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A FIXED POINT THEOREM FOR VECTORIAL MULTIVALUED SET FUNCTIONS

By

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0. Introduction

Fractals are often used to model various physical or chemical phenomena. An important step in the development of a rigorous mathematical theory of the fractals was done by HUTCHINSON [3]. According to his view fractals are certain invariant sets with respect to a family of continuous functions.

In order to recall HUTCHINSON's definition of an invariant set, let (X,d) be a complete metric space, let $\mathcal{K}(X)$ be the family of all compact nonempty subsets of X endowed with the Hausdorff metric, let $g_i : X \to X$ $(i=1,\dots,n)$ be continuous functions, and let $G:\mathcal{K}(X)\to\mathcal{K}(X)$ be the function defined by

$$G(K) = g_1(K) \cup \ldots \cup g_n(K).$$

Any fixed point of G is called by HUTCHINSON an *invariant* set with respect to the family $\{g_1,\dots,g_n\}$.

In the study of these invariant sets the following assertion plays an important role:

(A) *The function G has a unique fixed point and for every $K \in \mathcal{K}(X)$ the sequence $(G^\nu(K))_{\nu\in\mathbf{N}}$ converges to this point.*

In this assertion G^ν $(\nu \in \mathbf{N})$ denote the iteratives of G, i.e. the functions inductively defined by

$$\begin{cases} G^1 = G \\ G^{\nu+1} = G \circ G^\nu \ \forall \nu \in \mathbf{N}. \end{cases}$$

In the case when $g_1,\ldots,g_n$ are contractions HUTCHINSON [3] has proved that assertion (A) is valid. However, assertion (A) is true in a more general situation as well. Starting from Hutchinson–Barnsley's method for the study of the convergence of the iteratives of some multivalued set functions, MÁTÉ [5] has proved assertion (A) in the case when $g_1,\ldots,g_n$ are Lipschitz functions having a certain property. It must be pointed out that MÁTÉ did not use any fixed point theorem in his proof. Nevertheless, MÁTÉ's result can be applied if $g_1,\ldots,g_n$ are contractions.

HUTCHINSON's approach to the fractals seems to be too strict for the needs of practice. Therefore various authors have suggested modifications of it. For instance, MAULDIN and WILLIAMS [4] have generalized HUTCHINSON's method by considering graphs and by replacing HUTCHINSON's invariant set by an invariant family of sets with respect to a family of continuous functions. These invariant families of sets are the fixed points of a vectorial multivalued set function $H:(\mathcal{K}(X))^n \to (\mathcal{K}(X))^n$ defined by

$$H(K_1,\ldots,K_n) = \left(\left(\bigcup_{i=1}^{k_{1,1}} h_i(K_1)\right) \bigcup \ldots \bigcup \left(\bigcup_{i=k_{1,n-1}}^{k_{1,n}} h_i(K_n)\right),\ldots,\right.$$
$$\left.\left(\bigcup_{i=k_{n-1,n}}^{k_{n,1}} h_i(K_1)\right) \bigcup \ldots \bigcup \left(\bigcup_{i=k_{n,n-1}}^{k_{n,n}} h_i(K_n)\right)\right),$$

where $k_{i,j}$ $(i,j=1,\ldots,n)$ are natural numbers satisfying

$$k_{i,j} < k_{i,j+1} \text{ for all } i \in \{1,\ldots,n\} \text{ and all } j \in \{1,\ldots,n-1\},$$
$$k_{i,n} < k_{i+1,1} \text{ for all } i \in \{1,\ldots,n-1\},$$

and $h_i : X \to X$ $(i = 1,\ldots,k_{n,n})$ are continuous functions. MAULDIN and WILLIAMS have shown that an assertion analogous with (A) holds for H when the functions $h_1,\ldots,h_{k_{n,n}}$ are contractions.

The investigations by MAULDIN and WILLIAMS have been continued by BANDT [1] under the assumption that $h_1,\ldots,h_{k_{n,n}}$ are so-called cyclically contracting Lipschitz functions. The purpose of the present paper is to reveal that there exists a fixed point of H even when $h_1,\ldots,h_{k_{n,n}}$ are Lipschitz functions satisfying a weaker condition than that considered by BANDT. In the case $n=1$ this condition is exactly that considered by MÁTÉ [5]. It should be mentioned that our approach differs essentially from BANDT's one.

The paper is divided into four sections. In the first section we establish results concerning the powers of a square matrix with elements from an algebraical structure $(R,+,\cdot)$, which is more general than a ring and which we call prering. These results are the theoretical basis of a first fixed point theorem concerning H, which we derive in the second section under the assumption that $n = 2$. This theorem is proved without making use of any fixed point theorem. Under the same assumption concerning n we also prove a second fixed point theorem for H which is easier to be applied in practice than our first fixed point theorem. Some specializations of this second fixed point theorem are given in the third section. In the last section we extend both our fixed point theorems for the case of an arbitrary n in order to obtain results concerning Mauldin–Williams graphs.

1. Powers of a square matrix with elements from a prering

DEFINITION 1.1. Let R be a nonempty set, and let $+$ and $\cdot$ be two composition laws on R. The triple $(R,+,\cdot)$ is said to be a *prering* if the following conditions are satisfied:

(i) $(R,+)$ is an abelian semigroup;

(ii) $(R,\cdot)$ is a semigroup;

(iii) $x(y+z)=xy+xz$ and $(y+z)x=yx+zx$ whenever x, y, $z \in R$.

In what follows we suppose that $(R,+,\cdot)$ is a prering. If x is an element of R which is represented under the form $x_1+\ldots+x_k$, where $x_1, \ldots, x_k \in R$, then the set of all terms of this sum is denoted by $t(x)$, i.e. $t(x)=\{x_1,\ldots,x_k\}$.

Let $M_2(R)$ be the set of all square matrices with elements from R. In this set we define the multiplication in the usual manner:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} = \begin{pmatrix} xx'+yz' & xy'+yw' \\ zx'+wz' & zy'+ww' \end{pmatrix}.$$

Let m, n, p, q be integers satisfying $0<m<n<p<q$, let a_i $(i=1,\ldots,q)$ be elements of R, and let T be the matrix defined by

$$T = \begin{pmatrix} a_1+\ldots+a_m & a_{m+1}+\ldots+a_n \\ a_{n+1}+\ldots+a_p & a_{p+1}+\ldots+a_q \end{pmatrix}.$$

We will investigate the powers of this matrix T. Let $\nu \in \mathbf{N}$ be fixed. Accomplishing multiplication in $M_2(R)$, one finds that

$$T^\nu = \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix},$$

where A_ν, B_ν, C_ν and D_ν are unaccomplished sums whose terms are products of ν factors chosen from the elements $a_1,\dots,a_q$. It should be noted that A_ν, B_ν, C_ν and D_ν satisfy the following recurrence relations:

$$A_1 = \sum_{i=1}^{m} a_i, \quad A_{\nu+1} = \left(\sum_{i=1}^{m} a_i\right) A_\nu + \left(\sum_{i=m+1}^{n} a_i\right) C_\nu; \tag{1.1}$$

$$B_1 = \sum_{i=m+1}^{n} a_i, \quad B_{\nu+1} = \left(\sum_{i=1}^{m} a_i\right) B_\nu + \left(\sum_{i=m+1}^{n} a_i\right) D_\nu; \tag{1.2}$$

$$C_1 = \sum_{i=n+1}^{p} a_i, \quad C_{\nu+1} = \left(\sum_{i=n+1}^{p} a_i\right) A_\nu + \left(\sum_{i=p+1}^{q} a_i\right) C_\nu; \tag{1.3}$$

$$D_1 = \sum_{i=p+1}^{q} a_i, \quad D_{\nu+1} = \left(\sum_{i=n+1}^{p} a_i\right) B_\nu + \left(\sum_{i=p+1}^{q} a_i\right) D_\nu. \tag{1.4}$$

Next we introduce the following sets:

$$\Sigma_q = \{\sigma : \mathbf{N} \to \{1,\dots,q\}\},$$
$$S_1 = \{\sigma \in \Sigma_q \mid \forall \nu \in \mathbf{N} : a_{\sigma(1)} \dots a_{\sigma(\nu)} \in t(A_\nu) \cup t(B_\nu)\},$$
$$S_2 = \{\tau \in \Sigma_q \mid \forall \nu \in \mathbf{N} : a_{\tau(1)} \dots a_{\tau(\nu)} \in t(C_\nu) \cup t(D_\nu)\}.$$

PROPOSITION 1.1. *Let $\nu \in \mathbf{N}$. Then the following statements hold:*

1° *If $k_1,\dots,k_\nu \in \{1,\dots,q\}$ and $a_{k_1}\dots a_{k_\nu} \in t(A_\nu) \cup t(B_\nu)$, then there exists a $\sigma \in S_1$ such that $(k_1,\dots,k_\nu) = (\sigma(1),\dots,\sigma(\nu))$.*

2° *If $l_1,\dots,l_\nu \in \{1,\dots,q\}$ and $a_{l_1}\dots a_{l_\nu} \in t(C_\nu) \cup t(D_\nu)$, then there exists a $\tau \in S_2$ such that $(l_1,\dots,l_\nu) = (\tau(1),\dots,\tau(\nu))$.*

PROOF. By mathematical induction we show that the following property is true for all $k \in \mathbf{N}$:

$P_k =$ "If $i_1,\dots,i_k \in \{1,\dots,q\}$ and $a_{i_1}\dots a_{i_k} \in t(A_k) \cup t(B_k)$, then there exists a $\sigma \in S_1$ such that $(i_1,\dots,i_k) = (\sigma(1),\dots,\sigma(k))$; if $j_1,\dots,j_k \in \{1,\dots,q\}$ and $a_{j_1}\dots a_{j_k} \in t(C_k) \cup t(D_k)$ then there exists a $\tau \in S_2$ such that $(j_1,\dots,j_k) = (\tau(1),\dots,\tau(k))$."

STEP I. Let $k=1$, and let $i_1 \in \{1,\dots,q\}$ be such that $a_{i_1} \in t(A_1) \cup t(B_1)$.

CASE 1: $a_{i_1} \in t(A_1)$. Then $i_1 \in \{1,\dots,m\}$. We define $\sigma \in \Sigma_q$ by $\sigma(h) = i_1$ for all $h \in \mathbf{N}$. In view of (1.1) we have $\sigma \in S_1$. Obviously $\sigma(1) = i_1$.

CASE 2: $a_{i_1} \in t(B_1)$. Then $i_1 \in \{m+1,\dots,n\}$. We define $\sigma \in \Sigma_q$ by

$$\sigma(h) = \begin{cases} i_1 & \text{if } h = 1 \\ q & \text{if } h \geq 2. \end{cases}$$

In virtue of (1.2) and (1.4) we see that $\sigma \in S_1$. Obviously $\sigma(1)=i_1$.

Now let $j_1 \in \{1,\dots,q\}$ be such that $a_{j_1} \in t(C_1) \cup t(D_1)$.

CASE 3: $a_{j_1} \in t(C_1)$. Then $j_1 \in \{n+1,\dots,q\}$. We define $\tau \in \Sigma_q$ by

$$\tau(h) = \begin{cases} j_1 & \text{if } h = 1 \\ 1 & \text{if } h \geq 2. \end{cases}$$

In virtue of (1.1) and (1. 3) we see that $\tau \in S_2$. Obviously $\tau(1)=j_1$.

CASE 4: $a_{j_1} \in t(D_1)$. Then $j_1 \in \{p+1,\dots,q\}$. We define $\tau \in \Sigma_q$ by $\tau(h)=j_1$ for all $h \in \mathbf{N}$. In view of (1.4) we have $\tau \in S_2$. Obviously $\tau(1)=j_1$.

STEP II. Assume that property P_k is true for some $k \in \mathbf{N}$. We will show that P_{k+1} is also valid.

Let $i_1,\dots,i_{k+1} \in \{1,\dots,q\}$ be such that $a_{i_1} \dots a_{i_{k+1}} \in t(A_{k+1}) \cup t(B_{k+1})$. In view of (1.1) and (1.2) we have $i_1 \in \{1,\dots,n\}$.

CASE 1: $i_1 \in \{1,\dots,m\}$. From (1.1) and (1.2) it follows that

$$a_{i_2} \dots a_{i_{k+1}} \in t(A_k) \cup t(B_k).$$

Because P_k is valid, there exists a $\sigma' \in S_1$ such that

$$(i_2,\dots,i_{k+1}) = (\sigma'(1),\dots,\sigma'(k)).$$

Then we define $\sigma \in \Sigma_q$ by

$$\sigma(h) = \begin{cases} i_1 & \text{if } h = 1 \\ \sigma'(h-1) & \text{if } h \geq 2. \end{cases}$$

We claim that $\sigma \in S_1$. To prove this we note that $a_{\sigma(1)} \in t(A_1)$. Next we consider $h \geq 2$. Since $\sigma' \in S_1$, we have $a_{\sigma'(1)} \dots a_{\sigma'(h-1)} \in t(A_{h-1}) \cup t(B_{h-1})$. In view of (1.1) and (1.2) we obtain

$$a_{i_1} a_{\sigma'(1)} \dots a_{\sigma'(h-1)} \in t(A_h) \cup t(B_h);$$

therefore $a_{\sigma(1)} \dots a_{\sigma(h)} \in t(A_h) \cup t(B_h)$. Consequently $\sigma \in S_1$. Obviously

$$(i_1,\dots,i_{k+1}) = (\sigma(1),\dots,\sigma(k+1)).$$

CASE 2: $i_1 \in \{m+1,\ldots,n\}$. From (1.1) and (1.2) it follows that

$$a_{i_2}\ldots a_{i_{k+1}} \in t(C_k)\cup t(D_k).$$

Because P_k is valid, there exists a $\tau' \in S_2$ such that

$$(i_2,\ldots,i_{k+1}) = (\tau'(1),\ldots,\tau'(k)).$$

Then we define $\sigma \in \Sigma_q$ by

$$\sigma(h) = \begin{cases} i_1 & \text{if } h = 1 \\ \tau'(h-1) & \text{if } h \geq 2. \end{cases}$$

In order to prove that $\sigma \in S_1$, first we note that $a_{\sigma(1)} \in t(B_1)$. Next we consider $h \geq 2$. Since $\tau' \in S_2$, we have $a_{\tau'(1)}\ldots a_{\tau'(h-1)} \in t(C_{h-1})\cup t(D_{h-1})$. In view of (1.1) and (1.2) we obtain

$$a_{i_1}a_{\tau'(1)}\ldots a_{\tau'(h-1)} \in t(A_h)\cup t(B_h);$$

therefore $a_{\sigma(1)}\ldots a_{\sigma(h)} \in t(A_h)\cup t(B_h)$. Consequently $\sigma \in S_1$. Obviously

$$(i_1,\ldots,i_{k+1}) = (\sigma(1),\ldots,\sigma(k+1)).$$

Now let $j_1,\ldots,j_{k+1} \in \{1,\ldots,q\}$ be such that $a_{j_1}\ldots a_{j_{k+1}} \in t(C_{k+1})\cup$ $\cup\, t(D_{k+1})$. Then the relations (1.3) and (1.4) imply that $j_1 \in \{n+1,\ldots,q\}$. We distinguish two cases: firstly, when $j_1 \in \{n+1,\ldots,p\}$; secondly, when $j_1 \in \{p+1,\ldots,q\}$. Similar to the cases discussed above the existence of a $\tau \in S_2$ for which $(j_1,\ldots,j_{k+1})=(\tau(1),\ldots,\tau(k+1))$ can be proved.

From the steps I and II we conclude that property P_k holds for every $k \in \mathbf{N}$. By taking $k=\nu$, it follows that the assertions 1° and 2° are true. ∎

Now we endow the set Σ_q with the metric $d_c : \Sigma_q \times \Sigma_q \to \mathbf{R}_+$ given by the formula

$$d_c(\sigma,\tau) = \sum_{k=1}^{\infty} \frac{|\sigma(k)-\tau(k)|}{(q+1)^k}.$$

This metric is called the *series-metric* of Σ_q. It is easy to prove the following result.

PROPOSITION 1.2. *(Σ_q, d_c) is a compact metric space.*

In the sequel we assume that Σ_q is endowed with the series-metric d_c.

PROPOSITION 1.3. *The subsets S_1 and S_2 of Σ_q are nonempty and closed.*

PROOF. From Proposition 1.1 it follows that the sets S_1 and S_2 are not empty. Next we prove that S_1 is closed.

Let $(\sigma_\nu)_{\nu \in \mathbf{N}}$ be a sequence in S_1 that converges to an element $\sigma \in \Sigma_q$. Taking into account that for all $k \in \mathbf{N}$ the inequality

$$|\sigma_\nu(k) - \sigma(k)| \leq (q+1)^k d_c(\sigma_\nu, \sigma)$$

holds whenever $\nu \in \mathbf{N}$, it follows that

$$\lim_{\nu \to \infty} \sigma_\nu(k) = \sigma(k) \quad \text{for every } k \in \mathbf{N}. \tag{1.5}$$

Now fix any $h \in \mathbf{N}$. We prove that

$$a_{\sigma(1)} \ldots a_{\sigma(h)} \in t(A_h) \cup t(B_h). \tag{1.6}$$

Since $\sigma(\mathbf{N}) \subseteq \mathbf{N}$ and $\sigma_\nu(\mathbf{N}) \subseteq \mathbf{N}$ $(\nu \in \mathbf{N})$, it follows from (1.5) that for every $k \in \{1,\ldots,h\}$ there exists a $\nu_k \in \mathbf{N}$ such that $\sigma_\nu(k) = \sigma(k)$ for $\nu \geq \nu_k$. Put $\nu_0 = \max\{\nu_k \mid k = 1,\ldots,h\}$. Then we have $\sigma_{\nu_0}(k) = \sigma(k)$ for all $k \in \{1,\ldots,h\}$. Therefore

$$a_{\sigma(1)} \ldots a_{\sigma(h)} = a_{\sigma_{\nu_0}(1)} \ldots a_{\sigma_{\nu_0}}(h).$$

Because $\sigma_{\nu_0} \in S_1$, it follows that (1.6) is valid. Since h was arbitrarily fixed in $\mathbf{N}$, we have $\sigma \in S_1$. Consequently S_1 is closed.

Similarly it can be shown that S_2 is closed. ∎

2. A fixed point theorem for vectorial multivalued set functions

Let (X,d) be a complete metric space, and let $\mathcal{K}(X)$ be the family of all compact nonempty subsets of X. If M is a subset of X and ε is a positive real number, then we define $M(\varepsilon)$ by

$$M(\varepsilon) = \{x \in X \mid \exists y \in M : d(x,y) \leq \varepsilon\}.$$

If $f : X \to X$ is a Lipschitz function, then

$$\mathrm{Lip} f = \sup\left\{ \frac{d(f(x), f(y))}{d(x,y)} \,\middle|\, x,\ y \in X,\ x \neq y \right\}$$

denotes the smallest Lipschitz constant of f.

Let m, n, p, q be integers such that $0 < m < n < p < q$. Further let $f_i : X \to X$ $(i = 1, \ldots, q)$ be Lipschitz functions. We define the function

$$F : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X) \times \mathcal{K}(X)$$

by

$$F(K_1, K_2) = \begin{pmatrix} f_1 \cup \ldots \cup f_m & f_{m+1} \cup \ldots \cup f_n \\ f_{n+1} \cup \ldots \cup f_p & f_{p+1} \cup \ldots \cup f_q \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

i.e.

$F(K_1, K_2) =$

$$= \left(\left(\bigcup_{i=1}^{m} f_i(K_1) \right) \bigcup \left(\bigcup_{i=m+1}^{n} f_i(K_2) \right), \left(\bigcup_{i=n+1}^{p} f_i(K_1) \right) \bigcup \left(\bigcup_{i=p+1}^{q} f_i(K_2) \right) \right).$$

In our first fixed point theorem concerning F, which we shall establish, we shall use the so-called property *(P)* of F. In order to introduce this property we denote

$$g_\nu(\sigma, x) = f_{\sigma(1)} \circ \ldots \circ f_{\sigma(\nu)}(x) \text{ for all } \nu \in \mathbf{N} \text{ and } (\sigma, x) \in \Sigma_q \times X.$$

DEFINITION 2.1. Let S be a nonempty subset of Σ_q. We say that F *has property (P) with respect to* S if the following conditions are satisfied:

(i) there exists a function $\Gamma_S : S \to X$ such that for every $\sigma \in S$ and every $x \in X$ the sequence $(g_\nu(\sigma, x))_{\nu \in \mathbf{N}}$ converges to $\Gamma_S(\sigma)$;

(ii) for every $K \in \mathcal{K}(X)$ the convergence in

$$\Gamma_S(\sigma) \lim_{\nu \to \infty} g_\nu(\sigma, x)$$

is uniform on $S \times K$, i.e. for every $\varepsilon > 0$ there exists a $\nu_0 \in \mathbf{N}$ such that

$$d\,(\Gamma_S(\sigma), g_\nu(\sigma, x)) \leq \varepsilon \text{ whenever } \nu \geq \nu_0 \text{ and } (\sigma, x) \in S \times K.$$

The next theorem reveals a condition which assures that F has property *(P)* with respect to a nonempty subset of Σ_q.

THEOREM 2.1. *Let S be a nonempty subset of Σ_q, and let* $s^* = \overline{\lim_{\nu \to \infty}} (s_\nu)^{\frac{1}{\nu}} < 1$, *where*

$$s_\nu = \max\{\mathrm{Lip}\,(f_{\sigma(1)} \circ \ldots \circ f_{\sigma(\nu)}) \mid \sigma \in S\}.$$

Then F has property (P) with respect to S.

PROOF. Let $\alpha \in \mathbf{R}$ satisfy $s^* < \alpha < 1$. Then there exists a $\nu_0 \in \mathbf{N}$ such that

$$s_\nu < \alpha^\nu \quad \text{for all} \quad \nu \geq \nu_0. \tag{2.1}$$

Let $(\sigma,x)\in S\times X$ be arbitrarily chosen. We show that $(g_\nu(\sigma,x))_{\nu\in\mathbf{N}}$ is a Cauchy sequence. For this end we fix a number $j\in\mathbf{N}$. From the definition of s_j it follows that for all $y\in X$ the following inequality is valid:

$$d(f_{\sigma(1)}\circ\ldots\circ f_{\sigma(j)}(x), f_{\sigma(1)}\circ\ldots\circ f_{\sigma(j)}(y))\le s_j\, d(x,y).$$

If we take $y=f_{\sigma(j+1)}(x)$, then this inequality yields

$$d(g_j(\sigma,x), g_{j+1}(\sigma,x))\le s_j\, d(x,f_{\sigma(j+1)}(x))\le s_j\, c_x,$$

where

$$c_x=\max\{d(x,f_k(x))\mid k=1,\ldots,q\}.$$

In conclusion

$$d(g_j(\sigma,x), g_{j+1}(\sigma,x))\le s_j\, c_x \quad \text{for all} \quad j\in\mathbf{N}. \tag{2.2}$$

Now let $\varepsilon>0$ be arbitrary. From $\alpha<1$ and (2.1) it results that the series $\sum_{\nu=1}^{\infty} s_\nu$ is convergent. Therefore there exists a $\nu(\varepsilon,x)\in\mathbf{N}$ such that

$$\sum_{k=0}^{p-1} s_{\nu+k}<\frac{\varepsilon}{c_x+1} \quad \text{for every } \nu\ge\nu(\varepsilon,x) \text{ and every } p\in\mathbf{N}. \tag{2.3}$$

Fix any $\nu\ge\nu(\varepsilon,x)$ and $p\in\mathbf{N}$. By using the triangle inequality, we obtain

$$d(g_\nu(\sigma,x), g_{\nu+p}(\sigma,x))\le$$
$$\le d(g_\nu(\sigma,x), g_{\nu+1}(\sigma,x))+\ldots+d(g_{\nu+p-1}(\sigma,x), g_{\nu+p}(\sigma,x)).$$

In view of (2.2) this inequality implies that

$$d(g_\nu(\sigma,x), g_{\nu+p}(\sigma,x))\le c_x(s_\nu+\ldots+s_{\nu+p-1}).$$

Using (2.3), we finally obtain

$$d(g_\nu(\sigma,x), g_{\nu+p}(\sigma,x))<\varepsilon \text{ for every } \nu\ge\nu(\varepsilon,x) \text{ and every } p\in\mathbf{N},$$

i.e. $(g_\nu(\sigma,x))_{\nu\in\mathbf{N}}$ is a Cauchy sequence. Hence it is convergent, because X is complete. The limit of this sequence is independent of x. To see this select any $y\in X$. From the definition of s_ν it follows that

$$d(g_\nu(\sigma,x), g_\nu(\sigma,y))\le s_\nu\, d(x,y) \quad \text{for all } \nu\in\mathbf{N}.$$

Letting $\nu\to\infty$, we get

$$\lim_{\nu\to\infty} d(g_\nu(\sigma,x), g_\nu(\sigma,y))=0,$$

because $\lim_{\nu\to\infty} s_\nu = 0$ (this results from (2.1) and the fact that $\alpha < 1$). But, both of the sequences $(g_\nu(\sigma,x))_{\nu\in\mathbf{N}}$ and $(g_\nu(\sigma,y))_{\nu\in\mathbf{N}}$ are convergent. Consequently their limits must be equal. For $x \in X$ fixed we define $\Gamma_S : S \to X$ by

$$\Gamma_S(\sigma) = \lim_{\nu\to\infty} g_\nu(\sigma,x).$$

From above we conclude that the function Γ_S satisfies condition (i) from Definition 2.1. To see that condition (ii) of this definition is also valid, we select any $K \in \mathcal{K}(X)$ and any $\varepsilon > 0$. Since the function

$$x \in X \mapsto \max\{d(x, f_k(x)) \mid k = 1, \ldots, q\} = c_x \in \mathbf{R}$$

is continuous, there exists a number $c_K \in \mathbf{R}$ such that $c_x \leq c_K$ for all $x \in K$. From the convergence of the series $\sum_{\nu=1}^{\infty} s_\nu$ we conclude that there is a $\nu_0 \in \mathbf{N}$ such that

$$\sum_{k=0}^{p-1} s_{\nu+k} < \frac{\varepsilon}{c_K + 1} \quad \text{for all } \nu \geq \nu_0 \text{ and } p \in \mathbf{N}. \tag{2.4}$$

Let $\nu \geq \nu_0$ and $(\sigma,x) \in S \times K$ be arbitrary. Using similar arguments as before, it follows from (2.2) and (2.4) that

$$d(g_\nu(\sigma,x), g_{\nu+p}(\sigma,x)) < \varepsilon \quad \text{for all } p \in \mathbf{N}.$$

Letting $p \to \infty$, we obtain $d(g_\nu(\sigma,x), \Gamma_S(\sigma)) \leq \varepsilon$. Hence condition (ii) from Definition 2.1 is fulfilled. Therefore F has property *(P)* with respect to S. ∎

THEOREM 2.2. *If the function F has property (P) with respect to a closed nonempty subset S of Σ_q, then $\Gamma_S(S) \in \mathcal{K}(X)$.*

PROOF. Since $S \neq \emptyset$, we have $\Gamma_S(S) \neq \emptyset$. We claim that Γ_S is continuous on S. Indeed, fix $\sigma \in S$ and $\varepsilon > 0$ arbitrarily. Fix also $x \in X$. Since F has property *(P)* with respect to S, and $\{x\} \in \mathcal{K}(X)$, there is a $\nu_0 \in \mathbf{N}$ such that for all $\nu \geq \nu_0$ and all $\tau \in S$ the following inequality holds:

$$d(\Gamma_S(\tau), g_\nu(\tau,x)) < \frac{\varepsilon}{2}. \tag{2.5}$$

Set

$$\varrho = \frac{1}{(q+1)^{\nu_0}}.$$

After that choose any $\tau \in S$ such that $d_c(\sigma,\tau) < \varrho$. Then we have $\sigma(k) = \tau(k)$ for every $k \in \{1,\dots,\nu_0\}$. Indeed, if we assume that $\sigma(j) \neq \tau(j)$ for some $j \in \{1,\dots,\nu_0\}$, then we have $|\sigma(j) - \tau(j)| \geq 1$. Consequently it follows that

$$d_c(\sigma,\tau) \geq \frac{|\sigma(j) - \tau(j)|}{(q+1)^j} \geq \frac{1}{(q+1)^j} \geq \varrho$$

which contradicts the choice of τ. Thus we have $\sigma(k) = \tau(k)$ for every $k \in$ $\in \{1,\dots,\nu_0\}$ as claimed. Therefore

$$g_{\nu_0}(\sigma,x) = g_{\nu_0}(\tau,x). \tag{2.6}$$

Using (2.5) and (2.6), we obtain

$$d(\Gamma_S(\sigma),\Gamma_S(\tau)) \leq d(\Gamma_S(\sigma), g_{\nu_0}(\sigma,x)) + d(g_{\nu_0}(\sigma,x), g_{\nu_0}(\tau,x)) +$$
$$+ d(g_{\nu_0}(\tau,x),\Gamma_S(\tau)) < \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.$$

In other words, for all $\tau \in S$ satisfying $d_c(\sigma,\tau) < \varrho$, we have $d(\Gamma_S(\sigma),\Gamma_S(\tau)) <$ $< \varepsilon$, i.e. Γ_S is continuous at σ. But $\sigma \in S$ was arbitrarily chosen. Thus Γ_S is continuous on S. From this and the compactness of S (as a closed subset of a compact space) we obtain the compactness of $\Gamma_S(S)$. ∎

In what follows we consider the set $\mathcal{K}(X)$ endowed with the Hausdorff metric which we denote by h. The product space $\mathcal{K}(X) \times \mathcal{K}(X)$ is endowed with the product metric, which we denote by h'. It is known that $(\mathcal{K}(X), h)$ is a complete metric space, and hence $(\mathcal{K}(X) \times \mathcal{K}(X), h')$ is also a complete metric space.

When the functions $f_1,\dots,f_q$ are all contractions, one can easily verify that F is also a contraction. Applying Banach's fixed point theorem, one obtains that F has a unique fixed point and that for all $(K_1, K_2) \in \mathcal{K}(X) \times \mathcal{K}(X)$ the sequence $(F^\nu(K_1, K_2))_{\nu \in \mathbf{N}}$ converges to this point. The next theorem will show that this conclusion is also valid if $f_1,\dots,f_q$ are not necessarily contractions. Before stating this theorem we establish a formula which gives the expression of the iteratives of F. These iteratives will occur in the proof of the next theorem.

We endow the set $\mathcal{F} = \{f : \mathcal{K}(X) \to \mathcal{K}(X)\}$ with two composition laws $\cup$ and $\circ$, where $\cup$ is defined by

$$(f \cup g)(K) = f(K) \cup g(K) \quad \text{for} \quad f,\ g \in \mathcal{F} \text{ and } K \in \mathcal{K}(X),$$

and $\circ$ is the ordinary composition of functions. It is clear that $(\mathcal{F}, \cup, \circ)$ is a prering. Since $f_i \in \mathcal{F}$ for all $i \in \{1,\dots,q\}$, one obtains that

$$T = \begin{pmatrix} f_1 \cup \dots \cup f_m & f_{m+1} \cup \dots \cup f_n \\ f_{n+1} \cup \dots \cup f_p & f_{p+1} \cup \dots \cup f_q \end{pmatrix} \in M_2(\mathcal{F}).$$

With this notation, we can write

$$F(K_1, K_2) = T \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

Using mathematical induction, it can be shown that the iteratives of F are of the form

$$F^{\nu}(K_1, K_2) = T^{\nu} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}. \tag{2.7}$$

Accomplishing multiplication in $M_2(\mathcal{F})$, we get

$$T^{\nu} = \begin{pmatrix} A_{\nu} & B_{\nu} \\ C_{\nu} & D_{\nu} \end{pmatrix}, \tag{2.8}$$

where A_{ν}, B_{ν}, C_{ν} and D_{ν} have the same meaning as in Section 1, but the prering $(R,+,\cdot)$ is replaced by $(\mathcal{F},\cup,\circ)$ and the elements a_i $(i=1,\ldots,q)$ are replaced by the functions f_i $(i=1,\ldots,q)$. The relations (2.7) and (2.8) imply

$$F^{\nu}(K_1, K_2) = (A_{\nu}(K_1) \cup B_{\nu}(K_2), C_{\nu}(K_1) \cup D_{\nu}(K_2)). \tag{2.9}$$

We have proved in Section 1 that the powers of the matrix T can be written by using two nonempty subsets S_1 and S_2 of Σ_q. With these specifications we are now able to enunciate and to prove our first fixed point theorem concerning F.

THEOREM 2.3. *If the function F has property (P) with respect to the sets S_1 and S_2, then the following assertions hold:*

1° $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)) \in \mathcal{K}(X)\times\mathcal{K}(X)$.

2° *For all $(K_1,K_2)\in\mathcal{K}(X)\times\mathcal{K}(X)$ we have*

$$\lim_{\nu\to\infty} F^{\nu}(K_1, K_2) = (\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2)).$$

3° $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))$ *is the unique fixed point of F.*

PROOF. 1° From Proposition 1.3 it results that S_1 and S_2 are closed nonempty subsets of Σ_q. By using Theorem 2.2 for each of these subsets, we obtain the first assertion of the theorem.

2° Suppose that $(K_1,K_2) \in \mathcal{K}(X)\times\mathcal{K}(X)$. Then $K_1 \cup K_2 \in \mathcal{K}(X)$. Let $\varepsilon > 0$ be arbitrary. Since F has property *(P)* with respect to S_1 and S_2, there exist $\nu_1, \nu_2 \in \mathbf{N}$ such that

$$d(\Gamma_{S_1}(\sigma), g_{\nu}(\sigma,x)) \le \varepsilon \text{ for all } \nu \ge \nu_1 \text{ and all } (\sigma,x) \in S_1 \times (K_1 \cup K_2), \tag{2.10}$$

$$d(\Gamma_{S_2}(\tau), g_{\nu}(\tau,x)) \le \varepsilon \text{ for all } \nu \ge \nu_2 \text{ and all } (\tau,x) \in S_2 \times (K_1 \cup K_2). \tag{2.11}$$

Put $\nu_0 = \max\{\nu_1, \nu_2\}$ and let $\nu \in \mathbf{N}$ be such that $\nu \geq \nu_0$. Then the following inequality is valid

$$h(A_\nu(K_1) \cup B_\nu(K_2), \Gamma_{S_1}(S_1)) \leq \varepsilon. \tag{2.12}$$

Indeed, let $\sigma \in S_1$ be arbitrarily chosen. In view of the definition of S_1 we have

$$f_{\sigma(1)} \circ \ldots \circ f_{\sigma(\nu)} \in t(A_\nu) \cup t(B_\nu).$$

If $f_{\sigma(1)} \circ \ldots \circ f_{\sigma(\nu)} \in t(A_\nu)$, then we choose an $x_1 \in K_1$. Taking into account (2.10) one obtains

$$d(\Gamma_{S_1}(\sigma), g_\nu(\sigma, x_1)) \leq \varepsilon.$$

Since $g_\nu(\sigma, x_1) \in A_\nu(K_1)$, it follows that

$$\Gamma_{S_1}(S_1) \subseteq A_\nu(K_1)(\varepsilon). \tag{2.13}$$

If $f_{\sigma(1)} \circ \ldots \circ f_{\sigma(\nu)} \in t(B_\nu)$, then we choose an $x_2 \in K_2$. By similar arguments as above we can prove that

$$\Gamma_{S_1}(S_1) \subseteq B_\nu(K_2)(\varepsilon). \tag{2.14}$$

From (2.13) and (2.14) it follows that

$$\Gamma_{S_1}(S_1) \subseteq (A_\nu(K_1) \cup B_\nu(K_2))(\varepsilon). \tag{2.15}$$

Next we prove the inclusion

$$A_\nu(K_1) \cup B_\nu(K_2) \subseteq \Gamma_{S_1}(S_1)(\varepsilon). \tag{2.16}$$

Suppose that $i_1, \ldots, i_\nu \in \{1, \ldots, q\}$ and that $f_{i_1} \circ \ldots \circ f_{i_\nu} \in t(A_\nu)$. After that choose any element $x_1 \in K_1$. According to Proposition 1.1 there exists a $\sigma \in S_1$ such that $(i_1, \ldots, i_\nu) = (\sigma(1), \ldots, \sigma(\nu))$. This equality implies

$$f_{i_1} \circ \ldots \circ f_{i_\nu}(x_1) = g_\nu(\sigma, x_1).$$

From (2.10) it results that

$$d(\Gamma_{S_1}(\sigma), f_{i_1} \circ \ldots \circ f_{\nu_i}(x_1)) \leq \varepsilon,$$

and hence

$$A_\nu(K_1) \subseteq \Gamma_{S_1}(S_1)(\varepsilon). \tag{2.17}$$

By similar arguments it can be shown that

$$B_\nu(K_2) \subseteq \Gamma_{S_1}(S_1)(\varepsilon). \tag{2.18}$$

From (2.17) and (2.18) we obtain (2.16). Finally, (2.15) and (2.16) imply (2.12). Analogously it follows that

$$h(C_\nu(K_1)\cup D_\nu(K_2),\Gamma_{S_2}(S_2))\le\varepsilon. \tag{2.19}$$

On the other hand, (2.9) implies

$$h'(F^\nu(K_1,K_2),(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)))=$$
$$=\max\{h(A_\nu(K_1)\cup B_\nu(K_2),\Gamma_{S_1}(S_1)),h(C_\nu(K_1)\cup D_\nu(K_2),\Gamma_{S_2}(S_2))\}.$$

In virtue of (2.12) and (2.19), this equality yields

$$h'(F^\nu(K_1,K_2),(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)))\le\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this result means that

$$\lim_{\nu\to\infty}F^\nu(K_1,K_2)=(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)).$$

3° The definition of the iteratives of F implies

$$F(F^\nu(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)))=F^{\nu+1}(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)) \text{ for every } \nu\in\mathbf{N}. \tag{2.20}$$

But F is a Lipschitz function, because $f_1,\dots,f_q$ are Lipschitz functions. Hence the function F is continuous. In virtue of the assertions 1° and 2°, we conclude from (2.20) that

$$F(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))=(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2));$$

i.e. $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))$ is a fixed point of F.

To see that this fixed point of F is unique we assume that $(K_1,K_2)\in\in\mathcal{K}(X)\times\mathcal{K}(X)$ is another fixed point of F. From $F(K_1,K_2)=(K_1,K_2)$ it follows that

$$F^\nu(K_1,K_2)=(K_1,K_2)\quad\text{for all } \nu\in\mathbf{N}.$$

Using assertion 2°, it results that $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))=(K_1,K_2)$. Hence the pair $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))$ is the unique fixed point of F. ∎

By using Theorem 2.1 we can deduce from Theorem 2.3 the following second fixed point theorem concerning F. The assumptions of this theorem can be easier verified in practice than those of Theorem 2.3.

THEOREM 2.4. *Suppose that* $\overline{\lim_{\nu\to\infty}}(u_\nu)^{\frac{1}{\nu}}<1$ *and* $\overline{\lim_{\nu\to\infty}}(w_\nu)^{\frac{1}{\nu}}<1$, *where*

$$u_\nu=\max\{\operatorname{Lip}(f_{\sigma(1)}\circ\ldots\circ f_{\sigma(\nu)})\mid\sigma\in S_1\}$$

and

$$w_\nu = \max\{\mathrm{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(\nu)}) \mid \tau \in S_2\}.$$

Then the following assertions hold:

1° *The function F has property (P) with respect to the sets S_1 and S_2.*

2° $(\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2)) \in \mathcal{K}(X) \times \mathcal{K}(X)$.

3° *For all $(K_1, K_2) \in \mathcal{K}(X) \times \mathcal{K}(X)$ we have*

$$\lim_{\nu \to \infty} F^\nu(K_1, K_2) = (\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2)).$$

4° $(\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2))$ *is the unique fixed point of F.*

PROOF. 1° From Theorem 2.1 it results that F has property *(P)* with respect to the sets S_1 and S_2. By applying Theorem 2.3, we obtain the assertions 2°, 3° and 4° of the theorem. ∎

3. Specializations of Theorem 2.4

In this section we show that F satisfies property *(P)* if the functions $f_1,\ldots,f_q$ are contractions. After that by means of an example we point out that property *(P)* may be valid even if not all the functions $f_1,\ldots,f_q$ are contractions.

PROPOSITION 3.1. *Let $f, g : X \to X$ be Lipschitz functions. Then*

$$\mathrm{Lip}\,(f \circ g) \leq (\mathrm{Lip} f)(\mathrm{Lip}\, g).$$

PROOF. Let $x, y \in X$ be such that $x \neq y$. Then the following inequality holds

$$\frac{d((f \circ g)(x), (f \circ g)(y))}{d(x, y)} \leq (\mathrm{Lip} f)(\mathrm{Lip}\, g). \tag{3.1}$$

Indeed, if $g(x) = g(y)$, then (3.1) is obvious. If $g(x) \neq g(y)$, then we have

$$\frac{d((f \circ g)(x), (f \circ g)(y))}{d(x, y)} = \frac{d((f \circ g)(x), (f \circ g)(y))}{d(g(x), g(y))} \cdot \frac{d(g(x), g(y))}{d(x, y)}.$$

Obviously, this equality implies (3.1). ∎

COROLLARY 3.2. *If $f_1,\ldots,f_q$ are contractions, then the following assertions hold:*

1° *The function F has property (P) with respect to the sets S_1 and S_2.*

2° $(\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2)) \in \mathcal{K}(X) \times \mathcal{K}(X)$.

3° *For all* $(K_1,K_2)\in\mathcal{K}(X)\times\mathcal{K}(X)$ *we have*

$$\lim_{\nu\to\infty} F^{\nu}(K_1,K_2) = (\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)).$$

4° $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))$ *is the unique fixed point of* F.

PROOF. We have $\mathrm{Lip} f_i < 1$ for every $i \in \{1,\dots,q\}$. Consequently, the number

$$c = \max\{\mathrm{Lip} f_i \mid i \in \{1,\dots,q\}\}$$

satisfies $c<1$. By applying Proposition 3.1, we conclude that for all $\nu\in\mathbf{N}$ the following inequalities are true:

$$u_\nu = \max\{\mathrm{Lip}\,(f_{\sigma(1)}\circ\dots\circ f_{\sigma(\nu)}) \mid \sigma \in S_1\} \le c^{\nu},$$
$$w_\nu = \max\{\mathrm{Lip}\,(f_{\tau(1)}\circ\dots\circ f_{\tau(\nu)}) \mid \tau \in S_2\} \le c^{\nu}.$$

They imply

$$\overline{\lim_{\nu\to\infty}}\,(u_\nu)^{\frac{1}{\nu}} \le c \quad\text{and}\quad \overline{\lim_{\nu\to\infty}}\,(w_\nu)^{\frac{1}{\nu}} \le c.$$

Since $c<1$, we can apply Theorem 2.4. Consequently the four assertions of the corollary are true. ∎

Let $f_1,\dots,f_4 : X\to X$ be Lipschitz functions and define $F:\mathcal{K}(X)\times\mathcal{K}(X)$ by

$$F(K_1,K_2) = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

Keeping the notations from Section 2 for the expression of the powers of the matrix

$$\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix},$$

we can establish the following result.

THEOREM 3.3. *Let* s *and* t *be positive real numbers such that* $s<t$, $st<1$, $\mathrm{Lip} f_i \le s$ *for* $i\in\{1,2,4\}$ *and* $\mathrm{Lip} f_3\le t$. *Then the following assertions hold:*

1° *The function* F *has property (P) with respect to the sets* S_1 *and* S_2.

2° $(\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2))\in\mathcal{K}(X)\times\mathcal{K}(X)$.

3° *For all* $(K_1,K_2)\in\mathcal{K}(X)\times\mathcal{K}(X)$ *we have*

$$\lim_{\nu\to\infty} F^{\nu}(K_1,K_2) = (\Gamma_{S_1}(S_1),\Gamma_{S_2}(S_2)).$$

4° $(\Gamma_{S_1}(S_1), \Gamma_{S_2}(S_2))$ *is the unique fixed point of* F.

PROOF. We show that the assumptions of Theorem 2.4 are satisfied. For this end we prove by mathematical induction that the proposition P_ν is true for all $\nu \in \mathbf{N}$, where

$$P_\nu = \begin{cases} u_{2\nu-1} \leq s^\nu t^{\nu-1} \\ u_{2\nu} \leq s^\nu t^\nu \end{cases} \quad \text{and} \quad \begin{cases} w_{2\nu-1} \leq s^{\nu-1} t^\nu \\ w_{2\nu} \leq s^\nu t^\nu \end{cases}.$$

STEP I. Let $\nu = 1$. Then we have

$$u_1 = \max\{\mathrm{Lip} f_{\sigma(1)} \mid \sigma \in S_1\} = \max\{\mathrm{Lip} f_1, \mathrm{Lip} f_2\} \leq s.$$

Since

$$u_2 = \max\{\mathrm{Lip}\,(f_{\sigma(1)} \circ f_{\sigma(2)}) \mid \sigma \in S_2\} =$$
$$= \max\{\mathrm{Lip}\,(f_1 \circ f_1), \mathrm{Lip}\,(f_2 \circ f_3), \mathrm{Lip}\,(f_1 \circ f_2), \mathrm{Lip}\,(f_2 \circ f_4)\},$$

it follows from Proposition 3.1 that $u_2 \leq st$.

We have

$$w_1 = \max\{\mathrm{Lip} f_{\tau(1)} \mid \tau \in S_2\} = \max\{\mathrm{Lip} f_3, \mathrm{Lip} f_4\} \leq t.$$

Since

$$w_2 = \max\{\mathrm{Lip}\,(f_{\tau(1)} \circ f_{\tau(2)}) \mid \tau \in S_2\} =$$
$$= \max\{\mathrm{Lip}\,(f_3 \circ f_1), \mathrm{Lip}\,(f_4 \circ f_3), \mathrm{Lip}\,(f_3 \circ f_2), \mathrm{Lip}\,(f_4 \circ f_4)\},$$

it follows from Proposition 3.1 that $w_2 \leq st$. Consequently P_1 is true.

STEP II. Let $\nu \in \mathbf{N}$ be such that P_ν is true. We show that $P_{\nu+1}$ is also true. For this end we prove first the following inequality

$$u_{2\nu+1} \leq s^{\nu+1} t^\nu. \tag{3.2}$$

We know that $u_{2\nu+1} = \max\{\mathrm{Lip}\,(f_{\sigma(1)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \mid \sigma \in S_1\}$. Select any $\sigma \in S_1$. Then Proposition 3.1 implies

$$\mathrm{Lip}\,(f_{\sigma(1)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \leq \mathrm{Lip} f_{\sigma(1)} \mathrm{Lip}\,(f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)}). \tag{3.3}$$

In view of (1.1), (1.2) and the definition of S_1 it results that $\sigma(1) \in \{1, 2\}$.

CASE 1: $\sigma(1) = 1$. From (1.1) and (1.2) one obtains that

$$f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)} \in t(A_{2\nu}) \cup t(B_{2\nu}).$$

Accordingly assertion 1° of Proposition 1.1 implies the existence of a $\sigma' \in S_1$ such that $(\sigma(2),\ldots,\sigma(2\nu+1)) = (\sigma'(1),\ldots,\sigma'(2\nu))$. Taking into account the definition of $u_{2\nu}$, it follows that

$$\mathrm{Lip}\,(f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)}) = \mathrm{Lip}\,(f_{\sigma'(1)} \circ \ldots \circ f_{\sigma'(2\nu)}) \leq u_{2\nu}.$$

By the hypothesis of induction we then obtain

$$\mathrm{Lip}\,(f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \leq s^\nu t^\nu. \tag{3.4}$$

Together (3.3) and (3.4) yield

$$\mathrm{Lip}\,(f_{\sigma(1)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \leq s^{\nu+1} t^\nu. \tag{3.5}$$

CASE 2: $\sigma(1)=2$. From (1.1) and (1.2) one obtains that

$$f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)} \in t(C_{2\nu}) \cup t(D_{2\nu}).$$

Accordingly assertion 2° of Proposition 1.1 implies the existence of a $\tau' \in S_2$ such that $(\sigma(2),\ldots,\sigma(2\nu+1)) = (\tau'(1),\ldots,\tau'(2\nu))$. Taking into account the definition of $w_{2\nu}$, it results that

$$\mathrm{Lip}\,(f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)}) = \mathrm{Lip}\,(f_{\tau'(1)} \circ \ldots \circ f_{\tau'(2\nu)}) \leq w_{2\nu}.$$

By the hypothesis of induction we then obtain

$$\mathrm{Lip}\,(f_{\sigma(2)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \leq s^\nu t^\nu. \tag{3.6}$$

Together (3.3) and (3.6) yield

$$\mathrm{Lip}\,(f_{\sigma(1)} \circ \ldots \circ f_{\sigma(2\nu+1)}) \leq s^{\nu+1} t^\nu. \tag{3.7}$$

From (3.5) and (3.7) it follows that (3.2) is true.

Next we prove that

$$w_{2\nu+1} \leq s^\nu t^{\nu+1}. \tag{3.8}$$

We know that $w_{2\nu+1} = \max\{\mathrm{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(2\nu+1)}) \mid \tau \in S_2\}$. Select any $\tau \in S_2$. Then Proposition 3.1 implies

$$\mathrm{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(2\nu+1)}) \leq \mathrm{Lip} f_{\tau(1)} \mathrm{Lip}\,(f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)}). \tag{3.9}$$

In view of (1.3), (1.4) and the definition of S_2 it results that $\tau(1) \in \{3,4\}$.

CASE 3: $\tau(1)=3$. From (1.3) and (1.4) one obtains that

$$f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)} \in t(A_{2\nu}) \cup t(B_{2\nu}).$$

Accordingly assertion 1° of Proposition 1.1 implies the existence of a $\sigma' \in S_1$ such that $(\tau(2),\ldots,\tau(2\nu+1)) = (\sigma'(1),\ldots,\sigma'(2\nu))$. Taking into account the definition of $u_{2\nu}$, it follows that

$$\mathrm{Lip}\,(f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)}) \leq u_{2\nu}.$$

By the hypothesis of induction we then obtain

$$\text{Lip}\,(f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le s^{\nu} t^{\nu}. \tag{3.10}$$

Together (3.9) and (3.10) yield

$$\text{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le s^{\nu} t^{\nu+1}. \tag{3.11}$$

CASE 4: $\tau(1)=4$. From (1.3) and (1.4) one obtains that

$$f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)} \in t(C_{2\nu}) \cup t(D_{2\nu}).$$

Accordingly assertion 2° of Proposition 1.1 implies the existence of a $\tau' \in S_2$ such that $(\tau(2),\ldots,\tau(2\nu+1)) = (\tau'(1),\ldots,\tau'(2\nu))$. Taking into account the definition of $w_{2\nu}$, it follows that

$$\text{Lip}\,(f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le w_{2\nu}.$$

By the hypothesis of induction we then obtain

$$\text{Lip}\,(f_{\tau(2)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le s^{\nu} t^{\nu}. \tag{3.12}$$

Together (3.9) and (3.12) yield

$$\text{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le s^{\nu+1} t^{\nu}.$$

Since $s < t$, this inequality implies

$$\text{Lip}\,(f_{\tau(1)} \circ \ldots \circ f_{\tau(2\nu+1)}) \le s^{\nu} t^{\nu+1}. \tag{3.13}$$

From (3.11) and (3.13) it follows that (3.8) is true.

Using similar arguments as in the proof of the inequality (3.2), it can be shown that

$$u_{2\nu+2} \le s^{\nu+1} t^{\nu+1}, \tag{3.14}$$

but, instead of the hypothesis of induction, the inequalities (3.2) and (3.8) proved above have to be used.

Using similar arguments as in the proof of the inequality (3.8), it can be shown that

$$w_{2\nu+2} \le s^{\nu+1} t^{\nu+1}, \tag{3.15}$$

but this time the inequalities (3.2) and (3.8) are used instead of the hypothesis of induction.

The relations (3.2), (3.8), (3.14) and (3.15) express that $P_{\nu+1}$ is true.

From the steps I and II we conclude that P_ν is true for every $\nu \in \mathbf{N}$. Therefore the following inequalities hold:

$$\overline{\lim_{\nu\to\infty}}(u_{2\nu-1})^{\frac{1}{2\nu-1}} \le \lim_{\nu\to\infty}(s^\nu t^{\nu-1})^{\frac{1}{2\nu-1}} = \sqrt{st}, \tag{3.16}$$

$$\overline{\lim_{\nu\to\infty}}(u_{2\nu})^{\frac{1}{2\nu}} \le \lim_{\nu\to\infty}(s^\nu t^{\nu})^{\frac{1}{2\nu}} = \sqrt{st}, \tag{3.17}$$

$$\overline{\lim_{\nu\to\infty}}(w_{2\nu-1})^{\frac{1}{2\nu-1}} \le \lim_{\nu\to\infty}(s^{\nu-1} t^{\nu})^{\frac{1}{2\nu-1}} = \sqrt{st}, \tag{3.18}$$

$$\overline{\lim_{\nu\to\infty}}(w_{2\nu})^{\frac{1}{2\nu}} \le \lim_{\nu\to\infty}(s^\nu t^{\nu})^{\frac{1}{2\nu}} = \sqrt{st}. \tag{3.19}$$

From (3.16) and (3.17) it follows that

$$\overline{\lim_{\nu\to\infty}}(u_\nu)^{\frac{1}{\nu}} \le \sqrt{st},$$

while from (3.18) and (3.19) it follows that

$$\overline{\lim_{\nu\to\infty}}(w_\nu)^{\frac{1}{\nu}} \le \sqrt{st}.$$

But, $\sqrt{st} < 1$. Consequently the assumptions of Theorem 2.4 are fulfilled. By applying this theorem we obtain the four assertions of the theorem. ∎

REMARK. Theorem 3.3 shows that F has a unique fixed point even if f_1, $\dots, f_4$ are not all contractions. For example, if $s = \frac{1}{4}$ and $t = 2$, then the function f_3 must not be necessarily a contraction.

4. Connections with Mauldin–Williams graphs

Up to this point of our paper we have investigated only vectorial functions that were defined by means of square matrices. But fixed point results that are similar to those obtained in Section 2 can be stated also for functions which are defined by using matrices of degree greater than 2. After specifying some notations we confine us to enunciate these results, because their proofs are obtained by analogy to those from the previous section. At the end of our paper we will show that these results have applications in the theory of Mauldin–Williams graphs.

Let $n \in \mathbf{N}$ be fixed, and let

$$\begin{pmatrix} k_{1,1} & \dots & k_{1,n} \\ \vdots & \vdots & \vdots \\ k_{n,1} & \dots & k_{n,n} \end{pmatrix}$$

be a matrix with $k_{i,j} \in \mathbf{N}$ for all $i, j \in \{1, \dots, n\}$ such that

$$k_{i,j} < k_{i,j+1} \text{ for all } i \in \{1, \dots, n\} \text{ and all } j \in \{1, \dots, n-1\},$$
$$k_{i,n} < k_{i+1,1} \text{ for all } i \in \{1, \dots, n-1\}.$$

With the aid of given Lipschitz functions $h_i : X \to X$ $(i = 1, \dots, k_{n,n})$ we define the function $H : (\mathcal{K}(X))^n \to (\mathcal{K}(X))^n$ by

$$H(K_1, \dots, K_n) = \tag{4.1}$$
$$\begin{pmatrix} h_1 \cup \dots \cup h_{k_{1,1}} & \dots & h_{k_{1,n-1}} \cup \dots \cup h_{k_{1,n}} \\ \vdots & \vdots & \vdots \\ h_{k_{n-1,n}} \cup \dots \cup h_{k_{n,1}} & \dots & h_{k_{n,n-1}} \cup \dots \cup h_{k_{n,n}} \end{pmatrix} \begin{pmatrix} K_1 \\ \dots \\ K_n \end{pmatrix},$$

i.e.

$$H(K_1, \dots, K_n) = \left(\left(\bigcup_{i=1}^{k_{1,1}} h_i(K_1) \right) \bigcup \dots \bigcup \left(\bigcup_{i=k_{1,n-1}}^{k_{1,n}} h_i(K_n) \right), \dots \right.$$
$$\left. \dots, \left(\bigcup_{i=k_{n-1,n}}^{k_{n,1}} h_i(K_1) \right) \bigcup \dots \bigcup \left(\bigcup_{i=k_{n,n-1}}^{k_{n,n}} h_i(K_n) \right) \right).$$

Accomplishing multiplication in the set of matrices of degree n with elements from the prering $(\mathcal{F}, \cup, \circ)$, one finds that the powers of the matrix

$$T = \begin{pmatrix} h_1 \cup \dots \cup h_{k_{1,1}} & \dots & h_{k_{1,n-1}} \cup \dots \cup h_{k_{1,n}} \\ \vdots & \vdots & \vdots \\ h_{k_{n-1,n}} \cup \dots \cup h_{k_{n,1}} & \dots & h_{k_{n,n-1}} \cup \dots \cup h_{k_{n,n}} \end{pmatrix}$$

are

$$T^\nu = \begin{pmatrix} A_{1,1}(\nu) & \dots & A_{1,n}(\nu) \\ \vdots & \vdots & \vdots \\ A_{n,1}(\nu) & \dots & A_{n,n}(\nu) \end{pmatrix},$$

where $A_{i,j}(\nu)$ $(i, j \in \{1,\dots,n\})$ are unaccomplished sums whose terms are products of ν factors chosen from $h_1,\dots,h_{k_{n,n}}$.

We introduce the sets

$$\Sigma = \{\sigma : \mathbf{N} \to \{1,\dots,k_{n,n}\}\}$$

and

$$Q_i = \left\{\sigma \in \Sigma \mid \forall \nu \in \mathbf{N}: \ h_{\sigma(1)} \circ \dots \circ h_{\sigma(\nu)} \in \bigcup_{j=1}^{n} t(A_{i,j}(\nu))\right\}, \quad i = 1,\dots,n.$$

From Proposition 1.2 we know that if Σ is endowed with the series-metric, then it becomes a compact metric space. Following the same steps as in the proof of Proposition 1.3, one shows that Q_1, ..., Q_n are closed nonempty subsets of Σ. As in Section 2 we denote

$$g_\nu(\sigma,x) = h_{\sigma(1)} \circ \dots \circ h_{\sigma(\nu)}(x) \quad \text{for all } \nu \in \mathbf{N} \text{ and } (\sigma,x) \in \Sigma \times X.$$

The following definition is analogous with Definition 2.1.

DEFINITION 4.1. Let Q be a nonempty subset of Σ. We say that H *has property (P) with respect to* Q if the following conditions are satisfied:

(i) there exists a function $\Gamma_Q : Q \to X$ such that for every $\sigma \in Q$ and every $x \in X$ the sequence $(g_\nu(\sigma,x))_{\nu\in\mathbf{N}}$ converges to $\Gamma_Q(\sigma)$;

(ii) for every $K \in \mathcal{K}(X)$ the convergence in

$$\Gamma_Q(\sigma) = \lim_{\nu\to\infty} g_\nu(\sigma,x)$$

is uniform on $Q \times K$, i.e. for every $\varepsilon > 0$ there exists a $\nu_0 \in \mathbf{N}$ such that

$$d(\Gamma_Q(\sigma), g_\nu(\sigma,x)) \le \varepsilon \text{ whenever } \nu \ge \nu_0 \text{ and } (\sigma,x) \in Q \times K.$$

The analogue of Theorem 2.3 for the function H has the following formulation.

THEOREM 4.1. *If the function H has property (P) with respect to the sets Q_1, ..., Q_n, then the following assertions hold:*

1° $(\Gamma_{Q_1}(Q_1),\dots,\Gamma_{Q_n}(Q_n)) \in (\mathcal{K}(X))^n$.

2° *For all* $(K_1,\dots,K_n) \in (\mathcal{K}(X))^n$ *we have*

$$\lim_{\nu\to\infty} H^\nu(K_1,\dots,K_n) = (\Gamma_{Q_1}(Q_1),\dots,\Gamma_{Q_n}(Q_n)).$$

3° $(\Gamma_{Q_1}(Q_1),\dots,\Gamma_{Q_n}(Q_n))$ *is the unique fixed point of* H.

In general the assumptions of the next theorem can be easier verified in practice than those of Theorem 4.1.

THEOREM 4.2. *If the inequality*

$$\overline{\lim_{\nu\to\infty}} (q_{i,\nu})^{\frac{1}{\nu}} < 1$$

is satisfied for all $i\in\{1,\dots,n\}$, *where*

$$q_{i,\nu} = \max\{\mathrm{Lip}\,(h_{\sigma(1)}\circ\ldots\circ h_{\sigma(\nu)}) \mid \sigma\in Q_i\},$$

then the following assertions hold:

1° *The function* H *has property (P) with respect to the sets* $Q_1, \ldots, Q_n$.

2° $(\Gamma_{Q_1}(Q_1),\dots,\Gamma_{Q_n}(Q_n))\in(\mathcal{K}(X))^n$.

3° *For all* $(K_1,\dots,K_n)\in(\mathcal{K}(X))^n$ *we have*

$$\lim_{\nu\to\infty} H^{\nu}(K_1,\ldots,K_n) = (\Gamma_{Q_1}(Q_1),\ldots,\Gamma_{Q_n}(Q_n)).$$

4° $(\Gamma_{Q_1}(Q_1),\dots,\Gamma_{Q_n}(Q_n))$ *is the unique fixed point of* H.

REMARKS. 1) Assertion 3° of Theorem 4.2 can be used to approximate the unique fixed point of H.

2) When $n=1$, then the Theorems 4.1 and 4.2 become the results obtained by MÁTÉ [5].

3) As in the proof of Theorem 2.1 it can be shown that the functions $\Gamma_{Q_i}: Q_i\to X$ $(i=1,\dots,n)$ are defined by

$$\Gamma_{Q_i}(\sigma) = \lim_{\nu\to\infty} g_\nu(\sigma,x),$$

where $x\in X$ is fixed.

4) Theorem 3 given by BANDT in [1] is a corollary of Theorem 4.2. To see this put

$$c = \max\{\mathrm{Lip}\,h_i \mid i\in\{1,\ldots,k_{n,n}\}\}.$$

Next we note that the hypothesis of BANDT's theorem implies the existence of an $\alpha\in(0,1)$ such that

$$\mathrm{Lip}\,(h_{k_1}\circ\ldots\circ h_{k_t}) \le \alpha^{\left[\frac{t}{n}\right]} c^n \tag{4.2}$$

for each path $k_1,\dots,k_t$. From (4.2) we deduce that, for all $i\in\{1,\dots,n\}$, $\sigma\in Q_i$ and $\nu\in\mathbf{N}$, the following inequality holds

$$\mathrm{Lip}\,(h_{\sigma(1)}\circ\ldots\circ h_{\sigma(\nu)}) \le \alpha^{\left[\frac{\nu}{n}\right]} c^n. \tag{4.3}$$

Indeed, in view of $\sigma\in Q_i$, there exists a $j\in\{1,\dots,n\}$ such that

$$h_{\sigma(1)}\circ\ldots\circ h_{\sigma(\nu)} \in t(A_{i,j}(\nu)).$$

Taking into account the definition of $A_{i,j}$, we find numbers $j_1,\ldots,j_{\nu-1} \in \{1,\ldots$ $\ldots,n\}$ such that

$$h_{\sigma(1)} \in t(A_{i,j_1}(1)),\ h_{\sigma(2)} \in t(A_{j_1,j_2}(1)),\ \ldots,\ h_{\sigma(\nu)} \in t(A_{j_{\nu-1},j}(1)).$$

This means that $\sigma(1)\ldots\sigma(\nu)$ is a path in the sense of BANDT's definition. By applying (4.2) we get (4.3). From (4.3) it follows that

$$q_{i,\nu} \leq \alpha^{\left[\frac{\nu}{n}\right]} c^n.$$

Thus we have

$$(q_{i,\nu})^{\frac{1}{\nu}} \leq \alpha^{\left[\frac{\nu}{n}\right]\frac{1}{\nu}} c^{\frac{n}{\nu}}. \tag{4.4}$$

Since

$$\lim_{\nu\to\infty} \alpha^{\left[\frac{\nu}{n}\right]\frac{1}{\nu}} = \alpha^{\frac{1}{n}},$$

the inequality (4.4) implies

$$\overline{\lim_{\nu\to\infty}} (q_{i,\nu})^{\frac{1}{\nu}} \leq \alpha^{\frac{1}{n}}.$$

But we have $\alpha < 1$, and hence it is true that

$$\overline{\lim_{\nu\to\infty}} (q_{i,\nu})^{\frac{1}{\nu}} < 1.$$

By applying Theorem 4.2 and the preceding remark we obtain the conclusions of Theorem 3 given in [1].

Theorem 4.2 can be applied in the theory of Mauldin–Williams graphs. First we need some definitions from the theory of graphs.

DEFINITION 4.2. A *directed multigraph* is an ordered system (V,E,α,β), where V and E are nonempty sets, and α and β are functions from E to V.

The elements of V are called the *vertices* of the directed multigraph, while those of E are called the *edges* of the directed multigraph. If $e \in E$, then $\alpha(e)$ is the *initial vertex* of the edge e and $\beta(e)$ is the *final vertex* of the edge e.

If $u,\ w \in V$, then we denote by E_{uw} the set of all edges whose initial vertex is u and whose final vertex is w, i.e.

$$E_{uw} = \{e \in E \mid \alpha(e) = u,\ \beta(e) = w\}.$$

DEFINITION 4.3. A *Mauldin–Williams graph* is an ordered system $(V,E,\alpha,\beta,\gamma)$, where (V,E,α,β) is a directed multigraph and γ is a function from E to $]0,+\infty[$.

The function γ is called the *evaluation function* of the Mauldin–Williams graph.

DEFINITION 4.4. An *invariant list of representatives of a directed multigraph* (V,E,α,β) with respect to a family $(f_e : X \to X)_{e\in E}$ of continuous functions is any element $(K_v)_{v\in V} \in (\mathcal{K}(X))^V$ which satisfies

$$K_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} f_e(K_v) \quad \text{for every } u \in V.$$

DEFINITION 4.5. An *invariant list of representatives of a Mauldin–Williams graph* $(V,E,\alpha,\beta,\gamma)$ with respect to a family $(f_e : X \to X)_{e\in E}$ of continuous functions is an invariant list of representatives of the directed multigraph (V,E,α,β).

In the sequel let (V,E,α,β) be a directed multigraph with $V=\{1,\ldots,n\}$ and for all $i,j \in V$ let $E_{i,j}$ be a finite nonempty set. Without loss of the generality we can suppose that the sets $E_{i,j}$ are written with the aid of the elements of the matrix $(k_{i,j})_{1\le i,j\le n}$ in the following manner:

$$E_{i,j} = \begin{cases} \{1,\ldots,k_{1,1}\} & \text{if } (i,j) \in \{1\}\times\{1\} \\ \{k_{i,j-1},\ldots,k_{i,j}\} & \text{if } (i,j) \in \{1,\ldots,n\}\times\{2,\ldots,n\} \\ \{k_{i-1,n},\ldots,k_{i,1}\} & \text{if } (i,j) \in \{2,\ldots,n\}\times\{1\} \end{cases}.$$

Since

$$E = \bigcup_{i,j=1}^{n} E_{i,j},$$

it follows that $E=\{1,\ldots,k_{n,n}\}$.

The next theorem reveals the relationship between the invariant list of representatives of the directed multigraph (V,E,α,β) and the function H defined by (4.1).

THEOREM 4.3. *Let $(h_e : X \to X)_{e\in E}$ be a family of continuous functions. The element $(K_1,\ldots,K_n) \in (\mathcal{K}(X))^n$ is an invariant list of representatives of the directed multigraph (V,E,α,β) with respect to the family $(h_e : X \to \to X)_{e\in E}$ if and only if $(K_1,\ldots,K_n)$ is a fixed point of the function H defined by* (4.1).

PROOF. The element $(K_1,\ldots,K_n) \in (\mathcal{K}(X))^n$ is an invariant list of representatives of (V,E,α,β) with respect to the family $(h_e : X \to X)_{e\in E}$ if and only if

$$K_i = \bigcup_{j\in\{1,\ldots,n\}} \bigcup_{e\in E_{i,j}} h_e(K_j) \quad \text{for all } i \in \{1,\ldots,n\}. \tag{4.5}$$

Using the expression of the sets $E_{i,j}$, we see that (4.5) is equivalent to

$$(K_1,\dots,K_n) = H(K_1,\dots,K_n). \tag{4.6}$$

The relation (4.6) is obviously equivalent to the fact that $(K_1,\dots,K_n)$ is a fixed point of the function H defined by (4.1). ∎

REMARK. In the case when $n=1$, i.e. when the directed multigraph has a single vertex, then the function H is of the same type as the function G mentioned in the introduction of our paper. Hence the invariant list of representatives with respect to a finite family of continuous functions becomes the invariant set (with respect to the same family) defined in the introduction, while (4.5) becomes

$$K_1 = h_1(K_1) \cup \ldots \cup h_{k_{1,1}}(K_1). \tag{4.7}$$

This equality is the simplest way to obtain a fractal (namely the set K_1). Various constructions of "recurrent sets" given in the theory of fractals generalize this idea of constructing a fractal by means of (4.7). For instance, instead of a single equation (4.7), a system (4.5) of n equations involving $k_{n,n}$ mappings can be considered in order to obtain n components of a fractal. Therefore Theorem 4.3 characterizing the concept of invariant lists of representatives of a directed multigraph is a useful tool for investigating fractals.

Using Theorem 4.2, we are now able to state a sufficient condition for the existence and uniqueness of an invariant list of representatives of a Mauldin–Williams graph $(V,E,\alpha,\beta,\gamma)$ with respect to a family of Lipschitz functions. Moreover, taking into account the preceding remark, the next theorem reveals that, by means of Mauldin–Williams graphs, a relationship between the results of our paper and the theory of fractals can be stated.

THEOREM 4.4. *Let $(h_e : X \to X)_{e\in E}$ be a family of Lipschitz functions satisfying the following conditions:*

(C_1) $\operatorname{Lip} h_e \le \gamma(e)$ *for all* $e \in E$;

(C_2) $\overline{\lim_{\nu\to\infty}} (m_{i,\nu})^{\frac{1}{\nu}} < 1$ *for all* $i \in \{1,\dots,n\}$, *where*

$$m_{i,\nu} = \max\left\{ \prod_{k=1}^{\nu} \gamma(\sigma(k)) \mid \sigma \in Q_i \right\}.$$

Then the Mauldin–Williams graph $(V,E,\alpha,\beta,\gamma)$ has a unique invariant list of representatives with respect to the family $(h_e : X \to X)_{e\in E}$.

PROOF. Let $q_{i,\nu}$ $(i \in \{1,\dots,n\}, \nu \in \mathbf{N})$ be the numbers defined in Theorem 4.2. Using condition (C_1) and Proposition 3.1, it follows that

$$\operatorname{Lip}(h_{\sigma(1)} \circ \ldots \circ h_{\sigma(\nu)}) \le \prod_{k=1}^{\nu} \gamma(\sigma(k))$$

for all $i \in \{1,\dots,n\}$, $\sigma \in Q_i$, and $\nu \in \mathbf{N}$. Consequently we have $q_{i,\nu} \le m_{i,\nu}$ for all $i \in \{1,\dots,n\}$ and $\nu \in \mathbf{N}$. From this we conclude that

$$\overline{\lim_{\nu \to \infty}} (q_{i,\nu})^{\frac{1}{\nu}} \le \overline{\lim_{\nu \to \infty}} (m_{i,\nu})^{\frac{1}{\nu}} \quad \text{for all } i \in \{1,\ldots,n\}.$$

Using (C_2), it follows that

$$\overline{\lim_{\nu \to \infty}} (q_{i,\nu})^{\frac{1}{\nu}} < 1 \quad \text{for all } i \in \{1,\ldots,n\}.$$

In view of assertion 4° of Theorem 4.2 it results that H has a unique fixed point. Then, by Theorem 4.3, the graph $(V,E,\alpha,\beta,\gamma)$ has a unique invariant list of representatives with respect to the family $(h_e : X \to X)_{e\in E}$. ∎

COROLLARY 4.5. *Let $(h_e : X \to X)_{e\in E}$ be a family of contractions such that*

$$\operatorname{Lip} h_e \le \gamma(e) < 1 \quad \textit{for all } e \in E.$$

Then the Mauldin–Williams graph $(V,E,\alpha,\beta,\gamma)$ has a unique invariant list of representatives with respect to the family $(h_e : X \to X)_{e\in E}$.

PROOF. Since E is finite, there exists a real number $c<1$ such that $\gamma(e) \le c$ for all $e \in E$. Consequently we have $m_{i,\nu} \le c^{\nu}$ for all $i \in \{1,\dots,n\}$ and $\nu \in \mathbf{N}$, where $m_{i,\nu}$ are the numbers defined in Theorem 4.4. From these inequalities we obtain

$$\overline{\lim_{\nu \to \infty}} (m_{i,\nu})^{\frac{1}{\nu}} \le c < 1 \quad \text{for all } i \in \{1,\ldots,n\}.$$

In conclusion, the conditions (C_1) and (C_2) of Theorem 4.4 are fulfilled. By applying this theorem, we conclude that there exists a unique invariant list of representatives of the graph $(V,E,\alpha,\beta,\gamma)$ with respect to the family $(h_e : X \to X)_{e\in E}$. ∎

REMARK. Corollary 4.5 is already known in the theory of Mauldin–Williams graphs. Another proof for it can be given by using Banach's fixed point theorem.

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ON A FIXED POINT THEOREM FOR MULTIFUNCTIONS

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A space in this paper always means a topological space. If A is a subset of an ordered space Y, then $\inf A$ and $\sup A$ mean the greatest lower bound (infimum) and last least upper bound (supremum) of a set A, respectively. By a *multifunction* F on a set X we mean a relation on X onto Y; in other words, F assigns to each point $x \in X$ a nonempty subset $F(x)$ of Y. Thus, denoting by 2^Y the family of all nonempty subsets of Y, we write $F : X \to 2^Y$. It should be stressed that, even if Y is a space, no topology is considered on the family 2^Y. We shall denote multifunctions by upper case letters F, G, …, while lower case ones f, g, … will denote single-valued functions. The reader is referred to [1] and [6] for a general (and also detailed) information on multifunctions.

Denote by $K[0,1]$ the set of closed connected subsets of $[0,1]$; thus $K[0,1] \subset 2^{[0,1]}$. Let X be a connected space. The following definition is used in [7]. A multifunction $F : X \to K[0,1]$ is said to be *continuous* provided that if $F(x) = [f_0(x), f_1(x)]$, then the functions $f_0 : X \to [0,1]$ and $f_1 : X \to [0,1]$ are continuous.

The main result of [7] is the following theorem.

1. THEOREM (Szabó). *Let F, $G : X \to K[0,1]$ be continuous multifunctions and assume that*

$$\bigcup\{F(x) : x \in X\} = [0,1]. \tag{2}$$

Then there exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.

Further, a question is asked in [7, p. 198] how one can generalize the above theorem, in particular for other spaces than $[0,1]$. The aim of this paper is to answer the question. Namely, it is indicated that a) some superfluous assumptions can be eliminated in the result; b) if X is a Hausdorff space as well, then the closed unit interval $[0,1]$ can be replaced by a generalized arc; and c) no further generalizations concerning b) are possible. However, it should be stressed that in the present paper we only collect (or recall) some results already known in the literature (in fact in [4]) to get a result much stronger than Theorem 1.

We start with recalling a sequence of definitions and notation needed in the sequel.

Let there be given two spaces X and Y, and a multifunction $F: X \to 2^Y$. We use the following notation. If $A \subset X$ and $B \subset Y$, then:

$$F(A) = \bigcup\{F(x) : x \in A\},$$

$$F^{-}(B) = \{x \in X : F(x) \cap B\} \neq \emptyset,$$
$$F^{+}(B) = \{x \in X : F(x) \subset B\}.$$

3. Definition. ([6] and [1]). A multifunction $F: X \to 2^Y$ is said to be:

- *lower semicontinuous*, l.s.c., if for each open set $V \subset Y$ the set $F^{-}(V)$ is open in X;
- *upper semicontinuous*, u.s.c., if for each open set $V \subset Y$ the set $F^{+}(V)$ is open in X;
- *continuous*, if it is both lower and upper semicontinuous;
- *point closed*, if $F(x)$ is closed for each $x \in X$;
- *point connected*, if $F(x)$ is connected for each $x \in X$;
- *surjective*, if $\cup\{F(x): x \in X\} = Y$.

Therefore, if a multifunction $F : X \to K[0,1]$ is under consideration, we have two definitions, viz. the above mentioned definition as in [7] and Definition 3, of its continuity. It can be shown that the two concepts of continuity coincide, even if a more general range space is considered.

Indeed, let us consider, in place of the closed unit interval $[0,1]$, a *generalized* arc, that is, a continuum (not necessarily metrizable) having exactly two noncut points. It is well known that a generalized arc is an *arc* (i.e., a homeomorphic image of $[0,1]$) if and only if it is metrizable. The reader is referred to Chapter 3 of [8], where cut points are used to define a natural order on a connected set, [8, p. 41], and where it is proved that any generalized arc admits a natural order (see [8, Chapter 3, §6, (iv), p. 55]) which will be denoted by $\leq$. Given a generalized arc $[a,b]$, we denote by $K[a,b]$ the set of closed connected subsets of $[a,b]$. Thus each nondegenerate element of $K[a,b]$ is a generalized arc $[c,d]$ with $a \leq c < d \leq b$. Adopting the definition from [7] to this more general case we have the following.

4. DEFINITION. Let a generalized arc $[a,b]$ be fixed. A multifunction $F: X \to K[a,b]$ is said to be *continuous* provided that if $F(x) = [f_0(x), f_1(x)]$, then the functions $f_0: X \to [a,b]$ and $f_1: X \to [a,b]$ are continuous.

5. STATEMENT. *Let X be a space and $Y = [a,b]$ be a fixed generalized arc. Then a multifunction $F: X \to K[a,b] \subset 2^Y$ is continuous in the sense of Definition* 3 *if and only if it is continuous in the sense of Definition* 4.

PROOF. Assume that F is continuous according to Definition 3, i.e., that it is both l.s.c. and u.s.c., and let a point $x^* \in X$ be given. We have to show that the function $f_0 : X \to [a,b]$ is continuous at x^*, that is for each open subset V of $[a,b]$ with $f_0(x^*) \in V$ there is an open subset U of X with $x^* \in U$ and $f_0(U) \subset V$.

Since the open subintervals (c,d), $[a,c)$ and $(d,b]$ of $[a,b]$ with $a \leq c \subset \subset d \leq b$ form a basis for $[a,b]$, it is enough to take a member of this basis as V. Let $V = (c,d) \subset [a,b]$. Therefore the sets $[a,d)$ and $(c,b]$ are open in $[a,b]$. By lower and upper semicontinuity of F the sets $F^-([a,d))$ and $F^+((c,d])$ are open subsets of X both containing x^*, whence the intersection $U = F^-([a,d)) \cap F^+((c,d])$ is an open neighborhood of x^*. Further, for each point $x \in U$ we have

$$F(x) \cap [a,d) \neq \emptyset \qquad \text{and} \qquad F(x) \subset (c,d],$$

whence it follows that $\inf F(x) < d$ and $c < \inf F(x)$. Since $\inf F(x) = f_0(x)$, we have $f_0(x) \in (c,d) = V$. Thus $f_0(U) \subset V$. The cases when $V = [a,c)$ and $V = (d,b]$ run in a similar way. So, continuity of f_0 has been proved. A proof of continuity of f_1 is the same. Thus the multifunction F is continuous in the sense of Definition 4.

Assume now that F is continuous according to Definition 4, i.e. that the two functions f_0 and f_1 are continuous. To prove continuity of F as in Definition 3, i.e. its lower and upper semicontinuity, take an open set $V \in Y = [a,b]$. We have to show that the sets $F^-(V)$ and $F^+(V)$ are open. Again as in the previous part of the proof, it is enough to take V as an element of the basis, and we consider the case of $V = (c,d) \subset [a,b]$ only, since for the other two cases the proof is the same.

Suppose $F^{-}(V)$ is not open. Then there exists a point $x^* \in F^{-}(V)$ such that every open neighborhood U of x^* intersects $X \setminus F^{-}(V)$. Thus we have $U \setminus F^{-}(V) \neq \emptyset$, whence it follows that there is a point $x \in U \setminus F^{-}(V)$, i.e., $x \in U$ and $F(x) \cap V = \emptyset$. This equality means that either $F(x) \subset [a,c]$ or $F(x) \subset [d,b]$, whence we infer that

$$\text{either} \quad f_1(x) \leq c \quad \text{or} \quad d \leq f_0(x) \tag{6}$$

On the other hand, since $x^* \in F^{-}(V)$, we have $F(x^*) \cap V \neq \emptyset$, which is equivalent to

$$f_0(x^*) = \inf F(x^*) < d \quad \text{and} \quad c < \sup F(x^*) = f_1(x^*).$$

Since f_0 is continuous at x^*, for the open neighborhood $[a,d)$ of $f_0(x^*)$ in $[a,b]$ there is an open neighborhood U_0 of x^* such that for each point $x \in U_0$ we have $f_0(x) \in [a,d)$. Similarly, since f_1 is continuous at x^*, for the open neighborhood $(c,b]$ of $f_1(x^*)$ in $[a,b]$ there is an open neighborhood U_1 of x^* such that for each point $x \in U_1$ we have $f_1(x) \in (c,b]$. Taking $U = U_0 \cap U_1$, we have for each $x \in U$

$$f_0(x) < d \quad \text{and} \quad x < f_1(x),$$

contrary to (6).

Suppose now that $F^{+}(V)$ is not open. Then, arguing as previously, we see that there exists a point $x^* \in F$+(V) such that every open neighborhood U of x^* intersects $X \setminus F^{+}(V)$. Thus we have $U \setminus F^{+}(V) \neq \emptyset$, whence it follows that there is a point $x \in U \setminus F^{+}(V)$, i.e., $x \in U$ and $F(x)$ is not contained in V, so, $F(x) \setminus V \neq \emptyset$. This inequality means that

$$f_0(x) = \inf F(x) \leq c \quad \text{or} \quad d \leq \sup F(x) = f_1(x).$$

Now, using the same argument as in the previous part of the proof one can verify that continuity of the functions f_0 and f_1 at x^* leads to a similar contradiction. The proof is then complete.

Analyzing carefully assumptions of Theorem 1 we see that, in the light of Statement 5, the theorem can be reformulated as follows.

7. THEOREM. *Let X and Y be spaces and let multifunctions F, $G : X \to$ $\to 2^Y$ be given. Assume that*

(8) *X is connected;*

(9) *Y is a generalized arc;*

(10) *Y is metrizable;*

(11) *F is l.s.c.;*

(12) *F is u.s.c.;*

(13) *F is point closed;*

(14) *F is point connected;*

(15) *F is surjective;*

(16) *G is l. s.c.;*

(17) *G is u.s.c.;*

(18) *G is point closed;*

(19) *G is point connected.*

Then

(20) *there exists $x_0 \in X$ such that $F(x_0) \cap G(x_0) \neq \emptyset$.*

A possible generalization of Theorem 1 (or, equivalently, of Theorem 7) for Hausdorff spaces X can be formulated as the following proposition.

21. PROPOSITION. *If X is a Hausdorff space, then in Theorem* 7 *assumptions* (10), (12), (13), (17) *and* (18) *can be omitted.*

Before we prove Proposition 21, we have to recall some auxiliary concepts that are needed in the proof. The results related to nets (of points and of sets) can be found in (2), (3) and (5). In particular, we use the symbol $\mathrm{Ls}\{A_\sigma\}$ to denote the superior limit of the net $\{A\sigma\}$.

We say that a multifunction $F: X \to 2^Y$ defined on a Hausdorff space X is *componentwise continuous* (c.c.) if the condition $x = \lim x_\sigma$ implies that

(22) $\mathrm{Ls}\{C_\sigma\} \cap F\{x\} \neq \emptyset$, where C_σ is a component of $F(x_\sigma)$ for each σ;

(23) every component of $F(x)$ intersects $\mathrm{Ls}\{F(x_\sigma)\}$.

Observe that if F is point connected, then (22) and (23) are the same, and thus a point connected multifunction F is c.c. if and only if $x = \lim x_\sigma$ implies that $F(x) \cap \mathrm{Ls}\, F(x_\sigma) \neq \emptyset$.

PROOF OF PROPOSITION 21. If the space X is Hausdorff, then $\lim x_\sigma$ is uniquely determined (see [2, Proposition 1.6.7, p. 51]. Then assumptions (11) and (14), as well as (16) and (19), imply that the multifunctions F and G are c.c., according to Proposition 1.1 of [4, p. 228]. Thus F is a point connected surjection by (14) and (15), i.e., (i) of Theorem 3.1 of [4, p. 240] holds. Thereby assumptions (8) and (9) imply the existence of a coincidence point x_0 of (20) by the above quoted Theorem 3.1 of (4). The proof is complete.

Another modification of Theorem 1 is the following proposition.

24. Proposition. *If X is a Hausdorff space, then in Theorem* 7 *assumptions* (10)*,* (11) *and* (16) *can be omitted.*

Proof. Arguing as in the proof of Proposition 21 we again see that assumptions (12), (13) (14) and (17), (18) (19) imply by Proposition 1.1 of [4, p. 228] that the multifunctions F and G are c.c. Thus (20) follows as previously.

25. Question. Can the assumption that the space X is Hausdorff be omitted in Propositions 21 and 24?

26. Remark. Following the question in [7, p. 198], one can ask if further generalizations of the results are possible concerning the space Y, that is, whether we can get the conclusion (20) for spaces Y other than a generalized arc. For continua the answer is negative by Corollary 3.2 of [4, p. 241] which says that the continuum Y is a generalized arc if and only if for each continuum X and for every two c.c. point connected multifunctions $F,\ G : X \to 2^Y$ with $F(X) \subset G(X) = \operatorname{cl} G(X) \subset Y$ there is a coincidence point.

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MODULI OF MEAN SMOOTHNESS AND APPROXIMATION WITH A_p-WEIGHTS

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0. Introduction

Weights satisfying the so-called A_p-Condition (A_p-weights) introduced by MUCKENHOUPT [7] play an important role in different fields of mathematical analysis. In our paper [4] we investigated the best approximation by trigonometric polynomials in the spaces $L_u^p[2\pi]$ in the case u satisfies the A_p-Condition and has period 2π. In that paper Bernstein- and Jackson-type inequalities are given. But the problem of characterizing the best approximations by using any concept of moduli remained open. Since then we have some results, but only for special cases of A_p-weights (see e.g. [5]). One of the main results we want to present in this paper is a complete solution for this problem. For any weight u satisfying the A_p-condition on an arbitrary interval (a,b) $(-\infty \leq a < b \leq \infty)$, we introduce the so-called modulus of mean smoothness in the weighted space $L_u^p(a,b)$, and in the periodic case $L_u^p[2\pi]$ (Sect. 1). The characterization of weighted K-functionals in terms of the moduli is given in Sect. 2. Applications to best approximations will be based on this result and on some of earlier results in our [4], which will be presented in Sect. 3. Finally, in Sect. 4, the saturation of Fourier–Riesz means is given.

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1. Moduli of mean smoothness

Recall here the definition of A_p-weights. Let (a,b) be a finite or infinite interval, and let $1<p<\infty$ (this will be assumed throughout the paper). We say that a function u satisfies the A_p-Condition on (a,b) if it is non-negative and finite a.e. on (a,b) and for any finite subinterval J of (a,b) we have*

$$\left(\frac{1}{|J|}\int_J u(x)dx\right)\left(\frac{1}{|J|}\int_J u(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le c(p) \tag{1}$$

here $|J|$ denotes the length of J.

The class of such weights will be denoted by $\mathcal{A}_p(a,b)$. Furthermore let $\mathcal{A}_p[2\pi]$ be the class of all weights belonging to $\mathcal{A}_p(-\infty,\infty)$, which are 2π periodic. For $u\in\mathcal{A}_p(a,b)$ the weighted space $L^p_u(a,b)$ consists of all functions f defined on (a,b) for which

$$\|f\|_{L^p_u(a,b)} = \left(\int_a^b |f(x)|^p u(x)dx\right)^{\frac{1}{p}} < \infty.$$

If $u\in\mathcal{A}_p(2\pi)$, the space $L^p_u[2\pi]$ contains all 2π-periodic functions with a similar norm defined on an interval of length 2π. Sometimes we shall use the notation $X=X^p_u$ for all spaces defined above.

We need the following two properties of A_p-weights.

a) Denote by $L(X)$ the set of all measurable functions integrable on every finite subintervals of (a,b) in the case $X=L^p_u(a,b)$, and of $(-\infty,\infty)$ for $X=L^p_u(2\pi)$. We have for any A_p-weight u

$$\text{If}\quad f\in X^p_u = X, \quad \text{then}\quad f\in L(X). \tag{2}$$

b) Let $Mf(x)$ be the Hardy–Littlewood maximal function of $f\in X$. That is

$$Mf(x) = \sup_{x\in(c,d)\subset J(X)} \frac{1}{d-c}\int_c^d |f(t)|dt, \ \text{with } J(X)=(a,b) \text{ if } X=L^p_u(a,b)$$

and $=(-\infty,\infty)$ in the case $X=L^p_u[2\pi]$.

* Throughout the paper c, $C_k(x,\ldots)$, $c(x,\ldots)$ will denote absolute constants, and constants depending only on the variables specified in the brackets, resp.

Then (see [1], [7])

$$\|Mf\|_X \le c\|f\|_X. \tag{3}$$

In order to define moduli, we introduce the following means. Consider first the periodic case. Let for $f \in L_u^p[2\pi]$, $r \ge 1$

$$\Delta_t^r f(x) = \sum_{s=0}^{r} (-1)^{r+s+1} \binom{r}{s} f(x+st).$$

Introduce

$$\sigma_\delta^r f(x) = \frac{1}{\delta} \int_0^\delta |\Delta_t^r f(x)| dt. \tag{4}$$

(For $\delta < 0$ the integral should be understood as $\int_0^\delta = -\int_\delta^0$)

By (2) the integral on the right hand side of (4) exists. In the case $f \in L_u^p(a,b)$, the mean $\sigma_\delta^r f(x)$ is defined only for all $x \in (a,b)$ such that $x + r\delta \in (a,b)$. For other $x \in (a,b)$, let $\sigma_\delta^r f(x) = 0$.

Let now

$$0 < h < \begin{cases} \infty & \text{in the case } X = L_u^p[2\pi], \\ \frac{b-a}{r} & \text{if } X = L_u^p(a,b). \end{cases}$$

Define for any $f \in X$

$$\omega_r(f,h)_X = \sup_{|\delta| \le h} \|\sigma_\delta^r f(x)\|_X \tag{5}$$

The existence of the moduli follows from (3). More exactly,

$$\omega_r(f,h)_X \le c\|f\|_X \tag{6}$$

In order to distinguish between the new and traditional moduli we shall call the former moduli of mean smoothness. It turns out later in Theorem 1 that in the case $u \equiv 1$, the new moduli and the traditional ones are equivalent.

2. The equivalence theorem

We now introduce the weighted K-functionals and give the equivalence between them and the moduli. Let u be an A_p-weight. For $f \in X = X_u^p$, $r \geq 1$, $t > 0$, let

(7) $$K_r(f,t)_X = \inf_{g \in W^{(r)}(X)} \{\|f - g\|_X + t^r \|g^{(r)}\|_X\}$$

where $W^{(r)}(X)$ denotes the class of all functions g r-times locally absolutely continuous on (a,b) (if $X = L_u^p(a,b)$), on $(-\infty,\infty)$ (in the case $X = L_u^p[2\pi]$), for which g, $g^{(r)} \in X$.

THEOREM 1. *Let u be an A_p-weight. For any $f \in X = X_u^p$ we have for $0 < h \leq c$*

(8) $$\omega_r(f,h)_X \leq C_1 K_r(f,h)_X \leq C_2 \omega_r(f,h)_X$$

where $C_i = C_i(p,u,r)$ $(i = 1, 2)$, $c = c(r,u)$.

PROOF. First we give a detailed proof of the theorem for the periodic case. Most of the steps of the proof in the non-periodic cases are similar; they will be presented later.

A. The periodic case

Let $X = L_u^p[2\pi]$. Let $g \in W^{(r)}(X)$. By (2), $g^{(r)} \in L^1[2\pi]$, therefore

(9) $$\Delta_t^r g(x) = \int_0^t \dots \int_0^t g^{(r)}(x + t_1 + \dots + t_r)dt_1 \dots dt_r.$$

Consequently

$$\omega_r(g,h)_X = \sup_{|\delta| \leq h} \left\| \frac{1}{\delta} \int_0^\delta |\Delta_t^r g(x) dt| \right\|_X \leq$$

$$\leq \sup_{|\delta| \leq h} \frac{1}{\delta} \int_0^\delta \left\| \int_0^t \dots \int_0^t g^{(r)}(x + t_1 + \dots + t_r)dt_1 \dots dt_r \right\|_X dt \leq$$

$$\leq h^r \left\| \frac{1}{h^r} \int_0^h \dots \int_0^h |g^{(r)}(x + t_1 + \dots + t_r)| dt_1 \dots dt_r \right\|_X .$$

Hence substituting $t = t_1 + \ldots + t_r$ and using (3) we get

$$\omega_r(g,h)_X \le \left\| \frac{1}{h^{r-1}} \int_0^h \ldots \int_0^h \left\{ \frac{1}{h} \int_0^{h+t_1+\ldots+t_{r-1}} |g^{(r)}(x+t)dt \right\} dt_1 \ldots dt_r \right\|_X \le$$

$$\le h^r \left\| \frac{1}{h} \int_0^{rh} |g^{(r)}(x+t)dt \right\|_X \le c(r) h^r \|g^{(r)}\|_X \tag{10}$$

From (6), (10) we have for any $f \in X$, $g \in W^{(r)}(X)$

$$\omega_r(f,h) \le c(\|f - g\|_X + h^{(r)} \|g^{(r)}\|_X).$$

Therefore, by definition

$$\omega_r(f,h)_X \le c K_r(f,h)_X. \tag{11}$$

In order to prove the converse inequality of (11) we introduce a Steklov-type transform for $f \in X$, $r \ge 1$, $h > 0$

$$f_{r,h}(x) = \frac{2}{h} \int_{\frac{h}{2}}^{h} \left(\frac{1}{\delta^r} \int_0^\delta \ldots \int_0^\delta \sum_{s=0}^{r-1} (-1)^{r+s+1} \binom{r}{s} \cdot f\left(x + \frac{r-s}{r}(t_1 + \ldots + t_r)\right) dt_1 \ldots dt_r \right) d\delta. \tag{12}$$

We have

$$f_{r,h}(x) - f(x) = \frac{2}{h} \int_{\frac{h}{2}}^{h} \left(\frac{1}{\delta^r} \int_0^\delta \ldots \int_0^\delta \Delta^r_{\frac{t_1+\ldots+t_r}{t}} f(x) dt_1 \ldots dt_r \right) d\delta.$$

Hence

$$\|f_{r,h} - f\|_X \le \sup_{\frac{h}{2} \le \delta \le h} \left\| \frac{1}{\delta^r} \int_0^\delta \Delta^r_{\frac{t_1+\ldots+t_r}{t}} f(x) dt_1 \ldots dt_r \right\|_X .$$

Therefore substituting $t = t_1 + \ldots + t_r$ we get

(13) $\|f_{r,h} - f\|_X \le$

$$\le \sup_{\frac{h}{2}\le\delta\le h}\left\|\frac{1}{\delta^{r-1}}\int_0^{\delta}\dots\int_0^{\delta}\left(\frac{1}{\delta}\int_{t_1+\dots t_{r-1}}^{\delta+t_1+\dots+t_{r-1}}\Delta^r_{\frac{t}{r}}f(x)dt\right)dt_1\dots dt_{r-1}\right\|_X \le$$

$$\le \sup_{\frac{h}{2}\le\delta\le h}\left\|\frac{1}{\delta}\int_0^{r\delta}\left|\Delta_{\frac{t}{r}}f(x)\right|dt\right\|_X \le r\sup_{0<\delta\le h}\left\|\frac{1}{\delta}\int_0^{\delta}|\Delta^r_t f(x)|dt\right\|_X \le r\omega_r(f,h)_X.$$

On the other hand, by (12)

$$f^{(r)}_{r,h}(x) = \frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\Delta_{\frac{r-s}{r}}{}^r{}_{\delta}f(x)d\delta.$$

Hence

$$|f^{(r)}_{r,h}(x)| \le 2h^{-r}\frac{1}{h}\int_0^h\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\left|\Delta^r_{\frac{r-s}{r}\delta}f(x)\right|d\delta =$$

$$= 2h^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\frac{1}{h}\int_0^h\left|\Delta_{\frac{r-s}{r}\delta}f(x)\right|d\delta =$$

$$= 2h^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^r\frac{1}{\frac{r-s}{s}h}\int_0^{\frac{r-s}{r}h}|\Delta^r_u f(x)|du.$$

Consequently we have by the definition of the moduli

$$\|f^{(r)}_{r,h}\|_X \le c(r)h^{-r}\omega_r(f,h)_X, \tag{14}$$

with $c(r)=2(2r)^r$. Combining (13) and (14) we get

$$K_r(f,h)_X \le \|f-f_{r,h}\|_X + h^r\left\|f^{(r)}_{r,h}\right\|_X \le c\omega_r(f,h)_X.$$

We have proved the converse inequality of (11). By this the proof of Theorem 1 in the periodic case is complete.

B. Proof of Theorem 1 in the nonperiodic case.

Consider all weights $u(x)$ from the class $\mathcal{A}_p(a,b)$. By any linear transform T (of x), the class $\mathcal{A}_p(a,b)$ becomes the class $\mathcal{A}_p(a',b')$ with $a'=T(a)$,

$b' = T(b)$, and the moduli of mean smoothness as well as the weighted K-functionals of functions preserve their order. Therefore it is enough to prove our theorem for the following three cases:

a) $(a,b)=(0,\infty)$,

b) $(a,b)=(-\infty,\infty)$,

c) $(a,b)=(0,1)$.

In all three cases inequality (11) can be proved similarly to the periodic case. The proof of the converse inequality in the cases of the intervals $(0,\infty)$ and $(-\infty,\infty)$ is also similar, and is based on the Steklov-type transform (12). The case of the interval $(0,1)$ is somewhat, more complicated, since in this case the function $f_{r,h}$ in (12) is defined only on $(0,1-rh)$. Therefore in order to get a suitable transform we use the following method.

Let $f \in L_u^p(0,1)$. Let h' be a real number, $|h'|<1$. Denote by $f_{r,h'}$ the function defined as in (12). Let $\psi(x)$ be the function defined as:

$$\psi(x)=1 \quad \text{for} \quad 0 \le x \le \frac{1}{3}, \quad 0 \quad \text{for} \quad \frac{2}{3} \le x \le 1$$

and $\psi(x)$ is r-times continuously differentiable on $[0,1]$ with $|\psi^{(i)}(x)| \le c(r)$ $(i=1,\dots, r;\ \frac{1}{3} \le x \le \frac{2}{3})$.

Now, introduce for $0<h<\frac{1}{3r}$

$$f_{\langle r,h\rangle} = \psi f_{r,h} + (1-\psi) f_{r,-h} = f_{r,-h} + \psi(f_{r,h} - f_{r,-h}). \tag{15}$$

Repeating the techniques used in the proof of (13) and (14) we have

$$\|f - f_{r,h}\|_{L_u^p\left(0,\frac{2}{3}\right)} \le c\omega_r(f,h)_{L_u^p(0,1)} \tag{16}$$

$$\|f_{r,h}^{(r)}\|_{L_u^p\left(0,\frac{2}{3}\right)} \le ch^{-r}\omega_r(f,h)_{L_u^p(0,1)} \tag{17}$$

$$\|f - f_{r,-h}\|_{L_u^p\left(\frac{1}{3},1\right)} \le c\omega_r(f,h)_{L_u^p(0,1)} \tag{18}$$

$$\|f_{r,-h}^{(r)}\|_{L_u^p\left(\frac{1}{3},1\right)} \le ch^{-r}\omega_r(f,h)_{L_u^p(0,1)}. \tag{19}$$

Therefore by (15) and the definition of ψ

$$\|f - f_{\langle r,h\rangle}\|_{L_u^p(0,1)} \le \tag{20}$$

$$\le \|f - f_{r,h}\|_{L_u^p\left(0,\frac{2}{3}\right)} + \|f - f_{r,-h}\|_{L_u^p\left(\frac{1}{3},1\right)} \le c\omega_r(f,h)_{L_u^p(0,1)}.$$

On the other hand, again by (15), using the Leibniz formula and the properties of ψ we get

$$\left\|f^{(r)}_{\langle r,h\rangle}\right\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} \le c(r)\left\{\|f^{(r)}_{r,h}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} + \sum_{i=0}^{r}\|(f_{r,h}-f_{r,-h})^{(i)}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)}\right\}.$$

In order to continue the estimation, we notice that from the definition of A_p-weight (see (1)) we have $u^{-\frac{q}{p}},\ u \in L^1(0,1)$ $(\frac{1}{p}+\frac{1}{q}=1)$. Hence using an inequality between L^p_u-norm of the derivatives of different orders of functions (see [6, Theorem 3.3]) we get

$$\left\|f^{(r)}_{\langle r,h\rangle}\right\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} \le$$

$$\le c(r)\left\{\|f^{(r)}_{r,h}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} + \|f^{(r)}_{r,-h}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} + \|f_{r,h}-f_{r,-h}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)}\right\}.$$

Therefore, by (16)–(19)

$$\|f^{(r)}_{\langle r,h\rangle}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} \le c(r)h^{-r}\omega_r(f,h)_{L^p_u(0,1)}.$$

This together with (15), (17) and (19) implies

$$\|f^{(r)}_{\langle r,h\rangle}\|_{L^p_u(0,1)} \le \tag{21}$$

$$\le c(r)\left(\|f^{(r)}_{\langle r,h\rangle}\|_{L^p_u\left(0,\frac{1}{3}\right)} + \|f^{(r)}_{\langle r,h\rangle}\|_{L^p_u\left(\frac{1}{3},\frac{2}{3}\right)} + \|f^{(r)}_{\langle r,h\rangle}\|_{L^p_u\left(\frac{2}{3},1\right)}\right) \le$$

$$\le c(r)h^{-r}\omega_r(f,h)_{L^p_u(0,1)}.$$

The inequalities (20) and (21) imply the converse inequality of (11) in our case. We have completed the proof of Theorem 1.

3. Best approximation by trigonometric polynomials in the spaces $L^p_u[2\pi]$

Let $u \in \mathcal{A}_p[2\pi]$ $(1 < p < \infty)$. Let T_n be the set of all trigonometric polynomials of order at most n. For any $f \in X = L^p_u[2\pi]$, define

$$E_n(f)_X = \inf_{t_n \in T_n}\|f - t_n\|_X \qquad (n = 1, 2, \ldots).$$

This is finite because $u \in L^1[2\pi]$, hence $t_n \in L_u^p[2\pi]$. Using the results of [2] we have (for details see [4])

$$E_n(f)_X \to 0 \qquad (n \to \infty) \qquad (f \in X). \tag{22}$$

In [4] we proved

A. Bernstein-type inequality:

$$\|t_n^{(r)}\|_X \le cn^r \|t_n\|_X \qquad (t_n \in T_n) \tag{23}$$

and

B. Jackson-type inequality

$$E_n(f)_X \le \frac{c}{n^r}\|f^{(r)}\|_X \tag{24}$$

$(f \in W^{(r)}(X),\ n=1,2,\dots)$.

The following theorem follows from those inequalities and Theorem 1.

THEOREM 2 (direct and converse theorem). *Let* $u \in \mathcal{A}_p[2\pi]$ *(*$1<p<\infty$*),* $r \ge 1$*. For any* $f \in X = L_u^p[2\pi]$*,* $n \ge 1$ *we have*

$$E_n(f)_X \le c(r)\omega_r\left(f, \frac{1}{n}\right)_X \tag{25}$$

and

$$\omega_r(f,\delta)_X \le c(r)\delta^r \sum_{k=0}^{\left[\delta^{-1}\right]} (k+1)^{r-1} E_k(f)_X. \tag{26}$$

PROOF. Indeed, the same inequalities like (25) and (26) are true if we replace $\omega_r(f)$ by the K-functional $K_r(f)$. This fact follows from (23) and (24) by a well-known technique of the approximation theory (see also [8, Theorem 3.16]). Then using Theorem 1 we have completed the proof.

4. Saturation theorem for Riesz-mean

In this section we give the saturation of Riesz-means of Fourier-series by using the moduli of mean smoothness. Let $u \in \mathcal{A}_p[2\pi]$ $(1<p<\infty)$. Let $f \in L_u^p[2\pi]$. Then $f \in L[2\pi]$. It has the Fourier-series

$$f(x) \sim \frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i \cos ix + b_i \sin ix).$$

Let

$$A_k(x) = \frac{a_0}{2} + \sum_{i=1}^{k} (a_i \cos ix + b_i \sin ix).$$

Denote by $R_n^{[r]}$ the n-th Riesz-mean of order r of the Fourier-series of f. That is

$$R_n^{[r]}(f,x) = \sum_{k=0}^{n} \left[1 - \left(\frac{1}{k+1}\right)^r\right] A_k(x) \quad (r,\ n = 1, 2, \ldots).$$

We now prove

THEOREM 3. *Let $f \in X = L_u^p[2\pi]$ ($u \in \mathcal{A}_p(2\pi]$, $1 < p < \infty$). Let $r \geq 1$ be an integer. Then*

(i) *If $\|R_n^{[r]}(f) - f\|_X = o(n^{-r})$ then $f \equiv const.$*

(ii) *$\|R_n^{[r]}(f) - f\|_X = O(n^{-r})$ iff $\omega_r(f,\delta)_X = O(\delta^r)$.*

PROOF. The statement (i) was proved in our [3]. It was also proved in [3] that $\|R_n^{[r]}(f) - f\|_X = O(n^{-r})$ iff $f \in W^r(X)$. So, it remains to prove the equivalence

$$f \in W^r(X) \quad \text{iff} \quad \omega_r(f,\delta)_X = O(\delta^r). \tag{27}$$

The "only if" part of (27) follows from (10). To prove the "if" part we argue as follows. Since $u \in \mathcal{A}_p[2\pi]$ there exists $1 < p_1 < p$ such that $u \in \mathcal{A}_{p_1}[2\pi]$ (see [1, p.243]). If $f \in L_u^p[2\pi]$) then setting $p_0 = p/p_1$ (> 1) we have by Hölder inequality

$$\left(\int_0^{2\pi} |f(x)|^{p_0}\right)^{\frac{1}{p_0}} = \left(\int_0^{2\pi} |f(x)^{\frac{p}{p_1}} u(x)^{\frac{1}{p_1}} u(x)^{-\frac{1}{p_1}} dx\right)^{\frac{1}{p_0}} \leq \tag{28}$$

$$\leq \left(\int_0^{2\pi} |f(x)|^p u(x)\right)^{\frac{1}{p}} \left(\int_0^{2\pi} u(x)^{-\frac{p_1'}{p_1}} dx\right)^{\frac{1}{p_1' p_2}}$$

where $\frac{1}{p_1'} + \frac{1}{p_1} = 1$.

Observing that since $u \in \mathcal{A}_{p_1}[2\pi]$, the function $u(x)^{-\frac{p_1'}{p_1}}$ is integrable on $[0,2\pi]$, we obtain from (28)

$$\|f\|_{L^{p_0}[2\pi]} \le c(p_1,p)\|f\|_X. \tag{29}$$

Hence, denoting $X_0 = L^{p_0}[2\pi]$ we have

$$\omega_r(f,\delta)_{X_0} \le c(p_1,p)\omega_r(f,\delta)_X. \tag{30}$$

But we see from Theorem 1 that in the space $X_0 = L^{p_0}$ the traditional modulus (which is denoted usually by $\omega_r(f,\delta)_{p_0}$) is equivalent to the modulus of mean smoothness. Therefore from (30) we get

$$\omega_r(f,\delta)_{p_0} \le c(p_1,p)\omega_r(f,\delta)_X. \tag{31}$$

So, if $\omega_r(f,\delta)_X = O(\delta^r)$ then the same estimate holds for $\omega_r(f,\delta)_{p_0}$, too. Hence by a well-known property of the traditional moduli in $L^{p_0}[2\pi]$ we obtain that f is r-times locally absolutely continuous in $(-\infty,\infty)$ and for almost every x

$$\frac{1}{h^r}\Delta_h^r f(x) \to f^{(r)}(x) \qquad (h \to 0). \tag{32}$$

From (32) one has also for almost every x

$$\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{|\Delta_h^r f(x)|}{h^r}dh \to |f^{(r)}(x)| \qquad (\delta \to 0_+).$$

Hence by Fatou-Lemma

$$\|f^{(r)}\|_X \le \liminf_{\delta\to 0_+}\left\|\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta}\frac{|\Delta_h^r f(x)|}{h^r}dh\right\|_X \le$$

$$\le \limsup_{\delta\to 0_+}\frac{2^{r+1}}{\delta^r}\left\|\frac{1}{\delta}\int_0^{\delta}|\Delta_h^r f(x)|dh\right\|_X \le \limsup_{\delta\to 0_+} 2^{r+1}\frac{\omega_r(f,\delta)}{\delta^r} < \infty.$$

We have proved $f \in W^{(r)}(X)$. By this the proof of Theorem 3 is complete.

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WEIGTHED $(0;0,1,3)$-INTERPOLATION ON THE ZEROS OF HERMITE POLYNOMIALS

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In memory of Professor Arun Kumar Varma

1. Introduction

The first author and A. SHARMA [6] proved existence, uniqueness and explicit representation of $(0,2)$ and $(0,1,3)$ interpolations on infinite interval. Because of the complicated nature of explicite representation, one can not expect better results.

In order to get new results in the mentioned type of interpolations we apply and generalize the interpolation method of L. G. PÁL [8] for the case of Hermite polynomials. Originally L. G. PÁL could construct firstly polynomials $P_{2n-1}(x)$ of order $(2n-1)$ which satisfied the following interpolation properties: Let $\{x_k\})_k^n$ be arbitrarily chosen but fixed system of distinct nodal points, and let be given two systems $\{y_k\}_{k=1}^n$ and $\{y_k'\}_{k=1}^{n-1}$ of real numbers then the polynomials $P_{2n-1}(x)$ allow the equalities $P_{2n-1}(x_k)=y_k$; $(k=1, 2, \ldots, n)$ and $P_{2n-1}'(x_k^*)=y_k'$; $(k=1, 2, \ldots, n-1)$ where the points $\{x_k^*\}_{k=1}^{n-1}$ are the roots of $\omega'(x)=\frac{d}{dx}\left(\prod_{k=1}^{n}(x-x_k)\right)$, then there exists only one polynomial if we require a special additional condition $P_{2n-1}(a)=0$ for a fixed number $a\neq x_i$; $(k=1, 2, \ldots, n)$.

L. SZILI – who had investigated and proved convergence theorems with respect to the classical weighted $(0,2)$ interpolation on the roots of Hermite polynomials [11] – also applied the original simple method of Pál on the roots of Hermite polynomials [10].

Further, first author and R. B. SAXENA [7] extended the results of L. SZILI to the case of weighted $(0,1,3)$-interpolation on infinite interval. For more interesting result in this direction one is referred to a recent paper [12] due to I. JOÓ.

In this paper, we consider a special problem of mixed type weighted $(0;0,1,3)$-interpolation on the zeros of Hermite polynomials.

Let $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ be the zeros of $H_n(x)$ and $H_n'(x)$, where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \qquad n = 1, 2, \ldots, \tag{1.1}$$

The fundamental polynomials of Lagrange interpolation are given by

$$l_k(x) = \frac{H_n(x)}{H_n'(x_k)(x - x_k)}, \quad k = 1(1)n \tag{1.2}$$

and

$$L_k(x) = \frac{H_n'(x)}{H_n''(y_k)(x - y_k)}, \quad k = 1(1)n - 1 \tag{1.3}$$

In this paper, we resolve the following:

Weighted $(0;0,1,3)$-*Interpolation Problem:*

Let n be even, then for given arbitrary numbers $\{p_k\}_{k=1}^{n}$ $\{p_k^*\}_{k=1}^{n-1}$, $\{p_k^{**}\}_{k=1}^{n-1}$ and $\{p_k^{***}\}_{k=1}^{n-1}$, there exists a unique polynomial $S_n(x)$ of degree $\leq 4n-3$, such that

$$\begin{aligned} S_n(x_k) &= p_k, && k = 1(1)n,\\ S_n(y_k) &= p_k^*, && k = 1(1)n-1,\\ S_n'(y_k) &= p_k^{**}, && k = 1(1)n-1,\\ \left(e^{-x^2} S_n(x_k)\right)'''_{x=y_k} &= p_k^{***}, && k = 1(1)n-1, \end{aligned} \tag{1.4}$$

and

$$S_n''(0) = \sum_{k=1}^{n} \frac{2p_k H_n(0) H_n''(0) l_k^2(0)}{H_n'^2(x_k)}$$

For n odd, $S_n(x)$ does not exist uniquely. Precisely we shall prove the following:

THEOREM 1. *For n even,*

$$S_n(x)=\sum_{k=1}^{n}p_k\,U_k(x)+\sum_{k=1}^{n-1}p_k^{*}\,V_k(x)+\sum_{k=1}^{n-1}p_k^{**}\,W_k(x)+\sum_{k=1}^{n-1}p_k^{***}\,X_k(x) \tag{1.5}$$

such that

$$S_n''(0)=\sum_{k=1}^{n}\frac{2p_k H_n(0)H_n''(0)l_k^2(0)}{H_n'^2(x_k)},\qquad k=1(1)n,$$

where $U_k(x)$, $k=1(1)n$ and $V_k(x)$, $k=1(1)n-1$ are the fundamental polynomials of the first kind, $W_k(x)$, $k=1(1)n-1$ are the fundamental polynomials of second kind and X_k, $k=1(1)n-1$ are the fundamental polynomials of third kind of weighted $(0;0,1,3)$-interpolation. Each such fundamental polynomial of degree at most $4n-3$, is given by:

$$X_k(x)=\frac{e^{y_k^2}H_n(x)H_n'^2(x)}{24n^2H_n^3(y_k)}\int_0^x L_k(t)dt \tag{1.6}$$

$$\begin{aligned}W_k(x)=&\frac{-H_n(x)H_n'(x)L_k^2(x)}{2nH_n^2(y_k)}-\\&-\frac{H_n(x)H_n'^2(x)}{8n^3H_n^4(y_k)}\int_0^x\frac{2n(t-y_k)H_n(y_k)L_k'(t)+y_kH_n'(t)}{(t-y_k)^2}dt-\\&-2e^{-y_k^2}(2y_k^2-4n-1)X_k(x),\end{aligned} \tag{1.7}$$

$$\begin{aligned}V_k(x)=&\frac{H_n(x)L_k^3(x)\{1-3y_k(x-y_k)\}}{H_n(y_k)}-\frac{H_n(x)H_n'^2(x)}{H_n(y_k)H_n''^2(y_k)}\cdot\\&\cdot\int_0^x\frac{L_k'(t)\{1-3y_k(t-y_k)\}+L_k(t)\{-y_k+C_1(t-y_k)\}}{(t-y_k)^2}dt+\\&+4y_k\left[2y_k^2-2n-3\right]e^{-y_k^2}X_k(x),\end{aligned} \tag{1.8}$$

where

$$C_1=\frac{8y_k^2+2(n-2)}{3}$$

and

(1.9) $$U_k(x) = \frac{H_n'^2(x) l_k^2(x)}{H_n'^2(x_k)} + \\ + \frac{2H_n(x)H_n'^2(x)}{H_n'^4(x_k)} \int_0^4 \frac{l_k(t)H_n'(x_k) - \{1 - x_k(t - x_k)\}H_n'(t)}{(t - y_k)^2} dt,$$

where $l_k(x)$ *and* $L_k(x)$ *are given by* (1.2) *and* (1.3) *respectively.*

THEOREM 2. *Let the interpolated function* $f : \mathbf{R} \to \mathbf{R}$ *be continuously differentiable such that*

$$\lim_{|x|\to\infty} x^{2r} f(x)\varrho(x) = 0 \qquad (r = 0, 1, \ldots)$$

(1.10) *and*

$$\operatorname*{Lim}_{|x|\to\infty} \varrho(x) f'(x) = 0 \qquad \text{where } \varrho(x) = e^{-x^2}.$$

Further, let the numbers δ_k *satisfy the condition:*

(1.11) $$\delta_k = O\left((n)e^{\delta y_k^2}\omega\left(f'; \frac{1}{\sqrt{n}}\right)\right), \ k = 1(1)n-1 \text{ and } 0 < \delta < 1,$$

where ω *is the modulus of continuity of* f', *then*
(1.12)

$$S_n(f,x) = \sum_{k=1}^{n} f(x_k) U_k(x) + \sum_{k=1}^{n-1} f(y_k) V_k(x) + \sum_{k=1}^{n-1} f'(y_k) W_k(x) + \sum_{k=1}^{n-1} \delta_k X_k(x)$$

such that

$$S_n''(0) = 2\sum_{k=1}^{n} \frac{f(x_k) H_n''(0) l_k^2(0) H_n(0)}{H_n'^2(x_k)}$$

satisfies the relation:

(1.13) $$e^{-\nu x^2} |S_n(f,x) - f(x)| = O(\log n)\omega\left(f'; \frac{1}{\sqrt{n}}\right), \qquad \nu > 2,$$

which holds on the whole real line and O *does not depend on* n *and* x.

REMARK. $\omega(f,\delta)$ denotes the special form of modulus of continuity introduced by G. Freud [3] given by:

(1.14) $$\omega(f,\delta) = \sup_{0\le t\le\delta} \|\varrho(x+t)f(x+t) - \varrho(x)f(x)\| + \|\tau(\delta x)f(x)\varrho(x)\|$$

where

$$\tau(x) = \begin{cases} |x|, & \text{for } |x| \le 1 \\ 1 & |x| > 1 \end{cases}$$

and $\|\cdot\|$ denotes sup-norm in $C(\mathbf{R})$. If $f \in C(\mathbf{R})$ and

(1.15) $$\lim_{|x|\to\infty} \varrho(x)f(x) = 0 \quad \text{then} \quad \lim_{\delta\to 0} \omega(f,\delta) = 0.$$

2. Preliminaries

In this section, we shall give some well-known results, which we shall use in the sequel.

The differential equation satisfied by $H_n(x)$ is given by:

(2.1) $$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$$

(2.2) $$H_n'(x) = 2n H_{n-1}(x)$$

(2.3) $$l_k(x_j) = \begin{cases} 0 & j \ne k, \\ & \text{for} \quad , k = 1(1)n \\ 1 & j = k \end{cases}$$

(2.4) $$l_k'(x_j) = \begin{cases} \frac{H_n'(x_j)}{H_n'(x_k)(x_j - x_k)}, & j \ne k \\ x_k, & \text{j=k} \end{cases}$$

(2.5) $$l_k'(x_j) = -\frac{l_k(y_j)}{(y_j - x_k)}, \qquad j = 1(1)n-1.$$

From (1.3), one has

(2.6) $$L_k(y_j) = \begin{cases} 0 & j \ne k \\ & \text{for} \quad , j = 1(1)n-1 \\ 1 & j = k \end{cases}$$

(2.7) $$L_k'(y_j) = \begin{cases} \frac{H_n''(y_j)}{H_n''(y_k)(y_j - y_k)}, & j \ne k \\ & , \quad , j = 1(1)n-1 \\ y_k, & j = k \end{cases}$$

(2.8) $$L_k''(y_j) = \begin{cases} \frac{2H_n''(y_j)}{H_n''(y_k)(y_j - y_k)} \left\{ y_j - \frac{1}{(y_j - y_k)} \right\}, & j \ne k \\ & , j = 1(1)n-1 \\ \frac{4y_k^2 - 2(n-2)}{3}, & j = k \end{cases}$$

$$L_k'''(y_j) = \begin{cases} \frac{1}{(y_j - y_k)} \left\{ \frac{H_n^{\text{iv}}(x)}{H_n''(y_k)} - 3L_k''(y_j) \right\}, & j \neq k \\ & , j = 1(1)n-1 \\ y_k\{2y_k^2 - 2n + 5\}, & j = k \end{cases} \tag{2.9}$$

$$H_n^{(4)}(y_k) = [4y_k^2 - 2(n-2)]H_n''(y_k) \tag{2.10}$$

$$H_n^{(5)}(y_k) = 4y_k(2y_k^2 - 2n + 5). \tag{2.11}$$

K. K. MATHUR and R. B. SAXENA proved the order of estimate [7]:

$$\left|\varrho(x)Q_n'''(x)\right| = O(n)\omega\left(f'; \frac{1}{\sqrt{n}}\right) \tag{2.12}$$

for their polynomials $Q_n(x)$, that there are in the formula (5.3) of Lemma 4 in their paper [7] on page 43.

For the roots of $H_n(x)$, we have

$$x_k^2 \sim \frac{k^2}{n}. \tag{2.13}$$

L. SZILI proved,

$$|k_n^2 - 1| \leq e^{\beta x_k^2}, \qquad 0 < \beta < \frac{1}{2}, \tag{2.14}$$

$$|x\varrho(x)Q_n'(x)| = O(\sqrt{n})\omega\left(f'; \frac{1}{\sqrt{n}}\right) \tag{2.15}$$

and

$$\varrho(x)|Q_n''(x)| = O(\sqrt{n})\omega\left(f'; \frac{1}{\sqrt{n}}\right). \tag{2.16}$$

$$H_n(x) = O\left\{n^{-1}7\sqrt{2^n n!}\left(1 + \sqrt[3]{|x|}\right)e^{x^2/2}\right\}, \quad x \in \mathbf{R} \tag{2.17}$$

$$|H_n'(x_k)| \geq C2^{n+1}\left(\frac{n}{2}\right)!e^{\frac{\delta x_k^2}{2}}, \quad 0 < \delta < 1 \tag{2.18}$$

$$\sum_{i=1}^{n-1} \frac{H_i(y)H_i(x)}{2^i i!} = \frac{H_n(y)H_{n-1}(x) - H_{n-1}(y)H_n(x)}{2^n(n-1)!(y-x)}. \tag{2.19}$$

From (1.2) and (2.19) at $y = x_k$, we have

$$|l_k(x)| = \frac{O(1)2^{n+1}n!\sqrt{n}e^{\frac{\nu_1}{2}(x^2 + x_k^2)}}{H_n'^2(x_k)}, \quad \nu_1 > 1 \tag{2.20}$$

(2.21) $$\left|\int_0^x l_k(t)dt\right| = \frac{O(1)2^{n+1}n!\log n e^{\frac{\nu_1}{2}(x^2+x_k^2)}}{H_n'^2(x_k)}, \quad \nu_1 > 1$$

(2.22) $$\sum_{k=1}^{n} e^{-\varepsilon x_k^2} = O(\sqrt{n}), \qquad \text{where} \quad \varepsilon > 0,$$

(2.23) $$\sum_{k=1}^{n} e^{\delta x_k^2}(H_n'(x_k))^{-2} = O(2^{n+1}n!)^{-1}, \quad 0 < \delta < 1$$

and

(2.27) $$\frac{2^n\left[\left(\frac{n}{2}\right)!\right]^2}{(n+1)!} \sim n^{-1/2}, \qquad n = 1, 2, \ldots.$$

Proof of Theorem 1

Let

(3.1) $$S_n(x) = $$
$$= \sum_{k=1}^{n} p_k A_k(x) + \sum_{k=1}^{n-1} p_k^* B_k(x) + \sum_{k=1}^{n-1} p_k^{**} C_k(x) + \sum_{k=1}^{n-1} p_k^{***} D_k(x) + p_0 H_n^2(x),$$

where $A_k(x)$, $B_k(x)$, $C_k(x)$ and $D_k(x)$ are polynomials each of degree $\leq 4n-3$ and p_0 is a constant to be determined.

In the light of conditions (1.4), one can see that

(3.2)
$$A_k(x_j) = \begin{cases} 0 & \text{for } j \neq k \\ 1 & \phantom{\text{for }} j = k \end{cases}, j = 1(1)n,\ A_k(y_j) = 0, j = 1(1)n-1$$
$$A_k'(y_j) = 0,\ j = 1(1)n-1 \text{ and } \left(e^{-x^2}A_k(x)\right)'''_{x=y_j} = 0,\ j = 1(1)n-1.$$

(3.3)
$$B_k(x_j) = 0,\ j = 1(1)n,\ B_k(y_j) = \begin{cases} 0, & \text{for } j \neq k \\ 1 & \phantom{\text{for }} j = k \end{cases}, j = 1(1)n-1$$
$$B_k'(y_j) = 0,\ j = 1(1)n-1 \text{ and } \left(e^{-x^2}B_k(x)\right)'''_{x=y_j} = 0,\ j = 1(1)n-1.$$

(3.4)
$$C_k(x_j) = 0,\ j = 1(1)n,\ C_k(y_j) = 0,\ j = 1(1)n-1$$

$$C_k'(y_j) = \begin{cases} 0 & \text{for } \begin{array}{l} j \neq k \\ , j = 1(1)n-1 \\ \text{and } \left(e^{-x^2} C_k(x)\right)'''_{x=y_j} = 0 \; j = 1(1)n-1 \end{array} \\ 1 & j = k \end{cases}$$

(3.5)
$$D_k(x_j) = 0,\ j = 1(1)n,\ D_k(y_j) = 0,\ j = 1(1)n-1$$

$$D_k'(y_j) = 0,\ j = 1(1)n-1 \text{ and } \left(e^{-x^2} D_k(x)\right)''' x = y_j = \begin{cases} 0 & \text{for } \begin{array}{l} j \neq k \\ , j = 1(1)n-1 \end{array} \\ 1 & j = k \end{cases}$$

To determine $D_k(x)$, let

(3.6) $$D_x(k) = C_2 H_n(x) H_n'^2(x) \int_0^x L_k(t)dt, \quad \text{where } C_2 \text{ is a constant.}$$

It is a polynomial of degree $\leq 4n-3$. Obviously it satisfies conditions of (3.5) provided

$$C_2 = \frac{e^{y_k^2}}{24n^2 H_n^3(y_k)}.$$

Thus

$$D_k(x) \equiv X_k(x).$$

Let

(3.7) $$C_k(x) = C_3 H_n(x) H_n'(x) L_k^2(x) + \\ + C_4 H_n(x) H_n'^2(x) \int_0^x \frac{(t-y_k)L_k'(t) + C_5 H_n'(t)}{(t-y_k)^2} dt + C_6 X_k(x),$$

where C_3–C_6 are arbitrary constants, since it is a polynomial of degree $\leq 4n-$ -3, therefore, the expression under the integral sign must be a polynomial of degree $\leq n-2$, which implies

$$L_k'(y_k) + C_5 H_n''(y_k) = 0.$$

Using (2.1) at y_k and (2.7), we get

(3.8) $$C_5 = \frac{y_k}{2n H_n(y_k)}.$$

Obviously $C_k(x)$ satisfied the first, second and third condition of (3.4) which gives

$$C_3 = -\frac{1}{2nH_n^2(y_k)}. \tag{3.9}$$

For the fourth condition at $x = y_j$, $j \neq k$

$$\left(e^{-x^2} C_k(x)\right)'''_{x=y_j} = 0.$$

Using (2.6), (2.7) and (3.5), we get

$$C_4 = -\frac{1}{4n^2 H_n^3(y_k)}. \tag{3.10}$$

Similarly for $j = k$, using (2.6), (2.7), (2.8) and (3.5), we get

$$C_6 = -2\{2y_k^2 - 4n - 1\}e^{-y_k^2}. \tag{3.11}$$

substituting all the values of (3.8)–(3.11) in (3.7), we see that

$$C_k(x) \equiv W_k(x).$$

In order to prove (1.8), let

$$\begin{aligned} B_k(x) &= H_n(x)L_k^3(x)\{C_7 + C_8(x - y_k)\}+ \\ &+ \frac{C_9 H_n(x) H_n'^2(x)}{H_n(y_k)} \int_0^x \frac{L_k'(t)\{1 - 3y_k(t - y_k)\} + L_k(t)\{C_{10} + C_{11}(t - y_k)\}}{(t - y_k)^2} + \\ &+ C_{12} X_k(x), \end{aligned} \tag{3.12}$$

where C_7–C_{12} are arbitrary constants. Since $B_k(x)$ is a polynomial of degree $\leq 4n$–3, therefore, the expression under the integral sign must be a polynomial of degree $\leq n-2$, which implies

$$L_k'(y_k) + C_{10} L_k(y_k) = 0$$

and

$$L_k''(y_k) - 3y_k L_k'(y_k) + C_{10} L_k'(y_k) + C_{11} L_k(y_k) = 0.$$

Owing to (2.6), (2.7) and (2.8), one gets

$$c_{10} = -y_k$$

and

$$C_{11} = \frac{8y_k^2 + 2(n-2)}{3}$$

One can see that $B_k(x)$ satisfies all the four conditions of (3.3), by using (2.6), (2.7), (2.8), (2.9) and (3.5), gives

$$C_7 = \frac{1}{H_n(y_k)},$$

$$C_8 = -\frac{3y_k}{H_n(y_k)},$$

$$C_9 = -\frac{1}{H_n''^2(y_k)}$$

and

$$C_{12} = y_k\left[8y_k^2 - 8n - 12\right]e^{-y_k^2}.$$

Hence

$$B_k(x) \equiv V_k(x),$$

which gives (1.8).

Proof of (1.9) follows on the same lines as (1.8), so we omit the details.

If n is even, $H_n(0) \neq 0$, so (3.1) at $x=0$, owing to last condition of (1.4), gives $p_0=0$, which completes the proof of the theorem.

NOTE. If n is odd $H_n(0)=0$, so (3.1) at $x=0$, satisfies last condition of (1.4) for any value of p_0, other conditions hold good as above in the proof, so in this case, there are infinitely many solutions of $(0;0,1,3)$-interpolation problem.

4. Some lemmata

To prove theorem 2, we need

LEMMA 4.1. *For* $k=1(1)n$ *and* $x \in (-\infty,\infty)$, *we have*

$$\frac{1}{H_n'(x_k)}\int_0^x \frac{H_n'(x_k)l_k(t) - [1 - x_k(t-x_k)]H_n'(t)}{(t-x_k)^2}dt =$$

$$= -\frac{H_n(x) - H_n(0)}{H_n'(x_k)} + n\int_0^x l_k(t)dt + \frac{1}{2}[l_k'(x) - l_k'(0)],$$

where $l_k(x)$ *is given by* (1.2).

PROOF. From (1.2) we have

$$(t-x_k)l_k(t)=\frac{H_n(t)}{H_n'(x_k)}. \tag{4.1}$$

Differentiating it twice, we get

$$(t-x_k)l_k'(t)+l_k(t)=\frac{H_n'(t)}{H_n'(x_k)} \tag{4.2}$$

and

$$(t-x_k)l_k''(t)+2l_k'(t)=\frac{H_n''(t)}{H_n'(x_k)}. \tag{4.3}$$

From (4.2), we have

$$H_n'(x_k)l_k(t)-H_n'(t)=-(t-x_k)H_n'(x_k)l_k'(t). \tag{4.4}$$

Adding $x_k(t-x_k)H_n'(t)$ on both sides, we get

$$H_n'(x_k)l_k(t)-\{1-x_k(t-x_k)\}H_n'(t)=-(t-x_k)\{-x_kH_n'(t)+H_n'(x_k)l_k'(t)\}=$$

$$=-(t-x_k)[-x_k\{H_n'(t)-H_n'(x_k)l_k(t)\}+H_n'(x_k)\{l_k'(t)-x_kl_l(t)\}]. \tag{4.5}$$

Now multiplying (4.1) by $2n$ and (4.2) by $(-2t)$ and adding them to (4.3) and finally using (2.1), we get

$$l_k'(t)-x_kl_k(t)=-(t-x_k)\left[\frac{1}{2}l_k''(t)-tl_k'(t)+(n-1)l_k(t)\right]. \tag{4.6}$$

Putting (4.4) and (4.6) in (4.5) and using (4.2), we get,

$$\frac{H_n'(x_k)l_k(t)-[1-x_k(t-x_k)]H_n(t)}{H_n'(x_k)(t-x_k)^2}=-\frac{H_n'(t)}{H_n'(x_k)}+\frac{1}{2}l_k''(t)+n(l_k(t)),$$

which on integrating under limits 0 to x proves the lemma.

LEMMA 4.2. *For* $k=1(1)n$ *and* $x\in(-\infty,\infty)$, *we have*

$$|l_k'(x)|=O(1)2^{n+1}(n+1)![H_n'(x_k)]^{-2}e^{\frac{\nu_1}{2}(x^2+x_k^2)},\quad \nu_1>1$$

where $l_k(x)$ *is given by* (1.2).

PROOF. From (2.19) at $y=x_k$ and (1.2), we get

$$l_k(x)=\frac{2^{n+1}n!}{H_n'^2(x_k)}\sum_{i=0}^{n-1}\frac{1}{2^ii!}H_i(x_k)H_i(x), \tag{4.7}$$

differentiating it and using (2.2), we get

$$l'_k(x) = \frac{2^{n+1}n!}{H'^2_n(x_k)} \sum_{i=1}^{n-1} \frac{1}{2^{i-1}(i-1)!} H_i(x_k)H_{i-1}(x)$$

Further taking modulus and using (2.17), we get the lemma.

LEMMA 4.3. *Let* $\nu_1 > 1$ *and* $x \in (-\infty, \infty)$, *then*

$$|L_k(x)| = O\left(\frac{2^n n! e^{\nu_1\left(\frac{x^2+y_k^2}{2}\right)}}{\sqrt{n} H_n^2(y_k)}\right), \qquad k = 1(1)n-1,$$

where $L_k(x)$ *is given by* (1.3).

PROOF. From (2.19) at $y = y_k$ and using (1.3) and (2.2), we get

$$|L_k(x)| \le \frac{2^n(n-1)!}{H_n^2(y_k)} \sum_{i=0}^{n-1} \frac{1}{2^i i!} |H_i(x)||H_i(y_k)|. \tag{4.8}$$

On using (2.17), we get the required lemma.

LEMMA 4.4. *For* $k = 1(1)n-1$ *and* $x \in (-\infty, \infty)$, *we have*

$$\left|\int_0^x L_k(t)dt\right| = \frac{O(1)2^n(n-1)! e^{\frac{\nu_1}{2}(x^2+y_k^2)} \log n}{H_n^2(y_k)}, \quad \nu_1 > 1.$$

The proof of this lemma follows exactly on the same lines as Lemma 4.3, so we omit the proof.

LEMMA 4.5. *For* $x \in (-\infty, \infty)$ *and* $k = 1(1)n-1$, *we have*

$$|L'_k(x)| = \frac{O(1)2^n n! e^{\frac{\nu_1}{2}(x^2+y_k^2)}}{H_n^2(y_k)}, \qquad \nu_1 > 1,$$

where $L_k(x)$ *is given by* (1.3).

PROOF. From (4.8), we have

$$L_k(t) = \frac{2^n(n-1)!}{H_n^2(y_k)} \sum_{i=0}^{n-1} \frac{1}{2^i i!} H_i(t)H_i(y_k).$$

Differentiating and using (2.2), we get

$$L_k'(x) = \frac{2^n(n-1)!}{H_n^2(y_k)} \sum_{i=0}^{n-1} \frac{1}{2^{i-1}(i-1)!} H_i(y_k)H_{i-1}(x). \tag{4.9}$$

Further, taking modulus and using (2.17) we get the lemma.

LEMMA 4.6. *For* $x \in (-\infty,\infty)$ *and* $k = 1(1)n-1$, *we have*

$$|L_k''(x)| = O\left(\frac{2^n n!\sqrt{n}e^{\frac{\nu_1}{2}(x^2+y_k^2)}}{H_n^2(y_k)}\right).$$

The proof is similar to Lemma 4.5, so we omit the details.

LEMMA 4.7. *We have*

$$\frac{1}{2nH_n(y_k)} \int_0^x \frac{y_k H_n'(t) + 2n(t-y_k)H_n(y_k)L_k'(t)}{(t-y_k)^2} dt =$$

$$= -\frac{1}{2}(L_k'(x)-L_k'(0))+y_k(L_k(x)-L_k(0))+\frac{H_n'(x)-H_n'(0)}{H_n''(y_k)}-(n-1)\int_0^x L_k(t)dt,$$

where L_k *is given by* (1.3).

PROOF. From (1.3), we have

$$L_k(t)(t-y_k) = \frac{H_n'(t)}{H_n''(y_k)}. \tag{4.10}$$

Differentiating it twice, we get

$$L_k'(t)(t-y_k) + L_k(t) = \frac{H_n''(t)}{H_n''(y_k)} \tag{4.11}$$

and

$$L_k''(t)(t-y_k) + 2L_k'(t) = \frac{H_n'''(t)}{H_n''(y_k)}. \tag{4.12}$$

From the differential equation (2.1), we get

$$H_n'''(t) - 2tH_n''(t) + 2(n-1)H_n'(t) = 0. \tag{4.13}$$

Multiplying (4.12) by $(t-y_k)H_n''(y_k)$ and rearranging the terms,

$$(t-y_k)L_k'(t)H_n''(y_k) = \frac{1}{2}(t-y_k)H_n'''(t) - \frac{1}{2}(t-y_k)^2 L_k''(t)H_n''(y_k). \tag{4.14}$$

Similarly from (4.11), multiplying by y_k, we have

$$y_k L_k(t) = \frac{y_k H_n''(t)}{H_n''(y_k)} - y_k(t - y_k)L_k'(t),$$

on using (4.10), we get

$$y_k H_n'(t) = y_k(t - y_k)H_n''(t) - y_k(t - y_k)^2 H_n''(y_k)L_k'(t). \tag{4.15}$$

Subtracting (4.15) from (4.14), using (4.13) and (4.10), we get

$$\begin{aligned}(t - y_k)H_n''(y_k)L_k'(t) - y_k H_n'(t) = \\ = (t - y_k)^2 H_n''(y_k)\left[-\frac{1}{2}L_k''(t) + y_k L_k'(t) + \frac{H_n''(t)}{H_n''(y_k)} - (n-1)L_k(t)\right].\end{aligned} \tag{4.16}$$

Using (2.1) at y_k, we have

$$\begin{aligned}[2n(t - y_k)H_n(y_k)L_k'(t) + y_k H_n'(t)] = \\ = 2n(t - y_k)^2 H_n(y_k)\left[-\frac{1}{2}L_k''(t) + y_k L_k'(t) + \frac{H_n''(t)}{H_n''(y_k)} - (n-1)L_k(t)\right].\end{aligned} \tag{4.17}$$

Now, dividing the whole equation (4.17) by $2n(t-y_k)^2 H_n(y_k)$ and taking the integral, we get the required lemma.

LEMMA 4.8. *For* $x \in (-\infty, \infty)$ *and* $k = 1(1)n-1$, *we have*

$$\int_0^x \frac{L_k'(t)\{1 - 3y_k(t - y_k)\} + L_k(t)\left[-y_k + \frac{1}{3}(t - y_k))\{8y_k^2 + 2(n-2)\}\right]}{(t - y_k)^2} dt =$$

$$= (L_k''(x) - L_k''(0)) + 2n(L_k'(x) - L_k(0)) +$$

$$+8y_k(L_k'(x) - L_k'(0)) - 2xL_k'(x) + 16xy_kL_k(x) + 16y_k(n-1)\int_0^x L_k(t)dt,$$

where $L_k(x)$ *is given by* (1.3).

The proof runs analogous to the above Lemma 4.7, so it is omitted.

5. Estimation of the fundamental polynomials

LEMMA 5.1. *For* $k = 1(1)n$ *and* $x \in (-\infty, \infty)$, *we have*

$$\sum_{k=1}^{n} e^{x_k^2} |U_k(x)| = O(\sqrt{n} \log n) e^{\nu x^2}, \qquad \nu > 2,$$

where $U_k(x)$ *is given by* (1.9).

PROOF. From (1.9) and lemma 4.1, we get

$$\sum_{k=1}^{n} e^{x_k^2} |U_k(x)| \leq \sum_{k=1}^{n} \frac{e^{x_k^2} H_n'^2(x) l_k^2(x)}{H_n'^2(x_k)} + \\ + \sum_{k=1}^{n} \frac{e^{x_k^2} |H_n(x)| H_n'^2(x)[|l_k'(x) - l_k'(0)|]}{|H_n'^3(x_k)|} + \\ + \sum_{k=1}^{n} \frac{2e^{x_k^2} |H_n(x)| H_n'^2(x)[|H_n(x) - H(0)|]}{|H_n'^4(x_k)|} + \\ + \sum_{k=1}^{n} \frac{e^{x_k^2} 2n |H_n(x)| H_n'^2(x)}{|H_n'^3(x_k)|} \left| \int_0^x l_k(t) dt \right|.$$

On using (2.2), (2.17), (2.18), (2.20), (2.21), (2.23), (2.24) and lemma 4.2, we get the required lemma.

LEMMA 5.2. *For* $x \in (-\infty, \infty)$ *and* $k = 1(1)n - 1$, *we have*

$$\sum_{k=1}^{n-1} e^{y_k^2} |V_k(x)| = O(\sqrt{n} \log n) e^{\nu x^2}, \qquad \nu > 2.$$

where $V_k(x)$ *is given by* (1.8).

PROOF. From (1.8) and lemma 4.8, we get

$$\sum_{k=1}^{n} e^{y_k^2} |V_k(x)| \leq \sum_{k=1}^{n} \frac{e^{y_k^2} |H_n(x)||L_k^3(x)|[|1 - 3y_k(y - y_k)|]}{|H_n(y_k)|} + \\ + \sum_{k=1}^{n-1} \frac{e^{y_k^2} |H_n(x)| H_n'^2(x) |L_k''(x) - L_k''(0)|}{|H_n(y_k)| H_n''^2(y_k)} +$$

$$+\sum_{k=1}^{n-1}\frac{2ne^{y_k^2}|H_n(x)|H_n'^2(x)|L_k(x)|}{|H_n(y_k)|H_n''^2(y_k)}+$$

$$+\sum_{k=1}^{n-1}\frac{8|y_k|e^{y_k^2}|H_n(x)|H_n'^2(x)[|L_k'(x)-L_k'(0)|]}{|H_n(y_k)|H_n''^2(y_k)}+$$

$$+\sum_{k=1}^{n-1}\frac{2|x|e^{y_k^2}|H_n(x)|H_n'^2(x)|L_k'(x)-L_k'(0)|}{|H_n(y_k)|H_n''^2(y_k)}+$$

$$+\sum_{k=1} n-1\frac{e^{y_k^2}|y_k||18y_k^2-104n+961||H_n(x)H_n'^2(x)}{6|H_n(y_k)|H_n''^2(y_k)}\left|\int_0^x L_k(t)dt\right|+$$

$$+\sum_{k=1}^{n-1}\frac{16|x||y_k|e^{y_k^2}|H_n(x)+H_n'^2(x)|L_k(x)|}{|H_n(y_k)|H_n''^2(y_k)}.$$

Owing to (2.1) at y_k, (2.2), (2.13), (2.17), (2.18), (2.24), lemma 4.3, lemma 4.4, lemma 4.5 and lemma 4.6, we get the lemma.

LEMMA 5.3. *For* $k=1(1)n-1$ *and* $x\in(-\infty,\infty)$, *we have*

$$\sum_{k=1}^{n-1} e^{y_k^2}|W_k(x)| = O(\log n)e^{\nu x^2}, \qquad \nu>2,$$

where $W_k(x)$ *is given by* (1.7).

Lemma follows on the same lines as Lemma 5.2, so we omit its proof.

LEMMA 5.4. *For* $k=1(1)n-1$ *and* $x\in(-\infty,\infty)$, *we have*

$$\sum_{k=1}^{n-1}|X_k(x) = O\left(\frac{\log n}{n}\right)e^{\nu x^2}, \qquad \nu>2,$$

where $X_k(x)$ *is given by* (1.6).

PROOF. From (1.6), we have

$$\sum_{k=1}^{n-1}|X_k(x)|\le\sum_{k=1}^{n-1}\frac{e^{y_k^2}|H_n(x)|H_n'^2(x)}{24n^2|H_n^3(y_k)|}\left|\int_0^x L_k(t)dt\right|.$$

On using (2.2), (2.17), (2.22) and lemma 4.4, we get the required lemma.

6.

In this section, we mention certain results of G. FREUD and L. SZILI required in the proof of main Theorem 2.

G. Freud in [4], theorem 4 and in [3], Theorem 1, proved

Let $f : \mathbf{R} \to \mathbf{R}$ be continuously differentiable. Further, let

$$\lim_{|x|\to\infty} x^{2k}\varrho(x)f(x) = 0, \qquad k = 0, 1, 2, \ldots$$

and

$$\lim_{|x|\to\infty} \varrho(x)f'(x) = 0,$$

then there exists polynomial $Q_n(x)$ of degree $\leq n$ such that

$$\text{(6.1)} \qquad \varrho(x)|f(x) - Q_n(x)| = o\left(\frac{1}{\sqrt{n}}\right)\omega\left(f'; \frac{1}{\sqrt{n}}\right), \quad x \in \mathbf{R}$$

and

$$\text{(6.2)} \qquad \varrho(x)|f'(x) - Q_n'(x)| = o(1)\omega\left(f'; \frac{1}{\sqrt{n}}\right), \quad x \in \mathbf{R}$$

where ω stands for modulus of continuity defined by (1.14) and $\varrho(x)$ is the weight function.

SZILI ([11], lemma 4) established the following

$$\text{(6.3)} \qquad \varrho(x)|Q_n^{(r)}(x)| = o(1), \qquad r = 0, 1; \quad x \in \mathbf{R}.$$

7. Proof of theorem 2.

Since $S_n(x)$ given by (1.5) is exact for all polynomials $Q_n(x)$ of degree $\leq 4n - 3$, we have

$$\text{(7.1)}\; Q_n(x) = \sum_{k=1}^{n} Q_n(x_k)U_k(x) + \sum_{k=1}^{n-1} Q_n(y_k)V_k(x) + \sum_{k=1}^{n-1} Q_n'(y_k)W_k(x) +$$
$$+ \sum_{k=1}^{n-1}\left(e^{-x^2}Q_n(x)\right)'''_{x=y_k} X_k(x) + d_n H_n(x)H_n'^2(x),$$

where

$$d_n = \frac{1}{H_n(0)H_n''^2(0)}\left[Q_n''(0) - \sum_{k=1}^{n} \frac{2Q_n(x_k)H_n(0)H_n''(0)l_k^2(0)}{H_n'^2(x_k)}\right]. \tag{7.2}$$

One can easily see that

$$\begin{aligned} e^{-\nu x^2}|S_n(x) - f(x)| \le e^{-\nu x^2}&|f(x) - Q_n(x)| + \\ &+ e^{-\nu x^2}\left|\sum_{k=1}^{n} e^{-x_k^2}[f(x_k) - Q_n(x_k)]\,U_k(x)e^{x_k^2}\right| + \\ &+ e^{-\nu x_k^2}\left|\sum_{k=1}^{n-1} e^{-y_k^2}[f(y_k) - Q_n(y_k)]\,V_k(x)e^{y_k^2}\right| + \\ &+ e^{-\nu x_k^2}\left|\sum_{k=1}^{n-1} e^{-y_k^2}[f'(y_k) - Q_n'(y_k)]\,W_k(x)e^{y_k^2}\right| + \\ &+ e^{-\nu x_k^2}\left|\sum_{k=1}^{n-1}[\delta_k - (\varrho Q_n(x))'''_{(y_k)}]X_k(x)\right| + \\ &+ e^{-\nu x^2}|d_n H_n(x)H_n'^2(x)|. \end{aligned}$$

Thus (6.1), (6.2), (6.3), (1.11), (1.14), (1.15), (2.14), (2.15) (2.16), (2.12), (2.1), (2.2), (2.13), (2.17), (2.18), (2.20), (2.23), (2.24), lemma 5.1–5.4 complete the proof of theorem 2.

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ON TRANSPORTED INCLUSIONS

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1. Introduction

The principal aim of this paper is to analyze, for a given x^*, the existence of a solution y^* for the inclusion

$$x^* \in F^*(y^*), \tag{1.1}$$

where F^* is the transpose of the set-valued map F from the locally convex space X to the locally convex space Y. We shall be using the terminology of AUBIN and EKELAND [1]. In particular, given a non-empty set $M \subset X$, we denote by $\operatorname{cl} M$, $\operatorname{co} M$ and $\operatorname{cone} M$ the topological closure of M, the convex hull of M, and the convex cone spanned by M, respectively. The zero-vector is denoted by θ.

Our topic is closely related to the theory of infinite linear inequality systems. Let Z be a locally convex, real Hausdorff space, and let Z^* denote the dual space of Z. Let us consider the linear inequality system

$$\Sigma := \{\langle z_j, \varphi\rangle \geq c_j,\ j \in J\},$$

where J is an arbitrary non-empty index set, z_j belongs to Z, and c_j is a real number, for all $j \in J$. With the system Σ we associate the wedge

$$K := \operatorname{cone}\{(z_j, c_j),\ j \in J\},$$

named *moment cone* in GLASHOFF [6].

The theory of finite linear ineqality systems, for $Z = \mathbf{R}^n$, was first developed by FARKAS [5] and MINKOWSKI [13]. The pioneering contributions

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to the analysis of semi-infinite systems, assuming that either J is finite or Z is finite dimensional space, are due to HAAR [10]. For solvability of general linear inequality systems, fundamental results were obtained by FAN [3], [4] ZHU [15], GOBERNA and LÓPEZ [7], [8], [9], and others.

The system Σ is said to be *consistent* if there exists an element $\varphi \in Z^*$ such that

$$\langle z_j, \varphi \rangle \geq c_j \quad \text{for all} \quad j \in J. \tag{1.2}$$

The following theorem of FAN [4] is relevant in our context:

THEOREM 1.1. *The system Σ is consistent if, and only if, the element $(\theta, 1)$ of $Z \times \mathbf{R}$ is not in the closure of the moment cone K.*

This is a generalization of the well-known alternative theorem a GALE (see e.g. [12]).

2. The solvability conditions

In the following suppose that X and Y are two locally convex, real Hausdorff vector spaces, and X^* (respectively Y^*) denotes the dual space of X (respectively Y), i.e., the vector space of all continuous linear functionals on X (respectively Y). The set-valued map F from X to Y is said to be *proper* if its *domain*, $\mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}$, is nonvoid. A set-valued map F is characterized by its graph, $\mathrm{Graph}(F) := \{(x,y) \in X \times Y : y \in F(x)\}$. The map F is said to be *convex* if its graph is convex. A set-valued map whose graph is a cone is called *process*. A *closed convex process* is a set-valued map whose graph is a closed convex cone. We can adapt the concept of *transpose* to set-valued maps as follows (see e.g. [1]):

DEFINITION 2.1. Let F be a set-valued map from X to Y. We associate with F the convex process F^* from Y^* to X^* defined in for following way:

$$x^* \in F^*(y^*) \text{ if and only if } \sup_{x \in X} \sup_{y \in F(x)} \big(\langle x^*, x\rangle - \langle y^*, y\rangle\big) < +\infty.$$

If F is a process, then its transpose F^* is the closed convex process defined by

$$x^* \in F^*(y^*) \text{ if and only if } \forall x \in X,\ \forall y \in F(x),\ \langle x^*, x\rangle \leq \langle y^*, y\rangle .$$

For a linear operator A from X to Y, the transpose of the map F defined by $F(x) = \{Ax\}$, $x \in X$, coincides with F^* defined by $F^*(y^*) = \{A^*(y^*)\}$, $y^* \in Y^*$, where A^* is the transpose of A as a linear operator.

Our purpose is to give necessary and sufficient condition for the existence of a solution $y^* \in Y^*$ of the inclusion (1.1), if $x^* \in X^*$ is given. We mention that if $\mathrm{Dom}(F)$ is bounded, then $x^* \in F^*(0)$, hence the case of interest is where $\mathrm{Dom}(F)$ is unbounded.

THEOREM 2.1. *Let F be a proper set-valued map from X to Y. For a given $x^* \in X^*$, set*

$$M := \mathrm{cone}\,\{(y, 1, \langle x, x^* \rangle\} \in Y \times \mathbf{R} \times \mathbf{R} : \; (x, y) \in \mathrm{Graph}\,(F)\}.$$

Then the inclusion (1.1) *has a solution y^* if, and only if, the element* $(\theta, 0, 1)$ *of $Y \times \mathbf{R} \times \mathbf{R}$ is not in* $\mathrm{cl}\,M$*.*

PROOF. y^* is a solution of (1.1) if, and only if, there exists $a \in \mathbf{R}$ such that

$$\forall (x, y) \in \mathrm{Graph}\,(G), \qquad \langle x, x^* \rangle \le \langle y, y^* \rangle + a. \tag{2.1}$$

Let $Z := Y \times \mathbf{R}$ and $J := \mathrm{Graph}(F)$. For $j = (x, y) \in J$, we set $z_j := (y, 1)$ and $c_j := \langle x, x^* \rangle$. Then, for $\varphi := (y^*, a)$, condition (2.1) reduces to (1.2). By Theorem 1.1 the assertion follows. ∎

COROLLARY 2.1. *Let F be a convex process from X to Y. For a given $x^* \in X^*$ set*

$$P := \left\{ \left(y, \langle x, x^* \rangle\right) \in Y \times \mathbf{R} : \; (x, y) \in \mathrm{Graph}\,(F) \right\}.$$

Then the inclusion (1.1) has a solution y^* if, and only if, the element $(\theta, 1)$ *of $Y \times \mathbf{R}$ is not in* $\mathrm{cl}\,P$*.*

PROOF. For a convex process F we have

$$M = \{(y, \alpha, \langle x, x^* \rangle) : \; \alpha \in \mathbf{R}_+, \; (x, y) \in \mathrm{Graph}\,(F)\}.$$

Therefore $(\theta, 0, 1) \in \mathrm{cl}\,M$, if and only if $(\theta, 1) \in \mathrm{cl}\,P$. ∎

COROLLARY 2.2. *Let F be a convex process from X to Y, and let $x^* \in$ $\in X^*$ with $x^* \neq \theta$. Set*

$$Q := \{y \in Y : \exists x \in X \; \text{s.t.} \; \langle x, x^* \rangle = 1 \; \text{and} \; y \in F(x)\}.$$

Then either the inclusion (1.1) *has a solution, or $\theta \in \mathrm{cl}\,Q$, but never both.*

PROOF. If $\theta \in \mathrm{cl}\,Q$ then obviously $(\theta, 1) \in \mathrm{cl}\,P$, hence the inclusion (1.1) has no solution, by Corollary 2.1.

To prove the converse, we suppose that (1.1) has no solution. Then, by Corollary 2.1 $(\theta, 1) \in \mathrm{cl}\,P$, hence there exists a net of elements $(\overline{x}_n, \overline{y}_n) \in$ $\in \mathrm{Graph}(F)$ such that $a_n := \langle \overline{x}_n, x^* \rangle \to 1$ and $\overline{y}_n \to \theta$. We can suppose that

$a_n > 0$, for all n. Let $x_n = \frac{1}{a_n}\overline{x}_n$ and $y_n = \frac{1}{a_n}\overline{y}_n$. We have $y_n \in F(x_n)$, $y_n \to \theta$ and $\langle x_n, x^* \rangle = 1$, hence $\theta \in \operatorname{cl} Q$. ∎

COROLLARY 2.3. *Let $A : X \to Y$ be a (not necessarily continuous) linear operator, and let $x^* \in X^*$ with $x^* \neq \theta$. Set*

$$V := \{y \in Y : \ \exists x \in X \text{ s.t. } \langle x, x^* \rangle = 0 \text{ and } y = Ax\}.$$

Then either the equation

$$A^* y^* = x^* \tag{2.2}$$

has a solution, or there exists $v \in X$ with $\langle v, x^* \rangle = 1$ and $Av \in \operatorname{cl} V$, but never both.

PROOF. Suppose that $\langle v, x^* \rangle = 1$ and $Av = \operatorname{cl} V$. Then there exists a net of elements $\overline{x}_n \in X$ such that $\langle \overline{x}_n, x^* \rangle = 0$ and $A\overline{x}_n \to Av$. Then for $x_n = v - \overline{x}_n$ we have $Ax_n \in Q$ and $Ax_n \to 0$, hence $\theta \in \operatorname{cl} Q$. Therefore, by Corollary 2.2, the equation $A^* y^* = x^*$ has no solution.

Conversely, if where equation $A^* y^* = x^*$ has no solution, then $\theta \in \operatorname{cl} Q$, hence there exists a net of elements $x_n \in X$ with $\langle x_n, x^* \rangle = 1$ and $Ax_n \to 0$. Let $v \in X$ by such that $\langle v, x^* \rangle = 1$. Then, for $\overline{x}_n := v - x_n$, we have $\langle \overline{x}_n, x^* \rangle = 0$ and $A\overline{x}_n \to Av$, therefore $Av \in \operatorname{cl} V$. ∎

COROLLARY 2.4. *Let A and x^* be as in Corollary 2.3. Then the equation* (2.2) *has a solution if, and only if, there exists a function $f : Y \to \mathbf{R}$, continuous at θ, such that $f(\theta) = 0$ and*

$$|\langle x, x^* \rangle| \leq f(Ax) \quad \text{for all} \quad x \in X. \tag{2.3}$$

PROOF. If y^* is a solution of (2.2), then for any continuous seminorm p on Y there exists $m > 0$ such that

$$|\langle x, x^* \rangle| = |\langle x, A^* y^* \rangle| = |\langle Ax, y^* \rangle| \leq mp(Ax), \qquad x \in X,$$

hence one can choose $f = mp$.

Conversely, of a function f with the given properties exists, then we choose $v \in X$ such that $\langle v, x^* \rangle = 1$, and by (2.3) it follows

$$1 = |\langle v - x, x^* \rangle| \leq f(Av - Ax),$$

for all $x \in X$ with $\langle x, x^* \rangle = 0$. Therefore, $Av \notin \operatorname{cl} V$, hence by Corollary 2.3 the equation (2.2) has a solution.

REMARKS. 1. If A and x^* are as in Corollary 2.3, then the following two assertions are equivalent:

(i) There exists $x \in X$ such that $\langle v, x^* \rangle = 1$ and $Av \in \mathrm{cl}\, V$;

(ii) $Av \in \mathrm{cl}\, V$, for all $v \in X$ with $\langle v, x^* \rangle = 1$.

This follows immediately from the proof of Corollary 2.3.

2. Corollary 2.4 has been proved by EMBRY [2], for normed vector spaces X and Y, and continuous linear operator $A : X \to Y$. It was rediscovered later by KANTOROVITZ and HUGHES [11].

Corollary 2.3 has been proved by SEBESTYÉN [14] in case, when Y is a Hilbert space, X is a subspace of Y and $\mathrm{cl}\, X = Y$.

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GOOD IDEALS IN BLOCKED TRIANGULAR MATRIX RINGS OVER DIVISION RINGS

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1. Introduction

Every ring herein is associative with identity, subrings inherit the identity, and ideal means two-sided ideal.

In [4] an internal characterization of regular right duo rings was provided in the sense that a right duo ring R is regular if and only if every ideal A of R is a good ideal. An ideal A of a ring R is called a *good ideal* in [4] if it has the property that the coset product of any two cosets of A in the factor ring R/A is the same as their set product, denoted by $\circ$, i.e. if for any two cosets r_1+A and r_2+A of A,

$$r_1r_2 + A = \{(r_1 + a_1)(r_2 + a_2) : \ a_1, a_2 \in A\} =: (r_1 + A) \circ (r_2 + A)$$

A necessary condition for A to be a good ideal is thus that (with r_1, $r_2=0$) $A = \{a_1a_2 : \ a_1, a_2 \in A\} = A \circ A$. It is not known whether this condition is sufficient too. A number of equivalent characterizations of rings in which every ideal is good were provided in [4] for certain subclasses of the class of right duo rings.

Examples of good ideals, as well as of ideals which are not good, were exhibited in certain subrings of full matrix rings in [4]. It is the purpose of this note to give a complete characterization of the good ideals in the class of blocked triangular matrix rings over division rings. These matrix rings are not right duo, but the condition $A\circ A=A$ is shown to be indeed sufficient for an ideal A to be good in this large class of rings. In fact, the characterization is valid for blocked triangular matrix rings over simple rings, but because of the important application of blocked triangular matrix rings over division rings in the structure theory of rings in [5], which will be mentioned below, we state the characterization for such rings.

We first provide the relevant notation regarding blocked triangular matrix rings. We use $B = [b_{i,j}]$ to denote a reflexive and transitive $n \times n$ Boolean matrix, i.e. $b_{i,j} \in \{0,1\}$, $b_{i,i} = 1$ for every $i = 1, 2, \ldots, n$, and if $b_{i,j} = b_{j,k} = 1$, then $b_{i,k} = 1$. We call B *blocked triangular* if it is of the form

$$\begin{bmatrix} B_{1,1} & B_{1,2} & \ldots & B_{1,t} \\ 0 & B_{2,2} & \ldots & B_{2,t} \\ \ldots & \ldots & \vdots & \ldots \\ 0 & \ldots & 0 & B_{t,t} \end{bmatrix}, \tag{1}$$

where for every $i \le j$, $B_{i,j}$ is an $n_i \times n_j$ (Boolean) matrix with all its entries equal (i.e. $B_{i,j} = 0_{n_i \times n_j}$ or $1_{n_i \times n_j}$), and $n_1 + \ldots + n_t = n$. Here $0_{n_i \times n_j}$ (respectively $1_{n_i \times n_j}$) denotes the $n_i \times n_j$ matrix with every entry equal to 0 (respectively 1). Henceforth, if the size of the matrix is not decisive, then we shall merely write 0 instead of $O_{n_i \times n_j}$. Since we assume B to be reflexive, every entry of $B_{i,i}$ is 1 for $i = 1, 2, \ldots, t$. We call $B_{i,j}$ the (i,j)-th block and $B_{i,i}$ the i-th diagonal block of B. If $i \neq j$, then we call $B_{i,j}$ a non-diagonal block of B, and we then also call the i-th and j-th diagonal blocks the *parental blocks* of $B_{i,j}$. If every entry of $B_{i,j}$ is 1 for all $i \le j$, then B is called *complete blocked triangular*.

Let $\sigma \in S_n$ be a permutation of the set $\{1, 2, \ldots, n\}$. We denote by $\sigma(B)$ the blocked triangular matrix with $b_{i,j}$ in position $\left(\sigma^{-1}(i), \sigma^{-1}(j)\right)$, $i, j = 1, 2, \ldots, n$. For example,

$$B := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \sigma(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

if $\sigma \in S_6$ is the permutation (1 2 4 6 3 5). As mentioned in [2] and [7], it is easy to see that for any reflexive and transitive $n \times n$ Boolean matrix B there is a permutation $\sigma \in S_n$ such that $\sigma(B)$ is blocked triangular, and if $\varphi([r_{i,j}])$, $[r_{i,j}] \in \mathbb{M}(B,R)$, is the matrix with $r_{i,j}$ in position $(\sigma^{-1}(i), \sigma^{-1}(j))$, then $\varphi : \mathbb{M}(B,R) \to \mathbb{M}(\sigma(B),R)$ is a ring isomorphism. Here, $\mathbb{M}(B,R)$ is the structural matrix ring associated with B and R, i.e. the subring of $\mathbb{M}_n(R)$, R any ring, compromising all matrices having 0 in position (k,l)

whenever $b_{k,l} = 0$. The structural matrix ring associated with a (complete) blocked triangular Boolean matrix is called a (complete) blocked triangular matrix ring. The foregoing arguments imply that by studying the class of blocked triangular matrix rings, one is studying, up to isomorphism, the class of structural matrix rings, which has lately received considerable attention. See, for example, [1], [2], [3], [6], [7], [8] and [9]. The class of blocked triangular matrix rings over division rings as such has found applications in the structure theory of rings in [5], where, by considering suitably restricted primeness conditions, a characterization was provided of complete blocked triangular matrix rings over division rings by conditions analogous to those of the Artin–Wedderburn characterization of full matrix rings over division rings.

We recall the description in [8] of the ideals of a blocked triangular matrix ring $\mathbb{M}(B,R)$, with B as in (1). For a matrix X in $\mathbb{M}(B,R)$ we denote the submatrix of X comprising the entries corresponding to the (i,j)-th block $B_{i,j}$ of B by $X_{i,j}$, and we call $X_{i,j}$ the (i,j)-th block of X. Note that the $n\times n$ Boolean matrix $B=[b_{i,j}]$ determines and is determined by the binary relation c_B ("connected with respect to B") on the set $\{1,2,\ldots,n\}$ defined by

$$i \ \mathrm{c}_B \ j \quad \text{if and only if} \quad b_{i,j} = 1.$$

Since we have assumed that B is reflexive and transitive, it follows that c_B is a quasi-order relation (also called a preorder), and so it naturally gives rise to the equivalence relation $\sim_B$ (on the set $\{1,2,\ldots,n\}$) defined by

$$i \sim_B j \quad \text{if and only if} \quad i \ \mathrm{c}_B \ j \text{ and } j \ \mathrm{c}_B \ i.$$

Then (with B as in (1)) the number of equivalence classes induced by $\sim_B$ equals t (and $n_1,\ldots,n_t$ are the cardinalities of the t equivalence classes). Let *Rep* denote a set of representatives of the t equivalence classes. For $i,j \in Rep$ such that $i \,\mathrm{c}_B\, j$, we set

$$\Lambda_{i,j} = \{l : \ i \ \mathrm{c}_B \ l \ \mathrm{c}_B \ j\}.$$

Consider any set-inclusion preserving function

$$f : \ \{\Lambda_{i,j} : \ i \ \mathrm{c}_B \ j\} \to \{A : \ A \text{ is an ideal of } R\},$$

and let $\mathscr{A}_f$ be the set comprising all the matrices in $\mathbb{M}(B,R)$ with elements from $f(\Lambda_{i,j})$ in their (i,j)-th blocks. Then the $\mathscr{A}_f$'s are precisely the ideals of $\mathbb{M}(B,R)$.

It is clear that, by considering matrices with zeros in certain specified rows, a blocked triangular matrix ring is not right duo.

2. Good ideals in blocked triangular matrix rings over division rings

We show in Corollary 2.3 that in the class of blocked triangular matrix rings over division rings the condition $\mathscr{A}_f \circ \mathscr{A}_f = \mathscr{A}_f$ is sufficient for an ideal $\mathscr{A}_f$ to be a good ideal, and in Corollary 2.4 we conclude that the good ideals of a blocked triangular matrix ring $\mathbb{M}(B,D)$, D a division ring, are precisely the ideals having the property that whenever arbitrary elements of D are allowed in a non-diagonal block, then arbitrary elements of D are also allowed in at least one of its parental blocks. We wish to mention that this phenomenon, viz. that at least one of the parental blocks of a non-diagonal block imitates in some sense the non-diagonal block, also occurred in [3], where the particular version of this phenomenon was called the diagonal condition.

We first prove two preliminary theorems.

THEOREM 2.1. *Consider an ideal* $\mathscr{A}_f$ *of a blocked triangular matrix ring* $\mathbb{M}(B,D)$*, D a division ring. If* $f(\Lambda_{i,i}) = D$ *or* $f(\Lambda_{j,j}) = D$ *whenever* $f(\Lambda_{i,j}) = D$*, then* $\mathscr{A}_f$ *is a good ideal.*

PROOF. Suppose that

$$f(\Lambda_{i,i}) = D \quad \text{or} \quad f(\Lambda_{j,j}) = D \quad \text{whenever} \quad f(\Lambda_{i,j}) = D. \tag{2}$$

Let X, $Y \in \mathbb{M}(B,D)$, and let $L \in \mathscr{A}_f$. We shall show that $XY + L = (X + J)(Y + K)$ for some J, $K \in \mathscr{A}_f$, which will imply that $\mathscr{A}_f$ is a good ideal. Clearly, since we deal with cosets in the definition of a good ideal, we may assume, without loss of generality, that

$$X_{i,j} = 0 = Y_{i,j} \quad \text{whenever} \quad f(\Lambda_{i,j}) = D. \tag{3}$$

Let J be the matrix in $\mathscr{A}_f$ which differs possibly from L in having

$$J_{i,l} = \begin{cases} I_{n_i \times n_i}, & \text{if } l = i, \\ 0, & \text{if } l \neq i, \end{cases} \tag{4}$$

(5) for each i for which there is a j, $j \neq i$, such that $f(\Lambda_{i,j}) = D$ and $f(\Lambda_{j,j}) = \{0\}$.

Here $I_{n_i \times n_i}$ denotes the $n_i \times n_i$ identity matrix. Note that if i is as in (5), then (2) implies that $f(\Lambda_{i,i}) = D$, and so the first part of (4) does not violate our stipulation that J be in $\mathscr{A}_f$. Let K be the matrix in $\mathscr{A}_f$ which differs possibly from L in having

$$K_{i,j} = 0, \tag{6}$$

(7) $$K_{j,j} = I_{n_j \times n_j},$$

(8) $$K_{i,i} = I_{n_i \times n_i}, \quad \text{if } f(\Lambda_{i,i}) = D,$$

for each i for which there is a j, $j \neq i$, such that

(9) $$f(\Lambda_{i,j}) = D,$$

(10) $$\text{and } f(\Lambda_{j,j}) = D \text{ for every } j \text{ as in (9)},$$

and in having

(11) $$K_{i,i} = I_{n_i \times n_i}$$

for each i

(12) $$\text{such that } f(\Lambda_{i,j}) = \{0\} \text{ for every } j,\ j \neq i.$$

We are going to show that

(13) $$XK = 0 = JY \qquad \text{and} \qquad JK = L,$$

from which it will follow that $XY + L = (X + J)(Y + K)$.

First we show that $XK = 0$. Let i and j be such that $X_{i,j} \neq 0$. Then (3) implies that $f(\Lambda_{i,j}) = \{0\}$. Consider any k such that $B_{j,k} = 1_{n_j \times n_k}$ for some k. If $f(\Lambda_{j,k}) = \{0\}$, then $K_{j,k} = 0$, since $K \in \mathscr{A}_f$. Therefore assume that $f(\Lambda_{j,k}) = D$. Then $j \neq k$ and $f(\Lambda_{j,j}) = \{0\}$, otherwise it follows directly from the definition of $\mathscr{A}_f$ that $f(\Lambda_{i,j}) = D$. By (2) we have that $f(\Lambda_{k,k}) = D$. In fact, since $f(\Lambda_{j,j}) = \{0\}$, it follows from (2) that $f(\Lambda_{k',k'}) = D$ for every k', $k' \neq j$, such that $f(\Lambda_{j,k'}) = D$. Hence, by (6) and (9)–(10), $K_{j,k} = 0$, and so we conclude that $XK = 0$.

Second we show that $JY = 0$. Let $Y_{j,k} \neq 0$ for some j and k, and consider any i such that $B_{i,j} = 1_{n_i \times n_i}$ for some i. Then $f(\Lambda_{j,j}) = \{0\}$, otherwise $f(\Lambda_{j,k}) = D$, which contradicts (3). Similar to the proof of $XK = 0$ we assume that $f(\Lambda_{i,j}) = D$, otherwise we trivially have that $J_{i,j} = 0$. Since $f(\Lambda_{j,j}) = \{0\}$ and $f(\Lambda_{i,j}) = D$, it follows that $i \neq j$, and so (4) and (5) imply that $J_{i,j} = 0$, from which it follows that $JY = 0$.

Next we show that $JK = L$. Since J, $K \in \mathscr{A}_f$, we have that $JK \in \mathscr{A}_f$, and so if $f(\Lambda_{i,k}) = \{0\}$ for some i and k, then $(JK)_{i,k} = 0 = L_{i,k}$. Therefore we assume that i and k are such that

(14) $$f(\Lambda_{i,k}) = D.$$

Suppose first that i is as in (5), i.e. there is a j, $j \neq i$, such that $f(\Lambda_{i,j}) = D$ and $f(\Lambda_{j,j}) = \{0\}$. By (4), $J_{i,i} = I_{n_i \times n_i}$ and $J_{i,l} = 0$ if $l \neq i$. Furthermore, since i is not as in (9)–(10), we have that $K_{i,k} = L_{i,k}$. Consequently, $(JK)_{i,k} = J_{i,i} K_{i,k} = K_{i,k} = L_{i,k}$. (Note that the arguments in this paragraph are valid irrespective of whether $k = i$ or $k \neq i$ in (14).)

Second, let i be as in (9)–(10). We distinguish between the following values of k in (14): $k \neq i$ or $k = i$. Consider first the case $k \neq i$. Then by (10), $f(\Lambda_{k,k}) = D$, and by (6) and (7), $K_{i,k} = 0$ and $K_{k,k} = I_{n_k \times n_k}$. Moreover, since i is not as in (5), we have that $J_{i,l} = L_{i,l}$ for every l. We consider all j's such that $J_{i,j} \neq 0$ and $K_{j,k} \neq 0$. Suppose there is a j, with $j \neq i$ and $j \neq k$, for which $J_{i,j} \neq 0$ and $K_{j,k} \neq 0$. Since J, $K \in \mathcal{A}_f$, it follows that $f(\Lambda_{i,j}) = D$ and $f(\Lambda_{j,k}) = D$. Since we now have that $f(\Lambda_{j,k}) = D = f(\Lambda_{k,k})$, $k \neq j$ and $K_{j,k} \neq 0$, we conclude that there is a k', $k' \neq j$, such that $f(\Lambda_{j,k'}) = D$ and $f(\Lambda_{k',k'}) = \{0\}$, otherwise by (6) and (9)–(10) we would have had that $K_{j,k} = 0$. Since $f(\Lambda_{i,j}) = D = f(\Lambda_{j,k'})$, the definition of $\mathcal{A}_f$ implies that $f(\Lambda_{i,k'}) = D$. Since $f(\Lambda_{k',k'}) = \{0\}$ and $f(\Lambda_{i,k'}) = D$, it follows that $i \neq k'$. However, we have assumed that (9)–(10) holds for i, which thus contradicts the fact that k', $k' \neq i$, is such that $f(\Lambda_{i,k'}) = D$ and $f(\Lambda_{k',k'}) = \{0\}$. Therefore there is no j, with $j \neq i$ and $j \neq k$, for which $J_{i,j} \neq 0$ and $K_{j,k} \neq 0$. Consequently,

$$
\begin{aligned}
(JK)_{i,k} &= J_{i,i} K_{i,k} + \sum_{\substack{j \\ j \neq i \text{ and } j \neq k \\ i \, c_B \, j \, c_B \, k}} J_{i,j} K_{j,k} + J_{i,k} K_{k,k} = \\
&= 0 + 0 + J_{i,k} K_{k,k} = J_{i,k} = L_{i,k}.
\end{aligned}
$$

Next, let i be as in (9)–(10), and consider the case $k = i$ in (14). Then $J_{i,i} = L_{i,i}$, and by (8), $K_{i,i} = I_{n_i \times n_i}$. Moreover, the definition of $\sim_B$ implies that $(JK)_{i,i} = J_{i,i} K_{i,i}$, and so $(JK)_{i,i} = L_{i,i}$.

Finally, let i be as in (12). Since $f(\Lambda_{i,k}) = D$, we have that $i = k$. It follows as in the previous paragraph, using (11) in this case, that $(JK)_{i,i} = L_{i,i}$. This establishes (13), and hence completes the proof. ∎

THEOREM 2.2. *Consider an ideal* $\mathcal{A}_f$ *of a blocked triangular matrix ring* $\mathbb{M}(B, D)$*, D a division ring. If there are i and k for which* $f(\Lambda_{i,k}) = D$ *and* $f(\Lambda_{i,i}) = \{0\} = f(\Lambda_{k,k})$*, then* $\mathcal{A}_f \circ \mathcal{A}_f \subsetneqq \mathcal{A}_f$*, and so* $\mathcal{A}_f$ *is not a good ideal.*

PROOF. Assume that the ideal $\mathcal{A}_f$ is such that there are i and k for which

$$
f(\Lambda_{i,k}) = D \qquad \text{and} \qquad f(\Lambda_{i,i}) = \{0\} = f(\Lambda_{k,k}). \tag{15}
$$

Moreover, we may clearly assume that $\Lambda_{i,k}$ is a minimal element, with respect to set-inclusion, in the set $\{\Lambda_{i,k} : i \, c_B \, k\}$, having the property in (15). We shall show that

$$\{JK : J, K \in \mathscr{A}_f\} \subsetneqq \mathscr{A}_f. \tag{16}$$

For every

$$j, \; j \neq i \text{ and } j \neq k, \quad \text{such that} \quad f(\Lambda_{i,j}) = D = f(\Lambda_{j,k}), \tag{17}$$

we have that $\Lambda_{i,j} \subsetneqq \Lambda_{i,k}$, and so the minimality condition on $\Lambda_{i,k}$ implies that

$$f(\Lambda_{j,j} = D, \tag{18}$$

since $f(\Lambda_{i,i}) = \{0\}$. Let L be any matrix in $\mathscr{A}_f$ with

$$L_{i,k} \neq 0, \tag{19}$$

$$L_{j,j} = I_{n_j \times n_j}, \tag{20}$$

for every j in (17), and zeros elsewhere. Note that, by (18) and the first part of (15), the stipulation that L be in $\mathscr{A}_f$ is not violated.

Suppose that there are $J, K \in \mathscr{A}_f$ with $JK = L$. Let j be as in (17). First note that, since $L_{j,j} = (JK)_{j,j} = J_{j,j} K_{j,j}$, it follows from (20) that $J_{j,j}$ is invertible. Since c_B is a quasi-order relation on the set $\{1, 2, \ldots, n\}$, the set *Rep* of representatives of the equivalence classes (with respect to the equivalence relation $\sim_B$) is a partially ordered set with respect to c_B. We show now that

$$K_{j,k} = 0 \tag{21}$$

(for every j in (17)) by induction on the maximum length ℓ of the chains in the poset *Rep* which are maximal with respect to having j as minimum element and k as maximum element. If $\ell = 1$, then $j \, c_B \, k$ is the only (maximal) chain with j as minimum element and k as maximal element, and so

$$0 = L_{j,k} = J_{j,j} K_{j,k} + J_{j,k} K_{k,k}. \tag{22}$$

Since $f(\Lambda_{k,k}) = \{0\}$ and $K \in \mathscr{A}_f$, it follows that $K_{k,k} = 0$, and so (22) implies that $J_{j,j} K_{j,k} = 0$. The invertibility of $J_{j,j}$ then forces $K_{j,k}$ to be 0 too. Suppose now that $\ell \geq 2$, and that $K_{j',k} = 0$ for every j', $j' \neq i$ and $j' \neq k$, such $f(\Lambda_{i,j'}) = D = f(\Lambda_{j',k})$ and the maximum length of the chains in the poset *Rep* which are maximal with respect to having j' as minimum element and k as maximum element, is less than ℓ. Then

$$0 = L_{j,k} = J_{j,j} K_{j,k} + \sum_{\substack{j' \\ j' \neq j \text{ and } j' \neq k \\ j \, c_B \, j' \, c_B \, k}} J_{j,j'} K_{j',k} + J_{j,k} K_{k,k}. \tag{23}$$

The j''s in (23) are as in the induction hypothesis, and so $K_{j',k} = 0$ for all such j''s. Furthermore, we have already seen that $K_{k,k} = 0$. Therefore, by (23), $J_{j,j} K_{j,k} = 0$, and again the invertibility of $J_{j,j}$ forces $K_{j,k}$ to be 0. This proves (21).

We now have that

$$L_{i,k} = J_{i,i} K_{i,k} + \sum_{\substack{j \\ j \neq i \text{ and } j \neq k \\ i \subset_B j \subset_B k}} J_{i,j} K_{j,k}. \tag{24}$$

Consider the j's in (24). The induction proof in the previous paragraph shows that $K_{j,k} = 0$ for every such j, and so $L_{i,k} = J_{i,i} K_{i,k}$. Since $f(\Lambda_{i,i}) = \{0\}$ and $J \in \mathscr{A}_f$, it follows that $J_{i,i} = 0$, and so $L_{i,k} = 0$. However, this contradicts (19), and we conclude that there are no J, $K \in \mathscr{A}_f$ with $JK = L$. This establishes (16). ∎

Combining Theorems 2.1 and 2.2 we obtain the following two results

COROLLARY 2.3. *The condition $\mathscr{A}_f \circ \mathscr{A}_f = \mathscr{A}_f$ is sufficient for an ideal $\mathscr{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B, D)$ over a division ring D to be a good ideal.*

COROLLARY 2.4. *An ideal $\mathscr{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B, D)$ over a division ring D is good if and only if $f(\Lambda_{i,i}) = D$ or $f(\Lambda_{j,j}) =$ $= D$ whenever $f(\Lambda_{i,j}) = D$.*

It was shown in [4, Lemma 1] that a finitely generated right ideal in a right duo ring is a good ideal if and only if it is a principal (right) ideal generated by a central idempotent. In [4, Example 3] the ideal $\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ of the 2×2 upper triangular matrix ring over a field F was exhibited as an example of a good ideal (in a ring which is not right duo) which as a right ideal is not generated by an idempotent. However, this ideal is a principal left ideal generated by the idempotent $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We now give an example of a good ideal in a blocked triangular matrix ring which is not generated as a right ideal nor as a left ideal by an idempotent.

EXAMPLE 2.5. Consider the ideal $\mathcal{A}_f = \begin{bmatrix} D & D & 0 & D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & D \end{bmatrix}$ of the blocked triangular matrix ring $\mathbb{M}(B,D) := \begin{bmatrix} D & D & 0 & D \\ 0 & D & 0 & 0 \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{bmatrix}$, D a division ring. The only idempotents in $\mathcal{A}_f$ which have non-zero entries in positions $(1,1)$ and $(4,4)$ are of the form $\begin{bmatrix} 1 & r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$, r, $s \in D$, and so it is easily seen that $\mathcal{A}_f$ has the desired properties. Note however that as an ideal (i.e. as a two-sided ideal) $\mathcal{A}_f$ is generated by, for example, the idempotent $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

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THE DUAL OF THE VMO SPACE IN THE NON-REGULAR CASE

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1. Introduction

It is well-known in the martingale theory that the dual of the martingale Hardy space is the *BMO* space (functions of bounded mean oscillation), however the dual of the *BMO* space is *not* the Hardy space (see GARSIA [6], NEVEU [7], WEISZ [12]).

In many investigations a subspace of *BMO*, the so called *VMO* space (functions of vanishing mean oscillation) plays an important role. The dual of *VMO* is known in the case when the terms of the stochastic basis are generated by finite partitions of the basis probability space. In this case the dual of *VMO* is the Hardy space (see SCHIPP [10], WEISZ [12]).

In this paper we shall deal with the dual of *VMO* in the general case. It can be given as a factor space of a space consisting of some sequences of ℓ^2-valued vector measures. The construction is based on the theory of $\mathscr{B}$-orthogonality (SCHIPP [9]) and on some theorems about mixed norm spaces (CSÖRGŐ [4]).

In the last section we apply our general result for the finitely generated case, and we give a new proof for the above mentioned result: the dual of *VMO* is the Hardy space.

2. Definitions and preliminaries

Let $(\Omega,\mathscr{A},P)$ be a probability space, $\mathscr{B}\subset\mathscr{A}$ be a sub-sigma-field, and $E_{\mathscr{B}}: L^p \to L^p(\mathscr{B})$ be the conditional expectation operator (see NEVEU [7]). Let us recall some definitions and statements from the theory of $\mathcal{B}$-orthogonality (see SCHIPP [9]).

DEFINITION. Let $\Phi_i = \{\varphi \in L^2 : i \in I\}$ be a (not necessarily countable) function system. Φ is called

(i) $\mathcal{B}$-orthonormal ($\mathcal{B}$-*ONS*) if $E_{\mathcal{B}}(\varphi_i \overline{\varphi}_j) = 0$ for every $i, j \in I$, $i \neq j$ and there exists a set family $A_i \in \mathcal{B}$, $P(A_i) > 0$ $(i \in I)$ such that $E_{\mathcal{B}}|\varphi_i|^2 = 1_{A_i}$ $(i \in I)$. Here 1_{A_i} denotes the characteristic function of A_i.

(ii) $\mathcal{B}$-complete if $f \in L^2$ and $E_{\mathcal{B}}(f\overline{\varphi}_i) = 0$ $(i \in I)$ imply $f = 0$.

We note that $A_i = \operatorname{supp} E_{\mathcal{B}}|\varphi_i|^2 \supset \operatorname{supp}\varphi_i$ and that any system Φ can be made $\mathcal{B}$-normal by multiplication φ_i by $(E_{\mathcal{B}}|\varphi_i|^2)^{-1/2}$ on A_i and by 0 on $\Omega \setminus A_i$. One can easily prove—using Zorn's lemma—that there exists a $\mathcal{B}$-complete $\mathcal{B}$-orthonormal system in L^2.

The functions $E_{\mathcal{B}}(f\overline{\varphi}_i)$ $(i \in I)$ are called $\mathcal{B}$-Fourier coefficients of f, and are often denoted by $\hat{f}_i$. It is clear that $A_i \supset \operatorname{supp}\hat{f}_i$.

THEOREM A. (Bessel-identity and minimum property, SCHIPP [9]) *Let $\Phi = \{\varphi_i \in L^2 : i \in I\}$ be a $\mathcal{B}$-ONS in L^2, I_0 be a finite subset of I, $f \in L^2$. Then*

$$\inf\left\{ E_{\mathcal{B}}\left| f - \sum_{i\in I_0} \lambda_i \cdot \varphi_i \right|^2 \, : \, \lambda_i \in L^2(\mathcal{B}) \right\} =$$

$$= E_{\mathcal{B}}\left| f - \sum_{i\in I_0} \hat{f}_i \cdot \varphi_i \right|^2 = E_{\mathcal{B}}|f|^2 - \sum_{i\in I_0} |\hat{f}_i|^2.$$

THEOREM B. (Fourier-expansion and Conditional Parseval-formula, SCHIPP [9]) *Let $\Phi = \{\varphi_i \in L^2 : i \in I\}$ be a $\mathcal{B}$-complete $\mathcal{B}$-ONS in L^2. Then for each $f \in L^2$ the index set*

$$I_f = \{i \in I : \hat{f}_i \neq 0\}$$

is at most countably infinite ($I_f = \{i_k : k \in \mathbf{N}\}$, here $\mathbf{N}$ denotes the set of nonnegative integers) and

$$\lim_{N\to\infty} E_{\mathcal{B}}\left| f - \sum_{k=0}^{N} \hat{f}_{i_k} \cdot \varphi_{i_k} \right|^2 = 0$$

and

$$\sum_{i\in I}|\hat{f}_i|^2=\sum_{k=0}^{\infty}|\hat{f}_{i_k}|^2=E_{\mathcal{B}}|f|^2.$$

THEOREM C. (Conditional Riesz–Fischer Theorem, SCHIPP [9]) *Let* $\Phi = \{\varphi_i \in L^2 : i \in I\}$ *be a* $\mathcal{B}$*-ONS in* L^2 *and* $\lambda_i \in L^2(\mathcal{B})$ $(i \in I)$ *be a function system with*

(i) $\operatorname{supp}\lambda_i \subset A_i$ *and*

(ii) $\sum_{i\in I}\int_{A_i}|\lambda_i|^2 dP<\infty.$

Then there exists a function $f \in L^2$ *such that* $\hat{f}_i = \lambda_i$ $(i \in I)$.

Let I be a nonempty index set, and let us introduce the vector space of all $\ell^2(I)$-valued, $\mathcal{A}$-measurable and essentially bounded functions, the space $L^\infty(\Omega,\mathcal{A},P;\ell^2(I))$ or shortly $L^\infty(\mathcal{A};\ell^2(I))$ and the vector space of all $\ell^2(I)$-valued $\mathcal{A}$-measurable and P-integrable functions, the space $L^1(\Omega,\mathcal{A},P;\ell^2(I))$ or shortly $L^1(\mathcal{A};\ell^2(I))$ (cf. DIESTEL and UHL [5], IV.1.). These are Banach spaces under the norms

$$\|f\|_{L^\infty(\mathcal{A};\ell^2(I))}:=\sup_{\omega\in\Omega}\|f(\omega)\|_{\ell^2(I)}=\sup_{\omega\in\Omega}\left(\sum_{i\in I}|f_i(\omega)|^2\right)^{1/2}$$

and

$$\|f\|_{L^1(\mathcal{A};\ell^2(I))}:=\int_\Omega\|f(\omega)\|_{\ell^2(I)}dP(\omega)=\int_\Omega\left(\sum_{i\in I}|f_i(\omega)|^2\right)^{1/2}dP(\omega),$$

respectively. Let $L_0^\infty(\Omega,\mathcal{A},P;\ell^2(I))$, or shortly $L_0^\infty(\mathcal{A};\ell^2(I))$ be the closure of the subspace of the $\ell^2(I)$-valued $\mathcal{A}$-simple functions in $L^\infty(\mathcal{A};\ell^2(I))$ (cf. CSÖRGŐ [2]).

As in CSÖRGŐ [2] is pointed out, the dual space of $L_0^\infty(\mathcal{A};\ell^2(I))$ is $BA(\mathcal{A};\ell^2(I))$, the space of finitely additive, absolutely continuous $\ell^2(I)$-valued set functions (vector measures, see e.g. DIESTEL and UHL [5]) with finite variation, defined on $\mathcal{A}$. The norm on the space $BA(\mathcal{A};\ell^2(I))$ is the total variation:

$$\|\mu\|_{BA} := \sup\left\{\sum_{j=1}^{m} \|\mu(B_j)\|_{\ell^2(I)} \ : \ \{B_j\} \in \mathcal{P}\right\},$$

where $\mathcal{P}$ denotes the set of $\mathcal{A}$-measurable finite partitions of Ω. Remark that if $\mathcal{A}$ is generated by a finite partition of Ω, then $L_0^\infty(\mathcal{A};\ell^2(I))=L^\infty(\mathcal{A};\ell^2(I))$, so the dual of $L^\infty(\mathcal{A};\ell^2(I))$ is $BA(\mathcal{A};\ell^2(I))$, that is isomorphic to L^1 as one can see in CSÖRGŐ [4].

Let us fix the numbers $1\le p<\infty$, $1\le q\le\infty$, and define the quantity

$$\|f\|_{(\mathcal{B},p,q)} = \|E_{\mathcal{B}}|f|^p)^{1/p}\|_{L^q} \tag{1}$$

for any $\mathcal{A}$-measurable function $f:\Omega\to\mathbf{K}$ ($\mathbf{K}$ denotes the real of the complex number field.) Using the conditional Hölder's inequality (see NEVEU [7]) one can easily verify that (1) is a norm on the vector space

$$L_{(\mathcal{B},p,q)} = \{f \ : \ \Omega\to\mathbf{K} \ : \ f \text{ is } \mathcal{A}\text{-measurable}, \|f\|_{(\mathcal{B},p,q)} < \infty\}.$$

Let $\{\varphi_i \ : \ i\in I\}$ be a (not necessarily countable) $\mathcal{B}$-complete $\mathcal{B}$-*ONS* in L^2, and suppose that the corresponding sets A_i are equal to Ω ($i\in I$). Then the spaces $L_{(\mathcal{B},2,\infty)}$ and $L^\infty(\mathcal{B};\ell^2(I))$ are isometrically isomorphic by the identification $f\mapsto(\hat f_i, i\in I)=(E_{\mathcal{B}}(f\overline{\varphi}_i), i\in I)$. The subspace in $L_{(\mathcal{B},2,\infty)}$ corresponding to $L_0^\infty(\mathcal{B};\ell^2(I))$ by this identification is denoted by $L_{(\mathcal{B},2,\infty),0}$ and consists of all functions f for which

$$\lim_{N\to\infty}\left\|\left(E_{\mathcal{B}}\left|f-\sum_{k=0}^{N}\hat f_{i_k}\cdot\varphi_{i_k}\right|^2\right)^{1/2}\right\|_{L^\infty} = \lim_{N\to\infty}\left\|f-\sum_{k=0}^{N}\hat f_{i_k}\cdot\varphi_{i_k}\right\|_{(\mathcal{B},2,\infty)} = 0.$$

So the dual of $L_{(\mathcal{B},2,\infty),0}$ is $BA(\mathcal{B};\ell^2(I))$. Remark that if $\mathcal{B}$ is finitely generated, then $L_{(\mathcal{B},2,\infty),0}=L_{(\mathcal{B},2,\infty)}$, moreover it can be proved without the assumption $A_i=\Omega$ that the dual of $L_{(\mathcal{B},2,\infty)}$ is $L_{(\mathcal{B},2,1)}$ (see CSÖRGŐ [4]).

3. Hardy and *BMO* spaces

Let $(\Omega,\mathcal{A},P)$ be a probability space, $\mathcal{B}_0\subset\mathcal{B}_1\subset\ldots\subset\mathcal{A}$ be a monotone increasing sequence of sub-σ-fields, $\mathcal{B}_{-1}:=\mathcal{B}_0:=\{\emptyset,\Omega\}$, $\sigma\left(\bigcup_{n=0}^{\infty}\mathcal{B}_n\right)=\mathcal{A}$.

Denote by E_n the conditional expectation operator with respect to $\mathcal{B}_n$. Let us introduce the following spaces:

The Hardy spaces H_1 *and* $\mathcal{H}_1$ consists of all $f \in L^1$ for which

$$\|f\|_{H_1} := \left\| \sup_{n \in \mathbf{N}} |E_n f| \right\|_{L^1} < \infty$$

and

$$\|f\|_{\mathcal{H}_1} := \left\| \left(\sum_{n=1}^{\infty} E_{n-1} |E_n f - E_{n-1} f|^2 \right)^{1/2} \right\|_{L^1} < \infty,$$

respectively.

Remark that $L^p \subset H_1 \subset L^1$ for all $p > 1$ and that $L^2 \subset \mathcal{H}_1 \subset H_1 \subset L^1$ (see WEISZ [12]).

The *BMO* spaces are the duals of the Hardy spaces. *The spaces* BMO_2^- *and* BMO_2 consist of all $h \in L^1$ for which

$$\|h\|_{BMO_2^-} := \sup_{n \in \mathbf{N}} \left\| \left(E_n\, |h - E_{n-1} h|^2 \right)^{1/2} \right\|_{L^\infty} < \infty$$

and

$$\|h\|_{BMO_2} := \sup_{n \in \mathbf{N}} \left\| \left(E_n\, |h - E_n h|^2 \right)^{1/2} \right\|_{L^\infty} < \infty,$$

respectively.

Remark that $L^\infty \subset BMO_2^- \subset L^p$ for all $1 \le p < \infty$ and that $L^\infty \subset BMO_2 \subset$ $\subset L^2$. Moreover, the dual of H_1 is BMO_2^- and the dual of $\mathcal{H}_1$ is BMO_2 (see WEISS [12]).

The duals of the above defined *BMO* spaces are not the Hardy spaces (H_1 and $\mathcal{H}_1$ are not reflexive), but there are some subspaces of *BMO*, the so called *VMO* spaces whose duals in certain cases are the Hardy spaces. Let us denote by S the subspace of $\bigcup_{n=0}^{\infty} \mathcal{B}_n$-simple functions.

The spaces VMO_2^- *and* VMO_2 are defined as the closure of S in BMO_2^- norm and in BMO_2 norm, respectively. Remark that

(i) For every $h \in VMO_2^-$: $\lim_{n \to \infty} \|E_n |h - E_{n-1} h|^2)^{1/2}\|_{L^\infty} = 0$, and

(ii) For every $h \in VMO_2$: $\lim_{n\to\infty} \|E_n|h - E_n h|^2)^{1/2}\|_{L^\infty} = 0$.

In the finitely generated case (when every $\mathcal{B}_n$ is generated by a finite partition of Ω), the converse is also true. Moreover, in the finitely generated case the dual of VMO_2^- is H_1, and the dual of VMO_2 is $\mathcal{H}_1$ [see SCHIPP [10] and WEISZ [12]).

4. The dual of VMO_2^- and VMO_2

In this section two assumptions are supposed.

(i) For all $n \in \mathbf{N}$ $(\varphi_i^n,\ i \in I_n)$ is a $\mathcal{B}_n$-complete $\mathcal{B}_n$-*ONS* in L^2 such that the corresponding sets A_i $(= A_i^n)$ are equal to Ω.

(ii) $1_A - E_{n-1}(1_A) \in L_{(\mathcal{B}_n,2,\infty),0} \quad (n \in \mathbf{N},\ A \in \bigcup_{n=0}^{\infty} \mathcal{B}_n)$.

Remark that the second assumption would be true if we wrote $L_{(\mathcal{B}_n,2,\infty)}$ instead of $L_{(\mathcal{B}_n,2,\infty),0}$, because of

$$\|(E_n|1_A - E_{n-1}(1_A)|^2)^{1/2}\|_{L^\infty} \le \|1_A\|_{BMO_2^-} < \infty.$$

Furthermore, (ii) holds for $n \ge n_0+1$, where n_0 is an index for which $A \in \mathcal{B}_{n_0}$, because of

$$1_A - E_{n-1}(1_A) = 1_A - 1_A = 0 \in L_{(\mathcal{B}_n,2,\infty),0}.$$

To describe the dual of VMO_2^-, it is necessary to introduce the Banach-space

$$Y^1 = \left\{ \mu = (\mu_n, n \in \mathbf{N}) \ : \ \mu_n \in BA(\mathcal{B}_n; \ell^2(I_n)),\ \sum_{n=0}^{\infty} \|\mu_n\|_{BA} < \infty \right\}$$

with the norm $\|\mu\|_{Y^1} = \sum_{n=0}^{\infty} \|\mu_n\|_{BA}$. Define the equivalence relation $\mu \sim \nu$ on Y^1 as follows: $\mu \sim \nu$ if and only if

$$\sum_{n=0}^{\infty} \int_{\Omega} (E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n}),\ i \in I_n) d\mu_n =$$

$$= \sum_{n=0}^{\infty} \int_{\Omega} (E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n}),\ i \in I_n) d\nu_n$$

holds for any $h \in VMO_2^-$. It is well-known that the factor space $Y^1/\sim$ endowed with the factor norm

$$\|[\mu]\|_{Y^1/\sim} = \inf_{\nu\in[\mu]} \|\nu\|_{Y^1}$$

is a Banach-space.

THEOREM 1. *The dual of VMO_2^- is $Y^1/\sim$. The functionals of VMO_2^- can be given as*

$$(2)\quad F(h) = \sum_{n=0}^{\infty} \int_{\Omega} (E_n((h - E_{n-1}h)\cdot \overline{\varphi_i^n}),\ i \in I_n) d\mu_n$$

$$(h \in VMO_2^-,\ \mu = (\mu_n) \in Y^1).$$

PROOF. Let $X_n = L_{(\mathcal{B}_n,2,\infty),0}$ (endowed with the $\|\ .\ \|_{(\mathcal{B}_n,2,\infty)}$-norm) $(n \in \mathbf{N})$ and

$$X^0 = \left\{ g = (g_n, n \in \mathbf{N})\ :\ g_n \in X_n\ (n \in \mathbf{N}),\ \lim_{n\to\infty} \|g_n\|_{(\mathcal{B}_n,2,\infty)} = 0 \right\}$$

with the norm

$$\|g\|_{X^0} = \sup_{n\in\mathbf{N}} \|g_n\|_{(\mathcal{B}_n,2,\infty)} \qquad (g \in X^0).$$

First we embed VMO_2^- into X^0 by the linear isometry

$$\Phi h = (h - E_{n-1}h,\ n \in \mathbf{N}) \qquad (h \in VMO_2^-).$$

Let us fix $n \in \mathbf{N}$ and $h \in VMO_2^-$. From the estimation

$$\|(\Phi h)_n\|_{(\mathcal{B}_n,2,\infty)} = \|h - E_{n-1}h\|_{(\mathcal{B}_n,2,\infty)} =$$
$$= \|(E_n|h - E_{n-1}h|^2)^{1/2}\|_{L^\infty} \le \|h\|_{BMO_2^-} < \infty$$

it follows that $(\Phi h)_n \in L_{(\mathcal{B}_n,2,\infty)}$. To see $(\Phi h)_n \in L_{(\mathcal{B}_n,2,\infty),0}$ we need to show that for any $\varepsilon > 0$ there exists a finite index set $I_0 \subset I_n$ such that

$$(3)\qquad \left\| (\Phi h)_n - \sum_{i\in I_0} E_n\left((\Phi h)_n \cdot \overline{\varphi_i^n}\right) \cdot \varphi_i^n \right\|_{(\mathcal{B}_n,2,\infty)} < \varepsilon.$$

Since h can be approximated by $\bigcup_{n=0}^{\infty} \mathcal{B}_n$-step functions in BMO_2^- norm, there exists a step function $\psi \in S$ with the property

$$\|(\Phi h)_n - (\Phi\psi)_n\|_{(\mathcal{B}_n,2,\infty)} = \left\| \left(E_n|h - E_{n-1}h - \psi + E_{n-1}\psi|^2 \right)^{1/2} \right\|_{L^\infty} =$$

$$= \|E_n|(h-\psi) - E_{n-1}(h-\psi)|^2)^{1/2}\|_{L^\infty} = \|h-\psi\|_{BMO_2^-} < \varepsilon/2.$$

It is obvious by assumption (ii) that $(\Phi\psi)_n \in L_{(\mathcal{B}_n,2,\infty),0}$. So there exists a finite index set $I_0 \subset I_n$ such that

$$\left\| (\Phi\psi)_n - \sum_{i\in I_0} E_n((\Phi\psi)_n \cdot \overline{\varphi_i^n}) \right\|_{(\mathcal{B}_n,2,\infty)} < \varepsilon/2.$$

Applying the triangle inequality we obtain

$$\left\| (\Phi h)_n - \sum_{i\in I_0} E_n((\Phi\psi)_n \cdot \overline{\varphi_i^n}) \cdot \varphi_i^n \right\|_{(\mathcal{B}_n,2,\infty)} < \varepsilon. \tag{4}$$

Since the functions $E_n((\Phi\psi)_n \cdot \overline{\varphi_i^n})$ $(i \in I_n)$ are in $L^2(\mathcal{B}_n)$, one can use the minimum property of the $\mathcal{B}_n$-Fourier coefficients (Theorem A):

$$E_n \left| (\Phi h)_n - \sum_{i\in I_0} E_n((\Phi h)_n \cdot \overline{\varphi_i^n}) \cdot \varphi_i^n \right|^2 \le$$

$$\le E_n \left| (\Phi h)_n - \sum_{i\in I_0} E_n((\Phi\psi)_n \cdot \overline{\varphi_i^n}) \cdot \varphi_i^n \right|^2 .$$

After taking square root and L^∞-norm and using (4) we obtain (3). So $(\Phi h)_n \in L_{(\mathcal{B}_n,2,\infty),0}$. Since (see Section 3)

$$\lim_{n\to\infty} \|(\Phi h)_n\|_{(\mathcal{B},2,\infty)} = \lim_{n\to\infty} \|E_n|h - E_{n-1}h|^2)^{1/2}\|_{L^\infty} = 0,$$

so $\Phi h \in X^0$. The linearity is obvious. The isometry of Φ can be obtained from the equality

$$\|\Phi h\|_{X^0} = \sup_{n\in N} \|(E_n|h - E_{n-1}h|^2)^{1/2}\|_{L^\infty} = \|h\|_{BMO_2^-}.$$

Denote by M the range of Φ. It is a closed subspace of the Banach-space X^0, and Φ is an isometric isomorphism between VMO_2^- and M.

Referring to the well-known theorem about the dual of X^0 (see SCHIPP [10]) and to the theorem about the dual of $L_{(\mathcal{B}_n,2,\infty),0}$ (see CSÖRGŐ [4]) we can state that the dual of X^0 is Y^1 by the identification

$$F_1(g) = \sum_{n=0}^{\infty} \int_{\Omega} (E_n(g_n \cdot \varphi_i^n),\ i \in I_n) d\mu_n) \qquad (g \in X^0,\ \mu \in Y^1).$$

Finally, referring to the well-known theorem about the dual of a closed subspace of a Banach-space (see e.g. RUDIN [8]) we obtain that the dual of M is $Y^1/M^\perp$, where $M^\perp$ denotes the closed subspace of Y^1 consisting of functionals vanishing of M. One can easily see that $Y^1/M^\perp = Y^1/\sim$, and that the functionals have the form (2) indeed. ∎

An analogous theorem can be stated for VMO_2. The assumption (i) is unchanged, (ii) is modified to

(ii′) $$1_A - E_n(1_A) \in L_{(\mathcal{B}_n,2,\infty),0} \qquad (n \in \mathbf{N},\ A \in \bigcup_{n=0}^{\infty} \mathcal{B}_n).$$

Let us define the equivalence relation $\approx$ on Y^1 as follows: $\mu \approx \nu$ if and only if

$$\sum_{n=0}^{\infty} \int_{\Omega} (E_n((h-E_n h)\cdot\overline{\varphi_i^n}),\ i \in I_n) d\mu_n = \sum_{n=0}^{\infty} \int_{\Omega} (E_n((h-E_n h)\cdot\overline{\varphi_i^n}),\ i \in I_n) d\nu_n$$

holds for any $h \in VMO_2$.

THEOREM 2. *The dual of VMO_2 is the factor space $Y^1/\approx$. The functionals of VMO_2 can be given as*

$$F(h) = \sum_{n=0}^{\infty} \int_{\Omega} (E_n((h - E_n h) \cdot \overline{\varphi_i^n}),\ i \in I_n) d\mu_n$$

$$(h \in VMO_2,\ \mu = (\mu_n) \in Y^1).$$

5. Application for the finitely generated case

In this section we apply the Fourier-method to prove that the dual of VMO_2^- is H_1 and that the dual of VMO_2 is $\mathcal{H}_1$, in both cases provided that each $\mathcal{B}_n$ is generated by a finite partition $\{B_1^n,\dots,B_{m_n}^n\}$ of Ω. Here $P(B_j^n) > 0$ $(j = 1,\dots,m_n)$. (For another proof see SCHIPP [10] and WEISZ [12].) There are no more assumptions, since in the finitely generated case $L_{(\mathcal{B}_n,2,\infty),0} = L_{(\mathcal{B}_n,2,\infty)}$, so the assumption (ii) of the previous section is satisfied, and since the other assumption ($A_i^n = \Omega$) will not be used in the proof.

THEOREM 3. *In the finitely generated case the dual of VMO_2^- is H_1. The functionals of VMO_2^- can be given on the dense subspace S by*

$$F(h) = \int_\Omega h \cdot \overline{f}\, dP \qquad (h \in S,\ f \in H_1).$$

Furthermore, the functional-norm and the H_1-norm are equivalent:

$$\frac{1}{3}\|f\|_{H_1} \le \|F\| \le C \cdot \|f\|_{H_1}. \tag{5}$$

Here C is a positive constant.

PROOF. For every $n \in \mathbf{N}$ let $\{\varphi_i^n \in L^2 : i \in I_n\}$ be a $\mathcal{B}_n$-complete $\mathcal{B}_n$-ONS with the corresponding sets A_i^n, that are not necessarily equal to Ω. Using the construction in Theorem 1 and the results of CSÖRGŐ [4] we obtain that the functionals of VMO_2^- can be written in the form

$$F(h) = \sum_{n=0}^{\infty} \int_\Omega \lambda_n \, d\nu_n \qquad (h \in S), \tag{6}$$

where $\lambda_n = (E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n}),\ i \in I_n)$ is in $L^\infty(\mathcal{B}_n; \ell^2(I_n))$ and $\nu_n \in$ $\in BA(\mathcal{B}_n; \ell^2(I_n))$. They have the properties $\operatorname{supp}\lambda_{n,i} \subset A_i^n$ $(i \in I_n)$, $\lim_{n\to\infty} \|\lambda_n\|_{L^\infty(\mathcal{B}_n;\ell^2(I_n))} = 0$, $\nu_{n,i}(B_j^n) = 0$ if $B_j^n \cap A_i^n = \emptyset$ and $\sum_{n=0}^{\infty} \|\nu_n\|_{BA} < \infty$.

The integrals in (6) can be transformed into Lebesgue-integrals, namely, let

$$g_n = \sum_{j=1}^{m_n} \frac{\overline{\nu_n(B_j^n)}}{P(B_j^n)} \cdot 1_{B_j^n}.$$

This function is in $L^1(\mathcal{B}_n; \ell^2(I_n))$, $\operatorname{supp} g_{n,i} \subset A_i^n$ $(i \in I_n)$ (cf. CSÖRGŐ [4]) and

$$\int_\Omega \lambda_n d\nu_n = \int_\Omega \sum_{i \in I_n} \lambda_{n,i} \cdot \overline{g_{n,i}} dP.$$

The functions $g_{n,i}$ can be regarded as $\mathcal{B}_n$-Fourier coefficients of a function $f_n \in L_{(\mathcal{B}_n,2,1)}$ (cf. Theorem C and CSÖRGŐ [4]):

$$g_{n,i} = E_n(f_n \cdot \overline{\varphi_i^n}) \qquad (i \in I_n,\ n \in \mathbf{N}).$$

So we can rewrite (6) in the form

$$F(h) = \sum_{n=0}^{\infty} \int_\Omega \sum_{i \in I_n} E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n}) \cdot \overline{E_n(f_n \cdot \overline{\varphi_i^n})} dP. \tag{7}$$

The order of the sum by i and the integration can be interchanged by Lebesgue's theorem and by the estimation

$$\int_\Omega \sum_{i \in I_n} |E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n})| \cdot |\overline{E_n(f_n \cdot \overline{\varphi_i^n})}| dP \le$$

$$\le \int_\Omega \left(\sum_{i \in I_n} |E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n})|^2 \right)^{1/2} \cdot \left(\sum_{i \in I_n} |\overline{E_n(f_n \cdot \overline{\varphi_i^n})}|^2 \right)^{1/2} dP =$$

$$= \int_\Omega (E_n |h - E_{n-1}h|^2)^{1/2} \cdot \left(\sum_{i \in I_n} |E_n(f_n \cdot \overline{\varphi_i^n})|^2 \right)^{1/2} dP \le$$

$$\le \|h - E_{n-1}h\|_{(\mathcal{B}_n,2,\infty)} \cdot \|g_n\|_{L^1(\mathcal{B}_n; \ell^2(I_n))} < \infty.$$

We have

$$F(h) = \sum_{n=0}^{\infty} \sum_{i \in I_n} \int_\Omega E_n((h - E_{n-1}h) \cdot \overline{\varphi_i^n}) \cdot \overline{E_n(f_n \cdot \overline{\varphi_i^n})} dP. \tag{8}$$

The next considerations show that the n-th term of (8) is equal to

$$\int_{\Omega}(h-E_{n-1}h)\cdot\overline{f}_n\,dP. \tag{9}$$

Let $\{i_k : k\in\mathbf{N}\}$ be the countable subset of I_n (n is fixed) outside of which the terms in the inner sum of (8) are equal to 0 (cf. CSÖRGŐ [4]). For any $N\in\mathbf{N}$ we have

$$\left|\int_{\Omega}(h-E_{n-1}h)\cdot\overline{f}_n\,dP-\sum_{k=0}^{N}\int_{\Omega}E_n((h-E_{n-1}h)\cdot\overline{\varphi_{i_k}^n})\cdot\overline{E_n(f_n\cdot\overline{\varphi_{i_k}^n})}dP\right|=$$

$$=\left|\int_{\Omega}(h-E_{n-1}h)\cdot\overline{\left(f_n-\sum_{k=0}^{N}E_n(f_n\cdot\overline{\varphi_{i_k}^n})\cdot\varphi_{i_k}^n\right)}\,dP\right|\le$$

$$\le\int_{\Omega}(E_n|h-E_{n-1}h|^2)^{1/2}\cdot\left(E_n\left|f_n-\sum_{k=0}^{N}E_n(f_n\cdot\overline{\varphi_{i_k}^n})\cdot\varphi_{i_k}^n\right|^2\right)^{1/2}dP\le$$

$$\le\|h\|_{BMO_2^-}\cdot\int_{\Omega}\left(E_n\left|f_n-\sum_{k=0}^{N}E_n(f_n\cdot\overline{\varphi_{i_k}^n})\cdot\varphi_{i_k}^n\right|^2\right)^{1/2}dP. \tag{10}$$

The integrand in (10) tends to zero if $N\to\infty$ (see Theorem B) and it has the integrable majorant $E_n|f_n|^2$ (see Theorem A). So by Lebesgue's theorem (10) tends to zero if $N\to\infty$, which gives us (9).

Using this result one can write (7) in form

$$F(h)=\sum_{n=0}^{\infty}\int_{\Omega}(h-E_{n-1}h)\cdot\overline{f}_n\,dP=\sum_{n=0}^{\infty}\int_{\Omega}h\cdot\overline{(f_n-E_{n-1}f_n)}dP\qquad(h\in S).$$

Since the series $\sum_{n=0}^{\infty}(f_n-E_{n-1}f_n)$ converges in L^1-norm to a function $f\in L^1$ (see WEISZ [12]) and h is a step function, we can interchange the sum and the integral:

$$F(h)=\int_{\Omega}h\cdot\overline{\sum_{n=0}^{\infty}(f_n-E_{n-1}f_n)}dP=\int_{\Omega}h\cdot\overline{f}\,dP\qquad(h\in S).$$

So the functionals have the desired form. On the other hand, for the norm of the functional we can write (cf. Section 4 and CSÖRGŐ [4])

$$\|F\| = \sum_{n=0}^{\infty} \|\nu_n\|_{BA(\mathcal{B}_n;\ell^2(I_n))} = \sum_{n=0}^{\infty} \|g_n\|_{L^1(\mathcal{B}_n;\ell^2(I_n))} = \sum_{n=0}^{\infty} \|f_n\|_{(\mathcal{B}_n,2,1,)}.$$

From now on we refer to the usual way (see e.g. WEISZ [12], p. 50) to prove (5).

$$\|f\|_{H_1} \leq 3 \cdot \sum_{n=0}^{\infty} \|(E_n|f_n|^2)^{1/2}\|_{L_1} = 3 \cdot \sum_{n=0}^{\infty} \|f_n\|_{(\mathcal{B}_n,2,1)} = 3 \cdot \|F\| < \infty,$$

hence $f \in H_1$. Moreover,—by Feffermann's inequality (see GARSIA, [6])— there exists a constant $C > 0$ such that

$$|F(h)| \leq C \cdot \|f\|_{H_1} \cdot \|h\|_{BMO_2^-},$$

consequently

$$\|F\| \leq C \cdot \|f\|_{H_1}. \qquad \blacksquare$$

Using similar constructions an analogous theorem can be proved for VMO_2.

THEOREM 4. *In the finitely generated case the dual of* VMO_2 *is* $\mathcal{H}_1$. *The functionals of* VMO_2 *can be given on the dense subspace* S *by*

$$F(h) = \int_\Omega h \cdot \overline{f} \, dP \qquad (h \in S,\ f \in \mathcal{H}_1).$$

Furthermore, the functional norm and the $\mathcal{H}_1$*-norm are equivalent:*

$$\|f\|_{\mathcal{H}_1} \leq \|F\| \leq C \cdot \|f\|_{\mathcal{H}_1},$$

where C *is a positive constant.*

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SCALAR DERIVATIVES AND CONFORMITY

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0. Introduction

The Minty–Browder monotonicity notion [2] was an important achievement of the nonlinear analysis. It throws a new light on many new results and becomes a good instrument in various application oriented new investigations [6]. Monotonicity characterizes the global behavior of an operator. Looking for a simply local notion we introduced in [3] the so-called scalar derivative.

We are going to relate the existence of the scalar derivative with the notion of conformity.

We shall also relate the nonexpansive (expansive) maps with the decreasing (increasing) ones [3], giving this way a generalization of the well known Lie correspondence between antisymmetric maps and isometries. Only the finite dimensional case will be considered, which is enough to illustrate the principal ideas.

Let $\mathbf{R}^n$ be the n dimensional real Euclidean space with $\langle\ .\ ,\ .\ \rangle$ the scalar product and the norm of x defined by $\|x\|^2 = \langle x, x\rangle$. Consider the operator $f : \mathbf{R}^n \to \mathbf{R}^n$ and let x be in $\mathbf{R}^n$. If there exists the limit

$$\lim_{y \to x} \frac{\langle f(y) - f(x), y - x \rangle}{\|y - x\|^2}$$

then it is called the scalar derivative in x and it will be denoted by $f^{\#}(x)$. In this case f will be called scalarly differentiable at x. If $f^{\#}(x)$ exists for every x in $\mathbf{R}^n$, then f is said to be scalarly differentiable on $\mathbf{R}^n$, with scalar derivative

$f^{\#}$. Throughout this paper we shall use the following theorem (Theorem 2.4 of [3]) concerning the scalar differentiability in a point:

THEOREM 0.1. *Suppose that* $f:\mathbf{R}^n\to\mathbf{R}^n$, $f=(f_1,\dots,f_n)$ *is differentiable at* $x_0\in\mathbf{R}^n$. *Then the following statements are equivalent:*

1) f *is scalarly differentiable at* x_0.

2) a) $\frac{\partial f_1(x_0)}{\partial x^1}=\dots=\frac{\partial f_n(x_0)}{\partial x^n}$; b) $\frac{\partial f_i(x_0)}{\partial x^j}=-\frac{\partial f_j(x_0)}{\partial x^i}$;

$\forall\ i,j\in\{1,\dots,n\},\ i\neq j$.

If we identify f with a vector field and the components of f satisfy equations (2) (known in the literature as Killing equations [1]) then this vector field will generate a one parameter group of conformal transformations. So the above theorem shows that the scalar differentiability and conformity notions are similar up to a Lie correspondence. We are going to detailate this correspondence through this paper.

1. The coefficient of conformity and the conformal derivative

DEFINITION 1.1. Let $f:\mathbf{R}^n\to\mathbf{R}^n$ be a map and $p\in\mathbf{R}^n$. If there exists

$$f^c(p)=\lim_{q\to p}\frac{\|f(q)-f(p)\|}{\|q-p\|},$$

then f will be called "conformaly differentiable in p" and $f^c(p)$ is the "conformal derivative of f in p". If f is conformaly differentiable in each point of a subset U of $\mathbf{R}^n$ then we shall say that f is "conformaly differentiable on U".

LEMMA 1.2. *Let* $f:\mathbf{R}^n\to\mathbf{R}^n$ *be a differentiable* v *in* $p\in\mathbf{R}^n$. *Then* f *is conformaly differentiable in* p *if and only if* $\|df_p(x)\|=\lambda(p)\|v\|$ *for all vectors* v *where* $\lambda(p)$ *is some positive real valued function of* p *which does not depend on* v. *In this case* $f^c(p)=\lambda(p)$.

PROOF. "$\Rightarrow$" Let $q=p+tv$, with $t>0$, $t\to 0$ and v an arbitrary but fixed vector. Then $q\to p$,

$$f^c(p)=\lim_{t\to 0}\frac{\|f(p+tv)-f(p)\|}{|t|\|v\|}=\frac{1}{\|v\|}\|df_p(v)\|,$$

from where we obtain the required equality $\lambda(p)=f^c(p)$.

"$\Leftarrow$"

$$(1.1)\quad \frac{\|f(p+v)-f(p)\|}{\|v\|}=\sqrt{\frac{\|f(p+v)-f(p)\|^2}{\|v\|^2}}=\sqrt{\frac{\left\|df_p(v)+\omega(p,v)\|v\|\right\|^2}{\|v\|^2}}$$

$$=\sqrt{\frac{\|df_p(v)\|^2}{\|v\|^2}+2\frac{\langle df_p(v),\omega(p,v)\rangle}{\|v\|}+\|\omega(p,v)\|^2}=$$

$$=\sqrt{\lambda(p)^2+2\frac{\langle df_p(v),\omega(p,v)\rangle}{\|v\|}+\|\omega(p,v)\|^2},$$

where

$$(1.2)\qquad \omega(p,v)\to 0,\quad \text{whenever}\quad v\to 0.$$

On the other hand

$$\frac{|\langle df_p(v),\omega(p,v)\rangle|}{\|v\|}\le\frac{\|df_p(v)\|\|\omega(p,v)\|}{\|v\|}=\lambda(p)\|\omega(p,v)\|.$$

So

$$(1.3)\qquad \frac{\langle df_p(v),\omega(p,v)\rangle}{\|v\|}\to 0,\quad \text{whenever}\quad v\to 0$$

(1.1), (1.2) and (1.3) implies that

$$\lim_{v\to 0}\frac{\|f(p+v)-f(p)\|}{\|v\|}$$

exists and is equal to $\lambda(p)$. So f is scalarly differentiable at p and $f^c(p)=\lambda(p)$.

THEOREM 1.3. *Let $f:U\to\mathbf{R}^n$ be a map, where $U\subset\mathbf{R}^n$ is an open set. Then f is conformal if and only if it is conformaly differentiable on U. In this case the coefficient of conformity $\lambda(p)$ is equal to $f^c(p)$ at each p from U.*

The proof is a straightforward consequence of Lemma 1.2.

THEOREM 1.4. *Let $f:U\to\mathbf{R}^n$ be a map where $U\subset\mathbf{R}^n$ is an open set. Then f is conformal if and only if*

$$df_p^*\circ df_p=\lambda(p)^2I,$$

for all p and some real valued positive function $\lambda(p)$, where df_p denotes the adjoint operator of df_p and I the identity operator.

This theorem is an easy consequence of Lemma 1.2.

2. Monotone vector fields and expansive maps

The following theorem is a well known result from the theory of Lie Groups, which states that the Lie Algebra of $O(n)$ is the set of antisymmetric matrices. However, since we shall use the same idea in the foregoing that the proof of this theorem, and since this is an easy proof of a classical result we shall state and prove it.

THEOREM 2.1. *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ be the one parameter transformation group generated by v through p. Then v is antisymmetric as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if, $\psi(\varepsilon,p)$ is an isometry for every fixed ε.*

PROOF. Theorem 1.1 of [3] implies that

$$\left\langle v|_{\psi(\varepsilon,q)} - v|_{\psi(\varepsilon,p)}, \psi(\varepsilon,q) - \psi(\varepsilon,p)\right\rangle = 0$$

for all p, $q\in\mathbf{R}^n$ and $\varepsilon\in\mathbf{R}$. Hence

$$\frac{1}{2}\frac{d}{d\varepsilon}\|\psi(\varepsilon,q) - \psi(\varepsilon,p)\|^2 = 0,$$

since

$$\|\psi(\varepsilon,q) - \psi(\varepsilon,p)\|^2 = \langle\psi(\varepsilon,q) - \psi(\varepsilon,p), \psi(\varepsilon,q) - \psi(\varepsilon,p)\rangle$$

and

$$\frac{d}{d\varepsilon}\psi(\varepsilon,p) = v|_{\psi(\varepsilon,p)}.$$

Thus we have $\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|$=constant for p, q fixed. If we put $\varepsilon=0$ in this relation we obtain

$$\|\psi(\varepsilon,q) - \psi(\varepsilon,p)\| = \|q - p\|$$

for all p and q, since $\psi(0,p)=p$. The converse can be proved similarly.

THEOREM 2.2. *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ ($\varepsilon > 0$) the one parameter transformation group generated by v through p. Then v is increasing (decreasing) as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if, $\|\psi(\varepsilon,p)-$ $-\psi(\varepsilon,q)\|$ is increasing (decreasing) as a real function of ε for all p, q fixed.*

PROOF. "$\Rightarrow$"

$$\left\langle v|_{\psi(\varepsilon,q)} - v|_{\psi(\varepsilon,p)}, \psi(\varepsilon,q) - \psi(\varepsilon,p)\right\rangle \geq 0$$

$\forall\ p,\ q\in\mathbf{R}^n$ and $\forall\ \varepsilon\in\mathbf{R}$. Hence as before

$$\frac{1}{2}\frac{d}{d\varepsilon}\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|^2\geq 0,$$

from where it follows that $\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|$ is increasing.

"$\Leftarrow$" If $\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$ is increasing so is $\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|^2$, hence

$$\frac{d}{d\varepsilon}\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|^2\geq 0$$

which is equivalent to

$$\left\langle v|_{\psi(\varepsilon,q)}-v|_{\psi(\varepsilon,p)},\psi(\varepsilon,q)-\psi(\varepsilon,p)\right\rangle\geq 0$$

$\forall\ p,\ q\in\mathbf{R}^n$ and $\forall\ \varepsilon\in\mathbf{R}$. For $\varepsilon=0$ we have that

$$\left\langle v|_q-v|_q,q-p\right\rangle\geq 0$$

for all p and q. The case v decreasing can be treated similarly.

THEOREM 2.3. *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ be the one parameter transformation group generated by v through p. Then v is increasing (decreasing) as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if $\psi(\varepsilon,p)$ is an expansive (nonexpansive) function of p for all $\varepsilon>0$ fixed.*

PROOF. If v is increasing, then it follows from Theorem 2.2 that $\|\psi(\varepsilon,p)-$ $-\psi(\varepsilon,q)\|$ is increasing for all $p,\ q$ fixed. Particularly $\varepsilon>0$ yields

$$\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|\geq\|p-q\|$$

so $\psi(\varepsilon,p)$ is an expansive function of p.

Conversely, if $\psi(\varepsilon,p)$ is expansive for all $\varepsilon>0$ fixed, then we have

$$\|\psi(\delta-\varepsilon,\psi(\varepsilon,p))-\psi(\delta-\varepsilon,\psi(\varepsilon,q))\|\geq\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$$

for all $\delta>\varepsilon>0$ and all $p,\ q$ from $\mathbf{R}^n$. But $\psi(\delta-\varepsilon,\psi(\varepsilon,p))=\psi(\delta,p)$, since $\psi(\varepsilon,p)$ is an one parameter transformation group, so

$$\|\psi(\delta,p)-\psi(\delta,q)\|\geq\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$$

for all $\delta>\varepsilon>0$. So $\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$ is increasing as a real function of ε for all $p,\ q$ fixed. Hence by Theorem 2.2 v is increasing as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$.

2. The connection between the coefficient of conformity and scalar derivative

We have seen that the conformal derivative is exactly the coefficient of conformity for one parameter conformal transformation groups generated by Killing vector field, whose components are satisfying the Killing equations. These vector fields are exactly the scalar differentiable ones, regarded as functions from $\mathbf{R}^n$ to $\mathbf{R}^n$. We are going to compute for the one parameter transformation groups generated by such fields, the conformal derivative (or coefficient of conformity) in terms of the scalar derivative of f.

DEFINITION 3.1. A vector field on $\mathbf{R}^n$ is *scalarly differentiable* if it is scalarly differentiable as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$.

THEOREM 3.2. *Let v be a scalarly differentiable vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ $(\varepsilon > 0)$ the one parameter conformal transformation generated by v. Then we have*

$$\|d\psi_p(h)\| = \exp\left(\int_0^\varepsilon v^\#(\psi(\delta,p))d\delta\right).\|h\| \tag{3.1}$$

or equivalently

$$\psi^c(\varepsilon,p) = \exp\left(\int_0^\varepsilon v^\#(\psi(\delta,p))d\delta\right)$$

for all h, where the left hand side of (3.1) *is understood to be taken in ε.*

PROOF. From the chain rule we have that

$$\frac{d}{d\varepsilon}d\psi_p = d\frac{d}{d\varepsilon}\psi(\varepsilon,p) = d\left[v|_{\psi(\varepsilon,p)}\right] = dv_{\psi(\varepsilon,p)} \circ d\psi_p,$$

where $d\psi_p$ is taken in ε.

We have that

$$v^\#(\psi(\varepsilon,p)) = \lim_{t\to 0}\frac{\left\langle v|_{\psi(\varepsilon,p)+th} - v|_{\psi(\varepsilon,p)}, th\right\rangle}{\|th\|^2} =$$

$$= \frac{1}{\|h\|^2}\lim_{t\to 0}\left\langle \frac{v|_{\psi(\varepsilon,p)+th} - v|_{\psi(\varepsilon,p)}}{t}, h\right\rangle = \frac{\left\langle dv|_{\psi(\varepsilon,p)}(h), h\right\rangle}{\|h\|^2}$$

for all h.

If we put $d\psi_p(h)$ in this chain of relations instead of h, we obtain:

$$v^{\#}(\psi(\varepsilon,p)) = \frac{\left\langle (dv|_{\psi(\varepsilon,p)} \circ d\psi_p)(h), d\psi_p(h)\right\rangle}{\|d\psi_p(h)\|^2} = \frac{\left\langle \frac{d}{d\varepsilon} d\psi_p(h), d\psi_p(h)\right\rangle}{\|d\psi_p(h)\|^2} =$$

$$= \frac{1}{2}\frac{\frac{d}{d\varepsilon}\|d\psi_p(h)\|^2}{\|d\psi_p(h)\|^2}.$$

So

$$\text{(3.2)} \qquad \frac{d}{d\varepsilon}\ln\|d\psi_p(h)\| = v^{\#}(\psi(\varepsilon,p)).$$

Integrating (3.2) from $\delta=0$ to $\delta=\varepsilon$, we obtain

$$\ln\|d\psi_p(h)\| - \ln\|h\| = \int_0^{\varepsilon} v^{\#}(\psi(\delta,p))d\delta,$$

and from here the required relation. Here we used that $d\psi_p$ is the identical map for $\varepsilon=0$. This is an easy consequence of the Taylor expansion of ψ, see [5].

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SELF-DUAL HILBERT C^*-MODULES AND STATIONARY PROCESSES

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1. Introduction

Hilbert modules over a commutative $\mathbf{C}^*$-algebra A were introduced by I. KAPLANSKY in 1953 ([3]). A step forward in this theory was made by W. L. PASCHKE in 1973 by his doctoral thesis ([7]) and, almost at the same time, by M. A. RIEFFEL ([10]), but by another point of view, different of that of Paschke. Today Hilbert $\mathbf{C}^*$-modules represent an important instrument of study in $\mathcal{KK}$-theory ([4]), in the $\mathbf{C}^*$-algebraic approach to quantum group theory ([14]), but also in the prediction theory ([9], [6]).

A complete correlated action is, in fact a pre-Hilbert $\mathscr{L}(\mathscr{H})$-module ($\mathscr{H}$ being a Hilbert space). The existence of a self-adjoint projection in the parameter space in a complete correlated action which punctually corresponds to an orthogonal projection in the measurements space (to see [12] for a complete proof) suggests a certain structure of a complete correlated action in order to allow the complementability of the "orthogonal complement" of an $\mathscr{L}(\mathscr{H})$-submodule in the correlated action pre-Hilbert module. We shall prove that the structure we are looking for is self-duality and we shall specify the Hilbert submodule corresponding to the self-adjoint projection mentionned above.

In the development of the prediction theory an important role was played by a geometric type result, namely the Wold decomposition in its various variants ([13]).

Among the Wold type decompositions met in literature [2] (in smooth and reflexive Banach spaces), [8] and [9] (on Hilbert $\mathbf{C}^*$-modules), only the last seems to apply in this context. Hence, as a consequence of the results obtained and of the decomposition in [9] it will be finally given a short proof of the Wold decomposition for stationary processes in complete correlated actions.

2. Notations and Preliminaries

Let A be a $\mathbf{C}^*$-algebra and E be a right A-module having a compatible structure.

A map $\langle \cdot,\cdot \rangle : E \times E \to A$ linear in the second variable is said to be A-*valued inner product* on E if it satisfies

(i) $\langle x,x \rangle \geq 0, x \in E$;

(ii) $\langle x,x \rangle = 0$ if and only if $x = 0$;

(iii) $\langle x,y \rangle^* = \langle y,x \rangle$, x, $y \in E$;

(iv) $\langle x,ya \rangle = \langle x,y \rangle a$, $x \in E$, $a \in A$.

The couple $(E, \langle \cdot,\cdot \rangle)$ is said to be a *Hilbert A-module* if moreover the norm

$$\|x\| := \| \langle x,x \rangle \|^{1/2}, \quad x \in E.$$

is complete.

A pre-Hilbert A-module is said to be *self-dual* if every bounded A-module map $\tau : E \to A$ has the form

$$\tau(x) = \langle y,x \rangle, \quad x \in E,$$

$y \in E$ being uniquely determinated by τ. As it is seen in [7] every self-dual pre-Hilbert A-module is complete.

Let A be a von Neumann algebra. On a pre-Hilbert A-module we shall introduce two important topologies:

- s-generated on E by the family of seminorms $(p_\varphi)_\varphi$, $p_\varphi(x) = \varphi(\langle x,x \rangle)^{1/2}$, $x \in E$, where φ is running through the set of all positive linear functionals in the predual A_* of A;
- σ-defined with the convergence help: for $x_\alpha \in E$, $\alpha \in I$ we say that $x_\alpha \underset{\alpha \in I}{\overset{\sigma}{\to}} 0$ if for every $f_n \in A_*$, $x_n \in E (n \in \mathbf{N}^*)$ with the property that

$$\sum_{n=1}^{\infty} \|f_n\| \|x_n\|_E < \infty \text{ we have } \sum_{n=1}^{\infty} f_n(\langle x_n, x_\alpha \rangle) \overset{\alpha \in I}{\longrightarrow} 0.$$

It is not hard to see that s is stronger than σ, and if E is self-dual then $\sigma=\sigma(E,E_*)$ the s topology being compatible with the duality $E=(E_*)^*$ (we denoted by E_* the predual of E constructed in [7]). Consequently the convex sets (in particular the submodules) in a self-dual Hilbert module have the same closure with respect to these two topologies.

For a pre-Hilbert module collection $(E_\alpha)_\alpha$ over the von Neumann algebra A its *ultraweak direct sum* (according to [7]) is the pre-Hilbert A-module

$$\boxplus_{\alpha\in I} E_\alpha = \{x=(x_\alpha)_{\alpha\in I}\in\prod_{\alpha\in I}E_\alpha \mid \sup_F \|\sum_{\alpha\in F}\langle x_\alpha,x_\alpha\rangle\|<\infty\},$$

$$\langle (x_\alpha)_\alpha,(y_\alpha)_\alpha\rangle := \lim_F\sum_{k\in F}\langle x_k,y_k\rangle,\quad (x_\alpha)_\alpha,\quad (y_\alpha)_\alpha\in\boxplus_{\alpha\in I}E_\alpha,$$

where F is running through the set $\mathcal{F}$ of all finite parts of I, and the notation $\lim_F$ represents the ultraweak limit. Furthermore if $(E_\alpha)_\alpha$ is a family of parwise orthogonal (that is $\left\langle E_\alpha,E_\beta\right\rangle=0$ for $\alpha\neq\beta$) submodules in a pre-Hilbert A-module E then their *direct s-sum*

$$\boxplus_{\alpha\in I} E_\alpha := \{x=s-\lim_{F\in\mathcal{F}}\sum_{\alpha\in F}x_\alpha \,|\, x_\alpha\in E_\alpha\,(\alpha\in I) \text{ and } (x_\alpha)_{\alpha\in I} \text{ is } s\text{-summable}\}$$

is a submodule of E with the property

$$\langle x,y\rangle=\lim_F\sum_{\alpha\in F}\langle x_\alpha,y_\alpha\rangle,\ \text{for every } x=s-\lim_{F\in\mathcal{F}}\sum_{\alpha\in F}x_\alpha$$

and

$$y=s-\lim_{F\in\mathcal{F}}\sum_{\alpha\in F}y_\alpha \ \text{ in } \boxplus_{\alpha\in I}E_\alpha$$

(to see [9] for a complete proof).

A submodule E_0 of a Hilbert $\mathbf{C}^*$-module E is called *complementable* if there exists another submodule E_1 with the properties $E=E_0+E_1$ and $E_0\perp E_1$ (that is $\langle E_0,E_1\rangle=0$). For simplicity we shall use the notation $E=E_0\oplus E_1$.

An operator $T:E\to F$ between two Hilbert $\mathbf{C}^*$-modules is said to be *adjointable* if there exists $T^*:F\to E$ (called the *adjoint of* T) such that

$$\langle y,Tx\rangle=\langle T^*y,x\rangle,\quad x\in E, y\in F.$$

The set of all adjointable operators from E to F is denoted by $\mathcal{L}_A(E,F)$ (respectively $\mathcal{L}_A(E)$ if $E=F$), and for $T\in\mathcal{L}_A(E,F)$ it is used the notation $[E,F,T]$ (respectively $[E,T]$ if $E=F$).

With a classic case similar proof we mention.

PROPOSITION 2.1. *Every A-linear and bounded operator between two Hilbert $\mathbf{C}^*$-modules, the first one being self-dual, admits an adjoint.*

Let us suppose that E_0 is a self-dual submodule of a Hilbert module E. The natural embedding $i : E_0 \to E$ admits an adjoint i^*, and the range of $i \circ i^*$ is E_0. Furthermore $E = E_0 \oplus E_1$ where $E_1 = \ker(i \circ i^*)$, that is:

COROLLARY 2.2. *Every self-dual Hilbert submodule is complementable.*

A particular class of adjointable operators on a Hilbert module is represented by A-linear isometries with complementable range. Among these we can mention unitary operators and s-shifts.

DEFINITION 2.3. Let E and F be two Hilbert A-modules

- $U : E \to F$ is called a *unitary operator* if it is isometric and surjective;
- $S : E \to E$ is called an *s-shift* if there exists a Hilbert A-module F such that S and S_F are unitary equivalent (we denoted by S_F the operator on the ultraweak direct sum $\boxplus_{n=0}^{\infty} F$,

$$(x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots)).$$

In 1994 E. C. LANCE showed (in [5]) that unitary operators U admit adjoints U^* with the properties $UU^* = I_F$ and $U^*U = I_E$. The same results, but with another proof (shorter), can be found in [9].

3. Self-Duality of $\mathscr{L}(\mathscr{H}, \mathscr{K})$

For the beginning let us mention some general results of the Hilbert $\mathbf{C}^*$-modules theory, results which will found their applicability in this section.

Let A be a von Neumann algebra and E be a pre-Hilbert A-module. The next proposition has a detailed proof in [1].

PROPOSITION 3.1. *E is self-dual if and only if the closed unit ball of E (denoted by $(E)_1$) is σ-compact.*

Taking this characterization into account we can formulate:

COROLLARY 3.2. *If E is self-dual and $E_0 \subset E$ is an s-closed submodule (so σ-closed also) then E_0 is self-dual.*

Particularly the orthogonal complement $E_0^{\perp} = \{x \in E | \langle x, E_0 \rangle = 0\}$ of a submodule $E_0 \subset E$ is s-closed which allows the decomposition

$$E = E_0^{\perp} \oplus E_0^{\perp\perp}.$$

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ ($\mathcal{L}(\mathcal{H})$ if $\mathcal{H} = = \mathcal{K}$) the space of linear and bounded operators from $\mathcal{H}$ to $\mathcal{K}$. The right $\mathcal{L}(\mathcal{H})$-module (natural) structure

$$\mathcal{L}(\mathcal{H},\mathcal{K}) \times \mathcal{L}(\mathcal{H}) \ni (T, A) \mapsto TA \in \mathcal{L}(\mathcal{H},\mathcal{K})$$

and the $\mathcal{L}(\mathcal{H})$-valued inner product

$$\mathcal{L}(\mathcal{H},\mathcal{K}) \times \mathcal{L}(\mathcal{H},\mathcal{K}) \ni (S, T) \mapsto \langle S, T \rangle := S^* T \in \mathcal{L}(\mathcal{H})$$

equip $\mathcal{L}(\mathcal{H},\mathcal{K})$ with a pre-Hilbert $\mathcal{L}(\mathcal{H})$-module structure. This pre-Hilbert module is even Hilbert because the norm induced by the above defined inner product is exactly the operator norm.

THEOREM 3.3. *$\mathcal{L}(\mathcal{H},\mathcal{K})$ is a self-dual Hilbert $\mathcal{L}(\mathcal{H})$-module*

PROOF. Let $\tau : \mathcal{L}(\mathcal{H},\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ be linear and bounded A-module map. Equivalentely ([7]) τ verifies the inequality

$$\tau(T)^*\tau(T) \leq M\, T^* T, \quad T \in \mathcal{L}(\mathcal{H},\mathcal{K})$$

for a constant M independent of T.

In particular

(1) $$T\xi = 0, \xi \in \mathcal{H} \text{ implies } \tau(T)\xi = 0.$$

Fix $\xi \in \mathcal{H}$, $\xi \neq 0$. The map

$$\varphi_\xi : \mathcal{K} \to \mathbf{C}, \varphi_\xi(\eta) = \frac{1}{\|\xi\|^2} \left\langle \tau(\theta_{\eta,\xi})(\xi), \xi \right\rangle_{\mathcal{H}}, \eta \in \mathcal{K}$$

is linear (for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ we denoted by $\theta_{\eta,\xi}$ the operator $\zeta \mapsto \langle \zeta, \xi \rangle \eta$). Furthermore

$$|\varphi_\xi(\eta)| \leq \frac{1}{\|\xi\|} \|\tau(\theta_{\eta,\xi})(\xi)\| \leq \|\tau\| \|\theta_{\eta,\xi}\| = \|\tau\| \|\xi\| \|\eta\|, \xi \in \mathcal{H}, \eta \in \mathcal{K},$$

that is φ_ξ is bounded in norm with $\|\tau\|\|\xi\|$. The Riesz representation theorem insures the existence of a unique $S\xi \in \mathcal{K}$ which verifies

(2) $$\frac{1}{\|\xi\|^2} \left\langle \tau(\theta_{\eta,\xi})(\xi), \xi \right\rangle_{\mathcal{H}} = \langle \eta, S\xi \rangle_{\mathcal{K}}, \eta \in \mathcal{K}.$$

We define a map $S:\mathcal{H}\to\mathcal{K}$ (consider $S(0)=0$) such that $S\in\mathcal{L}(\mathcal{H},\mathcal{K})$. For $\lambda\in\mathbf{C}$, $\xi\in\mathcal{H}$ and $\eta\in\mathcal{K}$ we have

$$\langle\eta\,,S(\lambda\xi)\rangle_{\mathcal{K}}=\frac{1}{\|\lambda\xi\|^2}\left\langle\tau(\theta_{\eta,\lambda\xi})(\lambda\xi),\lambda\xi\right\rangle_{\mathcal{H}}=\frac{1}{\|\xi\|^2}\left\langle\tau(\theta_{\overline{\lambda}\eta,\xi})(\xi),\xi\right\rangle_{\mathcal{H}}=$$
$$=\left\langle\overline{\lambda}\eta\,,S(\xi)\right\rangle_{\mathcal{K}}=\langle\eta\,,\lambda S(\xi)\rangle_{\mathcal{K}}\,.$$

So S is homogeneous.

The additivity requires more preparations. If ξ', $\xi''\in\mathcal{H}^*=\mathcal{H}\setminus\{0\}$ with $\langle\xi',\xi''\rangle_{\mathcal{H}}=0$ then, for all $\eta\in\mathcal{K}$

$$\|\xi'\|^2\theta_{\eta,\xi''}(\xi'')=\|\xi'\|^2\|\xi''\|^2\eta=\theta_{\eta,\xi'+\xi''}(\|\xi'\|^2\xi'')$$

and

$$\|\xi''\|^2\theta_{\eta,\xi'}(\xi')=\|\xi''\|^2\|\xi'\|^2\eta=\theta_{\eta,\xi'+\xi''}(\|\xi''\|^2\xi').$$

Applying two times the relation (1), for $\eta\in\mathcal{K}$,

$$\|\xi'\|^2\tau(\theta_{\eta,\xi''})(\xi'')=\tau(\theta_{\eta,\xi'+\xi''})(\|\xi'\|^2\xi'')=\tau(\theta_{\eta,\xi'+\xi''})(\|\xi''\|^2\xi')=$$
$$=\|\xi''\|^2\tau(\theta_{\eta,\xi'})(\xi').$$

Furthermore, in the same conditions $\tau(\theta_{\eta,\xi'})(\xi'')=\tau(\theta_{\eta,\xi''})(\xi')=0$. The following calculus

$$\begin{aligned}
&\langle\eta\,,S(\xi'+\xi'')\rangle_{\mathcal{K}}=\\
&=\frac{1}{\|\xi'\|^2+\|\xi''\|^2}\left\langle\tau(\theta_{\eta,\xi'+\xi''})(\xi'+\xi''),\xi'+\xi''\right\rangle_{\mathcal{H}}=\\
&=\frac{1}{\|\xi'\|^2+\|\xi''\|^2}\left(\left\langle\tau(\theta_{\eta,\xi'})(\xi'),\xi'+\xi''\right\rangle_{\mathcal{H}}+\left\langle\tau(\theta_{\eta,\xi''})(\xi''),\xi''+\xi''\right\rangle_{\mathcal{H}}\right)=\\
&=\frac{1}{\|\xi'\|^2+\|\xi''\|^2}\left(\left\langle\tau(\theta_{\eta,\xi'})(\xi'),\xi'\right\rangle_{\mathcal{H}}+\frac{\|\xi'\|^2}{\|\xi''\|^2}\left\langle\tau(\theta_{\eta,\xi''})(\xi''),\xi''\right\rangle_{\mathcal{H}}+\right.\\
&\quad\left.+\frac{\|\xi''\|^2}{\|\xi'\|^2}\left\langle\tau(\theta_{\eta,\xi'})(\xi'),\xi'\right\rangle_{\mathcal{H}}+\left\langle\tau(\theta_{\eta,\xi''})(\xi''),\xi''\right\rangle_{\mathcal{H}}\right)=\\
&=\frac{1}{\|\xi'\|^2}\left\langle\tau(\theta_{\eta,\xi'})(\xi'),\xi'\right\rangle_{\mathcal{H}}+\frac{1}{\|\xi''\|^2}\left\langle\tau(\theta_{\eta,\xi''})(\xi''),\xi''\right\rangle_{\mathcal{H}}=\\
&=\langle\eta\,,S\xi'+S\xi''\rangle_{\mathcal{K}}\,,\qquad\eta\in\mathcal{K}
\end{aligned}$$

shows that $S(\xi'+\xi'')=S\xi'+S\xi''$ holds indeed.

More generaly, if $(e_\alpha)_\alpha$ is an orthonormal basis in $\mathcal{H}$, ξ_1 and ξ_2 two finite linear combinations of basis elements and $\lambda_1, \lambda_2 \in \mathbf{C}$ then, applying the results proved above

$$S(\lambda_1\xi_1+\lambda_2\xi_2) =$$

$$= S\left(\lambda_1 \sum_{\alpha\in I_1\setminus I_2} \lambda'_\alpha e_\alpha + \sum_{\alpha\in I_1\cap I_2} (\lambda_1\lambda'_\alpha+\lambda_2\lambda''_\alpha)e_\alpha + \lambda_2 \sum_{\alpha\in I_2\setminus I_1} \lambda''_\alpha e_\alpha\right) =$$

$$(3) = \lambda_1 S(\xi_1) + \lambda_2 S(\xi_2).$$

To pass to entire $\mathcal{H}$ we need some sort of continuity for S. Let $\xi_n, \xi \in \mathcal{H} (n \in \in \mathbf{N})$ with $\xi_n \to \xi$. Then

$$|\langle \eta, S\xi_n\rangle - \langle \eta, S\xi\rangle| = |\langle \eta, S(\xi_n - \xi)\rangle| \le \|\eta\|\|\tau\|\|\xi_n - \xi\|, \quad n \in \mathbf{N}, \eta \in \mathcal{K},$$

that is $\langle S\xi_n, \eta\rangle \xrightarrow{n} \langle S\xi, \eta\rangle$, $\eta \in \mathcal{K}$. Passing to limit in (3) we obtain that S is indeed linear.

Furthermore, for $\xi \in \mathcal{H}^*$,

$$\|S\xi\|^2 = \frac{1}{\|\xi\|^2}\langle \tau(\theta_{S\xi,\xi})(\xi), \xi\rangle \le \frac{1}{\|\xi\|}\|\tau\|\|\theta_{S\xi,\xi}(\xi)\| = \|S\xi\|\|\tau\|\|\xi\|,$$

that is $S \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $\|S\| \le \|\tau\|$.

Finally, for $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ fixed and $\xi \in \mathcal{H}^*$, using again (1),

$$T\xi = \frac{1}{\|\xi\|^2}\theta_{T\xi,\xi}(\xi) \text{ implies } \tau(T)(\xi) = \frac{1}{\|\xi\|^2}\tau(\theta_{T\xi,\xi})(\xi).$$

By replacing in (2)

$$\langle \tau(T)(\xi), \xi\rangle_{\mathcal{H}} = \langle T\xi, S\xi\rangle_{\mathcal{K}} = \langle S^*T(\xi), \xi\rangle_{\mathcal{H}}, \quad \xi \in \mathcal{H}$$

that is $\tau(T) = S^*T = \langle S, T\rangle$ and the proof is ended. ∎

The proof presented above seems to be more natural than the one given by M. A. RIEFFEL in [11] in which the author uses a Gram–Schmidt-type precess.

In the following we need a few notions.

DEFINITION 3.4. • Let $\mathcal{H}$ be a Hilbert space. The triplet $(\mathcal{H}, E, \langle\cdot,\cdot\rangle)$ is said to be the *correlated action of* $\mathcal{L}(\mathcal{H})$ *on* E *if* $(E, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert $\mathcal{L}(\mathcal{H})$-module. In this case E is called the *state space,* $\mathcal{H}$ *the parameter space*, and $\langle\cdot,\cdot\rangle$ *correlation of the* $\mathcal{L}(\mathcal{H})$ *action on* E.

- In particular $(\mathcal{H},\mathcal{L}(\mathcal{H},\mathcal{K}),\langle\cdot,\cdot\rangle)$ presented above is called *operator model.*

Every correlated action can be embedded in an operator model (to see again [12] for detailled proofs).

PROPOSITION 3.5. *Let $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ be a correlated action. Then there exist a Hilbert space $\mathcal{K}$ (called measurements space) and an algebraic embedding $E\ni x\overset{\varphi}{\mapsto}\varphi(x)\in\mathcal{L}(\mathcal{H},\mathcal{K})$ with the properties:*

(i) $\langle x,y\rangle=\varphi(x)^*\varphi(y)$, $x,\ y\in E$;

(ii) $\mathcal{K}=\overline{\{\varphi(x)\xi\,|\,x\in E,\xi\in\mathcal{H}\}}$.

This decomposition is umique (up to a unitary equivalence).

If φ is surjective then $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ is called *complete correlated action.*

REMARK 3.6. A complete correlated action E is a self-dual Hilbert $\mathcal{L}(\mathcal{H})$-module. Indeed, if $\tau:E\to\mathcal{L}(\mathcal{H})$ is $\mathcal{L}(\mathcal{H})$-linear and bounded then $\tau\circ\varphi^{-1}$ is an $\mathcal{L}(\mathcal{H})$-linear and bounded "functional" on $\mathcal{L}(\mathcal{H},\mathcal{K})$. Use now Theorem 3.3.

It is not difficult to observe that, having a submodule E_0 of E (a correlated action) and denoting by $\mathcal{K}_0=\bigvee_{x\in E_0}\varphi(x)\mathcal{H}$, the formula

$$\varphi(x_0)=P_{\mathcal{K}_0}\varphi(x),\quad x\in E$$

defines a projection of the $\mathbf{C}^*$-algebra $\mathcal{L}_A(E)$,

$$E\ni x\mapsto P_{E_0}(x):=x_0\in E.$$

Furthermore $\varphi(x_0)\xi\in\mathcal{K}_0$ and $\varphi(x-x_0)\xi\in\mathcal{K}_0^{\perp}$, $\xi\in\mathcal{H}$.

In the following we shall calculate this projection range and make the connection with the results at the beginning of this section.

PROPOSITION 3.7. *Let $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ be a complete correlated action and E_0 an $\mathcal{L}(\mathcal{H})$-submodule of E. Then P_{E_0} is the orthogonal projection corresponding to $E_0^{\perp\perp}=\overline{E_0}^s$ in the decomposition*

$$E=E_0^{\perp}\oplus E_0^{\perp\perp}.$$

Consequently $R(P_{E_0})=\overline{E_o}^s$.

PROOF. Let $x\in E$. Then $x\perp E_0$ if and only if $\varphi(x)^*\varphi(y)=0$, for every $y\in E_0$. Taking into account the definition of $\mathcal{K}_0$ this is equivalent to $\varphi(x)^*|\mathcal{K}_0 = 0$. So $\varphi(x)^* P_{\mathcal{K}_0} = 0$ and, by passing to adjoint $P_{\mathcal{K}_0}\varphi(x) = 0$. Hence

$$x \perp E_0 \text{ if and only if } P_{E_0}(x) = 0$$

and the proof is ended. ∎

COROLLARY 3.8. *In conditions of the previous proposition we have*

$$\left\langle x - P_{E_0}x, x - P_{E_0}x \right\rangle = \inf_{y \in E_0} \langle x - y, x - y \rangle, \quad x \in E,$$

the infimum being taken over the set of positive operators in $\mathcal{L}(\mathcal{H})$.

PROOF By a standard method we find that

$$\inf_{y \in E_0} \langle x - y, x - y \rangle = \inf_{y \in \overline{E_0}^s} \langle x - y, x - y \rangle, \quad x \in E.$$

It remains to prove that

$$\left\langle x - P_{E_0}x, x - P_{E_0}x \right\rangle \leq \langle x - y, x - y \rangle, \quad x \in E, y \in \overline{E_0}^s.$$

Indeed, for $x \in E$ and $y \in \overline{E_0}^s$,

$$\begin{aligned} \langle x - y, x - y \rangle &= \left\langle (x - P_{E_0}x) + (P_{E_0}x - y), (x - P_{E_0}x) + (P_{E_0}x - y) \right\rangle = \\ &= \left\langle x - P_{E_0}x, x - P_{E_0}x \right\rangle + \left\langle P_{E_0}x - y, P_{E_0}x - y \right\rangle \geq \\ &\geq \left\langle x - P_{E_0}x, x - P_{E_0}x \right\rangle. \end{aligned}$$

∎

4. Wold Structure

In this section we give some applications of the above results concerning the possibility of obtaining Wold-type decompositions.

DEFINITION 4.1. Let E be a Hilbert A-module. A closed A-submodule $L \subset E$ is said to be *wandering* for the A-linear isometry V if

$$V^n E \perp E, \text{ for all } n \in \mathbf{N}^*.$$

PROPOSITION 4.2. *([9]) Let E be a self-dual Hilbert A-module and S an A-linear and bounded isometric map on E. Then S is s-shift if and only if there exists a submodule $L \subset E$, wandering for S such that*

$$E = \boxplus_{n=0}^{\infty} S^n L.$$

PROOF. We give only a sketch. If S is s-shift then according to Definition 2.3 S is unitary equivalent to the standard shift S_F (F being a Hilbert A-module) by the unitary operator $U : E \to \boxplus_{n=0}^{\infty} F$. Denoting $L = U^*F$ one obtains that L is wandering for S and

$$E = \{U^*(x_n)_n | (x_n)_n = s - \lim_{m\to\infty} \sum_{k=0}^{m} (0, \dots, 0, x_k, 0, \dots)\} = \boxplus_{n=0}^{\infty} S^n L.$$

Conversely, by the equality $E = \boxplus_{n=0}^{\infty} S^n L$, we define $U : E \to \boxplus_{n=0}^{\infty} L$,

$$U(x) = (l_n)_n, \text{ where } x = s - \lim_{n\to\infty} \sum_{k=0}^{n} S^k l_k, x \in E, l_k \in L (k \in \mathbf{N}).$$

U makes an equivalence between S and S_F. ∎

Let E be a Hilbert A-module and $[E, V]$ be an isometry. It is not difficult to observe that, for all $n \in \mathbf{N}$,

$$E = \bigoplus_{k=0}^{n} V^k L \oplus V^{n+1} E, \text{ where } L = \ker V^*.$$

To this decomposition corresponds

$$x = \sum_{k=0}^{n} V^k l_k + V^{n+1} z_{n+1}, \text{ where } \{l_k\}_{k=0}^{n} \subset L, z_{n+1} \in E,$$

$$l_k = (I_E - V V^*) V^{*k} x, z_{n+1} = V^{n+1} V^{*(n+1)} x, n \in \mathbf{N}.$$

Taking into account these results and the properties of s-topology on self dual Hilbert modules it can be proved the following theorem obtained in [9] in a more general background.

THEOREM 4.3. *Let E be a self-dual Hilbert A-module. Every isometry $[E, V]$ admits a Wold-type decomposition, that is there exist two submodules E_0, $E_1 \subset E$ such that*

(i) $E = E_0 \oplus E_1$*;*

(ii) E_0 *(and consequently* E_1*) reduces* V*;*

(iii) $V | E_0$ *is a unitary operator and* $V | E_1$ *is an* s*-shift.*

This decomposition is unique.

COROLLARY 4.4. *Let E be a self dual Hilbert A-module and $[E, V]$ an isometry. Then V is s-shift if and only if V is completely non-unitary (that is the restriction to every non-null closed reducing submodule it is not a unitary operator).*

In the following E will be a complete correlated action having the parameter space $\mathcal{H}$. A *discrete stationary* process is a sequence $(f_n)_{n\in\mathbf{Z}}$ with the property that $\langle f_n, f_m\rangle$ depends only on the difference $m-n$ and not on m or n separately. The stationary process $(g_n)_{n\in\mathbf{Z}}$ is called *white noise* if $\langle g_n, g_m\rangle = 0$ for $m \neq n$. The stationary process $(f_n)_n$ *contains* the white noise $(g_n)_n$ if:

(i) $\langle f_n, g_m\rangle$ depends only on the difference $m-n$ and is equal to 0 for $m > n$;

(ii) $\varphi(g_0)\mathcal{H} \subset \bigvee_{k=-\infty}^{0} \varphi(f_k)\mathcal{H}$;

(iii) $Re\,\langle f_n - g_n, g_n\rangle \geq 0$, $n \in \mathbf{Z}$.

The stationary process $(f_n)_n$ is called *deterministic* if it contains no non-null white noise. $(f_n)_n$ is called the *moving average* for the white noise $(g_n)_n$ if

(i) $\{f_n\}_n$ contains $\{g_n\}_n$;

(ii) $\bigvee_{n=-\infty}^{\infty} \varphi(f_n)\mathcal{H} = \bigvee_{n=-\infty}^{\infty} \varphi(g_n)\mathcal{H}$.

We can give now a new proof for the Wold decomposition theorem for discrete stationary processes.

THEOREM 4.5. *Let $(f_n)_{n\in\mathbf{Z}}$ be a discrete stationary process. There exists a unique decomposition of the form*

$$f_n = u_n + v_u, \qquad n \in \mathbf{Z}$$

where

(a) $(u_n)_n$ *is the moving average of the maximal white noise contained in* $(f_n)_n$*;*

(b) $(v_n)_n$ *is a deterministic process;*

(c) $\langle u_n, v_m\rangle = 0$, *for all* m, $n \in \mathbf{Z}$.

PROOF. Let E_n^f, $n \in \mathbf{Z}$ the s-closed $\mathcal{L}(\mathcal{H})$-submodule generated by $(f_m)_{m \leq n}$. Extending this notion we obtain similarly E_∞^f. The relation

$$E_\infty^f \ni f_n \mapsto U_f(f_n) := f_{n+1} \in E_\infty^f, \qquad n \in \mathbf{Z}$$

defines a unitary operator on E_∞^f.

E_0^f is invariant to U_f^* and so we can build the isometry $V = U_f^* \mid E_0^f$. E_0^f being an s-closed submodule in the self-dual Hilbert $\mathcal{L}(\mathcal{H})$-module E it is also self-dual (Corollary 3.2). Using Theorem 4.3 $[E_0^f, V]$ admits a Wold-type decomposition

$$E_0^f = \bigcap_{n \geq 0} V^n E_0^f \oplus \boxplus_{n=0}^{\infty} V^n L \quad \text{where} \quad L \oplus V E_0^f = E_0^f.$$

This equality written in operator form becomes $I_{E_0^f} = P + Q$, where P and Q are self-adjoint projections. The relations

$$f_n = U_f^n Q f_0 + U_f^n P f_0, \qquad n \in \mathbf{Z}$$

represent the Wold decomposition that we are looking for. ∎

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THE CONJUGATE GRADIENT METHOD FOR A CLASS OF NON-DIFFERENTIABLE OPERATORS[1]

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Introduction

The conjugate gradient method (CGM) is one of the most frequently used methods for solving algebraic systems of equations, not only among the gradient type methods but also among all iterative methods (see e.g. [10]). It is used efficiently for boundary value problems of elliptic differential equations after discretization.

Both the gradient method and the CGM have been extended to Hilbert space setting. The results on the gradient method concern, on the one hand, bounded linear and Gâteaux differentiable nonlinear operators ([5], [7], [8]), on the other hand, certain unbounded linear and non-differentiable nonlinear operators that can be transformed to have the first mentioned properties on a suitable energy space ([6], [7], [12]). The latter methods allow direct application (without discretization) for differential equations in Sobolev spaces. Following the results of [1], [2], the CGM in Hilbert spaces was first given thorough investigation by [3]. The results for bounded linear and Gâteaux differentiable nonlinear operators have undergone further development (see e.g. [5], [9], [11]). The extension of the CGM to unbounded linear operators and its application to linear differential equations is also found already in [3]; however, for non-differentiable operators the method has not been established yet.

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Our aim is to show that the transformation which we used for the extension of the gradient method to certain non-differentiable operators in [12] also works for the CGM. This transformation is applied to the original method of [3] (the approach being apparently applicable to later modifications). The solution of the equation that contains a non-differentiable nonlinear operator is reduced to the successive solution of auxiliary linear equations. Then an example illustrates the main consequence of this result, i.e. how the obtained method can be applied to elliptic boundary value problems directly in the corresponding function spaces.

1. The classical CGM

In this section we recall the method in [3] which will be transformed.

The method is first proven to exhibit global convergence, then convergence estimates are given near the solution which show the greatest advantage of the CGM, i.e. the optimal rate of convergence. The results on the so-called basic CGM can be grouped into one theorem, given in the sequel. First we summarize the assumptions and the construction of the approximating sequence.

ASSUMPTIONS 1. (i) Let H be a real Hilbert space, $J:H\to H$ a continuous operator which is twice Gâteaux differentiable.

(ii) Let $A\geq a>0$ and $B>0$ be given constants; assume that there exist $u_0\in H$ and $R>0$ such that for any $u\in B(u_0,R):=\{u\in H:\|u-u_0\|\leq R\}$ we have $a\|h\|^2\leq\langle J'(u)h,h\rangle\leq A\|h\|^2$ and $\|J''(u)\|\leq B$.

(iii) Denote by $\phi:H\to\mathbf{R}$ the potential of J (which exists by the previous assumptions) and assume that $\{u\in H:\phi(u)\leq\phi(u_0)\}\subset B(u_0,R)$ holds for the level set corresponding to u_0.

(We have changed in two points the original assumptions of [3]. The assumption on $\langle J'(u)h,h\rangle$ in (ii) has been reduced from the whole space to $B(u_0,R)$, since this is sufficient together with (iii). Also, [3] assumed Frêchet differentiability of J but only used Gâteaux differentiability.)

THE CG ITERATION. Let $u_0\in H$ be as in assumption (ii), $p_0=r_0=-J(u_0)$. For $n\in\mathbf{N}=\{0,1,\dots\}$, successively, let $u_{n+1}:=u_n+c_np_n$ where c_n is the smallest positive root of $\langle J(u_n+cp_n),p_n\rangle=0$; set $r_{n+1}:=-J(u_{n+1})$, $p_{n+1}:=$ $=r_{n+1}+b_np_n$, where $b_n:=-\langle J'(u_{n+1})p_n,r_{n+1}\rangle/\langle J'(u_{n+1})p_n,p_n\rangle$.

FURTHER NOTATIONS. For any $n \in \mathbf{N}$ let $\varepsilon_n := \left\langle J'(u_n)^{-1} r_n, r_n \right\rangle^{1/2}$. Further, let $d := \frac{B}{A^3}\left(3 + \frac{A}{2a}\right)$, $\eta_n := \frac{\sqrt{A}B}{2a^2}\varepsilon_n$, $\sigma_n := \frac{4aA}{(A+a)^2}\frac{\eta_n}{1+\eta_n} + d\varepsilon_n$, $q := \frac{A-a}{A+a}$, $q_n := (q^2+\sigma_n)^{1/2}$, $R_n := \frac{\sqrt{A}}{a(1-q_n)}\varepsilon_n$.

Then we have

THEOREM 1 ([3]). *Under the above assumptions the following hold:*

(1) *The equation $J(u)=0$ has a unique solution $u^* \in H$, and the sequence (u_n) of the CG iteration converges strongly to u^*.*

(2) *Let $N_0 \in \mathbf{N}$ be such that $R_{N_0} < R$ and $\sigma_{N_0} < 1-q^2$. Then*

$$\|u_n - u^*\| \le R_{N_0} \cdot q_{N_0} \cdot q_{N_0+1} \cdots q_{n-1} \quad (n > N_0). \tag{1.1}$$

(Note that $\lim q_n = q$*).*

(3) *Let N_0 be as in (2). Then for any $m > N_0$ there exists $N_m \in \mathbf{N}$ such that*

$$\varepsilon_{n+m} \le \left(4\left(\frac{\sqrt{A}-\sqrt{a}}{\sqrt{A}+\sqrt{a}}\right)^{2m} + \delta_n\right)\varepsilon_n \quad (n > N_m) \tag{1.2}$$

where $\lim \delta_n = 0$. *(Note that ε_n is equivalent to $\|J(u_n)\|$ and $\|u_n - u^*\|$.)*

The above method encounters difficulties in the case of differential operators. First, they do not fulfil themselves the conditions of the method. Secondly, in the case of the corresponding generalized differential operators (which usually fulfil the assumptions) the steps of the iteration cannot be determined in general. The reason of this is that the generalized operators are not defined explicitly.

2. The CGM for a class of non-differentiable operators

The CGM is now established for non-differentiable operators T that can be transformed to have the required differentiability properties on the energy space of a suitable auxiliary linear operator. This linear operator can be regarded as a special kind of preconditioner. The solution of the equation containing T is thus reduced to the successive solution of auxiliary linear equations.

THEOREM 2.1 *Let H be a real Hilbert space with scalar product $\langle\ ,\ \rangle$. Let $D\subset H$ be a dense subspace and $T:D\to H$ a non-differentiable nonlinear operator. Let $B: D\to H$ be a strictly positive linear operator and denote by H_B the energy space of B, i.e. the completion of D with respect to the scalar product $\langle u,v\rangle_B := \langle Bu,v\rangle$ $(u,v\in D)$. (Denote by $\|\ \|$ and $\|\ \|_B$ the corresponding norms of H and H_B.) Assume that the following conditions hold:*

(i) $R(B)\supset R(T)$;

(ii) *the operator $B^{-1}T: D\to H_B$ has an extension $J: H_B\to H_B$ which fulfils Assumptions 1 on H_B.*

Let $u_0\in D$, $\hat{p}_0=\hat{r}_0=-B^{-1}T(u_0)$. For $n\in\mathbf{N}$, successively, let $\hat{u}_{n+1}:= = \hat{u}_n+\hat{c}_n\hat{p}_n$ where $\hat{c}_n$ is the smallest positive root of $\langle T(\hat{u}_n+c\hat{p}_n),\hat{p}_n\rangle=0$; set

$$\hat{r}_{n+1}:=-B^{-1}T(\hat{u}_{n+1}),\qquad \hat{p}_{n+1}:=\hat{r}_{n+1}+\hat{b}_n\hat{p}_n,$$

$$\text{where}\qquad \hat{b}_n:=-\left\langle J'(\hat{u}_{n+1})\hat{p}_n,\hat{r}_{n+1}\right\rangle_B \big/ \left\langle J'(\hat{u}_{n+1})\hat{p}_n,\hat{p}_n\right\rangle_B .$$

Then the following hold:

(1) *The equation $T(u)=0$ has a unique weak solution $u^*\in H_B$, i.e. for which*

$$\langle J(u^*),v\rangle_B=0\quad (v\in D), \tag{2.1}$$

and $(\hat{u}_n)$ converges to u^* strongly in the norm of H_B.

(2) *The linear convergence estimates* (1.1) *and* (1.2) *hold for* $(\hat{u}_n)$ *in H_B.*

(If B has positive lower bound then $(\hat{u}_n)$ converges to u^* also in the norm of H.)

PROOF. Owing to the assumptions, Theorem 1 applies to equation $J(u)= = 0$ in the space H_B. Hence there exists a unique $u^*\in H_B$ which satisfies $J(u^*)=0$. (This can be regarded as the generalized solution of $T(u)=0$ since for $u\in D$ the two equalities coincide.) Further, the sequence defined by the CG iteration in H_B converges to u^* according to (1.1)–(1.2).

It is clear that the obtained generalized solution $u^*\in H_B$ is the unique weak solution in the sense of (2.1) since D is dense by definition in the energy space H_B. Thus (1) holds.

We have to show that the CG iteration coincides with the sequence $(\hat{u}_n)$ defined in the present theorem. Indeed, if $u_0\in D$ then (by the definition of J) we have $J(u_0)=B^{-1}T(u_0)$, hence $p_0=r_0=-B^{-1}T(u_0)=\hat{p}_0=\hat{r}_0$; further, by the assumption $R(B)\supset R(T)$ this implies that $p_0\in D(B)=D$. From this we can see by induction that in each step of the CG iteration u_n,p_n are in D, $r_{n+1} = -B^{-1}T(u_{n+1})$, and $\langle J(u_n + c_n p_n), p_n \rangle_B = \langle T(u_n + c_n p_n), p_n \rangle$. Indeed, if $u_k, p_k \in D$ then $u_{k+1} = u_k + c_k p_k \in D$, hence (just as for $k = 0$) $J(u_{k+1}) = B^{-1}T(u_{k+1})$ and $J(u_{k+1}) \in D$, i.e. $r_{k+1} \in D$, and finally $p_{k+1} = r_{k+1} + b_k p_k \in D$. Further, for any $c > 0$ $\langle J(u_n + cp_n), p_n \rangle_B = \left\langle B^{-1}T(u_n + cp_n), p_n \right\rangle_B = \left\langle BB^{-1}T(u_n + cp_n), p_n \right\rangle = \langle T(u_n + cp_n), p_n \rangle$. These equalities mean that for any $n \in \mathbf{N}^+$ u_n, r_n, p_n and b_n coincide with $\hat{u}_n$, $\hat{r}_n$, $\hat{p}_n$ and $\hat{b}_n$, respectively.

Since (u_n) coincides with the CG iteration in H_B, the estimates (1.1)–(1.2) apply to (u_n), thus (2) holds. ∎

EXAMPLE. The following example illustrates how the obtained method can be applied to elliptic boundary value problems. To reduce the length of calculations, we consider the simple semilinear problem

$$\begin{cases} -\triangle u + u^3 = g \\ u_{|\partial\Omega} = 0 \end{cases} \tag{2.2}$$

where $\Omega := [0,\pi] \times [0,\pi] \subset \mathbf{R}^2$, $g \in L^2(\Omega)$. Denote by D the set

$$\left\{ \sum_{k+m \le n} a_{k,m} \sin kx \sin my \, : \, n \in \mathbf{N}^+, a_{k,m} \in \mathbf{R} \ (k+m \le n) \right\}$$

(sine polynomials). Then D is a dense subspace of $H := L^2(\Omega)$. We fix $\varepsilon > 0$ and approximate g by $\tilde{g} \in D$ such that $\|g - \tilde{g}\|_{L^2(\Omega)} < \frac{\varepsilon}{K_2}$ (where K_2 is defined in (2.4)).

Let $T : D \to L^2(\Omega)$, $T(u) := -\triangle u + u^3 - \tilde{g}$. The generalized differential operator $J : H_0^1(\Omega) \to H_0^1(\Omega)$ corresponding to T is defined by

$$\langle J(u), v \rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla u \nabla v + u^3 v - \tilde{g} v) \quad (v \in H_0^1(\Omega)).$$

Let $B : D \to L^2(\Omega)$, $B := -\triangle$. Then $H_B = H_0^1(\Omega)$ since the elements of D vanish on $\partial\Omega$. We will use the notations $\langle u, v \rangle_B := \langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \nabla v$ $(u, v \in H_0^1(\Omega))$ and $\|v\|_B$ for the corresponding norm.

LEMMA 2.1 *$R(B) \supset R(T)$ and the operator $J : H_B \to H_B$ is an extension of $B^{-1}T : D \to H_B$.*

PROOF. (i) $R(T) \subset D$ since for any sine polynomial u also $\triangle u$ and u^3 are sine polynomials. Further, $R(B) = D$ since for any $f(x,y) = \sum_{k+m\le n} a_{k,m} \sin kx \sin my$ the function

$$u(x,y) = \sum_{k+m\le n} \frac{a_{k,m}}{k^2+m^2} \sin kx \sin my \tag{2.3}$$

is the solution of $\triangle u = f$, $u_{|\partial\Omega} = 0$.

(ii) Let $u \in D$. Owing to the homogeneous Dirichlet BC we have

$$\left\langle B^{-1}T(u), v \right\rangle_B = \langle T(u), v\rangle_{L^2(\Omega)} = \int_\Omega (\nabla u \nabla v + u^3 v - \tilde{g} v) = \langle J(u), v\rangle_B$$

for any $v \in D$, hence for any $v \in H_0^1(\Omega)$ owing to density. I.e. J is an extension of $B^{-1}T$. ∎

LEMMA 2.2 *J fulfils assumptions 1 on $H_0^1(\Omega)$.*

PROOF. We are going to use throughout the proof the embeddings

$$H_0^1(\Omega) \subset L^p(\Omega), \quad \|w\|_{L^p(\Omega)} \le K_p \|w\|_B \quad (w \in H_0^1(\Omega)) \tag{2.4}$$

for all $p \ge 2$ with appropriate constants $K_p > 0$ (see [4]). Further, let $\mathcal{E} := \{v \in H_0^1(\Omega) : \|v\|_B \le 1\}$.

(i) For any $u \in H_0^1(\Omega)$ let $S(u) : H_0^1(\Omega) \to H_0^1(\Omega)$ be the bounded linear operator defined by

$$\langle S(u)h, v\rangle_B := \int_\Omega (\nabla h \nabla v + 3u^2 h v) \quad (h, v \in H_0^1(\Omega)).$$

Its existence is obtained from the estimate $|\int_\Omega (\nabla h \nabla v + 3u^2 hv)| \le (1 + 3K_4^4 \|u\|_B^2)\|h\|_B \|v\|_B$ and the Riesz theorem. Let $u, h \in H_0^1(\Omega)$ be fixed. Then, using notation $\delta(J)_{u,h}(t) := \|\frac{1}{t}\big(J(u+th) - J(u)\big) - S(u)h\|_B$ $(t \in \mathbf{R})$, we have

$$\delta(J)_{u,h}(t) = \sup_{v\in\mathcal{E}} \left\langle \frac{1}{t}(J(u+th) - J(u)) - S(u)h, v \right\rangle_B =$$

$$= \sup_{v\in\mathcal{E}} \int_\Omega (3tuh^2 v + t^2 h^3 v) \le K_4^4 (3t \|u\|_B \|h\|_B^2 + t^2 \|h\|_B^3) \;\to\; 0 \text{ if } t \to 0.$$

I.e. J is Gâteaux differentiable and $J'(u) = S(u)$ $(u \in H_0^1(\Omega))$.

Now for any $u \in H_0^1(\Omega)$ let $P(u) : H_0^1(\Omega) \times H_0^1(\Omega) \to H_0^1(\Omega)$ be the bounded bilinear operator determined by

$$\langle (P(u)h)v, w\rangle_B := 6\int_\Omega uhvw \quad (h, v, w \in H_0^1(\Omega)).$$

Its existence is now due to the estimate

$$\left|\int_\Omega uhvw\right| \leq K_4^4 \|u\|_B \|h\|_B \|v\|_B \|w\|_B .$$

For fixed $u, h \in H_0^1(\Omega)$ we now have

$$\delta(J')_{u,h}(t) := \|\frac{1}{t}\left(J'(u+th) - J'(u)\right) - P(u)h\|_B =$$

$$= \sup_{v,w\in\mathcal{E}} \left\langle \frac{1}{t}\left(J'(u+th) - J'(u)\right)v - (P(u)h)v, w\right\rangle_B =$$

$$= \sup_{v,w\in\mathcal{E}} \int_\Omega 3th^2vw \leq 3tK_4^4\|h\|_B^2 \to 0 \text{ if } t \to 0.$$

Hence J is twice Gâteaux differentiable and $J''(u) = P(u)$ $(u \in H_0^1(\Omega))$.
(ii) We have

$$\|h\|_B^2 \leq \langle J'(u)h, h\rangle_B = \int_\Omega (|\nabla h|^2 + 3u^2h^2) \leq \|h\|_B^2 (1 + 3K_4^4\|u\|_B^2),$$

hence $\langle J'(u)h, h\rangle_B / \|h\|_B^2$ has bounds $a = 1$ and $A = 1 + 3K_4^4(R + \|u_0\|_B^2)$ on any ball $B(u_0, R)$. Further,

$$\|J''(u)\| = \sup_{h,v,w\in\mathcal{E}} \langle (J''(u)h)v, w\rangle_B \leq 6K_4^4\|u\|_B,$$

i.e. on any ball $B(u_0, R)$ we obtain $\|J''\| \leq B = 6K_4^4(R + \|u_0\|_B)$.
(iii) The potential of J satisfies

$$\phi(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{u^4}{4} - \tilde{g}u\right) \geq \frac{1}{2}\|u\|_B^2 - \|\tilde{g}\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \geq$$

$$\geq \|u\|_B \left(\frac{1}{2}\|u\|_B - K_2\|\tilde{g}\|_{L^2(\Omega)}\right) \to +\infty \quad \text{if } \|u\|_B \to +\infty.$$

Hence any level set of ϕ is bounded. ∎

(Note that the lemma implies that Theorem 1 yields a theoretical approximation to equation $J(u) = 0$ on $H_0^1(\Omega)$. However, the functions $p_0 = r_0 = -J(u_0)$ and later, for any $n \in \mathbf{N}$, $r_{n+1} = -J(u_n)$ (and thus $p_{n+1} = r_{n+1} + b_n p_n$), cannot be given explicitly for general functions $u_0 \in H_0^1(\Omega)$ (and later for $u_n \in H_0^1(\Omega)$). On the other hand, the application of Theorem 2.1 gives an easily performed construction of the approximating sequence, which we are now going to summarize.)

Owing to lemmas 1.1–1.2, our example satisfies all the assumptions of Theorem 2.1. Thus we obtain the following result:

THEOREM 2.2 *Let $u_0 \in D$, $p_0 = r_0 = u_0 - \triangle^{-1}(u_0^3 - \tilde{g})$. For $n \in \mathbf{N}$, successively, let $u_{n+1} := u_n + c_n p_n$ where c_n is the smallest positive root of*

$$\alpha_n c^3 + \beta_n c^2 + \gamma_n c + \delta_n = 0,$$

using notation $\alpha_n := 3\int_\Omega p_n^4$, $\beta_n := 9\int_\Omega p_n^3 u_n$, $\gamma_n := \int_\Omega (9p_n^2 u_n + |\nabla p_n|^2$, $\delta_n :=$ $= \int_\Omega (3p_n u_n^3 + \nabla u_n \nabla p_n - \tilde{g} p_n)$. Set

$$r_{n+1} := u_{n+1} - \triangle^{-1}(u_{n+1}^3 - \tilde{g}),\ p_{n+1} = r_{n+1} + b_n p_n$$

$$\text{where} \quad b_n := -\int_\Omega (\nabla p_n \nabla r_{n+1} + 3u_{n+1}^2 p_n r_{n+1}) / \int_\Omega (|\nabla p_n|^2 + 3u^2 p_n^2).$$

Denote by $\tilde{u}$ the weak solution of equation $-\triangle u + u^3 = \tilde{g}$, $u_{|\partial\Omega} = 0$.

Then (u_n) converges to $\tilde{u}$ in $H_0^1(\Omega)$ such that the linear convergence estimates (1.1)–(1.2) hold in $H_0^1(\Omega)$.

REMARKS. 1. Denote by u^* the solution of (2.2). Then

$$\|\tilde{u} - u^*\|_B^2 \le \int_\Omega (|\nabla(\tilde{u} - u^*)|^2 + (\tilde{u}^3 - u^{*3})(\tilde{u} - u^*)) =$$

$$= \left\langle -\triangle \tilde{u} + \tilde{u}^3 + \triangle u^* - u^{*3}, \tilde{u} - u^* \right\rangle_B = \langle \tilde{g} - g, \tilde{u} - u^* \rangle_B \le$$

$$\le \|\tilde{g} - g\|_{L^2(\Omega)} \|\tilde{u} - u^*\|_{L^2(\Omega)} \le K_2 \|\tilde{g} - g\|_{L^2(\Omega)} \|\tilde{u} - u^*\|_B.$$

Hence the solution of the original equation is approximated by $\tilde{u}$ such that

$$\|\tilde{u} - u^*\|_B < K_2 \|\tilde{g} - g\|_{L^2(\Omega)} = \varepsilon.$$

2. The *numerical performance* of the method is made easy by keeping (u_n) in D. Namely, the inversion of $-\triangle$ means dividing the coefficients $a_{k,m}$ by $k^2 + m^2$ (see (2.4)); further, integration in all formulae that define u_{n+1}

means linear combination of the coefficients since all integrands are sine polynomials. (If $f(x,y)=\sum_{k+m\leq n} a_{k,m}\sin kx \sin my$, then $\int_\Omega f = 4 \sum_{\substack{k+m\leq n \\ k,m \text{ are odd}}} \frac{a_{k,m}}{km}$.)

3. *(Comparison to other methods.)* The main advantage of the method (besides optimal linear convergence) is the easy realization given in the preceding remark. This appears e.g. in comparison to the finite element and the Galerkin methods where the approximations are determined via nonlinear algebraic systems of equations. The advantage compared to general discretization methods is the analytic form of the approximations (u_n).

The price for this is the number of iterations to be executed for prescribed accuracy.

4. Our aim has been to give an example of the application of Theorem 2.1. Thus we only mention without proof that the method can be applied in a similar way to more general quasilinear equations $T(u) = -\operatorname{div} f(x, \nabla u) + q(x,u) = g$, $u_{|\partial\Omega} = 0$ on domains $\Omega \subset \mathbf{R}^2$ (transformable to cubes or with $\partial\Omega \in C^2$), supposed the matrices $\{\partial_{\eta_j} f_i(x,\eta)\}_{i,j=1,..,N}$ have eigenvalues between positive constants and q increases in u with at most polynomial growth. (Then the suitable choice $D := H^2(\Omega) \cap H_0^1(\Omega)$ yields $R(-\triangle) = L^2(\Omega) \supset R(T)$; further, the same kind of calculations can be used as in [12] to prove the required differentiability properties of the generalized differential operator.)

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ON THE SOLVABILITY OF NONLINEAR INEQUALITY SYSTEMS

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1. Introduction

Inequality systems (both linear and nonlinear) stayed in the middle of attention of many mathematicians of our century. Beginning with Farkas' and Minkowski's classical results and continuing till our days, there have been stated different compatibility conditions with applications in diverse branches of mathematics. One of the resolvability theorems significant applications domain is the optimization theory, especially concerning the necessary conditions of optimum and that of the duality ([1], [18], [12], [10], [4]). In the last decade the resolvability theorems have been extended for nonlinear systems and there have been given some applications for problems of nonsmooth optimization ([5], [22,23], [12], [9], [29], [30], [13], [15], [7,8]), for variational inequalities ([26], [21]), for minimax theory ([20], [11], [16], [19], [27]), and so on.

In resolvability theorems there occur two kinds of conditions: a convexity condition (a generalized one) and a so called closedness condition. In the recent paper [17], using a new convexity notion, introduced by Sebestyén [24] and a closedness condition used in [10], have been proved two theorems that generalize some old results ([2], [3]) and some new ones ([24-25], [14], [10]). These theorems are the generalizations of Gordan's classical theorem of the alternatives ([18]).

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In this paper the above mentioned results are extended for the case of a more general convexity condition than that of [17]. The closedness condition used here is also less restrictive than that in recent papers [14] and [6].

The used notions and the main results have been formulated in section 2. As applications, in section 3 have been given some generalizations of Mozkin's and Farkas' theorems of the alternatives.

2. Gordan type theorem of the alternatives

Let A and B be two nonempty sets and $\varphi : A\times B \to \mathbf{R}$ a function. We'll study the following two problems:

(P) Find an element $a_0 \in A$ satisfying the inequality

$$\varphi(a_0, b) \geq 0, \quad \forall\, b \in B.$$

(Q) Give conditions (both necessary and sufficient) under which the following inequality holds:

$$\sup_{a\in A} \inf_{b\in B} \varphi(a,b) \geq 0. \tag{2.1}$$

Clearly, if **(P)** admits a solution, then **(Q)** is also satisfied, the inverse affirmation being false. Problems **(P)** and **(Q)** cover many special problems as: minimization problems, saddle point problems, Kirszbraun's problem and so on. (See [14], [10], [17], [28]).

The following notations will be used. Let

$$\Delta_m := \{\lambda \in \mathbf{R^m} \mid \lambda = (\lambda_1, \ldots, \lambda_m),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\}$$

the $m-1$-dimensional standard simplex. If $b \in B$ and δ is a positive real number then

$$U(b,\delta) := \{a \in A \mid \varphi(a,b) + \delta < 0\}.$$

If $B_0 = \{b_1, \ldots, b_n\}$ is a finite subset of B and $\delta > 0$, then let us define the function $\psi = \psi(B_0;\delta) : A \to \mathbf{R^n}$ by

$$\psi(a) = (\varphi(a,b_1) + \delta, \ldots, \varphi(a,b_n) + \delta).$$

Put $\psi(A) = \{\psi(a) \mid a \in A\}$ and let $co(\psi(A))$ denote the convex hull of $\psi(A)$. Let $\emptyset$ denote the empty set and $\mathbf{R^n_+}$ the cone of elements from $\mathbf{R^n}$ with nonnegative components. The interior of this cone is $int(\mathbf{R^n_+})$.

The results obtained in the present paper are strongly related to the following convexity condition:

(CC) *For all* $\delta_0 > 0$ *and for all finite subsets* $B_0 \subset B$, *containing* n_0 *elements such that* $\psi(B_0,\delta_0)(A) \cap int(\mathbf{R}_+^{\mathbf{n_0}}) = \emptyset$, *there exist a finite subset* B_1 *of* B *and* $\delta_1 \in]0,\delta_0]$ *such that* $B_0 \subset B_1$ *and* $co(\psi(B_1,\delta_1)(A)) \cap int(\mathbf{R}_+^{\mathbf{n_1}}) = \emptyset$, *where* n_1 *is the number of the elements of* B_1.

If B is a finite set then in (CC) one can put $B_0 = B$ without loss of generality.

In [24] the following inequality has been considered:

$$\min_{1 \le j \le n} \sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \le \sup_{a \in A} \min_{1 \le j \le n} \varphi(a, b_j), \tag{2.2}$$

for all finite subsets $\{a_1,\ldots,a_m\} \subseteq A$ and $\{b_1,\ldots,b_n\} \subseteq B$, and all $\lambda \in \Delta_m$.

Clearly, if (2.2) is satisfied then (CC) holds. If A is a convex subset of a real vector space and $\varphi(.,b): A \to \mathbf{R}$ is a concave function for all $b \in B$, then (CC) holds. Even more, in [17,Proposition 3.1] it has been proved that if for any $\varepsilon > 0$, for every finite subets B_0, every $a_1, a_2 \in A$ and every $t \in]0,1[$ there exists $a_3 \in A$ such that

$$\varphi(a_3,b) \ge t\varphi(a_1,b) + (1-t)\varphi(a_2,b) - \varepsilon$$

holds for every $b \in B_0$ then inequality (2.2) holds for every $\{a_1,\ldots,a_m\} \subseteq A$, $\{b_1,\ldots,b_n\} \subseteq B$ and $\lambda \in \Delta_m$. Hence (CC) holds in this case, too.

The following example shows that (CC) is a less restrictive condition then (2.2). Let $A = B = \{1,2\}$, $\varphi(1,1) = \varphi(2,2) = 0$ and $\varphi(1,2) = \varphi(2,1) = -1$. For $\lambda_1 = \lambda_2 = \frac{1}{2}$ we have then

$$\min_{1 \le j \le 2} \sum_{i=1}^{2} \lambda_i \varphi(i,j) = -\frac{1}{2}$$

and

$$\max_{1 \le i \le 2} \min_{1 \le j \le 2} \varphi(i,j) = -1,$$

so (2.2) does not hold. For $B_0 = B$ and $\delta > 0$ we have $\psi(1) = (\delta, \delta - 1)$, $\psi(2) = (\delta - 1, \delta)$, therefore, one can choose $0 < \delta_1 \le \frac{1}{2}$ so that (CC) holds.

PROPOSITION 2.1. *Let* $B = \{b_1,\dots,b_n\}$ *be a finite set and suppose that* (CC) *is satisfied. Inequality* (2.1) *holds if and only if*

$$\inf_{\mu\in\Delta_n}\sup_{a\in A}\sum_{j=1}^{n}\mu_j\varphi(a,b_j)\geq 0. \tag{2.3}$$

PROOF. *Necessity:* If (2.1) holds, then for any $\varepsilon>0$ there exists $a_3\in A$ such that

$$\varphi(a_3,b)\geq -\varepsilon,\quad \forall\, b\in B$$

so,

$$\sup_{a\in A}\sum_{j=1}^{n}\mu_j\varphi(a,b_j)\geq -\varepsilon,$$

for every $\mu\in\Delta_n$. If $\varepsilon\to 0$ one can obtain (2.3).

Sufficiency: If inequality (2.1) does not hold, then there exists $\delta_0>0$ such that for each $a\in A$ we have the following inequality:

$$\min_{b\in B}\varphi(a,b)+\delta_0\leq -\varepsilon.$$

Defining the function $\psi=\psi(B,\delta_0)$ as above, we have $\psi(A)\cap int(\mathbf{R}^{\mathbf{n}}_+)=\emptyset$. (CC) implies the existence of a $\delta>0$ such that

$$co(\psi(B,\delta)(A)\cap int(\mathbf{R}^{\mathbf{n}}_+)=\emptyset.$$

In conformity with the separation theorem there exists $\mu\in\Delta_n$, $\mu=(\mu_1,\dots,\mu_n)$ such that

$$\sum_{j=1}^{n}\mu_j\varphi(a,b_j)+\delta\leq 0,\quad \forall a\in A,$$

holds. Therefore, we have

$$\inf_{\mu\in\Delta_n}\sup_{a\in A}\sum_{j=1}^{n}\mu_j\varphi(a,b_j)\leq -\delta<0,$$

in contradiction with (2.3). ∎

REMARKS. 1) In the special case when (CC) is substituted with (2.2), Proposition 2.1 has been proved in [17].

2) If $A=\mathbf{R}^{\mathbf{p}}$ $(p\in\mathbf{N})$ and $\varphi(.,b):A\to\mathbf{R}$ is an affine function for each $b\in B$, Proposition 2.1 is reduced to Gordan's classical theorem of the alternatives.

If B is an infinite set, in order to formulate an analogus result the previous one, we need certain closedness assumption.

DEFINITION 2.1. We say that φ satisfies the weak closedness condition if

$$\sup_{a\in A}\inf_{b\in B_0}\varphi(a,b)\geq 0, \tag{2.4}$$

holds for every finite subset B_0 of B then (2.1) holds also true.

Note, that the weak closedness condition can be formulated in several ways. Let us consider the space of functions $\mathbf{R}^{\mathbf{B}}=\{w:B\to\mathbf{R}\}$ endowed with the product topology π, where the neighborhood base of $w_0\in\mathbf{R}^{\mathbf{B}}$ is given by

$$W(B_0,\varepsilon,w_0)=\{w\in\mathbf{R}^{\mathbf{B}}\mid |w_0(b)-w(b)|<\varepsilon,\ \forall b\in B_0\},$$

where $\varepsilon>0$ andB_0 is a finite subset of B.

Let us introduce also, in $\mathbf{R}^{\mathbf{B}}$, the weaker topology τ, where the neighborhood base of $w_0\in\mathbf{R}^{\mathbf{B}}$ is given by the sets

$$V(B_0,\varepsilon,w_0)=\{w\in\mathbf{R}^{\mathbf{B}}\mid w_0(b)-w(b)<\varepsilon,\ \forall b\in B_0\},$$

where ε and B_0 are the same as above. Consider further the set

$$C=\{w\in\mathbf{R}^{\mathbf{B}}\mid \exists a\in A:\varphi(a,b)\geq w(b),\ \forall b\in B\}.$$

Let $cl\,C$ and $\bar{C}$ be the closure of the set C with respect to π and τ, respectively.

In [17, Proposition 2.2] it has been proved that the following four assertions are equivalent:

(i) The weak closedness condition is satisfied.

(ii) If for some $\delta_0>0$ the system $\{U(b,\delta_0)|b\in B\}$ covers A, then $\{U(b,\delta)|\ b\in B,\delta>0\}$ admits a finite subcover of A.

(iii) If $\underline{0}\in\overline{C}$ then $-\underline{\delta}\in C$ for all $\delta>0$, where $\underline{0}$ and $\underline{\delta}$ denotes the constant functions 0 and δ, respectively.

(iv) If $\underline{0}\in cl\,C$ then $-\underline{\delta}\in C$ for all $\delta>0$.

A variant of assumption (ii) first has been formulated within a minimax theorem in [24]. Assumption (iii) has been introduced in [10].

If B is a nonempty arbitrary set (finite or infinite) we state the following result:

THEOREM 2.1. *Suppose that* (CC) *is satisfied. Then inequality* (2.1) *holds if and only if the weak closedness condition is satisfied and inequality* (2.3) *holds for all finite subset* $\{b_1,\dots,b_n\}$ *of* B.

PROOF. The *necessity* can be proved as in case of Proposition 2.1, since (2.1) implies (iii).

Sufficiency: Let B_0 be a finite subset of B. As in case of Proposition 2.1, the weak closedness condition and (2.3) imply (2.1). ∎

Theorem 2.1 can be regarded as an extension of [10, Theorem 2] and [17, Theorem 2.1].

For the study of problem **(P)** we need the following closedness condition.

DEFINITION 2.2. We say that φ satisfies the strong closedness condition if the validity of the inequality (2.4), for all finite subsets B_0 of B, implies that problem **(P)** admits at least one solution.

Obviously,the strong closedness condition implies the weak closedness condition. The inverse statement does not hold, as the example of the function

$$\varphi :]0,1]\times]0,1] \to \mathbf{R},\ \varphi(a,b) = b - a$$

shows.

Observe, that if A is a compact topological space and if φ is upper semicontinuous with respect to the first variable, then the strong closedness condition holds true. In paper [10] there have been given some examples such that A was not compact but the strong closedness condition held. Even more, in [17, proposition 2.3] it has been proved that the following four assertions are equivalent:

(i′) The strong closedness condition holds.

(ii′) If the system $\{U(b,\delta)\,|\, b \in B, \delta > 0\}$ covers A, then it contains a finite subcover.

(iii′) If $\underline{0}\in\overline{C}\Rightarrow\underline{0}\in C$.

(iv′) If $\underline{0}\in cl\,C\Rightarrow\underline{0}\in C$.

Condition (ii′) has been used previously in [25] and (iii′) in [10]. In [14] and [6] the following closedness condition is used: (v) There exists a neighborhood U of the origin $\underline{0} \in \mathbf{R}^{\mathbf{B}}$, with respect to the topology π, such that the set $C \cap cl\,U$ is nonempty and closed. A variant of this condition has been introduced in [15].

Observe that (v) implies (iii′). Indeed, if $\underline{0} \in cl\,C$, then there exists a generalized sequence (w_k) in C such that $w_k \xrightarrow{\pi} \underline{0}$. Since the neighborhood

U satisfies the properties of (v), there exists a generalized subsequence (w_ν) such that $w_\nu \in C \cap U$ for all indices ν, and $\underline{0} = \lim_\nu w_\nu \in C \cap cl\, U$, we have $\underline{0} \in C$. ∎

The following example shows that in general condition (iii′) *does not* impliy (v). Indeed, let $A = \mathbf{N}^*$, $B = [0,1[$ and $\varphi(a,b) = 1 - b^a$, for all $(a,b) \in A \times B$. Then for any neighborhood U of the origin $\underline{0}$ of the space $\mathbf{R}^\mathbf{B}$ with respect to the π topology, there exists $\varepsilon \in]0,1[$ such that the function $w : B \to \mathbf{R}$ defined by

$$w(b) = \begin{cases} 0, & \text{if } b \in [0,\varepsilon] \\ 1, & \text{if } b \in]\varepsilon,1] \end{cases}$$

be contained in $cl(C \cap cl\, U)$ without belonging to $C \cap cl\, U$. Consequently, in this case property (v) does not hold true.

From the above statements we can conclude that though condition (iii′) is necessary for the existence of solutions for the problem **(P)**, property (v) is not necessary for that.

The following theorem is also a generalization of Gordan's theorem. It extends [17, Theorem 2.2] (see also Theorem 3.5 from [14], and Theorem2 from [10]).

THEOREM 2.2. *Suppose that* (CC) *is satisfied. Problem* **(P)** *has solution if and only if the strong closedness condition is satisfied and inequality (2.3) holds for all finite subsets* $\{b_1,\ldots,b_n\}$ *of* B.

The proof of this theorem is analogus to that of Theorem 2.1. ∎

3. Motzkin and Farkas type theorems of the alternatives

Let X and Y be two nonempty sets. We consider the functions $f : X \to \mathbf{R} \cup \{+\infty\}$ with $domf := \{x \in X \mid f(x) < +\infty\} \neq \emptyset$ and $g : X \times Y \to \mathbf{R}$. In this section there will be studied the resolvability of the following system:

$$x \in X : f(x) > 0,\ g(x,y) \geq 0 (\forall y \in Y). \tag{3.1}$$

This problem is reduced to **(P)**, if we put $A = X \times]0,+\infty[$, $B = \{\alpha\} \cup Y$, where α is any element not belonging to Y, and

$$\varphi(a,b) = \begin{cases} f(x) - \varepsilon, & \text{for } a = (x,\varepsilon) \text{ and } b = \alpha, \\ g(x,b), & \text{for } a = (x,\varepsilon) \text{ and } b \in Y. \end{cases} \tag{3.2}$$

In this case the set C is of the form

$$C = \{w = (u,r) \in \mathbf{R}^{\mathbf{Y}} \times \mathbf{R} \mid \exists (x,\varepsilon) \in X \times]0,+\infty[: f(x) \geq r + \varepsilon, \\ g(x,y) \geq u(y),\ \forall y \in Y\}.$$

So, the strong closedness condition can be formulated in the following way:

$$(\underline{0},0) \in cl\,C \ \Rightarrow\ (\underline{0},0) \in C. \tag{3.3}$$

This condition is verified, for example, if there exists a neighborhood U of the origin $\underline{0}$ of the space $\mathbf{R}^{\mathbf{Y}}$ with respect to the τ topology, and there exists $\gamma > 0$ such that the set $C \cap cl\,U \times [-\gamma,+\infty[$ is nonempty and closed in $\mathbf{R}^{\mathbf{Y}} \times \mathbf{R}$.

Like in [14] we say that the pair of functions (f,g) are *concavelike*, if

$$\exists t \in]0,1[:\ \forall\, x_1 x_2 \in X,\ \exists x_3 \in X : \\ f(x_3) \geq t f(x_1) + (1-t) f(x_2)\ and\ g(x_3,y) \geq t g(x_1,y) + (1-t) g(x_2,y), \\ \forall y \in Y.$$

It is easy to verify that if the pair (f,g) is concavelike then the function φ defined by (3.2) satisfies inequality (2.2) (see [17]). So, in this case property (CC) is satisfied.

From Theorem 2.2 we can deduce immediatelly Motzkin's theorem of the alternatives, which generalizes [14, Theorem 3.1].

THEOREM 3.1. *Suppose that the function φ, defined by* (3.2)*, satisfies property* (CC) *and condition* (3.3)*. Then one and only one of the following two assertions holds true:*

a) *The system (3.1) is compatible.*

b) *For all $\varepsilon > 0$ there exist a finite subset $\{y_1,\ldots,y_n\}$ of Y, an element $v \in \mathbf{R}^{\mathbf{n}}_{+}$ and a real number $\tau \geq 0$ such that $(v,\tau) \neq (0,0)$ and*

$$\tau(f(x) - \varepsilon) + \sum_{j=1}^{n} v_j g(x,y_j) < 0,\ \forall x \in X.$$

As a consequence of the last theorem we obtain the following generalization of Farkas' lemma.

THEOREM 3.2. *Suppose that the assumptions of Theorem* 3.1 *are satisfied. If there exists $x_0 \in X$ such that inequality $g(x_0,y) \geq 0$ holds true for all $y \in Y$, then the following two assertions are equivalent:*

α) *For all $x \in X$ with property $g(x,y) \geq 0$, for all $y \in Y$, inequality $f(x) \leq 0$ holds true.*

β) *For all* $\varepsilon > 0$ *there exists a finite subset* $\{y_1,\dots,y_n\}$ *of* Y *and an element* $\nu \in \mathbf{R}^{\mathbf{n}}_+$ *such that*

$$f(x) + \sum_{j=1}^{n} \nu_j g(x, y_j) < \varepsilon, \quad \textit{for all} \quad x \in X.$$

PROOF. Property α) means that the system (3.1) is incompatible, which is equivalent, by Theorem 3.1, with property b). Suppose that in b) we have $\tau = 0$. Then

$$\sum_{j=1}^{n} \nu_j g(x, y_j) < 0, \ \forall\, x \in X,$$

which is in a contradiction to the existence of an element $x_0 \in X$ with the property from the statement of the theorem. We can suppose, without a loss a generality, that $\tau = 1$, so property α) implies β). The inverse statement $\beta) \Rightarrow \alpha)$ is evident. ∎

REMARKS. 1) If we complete condition (CC) and the strong closedness condition with the assuption that the pair (f, g) is concavelike and there exists a neighborhood of $\underline{0} \in \mathbf{R}^{\mathbf{Y}}$, respectively a number $\gamma > 0$ such that the set $C \cap cl\, U \times [-\gamma, +\infty[$ is nonempty and closed in $\mathbf{R}^{\mathbf{Y}} \times \mathbf{R}$, then by Theorem 3.2 we obtain [14;Theorem 3.2].

2) In case when (CC) holds and there exists an element $x_0 \in X$ such that $g(x_0, y) \geq 0$ for all $y \in Y$, *property* β) *is equivalent to the following one ([14]):*
γ) If $(\underline{0}, \sigma) \in cl\, C$, then $\sigma \leq 0$.

Indeed, by assumption γ) it follows that $(\underline{0}, 0) \notin cl\, C$, so $(\underline{0}, 0) \notin C$. This means that the system (3.1) is incompatible. By Theorem 3.1, the existence of an element x_0 with the above mentioned property implies β). Conversely, if we suppose that β) holds whilest γ) doesn't hold, we will come to contradiction. Let $(\underline{0}, 0) \in C$ and $\varepsilon > 0$. By assumption β) there exists a finite subset $\{y_1, \dots, y_n\}$ of Y and an element $\nu \in \mathbf{R}^{\mathbf{n}}_+$ such that inequality

$$f(x) + \sum_{j=1}^{n} \nu_j g(x, y_j) < \varepsilon, \ \forall\, x \in X \tag{3.4}$$

holds true.

By the definition of $cl\,C$ we can conclude that for any $\delta > 0$ there exists $x \in X$ such that

$$f(x) \geq \frac{3}{4}\varepsilon \ and\ g(x, y_j) \geq -\delta,\ \forall j \in \{1, \dots, n\}.$$

If we choose δ such that $\delta \sum_{j=1}^{n} \nu_j \leq \varepsilon$, then we have

$$f(x) + \sum_{j=1}^{n} \nu_j g(x, y_j) \geq \varepsilon,$$

which contradicts (3.4).

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ON A CLASS OF DISCONTINUOUS NONLINEAR ELLIPTIC EQUATIONS

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1. Introduction

In this paper we study a class of quasilinear problems with discontinuous nonlinearities. It is well-known that such problems need not have a solution and in order to develop an adequate existence theory, we need to consider instead a multivalued version of the original problem. In general this multivalued version is obtained by, roughly speaking, filling in the gaps at the discontinuity points (see CHANG [2]) and then use some known method (like nonsmooth critical point theory or the method of upper and lower solutions) to obtain existence theorems.

In this paper following the lead of STUART–TOLAND [7], we consider a multivalued version of the original problem, by filling in only the downwards jumps of the nonlinearity $f(\cdot)$. This leads to a more restrictive (but more interesting) problem which approximates better the original single valued one. This formulation requires $f(\cdot)$ to be locally of bounded variation but otherwise we do not need a subcritical growth condition like the one assumed by CHANG [2]. Our approach is variational and utilizes a nonresonance condition below the first eigenvalue $\lambda_1 > 0$ of the p-Laplacian. Our existence theorem extends theorem 3.2 of STUART–TOLAND [7], who study semilinear equations (i.e. $p=2$).

2. Preliminaries

Let $Z \subseteq R^N$ be a bounded domain with a C^1-boundary Γ. We consider the following quasilinear elliptic differential equation.

$$(1) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = f(x(z)) + u(z) \quad \text{in} \quad Z, \\ x|_\Gamma = 0, \quad 2 \le p \le \infty. \end{array} \right\}$$

Here $f : R \to R$ is locally of bounded variation. We do not make any continuity hypotheses on $f(\cdot)$. It is well-known that $f(\cdot)$ has a countable number of jump discontinuities. We require that $f(r) \in \overline{\text{conv}}(f(r^+), f(r^-))$ for every $r \in R$ (i.e. $f \in BN$ in the notation of STUART–TOLAND [7]). Let $U(f) = \{r \in R : f(r^-) < f(r^+)\}$ (the set of upward jumps) and $D(f) = \{r \in \in R : f(r^+) < f(r^-)\}$ (the set of downward jumps). Then $f(\cdot)$ is everywhere continuous except at the set $U(f) \cup D(f)$. Using these two sets we can define two multifunctions:

$$\hat{f}(r) = \begin{cases} (f(r^-), f(r^+)) & \text{if } r \in R \setminus D(f) \\ (f(r^+), f(r^-)) & \text{if } r \in D(f), \end{cases}$$

$$\hat{f}_0(r) = \begin{cases} f(r^-) & \text{if } r \in R \setminus D(f) \\ (f(r^+), f(r^-)) & \text{if } r \in D(f). \end{cases}$$

Evidently $f(\cdot)$ fills in all the jump discontinuities of the original function while $\hat{f}_0(\cdot)$ fills in only the downward ones. In our multivalued reformulation of problem (1) we will use $\hat{f}_0(\cdot)$. More precisely the multivalued problem that we will study instead of (1) is the following:

$$(2) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \hat{f}_0(x(z)) + u(z) \quad \text{in} \quad Z \\ x|_\Gamma = 0, \quad 2 \le p < \infty \end{array} \right\}$$

DEFINITION By a solution of (2) we mean a function $x \in W_0^{1,p}(Z)$ such that $x|_\Gamma = 0$ and

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = v(z) + u(z) \quad \text{a.e. on } Z$$

$$\text{with} \quad v \in L^2(Z),\ v(z) \in \hat{f}_0(x(z)) \quad \text{a.e. on } Z.$$

In our considerations we will need the first (principal) eigenvalue of the following non-linear eigenvalue problem:

$$(3) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda |x(z)|^{p-2}x(z) \quad \text{in} \quad Z \\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{array} \right\}$$

From LINDQVIST [3] we know that the first eigenvalue λ_1 exists, is positive and isolated. A corresponding eigenfunction $u_1 \in W_0^{1,p}(Z) \cap L^\infty(Z)$ can be chosen so that $u_1(z) > 0$ a.e. on Z. In fact under additional smoothness assumptions on the boundary Γ, we can have that $u_1 \in C^{1,\beta}$, $\beta \in (0,1)$ and $\frac{\partial u_1}{\partial n} < 0$ a.e. on Γ. Moreover, we know that λ_1 is given as the minimum of the Rayleigh quotient; i.e.

$$\lambda_1 = \inf \left\{ \frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z) \right\}$$

As we already mentioned, our approach will be variational. Since the functionals are convex and nondifferentiable in general, we will use the notion of the convex subdifferential. So let us recall very briefly some basic definitions and results of the subdifferential theory. Let X be a Banach space and $\varphi : X \to \overline{R} = R \cup \{\infty\}$ be a proper, convex function (proper means that $\operatorname{dom}\varphi = \{x \in X : \varphi(x) < +\infty\} \neq \emptyset\}$. By definition the "subdifferential" of φ is the (possibly multivalued) operator $\partial\varphi : X \to X^*$ defined by

$$\partial\varphi = \{x^* \in X^* : (x^*, y - x) \le \varphi(y) - \varphi(x) \quad \text{for all} \quad y \in X\}, \quad x \in X.$$

Let $D(\partial\varphi) = \{x \in X : \partial\varphi(x) \neq \emptyset\}$. We have that $D(\partial\varphi) \subseteq \operatorname{dom}\varphi$ and $\partial\varphi$ is a maximal monotone operator (in fact cyclically maximal monotone) If $\varphi(\cdot)$ is Gateaux differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$ (i.e. $\partial\varphi$ is a singleton, namely the Gateaux derivative of $\varphi(\cdot)$ at x). Moreover, we have $\overline{D(\partial\varphi)} = \overline{\operatorname{dom}\varphi}$ and $\operatorname{int}D(\partial\varphi) = \operatorname{int}\operatorname{dom}\varphi$. Let X be a Hilbert space identified with its dual (pivot space). Since $\partial\varphi$ is a maximal monotone operator, for every $\lambda > 0$ we can define a regular single valued approximation of it, known as the Yosida approximation, by setting $(\partial\varphi)_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ with I being the identity operator and J_λ the resolvent of $\partial\varphi$, i.e. $J_\lambda = (I + \lambda\partial\varphi)^{-1}$. Recall that $D(J_\lambda) = D(\partial\varphi)_\lambda = X$ for all $\lambda > 0$, both operators are single valued, J_λ is nonexpansive, $(\partial\varphi)_\lambda$ is monotone and Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$ hence maximal monotone and $(\partial\varphi)_\lambda(x) \in \partial\varphi(J_\lambda(x))$ for all $x \in X$. For the function $\varphi(\cdot)$ we can define the Moreau–Yosida regularization φ_λ by

$$\varphi_\lambda(x) = \inf \left\{ \frac{\|x - y\|^2}{2\lambda} + \varphi(y) : y \in X \right\}, \qquad x \in X, \ \lambda > 0.$$

Then φ_λ is convex and Frechet differentiable, $(\partial\varphi)_\lambda = \partial\varphi_\lambda$ for all $\lambda > 0$, $\varphi_\lambda = \frac{\|x-J_\lambda(x)\|^2}{2\lambda} + \varphi(J_\lambda(x))$ for all $x \in H$ and all $\lambda > 0$, and $\varphi_\lambda \uparrow \varphi$ as $\lambda \downarrow 0$. Finally if $\Phi : L^2(Z) \to \overline{R} = R \cup \{+\infty\}$ is defined by

$$\Phi(x) = \begin{cases} \int_Z \varphi(x(z))dz & \text{if } \varphi(x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Then Φ is proper convex and $v \in \partial\Phi(x)$ if and only if $v(x) \in \partial\varphi(x(z))$ a.e. on Z. Moreover, $(\partial\Phi)_\lambda(x) = \partial\Phi_\lambda(x) = \int_Z \partial\varphi_\lambda(x(z))dz$ for all $\lambda > 0$ and all $x \in L^2(Z)$. For more details and additional results on these and related issues, we refer to Barbu [1].

In the sequel we will use the following elementary inequality

$$(\alpha|\alpha|^{p-2} - \beta|\beta|^{p-2})(\alpha - \beta) \geq 2^{2-p}|\alpha - \beta|^p$$

for every $\alpha, \beta \in R$ and $p \geq 2$.

3. Auxiliary results

In our analysis of problem (2), we will need the following decomposition of $f(\cdot)$ proved by Stuart–Toland [7] (lemma 2.1). It is a refinement of a well-known result from analysis which says that a function which is locally of bounded variation can be written as the difference of two nondecreasing functions. Recall (see section 2) that $BN = \{f : R \to R : f(\cdot)$ is locally of bounded variation and for every $r \in R$ $f(r) \in \text{conv}\{f(r^+), f(r^-)\}\}$.

Proposition 1. *If $f \in BN$, then there exists two nondecreasing functions $g, h : R \to R$ such that*

1. *$f(r) = g(r) - h(r)$ for every $r \in R$;*
2. *$g(\cdot)$ is continuous on $R \setminus U(f)$;*
3. *$h(\cdot)$ is continuous on $R \setminus D(f)$;*
4. *$\hat{f}(r) = \hat{g}(r) - \hat{h}(r)$ and $\hat{f}_0(r) = g(r) - \hat{h}(r)$ for all $r \in R$.*

We introduce the potential functions G, $H : R \to R$ corresponding to g, h respectively, i.e.

$$G(r) = \int_0^r g(s)ds \qquad \text{and} \qquad H(r) = \int_0^r h(s)ds, \qquad r \in R.$$

Evidently $G(\cdot)$ and $H(\cdot)$ are continuous convex functions. Using them we define the following two integral functionals on the Hilbert space $L^2(Z)$

$$V_1(x) = \begin{cases} \int_Z \int_0^{x(z)} g(s)ds\,dz + \int_Z u(z)x(z)dz & \text{if } x \in W_0^{1,p}(Z), \\ & \quad G(x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise,} \end{cases}$$

$$V_2(x) = \begin{cases} \frac{1}{p}\|Dx\|_p^p + \int_Z \int_0^{x(z)} h(s)ds\,dz & \text{if } x \in W_0^{1,p}(Z),\ G(x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise,} \end{cases}$$

In the next proposition we have gathered some basic properties of V_1, V_2 and their subdifferentials.

PROPOSITION 2. 1. $V_1,\ V_2 : L^2(Z) \to = \hat{R} \cup \{+\infty\}$ *are proper, convex and lower semicontinuous;*

2. $v_2 \in \partial V_2(x)$ *if and only if* $v_2(z) \in -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) + \hat{h}(x(z))$ *a.e. on* Z*;*

3. $v_1 \in \partial V_1(x)$ *if and only if* $v_1(x) \in \hat{g}(x(z)) + u(z)$ *a.e. on* Z*;*

4. *for every* $x \in L^2(Z)$, $\partial V_1(x) \cap \partial V_2(x)$ *is empty or a singleton.*

PROOF. 1. It is clear that V_1, V_2 are proper, convex. We will prove the lower semicontinuity of $V_2(\cdot)$, the proof for $V_1(\cdot)$ being similar. To prove the lower semicontinuity of $V_2(\cdot)$, we need to show that for every $\lambda \in R$ the sublevel set $L_\lambda = \{x \in L^2(Z) : V_2(x) \le \lambda\}$ is closed. To this end let $\{x_n\}_{n\ge1} \subseteq L_\lambda$ and assume that $x_n \to x$ in $L^2(Z)$ as $n \to \infty$. By passing to a subsequence if necessary, we may also assume that $x_n(z) \to x(z)$ a.e. on Z. The function $H(\cdot)$ being convex, continuous, is minorized by an affine function $\alpha r + \beta$, α, $\beta \in R$. So $0 \le H(x_n(z)) - \alpha x_n(z) - \beta$ a.e. on Z and so by Fatou's lemma we have $\int_z (H(x(z)) - \alpha x(z) - \beta)dz \le$
$\le \underline{\lim} \int_Z (H(x_n(z)) - \alpha x_n(z) - \beta dz \Rightarrow \int_Z H(x(z))dz < \underline{\lim} \int_Z (H(x_n(z))dz$. Also $\|Dx\|_p^p + \int_Z H(x_n(z))dz \le \lambda \Rightarrow \|Dx_n\|_p^p \le c_1 + c_2\|x_n\|_2$ for some c_2, $c_1 > 0$ and for all $n \ge 1$. Hence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ is bounded and by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ as

$n \to \infty$. Recalling that the norm functional in a Banach space is weakly lower semicontinuous, we have $\|Dx\|_p^p \leq \underline{\lim}\|Dx_n\|_p^p$. Thus in the limit as $n \to \infty$ we have $\frac{1}{p}\|Dx\|_p^p + \int_Z H(x(z))dz \leq \lambda$, hence $x \in L_\lambda$ and so it follows that $V_2(\cdot)$ is lower semicontinuous.

2. Let $A: W_0^{1,p}(Z) \to W_0^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N} dz.$$

Here by $\langle \cdot, \cdot\rangle$ we denote the duality brackets for the pair

$$(W_0^{1,p}(Z), W^{-1,q}(Z)).$$

It is easy to check that $A(\cdot)$ is demicontinuous. Moreover, we have

$$\langle A(x) - A(y), y - x\rangle =$$

$$= \int_Z (\|Dx\|^{p-2}(Dx(z), Dx(z) - Dy(z))_{R^N}) -$$

$$-\|Dy(z)\|^{p-2}(Dy(z), Dx(z) - Dy(z))_{R^N})dz \geq$$

$$\geq \|Dx\|_p^p + \|Dy\|_p^p - \|Dx\|_p^{p-1}\|Dy\|_p - \|Dx\|_p\|Dy\|_p^{p-1} =$$

$$= (\|Dx\|_p^{p-1} - \|Dy\|_p^{p-1})(\|Dx\|_p - \|Dy\|_p).$$

Using the inequality mentioned at the end of section 2, we obtain

$$\langle A(x) - A(y), x - y\rangle \geq \int_Z |\|Dx(z)\|^p - \|Dy(z)\|^p|\, dz \qquad \text{for some } c > 0.$$

From this it follows that $A(\cdot)$ is monotone (in fact strictly monotone). So $A(\cdot)$ being monotone, demicontinuous and everywhere defined is maximal monotone (see PASCALI–SBURLAN [5], p. 106).

Let $D = \{x \in W_0^{1,p}(Z) : A(x) \in L^2(Z)\}$ and let $\hat{A} = A|_D$, *i.e.* $\hat{A} : D \subseteq \subseteq L^2(Z) \to L^2(Z)$. We claim that $\hat{A}$ is maximal monotone. To this end, it suffices to show that $R(\hat{A}+I) = L^2(Z)$ (i.e. $\hat{A}+I$ is surjective; see PASCALI–SBURLAN [5], theorem 2.11, p. 123). Here I is the identity operator on $L^2(Z)$. Let $\iota : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the embedding map. Then $\iota = I|_{W_0^{1,p}(Z)}$.

From corollary 2.9, p. 120 of PASCALI–SBURLAN [5], we know that $R(A+\iota)= = W^{-1,q}(Z)$. So given $v \in L^2(Z) \subseteq W^{-1,q}(Z)$ we can find $x \in W^{1,p}(Z)$ such that $A(x)+\iota(x)=v \Rightarrow A(x)=v-\iota(x) \in L^2(Z)$ (since $W_0^{1,p}(Z) \subseteq L^2(Z)$). So $A(x)=\hat{A}(x)$ and this shows that $v \in R(\hat{A}+I)$. Since $v \in L^2(Z)$ was arbitrary, we conclude that $R(\hat{A}+I)=L^2(Z)$ and this establishes the maximality of the monotone operator $\hat{A}(\cdot)$. In fact it is easy to see that $A(x)=\partial\psi(x)$, where $\psi : L^2(Z) \to \overline{R}=R \cup \{+\infty\}$ is defined by

$$\psi(x)=\begin{cases} \frac{1}{p}\|Dx\|_p^p & \text{if } x \in W_0^{1,p}(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Also let $W : L^2(Z) \to R$ be defined by

$$W(x)=\begin{cases} \int\limits_Z H(x(z))dz & \text{if } H(x) \in L^1(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

As before we can check that $W(\cdot)$ is proper, convex and lower semicontinuous. Moreover, we know (see section 2) that $\partial W(x)=\{w \in L^2(Z) : w(z) \in \in \partial H(x(z))$ a.e. on $Z\}$. For every $\lambda > 0$ we have $(\partial W)_\lambda = \partial W_\lambda$ and $W_\lambda(x)=\int\limits_Z H_\lambda(x(z))dz$ for all $x \in L^2(Z)$. Hence $\partial W_\lambda(x)=H'_\lambda(x(\cdot))=\hat{h}_\lambda(x(\cdot))$. If by $(\cdot,\cdot)_2$ we denote the inner product for the Hilbert space $L^2(Z)$, we have for all $x \in D$

$$\big(\hat{A}(x), \partial W_\lambda(x)\big)_2 = \langle \hat{A}(x), \partial W_\lambda(x)\rangle =$$

$$= \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\hat{h}_\lambda(x(z)))_{R^N} dz =$$

$$= \int_Z \|Dx(z)\|^{p-2}(Dx(z), \hat{h}'_\lambda(x(z))Dx(z))_{R^N} dz =$$

(chain rule, see MARKUS–MIZEL [4])

$$= \int_Z \hat{H}'_\lambda(x(z))\|Dx(z)\|^p dz.$$

But recall (see section 2) that $\hat{h}_\lambda(\,\cdot\,)$ is a nondecreasing, Lipschitz function from R into R. Hence $\int_Z \hat{h}'_\lambda(x(z))\|Dx(z)\|^p dz \geq 0$. So we conclude that

$$\big(\hat{A}(x), \partial W_\lambda(x)\big)_2 \geq 0 \qquad \text{for all } x \in D.$$

Invoking theorem 3.8, p. 144 of PASCALI–SBURLAN [5], we have that $\hat{A}+\partial W$ is a maximal monotone operator in the Hilbert space $L^2(Z)$. Let $x \in D \cap \cap \operatorname{dom} \partial W$ and let $w \in \partial W(x)$ and $r \in L^2(Z)$. We have

$$\big(\hat{A}(x)+w, r\big)_2 = \big\langle \hat{A}(x)+w, r\big\rangle =$$

$$= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dr(z))_{R^N} dz + \int_Z w(z)r(z)dz \leq$$

$$\leq \frac{1}{p}\|D(x+r)\|_p^p - \frac{1}{p}\|Dx\|_p^p + W(x+r) - W(x).$$

Here we have used the definition of the convex subdifferential (see section 2) and the fact that the map $\eta : W_0^{1,p}(Z) \to R$ defined by $\eta(x) = \frac{1}{p}\|Dx\|_p^p$ is Gateaux differentiable with derivative equal to $\eta'(x)(\,\cdot\,) = \|Dx(\,\cdot\,)\|^{p-2}Dx(\,\cdot\,)$. So we have

$$\big(\hat{A}(x)+w, r\big)_2 \leq V_2(x+r) - V_2(x) \qquad \text{for all } r \in L^2(Z),$$

$$\Rightarrow \; \hat{A} + \partial W \subseteq \partial V_2.$$

Since both operators are maximal monotone, we deduce that $\hat{A}+\partial W = \partial V_2 \subseteq \subseteq L^2(Z)$. So $v_2 \in \partial V_2(x)$ if and only if $v_2(z) = \hat{A}(x(z)) + w(z)$ a.e. on Z with $w \in \partial W(x) = \{w \in L^2(Z) : w(z) \in \hat{h}(x(z))$ a.e. on $Z\}$. Hence for all $\theta \in C_0^\infty(Z)$ we have

$$(v_2 - w, \theta) = (\hat{A}(x), \theta)_2 = \langle A(x), \theta\rangle$$

$$\Rightarrow \; \int_Z (v_2(z) - w(z))\theta(z)dz = \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\theta(z))_{R^N} dz.$$

From the definition of the distributional derivative, it follows that

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) = v_2(z) - w(z) \quad \text{a.e. on } Z$$

$$\Rightarrow \; v_2(z) \in -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) + \hat{h}(x(z)) \quad \text{a.e. on } Z,\ x|_\Gamma = 0.$$

3. This is a well-known result from the theory of maximal monotone operators of the subdifferential type (see BARBU [1] and section 2).

4. Let $v,\ v_1 \in \partial V_1(x) \cap \partial V_2(x)$. Then from parts (2) and (3), we have

$$v(z),\ v_1(z) \in -\mathrm{div}\,(\|Dx(z)\|^{p-2}Dx(z)) + \hat{h}(x(z)) \quad \text{for all} \quad z \in Z \setminus N_1,\ |N_1| = 0 \qquad \text{(part 2)}$$

and

$$v(z),\ v_1(z) \in \hat{g}(x(z)) \quad \text{for all } z \in Z \setminus N_2,\ |N_2| = 0 \qquad \text{(part 3)}$$

Let $x \in Z \setminus N,\ N = N_1 \cup N_2,\ |N| = 0$. If $x(z) \in U(f)$, then $v(z) = v_1(z) = = -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z) + h(x(z))$ (see proposition 1 (3)). If $x(z) \in D(f)$, then $v(z) = v_1(z) = g(x(z))$ (see proposition 1 (2)). Finally if $x(z) \in R \setminus (U(f) \cup \cup D(f))$, then $v(z) = v_1(z) = -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) + h(x(z)) = g(x(z))$. So for almost all $z \in Z$, $v(z) = v_1(z)$. ∎

Now we are ready to state and prove our existence theorem for problem (2). Our hypotheses on the discontinuous nonlinearity $f(r)$ are the following:

H(f): $f : R \to R$ is locally of bounded variation and $\overline{\lim} \frac{f(r)}{|r|^{p-2}r} < \lambda_1$ as $|r| \to \infty$.

THEOREM 3. *If hypotheses $H(f)$ hold and $u \in L^2(Z)$, then problem* (2) *has a solution.*

PROOF. Let $g - h$ be a decomposition of f according to proposition 1 and let $V_1,\ V_2,\ W : L^2(Z) \to \overline{R}$ be defined as before. We set $R(x) = V_2(x) - V_1(x)$, $x \in L^2(Z)$. By virtue of hypothesis $H(f)$ we can find $\mu \in (0, \lambda_1)$ and $M > 0$ such that $f(r) \le \mu |r|^{p-2}r$ for $r \ge M$ and $f(r) \ge \mu |r|^{p-2} r$ for $r < -M$. Also for $|r| \le M$ there exists $M_1 > 0$ such that $|f(r)| \le M_1$. Hence $F(r) \le \frac{\mu}{p}|r|^p + M_1|r|$ for all $r \in R$. Thus for all $x \in W_0^{1,p}(Z)$ we have

$$R(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F(x(z))dz \ge$$

$$\ge \frac{1}{p}\|Dx\|_p^p - \frac{\mu}{p}\int_Z |x(z)|^p dz - M_2\|x\|_p, \quad M_2 > 0.$$

Recall (see section 2) that $\lambda_1 \|x\|_p^p \le \|Dx\|_p^p$. So we have

$$R(x) \ge \frac{1}{p}\left(1 - \frac{\mu}{\lambda}\right)\|Dx\|_p^p - M_2\|x\|_p.$$

Since $\mu < \lambda_1$ it follows that $R(\cdot)$ is coercive.

Also we claim that $R(\cdot)$ is weakly sequentially lower semicontinuous on $W_0^{1,p}(Z)$. Indeed let $x_n \in \{x \in W_0^{1,p}(Z) : R(x) \leq \lambda\}$, $n \geq 1$, and assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. Since $W_0^{1,p}(Z)$ is embedded compactly in $L^2(Z)$, we have $x_n \to x$ in $L^2(Z)$ as $n \to \infty$. Note that $F(r) = G(r) - H(r)$ for all $r \in R$ and H, G are continuous. So assuming without loss of generality that $x_n(z) \to x(z)$ a.e. on Z, we have $\lim F(x_n(z)) = G(x(z)) - H(x(z)) = F(x(z))$ a.e. on Z. Also $-F(r) \geq -\frac{\mu}{p}|r|^p - M_1|r|$ and so via Fatou's lemma, we have

$$\underline{\lim} \int_Z F(x_n(z))dz \geq \int_Z -F(x(z)dz.$$

Since $\|Dx\|_p^p \leq \underline{\lim}\|Dx_n\|_p^p$ we have

$$R(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F(x(z))dz \leq \underline{\lim} R(x_n) \leq \lambda,$$

$$\Rightarrow \; L_\lambda = \{x \in W_0^{1,p}(Z) : R(x) \leq \lambda\} \quad \text{is weakly sequentially closed,}$$

$$\Rightarrow \; R(\cdot) \quad \text{is weakly sequentially lower semicontinuous.}$$

Thus we can apply Weierstrass theorem and obtain $x \in W_0^{1,p}(Z)$ a minimizer of $R(\cdot)$. We have

$$0 \leq R(x+h) - R(x) \qquad \text{for all} \quad h \in W_0^{1,p}(Z)$$

$$\Rightarrow \; 0 \leq \frac{1}{p}\|D(x+h)\|_p^p + W(x+h) - V_1(x+h) - \frac{1}{p}\|Dx\|_p^p - W(x) + V_1(x)$$

$$\Rightarrow \; V_1(x+h) - V_1(x) \leq$$

$$\leq \frac{1}{p}\|D(x+h)\|_p^p + W(x+h) - \frac{1}{p}\|Dx\|_p^p - W(x) = V_2(x+h) - V_2(x).$$

From the last inequality it follows that $\partial V_1(x) \subseteq \partial V_2(x)$. Invoking theorem 6.2 of STAMPACCHIA (6), we have $x \in L^\infty(Z)$. Recall that

$$\partial V_1(x) = \{v \in L^2(Z) : g(x(z)^-) \leq v(z) \leq g(x(z)^+) \quad \text{a.e. on} \quad Z\} + u.$$

Since $g(\cdot)$ is nondecreasing and $x \in L^\infty(Z)$, we have

$$g(-\|x\|_\infty) \leq g(x(z)^-) \leq g(x(z)^+) \leq g(\|x\|_\infty)$$

a.e. on, Z and so $\partial V_1(x) \neq \emptyset$. Let $v_1 \in \partial V_1(x)$, $\chi_U(s)$ is defined by

$$\chi_U(x) = \begin{cases} 1 & \text{if } s \in U(f) \\ 0 & \text{otherwise} \end{cases}$$

(the characteristic function of the set $U(f)$ and define

$$\varrho_{\pm}(\,\cdot\,) = (1-\chi_U(x(\,\cdot\,)))v_1(\,\cdot\,) + \chi_U(x(\,\cdot\,))\left(g\left(x(\,\cdot\,)^{\pm}\right)+u(\,\cdot\,)\right).$$

Then $\varrho \in \partial V_1(x)$.

From proposition 2 (4) we know that $\partial V_1(x) \cap \partial V_2(x)$ is either empty or a singleton. Thus since $\partial V_1(x) \subseteq \partial V_2(x)$ we infer that $\varrho_+ = \varrho_- = v_1$ and this means that $\chi_U(x(z)) = 0$ a.e. on Z or otherwise $\varrho_- \neq \varrho_+$ since by proposition 1 $g(\cdot)$ is discontinuous on $U(f)$. Hence we deduce that $\hat{g}(x(z)) = \{g(x(z))\}$ a.e. on Z. Thus by proposition 2, we have

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in g(x(z)) - \hat{h}(x(z)) + u(z) \quad \text{a.e. in } Z,\ x|_\Gamma = 0$$
$$\Rightarrow\ -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \hat{f}_0(x(z)) + u(z) \quad \text{a.e. on } Z,\ x|_\Gamma = 0,$$
$$\Rightarrow\ x(\,\cdot\,) \in W_0^{1,p}(Z) \qquad \text{solves problem (2).} \qquad \blacksquare$$

If $f(\cdot)$ exhibits only upward jump discontinuities, then we have a solution for the original problem (1).

H(f)$_1$: $f : R \to R$ is locally of bounded variation, $f(r^-) \leq f(r^+)$ for all $r \in R$ and $\overline{\lim}\frac{f(r)}{|r|^{p-2}r} < \lambda_1$ as $|r| \to \infty$.

Then combining theorem 3 with proposition 1, we obtain

COROLLARY 4. *If hypotheses $H(f)_1$ hold and $u \in L^2(Z)$, then problem* (1) *has a solution $x \in W_0^{1,p}(Z)$.*

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SWITCHBOX ROUTING IN THE MULTILAYER MANHATTAN MODEL

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1. Introduction

The first classic result in the topic of VLSI routing is probably Gallai's linear time algorithm that gives a solution with optimal width in the 2-layer Manhattan model if the terminals are situated on a single line on one side of the board. However, it is NP-complete to decide whether a channel routing problem is at all solvable in the same model [5]. It is true, on the other hand, that every channel routing problem is solvable in the unconstrained 2-layer model [3] and, moreover, also in linear time [2].

There is no fixed number of layers that would suffice for every specification in the case of switchbox routing. At least $m+1$ layers are needed in the worst case, where $m = \max(\frac{h}{w}, \frac{w}{h})$, h and w are the height and width of the switchbox, respectively [4]. According to the result of E. Boros, A. Recski and F. Wettl [1], however, every switchbox is solvable on $2m+14$ layers if $m \geq 2$ and on 18 layers if $m \leq 2$ (their algorithm is also linear). The authors conjecture in the same article that in fact $m+3$ layers also suffice (in the unconstrained model).

In this paper we give an improvement of the result of [1]. We present a linear time algorithm that solves any switchbox routing problem on $2\lceil m \rceil + 4$ layers. Furthermore, our solution will be in the Manhattan model.

2. Basic definitions

A *switchbox* is a rectangular grid G consisting of horizontal *tracks* (numbered from 0 to $w+1$) and vertical *columns* (numbered from 0 to $h+1$), where w is the *width* and h is the *height* of the switchbox. The boundary points of G are called *terminals*. Depending on which boundary of the switchbox they are situated on, the terminals are called *Northern*, *Southern*, *Eastern* or *Western*. The "corners" of the switchbox are not regarded as terminals, routings must not use them either.

A *net* is a collection of terminals. A *switchbox routing problem* is a set $\mathcal{N} = \{N_1, \ldots, N_n\}$ of pairwise disjoint nets. If all the terminals of every net are southern, we speak of a *single row routing problem*, if all of them are southern or northern, we speak of a *channel routing problem*.

The *solution* of a routing problem in the *unconstrained k-layer model* is a set $\mathcal{H} = \{H_1, \ldots, H_n\}$ of pairwise vertex disjoint subgraphs (also called *wires*, these are usually Steiner trees) of the k-layer rectangular grid such that H_i connects the terminals of N_i for $i = 1, \ldots, n$. The wires can access the terminals on any layer. Edges of the wires that join adjacent vertices of two consecutive layers are called *vias*.

In the multilayer *Manhattan model* consecutive layers must contain wire segments of different directions. Thus layers with horizontal (east-west) and with vertical (north-south) wire segments alternate.

3. The square-shaped switchbox

In this section we consider the square-shaped switchbox first, that is, for which $m = 1$. We shall prove that the necessary number of layers to lay out such a switchbox is between 4 and 6 in the worst case.

3.1. The necessary number of layers

It is easy to prove by repeating the argument of [4] that at least 4 layers are needed in the worst case to lay out a square-shaped switchbox in the Manhattan model. In the switchbox of Figure 1 all the $2n$ nets have terminals on both sides of the lines e and f. Thus at least $2n$ horizontal wire segments must intersect e and at least $2n$ vertical ones must intersect f. Therefore at least two layers are needed for the horizontal wire segments and at least another two are needed for the vertical ones.

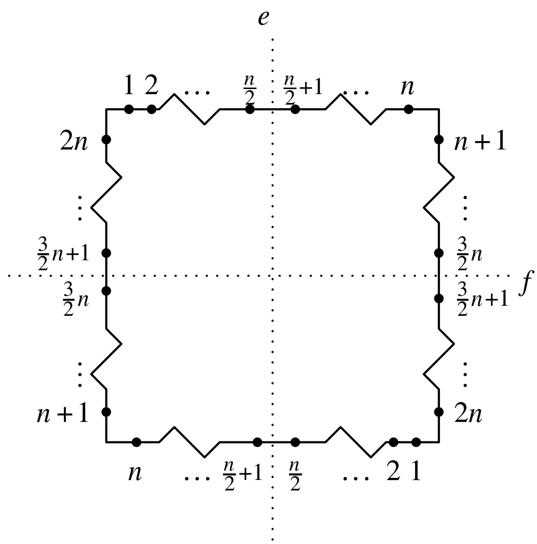

Figure 1

LEMMA 3.1. *Every square-shaped switchbox can be laid out in linear time on* 6 *layers in the Manhattan model.*

PROOF. We shall illustrate the proof on the switchbox routing problem of Figure 2. Terminals belonging to the same net are marked with the same number.

Obviously, one can assume without loss of generality that every terminal belongs to some net and that every net has at least two elements.

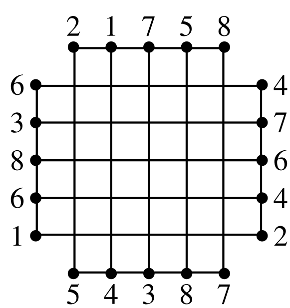

Figure 2

In order to simplify reference to them, we classify the nets according to the boundaries of the switchbox on which they occur and the number of occurrences on the boundaries. We say, for example, that a net is SE type if it has terminals on the southern and eastern boundaries, but not on the other two ones; a net is N type if it has only northern terminals; a net is NSW type if it has terminals on the northern, southern and western boundaries, but not on the eastern one, etc. Furthermore we say, for example, that a net is $\mathsf{S}_1\mathsf{W}_1$ type if it has exactly two terminals, one of which is on the southern boundary and the other is on the western one; a net is $\mathsf{N}_1\mathsf{W}$ type if it has one terminal on the north, some (maybe one) on the west and none elsewhere; a net is $\mathsf{S}_{\geq 2}\mathsf{W}_1$ type if it has at least two terminals on the south, exactly one on the west and none elsewhere, etc.

Getting on to the proof at last, we may obviously assume without loss of generality that out of NE, NW, SE, SW type nets NE nets are (one of) the greatest in number. (In the example of Figure 2 there is exactly one of all the four types, so the condition holds.) Having this in mind, the sketch of the construction is shown in Figure 3. The term S-comb means, for example, that on that layer a wire segment leads from each southern terminal to (for the sake of simplicity) the opposite boundary (unnecessary wire ends can be removed later on). The 2nd layer will contain horizontal wire segments while the 5th will contain vertical ones, thus the construction will indeed be in the Manhattan model.

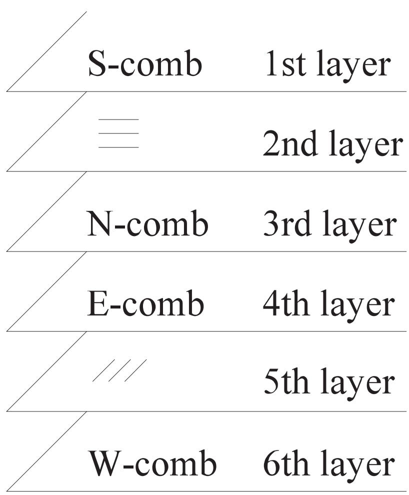

Figure 3

We associate tracks of the 2nd layer and columns of the 5th layer with the nets: these tracks and columns may only contain wire segments belonging to the net with which they are associated. The following nets are given a track on the 2nd layer (each listed net is given one track):

- nets having altogether at least two terminals on the northern and southern boundaries, but which are not NE type;
- N_1W type nets;
- half of the S_1W_1 type nets (if there is an odd number of these then we mean the upper integer part).

Similarly, the following nets are given a column on the 5th layer:

- nets having altogether at least two terminals on the eastern and western boundaries, but which are not NE type;
- SE_1 type nets;
- the remaining half of the S_1W_1 type nets.

Consequently, in the example of Figure 2 nets 1,3,5,7 and 8 are given a track on the 2nd layer, nets 4 and 6 are given a column on the 5th layer. (The rest of the columns of the 5th layer are not associated with any net, these columns remain empty.)

Let us accept for a while that there is indeed a sufficient number of tracks and columns on the 2nd and 5th layers to accomodate the listed nets; we will come back to this at the end of the proof. Instead we make rules for how the nets that are given a track (or column) on the 2nd (or 5th) layer should share these. If a net is given a track on the 2nd layer and there is a terminal belonging to this net on the **western** boundary then this net must be given a track next to (one of) its western occurrence(s). (This obviously can always be managed.) Such tracks will further on be called *fixed tracks*. If a net does not have a terminal on the western boundary then there is no restriction as to which track it should be given. Similarly, if a net is given a column on the 5th layer and there is a terminal belonging to this net on the **southern** boundary then this net must be given a track above (one of) its southern occurrence(s) (and such columns are called *fixed columns*). A possible distribution of the tracks and columns on the 2nd and 5th layers of the illustrative example is shown in Figure 4. The tracks and columns are marked with the number of the net with which they are associated, fixed tracks and columns are extended to the western and southern boundaries, respectively.

After all this preparation the desired routing can at last be given. Vertical wire segments on the 1st and 3rd layers coming from the northern and southern members of a net N_i can be interconnected by means of a single wire segment with the necessary vias in the track of the 2nd layer associated with the net N_i. If the track in question is fixed then the western terminal to

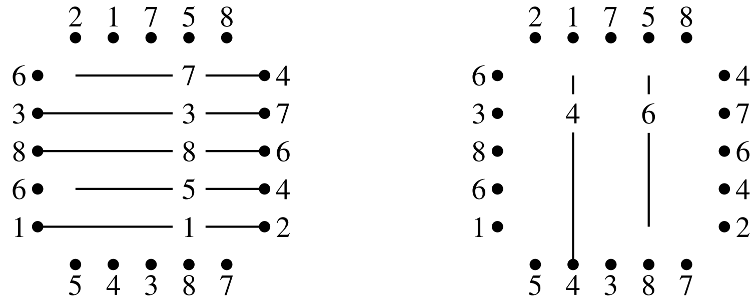

Figure 4

its left also belongs to the net N_i, so the wire segment in the track should be extended to reach this terminal as well. Similarly, wire segments (on the 4th and 6th layers) coming from the western and eastern members of a net can be interconnected in the columns of the 5th layer. Wire segments in the fixed columns are likewise extended to reach the southern members of the nets. Finally, we introduce extra vias between the 3rd and 4th layers suitably: if a northern and an eastern terminal belong to the same net then the vertical wire segment coming from the northern terminal on the 3rd layer and the horizontal wire segment coming from the eastern terminal on the 4th layer can be connected through an appropriate via. (Evidently, it is very often unnecessary to take advantage of all the possibilities to introduce extra vias. Moreover, it is not only unnecessary, but with certain specifications the number of (mainly needless) extra vias would thus become so high that the size of the output would be proportional with the square of the size of the input, so our algorithm would not be linear. Therefore let us settle this problem temporarily by saying that northern and eastern members of the nets can be connected through suitably chosen vias between the 3rd and 4th layers; we will come back to the necessary number of vias later on.)

The final routing of our illustrative example is shown in Figure 5 (unnecessary wire ends have also been removed from the "comb layers"). Vias going up and down from a layer are denoted by squares and circles, respectively.

We claim that the above described routing is good. This must be verified by distinguishing cases. If a net is of N, S or NS type then it was given a track on the 2nd layer (every net has at least two members) so its members became interconnected in the above described way. The situation is the same with E, W and EW type nets which were given a column on the 5th layer. If a net is NW type then it was surely given a track on the 2nd layer, through

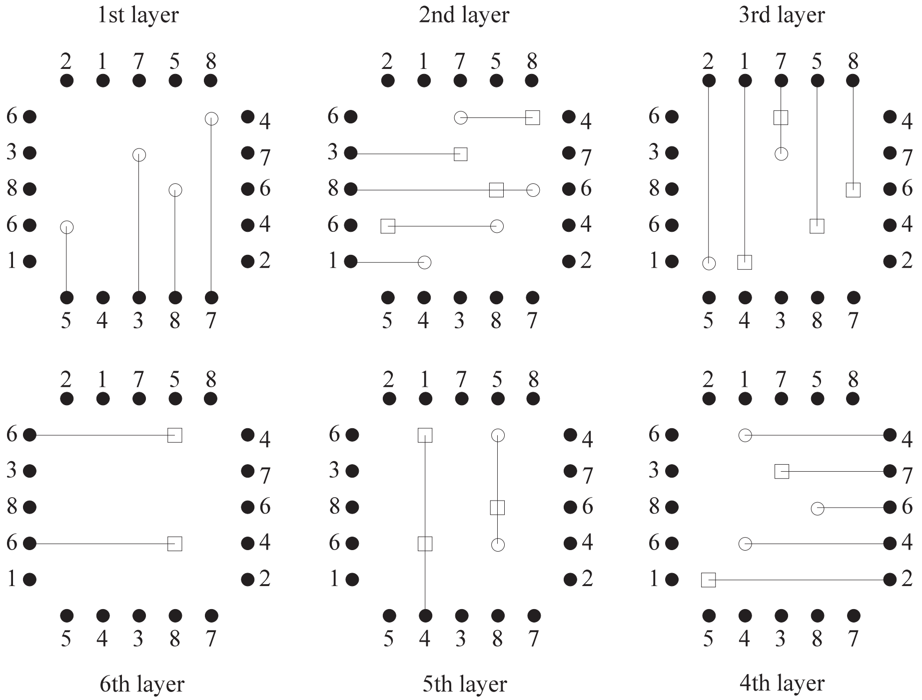

Figure 5

which northern members of the net became interconnected. In addition, since this track is fixed according to our rule, the northern members also became connected with one of the western terminals. If, on the other hand, the net has more than one western terminal then it was also given a column on the 5th layer through which western members of the net also became interconnected. Analogously, SE type nets were surely given a (fixed) column on the 5th layer and if such a net has more then one southern terminal then it was also given a track on the 2nd layer. The wiring of NE type nets is solved simply by the extra vias between the 3rd and 4th layers. The most difficult is the case of SW type nets. $\mathsf{S}_{\geq 2}\mathsf{W}_{\geq 2}$ nets were both given a track on the 2nd and a column on the 5th layer, thus both the southern and the western members of these nets became interconnected within themselves. Moreover, since both the track and the column are fixed according to our rules, the two groups also became (doubly) connected with each other. $\mathsf{S}_{\geq 2}\mathsf{W}_1$ nets were only given a track on the 2nd layer, but this track is fixed, thus the southern terminals are connected with each other and the single western terminal. Wiring of the $\mathsf{S}_1\mathsf{W}_{\geq 2}$ nets is analogous. Finally, whether a $\mathsf{S}_1\mathsf{W}_1$ type net was given a track on the 2nd or a column on the 5th layer, its wiring is solved through the fixed track or column. We can start to deal with nets appearing on at least 3 boundaries. NSW nets were surely given a (fixed) track on the 2nd layer, which solves the interconnection of the southern and eastern members as well as the connection of all these to (one of) the western one(s). If there is more than one western member then the net was also given a column on the 5th layer, thus the wiring is complete. In case of NSE nets the only difference is that the track given on the 2nd layer is not fixed, but this is compensated by the extra vias between the 3rd and 4th layers which interconnect the northern and eastern terminals. Wiring of the SEW and NEW type nets is analogous to the above. Finally, NSWE type nets were both given a track on the 2nd and a column on the 5th layer, both of these are fixed, thus the wiring of these nets is more than complete.

We still owe the proof that there is a sufficient number of tracks and columns on the 2nd and 5th layers, respectively, to be enough for the listed nets. We prove this only for the 2nd layer, the proof for the 5th layer is similar. Let n denote the width (and at the same time height) of the switchbox; let x denote the number of nets having altogether at least 2 terminals on the northern and southern boundaries but which are not NE type; let y denote the number of $\mathsf{N}_1\mathsf{W}$ type nets; let z denote the number of $\mathsf{S}_1\mathsf{W}_1$ nets; finally, let t denote the number of NE nets. Using these notations, $x+y+\lceil \frac{z}{2} \rceil \le n$ is to be proved. Since there are altogether $2n$ terminals on the northern and southern boundaries, obviously $2x+y+z+t \le 2n$. Because of the assumption made at the beginning of the proof $y \le t$ holds, thus $2x+2y+z \le 2n$ is also true. Divided by 2 we get $x+y+\frac{z}{2} \le n$ and since x, y and n are integers, the proof is complete. ∎

3.2. Linear time algorithm

It is not too difficult to verify that the above presented construction can be realized by a linear time algorithm. The algorithm should consider the nets one by one and decide whether the examined net should be given a track on the 2nd layer and/or a column on the 5th one (this requires nothing but scanning the terminals belonging to the net). It can at the same time assign the possibly awarded tracks and/or columns to the nets and place the necessary wire segments and the vias in the tracks/columns. For all this, it is enough to scan the terminals of the nets a second time. Meanwhile, the algorithm must make sure to give fixed tracks and columns to the nets that require one. If this meets with difficulties because the algorithm has already given a necessary track or column to a formerly examined net then it should simply remove the wire segment in way to a still empty track or column (preserving the position of the wire segment and the vias in the track or column). During the repeated scanning of the nets, the algorithm has dealt twice with every terminal ($4n$ in number), so it has made a linear number of steps so far.

In the next phase the extra vias are placed between the 3rd and 4th layers. We only prove here that the necessary number of vias is indeed linear, the realization will then be trivial. We only need vias here for nets that have terminals both on the northern and the eastern boundaries. It is easy to see that in case of NE type nets the necessary number of vias is not more than the number of terminals of the net minus 1 (we only need to place vias into the column of one of the northern terminals and into the track of one of the eastern terminals). In case of NSE nets we do not need more vias than the number of eastern terminals of the net (since the northern terminals are interconnected through the 2nd layer anyway, we only need to connect the eastern terminals with one of the northern ones). Similarly, for NEW type nets not more vias are needed than the number of northern terminals. Finally, no vias at all are needed in fact between the 3rd and 4th layers for NSWE type nets. Consequently, the necessary number of vias between the 3rd and 4th layers is not more than the number of northern and eastern terminals together, that is, $2n$.

The "combs" of the 1st, 3rd, 4th and 6th layers are made in the last phase. If we want to avoid unnecessary wire ends, there is no need to draw the wires to the opposite boundary, it is enough to draw them to the last via (which we already know).

4. The switchbox of arbitrary shape

We can now easily prove the theorem promised in the introduction based on the above results. We denote, as we have done so far, the quantity $\max(\frac{h}{w}, \frac{w}{h})$ by m, where h and w are the height and width of the switchbox, respectively.

THEOREM 4.1. *Every switchbox can be laid out in linear time on* $2\lceil m\rceil+4$ *layers in the Manhattan model.*

PROOF. $w \geq h$ can be assumed without loss of generality. We shall suitably modify the proof of Lemma 3.1 employing the idea of [1, Lemma

3]. The only difference is that there are not necessarily enough tracks on the 2nd layer for all the nets that would require one. (There is a column surplus on the 5th layer on the other hand.) Therefore we add further layers to the construction of Figure 3: a new layer is placed above the 1st one to hold horizontal wire segments, a N-comb comes on top of this one, again a new layer for horizontal wire segments, then a S-comb, etc. In other words, new layers are sandwiched between N-combs and S-combs. The horizontal tracks on these new layers are going to serve as the tracks of the 2nd layer did in Lemma 3.1. With this addition both the routing and the proof are the same as in Lemma 3.1.

It is easy to see by repeating the calculation of the last paragraph of Lemma 3.1 that not more than w tracks are needed on the "sandwiched" layers (including the original 2nd layer). There are h tracks on such a layer, thus $\lceil \frac{h}{w} \rceil = \lceil m \rceil$ sandwiched layers are needed altogether. Therefore, as it can be seen from the construction, $2\lceil m \rceil + 4$ layers are used altogether. ∎

Moreover, it is obvious that the algorithm realizing the above construction is also linear as it was in case of Lemma 3.1.

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A NOTE ON MULTIPLIERS FOR WALSH SYSTEM

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1. Introduction

We consider multiplier operators from dyadic Hardy spaces H^p $(0<p\leq$ $\leq 1)$ into L^q $(q>0)$ with respect to the one- and two-parameter Walsh–Paley system. These multipliers and some Paley type inequalities were partly investigated in earlier papers Simon and Weisz [4] and Simon [4], [6], [7]. Thus e.g. the boundedness of multipliers from H^p into L^p for some p were shown. In the present work we give a necessary and sufficient condition for a multiplier to be bounded from H^1 into L^2 and partly generalize this result for H^p spaces. These observations lead to a modified version of a Paley type estimation for functions from H^p $(0<p\leq 1)$ (see Simon and Weisz [4]). Some consequences will be formulated, thus also a Khinchin type inequality.

In this section the most important concepts, notations of the Walsh–Fourier analysis used in the further investigations will be formulated. (For details see the book Schipp–Wade–Simon [3].) Let r be the function defined on $[0,1)$ by

$$r(x):=\begin{cases}1 & (0\leq x<1/2)\\ -1 & (1/2\leq x<1)\end{cases}$$

extended to the real line by periodicity of period 1. If $n=0,1,\ldots$ is an arbitrary natural number then denote r_n the n-th Rademacher function:

$$r_n(x):=r(2^nx) \qquad (0\leq x<1).$$

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It is well-known that $r_n(x)=(-1)^{x_n}$ where

$$x=\sum_{k=0}^{\infty}\frac{x_k}{2^{k+1}}\qquad (x_k=0,1;k=0,1,\dots)$$

is the dyadic expansion of $x\in[0,1)$. (In the case there are two different expansions, we choose the one for which $x_k=0$ if k is large enough.)

The (one-parameter) Walsh–Paley system $(w_n, n=0,1,\dots)$ is the product system generated by the functions r_n $(n=0,\ 1,\ \dots)$, i.e.

$$w_n:=\prod_{k=0}^{\infty}r_k^{n_k},$$

where $n=\sum_{k=0}^{\infty}n_k2^k$ $(n_k=0,\ 1;\ k=0,\ 1,\ \dots)$ is the binary decomposition of n. $(w_n, n=0,1,\dots)$ is an orthonormal and complete system of functions with respect to the usual Lebesgue measure of $[0,1)$. If $m=\sum_{k=0}^{\infty}m_k2^k$ is the binary decomposition of $m=0,1,\dots$ then let $n\oplus m:=\sum_{k=0}^{\infty}|n_k-m_k|2^k$ the dyadic sum of n and m. It is well-known that $w_{n\oplus m}=w_nw_m$. So e.g. $w_{2^n+k}=w_{2^n}w_k=r_nw_k$ holds for all $k=0,\dots,2^n-1$, because in this case $2^n\oplus k=2^n+k$. Furthermore, let $x,y\in[0,1)$ and $x=\sum_{k=0}^{\infty}x_k2^{-k-1}, y=$ $=\sum_{k=0}^{\infty}y_k2^{-k-1}$ be their dyadic expansions. Then the dyadic sum $x\dot{+}y$ of x and y is defined by $x\dot{+}y:=\sum_{k=0}^{\infty}|x_k-y_k|2^{-k-1}$. It is not hard to see that $w_k(x\dot{+}y)=w_k(x)w_k(y)$ holds for all $k=0,1,\dots$ and $x,y\in[0,1)$.

If $f\in L^1[0,1)$ then define the n-th Walsh–Fourier coefficient of f as $\hat{f}(n):=\int_0^1 fw_n$ $(n=0,\ 1,\ \dots)$. Let $\hat{f}$ be the sequence of the numbers $\hat{f}(k)$ $(k=0,$ $1,\ \dots)$. We denote by S_nf the n-th partial sum of the Walsh–Fourier series $\sum_{k=0}^{\infty}\hat{f}(k)w_k$, that is

$$S_nf:=\sum_{k=0}^{n-1}\hat{f}(k)w_k\qquad (n=1,2,\dots).$$

The functions

$$D_n:=\sum_{k=0}^{n-1}w_k,\qquad (n=1,\ 2,\ \dots)$$

are the exact analogues of the well-known (trigonometric) kernel functions of Dirichlet's type. We mention a simple result with respect to D_{2^n}'s, which plays a central role in the Walsh–Fourier analysis, as well in our investigations:

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \le x < 2^{-n}) \\ 0 & (2^{-n} \le x < 1) \end{cases} \qquad (n = 0, 1, \ldots). \tag{1}$$

For the functions $f, g \in L^1$ let $f * g$ be their dyadic convolution given as

$$f * g(x) := \int_0^1 f(t) g(x \dot{+} t)\, dt \qquad (x \in [0,1)).$$

Then $\widehat{f * g} = \hat{f}\hat{g}$ and $S_n f = f * D_n$ $(n = 1, 2, \ldots)$. In the special case $n = 2^k$ $(k = 0, 1, \ldots)$ we have by (1)

$$S_{2^k} f(x) = \frac{1}{|I_k(x)|} \int_{I_k(x)} f = 2^k \int_{I_k(x)} f \qquad (x \in [0,1)),$$

where $I_k(x)$ stands for the (unique) dyadic interval $[\nu 2^{-k}, (\nu+1)2^{-k})$ $(\nu = 0, \ldots, 2^k - 1)$ containing x and $|I_k(x)|$ is the Lebesgue measure of $I_k(x)$.

The Kronecker product $w_{n,m}$ $(n, m = 0, 1, \ldots)$ of two Walsh systems is said to be the two-dimensional (or two-parameter) Walsh system. Thus

$$w_{n,m}(x, y) := w_n(x) w_m(y) \qquad (x, y \in [0,1)).$$

For the two-parameter Walsh–Fourier coefficients of a function $f \in L^1[0,1)^2$ the same notations will be used as in the one-dimensional case. That is, let

$$\hat{f}(n, m) := \int_0^1 \int_0^1 f(x, y) w_{n,m}(x, y)\, dx\, dy \qquad (n, m = 0, 1, \ldots)$$

and $\hat{f} := (\hat{f}(n, m); n, m = 0, 1, \ldots)$. Furthermore, let

$$S_{n,m} f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \hat{f}(k, l) w_{k,l} \qquad (n, m = 1, 2, \ldots)$$

be the (n, m)-th (rectangular) partial sum of the two-parameter Walsh–Fourier series $\sum_{k,l=0}^{\infty,\infty} \hat{f}(k, l) w_{k,l}$ of $f \in L^1[0,1)^2$. It is easy to show that

$$S_{n,m} f(x, y) = \int_0^1 \int_0^1 f(t, u) D_n(x \dot{+} t) D_m(y \dot{+} u)\, dt\, du \qquad (x, y \in [0,1)).$$

In the special case $n = 2^k$, $m = 2^l$ $(k,\ l = 0,\ 1,\ \dots)$ it follows as for the one-parameter partial sums that

$$S_{2^k,2^l}f(x,y) = \frac{1}{|I_{k,l}(x,y)|}\int\limits_{I_{k,l}(x,y)} f = 2^{k+l}\int\limits_{I_{k,l}(x,y)} f \qquad (x,y \in [0,1)),$$

where the dyadic rectangle $I_{k,l}(x,y)$ is defined by the Descartes product

$$I_{k,l}(x,y) := I_k(x) \times I_l(y)$$

and $|I_{k,l}(x,y)| = |I_k(x)|\cdot|I_l(y)|$. If $k = l$ then $I_{k,k}(x,y)$ is a so-called dyadic square.

2. Hardy spaces

The Hardy spaces play important role in the Fourier analysis, especially in the theory of Walsh–Fourier series. The dyadic analogues of them can be defined as follows. (For details see the book of Weisz [8].) Let $n=0,1,\dots$ and denote $\mathcal{F}_n$ the σ-algebra generated by the dyadic intervals $[k2^{-n},(k+1)2^{-n})$ $(k = 0,\ 1,\ \dots,\ 2^n - 1)$. Obviously, the sequence $\mathcal{F} := (\mathcal{F}_n, n = 0,1,\dots)$ of σ-algebras is non-decreasing, i.e. $\mathcal{F}_n \subset \mathcal{F}_m$ if $n < m$. If g is an integrable real function defined on $[0,1)$ then $S_{2^n}g$ is the conditional expectation of g relative to $\mathcal{F}_n$ $(n = 0,\ 1,\ \dots)$. A sequence $f = (f_n, n = 0,1,\dots)$ of integrable functions is said to be a (dyadic) martingale with respect to $\mathcal{F}$, if each f_n is $\mathcal{F}_n$ measurable and $S_{2^n}f_m = f_n$ for all $n \le m$ $(n,\ m = 0,\ 1,\ \dots)$. This implies the existence of a sequence $(\alpha_k, k=0,1,\dots)$ of real numbers such that

$$f_n = \sum_{k=0}^{2^n-1} \alpha_k w_k \qquad (n = 0,1,\dots).$$

Conversely, if the real numbers β_k $(k=0,\ 1,\ \dots)$ are arbitraly given, then the Walsh polynomials $\sum_{k=0}^{2^n-1}\beta_k w_k$ $(n=0,\ 1,\ \dots)$ form evidently a martingale. Thus it is clear that for every $f \in L^1$ the sequence $(S_{2^n}f, n = 0,1,\dots)$ is a martingale (called martingale obtained from f and denoted likewise by f). Furthermore, the definition of $\hat{f}(k)$ $(k = 0,\ 1,\ \dots)$ can be extended to a martingale f given by a real sequence β_k $(k = 0,\ 1,\ \dots)$ in the above sense as $\hat{f}(k) := \beta_k$ $(k = 0,\ 1,\ \dots)$. Consequently the Walsh–Fourier coefficients of $f \in L^1$ are the same as those of the martingale obtained from f.

We say that a martingale $f = (f_n, n = 0,1,\ldots)$ is L^p-bounded for some $0<p<\infty$ if

$$\|f\|_p := \sup_n \|f_n\|_p < \infty.$$

(The symbol $\|f_n\|_p = \left(\int_0^1 |f_n|^p\right)^{1/p}$ denotes the usual p-norm or quasi-norm of f_n.) It is well-known that for $1<p<\infty$ the assumption $\|f\|_p<\infty$ is equivalent to the existence of a real function from the space $L^p[0,1)$, from which f is obtained. For all $0<p<\infty$ L^p will denote the set of the L^p-bounded martingales. Hence, if $1<p<\infty$ then L^p and $L^p[0,1)$ can be identified.

To the definition of (dyadic) Hardy spaces let the quadratic variation of a martingale $f=(f_n, n=0,1,\ldots)$ be denoted by

$$Qf := \left(\sum_{n=0}^{\infty} |f_n - f_{n-1}|^2\right)^{1/2}$$

(where $f_{-1} := 0$). In particular, the quadratic variation of the martingale obtained from $f \in L^1[0,1)$ is not another as

$$Qf := \left(|\hat{f}(0)|^2 + \sum_{n=1}^{\infty} |S_{2^n}f - S_{2^{n-1}}f|^2\right)^{1/2}.$$

Let $\Delta_n f := S_{2^{n+1}}f - S_{2^n}f$ $(n=0,\ 1,\ \ldots)$. We introduce the Hardy spaces H^p for $0<p<\infty$ as follows: denote H^p the set of all martingales f for which $\|f\|_{H^p} := \|Qf\|_p < \infty$. We remark that for $1<p<\infty$ the spaces H^p, L^p are the same.

The atomic decomposition is a useful characterization of some Hardy spaces. To demonstrate this we give first the concept of atoms: let $0<p\le 1$, then a function $a \in L^\infty[0,1)$ is called a p-atom if either a is identically equal to 1 or there exists a dyadic interval I for which

$$\text{(2)} \qquad \operatorname{supp} a \subset I\ ,\ \|a\|_\infty \le |I|^{-1/p} \quad \text{and} \quad \int_0^1 a = 0.$$

We shall say that a is supported on I. Then a martingale $f=(f_n, n=0,1,\ldots)$ belongs to H^p for $0<p\le 1$ if and only if there exist a sequence $(a_k,\ k=0,$

1, ...) of p-atoms and a sequence $(\mu_k,\ k=0,\ 1,\ \ldots)$ of real numbers such that $\sum_{k=0}^{\infty}|\mu_k|^p<\infty$ and

$$f=\sum_{k=0}^{\infty}\mu_k a_k. \tag{3}$$

Hence, in the p-quasi-norm and a.e. $\sum_{k=0}^{\infty}\mu_k S_{2^n}a_k=f_n \quad (n=0,1,\ldots)$. Moreover, the following equivalence of norms holds:

$$c_p\|f\|_{H^p}\le\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}\le C_p\|f\|_{H^p}\qquad (f\in H^p),$$

where the infimum is taken over all decompositions of f of the form (3). (Here and later c_p, C_p will denote positive constants depending only on p although not always the same in different occurences.)

The so-called (dyadic) BMO space is defined as the set of all martingales f such that

$$\|f\|_{\mathrm{BMO}}:=|\hat f(0)|+\|\sup_k\left(S_{2^{k+1}}|f-S_{2^k}f|^2\right)^{1/2}\|_\infty<\infty.$$

It is known that BMO is the dual of H^1 in the following sense: Φ is a bounded linear functional on H^1 if and only if there exists a unique $g\in\mathrm{BMO}$ such that $\Phi(f)=\lim_k\int_0^1 S_{2^k}f\,S_{2^k}g\ \ (f\in H^1)$. Moreover, the norm of Φ is equivalent to $\|g\|_{\mathrm{BMO}}$.

Let us introduced a new space of Hardy type as follows. If $f=(f_n,n=$ $=0,1,\ldots)$ is a martingale then define $\|f\|_*$ as

$$\|f\|_*:=\left(\sum_{n=0}^{\infty}\|f_n-f_{n-1}\|_1^2\right)^{1/2}.$$

Furthermore, let H^* be the set of all martingales f such that $\|f\|_*<\infty$. We remark that H^* is evidently a vector space and $\|.\|_*$ is a quasi-norm on H^*. It is not hard to see that the sets $H^*\setminus L^1, L^1\setminus H^*$ are not empty. Indeed, if

$$f_n(x):=\sum_{j=1}^{n-1}j^{-1}\sum_{k=2^{j-1}}^{2^j-1}w_k(x\dot{+}2^{-j+1})\qquad (x\in[0,1),n=2,3,\ldots) \tag{4}$$

then by (1)

$$\|f\|_*^2 = \sum_{n=1}^{\infty} \|n^{-1} \sum_{k=2^{n-1}}^{2^n-1} w_k(2^{-n+1})w_k\|_1^2 =$$

$$= \sum_{n=1}^{\infty} n^{-2}\|r_{n-1}D_{2^{n-1}}\|_1^2 = \sum_{n=1}^{\infty} n^{-2} < \infty,$$

i.e. $f \in H^*$. On the other hand the supports $[2^{-j+1}, 2^{-j+2})$ $(j = 1, \ldots, n-1)$ of $j^{-1}\sum_{k=2^{j-1}}^{2^j-1} w_k(2^{-j+1})w_k$ are pairwise disjoint, therefore

$$\|f_n\|_1 = \sum_{j=1}^{n-1} \|j^{-1} \sum_{k=2^{j-1}}^{2^j-1} w_k(2^{-j+1})w_k\|_1 = \sum_{j=1}^{n-1} j^{-1} \qquad (n = 1, 2, \ldots).$$

Thus $\sup_n \|f_n\|_1 = \infty$, hence $f \notin L^1$.

Let $f := \sum_{n=1}^{\infty} n^{-3/2} D_{2^n}$ then by (1) the last serie converges in L^1-norm, i.e. f belongs to L^1. We show that $f \notin H^*$. First of all observe that

$$f_n = S_{2^n} f = \sum_{k=1}^{n} k^{-3/2} D_{2^k} + D_{2^n} \sum_{k=n+1}^{\infty} k^{-3/2},$$

from which

$$f_n - f_{n-1} = n^{-3/2}(D_{2^n} - D_{2^{n-1}}) + (D_{2^n} - D_{2^{n-1}}) \sum_{k=n+1}^{\infty} k^{-3/2}$$

follows. This implies that

$$\|f_n - f_{n-1}\|_1^2 =$$

$$= \|D_{2^n} - D_{2^{n-1}}\|_1^2 \left(\sum_{k=n}^{\infty} k^{-3/2}\right)^2 = \left(\sum_{k=n}^{\infty} k^{-3/2}\right)^2 \geq \left(\frac{2}{\sqrt{n}}\right)^2 = \frac{4}{n},$$

that is, $\|f\|_* = \infty$. Therefore $f \notin H^*$.

Now, we shall show that $H^1 \subset H^*$, moreover, this inclusion is also strict. To this end let $f = (f_n, n = 0, 1, \ldots) \in H^1$. Then

$$\|f\|_* = \left\|(\|f_j - f_{j-1}\|_1, j = 0, 1, \ldots)\right\|_{\ell_2},$$

where $\|(b_j, j=0,1,\dots)\|_{\ell_2} := \left(\sum_{j=0}^{\infty} b_j^2\right)^{1/2}$ denotes the usual ℓ_2-norm of the sequence $(b_j, j=0,1,\dots)$ of real numbers. Applying the well-known inequality of Minkowski on the change of integrals (see e.g. Zygmund [9]) it follows that

$$\|f\|_* \leq C \left\| \|(f_j - f_{j-1}, j=0,\ 1,\ \dots)\|_{\ell_2} \right\|_1 = \|Q(f)\|_1 = \|f\|_{H^1},$$

where C is an absolute constant (not always the same in different occurences). This proves that $H^1 \subset H^*$. To show that $H^1 \neq H^*$ let f_n $(n=2,\ 3,\ \dots)$ be the martingale given by (4) then for all $k=1,\ 2,\ \dots$ and $x \in [2^{-k+1}, 2^{-k+2})$ we get $Qf(x) = 2^{k-1}/k$. Therefore

$$\|Qf\|_1 \geq \sum_{k=1}^{\infty} \int_{2^{-k+1}}^{2^{-k+2}} Qf = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

that is, $f \notin H^1$. On the other hand we have been seen above that $\|f\|_* < \infty$.

We recall the known convolution inequality (see Schipp–Wade–Simon [3])

$$\|f * g\| \leq \|f\|_1 \|g\| \qquad (f \in L^1, g \in X), \tag{5}$$

where $X = L^p[0,1), \|.\| = \|.\|_p$ $(p \geq 1)$ or $X = H^1 \cap L^1[0,1), \|.\| = \|.\|_{H^1}$, respectively. A simple calculation shows that the following H^*-version of (5) is also true: $\|f * g\|_* \leq \|f\|_1 \|g\|_*$ $(f \in L^1[0,1),\ g \in L^1[0,1) \cap H^*)$. Indeed, by (5) we can write that

$$\|\Delta_n(f * g)\|_1 = \|f * (\Delta_n g)\|_1 \leq \|f\|_1 \|\Delta_n g\|_1 \qquad (n = 0, 1, \dots),$$

i.e.

$$\|f * g\|_* = \left(\sum_{n=0}^{\infty} \|\Delta_n(f * g)\|_1^2\right)^{1/2} \leq \|f\|_1 \left(\sum_{n=0}^{\infty} \|\Delta_n g\|_1^2\right)^{1/2} = \|f\|_1 \|g\|_*.$$

In the two-dimensional case the above concepts will be defined in analoguous way. Let $\mathcal{F}_{n,m}$ $(n,m = 0,1,\dots)$ be the σ-algebra generated by the dyadic rectangles $I_{n,m}(x,y)$ $(x,\ y \in [0,1))$. Then the conditional expectation operator relative to $\mathcal{F}_{n,m}$ is not another as $S_{2^n,2^m}$. A sequence of integrable functions $f = (f_{n,m}; n,m=0,1,\dots)$ is said to be a martingale if

i) $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n,m=0,1,\dots$

and

ii) $S_{2^n,2^m} f_{k,l} = f_{n,m}$ for all $n, m, k, l = 0, 1, \ldots$ such that $n \le k$ and $m \le l$.

For example, if $f \in L^1[0,1)^2$ then the sequence $(S_{2^n,2^m} f; n, m = 0, 1, \ldots)$ is evidently a martingale (called martingale generated by f).

The concept of the Walsh–Fourier coefficients can be extended to the martingales as in the one-parameter case. That is, $\hat{f}$ will denote the sequence of the Walsh–Fourier coefficients of the function or martingale f.

Let $\|g\|_p = \left(\int_0^1 \int_0^1 |g(x,y)|^p \, dx \, dy \right)^{1/p}$ $(0 < p < \infty)$ be the usual L^p-norm (or quasi-norm) of a measurable function g defined on the unit square $[0,1)^2$. We shall say that a martingale $f = (f_{n,m}; n, m = 0, 1, \ldots)$ is L^p-bounded if

$$\|f\|_p := \sup_{n,m} \|f_{n,m}\|_p < \infty.$$

The symbol L^p will denote the set of the L^p-bounded two-parameter martingales. If $p > 1$ then L^p and $L^p[0,1)^2$ can be identified.

The quadratic variation Qf of a martingale $f = (f_{n,m}; n, m = 0, 1, \ldots)$ are defined by

$$Qf := \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}|^2 \right)^{1/2},$$

where $f_{-1,k} := f_{k,-1} := 0$ $(k = -1, 0, 1, \ldots)$. To define (dyadic) Hardy spaces H^p for $0 < p < \infty$ we use also in the two-dimensional case the quadratic variation: denote H^p the space of all martingales f for which

$$\|f\|_{H^p} := \|Q(f)\|_p < \infty.$$

The so-called diagonal quadratic variation $Q_\diamond f$ of a martingale $f = (f_{n,m}; n, m = 0, 1, \ldots)$ is given as follows:

$$Q_\diamond f := \left(\sum_{n=0}^{\infty} |f_{n,n} - f_{n-1,n-1}|^2 \right)^{1/2}.$$

Moreover, let the spaces of Hardy type $H_\diamond^p$ $(0 < p < \infty)$ be defined as the set of all martingales f such that

$$\|f\|_{H_\diamond^p} := \|Q_\diamond f\|_p < \infty.$$

For $1<p<\infty$ the spaces $H^p, H^p_\diamond, L^p$ are the same.

The atomic characterization of $H^p_\diamond$ $(0<p\leq 1)$ is similar to the one-dimensional case. Namely, a bounded measurable function a is an $H^p_\diamond$-atom if $a\equiv 1$ or there exists a dyadic square I such that

$$\operatorname{supp} a \subset I\ ,\ \|a\|_\infty \leq |I|^{-1/p}\ ,\ \int_0^1\int_0^1 a = 0. \tag{6}$$

We shall say also in this case that a is supported on I. Then a martingale $f=(f_{n,m};\ n,m=0,1,\ldots)$ is in $H^p_\diamond$ if and only if there exist a sequence $(a_k, k=0,1,\ldots)$ of $H^p_\diamond$-atoms and a sequence $(\lambda_k, k=0,1,\ldots)$ of real numbers such that $\sum_{k=0}^\infty |\lambda_k|^p<\infty$ and

$$\sum_{k=0}^\infty \lambda_k S_{2^n,2^n} a_k = f_{n,n} \qquad (n=0,1,\ldots). \tag{7}$$

Moreover,

$$c_p \inf\left(\sum_{k=0}^\infty |\lambda_k|^p\right)^{1/p} \leq \|f\|_{H^p_\diamond} \leq C_p\left(\sum_{k=0}^\infty |\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (7).

Unfortunatly, the atomic characterization of H^p is much more complicated in the two-dimensional case than in the one-dimensional. Indeed, in the two-dimensional case the support of an atom in H^p is not a dyadic rectangle but an open set (see Weisz [8]). However, a finer atomic decomposition can be given, that is, the atoms can be decomposed into elementary rectangle particles. In some investigations this makes possible to examine only atoms supported on dyadic rectangles. To their definition let $0<p\leq 1$. A function $a\in L^\infty[0,1)^2$ is called a rectangle H^p-atom if either a is identically equal to 1 or there exists a dyadic rectangle I such that

$$\operatorname{supp} a \subset I\ ,\qquad \|a\|_2 \leq |I|^{1/2-1/p}\ ,$$

$$\int_0^1 a(x,t)\,dt = \int_0^1 a(u,y)\,du = 0 \quad (x,\ y\in[0,1)). \tag{8}$$

We shall say that a is supported on I. As we remarked above a characterization by means of H^p-atoms of H^p $(0<p\leq 1)$ similar to (7) fails to hold.

Finally, the two-parameter space H^* is given similarly to the one-parameter case. Namely, H^* is the set of all martingales $f = (f_{n,m}; n, m = =0, 1, \ldots)$ such that

$$\|f\|_* = \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \|f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}\|_1^2\right)^{1/2} < \infty.$$

Furthermore, let $H_\diamond^* := \{f : \|f\|_{H_\diamond^*} < \infty\}$, where

$$\|f\|_{H_\diamond^*} := \left(\sum_{n=0}^{\infty} \|f_{n,n} - f_{n-1,n-1}\|_1^2\right)^{1/2}.$$

The strict inclusions $H^1 \subset H^*$ and $H_\diamond^1 \subset H_\diamond^*$ can be shown analoguously as in the one-dimensional case, respectively.

3. Multipliers

Let $\lambda = (\lambda_n, n = 0, 1, \ldots)$ be a sequence of real numbers and define formally the so-called (one-parameter) multiplier transformation T_λ by $\widehat{T_\lambda f} = \lambda \hat{f}$, i.e. $\widehat{T_\lambda f}(n) = \lambda_n \hat{f}(n)$ $(n = 0, 1, \ldots)$. This always makes sense if here f stands for a Walsh polynomial, that is, if $\hat{f}(k) = 0$ $(k = N, N+1, \ldots)$ with a suitable natural number N. For example, if λ is bounded then by the well-known theorem of Riesz–Fischer for all $f \in L^2$ there exists a unique $g \in L^2$ such that $\hat{g} = \lambda \hat{f}$ and $\|g\|_2 \le \sup_n |\lambda_n| \|f\|_2$. In other words, in this case T_λ can be extended to a bounded operator from L^2 to L^2. If λ is bounded and $\lambda_{2^n+k} = \lambda_{2^n+l}$ $(k, l = =0, \ldots, 2^n - 1; n = 0, 1, \ldots)$ then T_λ is a special martingale transform. It is known (see e.g. Schipp–Wade–Simon [3]) that in this case $T_\lambda f$ can be defined for all $f \in H^p$ $(0 < p < \infty)$ and $T_\lambda : H^p \to L^p$ is bounded. In general, if the "exponents" $p, q > 0$ are given, $X, Y \in \{H^*, H^p, L^q, BMO\}$ and T_λ extends to a bounded operator from X to Y then the sequence λ will be called an (X, Y)-multiplier. Thus e.g. a bounded λ is an (L^2, L^2)-multiplier (and it is easy to see that the boundedness of λ is also necessary for this).

If $0 < p \le 1, q > 0$ then a simple necessary condition can be formulated for λ to be (H^p, L^q)-multiplier. Indeed, the function $f_n := 2^{n(1/p-1)} r_n D_{2^n}$

is a p-atom for all $n = 0,1,\ldots$ and $T_\lambda f_n = 2^{n(1/p-1)}\Lambda_n$, where $\Lambda_n := = \sum_{k=2^n}^{2^{n+1}-1}\lambda_k w_k$ $(n=0,1,\ldots)$. Therefore, if $T_\lambda : H^p \to L^q$ is bounded then

$$\|\Lambda_n\|_q = O(2^{n(1-1/p)}) \qquad (n\to\infty).$$

For $p=1$ we get the assumption $\sup_n \|\Lambda_n\|_q < \infty$. Hence $\|\Lambda_n\|_1 = O(1)$ $(n\to\infty)$ is a necessary condition for λ to be (H^1,L^1)-multiplier.

A sufficient condition for this purpose can be easy formulated. Namely, if

$$\sum_{n=0}^{\infty}\|\Lambda_n\|_1 < \infty$$

then λ is an (H^1,L^1)-multiplier. Moreover, the last condition implies that λ is also an (H^1,H^1)-multiplier (see Onneweer–Quek [2]). Indeed, by the convolution inequality (5) we get for all $f\in H^1$ that

$$\|T_\lambda f\|_{H^1} \le \sum_{n=0}^{\infty}\|\Lambda_n * f\|_{H^1} \le \sum_{n=0}^{\infty}\|\Lambda_n\|_1\|f\|_{H^1} \le C\|f\|_{H^1}.$$

The simple example $\lambda_k = 1$ $(k=0,1,\ldots)$ shows that the sufficient condition in question is not necessary, because in this case by (1) $\sum_{n=0}^{\infty}\|\Lambda_n\|_1 = \sum_{n=0}^{\infty}\|r_n D_{2^n}\|_1 = \sum_{n=0}^{\infty}1 = \infty$ but the oparator T_λ given by $T_\lambda f = f$ $(f\in H^1)$ is evidently (H^1,H^1)-bounded.

A theorem of Daly–Phillips [1] improves the condition $\sum_{n=0}^{\infty}\|\Lambda_n\|_1 < < \infty$ to

$$\sup_N \sum_{j=N}^{\infty}\int_{2^{-N}}^{1}|\Lambda_j| < \infty.$$

Furthermore, in Simon [4] we extended the result of Daly–Phillips and gave the following sufficient condition for λ to be (H^p,L^p)- as well (H^p,H^p)-multiplier $(0<p\le 1)$:

$$\sup_N \sum_{j=N}^{\infty} 2^N \int_{2^{-N}}^{1}\left(\int_0^{2^{-N}}|\Lambda_j(x\dot{+}t)|\,dt\right)^p dx < \infty.$$

(See also Simon [6], [7].)

If $q=2$ then $\|\Lambda_n\|_q = \left(\sum_{k=2^n}^{2^{n+1}-1} \lambda_k^2\right)^{1/2}$ $(n=0,1,\ldots)$ and $\sup_n \|\Lambda_n\|_q < \infty$ is equivalent to

$$\sup_n \sum_{k=2^n}^{2^{n+1}-1} \lambda_k^2 < \infty. \tag{9}$$

Because $\|r_n D_{2^n}\|_{H^*} = 1$ $(n=0,1,\ldots)$, it follows analoguously that (9) is also necessary for λ to be (H^*, L^2)-multiplier. Moreover, the following theorem is true.

THEOREM 1. *The assumption (9) is a necessary and sufficient condition for λ to be (H^*, L^2)-multiplier.*

PROOF. We need to prove only the sufficiency of (9). To this end let $f \in H^*$ ($\hat{f}(0)=0$ can be evidently assumed) then Parseval's equality implies that

$$\|T_\lambda f\|_2 = \left(\sum_{j=0}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} \lambda_k^2 |\hat{f}(k)|^2\right)^{1/2} =$$

$$= \left(\sum_{j=0}^{\infty} \|\Lambda_j * f\|_2^2\right)^{1/2} = \left(\sum_{j=0}^{\infty} \|\Lambda_j * (\Delta_j f)\|_2^2\right)^{1/2}.$$

From this we get by (5) and (9)

$$\|T_\lambda f\|_2 \le \left(\sum_{j=0}^{\infty} \|\Lambda_j\|_2^2 \|\Delta_j f\|_1^2\right)^{1/2} \le C \left(\sum_{j=0}^{\infty} \|\Delta_j f\|_1^2\right)^{1/2} = C\|f\|_{H^*},$$

i.e. $T_\lambda : H^* \to L^2$ is bounded. This proves our statement.

We remark that the same statement holds for λ to be (H^1, L^2)-multiplier. Furthermore, a usual duality argument shows that (9) is necessary and sufficient condition for λ to be (L^2, BMO)-multiplier.

Theorem 1 implies a Paley type inequality in the following way. Let $n=0,1,\ldots$ and denote A_n a set of indices such that $A_n \subset [2^n, 2^{n+1}-1)$. If $[A_n]$ stands for the cardinality of A_n then the next corollary holds.

COROLLARY 1. *The following assumptions are equivalent:*

i) *there exists a constant* $C>0$ *such that for all* $f\in H^*$ *the inequality*

$$\left(\sum_{n=0}^{\infty}\sum_{k\in A_n}|\hat{f}(k)|^2\right)^{1/2}\le C\|f\|_{H^*}$$

is true;

ii) $\sup_n[A_n]<\infty$.

Indeed, if

$$\lambda_k:=\begin{cases}1 & (k\in A_n, n=0,1,\dots)\\ 0 & (\text{otherwise}, k=0,1,\dots)\end{cases}$$

then $\sup_n[A_n]<\infty$ implies (9). Conversely, if i) holds then λ is an (H^*,L^2)-multiplier, i.e. by (9) we get $\sup_n[A_n]=\sup_n\sum_{k=2^n}^{2^{n+1}-1}\lambda_k^2<\infty$. Therefore Corollary 1 follows directly from Theorem 1.

As in Theorem 1 H^* can be replaced in Corollary 1 by H^1. In the special case $A_n=\{2^n\}$ $(n=0,1,\dots)$ Corollary 1 is the classical Paley's theorem:

$$\sum_{n=0}^{\infty}|\hat{f}(2^n)|^2<\infty \qquad (f\in H^1).$$

In this connection we recall a result of Simon and Weisz [4], namely: if $0<p\le 1$ and $f\in H^p$ then

$$\left(\sum_{n=0}^{\infty}2^{2n(1-1/p)}|\hat{f}(2^n)|^2\right)^{1/2}\le C_p\|f\|_{H^p}. \tag{10}$$

In other words the sequence λ given by

$$\lambda_k:=\begin{cases}2^{n(1-1/p)} & (k=2^n, n=0,1,\dots)\\ 0 & (\text{otherwise}, k=0,1,\dots)\end{cases}$$

is an (H^p,L^2)-multiplier. By means of a simple modification of the proof in Simon and Weisz [4] we get the following extension of (10).

THEOREM 2. *Let* $0<p\le 1$ *and* $A_n\subset[2^n,2^{n+1}-1)$ $(n=0,1,\dots)$ *be sets of indices. Then the following assumptions are equivalent:*

i) *there exists a constant* $C_p > 0$ *depending only on* p *such that for all* $f \in H^p$ *the inequality*

$$\left(\sum_{n=0}^{\infty} 2^{2n(1-1/p)} \sum_{k \in A_n} |\hat{f}(k)|^2\right)^{1/2} \leq C_p \|f\|_{H^p}$$

is true;

ii) $\sup_n [A_n] < \infty$.

Hence, the sequence λ defined as

$$\lambda_k := \begin{cases} 2^{n(1-1/p)} & (k \in A_n, n = 0, 1, \ldots) \\ 0 & (\text{otherwise}, k = 0, 1, \ldots) \end{cases}$$

is an (H^p, L^2)-multiplier if and only if $\sup_n [A_n] < \infty$. Theorem 2 is a generalization of Corollary 1 since for $p = 1$ and for H^1 instead of H^* they coincide.

For the sake of completeness we give the proof of Theorem 2. Namely, if $\sup_n [A_n] < \infty$ then by the atomic structure (3) of H^p it is enough to show that

$$\sup \sum_{n=0}^{\infty} 2^{2n(1-1/p)} \sum_{k \in A_n} |\hat{a}(k)|^2 < \infty,$$

where the supremum is taken over all p-atoms a. To this end let $a \neq 1$ be a p-atom supported on the dyadic interval $I := [k2^{-N}, (k+1)2^{-N})$ for some $N = 0, 1, \ldots$ and $k = 0, \ldots, 2^N - 1$ (see (2)). Then $\hat{a}(j) = 0$ $(j = 0, \ldots, 2^N - 1)$, because the Walsh functions $w_0, \ldots, w_{2^N-1}$ are constant on I and $\int_0^1 a = \int_I a = 0$.

Furthermore, if $n = N, N+1, \ldots$ and $k \in A_n$ then for $l = 0, \ldots, 2^N - 1$ it follows that

$$|\hat{a}(k \oplus l)| = \left| \int_I a w_l w_k \right| = \left| w_l(k2^{-N}) \int_I a w_k \right| = |\hat{a}(k)|.$$

We remark that for all $n = N$, $N+1$, $\ldots$ and $k \in A_n$ the indices $k \oplus l$ $(l = 0, \ldots, 2^N - 1)$ are pairwise distinct and belong to A_n. Therefore

$$\sum_{k \in A_n} |\hat{a}(k)|^2 = 2^{-N} \sum_{k \in A_n} \sum_{l=0}^{2^N-1} |\hat{a}(k \oplus l)|^2 \leq$$

$$\leq 2^{-N} \sum_{k \in A_n} \sum_{j=2^n}^{2^{n+1}-1} |\hat{a}(j)|^2 \leq C2^{-N} \sum_{j=2^n}^{2^{n+1}-1} |\hat{a}(j)|^2,$$

where C is an absolute constant such that $[A_n] \leq C \quad (n = 0, 1, \ldots)$. These observations lead to the following estimation:

$$\sum_{n=0}^{\infty} 2^{2n(1-1/p)} \sum_{k \in A_n} |\hat{a}(k)|^2 \leq C2^{-N} \sum_{n=0}^{\infty} 2^{2n(1-1/p)} \sum_{j=2^n}^{2^{n+1}-1} |\hat{a}(j)|^2 \leq$$

$$\leq C2^{-N} 2^{2N(1-1/p)} \sum_{n=0}^{\infty} \sum_{j=2^n}^{2^{n+1}-1} |\hat{a}(j)|^2 \leq$$

$$\leq C2^{N-2N/p} \|a\|_2^2 \leq C2^{N-2N/p} |I|^{1-2/p} =$$

$$= C2^{N-2N/p} 2^{-N+2N/p} = C.$$

On the other hand, if (i) holds and we apply it for the p-atoms

$$f_j := 2^{j(1/p-1)} r_j D_{2^j} \qquad (j = 0, 1, \ldots)$$

then

$$\left(\sum_{n=0}^{\infty} 2^{2n(1-1/p)} \sum_{k \in A_n} |\hat{f}_j(k)|^2 \right)^{1/2} = \left(2^{2j(1-1/p)} \sum_{k \in A_j} |\hat{f}_j(k)|^2 \right)^{1/2} =$$

$$= \sqrt{[A_j]} \leq C_p \|f_j\|_{H^p} \leq C_p \qquad (j = 0, 1, \ldots),$$

which was to be proved.

From Theorem 2 we get by usual interpolation arguments the next corollary (for details see Simon and Weisz [4] and Weisz [8]).

COROLLARY 2. *Let* $1 < p < 2$. *If* $A_n \subset [2^n, 2^{n+1})$ $(n = 0, 1, \ldots)$ *are uniformly bounded sets of indices then*

$$\left(\sum_{n=0}^{\infty} \sum_{k \in A_n} |\hat{f}(k)|^2 \right)^{1/2} \leq C_p \|f\|_p \qquad (f \in L^p).$$

Furthermore, Theorem 2 implies some dual inequalities.

COROLLARY 3. *Let* $A_n \quad (n = 0, 1, \ldots)$ *be the sets as in Corollary 2 and denote* $(\alpha_n, n = 0, 1, \ldots)$ *a sequence of real numbers such that*

$\sum_{n=0}^{\infty}\sum_{k\in A_n}|\alpha_k|^2<\infty$. *Then the function* $f:=\sum_{n=0}^{\infty}\sum_{k\in A_n}\alpha_k w_k$ *belongs to BMO. Moreover, there exists a constant* C *independent on* f *such that*

$$\|f\|_{\mathrm{BMO}}\le C\left(\sum_{n=0}^{\infty}\sum_{k\in A_n}|\alpha_k|^2\right)^{1/2}.$$

We recall that $\|.\|_p\le\|.\|_{\mathrm{BMO}}$ $(1<p<\infty)$ and $\|.\|_q\le\|.\|_s$ $(0<q\le s)$. Taking into account these inequalities Corollary 2 and Corollary 3 lead to

COROLLARY 4. *Under the assumptions of Corollary 3 the equivalences*

$$\|f\|_p\sim\left(\sum_{n=0}^{\infty}\sum_{k\in A_n}|\alpha_k|^2\right)^{1/2}\sim\|f\|_{\mathrm{BMO}}\qquad(0<p<\infty)$$

hold for $f:=\sum_{n=0}^{\infty}\sum_{k\in A_n}\alpha_k w_k$.

Note that the first $\sim$ in Corollary 4 is a generalization of the well-known Khinchin's inequality (see e.g. Schipp–Wade–Simon [3]).

The concept of multipliers can be defined also in the two-parameter case, similarly to the one-parameter multipliers. Namely, if $\lambda=(\lambda_{n,m};n,m=0,1,\ldots)$ is a sequence of real numbers then let T_λ be defined formally as $\widehat{T_\lambda f}=\lambda\hat{f}$. The concept of (X,Y)-multiplier $(X,Y\in\{H^*,H_\diamond^*,H^p,H_\diamond^p,L^q,BMO\}$ $(p,q>0)$ is taken analoguously as in the one-parameter case. In Simon [6], [7] we are concerned with special two-parameter multipliers. Furthermore, let $\Lambda_{n,m}:=\sum_{k=2^n}^{2^{n+1}-1}\sum_{j=2^m}^{2^{m+1}-1}\lambda_{k,j}w_{k,j}$ $(n,m=0,1,\ldots)$. Then

$$\|\Lambda_{n,m}\|_q=O(2^{(n+m)(1-1/p)})\ \text{ or }\ \|\Lambda_{n,n}\|_q=O(2^{(2n)(1-1/p)})\qquad(n,m\to\infty)$$

is necessary for λ to be (H^p,L^q)- or $(H_\diamond^p,L^q)$-multiplier $(0<p\le 1,q>0)$, respectively. Moreover, the proof of Theorem 1 can be repeated also in the two-parameter case, that is,

$$\sup_{n,m}\sum_{k=2^n}^{2^{n+1}-1}\sum_{j=2^m}^{2^{m+1}-1}\lambda_{k,j}^2<\infty$$

is necessary and sufficient for λ to be (H^*, L^2) (or (H^1, L^2))-multiplier. If $T_{\lambda,\diamond}$ is given by $T_{\lambda,\diamond} f := \sum_{n=0}^{\infty} \Lambda_{n,n} * f$ then the following diagonal version of Theorem 1 holds:

$$\sup_n \sum_{k=2^n}^{2^{n+1}-1} \sum_{j=2^n}^{2^{n+1}-1} \lambda_{k,j}^2 < \infty$$

is necessary and sufficient for $T_{\lambda,\diamond}$ to be bounded from $H_\diamond^*$ (or $H_\diamond^1$) to L^2, respectively.

The exact analogue of Corollary 1 is true also for H^* (or H^1) in the two-parameter case: let be assumed that the cardinalities of the sets of indices $A_{n,m} \subset [2^n, 2^{n+1}) \times [2^m, 2^{m+1})$ $(n, m = 0, 1, \ldots)$ are uniformly bounded. Then for all $f \in H^*$ the following inequality can be proved:

$$\left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{(k,l) \in A_{n,m}} |\hat{f}(k,l)|^2 \right)^{1/2} \leq C \|f\|_{H^*}. \tag{11}$$

Similarly, for $H_\diamond^*$ (or $H_\diamond^1$) we get

$$\left(\sum_{n=0}^{\infty} \sum_{(k,l) \in A_{n,n}} |\hat{f}(k,l)|^2 \right)^{1/2} \leq C \|f\|_{H_\diamond^*}.$$

However, Theorem 2 fails to hold for $H_\diamond^p$ $(0 < p \leq 1)$. To this end let $n = 1, 2, \ldots$ and F_n be the function given by

$$F_n(x, y) := 2^{2n(1/p-1)} r_n(x) D_{2^n}(x) D_{2^n}(y) \quad (x, y \in [0, 1)).$$

Then F_n is evidently an $H_\diamond^p$-atom, i.e. $\|F_n\|_{H_\diamond^p} \leq 1$ and

$$F_n = 2^{2n(1/p-1)} \left(\sum_{i=0}^{n-1} \sum_{k=2^n}^{2^{n+1}-1} \sum_{j=2^i}^{2^{i+1}-1} w_{k,j} + \sum_{k=2^n}^{2^{n+1}-1} w_{k,0} \right).$$

On the other hand, from the last equality it follows that

$$\left(\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} 2^{(u+v)(1-1/p)} |\hat{F}_n(2^u, 2^v)|^2 \right)^{1/2} =$$

$$= \left(\sum_{i=0}^{n-1} 2^{(n+i)(1-1/p)} |\hat{F}_n(2^n, 2^i)|^2 \right)^{1/2} \geq$$

$$\geq C_p \begin{cases} \sqrt{2^{3n(1/p-1)}} & (p < 1) \\ \sqrt{n} & (p = 1). \end{cases}$$

It remains open whether Theorem 2 is true for for H^p $(0 < p < 1)$ in the two-parameter case. We remark that the argument from the proof of Theorem 2 cannot be applied for H^p or $H^p_\diamond$ $(0 < p \leq 1)$. Indeed, if $A_{n,m} \subset [2^n, 2^{n+1}) \times \times [2^m, 2^{m+1})$ $(n, m = 0, 1, \ldots)$ are sets of indices and $\sup_{n,m} [A_{n,m}] < \infty$ then we get similarly as in the proof of Theorem 2 that

$$\sup \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l) \in A_{n,m}} |\hat{a}(k,l)|^2 < \infty,$$

where the supremum is taken over all H^p-atoms a. Unfortunately, in the two-parameter case this is not enough for

(12)
$$\left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l) \in A_{n,m}} |\hat{f}(k,l)|^2 \right)^{1/2} \leq C_p \|f\|_{H^p} \quad (f \in H^p).$$

Let H^p_1 be the set of all martingales f such that $f = \sum_{n,m=0}^{\infty} \mu_{n,m} a_{n,m}$, where $a_{n,m}$'s are rectangular p-atoms and for the real coefficients $\mu_{n,m}$ $(n, m = = 0, 1, \ldots)$ the inequality $\sum_{n,m=0}^{\infty} |\mu_{n,m}|^p < \infty$ holds. Define the quasi-norm $\|f\|_{H^p_1}$ of f as infimum of $\left(\sum_{n,m=0}^{\infty} |\mu_{n,m}|^p \right)^{1/p}$'s, where the infimum is taken over all representations of f mentioned above. Then H^p_1 is a (proper) subspace of H^p and (12) will be true for H^p_1 instead of H^p.

Furthermore, if a is an $H^p_\diamond$-atom supported on the dyadic square I and $|I| = 2^{-2N}$ for some $N = 0, 1, \ldots$ then we can state $\hat{a}(k,l) = 0$ only for k, l mutually less then 2^N. Hence, the basic idea from the proof of Theorem 2 cannot be applied.

However, a restricted version of Theorem 2 is true also in the two-parameter case.

THEOREM 3. *Let* $0<p\le 1$ *and* $A_{n,m}\subset[2^n,2^{n+1}-1)\times[2^m,2^{m+1}-1)$ $(n,m=$ $=0,1,\dots)$ *be uniformly bounded sets of indices. Then for all non-negative real number* α *there exists a constant* $C_{p,\alpha}>0$ *depending only on* p *and* α *such that for all* $f\in H^p_\diamond$ *the next inequality holds:*

$$\left(\sum_{n,m=0,|n-m|\le\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat f(k,l)|^2\right)^{1/2} \le C_{p,\alpha}\|f\|_{H^p_\diamond}.$$

Because the proof is similar to the one-parameter case we formulate only the most important steps of the proof. Namely, taking into consideration the atomic structure of $H^p_\diamond$ (see (6), (7)) we need to show that

$$q := \sup \sum_{n,m=0,|n-m|\le\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat a(k,l)|^2 < \infty, \tag{13}$$

where the supremum is taken over all $H^p_\diamond$-atoms a. Hence let a be an $H^p_\diamond$-atom supported on a dyadic square I. If $|I|=2^{-2N}$ for some $N=0,1,\dots$ then (as we remarked above) $\hat a(k,l)=0$ for $k,l\in\{0,1,\dots,2^N-1\}$. Therefore it can be assumed in (13) that $n\ge N$ or $m\ge N$, that is,

$$\begin{aligned} q \le{}& \sup\sum_{n=N}^{\infty}\sum_{m\ge N-\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat a(k,l)|^2 + \\ &+\sup\sum_{m=N}^{\infty}\sum_{n\ge N-\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat a(k,l)|^2 \le \\ \le{}& \sup\sum_{n=N}^{\infty}\sum_{m=N+1-\tilde\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat a(k,l)|^2 + \\ &+\sup\sum_{m=N}^{\infty}\sum_{n=N+1-\tilde\alpha}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{(k,l)\in A_{n,m}} |\hat a(k,l)|^2, \end{aligned}$$

where $\tilde\alpha$ is the integer part of α. As in the one-parameter case if $n=N,N+1,\dots$ and $m=N+1-\tilde\alpha,N+2-\tilde\alpha,\dots$ or $m=N,N+1,\dots$ and $n=N+1-\tilde\alpha,N+2-\tilde\alpha,\dots$ we get $|\hat a(k\oplus j,l\oplus t)|=|\hat a(k,l)|$ or $|\hat a(k\oplus t,l\oplus j)|=|\hat a(k,l)|$ for $(k,l)\in A_{n,m}$;

$j=0,\dots,2^N-1; t=0,\dots,2^{N+1-\tilde{\alpha}}-1$, respectively. From this it follows that

$$q \le C2^{-2N-1-\tilde{\alpha}} \sum_{n,m=N+1-\tilde{\alpha}}^{\infty} 2^{2(n+m)(1-1/p)} \sum_{k=2^n}^{2^{n+1}-1} \sum_{l=2^m}^{2^{m+1}-1} |\hat{a}(k,l)|^2 \le$$

$$\le C2^{-2N-1-\tilde{\alpha}} 2^{4(N+1-\tilde{\alpha})(1-1/p)} \|a\|_2^2 \le$$

$$\le C2^{-2N-1-\tilde{\alpha}} 2^{4(N+1-\tilde{\alpha})(1-1/p)} |I|^{1-2/p} =$$

$$= C2^{-2N-1-\tilde{\alpha}} 2^{4(N+1-\tilde{\alpha})(1-1/p)} 2^{2N(2/p-1)} = C2^{4(1-\tilde{\alpha})(1-1/p)-1-\tilde{\alpha}}.$$

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A NOTE ON A MULTIPLICATIVE PROBLEM

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1. Introduction

We will use the notations of [1] (J.-L. Nicolas, I. Z. Ruzsa and A. Sárközy). $\mathcal{A}$ will denote a subset of **N**: $\mathcal{A} = \{a_1, a_2, \ldots\} \subset \mathbf{N}$ $(a_1 < a_2 < \ldots)$. $p(\mathcal{A}, n)$ denotes the number of solutions of the equation $a_1x_1 + a_2x_2 + \ldots = n$, where the x_i's are non-negative integers. $p(n)(=p(\mathbf{N}, n))$ denotes the number of unrestricted partitions of n. The number of solutions of the equation $a_i + a_j = n$, $i \le j$ will be denoted by $r(\mathcal{A}, n)$. Ramanujan initiated the study of the parity of the numbers $p(n)$. It was shown that $p(n)$ assumes both even and odd values infinitely often. In [1] further improvements of the earlier results can be found.

A. Sárközy ([2]) suggested to study two related problems similar to the one mentioned above.

$\mathcal{B} = \{b_1, b_2, \ldots\}$, $b_i < b_j$ if $i < j$ will denote a subset of **N**. The problems concern the parity of the number of the solutions of the equation

$$(1) \qquad b_i \cdot b_j = m, \qquad i \le j. \quad (m \text{ is a positive integer})$$

The first problem is about the number of m's which have an even number of solutions while the second is about the number of those which have an odd number of solutions:

PROBLEM 1. Is it true that there is a set $\mathcal{B}$ for which the number of solutions of (1) is odd for all m? (Or the number of solutions of (1) is odd for all but finitely many m.)

PROBLEM 2. Is it true that there is a set $\mathcal{B}$ for which the number of solutions of (1) is even for all m? (Or the number of solutions of (1) is even for all but finitely many m.)

The connection between the additive and the multiplicative problems can be seen: if there is a set $\mathcal{A}$ such that the number of solutions of $a_i + a_j = m$ is always even, then the set $\mathcal{B} = \{p^{a_i}\}$ (for a fixed p prime) yields a similar property with the additional fact that if m has no solution in $\mathcal{B}$ it still has an even number of solutions.

2. Even number of solutions

LEMMA 1. *There are at least two numbers m_1, m_2 for which $\mathcal{B}$ gives an odd number of solutions.*

PROOF. Let $\mathcal{B} = \{b_1, b_2, \ldots\}$, $b_1 < b_2 < \ldots$. $m_1 = b_1^2$ has exactly one solution in $\mathcal{B}$, since $m_1 = b_1 \cdot b_1$ and m_1 has no other divisor less than b_1. $m_2 = b_1 \cdot b_2$ too, has exactly one solution in $\mathcal{B}$, since there are no other divisors of m less than b_2 but b_1. ∎

THEOREM 1. *There is a set $\mathcal{B}$ such that the equation* (1) *has an even number of solutions for all m's except for $b_1 \cdot b_1$ and $b_1 \cdot b_2$.*

PROOF. Let $b_1 = 1$ and $b_2 = p$ (a prime) be in $\mathcal{B}$. $\mathcal{B}$ will consist of powers of p. We will decide one-by-one whether to include a power of p or not. $m = p^2$ has one solution in $\mathcal{B}$ and if we include p^2, it will have two solutions: $m = p \cdot p = 1 \cdot p^2$. $m = p^3$ has one solution and if we include p^3 it will have two solutions: $m = p \cdot p^2 = 1 \cdot p^3$. $m = p^4$ has two solutions: $m = p^2 \cdot p^2 = p \cdot p^3$, so we do not want to include p^4.

In general, if $m = p^k$ has an even number of solutions in $\mathcal{B}$, we do not include it, otherwise we do.

The given $\mathcal{B}$ will satisfy the conditions of the theorem.

1. If m is not a power of p, it has no solutions in $\mathcal{B}$.

2. If m is a power of p, then by the construction we get an even number of solutions. (Once we have decided about including a power of p, we will never have to decide about it again.)

3. The set $\mathcal{B}$ is infinite: let us suppose that p^r is the last number in $\mathcal{B}$, then $m = p^{2r}$ would have one solution, a contradiction. ∎

REMARK. It is clear that if we defined $\mathcal{A}$ as the set of the powers of p in $\mathcal{B}$, we got a set which satisfies the condition that the equation $a_i + a_j = m$ has an odd number of solutions for nearly all m. ($m \neq 0$, $m \neq 1$.) There is just one problem: $0 \in \mathcal{A}$. It raises another problem: the 0 comes from the 1. Could we leave that 1 out of $\mathcal{B}$?

With a slight change of the construction we get an answer:

COROLLARY 1. *There is a set $\mathcal{B}_*$ such that $1 \notin \mathcal{B}$ and the equation*

$$b_{i*} b_{j*} = m$$

has an even number of solutions for all m's except for $b_{1*} \cdot b_{1*}$ and $b_{1*} \cdot b_{2*}$.

PROOF. Let $\mathcal{B}_* = p_* \cdot \mathcal{B}$ where p_* is a prime other than p. The two least elements of $\mathcal{B}_*$ are p_* and $p_* p$. There is an odd number of solutions for $m = (p_*)^2$ and $m = p_* p$ (because of Lemma 1). Some m's have no solution at all in $\mathcal{B}$. (This is even.) Those which have, can be written in the form: $p_i p_* \cdot p_j p_*$, that is $p_i \cdot p_j \cdot p_*^2$. From $\mathcal{B}$ $p_i \cdot p_j$ has an even number of solutions, which means that $p_1 \cdot p_2 \cdot p_*^2$ has also an even number of solutions. ∎

REMARK. We could have chosen p_* as p, since $p^k = p^2 \cdot p^{k-2}$ and p^{k-2} has an even number of solutions in $\mathcal{B}$. For each solution $p^{k_1} \cdot p^{k_2}$ ($k_1 + k_2 = k-2$) we have to write $pp^{k_1} \cdot pp^{k_2}$. This way we get a set $\mathcal{B}$ for which it is really true that in the set of the powers ($\mathcal{A}$) the number of solutions of $a_i + a_j = m$ is nearly always even.

3. Odd number of solutions

Now we would like to find a set $\mathcal{B}$ such that the equation (1) has an even number of solutions for only finitely many m's.

LEMMA 2. *Such a $\mathcal{B}$ set must contain the number* 1.

PROOF. If $1 \notin \mathcal{B}$ then the equation $m = p$ would have no solutions that is an even number of solutions for infinitely many m's (no matter if p is in $\mathcal{B}$ or not). ∎

THEOREM 2. *There is a set $\mathcal{B}$ such that the equation* (1) *has an odd number of solutions for all m's.*

PROOF. By Lemma 2 we have to include 1 in $\mathcal{B}$.

We define classes on the natural numbers: class k will contain the numbers $n = \prod p_i^{\alpha_i}$ for which $\sum \alpha_i = k$. Class 0 has one number in it: 1, class 1 contains the prime numbers, etc.

The construction of B will be similar to the one in Theorem 1.

We consider the elements of a class at the time: if a number in class k has an even number of solutions in $\mathcal{B}$, we include it, otherwise not.

Let m_{k_i} be in class k. If it has an even number of solutions in the set we have already assembled, we include it, otherwise not. It cannot have a divisor in class k other than m_{k_i}, so we can decide about it independently of the other elements in the same class.

The set $\mathcal{B}$ constructed this way satisfies the conditions of the theorem. It is infinite (as the $\mathcal{B}$ constructed earlier) and each m has an odd number of solutions, since m has to belong to a class and we decided about each element of each class. ∎

4. Concluding remarks

In both the even and the odd case we constructed the sets element-by-element or class-by-class. If in such a construction we decide to leave an element out while by condition we should include it we can still continue the construction of the sets. The result will be a set where there would be one more m which does not satisfy the condition. Therefore we can state the following two corollaries:

COROLLARY 2. *For any $n \geq 2$ we can give a set $\mathcal{B}$ for which there are exactly n numbers ($m_1, m_2, \ldots, m_n$) for which the equation* (1) *has an odd number of solutions.*

COROLLARY 3. *For any $n \geq 0$ we can give a set $\mathcal{B}$ for which there are exactly n numbers for which the equation* (1) *has an even number of solutions.*

In **N** equation (1) has both even and odd number of solutions for infinitely many numbers. If m is a prime number it has an odd number of solutions and the prime squares have an even number of solutions. Only for the sake of complexity we state the following trivial corollary:

COROLLARY 4. *There is a set $\mathcal{B}$ for which equation* (1) *has both even and odd solutions infinitely many times.*

References

[1] J.-L. NICOLAS, I. Z. RUZSA, A. SÁRKÖZY, On the parity of additive representation functions to appear in *Number Theory*.

[2] A. SÁRKÖZY, Private conversation.

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A kiadásért felelős: az Eötvös Loránd Tudományegyetem rektora
A kézirat a nyomdába érkezett: 1998. január. Megjelent: 1998. február
Terjedelem: 17,6 A/4 ív. Példányszám: 500
Készült az EMTEX szedőprogram felhasználásával
az MSZ 5601–59 és 5602–55 szabványok szerint
Az elektronikus tipografálás Juhász Lehel és Fried Katalin munkája
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