$

und somit eine Metrik definieren. Wenn wir eine reguläre Polarität haben (Det $(b^{ij}) \neq 0$), dann können wir auch in $\mathbf{V}^4$ mit Hilfe der Matrix $(a_{ij}) = {=}(b^{ij})^{-1}$ (nach (2.5) und (2.6)) ein Skalarprodukt definieren:

$$\langle\ ,\ \rangle : \mathbf{V}^4 \times \mathbf{V}^4 \to \mathbf{R}, \quad \langle \mathbf{x}, \mathbf{y} \rangle := \left\langle x^i \mathbf{e}^i, y^j \mathbf{e}_j \right\rangle = x^i a_{ij} y^j. \tag{2.8}$$

Ferner gelten die folgenden Gleichungen:

$$\begin{aligned} \mathbf{u} = \boldsymbol{u}_* = (\boldsymbol{e}^i u_i)_* = u_i \boldsymbol{e}_*^i = u_i b^{ij} \mathbf{e}_j, \\ \boldsymbol{x} = \mathbf{x}^* = (x^i \mathbf{e}_i)^* = \mathbf{e}_i^* x^i = \boldsymbol{e}^k a_{ki} x^i, \\ \mathbf{x}\boldsymbol{u} = \langle \boldsymbol{x}, \boldsymbol{u} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle. \end{aligned} \tag{2.9}$$

Unter den Elementen des Raumes $\mathbf{V}^4$ (bzw. des dualen Raumes $\boldsymbol{V}_4$) definieren wir die folgende Relation:

$$\mathbf{x} \sim \mathbf{y} \quad (\boldsymbol{u} \sim \boldsymbol{v}) \iff \exists\, c \in \mathbf{R} \setminus \{0\}, \quad \mathbf{y} = c\mathbf{x} \quad \left(\boldsymbol{v} = \boldsymbol{u}\frac{1}{c}\right). \tag{2.10}$$

Über der Teilraumstruktur von $\mathbf{V}^4$ bzw. $\boldsymbol{V}_4$ gewinnen wir nach (2.7), (2.8) und nach der Relation (2.10) den dreidimensionalen projektiv–metrischen

Raum $\mathtt{P}^3(\mathbf{V}^4, \boldsymbol{V}_4, \langle\ \rangle)$. Es lohnt sich auch die projektiv–metrische Sphäre $\mathtt{P}^3\mathtt{S}^3(\mathbf{V}^4, \boldsymbol{V}_4, \langle\ \rangle)$ mit der "Halbstrahlrelation"

$$\text{(2.10.a)} \qquad \mathbf{x} \sim \mathbf{y} \quad (\boldsymbol{u} \sim \boldsymbol{v}) \iff \exists\, c \in \mathbf{R},\ 0 < c,\ \mathbf{y} = c\mathbf{x} \quad \left(\boldsymbol{v} = \boldsymbol{u}\frac{1}{c}\right)$$

einzuführen. Dann wird $\mathtt{P}^3$ ein Sonderfall, wenn wir die ergänzenden Halbstrahlen und so die gegenüberliegenden Punkten von $\mathtt{P}^3\mathtt{S}^3$ identifizieren. Der affine Raum $\mathtt{A}^3$ ist noch spezieller, wenn eine ideale Ebene, z.B. mit der Gleichung $0 = \mathbf{x}\boldsymbol{e}^0 = x^0$ in $\mathtt{P}^3$ ausgezeichnet wird, und sie mit ihren Punkten (idealen Punkten von $\mathtt{P}^3$) weggelassen werden [9].

2.2. Es sei $\varphi : \mathbf{V}^4 \to \mathbf{V}^4$, $\varphi : \mathbf{x} \mapsto \mathbf{y} := \mathbf{x}\varphi$ eine reguläre lineare Abbildung. Diese Abbildung erzeugt im dualen Raum eine duale Abbildung φ^* durch

$$\text{(2.11)} \qquad \begin{gathered} \varphi^* : \boldsymbol{V}_4 \to \boldsymbol{V}_4, \quad \varphi^* = \boldsymbol{v} \mapsto \varphi^*\boldsymbol{v}, \\ \mathbf{x}\boldsymbol{v} = (\mathbf{x}\varphi)(\varphi^*\boldsymbol{v}). \end{gathered}$$

Die Matrizen der Abbildungen φ bzw. φ^* werden durch die folgenden Gleichungen (2.12), (2.13) mit t_i^r bzw. T_j^s bezeichnet

$$\text{(2.12)} \qquad \mathbf{e}_i\varphi = t_i^r\mathbf{e}_r,$$

$$\text{(2.13)} \qquad \varphi^*\boldsymbol{e}^s = \boldsymbol{e}^j\, T_j^s.$$

Man kann aus der Gleichung (2.14) nach der Formel (2.11) sehen, daß die Matrix t_i^r die inverse Matrix der Matrix T_j^s ist:

$$\text{(2.14)} \qquad x^i u_i = (x^i\mathbf{e}_i)(\boldsymbol{e}^j u_j) = (x^i t_i^r \mathbf{e}_r)(\boldsymbol{e}^j\, T_j^s u_s) \ \Rightarrow\ \delta_i^s = t_i^j\, T_j^s.$$

Weiterhin bezeichnen wir eine reguläre lineare Abbildung mit $\varphi(T^{-1}, T)$, wo die Matrix T^{-1} auf $\mathbf{V}^4$ und die Matrix T auf $\boldsymbol{V}_4$ wirkt. Das Produkt der Transformationen $\varphi_1(T^{-1}, T)$ und $\varphi_2(Z^{-1}, Z)$ ist durch die Formeln (2.15) auszurechnen:

$$\text{(2.15)} \qquad \begin{gathered} \varphi_1\varphi_2(T^{-1}Z^{-1}, ZT) \\ \varphi_1\varphi_2 : \mathbf{x} \mapsto \mathbf{x}T^{-1}Z^{-1}, \quad \boldsymbol{u} \mapsto ZT\boldsymbol{u}. \end{gathered}$$

2.3. Basistransformationen in Räumen $\mathbf{V}^4$ und $\boldsymbol{V}_4$: Es seien $\{\mathbf{e}_i\}$ die ursprüngliche Basis im Raum $\mathbf{V}^4$ und $\{\boldsymbol{e}^k\}$ die zu ihr duale Basis im Raum $\boldsymbol{V}_4$. Die Vektoren $\{\mathbf{e}_{i'}\}$ werden die neue Basis in $\mathbf{V}^4$ bilden und wir bezeichnen die zu $\{\mathbf{e}_{i'}\}$ duale Basis in $\boldsymbol{V}_4$ mit $\{\boldsymbol{e}^{k'}\}$:

$$\text{(2.16)} \qquad \mathbf{e}_{i'} = e_{i'}^i\mathbf{e}_i.$$

(2.17) $$\boldsymbol{e}^{k'} = \boldsymbol{e}_k e_k^{k'}.$$

(2.18) $$\delta_{i'}^{k'} = \mathbf{e}_{i'}\boldsymbol{e}^{k'} = (e_{i'}^{i}\mathbf{e}_i)(\boldsymbol{e}_k e_k^{k'}) = e_{i'}^{i}(\mathbf{e}_i\boldsymbol{e}^k)e_k^{k'} = e_{i'}^{i}\delta_i^k e_k^{k'} = e_{i'}^{k}e_k^{k'}.$$

Die Matrizen der Transformation $\varphi(T^{-1}, T)$ bei diesen Basistransformationen sind die Folgenden:

(2.19) $$t_{i'}^{k'} = e_{i'}^{i} t_i^k e_k^{k'}, \quad T_{j'}^{s'} = e_{j'}^{j} T_j^s e_s^{s'} \quad \text{mit} \quad t_{i'}^{k'} T_{k'}^{j'} = \delta_{i'}^{j'}.$$

BEMERKUNG. i. Wenn $(e_{i'}^{i})$ und $(e_k^{k'})$ spezielle diagonale Matrizen sind,

$$(e_{i'}^{i}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{{}_jc} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (e_k^{k'}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & {}_jc & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

d.h., wenn wir nur von $\mathbf{e}_j$ zum neuen Basisvektor $\mathbf{e}_{j'} = \frac{1}{{}_jc}\mathbf{e}_j$ übergehen (und so von Form $\boldsymbol{e}^j$ zu $\boldsymbol{e}^j{}_jc$, hier steht keine Summierung), die übrigen sich nicht ändern, dann ist (2.19) speziell:

(2.20) $$t_{i'}^{k'} = \left(\frac{1}{{}_jc}\right)_{i'}^{i} t_i^k ({}_jc)_k^{k'}.$$

Das liefert uns eine Regelung für eine geeignete Basistransformation. Man darf die j-te Reihe von (t_i^k) mit ${}_jc$ dividieren und zugleich die j-te Spalte mit ${}_jc$ multiplizieren.

2.3. Wir betrachten zunächst diejenigen kombinatorischen Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ $(a \geq 1,\ b \geq 1)$, deren zukünftige erzeugende Bewegungen r_0, r_1, r_2, r_3 für die Gruppe $\Gamma^{24}_{(a,b)} := \Gamma_{18}(4a, 8b)$ Halbdrehungen sind, wie es die Abbildung 1 zeigt. Unter diesen Erzeugenden bestehen die folgenden definierenden Relationen in der Gruppe:

$$\Gamma_{18}(4a, 8b) \quad a \geq 1,\ b \geq 1,$$
$$r_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_0r_1r_3r_1)^a = (r_1r_2r_3r_0r_2r_0r_3r_2)^b = 1.$$

Ferner betrachten wir das Tetraeder $A_0A_1A_2A_3$, das ein Fundamentalbereich für die Gruppe Γ_{18} ist. Im Fall $(a,b) = (1,1)$ würden die Eckpunkte des Polyeders der Polyederpflasterung auf dem Absolutgebilde liegen, und in den Fällen, wenn $a > 1$ oder $b > 1$ ist, sind die Eckpunkte der Pflastersteine außerhalb des Absolutgebildes eines geeigneten Raumes. Die Stabilisatorgruppe eines Eckpunktes ist nämlich eine euklidische ebene kristallographische Gruppe im ersten Fall und sie ist hyperbolisch in den übrigen Fällen, wie ihr Fundamentalbereich $\mathbf{F}^0$ und Flächendiagramm in Abbildung 1 zeigen.

Die Formen $\boldsymbol{b}^0$, $\boldsymbol{b}^1$, $\boldsymbol{b}^2$, $\boldsymbol{b}^3$ des dualen Raumes $\boldsymbol{V}_4$ stellen die Flächenebenen dieses Tetraeders dar und bilden eine Basis. Die Vektoren $\mathbf{a}_i \in \mathbf{V}^4$, $i \in \{0,1,2,3\}$, bilden die duale Basis mit der Gleichung $\mathbf{a}_i \boldsymbol{b}^j = \delta_i^j$. Sie kennzeichnen die Eckpunkte unseres Tetraeders $A_0A_1A_2A_3$.

Wir bestimmen, bezüglich des dualen Raumes die Matrizen der Transformationen $r_0({}^0R,{}_0R)$, $r_1({}^1R,{}_1R)$, $r_2({}^2R,{}_2R)$, $r_2({}^2R,{}_2R)$ wobei die Matrizen ${}_iR$, $i = \{0,1,2,3\}$ in $\boldsymbol{V}_4$ und die Matrizen ${}^iR$, $i = \{0,1,2,3\}$ im Raum $\mathbf{V}^4$ wirken. In unserem Fall sind die Transformationen r_0, r_1, r_2, r_3 involutorisch, deshalb ist ${}_iR = {}^iR$, $i = \{0,1,2,3\}$. Zum Beispiel sind die Bilder der Flächenebenen $\boldsymbol{b}^0$, $\boldsymbol{b}^1$, $\boldsymbol{b}^2$, $\boldsymbol{b}^3$ bei der Transformation r_2, die folgenden:

$$\begin{aligned} \boldsymbol{b}^0 &\xrightarrow{r_2} \boldsymbol{b}^0 r_0^0 + \boldsymbol{b}^2 r_2^0 \\ \boldsymbol{b}^1 &\xrightarrow{r_2} \boldsymbol{b}^2 r_2^1 + \boldsymbol{b}^3 r_3^1 \\ \boldsymbol{b}^2 &\xrightarrow{r_2} \boldsymbol{b}^2 r_2^2 \\ \boldsymbol{b}^3 &\xrightarrow{r_2} \boldsymbol{b}^1 r_1^3 + \boldsymbol{b}^2 r_2^3 \end{aligned} \tag{2.21}$$

$${}_2R(\boldsymbol{b}^0,\boldsymbol{b}^1,\boldsymbol{b}^2,\boldsymbol{b}^3) = (\boldsymbol{b}^0,\boldsymbol{b}^1,\boldsymbol{b}^2,\boldsymbol{b}^3)\begin{bmatrix} r_0^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1^3 \\ r_2^0 & r_2^1 & r_2^2 & r_2^3 \\ 0 & r_3^1 & 0 & 0 \end{bmatrix}.$$

Da die Transformation r_2 involutorisch ist, folgt

$$\begin{bmatrix} r_0^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1^3 \\ r_2^0 & r_2^1 & r_2^2 & r_2^3 \\ 0 & r_3^1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} (r_0^0)^2 & 0 & 0 & 0 \\ 0 & r_1^3 r_3^1 & 0 & 0 \\ r_2^0 r_0^0 + r_2^2 r_2^0 & r_2^2 r_2^1 + r_2^3 r_3^1 & (r_2^2)^2 & r_2^1 r_1^3 + r_2^2 r_2^3 \\ 0 & 0 & 0 & r_3^1 r_1^3 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} (r_0^0)^2 = r_3^1 r_1^3 = (r_2^2)^2 := 1 \\ r_2^0(r_0^0 + r_2^2) = 0 \\ r_2^2 r_2^1 + r_2^3 r_3^1 = 0 \\ r_2^1 r_1^3 + r_2^2 r_2^3 = 0 \end{matrix} \Rightarrow \begin{matrix} r_0^0 = 1,\ r_2^2 = -1 \\ r_2^3 = -\frac{r_2^2 r_2^1}{r_3^1} = \frac{r_2^1}{r_3^1} \end{matrix} \Rightarrow$$

$$\Rightarrow_2 R \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1^3 \\ 0 & 0 & 0 & r_1^3 \\ r_2^0 & r_2^1 & -1 & r_2^1 r_1^3 \\ 0 & \frac{1}{r_1^3} & 0 & 0 \end{bmatrix} =: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ r & s & -1 & st \\ 0 & \frac{1}{t} & 0 & 0 \end{bmatrix} \quad \text{mit} \quad t > 0.$$

Analog erhalten wir die Matrizen für die Transformationen r_0, r_1, r_3. So ergeben sich zunächst 12 Parameter. Wir können zwei Parameter mittels der Basistransformation nach der Formel (2.20) gleich eliminieren. Endlich gewinnen wir die folgenden Matrizen für die Transformationen r_0, r_1, r_2, r_3:

$$
{}_0R \sim \begin{bmatrix} -1 & x & y & yz \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & \frac{1}{z} & 0 \end{bmatrix}, \qquad {}_1R \sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ v & -1 & u & v \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
$$

(2.22)

$$
{}_2R \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ r & s & -1 & st \\ 0 & \frac{1}{t} & 0 & 0 \end{bmatrix}, \qquad {}_3R \sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ p & n & p & -1 \end{bmatrix}
$$

$$\text{mit } 0 < t, \quad 0 < z.$$

2.4. Weitere Bedingungen lassen sich für die Parameter aus der definierenden Relationen bekommen. Von der ersten Relation $(r_1 r_0 r_1 r_3)^a = 1$, $(a \geq 1)$ die eine a-zählige Drehung um die Achse $A_0 A_2$ beschreibt, erhalten wir, als Produkt der Matrizen, nach der Formel (2.15) die folgende Matrix:

$$
R_i^j := \begin{bmatrix}
z & 0 & 0 & 0 \\
(vx-1)v + \\ +(vzy+uz) & -vz+1 & (vx-1)u + \\ +vy+\frac{v}{z} & v(vx-2) \\
0 & 0 & \frac{1}{z} & 0 \\
(vx-1)vn + \\ +(vzy+uz)n + \\ +pz - xv - yz & -(vx-1)n + x & \frac{p}{z} + (vx-1)un + \\ +vyn + \frac{vn}{z} - \\ -xu - y & vn(vx-2) - \\ -vx+1
\end{bmatrix}
$$

Diese Trasformation ist für die Punkte der Strecke $A_0 A_2$ eine Identität, deshalb folgt die Gleichung $z = \frac{1}{z} \Rightarrow z = 1$. Weitere Bedingungen für die Parameter bekommen wir aus dem Eigenwertproblem der Transformation $(r_1 r_0 r_1 r_3)$.

So erhalten wir die folgende Gleichung:

(2.24) $$2\cos\frac{2\pi}{a} = 2 - 2vx + vn(vx-2) = -2 + (vx-2)(vn-2).$$

In den Fällen $a \geq 3$ ist die erste Relation, $(r_1 r_0 r_1 r_3)^a = 1$ equivalent mit der Gleichung (2.24). Wenn $a = 1$ bzw. $a = 2$ ist, dann gewinnen wir weitere Bedingungen:

(2.25) $$a = 1 \;\Rightarrow\; p = y,\; u = 0,\; v = 0,\; n = -x,$$

(2.26) $$a = 2 \;\Rightarrow\; v = \frac{2}{x},\; n = x.$$

Analog ergibt sich aus der zweiten Ralation $(r_1 r_2 r_3 r_0 r_2 r_0 r_3 r_2)^b = 1$, $(b \geq 1)$, für die weiteren Parameter ein sehr kompliziertes Gleichungssystem, aus dem wir mit Hilfe der hier nicht zu detaillierenden “Maple” Rechnungen die folgende Gleichung für $b=1$ bekommen.

$$\begin{aligned}(2.27)\qquad &-2-4r-r^2+2r^3+y^2(2+10r+r^2-r^3)+y(r^3-r^2+4r+2)+\\ &+y^3(-3r^3-7r^2-6r-6)+y^4(r^3+4r^2+4r)+\\ &+2\cos\frac{\pi}{a}\Big[r^3-r^2-2r-1+y(1+4r+r^2+r^3)+\\ &+y^2(-1+r-3r^2-2r^3)+y^3(-1+r+r^2)\Big]=0.\end{aligned}$$

2.5. Ferner betrachten wir die Polarität (2.3)

$$(_*): \boldsymbol{V}_4 \to \mathbf{V}^4 : \boldsymbol{u} \mapsto \boldsymbol{u}_* =: \mathbf{u}, \quad \text{wobei nach der Formel (2.9)}$$
$$\mathbf{u} = \boldsymbol{u}_* = (\boldsymbol{b}^i u_i)_* = u_i \boldsymbol{b}^i * = u_i b^{ij} \mathbf{a}_j .$$

Es sei $r({}^i R,{}_i R)$, $i \in \{1,2,3,4\}$, eine der Transformationen r_0, r_1, r_2, r_3. Die Vektoren $\boldsymbol{u}=\boldsymbol{b}^i u_i$ und $\mathbf{u}=\boldsymbol{u}_*=(\boldsymbol{b}^i u_i)_*=u_i\boldsymbol{b}^i*=u_i b^{ij}\mathbf{a}_j$ bzw. ${}_k\boldsymbol{R}\boldsymbol{u}$ und $\boldsymbol{u}_*{}^k R$ $k \in \{1,2,3,4\}$ bilden bei dieser Polarität ein polar–pol Paar, deshalb gelten die folgenden Gleichungen:

$$(2.28)\qquad {}_k\boldsymbol{R}\boldsymbol{u} = \boldsymbol{b}^i\ {}_kR_i^j u_j \mapsto u_j\ {}_kR_i^j b^{is}\mathbf{a}_s = u_i b^{ij}\ {}^kR_j^s \mathbf{a}_s$$
$$\text{wobei}\qquad {}^kR_i^j\ {}_kR_j^s = \delta_i^s, \quad k \in \{1,2,3,4\}.$$

In unserem Fall sind die Transformationen r_0, r_1, r_2, r_3 speziell involutorisch. Im allgemeinen gewinnen wir nach der Formel (2.28) für (b^{ij}) die Gleichungen

$$(2.29)\qquad {}_kR_i^j b^{is}\ {}_kR_s^m = b^{jm}c, \qquad \text{mit einem Konstant } c>0$$
$$\text{jetzt}\quad k \in \{1,2,3,4\} \text{ und } c=1.$$

Im allgemeinen ist die Anzahl der unabhängigen Gleichungen von den Erzeugenden bestimmt.

3. Metrische Realisierungen der Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a \geq 2,\ b \geq 1)$

SATZ 3.1. *Die körpertransitiven Symplexpflasterungen* $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ *und* $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, *(*$a \geq 2$, $b \geq 1$*) sind im hyperbolischen Raum* H^3 *metrisch realisierbar. Die Ecken der Symplexen der Pflasterungen liegen außerhalb des*

Absolutgebildes, ferner sind die Bewegungsgruppen $\Gamma^{24}_{(a,b)}$ *und* $g^{31}_{(a,b)}$ *in den Fällen* $a \neq 2b$ *maximal.*

BEWEIS.

3.1. *Die Symplexpflasterung* $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, *(*$a=2$, $b=1$*)*

Alle Parameter (n, p, s, t, u, y, v, b^{ij}) sind durch das Gleichungssystem (2.29) und durch die Gleichungen (2.26) und (2.27) festgelegt, und so gewinnen wir die Matrix (b^{ij}):

$$(b^{ij}) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \tag{3.1}$$

Das bedeutet, daß die Matrix (b^{ij}) die Signatur $(-,+,+,+)$ hat und diese Pflasterung im hyperbolischen Raum wirklich realisierbar ist ([9], [12]). Die Keilwinkel β^{ij} zwischen den eigentlichen Flächenebenen ($\boldsymbol{b}^i$) und ($\boldsymbol{b}^j$) i, $j \in \{1,2,3,4\}$ werden durch die Formel (3.2) und durch die Matrix (3.1) gekennzeichnet. Folglich ergibt sich, daß die Keilwinkel bei allen Kanten $\frac{\pi}{4}$ sind,

$$\cos\beta^{ij} = \frac{-\left\langle \boldsymbol{b}^i, \boldsymbol{b}^j \right\rangle}{\sqrt{\left\langle \boldsymbol{b}^i, \boldsymbol{b}^i \right\rangle \left\langle \boldsymbol{b}^j, \boldsymbol{b}^j \right\rangle}}. \tag{3.2}$$

3.2. *Die Symplexpflasterungen* $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, *(*$a \geq 3$, $b=1$*).*

Wir können mit Hilfe der Gleichungssysteme (2.27), (2.29) und mit der Gleichung (2.24) die folgenden Parameter eliminieren

$$\begin{gathered} t = z = 1, \\ r = \frac{-y^2 + \sqrt{y^4 + 4y(y^2-1)(y+2)}}{2(y^2-1)}, \quad s = \frac{yr(r+1) - y - r^2}{(r+1)(y-1)}, \\ n = \frac{(y-1)(2s-r)}{(r-1)}, \qquad u = \frac{4(y+1)(r-1)\cos^2\frac{\pi}{2a}}{(yr-1)n}, \\ v = \frac{4\cos^2\frac{\pi}{2a}}{n}, \qquad p = y \qquad x = \frac{2\left(\cos\frac{2\pi}{a} + 1\right)}{(nv-2)v} + \frac{2}{v} = n. \end{gathered} \tag{3.3}$$

Die Elemente der Matrix (b^{ij}) werden durch die obigen Parameter ausgedrückt:

$$b^{00} = \frac{b^{33}(nv-2)^2}{2\left(\cos\frac{2\pi}{a}+1\right)} = b^{33}, \quad b^{20} = b^{02} = b^{23}, \quad b^{33} = b^{22}\frac{yr-1}{y^2-1},$$

$$b^{11} = \frac{b^{33}\left(n^2v^2 + 2\left(\cos\frac{2\pi}{a}-1\right)(nv-1)\right)}{2v^2\left(\cos\frac{2\pi}{a}+1\right)} = b^{33}\frac{n}{v},$$

$$(3.4)\quad b^{01} = b^{10} = -\frac{b^{00}x}{2}, \quad b^{12} = b^{21} = -\frac{b^{11}u}{2}, \quad b^{13} = b^{31} = -\frac{b^{33}n}{2},$$

$$b^{03} = b^{30} = \left(\frac{b^{22}r}{2} - b^{00}y\right), \quad b^{23} = b^{32} = -\frac{1}{2}(uxb^{00} + rb^{22} - vub^{11}).$$

Wir dürfen noch ein Element aus den Elementen der Matrix (b^{ij}) festlegen. Es sei $b^{22} = 1$. Dabei ergibt sich die Gleichung (2.27) für den Parameter y. Die weiteren Parameter und die Elemente der Matrix (b^{ij}) sind durch den Parameter y schon festgelegt und die Gleichungen (2.24), (2.29) sind erfüllt.

Es ist zu zeigen, daß die Gleichung (2.27) im Intervall $y \in [\sqrt{2}, 1.555]$ für alle $a \geq 3$, wenigstens einen Wurzel hat. Die Werte dieser Wurzeln sind für alle $a \geq 3$ beliebig genau aus der Gleichung (2.27) festzulegen:

$$(3.5)\quad \begin{array}{lll} a = 3 & r = 1.343683011\ldots, & y = 1.493200411\ldots, \\ a = 4 & r = 1.322909689\ldots, & y = 1.520146775\ldots, \\ a = 5 & r = 1.313848712\ldots, & y = 1.532507748\ldots, \\ a = 6 & r = 1.309062592\ldots, & y = 1.539193970\ldots, \\ a = 10 & r = 1.302263777\ldots, & y = 1.548884609\ldots, \\ a = 100 & r = 1.298558406\ldots, & y = 1.554263558\ldots, \\ a \to \infty & r \to 1.298521266\ldots, & y \to 1.554317827\ldots, \end{array}$$

Daraus folgt:

$$(3.6)\quad \begin{array}{c} r \in [1.343683011\ldots, 1.298521266\ldots] \text{ und} \\ y \in [1.493200411\ldots, 1.554317827\ldots] \end{array}$$

Die Matrizen (b_a^{ij}) sind durch die Parameter (3.3), (3.4) und (3.7) festgelegt. Diese Matrizen sind in jedem Fall anzugeben, z.B.:

$$(b_3^{ij}) = \\ = \begin{bmatrix} 0.8184362177 & -0.6371510920 & -0.6718415055 & -0.5502477915 \\ -0.6371510920 & 0.6613612736 & -0.5424906120 & -0.6371510920 \\ -0.6718415055 & -0.5424906120 & 1 & -0.6718415055 \\ -0.5502477915 & -0.6371510920 & -0.6718415055 & 0.8184362177 \end{bmatrix},$$

$$\mathrm{Det}\,(b_3^{ij}) = -3.065035068,$$

$(b_\infty^{ij}) =$

$$= \begin{bmatrix} 0.7191976425 & -0.5989496920 & -0.6492606330 & -0.4686010840 \\ -0.5989496920 & 0.4988068820 & -0.4484959440 & -0.5989496920 \\ -0.6492606330 & -0.4484959440 & 1 & -0.6492606330 \\ -0.4686010840 & -0.5989496920 & -0.6492606330 & 0.7191976425 \end{bmatrix},$$

$$\mathrm{Det}\,(b_\infty^{ij}) = -2.091781585.$$

Bei diesen Werten der Parameter y und r ist Det $(b^{ij}) < 0$, für alle $a \geq 3$, (auch im Fall $a \to \infty$), wie es man aus (3.7) sehen kann

(3.7)

$$\mathrm{Det}\,(b^{ij}) = \frac{-(r-2+yr)^2(yr-r+2y)^2 \begin{pmatrix} 4ry^2+2ry-6y+2r-2+ \\ +2\cos\frac{\pi}{a}(3yr-y+r-3) \end{pmatrix}}{4\cos^2\frac{\pi}{2a}(y-1)^3(y+1)^2(r+1)}$$

Nach den Formeln (3.2), (3.3), (3.4) ergibt sich, daß

$$\beta^{01} = \beta^{13} \text{ und } \cos\beta^{01} = \cos\beta^{13} = \frac{\sqrt{\sqrt{2\cos\frac{2\pi}{a}+2}+2}}{2} = \cos\frac{\pi}{2a}.$$

Das Gleichungssystems (2.27) bedeutet zunächst, daß $\beta^{03}+\beta^{02}+\beta^{23}+\beta^{12}=k\pi$ ($k \in \mathbf{N}$) gilt. Man kann weiter aus den Formeln (3.2), (3.3), (3.4) und (3.7) sehen, daß $\beta^{03}+\beta^{02}+\beta^{12}+\beta^{23}=\pi$ ist. Ferner haben die Matrizen (b^{ij}) für alle $a \geq 3$ die Signatur $(-,+,+,+)$ ([9], [12]).

Damit ist abschließend gezeigt, daß die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, $(a \geq 3,\ b=1)$ im hyperbolischen Raum wirklich realisierbar sind.

3.3. *Die Symplexlpflasterungen* $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, *$(a \geq 2,\ b \geq 2)$.*

Wie wir sehen werden, lohnt es sich, für die Kennzeichnung der metrischen Realisierbarkeit in einigen Fällen eine andere Methode nutzen, die auch auf der Theorie der projektiv-metrischen Geometrie beruht [13].

In diesen Fällen gehen wir von der Coxeter–Schläfli Matrix (3.8) des Tetraeders $A_0A_1A_2A_3$ (Abb. 1) aus. Die Flächenebenen dieses Tetraeders werden durch die Formen $\boldsymbol{b}^i \in V_4 \quad i \in \{0,1,2,3\}$ gekennzeichnet (siehe noch die Arbeit von M. Stojanovic [13]). Die Matrixelemente $b^{ij} =$

$= \begin{cases} 1, & i=j \\ -\cos\beta^{ij}, & i \neq j \end{cases}$ kennzeichnen die Keilwinkel β^{ij} zwischen den Seitenflächen $(\boldsymbol{b}^i)$, $(\boldsymbol{b}^j)$:

$$(3.8) \qquad (b^{ij}) = \begin{bmatrix} 1 & -\cos\beta^{01} & -\cos\beta^{02} & -\cos\beta^{03} \\ -\cos\beta^{01} & 1 & -\cos\beta^{12} & -\cos\beta^{13} \\ -\cos\beta^{02} & -\cos\beta^{22} & 1 & -\cos\beta^{23} \\ -\cos\beta^{03} & -\cos\beta^{13} & -\cos\beta^{23} & 1 \end{bmatrix}.$$

Wenn diese Pflasterungen metrisch realisierbar sind, dann haben wir für die Keilwinkel und für die Kanten aus dem Poincaréschen Algorithmus und aus der Symmetrie des Tetraeders $A_0A_1A_2A_3$ die folgenden Gleichungen:

$$\beta^{01} = \beta^{13} = \frac{\pi}{2a},\ \beta^{23} = \beta^{02},\ \beta^{02} + \beta^{03} + \beta^{12} + \beta^{23} = \frac{\pi}{b},\ (a \geq 2,\ b \geq 2),$$

$$(3.9) \qquad \Rightarrow\ \beta^{03} = \frac{\pi}{b} - \beta^{12} - 2\beta^{02}, \quad \text{ferner}$$

$$A_0A_1 = A_1A_3 = A_0A_3 = A_1A_2, \qquad A_0A_2 = A_2A_3.$$

Wir werden die Veränderlichen $x = \cos\beta^{12}$, $y = \cos\beta^{02} = \cos\beta^{23}$ für $a \geq 2$, $b \geq 2$ so festlegen, daß wir schließlich zu einer hyperbolischen Metrik gelangen. Wenn eine Pflasterung im hyperbolischen Raum existieren würde, dann hätte ihre Coxeter–Schläflische Matrix die Signatur $(-,+,+,+)$, und so könnten wir die inverse Matrix $(a_{ij}) = (b^{ij})^{-1}$ mit $a_{ik}b^{kj} = \delta_i^j$ bilden. Die Entfernung d_{ik} zwischen den eigentlichen Ecken $(\mathbf{a}_i)$, $(\mathbf{a}_k)$ ist:

$$(3.10) \qquad \cosh d_{ik} = \frac{-\langle \mathbf{a}_i, \mathbf{a}_k \rangle}{\sqrt{\langle \mathbf{a}_i, \mathbf{a}_i \rangle \langle \mathbf{a}_k, \mathbf{a}_k \rangle}} = \frac{-a_{ik}}{\sqrt{a_{ii}a_{kk}}}.$$

Wir gewinnen aus den Forderungen (3.9) nach der Entfernungsformel (3.10) der hyperbolischen Geometrie für den Unbekannten x, y ein Gleichungssystem (3.12) wobei wir noch die folgenden Bezeichnungen (3.11) nutzen

$$A = \cos\frac{\pi}{2a},\ B = \cos\frac{\pi}{b},\ (a \geq 2,\ b \geq 2),$$

$$(3.11) \qquad z = \cos\left(\frac{\pi}{b} - \beta^{12} - 2\beta^{02}\right) \ \Rightarrow$$

$$z = (2y^2 - 1)\left(Bx + \sqrt{1-B^2}\sqrt{1-x^2}\right) + \\ +2y\sqrt{1-y^2}\left(x\sqrt{1-B^2} - B\sqrt{1-x^2}\right).$$

$$A_0A_1 = A_0A_3 \ \Rightarrow\ w1(x,y):$$

$$z(x^2-1)(z+2y^2) + xy(xy - 2Ay^2 + 2A) - y^4 - A^2(y^2-1) = 0.$$

$$(3.12) \qquad A_0A_1 = A_1A_2 \ \Rightarrow\ w2(x,y):$$

$$2A^2y^2 + 2Axy(1-z) + x^2 - A^2 - z(x^2 + A^2) = 0.$$

Ferner definieren wir durch die Ungleichungen (3.13) einen Bereich $\boldsymbol{G}^{(a,b)}$ (Abb. 3).

$$B \le x \le 1,$$
$$\cos\frac{\pi}{2b} = \sqrt{\frac{B+1}{2}} \le y \le 1, \tag{3.13}$$
$$(x+B)^2 - 4Bxy^2 - 4y^2 - 4y^4 \ge 0 \iff \frac{\pi}{b} - 2\beta^{02} - \beta^{12} \ge 0.$$

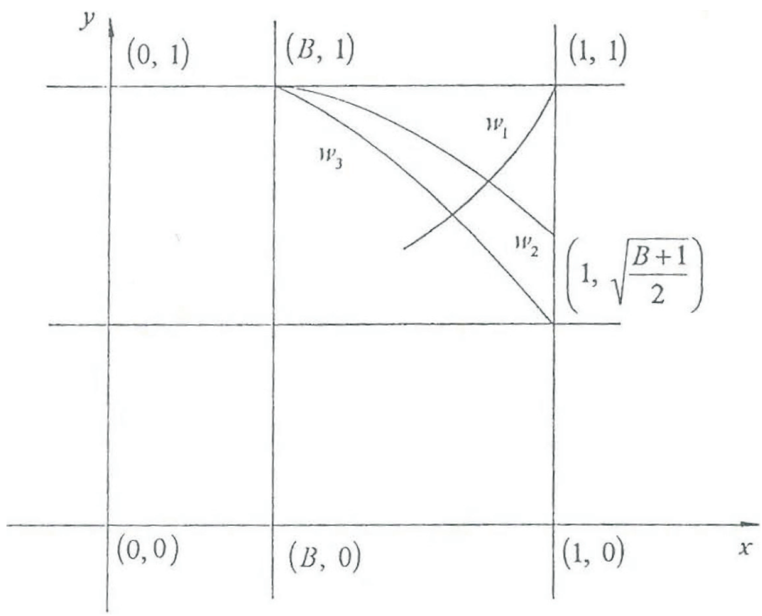

Abb. 3.

Nun haben wir für die Unbekannten x und y zwei komplizierte Gleichungen $w1(x,y) = 0$ und $w2(x,y) = 0$ mit den Parametern A und B. Um eine hyperbolische Metrik für jeden Parameter A und B zu gewinnen, müssen wir die Lösung des Gleichungssystems (3.12) im Bereich $\boldsymbol{G}^{(a,b)}$ suchen. Wenn es in $\boldsymbol{G}^{(a,b)}$ eine Lösung des Gleichungssystems (3.12) gibt, dann hat die Matrix (b^{ij}) für alle Parameter die Signatur $(-,+,+,+)$, wie man aus (3.14) sehen kann:

$$\operatorname{Det}(b^{ij}) = (z+1)(z(x^2-1) + 1 - 2y^2 - x^2 - 4xAy - 2A^2) < 0, \tag{3.14}$$
$$B \le x \le 1,$$
$$\cos\frac{\pi}{2b} = \sqrt{\frac{B+1}{2}} \le y \le 1.$$

Die Gleichungen $w1(x,y)=0$ und $w2(x,y)=0$ bestimmen auch zwei Kurven in einem Koordinatensystem (x,y), wobei die Kurven von den Parametern A und B hängen. Wir betrachten die einfache ebene Kurve

$$w3(x,y) : (x+B)^2 - 4Bxy^2 - 4y^2 - 4y^4 = 0,$$

wofür die folgenden Gleichungen gelten:

$$w3(B,1)=0 \quad \text{und} \quad w3\left(1, \sqrt{\frac{B+1}{2}}\right) = 0, \quad \text{(Abb. 3)}.$$

Es ist leicht zu sehen, daß die Gleichungen $w2(B,1)=0$ und $w1(1,1)=0$ erfüllen. Man kann sehen, daß die Ungleichungen $w2\left(1,\sqrt{\frac{B+1}{2}}\right) \geq 0$ und $w2(1,1) \leq 0$ gelten d.h. der Schnittpunkt der Kurve $w2(x,y)=0$ und der Linie $x=1$ liegt zwischen den Punkten $(1,1)$ und $\left(1,\sqrt{\frac{B+1}{2}}\right)$. Analog ist zu sehen, daß die Kurve $w2(x,y)=0$ im Bereich $\boldsymbol{G}^{(a,b)}$ liegt, wenn $B \leq x \leq 1$ gilt. Aus der Gleichung $w1(x,y)=0$ ergibt sich, daß $w1(B,1) \leq 0$ und $w1(1,y) \geq$ ≥ 0, wenn $\sqrt{\frac{B+1}{2}} \leq y \leq 1$ gilt, d.h., daß es einen Schnittpunkt der Kurven $w1(x,y)=0$ bzw. $w2(x,y)=0$ gibt, der im Bereich $\boldsymbol{G}^{(a,b)}$ ist (Abb. 3). Das bedeutet nach (3.1), daß die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, $(a \geq 2,$ $b \geq 2)$ im hyperbolischen Raum wirklich metrisch realisierbar sind. Wir bekommen z.B im Fall $(a=2,\ b=2)$ die folgenden Resultate

$$x = 0.933029155, \qquad y = 0.9128826398,$$

$$(b^{ij}) =$$
$$= \begin{bmatrix} 1 & -1\sqrt{2} & -0.9128826398 & -0.9352857513 \\ -\frac{1}{\sqrt{2}} & 1 & -0.9330291550 & -\frac{1}{\sqrt{2}} \\ -0.9128826398 & -0.9330291550 & 1 & -0.9128826398 \\ -0.9352857513 & -\frac{1}{\sqrt{2}} & -0.9128826398 & 1 \end{bmatrix},$$
$$\text{Det}\,(b^{ij}) = -9.806931975.$$

3.4. *Die Symplexpflasterungen* $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$*,* $(a \geq 2,\ b \geq 1)$*.*

Wie man aus den Punkten **3.1.**, **3.2.**, **3.3.** sieht, existieren die Tetraederpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, $(a \geq 2,\ b \geq 1)$ und so auch die Tetraeder $A_0A_1A_2A_3$ (Abb. 1) im hyperbolischen Raum. Daraus folgt, daß die Tetraeder $\overline{A}_0\overline{A}_1\overline{A}_2\overline{A}_3$ (Abb. 2) der Polyederpflasterungen $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a \geq 2,\ b \geq 1)$ im hyperbolischen Raum metrisch existieren, denn das

Tetraeder $A_0A_1A_2A_3$ ist deckungsgleich mit dem Tetraeder $\overline{A}_2\overline{A}_0\overline{A}_1\overline{A}_3$ für alle $(a \geq 2,\ b \geq 1)$.

Der Satz wird durch die Punkte **3.1.**, **3.2.**, **3.3.**, **3.4.** bewiesen.

4. Zwei unendliche Serien der metrisch nicht realisierbaren Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a = 1,\ b \geq 2)$

SATZ 4.1. *Die körpertransitiven kombinatorischen Symplexpflasterungen* $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ *und* $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a = 1,\ b \geq 2)$ *sind nicht zu realisieren, d.h. daß man keine maximale homogene Geometrie (von den* 8 *Geometrien von Thurston) finden kann, in deren Raum die obigen Pflasterungen metrisch realisierbar sind.*

BEWEIS. **4.1.** Im Fall $a = 1$ bekommen wir aus der ersten Relation $(r_1r_0r_1r_3)^a = 1$ für die Parameter die folgenden Resultaten (siehe noch (2.25)):

(4.1) $$p = y, \quad u = 0, \quad v = 0, \quad n = -x,$$

ferner gilt $z = 1$.

Dabei betrachten wir das Gleichungssystem (2.29), aus dem wir für den Parameter b^{12} die folgende Gleichung erhalten:

(4.2) $$b^{12} = 0.$$

Daraus folgt nach der Formel (3.2), daß der Keilwinkel β^{12} bei der Kante A_0A_2 ein rechter Winkel ist. Wenn die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ metrisch realisierbar wären, dann wäre die folgende Gleichung:

(4.3) $$\beta^{03} + \beta^{02} + \beta^{23} + \beta^{12} = \frac{\pi}{b} \qquad (b \geq 2)$$

gültig. In unseren Fällen kann die Gleichung (4.3) wegen der obigen Überlegungen nicht bestehen, deshalb lassen sich die Symplexpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$, $(a = 1,\ b \geq 2)$ metrisch nicht realisieren.

4.2. Analog ist zu sehen, daß die Tetraederpflasterungen $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$, $(a = 1,\ b \geq 2)$ sich metrisch nicht realisieren lassen.

BEMERKUNG. Im Fall $(a = 1,\ b = 1)$ sind die Tetraederpflasterungen $\left(T^{24}_{(a,b)}, \Gamma^{24}_{(a,b)}\right)$ und $\left(T^{31}_{(a,b)}, \Gamma^{31}_{(a,b)}\right)$ auch nicht realisierbar, wie es in einer kommenden Arbeit von E. MOLNÁR und J. WEEKS gezeigt werden wird.

Der Satz 4.1 ist bewiesen

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CONTROLLABILITY OF LIPSCHITZIAN DISCRETE-TIME MULTIFUNCTION SYSTEM WITH RESTRAINED CONTROLS IN FINITE-DIMENSIONAL SPACE

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1. Introduction

This paper is concerned with the local reachability of Lipschitzian discrete-time multifunction system described by

$$x_{k+1} \in f(x_k, u_k), \qquad x_k \in \mathbb{R}^n, \qquad u_k \in \Omega \subset \mathbb{R}^m,$$

where Ω is an arbitrary subset for which $0 \in \operatorname{int}\Omega$, $f : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R}^n$ is pseudo Lipschitzian multifunction around $(\mathbb{O}, \mathbb{O}) \in \mathbb{R}^n \times \mathbb{R}^m$, moreover we put further in our case that the multifunction f is locally bounded around $(\mathbb{O}, \mathbb{O})$.

In recent years, the controllability problem of nonsmooth system have been studied by many authors. N. D. YEN in [3], I. JOÓ and NG. VAN SU in [4], [5], for example, have studied the controllability of the Lipschitzian discrete-time systems with restrained controls and some sufficient conditions for the controllability of such systems were obtained. Remark that, in the both case, when the phasis spaces are finite or infinite dimensional, the generalized Jacobian concept (in finite dimensional spaces) and the strict prederivative concept (in finite dimensional spaces) can be defined and we can use the results of the Open Mapping Theorems and the Implicit Function Theorem for studying Lipschitzian discrete-time systems. Notice that these concepts and many results on Lipschitzian functions (single-valued functions) can be found detailly in CLARKE's work [1] and in IOFFE's work [2].

The main purpose of this paper is to continue investigating the controllability of Lipschitzian discrete-time multifunction system (set-valued systems). The derivative-like construction (so called coderivative) for multifunctions has been introduced by MORDUKHOVICH, for more detail about this concept

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and the properties of the coderivative we refer the reader to [6], [7], [8], [9], [10]. The useful result about the Lipschitzian property of multifunction, the Lipschitzian property of the composition of two Lipschitzian multifunction can be referred to reader to Mordukhovich's works and to ROCKAFELLAR [11]. Like in our previous work, we use here the openness of the multifunction to study the controllability of Lipschitzian multifunction systems. Notice that the main result about the controllability of Lipschitzian discrete-time multifunction systems is mainly base on the theorem about the openness of the multifunctions.

2. Basic definitions, notations and preliminaries

In this section we introduce the basic generalized differential object for non-smooth set-valued mapping. For our study of controllability we present some properties of Lipschitzian multifunction, the chain rules for the coderivative of composition of multifunctions, and finally give the characterization of openness of non-smooth set-valued mapping. Notice that most of this material with the proofs and detailed discussion can be found in [6], [7], [8], [9], [10].

For an arbitrary multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote by

$$\text{Dom}\,\Phi = \{x : \Phi(x) \neq \emptyset\}$$
$$\text{Im}\,\Phi = \{y \in \Phi : x \in \text{Dom}\,\Phi\}$$
$$\text{Ker}\,\Phi = \{x : 0 \in \Phi(x)\}$$

its domain, image and kernel.

The set

$$\limsup_{x \to \overline{x}} \Phi(x) := \{y \in \mathbb{R}^m : \exists \text{ sentences } x_k \to \overline{x},\ y_k \to y \text{ with } y_k \in \Phi(x_k),\ k = 1, 2, \ldots\}$$

is called the Kuratowski–Rainlevé upper limit of the multifunction $\Phi(x)$ as $x \to \overline{x}$.

The mapping $\Phi^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$

$$\Phi^{-1}(y) := \{x \in \mathbb{R}^n : y \in \Phi(x)\}$$

is called the inverse to Φ.

Let Ω be a nonempty subset of $\mathbb{R}^n$ and let

$$P(x, \Omega) := \{\omega \in \text{cl}\,\Omega : \|x - \omega\| = \text{dist}\,(x, \Omega)\}$$

be the set of best approximations of x in $\text{cl}\,\Omega$ with respect to the Euclidean distance function $\text{dist}(x, \Omega)$.

2.1. DEFINITION. Given $\overline{x} \in \overline{\Omega}$, the closed cone

$$N(\overline{x} \mid \Omega) := \limsup_{x \to \overline{x}} [\text{cone}\,(x - P(x, \Omega))]$$

is called the normal cone to Ω at the point $\overline{x} \in \Omega$.

The graph of multifunction $\Phi(x)$ is denoted by

$$\text{gph}\,\Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Phi(x)\}.$$

If A is a matrix, then we denote by A^* the adjoint matrix to A.

2.2. DEFINITION. Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph. The multifunction $D^*\Phi(\overline{x}, \overline{y}) := \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$D^*\Phi(\overline{x}, \overline{y})(y^*) := \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N((\overline{x}, \overline{y}) \mid \text{gph}\,\Phi)\}$$

is said to be the *coderivative* of Φ at the point $(\overline{x}, \overline{y}) \in \text{gph}\,\Phi$.

2.3. DEFINITION. The multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the closed graph is said to be *pseudo-Lipschitzian* around $(\overline{x}, \overline{y}) \in \text{gph}\,\Phi$ is there exists a neighborhood $\mathcal{U}$ of $\overline{x}$, a neighborhood $\vee$ of $\overline{y}$, and a constant $l \geq 0$ such that

$$\Phi(x') \cap \vee \subset \Phi(x) + l\|x' - x\|B \quad \text{for any} \quad x,\ x' \in \mathcal{U}, \tag{2.1}$$

where $B \subset \mathbb{R}^n$ is the unit closed ball.

If for every compact set $\vee \subset \mathbb{R}^n$ there exist a neighborhood $\mathcal{U}$ of $\overline{x}$ and a number $l \geq 0$ such that (2.1) holds, then the multi-function Φ is called *sub-Lipschitzian* around $\overline{x} \in \text{Dom}\,\Phi$. Finally, the multifunction Φ is said to be locally *Lipschitzian*, around $\overline{x} \in \text{Dom}\,\Phi$ if there exist a neighborhood $\mathcal{U}$ of $\overline{x}$ and a number $l \geq 0$ such that (2.1) is fulfilled with $\vee = \mathbb{R}^m$.

The multifunction Φ is said to be *locally bounded* around $\overline{x}$ if there is a neighborhood $\mathcal{U}$ of $\overline{x}$ such that the set $\Phi(\mathcal{U})$ is bounded

The following assertion can be found in ROCKAFELLAR [11].

2.4 PROPOSITION. *For any closed-graph multifunction* Φ *we have:*

(i) Φ *is sub-Lipschitzian around* $\overline{x}$ *if and only if* Φ *is pseudo-Lipschitzian around* $(\overline{x}, \overline{y})$ *for every point* $\overline{y} \in \Phi(\overline{x})$*;*

(ii) Φ *is locally Lipschitzian around* $\overline{x}$ *if and only if* Φ *is locally bounded and sub-Lipschitzian around this point.*

The following criterion for the pseudo-Lipschitzian property of multifunction is proved by MORDUKHOVICH in [6].

2.5. PROPOSITION. *Let $\Phi : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ be a multifunction with the closed graph. Then Φ is pseudo-Lipschitzian around $(\overline{x},\overline{y})$ if and only if*

$$D^*\Phi(\overline{x},\overline{y})(0) = \{0\}.$$

The next result for the sub-Lipschitzian and locally Lipschitzian properties of multifunctions follow directly from the two above propositions

2.6. COROLLARY. *For any closed-graph multifunction Φ being sub-Lipschitzian around $\overline{x}$, it is necessary and sufficient that*

$$D^*\Phi(\overline{x},\overline{y})(0) = \{0\} \quad \text{defin every} \quad \overline{y} \in \Phi(\overline{x}). \tag{2.2}$$

If Φ is locally bounded around $\overline{x}$, then condition (2.2) *is necessary and sufficient for Φ being locally Lipschitzian around this point.*

In the next we state the chain rules for the generalized differentiation of compositions of multifunctions. Let $\Phi : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ and $F : \mathbb{R}^m \Rightarrow \mathbb{R}^q$ be arbitrary multifunctions and let the multifunction $F \circ \Phi : \mathbb{R}^n \Rightarrow \mathbb{R}^q$ defined by

$$(F \circ \Phi)(x) := \bigcup_{y \in \Phi(x)} F(y)$$

be their composition.

From MORDUKHOVICH [6] we have the following results

2.7. THEOREM. *Let Φ and F have closed graphs and let $\overline{z} \in (F \circ \Phi)(\overline{x})$. Assume that the multifunction $M : \mathbb{R}^n \times \mathbb{R}^q \Rightarrow \mathbb{R}^m$ defined by*

$$M(x,z) := \Phi(x) \cap F^{-1}(z) = \{y \in \Phi(x) : z \in F(y)\}$$

is locally bounded around $(\overline{x},\overline{z})$ and the qualification condition

$$D^*F(y,\overline{z})(0) \cap \ker D^*(\overline{x},y) = \{0\} \quad \forall y \in \Phi(\overline{x}) \cap F^{-1}(\overline{z})$$

is fulfilled. Then one has

$$D^*(F \circ \Phi)(\overline{x},\overline{z}) \subset \bigcup_{y \in \Phi(\overline{x}) \cap F^{-1}(\overline{z})} \left[D^*\Phi(\overline{x},y) \circ D^*F(y,\overline{z})\right]. \tag{2.3}$$

2.8. COROLLARY. *For each $y \in \Phi(\overline{x})$, let F be pseudo-Lipschitzian around $(y,\overline{x})$. Then the chain rule* (2.3) *holds if the above defined multifunction M is locally bounded around $(\overline{x},\overline{z})$.*

Now we return to the openness of multifunction. This characterization of openness can be found detailed by MORDUKHOVICH [7]. Note that the openness of multifunction plays important rule in the later our study — the controllability of the multifunction.

2.9. DEFINITION. A multifunction Φ is said to be *open at a linear rate* around $(\overline{x},\overline{y}) \in \operatorname{gph}\Phi$ if there exists a number $a>0$, a neighborhood $\mathcal{U}$ of $\overline{x}$, and a neighborhood $\vee$ of $\overline{y}$ such that

$$B_{ar}(\Phi(x)\cap \vee) \subset \Phi(B_r(x)) \quad \text{for any} \quad (x,r) \quad \text{with} \quad B_r(x) \subset \mathcal{U},$$

where $B_{\varrho}(z)$ means the closed ball with center z and radius ϱ

Due to MORDUKHOVICH [7] we have the following result.

2.10. THEOREM. *The following conditions are equivalent*

(a) Φ *is open at a linear rate around* $(\overline{x},\overline{y})$.

(b) *There is a number* $c>0$*, a neighborhood* $\mathcal{U}$ *of* $\overline{x}$*, and a neighborhood* $\vee$ *of* $\overline{y}$ *such that*

$$\|y^*\| \le c\|x^*\| \text{ for all } x^* \in D^*\Phi(x,y)(y^*),\ x \in \mathcal{U} \text{ and } y \in \Phi(x)\cap\vee.$$

(c) *There exists neighborhood* $\mathcal{U}$ *of* $\overline{x}$*, and* $\vee$ *of* $\overline{y}$ *such that*

$$\operatorname{Ker} D^*(x,y) = \{0\} \quad \text{for all} \quad x \in \mathcal{U} \quad \text{and} \quad y \in \Phi(x)\cap\vee.$$

(d) $\operatorname{Ker} D^*\Phi(\overline{x},\overline{y}) = \{0\}$.

3. Main results

Consider the following controlled system

$$(1) \qquad x_{k+1} \in f(x_k,u_k), \qquad x_k \in \mathbb{R}^n, \qquad u_k \in \Omega \subset \mathbb{R}^m,$$

where $\Omega \subset \mathbb{R}^m$ is an arbitrary subset for which $\mathbb{O} \in \operatorname{int}\Omega$, $f:\mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded and pseudo Lipschitzian around $(\mathbb{O},\mathbb{O}) \in \mathbb{R}^n \times \mathbb{R}^m$ with the closed graphs, moreover $\mathbb{O} \in f(\mathbb{O},\mathbb{O})$.

3.1. DEFINITION. System (1) is said to be *locally reachable* from the origin (LR) after M steps if there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin $\mathbb{O} \in \mathbb{R}^n$ with the property that for any $x \in \mathcal{U}$ there are M vectors $u_0, u_1, \ldots, u_{M-1} \in \Omega$ such that

$$\begin{aligned}
&\mathbb{O} \in f(\mathbb{O},\mathbb{O})\\
&x_1 \in f(\mathbb{O},u_0)\\
&x_2 \in f(x_1,u_1) \quad \text{where} \quad x_1 \in f(\mathbb{O},u_0)\\
&x_3 \in f(x_2,u_2) \quad \text{where} \quad x_2 \in f(x_1,u_1)\\
&\vdots\\
&x \in f(x_{M-1},u_{M-1}) \quad \text{where} \quad x_{M-1} \in f(x_{M-2},u_{M-2}).
\end{aligned}$$

The question of the controllability problem as follows: under which condition about the multifunction f, the system (1) is locally reachable?

Let $u \in (\mathbb{R}^m)^M$, $u = (u_0, u_1, \dots, u_{M-1})$, $u_i \in \mathbb{R}^m$, $i = 0, 1, \dots, M-1$. We define the following projections

$$I^k_{\mathbb{N}} : (\mathbb{R}^m)^{\mathbb{N}} \to (\mathbb{R}^m)^k$$

$$I^k_{\mathbb{N}}(u_0, u_1, \dots, u_{\mathbb{N}-1}) := (u_0, u_1, \dots, u_{k-1}) \quad \text{for} \quad k = 1, 2, \dots, \mathbb{N}-1,$$

and

$$\hat{I}^k_N : (\mathbb{R}^m)^N \Rightarrow \mathbb{R}^m$$

$$\hat{I}^k_N(u_0, u_1, \dots, u_{N-1}) = u_k \quad k = 0, 1, 2, \dots, N-1$$

$$\hat{I}^N_N(u_0, u_1, \dots, u_{N-1}) = u_{N-1},$$

where N is a natural number.

Let us define step by step the following multifunctions

Define

$$F_1 : \mathbb{R}^n \Rightarrow \mathbb{R}^n$$

$$F_1(u_0) := f(\mathbb{O}, u_0) \quad \text{for} \quad u_0 \in \mathbb{R}^m$$

$$F_2 : (\mathbb{R}^m)^2 \Rightarrow \mathbb{R}^n$$

$$F_2(u_0, u_1) := f(f(\mathbb{O}, u_0), u_1) = \bigcup_{y \in f(\mathbb{O}, u_0)} f(y, u_1) = \bigcup_{y \in F_1(u_0)} f(y, u_1)$$

$$\vdots$$

$$F_M : (\mathbb{R}^m)^M \Rightarrow \mathbb{R}^n$$

$$F_M(u_0, u_1, \dots, u_{M-1}) := f(F_{M-1}(u_0, \dots, u_{M_2}), u_{M-1}) = \bigcup_{y \in F_{M-1}(u_0, u_1, \dots, u_{M-2})} f(y, u_{M-1}).$$

From the above notations and the definition of F_M we have

$$F_M(u) = f\left(F_{M-1}\left(I^{M-1}_M(u)\right), \hat{I}^{M-1}_M(u)\right) = \bigcup_{y \in F_{M-1}\left(I^{M-1}_M(u)\right)} f\left(y, \hat{I}^{M-1}_M(u)\right)$$

where $u = (u_0, u_1, \dots, u_{k-1}) \in (\mathbb{R}^m)^M$.

Let us define the multifunctions $T_k : (\mathbb{R}^m)^k \Rightarrow \mathbb{R}^n \times \mathbb{R}^m$ as follows for $u = (u_0, u_1, \dots, u_{k-1}) \in (\mathbb{R}^m)^k$

$$T_k(u) := (F_{k-1}(I^{k-2}_k(u)), I^{k-1}_k(u)), \tag{3.1}$$

where $k=2, 3, \ldots, M$.

Then from the aobve we have

$$F_k = f \circ T_k \qquad \text{for} \quad k = 2, 3, \ldots, M. \tag{3.2}$$

Since $f : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R}^n$ is locally bounded and pseudo Lipschitzian around $(\mathbb{O}, \mathbb{O}) \in \mathbb{R}^n \times \mathbb{R}^m$ with the closed graph we obtain that $F_k : (\mathbb{R}^m)^k \Rightarrow \mathbb{R}^n$ is locally bounded and pseudo Lipschitzian around $\mathbb{O} \in (\mathbb{R}^m)^k$ with the closed graph (see ROCKAFELLAR [11], Theorem 4.1) for $k=2, 3, \ldots, M$. Now we are in position to calculate the coderivatives of F_k, $k=2, 3, \ldots, M$.

From the formula (3.1) and Theorem 5.2 (MORDUKHOVICH [10]) it is easy to verify that

$$D^* T_k(\mathbb{O}, y)(z^*) \subset \bigcup_{(\overline{y}_1, \overline{y}_2) \in M_k(\mathbb{O}, y)} \Big[D^* F_{k-1}(I_k^{k-2})(\mathbb{O}, \overline{y}_1) z^*) + $$
$$ + D^* I_k^{k-1}(\mathbb{O}, \overline{y}_2) z^* \Big], \qquad (y \in \mathbb{R}^m \times \mathbb{R}^n),$$

where

$$M_k(\mathbb{O}, y) := \Big\{ (\overline{y}_1, \overline{y}_2) \in \mathbb{R}^n \times \mathbb{R}^m : \overline{y}_1 \in F_{k-1}\left(I_k^{k-2}(\mathbb{O}\right),$$
$$\overline{y}_2 = \mathbb{O},\ (\overline{y}_1, \overline{y}_2) = y \Big\} = \{(y_1, 0)\},$$

where $y = (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^m$. From Proposition 2.4 (MORDUKHOVICH [6] we have

$$D^* T_k(\mathbb{O}, y)(z^*) \subset D^* F_{k-1}\left(I_k^{k-2} \right)(\mathbb{O}, y_1) z^* + z^*, \tag{3.3}$$

where $y = (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^m$). Since the multifunctions f and F_k locally bounded and pseudo Lipschitzian around $(\mathbb{O}, \mathbb{O}) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{O} \in (\mathbb{R}^m)^k$, respectively; we obtain that (see Corollary 5.2 (MORDUKHOVICH [6] and formula (3.2)).

$$D^* F(\mathbb{O}, y) \subset \bigcup_{z \in N_k(\mathbb{O}, y)} D^* T_k(\mathbb{O}, z) \circ D^* f(z, \mathbb{O}), \tag{3.4}$$

where

$$N_k(\mathbb{O}, y) := \left(F_{k-1}(I_k^{k-2}(\mathbb{O})), \mathbb{O} \right) \cap f^{-1}(y),$$

and

$$D^* F_k(\mathbb{O}, \mathbb{O}) \subset \bigcup_{z \in F_{k-1}(I_k^{k-2}(\mathbb{O})), \mathbb{O}) \cap \ker f = N_k(\mathbb{O}, \mathbb{O})} D^* T_k(\mathbb{O}, z) \circ D^* f(z, \mathbb{O}) \tag{3.5}$$

for $k = 1, 2, \ldots, M$.

From (3.3) and (3.5) it follows that

$$D\overset{*}{\underset{k}{F}}(\mathbb{O},\mathbb{O}) \subset \tag{3.6}$$

$$\subset \bigcup_{\substack{z\in F_{k-1}(I_k^{k-2}(\mathbb{O})),\mathbb{O})\cap\ker f\\ z=(z_1,z_2)}} (D^*F_{k-1}(I_k^{k-2})(\mathbb{O},z_1)+id)\circ D^*f(z,\mathbb{O}),$$

where id means the identity in the space $(\mathbb{R}^n\times\mathbb{R}^m)^*$.

By induction with step $k=1$ to M we obtain from formula (3.6) the following inclusion:

$$D^*F_M(\mathbb{O},\mathbb{O})\subset \bigcup_{\substack{z_{M-1}\in N_{M-1}(\mathbb{O},\mathbb{O})\\ z_{M-1}=(z_{M-2},z'_{M-2})}} \left(\left(\bigcup_{\substack{z_{M-3}\in N_{M-2}(0,z_{M-2})\\ z_{M-3}=(z_{M-4},z'_{M-4})}} \ldots\right)+id\right)\circ$$

$$\circ(D^*f(z_{M-2},\mathbb{O})+id)\circ D^*f(z_{M-1},\mathbb{O});$$

$$D^*F_M(\mathbb{O},\mathbb{O})\subset \bigcup_{\substack{z_{M-2}\in N_{M-1}(\mathbb{O},\mathbb{O})\\ z_{M-2}=(z_{M-2},z'_{M-2})}} \left(\left(\ldots\left(\bigcup_{z_0\in N_2(\mathbb{O},z_1)} (D^*f(z_0,\mathbb{O})+id)\right)\circ\right.\right.$$

$$\left.\left.\circ D^*f(z_1,\mathbb{O})\ldots\right)+id\right)\circ D^*f(z_{M-1},\mathbb{O}). \tag{3.7}$$

REMARK. The right part of the above inclusions looks like the Kalman-controllability condition in the classic case.

Let us define the multifunction $S(\mathbb{O},\mathbb{O}):(\mathbb{R}^n)^*\Rightarrow((\mathbb{R}^m)^M)^*$ as follows

$$S(\mathbb{O},\mathbb{O}):=\bigcup_{\substack{z_{M-1}\in N_{M-1}(\mathbb{O},\mathbb{O})\\ z_{M-1}=(z_{M-2},z'_{M-2})}} \left(\left(\ldots\left(\bigcup_{z_0\in N_2(\mathbb{O},z_1)} D^*f(z_0,\mathbb{O})+id\right)\right)\circ\right.$$

$$\left.\circ(D^*f(z_1,\mathbb{O})\ldots)+id\right)\circ D^*f(z_{M-1},\mathbb{O}).$$

THEOREM. *Consider the controlled system* (1). *Assume that* $\mathbb{O}\in\operatorname{int}\Omega$, $\mathbb{O}\in f(\mathbb{O},\mathbb{O})$ *and the multifunction* $f:\mathbb{R}^n\times\mathbb{R}^m\Rightarrow\mathbb{R}^n$ *is locally bounded and pseudo Lipschitzian around* $(\mathbb{O},\mathbb{O})$. *If* $\operatorname{Ker}S(\mathbb{O},\mathbb{O})=\{\mathbb{O}\}$ *then the system* (1) *is locally reachable* (LR) *from* $\mathbb{O}$ *after* M *steps.*

PROOF. From the formula (3.7) and the condition $\operatorname{Ker}S(\mathbb{O},\mathbb{O})=\{\mathbb{O}\}$ we have $\operatorname{Ker}D^*F_M(\mathbb{O},\mathbb{O})=\{\mathbb{O}\}$. From Theorem 3.7 (see MORDUKHOVICH [7] it follows that the multifunction $F_M : (\mathbb{R}^m)^M \rightrightarrows \mathbb{R}^n$ is open at a linear rate around $(\mathbb{O}, \mathbb{O})$. It follows that there exists a number $a > 0$, a neighborhood $\mathcal{U}$ of $\mathbb{O} \in (\mathbb{R}^m)^M$ and a neighborhood $\vee$ of $\mathbb{O} \in \mathbb{R}^n$ such that

$$B_{ar}(F_M(u) \cap \vee) \subset F_M(B_r(u)) \text{ for any } (u, r) \text{ with } B_r(u) \subset \mathcal{U}.$$

Since $\mathbb{O} \in \operatorname{int} \Omega$ it follows that

$$(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \in \operatorname{int} \Omega^M.$$

Let r be such that $B_r(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \subset \mathcal{U} \cap \Omega^M$ then we have from the above

$$B_{ar}(F_M(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \cap \vee) \subset F_M(B_r(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}})).$$

From condition $\mathbb{O} \in F_M(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}})$ we have that $\mathbb{O} \in F_M(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \cap \vee$.

Putting $\overline{\mathbb{V}} := B_{ar}(F_M(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \cap \vee)$, then $\overline{\mathbb{V}}$ is a neighborhood of $\mathbb{O} \in \mathbb{R}^n$. We will prove that every $x \in \overline{\vee}$ is reachable from the origin $\mathbb{O}$ after M steps. Indeed, if $x \in \overline{\vee}$ then $x \in F_M(B_r(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}))$. This means that there exists $(u_0, u_1, \ldots, u_{M-1} \in B_r(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}})$ such that

$$x \in F_M(u_0, u_1, \ldots, u_{M-1}).$$

Hence $B_r(\underbrace{\mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}}_{M \text{ times}}) \subset \Omega^M$, therefore we can say that there u_0, u_1, …, $u_{M-1} \in \Omega$ such that

$$x \in F_M(u_0, u_1, \ldots, u_{M-1}).$$

This means that there exist u_0, u_1, …, $u_{M-1} \in \Omega$ such that

$$\begin{aligned}
&x \in f(x_{M-1}, u_{M-1}) \quad \text{where} \quad x_{M-1} \in f(x_{M-2}, u_{M-2})\\
&x_{M-1} \in f(x_{M-2}, u_{M-2}) \quad \text{where} \quad x_{M-2} \in f(x_{M-3}, u_{M-3})\\
&\quad \vdots\\
&x_2 \in f(x_1, u_1) \quad \text{where} \quad x_1 \in f(\mathbb{O}, u_0)\\
&x_1 \in f(\mathbb{O}, u_0).
\end{aligned}$$

So x is reachable from the origin $\mathbb{O}$ after M steps, thus the system (1) is locally reachable from the origin $\mathbb{O}$ after M steps. The proof is complete.

REMARK. As in [4] and [5] we can study the case, when $\mathbb{O} \notin \operatorname{int} \Omega$ but $\mathbb{O} \in \Omega$.

EXAMPLE. Let is consider for following controlled munltifunction system

$$x_{k+1} \in f(x_k, u_k), \quad x_k \in \mathbb{R}, \quad u_k \in \Omega \subset \mathbb{R}$$

where the multifunction $f : \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$ is given as follows:

$$\begin{aligned}
&f(x,u) := [0,1) \quad &&\text{for} \quad 0 \le x \le \frac{1}{2} \quad \text{and} \quad u \ge 0\\
&f(x,u) := (-1,0) \quad &&\text{for} \quad \frac{1}{2} < x \le 1 \quad \text{and} \quad u \ge 0\\
&f(x,u) := [0,1) \quad &&\text{for} \quad 0 \le x \le \frac{1}{2} \quad \text{and} \quad u < 0\\
&f(x,u) := (-1,0) \quad &&\text{for} \quad \frac{1}{2} < x \le 1 \quad \text{and} \quad u < 0\\
&f(x,u) := [0,1) \quad &&\text{for an other case.}
\end{aligned}$$

It is easy to see that the above multifunction $f : \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$ is locally bounded and locally Lipschitzian around the point $(\mathbb{O}, \mathbb{O}) \in \mathbb{R} \times \mathbb{R}$.

We can verify easily that the $f(\mathbb{O}, u)$ takes values $[0,1)$ or $(-1,0)$ only, this means that the above controlled multifunction system is not locally reachable from the origin after 1 step.

We will show that this system os locally reachable from the origin after 2 steps. Indeed

$$f(f(\mathbb{O}, \Omega), \Omega) \supset f\left(f\left(\mathbb{O}, \frac{1}{2}\right), \Omega\right),$$

but by the definition of the function f we have

$$f\left(\mathbb{O}, \frac{1}{2}\right) = \left[0, \frac{1}{2}\right).$$

So it follows that

$$f(f(\mathbb{O}, \Omega), \Omega) \supset f\left(f\left(\mathbb{O}, \frac{1}{2}\right), \Omega\right) \supset \bigcup_{\substack{x \in [0,1) \\ u \in [-1,1]}} f(x,u) \supset$$

$$\supset f\left(\mathbb{O}, \frac{1}{2}\right) \cup f\left(\frac{3}{4}, 1\right) \supset [0,1) \cup (-1,0).$$

This means that

$$(-1,1) \subset f(f(\mathbb{O}, \Omega), \Omega).$$

That is, the above controlled multifunction system is locally reachable from the origin adfter 2 steps.

Now let us consider the multifunction

$$F_1 : \mathbb{R} \rightrightarrows \mathbb{R}, \qquad F_1(u) := f(\mathbb{O}, u).$$

We know that $S_1(\mathbb{O},\mathbb{O}) = D^* F_1(\mathbb{O},\mathbb{O})$. By the definition of the multifunction f it follows that $\operatorname{gph} F_1 = (-1,1) \times \mathbb{R}$.

It is easy to see that

$$N((\mathbb{O},\mathbb{O}) \mid \text{gph } F_1) :=$$
$$:= \limsup_{(u,v)\to(\mathbb{O},\mathbb{O})} [\text{cone } ((u,v) - P((u,v)), \text{gph } F_1) = \mathbb{R} \times \mathbb{R}.$$

By the definition of coderivative of the multifunction F_1 we obtain

$$D^* F_1(\mathbb{O},\mathbb{O})(y) :=$$
$$:= \{x \in \mathbb{R} : (x, -y) \in N((\mathbb{O},\mathbb{O}) \mid \text{gph } F_1)\} = \mathbb{R} \quad \text{for all } y \in \mathbb{R}.$$

It shows that for 1 step

$$\text{Ker } S_1(\mathbb{O},\mathbb{O}) \neq \{\mathbb{O}\}.$$

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A NOTE ON MINIMAX THEOREMS

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To the memory of Paul Erdős

This aim of this paper is to prove the following convex type theorem:

THEOREM 1. *Let* $I_1 = I_2 = \ldots = I_n = [0,1]$, $f_k : I_1 \times I_2 \times \ldots \times I_n \to \mathbf{R}$ *(*$k = 1, \ldots, n$*) be continuous functions. Suppose that the following property* (A) *holds:*

If $f_k' : I_1 \times I_2 \times \ldots \times I_n \to \mathbf{R}$ *(*$k = 1, \ldots, n$*) is continuous and partially concave in the* k*-th variable*

$$\text{(A)} \qquad \begin{array}{l} \text{then the functions } g_k = f_k + f_k' \text{ have a saddle point } x^* \in I_1 \times \\ \times I_2 \times \ldots \times I_n : g_k(x^*) \geq g_k(x_1^*, x_2^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_n^*) \\ \text{for } x_k \in I_k \ (k = 1, \ldots, n) \end{array}$$

If property (A) *holds, then* f_k *is partially concave in its* k*-th variable (*$k = 1, \ldots, n$*).*

REMARKS. I) The converse implication (that the partially concavity of f_k implies property (A)) is the well known Nikaido–Isoda theorem [1].

II) The special case $n = 2$ of Theorem 1 is given in the paper [2].

PROOF OF THEOREM 1. Suppose indirectly that e.g. f_1 is not concave in x_1; this means that there exists a $y_0 = (y_{0,2}, \ldots, y_{0,n}) \in I_2 \times \ldots \times I_n$ for which $f_1(., y_0)$ is not concave in I_1. We can suppose that

$$(1) \qquad 0 < y_{0,i} < 1 \qquad i = 2, \ldots, n$$

since the property of not being concave is invariant under small perturbations and f_1 is supposed to be continuous, thus y_0 can be moved within a small distance if necessary. Further we can substitute f_1 by $f_1 + cx_1$ since it alters neither the truth of (A) nor the concavity of f_1. Therefore the indirect assumption yields that, there exists $0 \le a < b \le 1$ such that

(2)

$$f(a, y_0) = f(b, y_0) = m, \quad f(x_1, y_0) < m \text{ for } a < x_1 < b, \quad m = \max(f\ .\ , y_0)$$

(see in [2] for more details).

Introduce further the function $f_1'(x_1, y)$ for $x_1 \in I_1$, $y \in (y_2, \dots, y_n) \in I_2 \times \times \dots \times I_n$ by

$$f_1'(x_1, y) = \begin{cases} -\alpha|x_1 - a||y_2 - y_{0,2}|, & \text{if } y_2 \ge y_{0,2} \\ -\alpha|x_1 - b||y_2 - y_{0,2}|, & \text{if } y_2 \le y_{0,2} \end{cases}$$

and let $g_1 = f_1 + f_1'$. Clearly f_1' satisfies the requirements of (A). Collect the maximum places of g_1 in the variable x_1 into the set M_1

$$M_1 = \{(x_1^*, y) : y \in I_2 \times \dots \times I_n, \quad g_1(x_1^*, y) = \max g_1(\ .\ , y)\}.$$

Finally draw the line L by

$$L = \left[(0, \dots, 0), \left(\frac{a+b}{2}, y_0\right)\right] \cup \left[\left(\frac{a+b}{2}, y_0\right), (1, \dots, 1)\right]$$

i.e. we join the points $(0,\dots,0)$, $\left(\frac{a+b}{2}, y_0\right)$, $(1,\dots,1) \in I_1 \times I_2 \times \dots \times I_n$ by straight lines. Are basic idea of our proof if the following:

LEMMA. *If $\alpha > 0$ is large enough, $M \cap L = \emptyset$.*

PROOF OF THE LEMMA. First of all, the continuity of g_1 implies that the set M_1 is closed, since the property of not being a maximum point in x_1 is invariant under small perturbation. By the definition of a and b, we have that $\left(\frac{a+b}{2}, y_0\right) \notin M_1$. Since M_1 is closed, there exists $\delta > 0$ such that, no points $(x_1, y) \in L$ with $|y_2 - y_{0,2}| \le \delta$ can belong to M_1. Now if $y_2 > y_{0,2} + \delta$, then $g_1 = f_1 - \alpha|x_1 - a||y_2 - y_{0,2}|$ and for the points $(x_1, y) \in L$, we have $x_1 > \frac{a+b}{2}$ (since along L all coordinates are monotone increasing). For $x_1 = a$, $g_1 = f_1$ and for $x_1 > \frac{a+b}{2}$, $g_1 \le f_1 - \alpha\frac{b-a}{2}\delta$. The function f_1 being bounded, we can choose α so large that $g_1(.,y)$ can not be maximal for $x_1 > \frac{a+b}{2}$ i.e. in a point of L. Finally if $y_2 < y_{0,2} - \delta$, then in L we have by monotonity $x_1 < \frac{a+b}{2}$ and then $g_1 = f_1 - \alpha|x_1 - b||y_2 - y_{0,2}| \le f_1 - \alpha\frac{b-a}{2}\delta$ can not be maximal for large α.

The lemma is proved.

Returning to the proof of the Theorem, construct the functions f'_k $k \geq 2$ as follows: We parametrize L by its first coordinate. Let

$$L = \{(x_1, y_2(x_1), \dots, y_n(x_1)) : x_1 \in I_1\}.$$

We do not need the concrete form of the piecewise linear functions $y_k(x_1)$. Now let for $x_1 \in I_1$ and $y = (y_2, \dots, y_n) \in I_2 \times \dots \times I_n$

$$f'_k(x_1, y) = -\alpha |y_k - y_k(x_1)|, \qquad k = 2, \dots, n.$$

They satisfy the conditions given in (A).

Let now $g_k = f_k + f'_k$ and

$$\begin{aligned} M_k = \{(y_1, \dots, y_{k-1}, x_k^*, y_{k+1}, \dots, y_n) :& \\ y_i \in I_i ; g_k(y_1, \dots, y_{k-1}, x_k^*, y_{k+1}, \dots, y_n) &= \\ = \max_{I_k} g_k(y_1, \dots, y_{k-1}, \ \cdot\ , y_{k+1}, \dots, y_n)\}&. \end{aligned}$$

We see from the construction that

$$g_k = f_k \quad \text{if} \quad x_k = y_k(x_1) \qquad\qquad g_k \leq f_k - \alpha\delta \quad \text{if} \quad |x_k - y_k(x_1)| \geq d.$$

Since f_k is bounded, for every $\delta > 0$ there exists $\alpha > 0$ large enough such that g_k can be maximal in x_k if $|x_k - y_k(x_1)| \geq \delta$. Consequently the points $x \in M_2 \cap \dots \cap M_n$ satisfy $|x_k - y_k(x_1)| \leq \delta$, $k = 2, \dots, n$. In other words this means that the points of $M_2 \cap \dots \cap M_n$ are close to the points of L.

On the other hand $M_1 \cap L = \emptyset$, M_1 and L are compact sets, hence M_1 can not have points arbitrarily close to L. Since $\delta > 0$ is arbitrary above, this implies

$$M_1 \cap M_1 \cap \dots \cap M_n = \emptyset.$$

But this means that $g_1, \dots, g_n$ can not have a saddle point (by the definition, given in (A), saddle points are exactly the points of $M_1 \cap M_1 \cap \dots \cap M_n$).

The contradiction proves Theorem 1.

For the proof of a genaralization of Theorem 1 we need its statement and also the following one:

PROPOSITION. *Let $\Omega \subset \mathbf{R}^N$ be an open convex and bounded set, $f : \overline{\Omega} \to \mathbf{R}$ be a concave and continuous function. Then there exists concave functions $f_n : \mathbf{R}^N \to \mathbf{R}$ which converge to f uniformly on the set $\overline{\Omega}$.*

PROOF. Denote

$$\Omega_n = \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{n} \right\}.$$

Since f is concave, for every $x \in \Omega$ there exists a hyperplane $h_x \subset \mathbf{R}^N \times \mathbf{R}$ containing the point $(x, f(x))$ such that all points of the graph of f lie under h_x. Now let

$$f_n(y) = \inf\{h_x(y) : x \in \Omega_n\}.$$

Since the h_x are concave, the function f_n is concave in $\mathbf{R}^N$ as well, and by definition

$$f_n(x) = f(x) \quad (x \in \Omega_n), \qquad f_n(x) \geq f(x) \quad (x \in \overline{\Omega}).$$

Being concave in $\mathbf{R}^N$, the functions f_n are continuous.

Suppose indirectly that the convergence of f_n to f is not uniform i.e. that there exist points $x_n \in \overline{\Omega}$ with

$$f_n(x_n) - f(x_n) \geq \varepsilon_0 > 0 \qquad \forall\, n.$$

Since $\overline{\Omega}$ is compact, the sequence x_n has an accumulation point x^*. From $\operatorname{dist}(x_n, \partial\Omega) \leq \frac{1}{n}$ we get $x^* \in \partial\Omega$.

Now take any point $x^{(0)} \in \Omega$ and $\delta > 0$. Denote $y_n \in \Omega$ the point of the segment $[x^{(0)}, x_n]$ being at a distance $\varepsilon_0\delta$ from x_n. If we suppose by taking subsequences that

$$x_n \to x^*$$

then the points y_n converge to y^*, the point of the segment $[x^{(0)}, x^*]$ for which $|y^* - x^*| = \varepsilon_0\delta$. Consequently for large n,

$$f(x^{(0)}) = f_n(x^{(0)}), \quad f(y_n) = f_n(y_n), \quad f_n(x_n) - f(x_n) \geq \varepsilon_0.$$

Now if $\delta > 0$ is small enough then both $f(y_n)$ and $f(x_n)$ is nearly $f(x^*)$; more precisely

$$|f(y_n) - f(x_n)| < \frac{\varepsilon_0}{2} \qquad \text{if } \delta > 0 \text{ is small.}$$

Hence in this case

$$f_n(x_n) - f_n(y_n) > \frac{\varepsilon_0}{2}, \qquad |x_n - y_n| = \varepsilon_0\delta.$$

By the concavity of f_n, $f_n(x^0)$ can not be larger than the height of the line through $(x_n, f_n(x_n))$ and $(y_n, y_n(y_n))$ over x^0. In the endpoint x_n the height $f_n(x_n)$ is bounded since $f_1(x_n) \geq f_n(x_n) \geq f(x_n)$. The rate of loosing height is $\geq \frac{\varepsilon_0}{2}/(\varepsilon_0\delta) = \frac{1}{2\delta}$ which is very high for $\delta > 0$ small; consequently the height of the other endpoint, $f_n(x^{(0)})$ is extremely low if δ is small enough. But this is nonsense since $f_n(x^{(0)}) = f(x^{(0)})$ is constant. The contradiction proves the Proposition. ∎

Now we are in the position to prove the

THEOREM 2. *Let* $K_1, \ldots, K_n \subset \mathbf{R}^N$ *be compact convex sets and* $f_k : K_1 \times \times \ldots \times K_n \to \mathbf{R}$ *be continuous functions. Suppose that*

(A$_1$) *If* $f'_k : K_1 \times \ldots \times K_n \to \mathbf{R}$ *is continuous and partially concave in the* k*-th variable on* K_k *then the functions* $g_k = f_k + f'_k$ *have a saddle point in* $K_1 \times \ldots \times K_n$.

If the property (A$_1$) *holds then* f_k *is partially concave in* K_k.

PROOF. Let $L_k \subset K_k$ be any segment in K_k. It is enough to show that $\hat{f}_k = f_k\Big|_{L_1 \times \ldots \times L_n}$ is partially concave in the k-th variable. Let $\hat{f}'_k : L_1 \times \ldots \times \times L_n \to \mathbf{R}$ be a continuous and partially concave in L_k. Extend $\hat{f}_k$ to $\mathbf{R}\times \ldots \times\mathbf{R}$ by the above Proposition and then extend $\hat{f}_k$ to $K_1 \times \ldots \times K_n$ such that in the directions in K_i orthogonal to L_i the functions $\hat{f}_k$ be constant. By (A$_1$) the functions $f_k + \hat{f}_k$ have a saddle point, hence the condition (A) of Theorem 1 fulfills; by Theorem 1 the $\hat{f}_k$ are partially concave and it was to be proved. ∎

REMARK. Theorem 2 holds also for the case when $K_1, \ldots, K_n$ are compact convex subsets of Banach spaces.

The proof is the same with the only exception that the extension of $\hat{f}_k$ from $\mathbf{R} \times \ldots \times \mathbf{R}$ to $K_1 \times \ldots \times K_n$ goes without using orthogonality, by the following way. Denote $B_i \supset K_i$ the corresponding Banach space and $\mathbf{R} \subset B_i$ the line containing the segment L_i. Then there exists a subspace $V_i \subset B_i$ not containing $\mathbf{R}$ such that

$$B_i = \mathbf{R} \dotplus V_i$$

and let

$$\hat{f}_k(t_1 + v_1, \ldots, t_n + v_n) = \hat{f}_k(t_1, \ldots, t_n)$$

be the desired partially concave extension of $\hat{f}_k$ to $K_1 \times \ldots \times K_n$. The other parts of the proof of Theorem 2 remain unchanged.

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