ANNALES

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SECTIO MATHEMATICA

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THE KOROVKIN CLOSURE IN SOME SPECIAL CASE

By

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In this paper we are going to give a short proof for a theorem due to BAUER on the Korovkin Closure and to use its results in some special case for identifying precisely the Korovkin shadow or at least a subset of it. At the end of the paper we give a proof for the standard Stone–Weierstrass theorem.

1. Basic definitions and results

Let X be a locally compact Hausdorff space, and let us denote by $C(X)$ the set of all continuous real-values functions defined on X. X is said to be separated by $F \subset C(X)$ (or F separates X) if for all $x, y \in X$ there exists an $f \in F$ such that $f(x) \neq f(y)$. Given a linear subspace $L \subset C(X)$, we denote by L_b the following subset of $C(X)$:

$$L_b = \{f \in C(X) \ : \ \exists \ h_1; h_2 \in L \text{ such that } h_1 \leq f \leq h_2\}.$$

Now we can introduce the following definition after [3].

DEFINITION 1.1. The Korovkin closure (or shadow) of L is the set of all continuous function $f \in L_b$ having the following property: for every net $(M_i)_{i \in I}$ of positive linear maps, where $M_i : L_b \to L_b$ such that $\lim_{i \in I} M_i h = h$ pointwise on X for all $h \in L$, then $\lim_{i \in I} M_i f = f$ pointwise on X. We denote this closure by Kor(L). If L is not linear then Kor(L) is defined as Kor(Lin$\{L\}$).

REMARK 1.1. The following inclusions are obvious:

$$L \subset \mathrm{Kor}(L) \subset L_b \subset C(X).$$

Our results are based on the following well known theorem which was proved by H. BAUER (in [1]):

THEOREM 1.1.

$$\mathrm{Kor}(L) = \{f \in L_b \ : \ \sup\{h \in L \ : \ h \leq f\} = \inf\{h \in L \ : \ h \geq f\}\}.$$

Let the right-side of the equality above be denoted by $\hat{L}$. Before proving this theorem we shall prove the following lemma which we will use in the theorem later.

LEMMA 1.2. *Let g be in L_b, x be in X and $c \in \mathbf{R}$ such that*

$$\sup f\{(x) \ : \ g \geq f \in L\} \leq c \leq \inf\{f(x) \ : \ g \leq f \in L\}.$$

Then there exists a positive linear functional $\mu : L_b \to \mathbf{R}$ with

(1) $\mu(g) = c$ *and*

(2) $\mu(f) = f(x)$ *are all f in L.*

PROOF. On L_b the map $p : h \to \inf\{f(x) \ : \ h \leq f \in L\}$ is a positive sublinear functional, which obviously majorizes the linear form $\Lambda \ : \ g \to \to cg$ defined on the linear space generated by g. Therefore the Hahn–Banach Theorem provides a linear form, namely μ, on L_b with $\mu \leq p$. Hence (1) is satisfied by μ and we claim that (2) holds as well. Indeed,

$$\mu(x) \leq p(h) = h(x)$$

and on the other hand

$$\mu(h) = \mu(-(-h)) = -\mu(-h) \geq -p(-h) = h(x),$$

which shows (2). If $f \leq 0$ then $p(f) \leq 0$ because $0 \in L$, considering, that $\mu \leq p$ again, so we arrived at the positivity property. ∎

In the proof of the theorem we use a property of locally compact Hausdorff spaces what is very similar to the complete regularity property. Therefore we prove the following

LEMMA 1.3. *Let X be a locally compact topological space, and $F \subset X$ closed set such that $F \subset \operatorname{int} K$ for a given compact set K.*

Then F and $X \setminus K$ can be separated by function.

PROOF. Assign to each $k \in K$ an F_k compact set that is a neighbourhood of k. Let $U_k \subset F_k$ be an open neighbourhood of k, arbitrary. Obviously $\bigcup\limits_{k \in I} U_k \supset K$. Since K is compact we can find finitely many $k \in K$, namely $k_1, \ldots, k_n$, such that

$$K \subset \bigcup_{i=1}^{n} U_{k_i}.$$

Moreover

$$K \subset \bigcup_{i=1}^{n} \operatorname{int} F_{k_i}$$

and the right-hand side of the inclusion is clearly relatively compact. Let us denote $\bigcup_{i=1}^{n} F_{k_i}$ by Y. Y is a compact Hausdorff space, therefore it is normal. In Y, F and $Y \setminus K$ can be separated by function, provided by Uryshon's Lemma, which might be extended to X continuously as constant on $X \setminus Y$. ∎

Using the lemmata above the theorem can be proved:

PROOF OF THEOREM 1.1. First we show that Kor(L) $\supset \hat{L}$. Let $(M_i)_{i\in I}$ be an arbitrary not of positive linear maps such that $(M_i l)_{i\in I}$ converges to 1 on X pointwise. We have to show that for any $f \in \hat{L}$ $(M_i f)_{i\in I}$ also converges to f pointwise on X. Let $x \in X$ and $l_1; l_2 \in L$ such that $l_1 \leq f$ but $l_1(x) > f(x) - \varepsilon$ and $l_2 \geq f$ but $l_2(x) < f(x) + \varepsilon$. There exists an $i_0 \in I$ with $|M_i l_j(x) - l_j(x)| < \varepsilon$ if $i \geq i_0$ and $j = 1;2$. By the positivity of M_i we know that $M_i l_1 \leq M_i f \leq M_i l_2$, which implies that $l_1 - \varepsilon \leq M_i f \leq l_2 + \varepsilon$, which means that $|M_i f(x) - f(x)| < 4\varepsilon$. If ε converges to 0 we obtain the convergence of $(M_i f)$ in x.

We shall prove that Kor(L) $\subset \hat{L}$. Obviously it is enough to show that any $g \in L_0 \setminus \hat{L}$ does not belong to Kor(L). Using the fact that $g \notin \hat{L}$, there exist $c; x$ such as in Lemma 1.2. and μ satisfying (1) and (2). One can find h_1 and h_2 in L with $h_1 \leq g \leq h_2$ be the definition of L_b. $h_2 - h_1 \geq 0$ and in x strictly greater than 0. Multiplying $h_2 - h_1$ by a proper constant we find a h_0 in L such that $h_0 \geq 0$ and $g_0(x) > 1$. Fix a neighbourhood base B of x in the weak topology induced by $C(X)$ such that the following two properties hold:

(i) $h_0(t) > 1$ for any $t \in U \in B$ and

(ii) L is bounded on U.

To each $U \in B$ we assign a $q_U \in C(X)$ such that

(iii) $0 \leq q_U \leq 1$, $q_0(x) = 1$ but $q_U(t) = 0$ for every $t \notin U$.

Now let us define M_U on $C(X)$ as follows:

$$M_U f \equiv \mu(f) q_U + f - f q_U.$$

M_U is clearly positive and we claim that the range of M_U lies in L_b. Indeed, q_U is in L_b, h_0 dominates it, and $f q_U$ is in L_b too, because of (iii). Finally, take the net $(M_U)_{U\in B}$ of positive linear maps. An easy computation shows that $(M_U h)_{U\in B}$ converges to h pointwise for all $h \in L$. (At x: $M_U h(x) = h(x)$ because $\mu(h) = h(x)$ and if $x \neq y$, there exists a $U \in B$ with $y \notin U$, so $M_U(y) = h(y)$.) The net $(M_U(g))_{U\in B}$ does not converge to g pointwise, because $M_U g(x) = c$ and $c \neq g(x)$.

REMARK 1.2. If X is compact and L is a lattice, then for all $f \in \mathrm{Kor}(L)$ and $\varepsilon > 0$ one can find an l_ε^f in L so that:

$$\sup_{x\in X} \left| f(x) - l_\varepsilon^f \right| \leq \varepsilon .$$

PROOF. By the previous theorem, it is easy to verify that

$$\forall\, x \in X,\ \forall\, \varepsilon > 0 : \quad \exists\, l_{\varepsilon,x}^f \quad \text{such that} \quad l_{\varepsilon,x}^f(x) - f(x) < \frac{\varepsilon}{2} \quad \text{and}$$

$$f(y) \leq l_{\varepsilon,x}^f(y) \quad \text{for all} \quad y \in X.$$

Let $U_x = \{y \in X : l_{\varepsilon,x}^f(y) - f(y) < \varepsilon\}$. It is obvious that $X = \bigcup_{x\in X} U_x$ because $x \in U_x$. Using the compactness argument one can find finitely many $x \in X$ denoted by $x_1, \ldots, x_n$, such that

$$X = \bigcup_{i=1}^{n} U_{x_i} .$$

Let us define l_ε^f as follows:

$$l_\varepsilon^f(y) = \bigwedge_{i=1}^{n} l_{\varepsilon,x_i}^f(y).$$

This function clearly satisfies the property above. ∎

2. Separating a compact set from a single point

LEMMA 2.1. *Let X be a topological space and $K \subset X$ be a compact set. Let $F \subset C(X)$ with the following property:*

(∗) $\quad \forall\, x; y \in X \quad \exists\, f \in F \quad$ *such that* $\ f(x) = 0\ $ *but* $\ f(y) \neq 0.$

Then for all $z \notin K$ there exists $f \in \mathrm{Lin}\{f^2 : f \in F\}$ such that

(1) $f(z) = 0$

(2) $f(x) > 1$ *for all $x \in K$.*

PROOF. We know from $(*)$ that for all $x \in X$ there exists an $f \in F$ such that, $f(z) = 0$ but $f(x) \neq 0$. So one can find an $f_x^2 \in \mathrm{Lin}\{F^2\}$ such that $f_x^2(z) = 0$ but $f_x^2(x) = 1.5$. L et us define U_x for all $x \in K$ as the following set:

$$U_x = \left\{ y \in K : \left| f_x^2(y) - 1.5 \right| < \frac{1}{2} \right\}.$$

U_x is obviously an open set in K and $K = \bigcup_{x \in K} U_x$. Using the compactness argument, there can be found finitely many $x \in X$, namely $x_1, \ldots, x_n$ with

$$K = \bigcup_{i=1}^{n} U_{x_i}.$$

Now we are ready to define $\hat{f}$, which separates K from z

$$\hat{f}(y) = \sum_{i=1}^{n} f_{x_i}^2(y).$$

$\hat{f}(x) = 0$ because $f_{x_i}^2(z) = 0$ for all $i = 1, \ldots, n$, and for any $x \in K$ there exists an $i \in \{1, \ldots, n\}$ such that $x \in U_{x_i}$, which means

$$\hat{f}(x) \geq f_{x_i}^2(x) > 1.$$

So $\hat{f}$ satisfies properties (1) and (2). ∎

Sometimes it can be difficult to check whether property $(*)$ holds, therefore the following lemma might be useful.

LEMMA 2.2. *Let X, K, z be as in the previous lemma and let $G \subset C(X)$ be a linear subspace with the following properties:*

(i) $\forall\, x \in X\ \exists\, f \in G$ *such that* $f(x) \neq 0$

(ii) G *separates* X.

Then there exists a positive function $f \in \mathrm{Lin}\{G^2, G^3, G^4\}$ *such that*

(1) $f(z) = 0$

(2) $f(x) \geq 1$ *for all* $x \in K$.

PROOF. We have to prove, that there exists an $f_{x,y} \in \mathrm{Lin}\{G \cup G^2\}$ which satisfies property $(*)$. By (1) and (2) we know that there exist $f, f_y \in F$ with $f(x) \neq f(y)$ and $f_y(y) \neq 0$. We can find c_1 and c_2 in $\mathbb{R}$ such that

$$c_1 f(x) + c_2 f_y(x) \neq c_1 f(y) + c_2 f_y(y) \quad \text{and} \quad c_1 f(y) + c_2 f_y(y) \neq 0.$$

If $c_1 f(x) + c_2 f_y(x) = 0$ let us define $f_{x,y}$ as $c_1 f + c_2 f_y$, otherwise let us define $f_{x,y}$ as

$$c_1 f + c_2 f_y - \frac{c_1 f + c_2 f_y}{c_2 f(x) + c_2 f_y(x)}.$$

COROLLARY 2.3. *If $1_x \in F$ and F is a linear space which separates X then $(*)$ holds. In this case* $\mathrm{Lin}\{f^2 : f \in F\}$ *can be replaced by* $\mathrm{Lin}\{|f| : f \in \in F\}$, *and* $f_{x_i}^2$ *by* $|f_{x_i}|$.

3. Approximation at one point

In this section we try to approximate functions at one point so that the function approximating is everywhere greater (or smaller) than the function. It can be convenient to introduce the following notation:

$$L_0 = \Big\{ g \in C(x) : \ \exists\, f \in F^+ \text{ such that } \forall\, \varepsilon > 0 \\ \{x \in X : \ |g(x)| > \varepsilon f(x)\} \text{ is compact} \Big\},$$

where F^+ denotes the set of all positive functions in F.

LEMMA 3.1. *Let X be as in the previous lemmas, and $F \subset C(X)$ be linear space with the following properties:*

(i) $\forall\, x \in X \ \exists\, f \in L_0 \cap F^+$ *s.t.* $f(x) \neq 0$.

(ii) $\forall\, x \in X \setminus K$ *where K is a compact set* $\exists\, f_{x,K} \in F^+$ *function s.t.* $f_{x,K}(x) = 0$ *but* $f_{x,K} > 0$ *on K.*

Then $\forall\, \varepsilon > 0\ \forall\, z \in X\ \forall g \in L_0\ \exists l \in \mathrm{Lin}\{F\}$ *such that*

$$\forall\, x \in X \qquad l(x) > g(x) \quad \text{and} \quad l(z) < g(z) + \varepsilon.$$

PROOF. Let f_z be in $L_0 \cap F^+$ such that $f_z(z) \neq 0$ by (i). We can assume that $g(z) + f_z(z) > 0$. Now we can define $l_1(x)$ as follows:

$$l_1(x) = \delta_1 f_g(x) + \delta_2 f_{f_z}(x) + \delta_3 f_z(x),$$

where f_g, f_{f_z} correspond to g and f_z by the definition of L_0, and $\delta_1 f_g(z) + \delta_2 f_{f_z}(z) < \frac{\varepsilon}{2}$ but $\delta_3 f_z(z) = g(z) + f_z(z) + \frac{\varepsilon}{4}$. We shall show that the set of those points where $l_1 < g + f_z$ is precompact. Indeed,

$$K_1 = \{x \in X : \ l_1(x) < g(x) + f_z(x)\} \subset \{x \in X : \ \delta_1 f_g(x) < |g(x)|\} \subset \\ \subset \{x \in X : \ \delta_2 f_{f_z}(x) < (1 + |\delta_3|) f(z)\}.$$

The last two sets are precompact, which means that K_1 is also precompact. Denote that closure of K_1 by K. Obviously, K is a compact set and does not contain z. We are ready to define l_2:

$$l_2(x) = l_1(x) + \max_{y \in K} \left(|g(y)| + f_z(y)\right) \cdot f_{z,K},$$

where $f_{z,K}$ separates K and z by (ii) and can be found in F.

(a) In z: $l_2(z) = l_1(z) > g(z) + f_z(z)$, but $l_1(z) < g(z) + f_z(z) + \varepsilon$.

(b) If $y \notin K$: $l_2(y) \geq l_1(y) > g(y) + f_z(y)$.

(c) If $y \in K$: $l_2(y) > \max_{t \in K} \left(|g(t)| + f(t)\right) \geq g(y) + f_z(y)$.

Now let us define l as $l_2 - f_z$. This function obviously satisfies the conditions of the theorem. ∎

COROLLARY 3.2. *If* $1_x \in L_0 \cap F^+$ *then* (i) *is automatically satisfied.*

COROLLARY 3.3. *According to the previous section, of* G *separates* X *then the following sets satisfy* (ii)*:*

(a) *the linear lattice generated by* $1_x \cup G$*,*

(b) *if* $1_x \in G$ *and* G *is linear then* $\mathrm{Lin}\{G^2\}$*,*

(c) *the vector space generated by* $G^2 \cup G^3 \cup G^4$ *if for all* $x \in X$ *there exists* $g \in G$ *with* $g \neq 0$.

4. Conclusions

Using Theorem 1.1 and Lemma 3.1 we can always say that $L_0 \subset \mathrm{Kor}(F)$ if F satisfies (i) and (ii) in Lemma 3.1. If X is compact then $L_0 = C(X)$. Using the notations of Corollary 3.3 we can arrive at the following results:

(1) If $0 \notin X \subset \mathbb{R}$, $G = \{\mathrm{id}_x\}$ and $F = \mathrm{Lin}\{\mathrm{id}_x^2, \mathrm{id}_x^3, \mathrm{id}_x^4\}$ then

$$\mathrm{Kor}\left(\{\mathrm{id}_x^2, \mathrm{id}_x^3, \mathrm{id}_x^4\}\right) \supset \{g(x) \in C(X) :\ g(x) = O(x^4)\}.$$

In the special case, when X is compact: $\mathrm{Kor}\left(\{\mathrm{id}_x^2, \mathrm{id}_x^3, \mathrm{id}_x^4\}\right) = C(X)$.

(2) If $X \subset \mathbb{R}$, $G = \{\mathrm{id}_x\}$ and $F = \mathrm{Lin}\{1_x, \mathrm{id}_x, \mathrm{id}_x^2\}$ then

$$\mathrm{Kor}\left(\{i_x, \mathrm{id}_x, \mathrm{id}_x^2\}\right) \supset \{g(x) \in C(X) :\ g(x) = O(x^2)\}.$$

Moreover, if X is compact $\mathrm{Kor}(\{1_x, \mathrm{id}_x, \mathrm{id}_x^2\}) = C(X)$.

By remark 1.2 we can easily prove the classical Stone–Weierstrass Theorem.

THEOREM 4.1. *Let X be a locally compact Hausdorff space, and $A \subset$ $\subset C_0(X)$ be a closed subalgebra in the space $(C_0(X), |\cdot|)$ with the following properties:*

(i) $\forall\, x \in X\ \exists f \in A$ *such that* $f(x) \neq 0$ *and*

(ii) X *is separated by* A.

Then $A = C(X)$.

PROOF. Let $X_1 = X \cup \omega$ be the one-point compactification of X. Let us extend all $f \in A$ into ω with $f(\omega) = 0$. Let F be the algebra generated by $A \cup 1_x$. In fact, on compact spaces closed algebras containing neutral elements are lattices (see [4]). It means that we can apply Remark 1.2 considering Corollary 3.2 and Corollary 3.3. Hence, for every $h \in C(X_1)$ (where $h(\omega) = 0$) and $\varepsilon > 0$, one can find $f_1 \in F$ such that

$$\sup_{x \in X_1} |f_1(x) = h(x)| < \frac{\varepsilon}{2}.$$

Especially in ω: $|f_1(\omega)| < \frac{\varepsilon}{2}$. By definition of F: $f_1(x) = f(x) + c$ where $f \in A$ and c is a constant smaller than $\frac{\varepsilon}{2}$. Therefore $|f_1(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in X_1$. Using the triangle inequality $\sup\limits_{x \in X} |f(x) - h(x)| < \varepsilon$ which implies that $h \in \overline{A} = A$. ∎

References

[1] H. BAUER, Theorems of Korovkin type for adapted spaces, *Ann. Inst. Fourier,* **23** (1973), 245–260.

[2] H. BAUER, Approximations and abstract boundaries, *Amer. Math. Monthly,* **85** (1987), 328–332.

[3] K. DONNER, Korovkin closure for positive linear operators, *J. of Approximation Theory,* **26** (1979), 14–25.

[4] M. H. STONE, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Soc.,* **41** (1937), 375–481.

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REMARK ON THE SPACE OF RESTRICTED DERIVATIVES

By

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Let $H \subset [0,1]$ be a nowhere dense closed subset of positive measure. We denote by $\mathcal{D}_H$ and $\mathcal{B}^1_H$ the classes of derivates and Baire-1 functions restricted on H. In [1] it is proved that $\mathcal{D}_H$ is *not* a G_δ subset in the space $\mathcal{B}^1_H$ with the unifom convergence. G. PETRUSKA asked whether it is an F_σ subset.

THEOREM. *$\mathcal{D}_H$ is not an F_σ subset of $\mathcal{B}^1_H$ with respect to the topology of uniform convergence.*

PROOF. Let $b\mathcal{D}_H$ and $b\mathcal{B}^1_H$ be the set of the bounded elements of $\mathcal{D}_H$ and $\mathcal{B}^1_H$. It is enough to show that $b\mathcal{D}_H$ is not an F_σ subset of $b\mathcal{B}^1_H$.

Suppose indirectly that the complement set of $b\mathcal{D}_H$ is G_δ in $b\mathcal{B}^1_H$. Since $b\mathcal{B}^1_H$ is a complete metric space, the subspace topology of the complement can be generated by a metric ϱ such that $(b\mathcal{D}_H \setminus b\mathcal{B}^1_H, \varrho)$ is complete, and given a function f, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\sup|f-g| < \delta(\varepsilon)$ implies $\varrho(f,g) < \varepsilon$.

We can suppose that $\inf H = 0$ and $\sup H < 1$. Let $1 = x_1 > x_2 > \dots$ be a sequence of $[0,1] \setminus H$ tending to 0, and let $I_n = [x_{n+1}, x_n] \cap H$. We define a sequence of functions $f_n \in b\mathcal{B}^1_H \setminus b\mathcal{D}_H$ by induction.

We choose $f_0 \in b\mathcal{B}^1_H \setminus b\mathcal{D}_H$ arbitrarily. If f_{n-1} has been defined, then let f_n be a function for which

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(i) $f_n|_{I_k} = f_{n-1}|_{I_k} \quad k = 1, 2, \ldots, n-1$;

(ii) $f_n|_{I_n} \in b\mathcal{D}_{I_n}$;

(iii) $f_n \in b\mathcal{B}^1_H \setminus b\mathcal{D}_H$;

(iv) $\sup |f_n - f_{n-1}| < \delta(1/2^n)$;

(v) $f_n(0) = 0$.

In [2] it is proved that $b\mathcal{D}_H$ is everywhere dense in $b\mathcal{B}^1_H$, from this the existence of a function f_n satisfying (i)-(v) follows. Conditions (iii) and (iv) imply that there exists $\lim f_n = f \in b\mathcal{B}^1_H \setminus b\mathcal{D}_H$. According to (i) and (ii) there exist differentiable functions $g_n : I_n \to \mathbf{R}$ with $g_n' = f|_{I_n}$. Now we construct a differentiable function $h : H \to \mathbf{R}$ for which $h' = f$, and extending h to the interval [0,1] in the usual way to be a differentiable function it contradicts $f \in b\mathcal{B}^1_H \setminus b\mathcal{D}_H$. (Putting $h(u) \stackrel{\text{def}}{=} g_n(u)$ for every n and $u \in I_n$ we would only have $h'(u) = f(u)$ for every $u \neq 0$.)

Since g_n is continuous, there exist finitely many pairwise disjoint intervals (y_1, z_1), (y_2, z_2), $\ldots$, (y_m, z_m) of real ordering, for which y_i, $z_i \in I_n$, $(z_i, y_{i+1}) \cap H = \emptyset$, $\cup_i (y_i, z_i)$ covers I_n, and for every $u \in (y_i, z_i) \cap I_n$ we have $|g_n(u) - g_n(y_i)| < x_n^2$. Let $h_n(u) \stackrel{\text{def}}{=} g_n(u) - g_n(y_i)$ for every $1 \leq i \leq m$ and $u \in (y_i, z_i) \cap I_n$. Now it is immediate that for every $u \in I_n$ we have $h_n'(u) = f(u)$, and $|h_n(u)| < x_n^2$. Putting

$$h(u) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } u = 0 \\ h_n(u) & \text{if } u \in I_n \end{cases}$$

we have a function $h : H \to \mathbf{R}$ for which $h'(u) = f(u)$ for every $u \neq 0$ and according to $|h_n(u)| < x_n^2$ for every $u \in I_n \subset (x_{n+1}, x_n)$ we have $h'(0) = 0 = = f(0)$. ∎

References

[1] Petruska, G., On the space of restricted derivatives, *Annales Univ. Sci. Budapest Eötvös Nom., Sectio Math.,* **24** (1981), 253–254.

[2] Petruska, G., Laczkovich, M., Baire 1 functions, approximately continuous functions and derivatives, *Acta Math. Acad. Sci. Hung.*, **25** (1974), 189–212.

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SOME REMARKS ON EQUICONTINUOUS FOLIATIONS

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In an appendix to P. MOLINO's book *Riemannian Foliations* E. GHYS proposed the study of foliations whose pseudogroup is equicontinuous for some Riemannian metric on the transverse manifold. In this short note we propose to look at this problem in the case of $\nabla - G$-foliations, the class of foliations which contains transversely affine foliations.

Let $\mathcal{F}$ be a foliation on a manifold M. The foliation $\mathcal{F}$ is given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modelled on a manifold N_0, i.e.

i) $\{U_i\}$ is an open covering of M,

ii) $f_i : U_i \to N_0$ are submersions with connected fibres defining $\mathcal{F}$,

iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} \circ f_j = f_i$ on $U_i \cap U_j$.

The manifold $N = \coprod f_i(U_i)$ we call the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$ and the pseudogroup $\mathcal{H}$ generated by g_{ij} the holonomy pseudogroup (representative) on the transverse manifold N.

The foliation $\mathcal{F}$ is called a $\nabla - G$-foliation if on the transverse manifold N there exists an $\mathcal{H}$-invariant G-structure with a G-connection ∇ of which the holonomy pseudogroup $\mathcal{H}$ is a pseudogroup of local affine transformations. The existence of such a G-structure is equivalent to the existence of a foliated G-reduction $B(M, G; \mathcal{F})$ of the bundle $L(M; \mathcal{F})$ of transverse linear frames of the normal bundle $N(M; \mathcal{F})$ of the foliation $\mathcal{F}$. To the connection ∇ corresponds a transversely projectable connection ω in the bundle $B(M, G; \mathcal{F})$.

Our main result is the following (all new notions are explained below):

THEOREM 1. *Let $\mathcal{F}$ be a transversely complete $\nabla - G$ foliation on a compact manifold M. If $\mathcal{F}$ has a holonomy pseudogroup representative which is equicontinuous for some Riemannian metric on the corresponding transverse* *manifold, then there exists a bundle-like Riemannian metric on* $(M, \mathcal{F})$*, i.e. the foliation* $\mathcal{F}$ *is Riemannian.*

The above theorem is a consequence of a more technical result:

THEOREM 2. *Let* $\mathcal{F}$ *be a transversely complete* $\nabla - G$ *foliation on a compact manifold* M*. If the closures of leaves of* $\mathcal{F}_1$ *are compact then* $\mathcal{F}$ *is a Riemannian foliation.*

The author would like to express his deep gratitude to Pierre Molino for his invaluable suggestions.

1. Preliminaries

We recall some basic information about foliations and introduce notions which will be useful for us.

The choice of a supplementary subbundle Q to $T\mathcal{F}$ fixes our choice of a supplementary subbundle $\tilde{Q}$ to $T\mathcal{F}_1$, (the natural foliation of $B(M, G; \mathcal{F})$ i.e. $\tilde{Q} = (d\pi)^{-1}(Q)$. Therefore the corresponding fundamental horizontal vector fields $B(\xi)$ and the fundamental vertical vector fields A^* form a transverse parallelism of $\mathcal{F}_1$. This transverse parallelism is complete iff the connection ω is transversely complete, i.e. its geodesics tangent to Q are globally defined. This results from the following simple lemma.

LEMMA 1. *The projections onto* M *of integral curves of the vector fields* $B(\xi)$ *are geodesics tangent to* Q *of the connection* ω*.*

PROOF. Let $\tilde{\omega}$ be the extension of the connection ω. $\tilde{\omega}$ is a connection in the $GL(p) \times G$-structure $B(M, GL(p) \times G)$ which can be written as the fibre product $L(T\mathcal{F}) \times_M B(M, G; \mathcal{F})$. The geodesics of $\tilde{\omega}$ which are tangent to Q are precisely the "transverse geodesics" of ω, i.e. solutions of the equation of the geodesic of ω, cf. [12].

The fundamental horizontal vector fields $B(\xi)$, $\xi \in R^q$, of $B(M, G; \mathcal{F})$ can be lifted to $B(M, GL(p) \times G)$. The lift of $B(\xi)$ is precisely the vector field $B((0, \xi))$ for $(0, \xi) \in R^p \times R^q = R^n$. The projection of an integral curve of $B((0, \xi))$ is a geodesic of $\tilde{\omega}$, which must be tangent to Q. Since $B((0, \xi))$ is the lift of $B(\xi)$ the projections on M of integral curves of these vector fields are the same. ∎

The above considerations lead to the following definition.

DEFINITION 1. A $\nabla - G$ foliation is transversely complete if for some choice of a supplementary subbundle Q the corresponding equation of the geodesic is transversely complete, or equivalently if the corresponding transverse parallelism is complete.

With this definition in mind we have the following structure theorem for $\nabla - G$-foliations, cf. [11]:

THEOREM 3. *Let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation on a manifold M. Then the closures of leaves of the foliation $\mathcal{F}_1$ of the foliated G-structure $B(M, G; \mathcal{F})$ are fibres of a locally trivial fibre bundle, called the basic fibration. The foliation of the closure of a leaf of $\mathcal{F}_1$ by leaves of $\mathcal{F}_1$ is a Lie foliation with the same model Lie group for any leaf.*

PROOF. It is a direct consequence of our considerations and Molino's structure theorem for complete transversely parallelisable (T.P.) foliations, cf. [6, 9]. ∎

For $\nabla - G$-foliations we can define, following P. Molino, the commuting sheaf, cf. [8, 9]. Let $\mathcal{C}_1$ be the sheaf of germs of foliated vector fields X on B commuting with all global foliated vector fields of $(B, \mathcal{F}_1)$, thus in particular the transverse parallelism of $\mathcal{F}_1$. This last condition is equivalent to $L_X\theta = L_X\omega = 0$. Let $\overline{X}$ be the corresponding vector field on the total space of $B(N, G)$. Then $L_{\overline{X}}\overline{\theta} = L_{\overline{X}}\overline{\omega} = 0$ where $\overline{\omega}$ is the connection form of ∇. This means that $\overline{X}$ is the lift of a local infinitesimal affine transformation of ∇. Thus the sheaf $\mathcal{C}_1$ defines the sheaf $\mathcal{C}$ of germs of foliated vector fields which are also local infinitesimal affine transformations of the transversely projectable connection ω. We call $\mathcal{C}$ the commuting sheaf of $\mathcal{F}$.

DEFINITION 2. We say that the commuting sheaf $\mathcal{C}$ is of compact type if the orbits of the sheaf $\mathcal{C}_1$ are compact.

The following proposition is an immediate consequence of the definition and of the properties of transversely parallelisable (T.P.) foliations, cf. [6, 9].

PROPOSITION 1. *Let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation. If its commuting sheaf is of compact type, then the closures of leaves are compact and they are integral submanifolds of a regular distribution of non-constant dimension defined by the commuting sheaf.*

Let us assume that the foliation $\mathcal{F}$ is a transversely complete $\nabla - G$-foliation. Since the foliation $\mathcal{F}_1$ is complete T.P., it determines a locally trivial fibration (the basic fibration) $p : B \to W$ whose fibres are the closures of leaves of the foliation $\mathcal{F}_1$. As this foliation is G-invariant, the group G acts on the basic manifold W. This leads us to the following lemma whose proof is trivial.

LEMMA 2. *If the commuting sheaf $\mathcal{C}$ is of compact type, then the action of the group G on the basic manifold W is proper.*

Now let us turn our attention back to pseudogroups. We introduce two key definitions.

DEFINITION 3. A pseudogroup $\mathcal{H}$ of local homeomorphisms of a topological space N is called equicontinuous for a metric d on N if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $h \in \mathcal{H}$ $d(x,y) < \delta$ implies $d(h(x),h(y)) < \epsilon$ whenever $h(x)$ and $h(y)$ are defined.

Now we shall recall the key notion 'of compact type', cf. [13].

Let $\mathcal{H}$ be a pseudogroup of local affine transformations of a connection ∇ in a G-structure $B(S,G)$. To any element h of $\mathcal{H}$ corresponds a local diffeomorphism h^1 of B which preserves the parallelism of B. And viceversa, any such a local diffeomorphism of B of connected domain is defined by a local affine transformation. The correspondence $j^1 : h \mapsto h^1$ associates to the pseudogroup $\mathcal{H}$ a pseudogroup $\mathcal{H}^1$ of local diffeomorphisms of B preserving the parallelism.

For our purposes we need also the following definition.

DEFINITION 4. We say that a pseudogroup $\mathcal{H}$ of local affine transformations of a connection in a G-structure $B(S,G;\pi)$ is of compact type if for any compact subset K of S and any point x of B the set $\mathcal{H}^1 x \cap \pi^{-1}(K)$ is relatively compact.

REMARK. Both notions "of compact type" are 'invariant' under compactly generated equivalences of pseudogroups.

Moreover, the following is true.

LEMMA 3. *Let $\mathcal{F}$ be a $\nabla - G$-foliation on a compact manifold M. The closures of leaves of the lifted foliation $\mathcal{F}_1$ are compact iff for any finite cycle the corresponding representative of the holonomy pseudogroup is of compact type.*

PROOF. Let $\{U_i, f_i, g_{ij}\}$ be a finite cocycle defining the foliation $\mathcal{F}$. Since M is compact we can assume that there exists a cocycle $\{W_i, k_i, l_{ij}\}$ defining $\mathcal{F}$ such that $\overline{U_i} \subset W_i$, $k_i \,|\, U_i = f_i$, $l_{ij} \,|\, f_i(U_i) = g_{ij}$. (One can shrink slightly U_i

and put $U_i = W_i$.) Then $\{V_i, \bar{f}_i, \bar{g}_{ij}\}$ where $V_i = \pi^{-1}(U_i)$ and the mappings $\bar{f}_i$ and $\bar{g}_{ij}$ are the mappings induced on the level of G-structures by f_i and g_{ij}, respectively, is a cocycle defining the foliation $\mathcal{F}_1$ on B. Let L be a leaf of $\mathcal{F}_1$. It corresponds to the $\mathcal{H}^1$-orbit of a point x_L. As the cocycle is finite and sets U_i are relatively compact the closure of L is equal to $\bigcup_i f_i^{-1}(\overline{\mathcal{H}_1 x_L})$. Having this in mind it is not difficult to conclude the proof. ∎

2. The proof of the main theorem

First of all we shall prove Theorem 2, which is the main step in the proof of Theorem 1.

REMARK. A version of Theorem 2 was proved in [13] using highly complicated theory of pseudogroups. Presently we shall give a proof using only classical notions of differential geometry which makes it more readable and simpler.

PROOF OF THEOREM 2. Under the assumptions of the theorem we shall construct a suitable Riemannian metric on the manifold M.

The foliation $\mathcal{F}_1$ on the total space B of the foliated G-structure $B(M, G; \mathcal{F})$ has compact closures of leaves. These closures form a foliation (the basic foliation) which is also G-invariant. The space of leaves of the basic foliation is a G-manifold W. The action of the Lie group G on W is proper so it admits a G-invariant Riemannian metric.

A foliated vector field tangent to the closure of a leaf of $\mathcal{F}_1$ at one point is tangent to it at any point of this closure; i.e. there exists a vector bundle of finite dimension $E \to W$; its fibre over $w \in W$ consists of foliated vector fields tangent to the leaf closure $p^{-1}(w)$, cf. [6, 9].

The group G sends foliated vector fields to foliated vector fields, thus G acts on E and the action is compatible with the action of G on W via the projection $E \to W$; i.e. $E \to W$ is G-equivariant.

The standard proof of the existence of a G-invariant Riemannian metric for proper G-actions, cf. [10], can be adapted to prove the following theorem.

THEOREM 4. *Let $E \to W$ be a G-equivariant vector bundle of finite dimension. If the action of the Lie group G on W is proper then there exists a G-invariant Riemannian tensor field in E.*

Now we can define a G-invariant bundle-like metric on $(B, \mathcal{F}_1)$. Let $\tilde{Q}$ be a G-invariant subbundle supplementary to $T\mathcal{F}_1$ such that $\tilde{Q} = Q_1 \oplus Q_2$ and $T\mathcal{F}_1 \oplus Q_1 = T\overline{\mathcal{F}_1}$, with G-invariant bundle Q_2 isomorphic to the pullback of the tangent bundle TW. We lift the G-invariant Riemannian metric from W to Q_2. It is base-like for $\mathcal{F}_1$. It remains to define a Riemannian metric on Q_1. Let $X,\ Y \in Q_{1x}$. Then there exist unique foliated vector fields $\overline{X}$, $\overline{Y}$ along E_x such that $\overline{X}_x = X$ and $\overline{Y}_x = Y$. We put $\overline{h}(X, Y) = h(\overline{X}, \overline{Y})$ where h is the tensor field from Theorem 4. It is obvious that $\overline{h}$ on Q_1 is $\mathcal{F}_1$ base-like and G-invariant so the whole metric $\overline{h}$ is a G-invariant $\mathcal{F}_1$ base-like metric. This G-invariant Riemannian metric, when restricted to the horizontal bundle of the connection induces a bundle-like metric on the foliated manifold $(M, \mathcal{F})$. ∎

3. Applications

For general pseudogroups the notion of a complete pseudogroup is not invariant under equivalences as the following example illustrates well this fact.

EXAMPLE. Let $N = \mathbf{R}$ and $\mathcal{H}$ be a pseudogroup generated by the homothety $h_\lambda : x \mapsto \lambda x$ for $0 < \lambda < 1$ and the translation $\tau_1 : x \mapsto x + 1$. $(N, \mathcal{H})$ is a complete pseudogroup. It is equivalent to its restriction $\mathcal{H}'$ to the interval $(-1/2, 1/2)$. However this second pseudogroup is not complete.

To obtain a notion which is invariant under equivalences of pseudogroups we should demand more.

DEFINITION 5. A pseudogroup $(N, \mathcal{H})$ is strongly complete if for any two points x and y of N and any neighbourhood V_0 of y there exist neighbourhoods U and V of x and y, respectively, $V \subset V_0$, such that any element h of $\mathcal{H}$ with domain in U and target in V can be extended to an element h' of $\mathcal{H}$ defined on the whole set U and whose image is contained in V_0.

For pseudogroups of local isometries and for "equicontinuous" pseudogroups the notions of completeness and strong completeness are equivalent. It is not difficult to check that a pseudogroup equivalent to a strongly complete pseudogroup is itself strongly complete. Moreover it is obvious that a strongly complete pseudogroup is complete. The pseudogroup $(N, \mathcal{H})$ of Example is not strongly complete although it is complete.

The assumption of strong completeness seems to impose very strong restrictions on the pseudogroup. Let us look into this problem. Let $\mathcal{V}$ be any cocycle defining $\mathcal{F}$ and $\mathcal{U}$ be a relatively compact cocycle associated to it. We denote the transverse manifold associated to the cocycle $\mathcal{U}$ by N' and the holonomy pseudogroup representative on N' by $\mathcal{H}'$. The transverse manifold associated to the cocycle $\mathcal{V}$ is denoted by N and the holonomy pseudogroup representative on N by $\mathcal{H}$. The subset N' of N is relatively compact and the pseudogroup $\mathcal{H}'$ is the restriction of $\mathcal{H}$ to N'.

Since N' is a relatively compact subset of N there exists $\epsilon > 0$ such that for any point $x \in \overline{N}'$ there is an open relatively compact neighbourhood V_x of x with the following properties:

i) for any $z \in V_x$ $\exp_z |B(0_z, \epsilon)$ is a diffeomorphism onto the image;

ii) $\exp_z(S(0_z, \epsilon)) \cap V_x = \emptyset$;

where $B(0_z, \epsilon) = \{v \in TN_z : \|v\| < \epsilon\}$ and $S(0_z, \epsilon) = \{v \in TN_z : \|v\| = \epsilon\}$, for some Riemannian metric on N. This fact results easily from a slight refinement of the classical argument about geodesically convex neighbourhoods, cf. [4].

Having established these technical details we return to the strong completeness. Let us take a pair of points x and y with $V_0 = V_y$. Then the set U can be equal to $B(x, \delta) = \exp_x(B(0_x, \delta))$ for some $\delta > 0$, and $V \subset V_y$. Then for any element h of $\mathcal{H}$ defined on U we have

$$h \circ \exp_x |B(0_x, \delta) = \exp_{h(x)} \circ d_x h | B(0_x, \delta)$$

where exp is the exponential mapping defined by the connection. As $h(U) \subset \subset V_y$, the set $d_x h(B(0_x, \delta))$ must be contained in $B(0_{h(x)}, \epsilon)$. This means precisely that the set $\mathcal{H}^1(x, V) = \{j^1_x h : h \in \mathcal{H},\ h(x) \in V\}$ is relatively compact. Hence for any relatively compact subset K of N' the set $\mathcal{H}^1(x, K) = = \{j^1_x h : h \in \mathcal{H},\ h(x) \in K\}$ is relatively compact. Thus we have proved the following lemma.

LEMMA 4. *A strongly complete pseudogroup of local affine transformations of an affine connection is of compact type.*

Combining Lemmas 4 and 3 with Theorem 2 we get the following theorem.

THEOREM 5. *Let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation on a compact manifold. If the holonomy pseudogroup of $\mathcal{F}$ is strongly complete, then $\mathcal{F}$ is a Riemannian foliation.*

For equicontinuous foliations we have the following.

THEOREM 6. *Let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation on a compact manifold. If $\mathcal{F}$ has a representative of the holonomy pseudogroup which is equicontinuous for some metric inducing the natural topology on the transverse manifold, then $\mathcal{F}$ is a Riemannian foliation.*

PROOF. According to the result of [13] our representative is complete and thus strongly complete. Therefore Lemma 4 ensures that our holonomy pseudogroup is of compact type, so the closures of leaves of the foliation $\mathcal{F}_1$ are compact, (Lemma 3). An application of Theorem 2 completes the proof. ∎

For transversely affine flows we have an even stronger result.

COROLLARY 1. *Let $\mathcal{F}$ be a transversely complete transversely affine flow on a compact manifold. If for some finite cocycle defining $\mathcal{F}$ the representative of its holonomy pseudogroup is equicontinuous and the commuting sheaf is non-trivial, then the closures of its orbits are diffeomorphic to tori and the flow is Riemannian.*

PROOF. As the foliation of the closures of leaves of the lifted foliation is Lie, Carriere's result, cf. [9], assures that either the closure is a torus or the leaf is a closed one. The assumption on the commuting sheaf excludes the second possibility. According to Lemma 3 the representative of the holonomy pseudogroup is of compact type; Lemma 4 of [13] ensures that this pseudogroup is complete. Theorem 5 assures that the foliation is Riemannian. ∎

REMARK. It is impossible to prove that the flow is distal and then apply the result of Ellis, cf. [1]. There are 1-dimensional transversely affine flows with distal holonomy group which are induced by both distal and non-distal flows. Such an example has been constructed by E. Ghys.

References

[1] R. ELLIS, Distal transformation groups, *Pacific J. Math.,* **8** (1958), 401–405.

[2] E. GHYS, Riemannian foliations; examples and problems, Appendix E in [9].

[3] A. HAEFLIGER, Pseudogroups of local isometries, *Differential Geometry,* L. A. Cordero ed., Proceedings Vth International Colloquium on Differential Geometry, Santiago de Compostela 1984, Pitman 1985.

[4] S. KOBAYASHI, K. NOMIZU, *Foundations of Differential Geometry,* Interscience Publ., New York 1963, 1969.

[5] J. L. KOSZUL, *Lectures on Groups of Transformations,* Tata Institute of Fundamental Research, Bombay, 1960.

[6] P. MOLINO, Etude des feuilletages transversalement complets et applications, *Ann. Sci. Ecole Norm. Sup.,* **10** (1977), 289–307.

[7] P. MOLINO, Feuilletages riemanniens sur les variétés compactes: champs de Killing transverse, *C. R. Acad. Sc. Paris,* **289** (1979), 421–423.

[8] P. MOLINO, Géométrie global des feuilletages riemanniens, *Proc. Kon. Neder. Akad.,* **85** (1982), 45–76.

[9] P. MOLINO, *Riemannian Foliations,* Progress in Math. vol. **73**, Birkhäuser 1988.

[10] R. PALAIS, On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.,* **73** (1961), 295–323.

[11] R. A. WOLAK, On $\nabla - G$-foliations, *Suppl. Rend. Cir. Mat. Palermo,* **6** (1984), 329–341.

[12] R. A. WOLAK, The structure tensor of a transverse G-structure on a foliated manifold, *Boll. U. M. I.,* **4** (1989), 1–15.

[13] R. A. WOLAK, Foliated G-structures and Riemannian foliations, *Manus. Math.,* **66** (1989), 45–59.

[14] R. A. WOLAK, Transverse completeness of foliated systems of differential equations, *Proc VIth Inter. Coll. on Differential Geometry,* Santiago de Compostela 1988, ed. L. A. Cordero, Santiago de Compostela, 1989, 253–262.

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EXTREMAL PERIODIC SOLUTIONS FOR QUASILINEAR DIFFERENTIAL EQUATIONS

By

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1. Introduction

In this paper we study the following nonlinear periodic problem:

$$(1)\qquad \left\{\begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' = f\left(t, c(t), x'(t)\right) \text{ a.e. on } T=[0,b] \\ x(0)=x(b),\ x'(0)=x'(b), \qquad 2 \le p. \end{array}\right\}$$

Assuming the existence of an upper solution φ and a lower solution ψ, we prove the existence of a greatest solution and of a least solution in the order interval $[\psi,\varphi]$ (extremal periodic solutions). The problem has been studied only in its semilinear form (i.e. $p=2$), using different techniques and under more restrictive hypotheses on the data. More specifically NIETO [8] has a right hand side term which is independent of the derivative. OMARI–TROMBETTA [9] assume that $f(t,x,y)=f_1(t,x)-cy$, $c>0$. More recently WANG–CABADA–NIETO [11] and GAO–WANG [6], assume a general nonlinear right hand side, but establish the existence of extremal solutions under a restrictive one-sided Lipschitz condition. Finally there is also the very recent work of PAPAGEORGIOU–PAPALINI [10], where the hypotheses on $f(t,x,y)$ are different; they assume a monotonicity type condition in x which allows discontinuities in that variable, but they require Lipschitz continuity in the y variable. In all these works $p=2$ and the methods employed are different from the one used in this paper. Here our approach uses the theory of operators of monotone type as this was developed by BROWDER–HESS [2], combined with truncation and penalization techniques. Our proof of the existence of extremal solutions uses a special test function technique which is different from the approach of PAPAGEORGIOU–PAPALINI [10] (for $p=2$) and permits the relaxation of some restrictive hypotheses on $f(t,x,y)$. Also our notions of upper and lower solutions are weaker.

2. Preliminaries

In this section we briefly recall some basic notions and facts from the theory of operators of monotone type. Our basic reference is the paper of BROWDER–HESS [2].

So let X be a reflexive Banach space. An operator $A : D \subseteq X \to X^*$ is said to be "monotone", if $\langle A(x) - A(y), x - y\rangle \geq 0$ for all $x, y \in D$. Furthermore, A is said to be "maximal monotone", if A is monotone and it follows from $(y, y^*) \in X \times X^*$ and $\langle A(x) - y^*, x - y\rangle \geq 0$ for all $x \in D$, that $y \in D$ and $y^* = A(y)$. Note that here by $\langle \cdot , \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). The operator A is said to be "demicontinuous", if $D = X$ and $x_n \to x$ in X implies $A(x_n) \xrightarrow{w} A(x)$ in X^* as $n \to \infty$. A monotone demicontinuous map A, is maximal monotone.

A map $A : X \to X^*$ is said to be "pseudomonotone", if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\, \langle A(x_n), x_n - x\rangle \leq 0$ implies $\langle A(x), x - y\rangle \leq \underline{\lim}\, \langle A(x_n), x - y\rangle$ for all $y \in X$. If $A(\cdot)$ is bounded (i.e. maps bounded sets of X into bounded sets of X^*), then the above definition of pseudomonotonicity of $A(\cdot)$ is equivalent to saying that if $x_n \xrightarrow{w} x$ in X, $A(x_n) \xrightarrow{w} u^*$ in X^* as $n \to \infty$ and $\overline{\lim}\, \langle A(x_n), x_n - x\rangle \leq 0$, then $A(x) = u^*$ and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ as $n \to \infty$ ("generalized pseudomonotonicity"; see BROWDER–HESS [2]). Any maximal monotone map A, with $D = X$, is a pseudomonotone one. So in particular a monotone demicontinuous map, is pseudomonotone. The property of pseudomonotonicity is preserved by addition.

A map $A : D \subseteq X \to X^*$ is said to be "weakly coercive", if either D is bounded or D is unbounded and $\|A(x)\|_* \to \infty$ as $\|x\| \to \infty$. Here by $\|\cdot\|$ (resp. $\|\cdot\|_*$) we denote the norm of X (resp. of X^*). Any pseudomonotone, bounded, weakly coercive map is surjective (see BROWDER–HESS [2], theorem 3, p. 269). Also a maximal monotone, weakly coercive map $A : D \subseteq X \to X^*$ is surjective too. It should be mentioned that in BROWDER–HESS [2] all these notions are defined for set-valued maps. However in this work we will not need this generality and so everything is defined in the context of single-valued maps.

Next let us define what we mean by solution, upper solution and lower solution for problem (1). In what follows

$$W^{1,p}_{per}(T) = \left\{ y \in W^{1,p}(T) : \; y(0) = y(b) \right\}$$

(recall $W^{1,p}_{per}(T) \subseteq C(T)$).

DEFINITION. By a solution of problem (1), we mean a function $x(\cdot) \in$ $\in C^1(T)$ such that

$$|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T) \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

and it satisfies (1).

DEFINITION. By an "upper solution" of problem (1), we mean a function $\varphi(\cdot) \in W^{1,p}_{per}(T)$ such that

$$\left.\begin{array}{l} \int_0^b |\varphi'(t)|^{p-2}\varphi'(t)y'(t)dt \geq \int_0^b f\left(t, \varphi(t), \varphi'(t)\right) y(t)dt \\ \text{for all } y \in W^{1,p}_{per}(T) \cap L^p(T)_+. \end{array}\right\}$$

Similarly by a "lower solution" of problem (1), we mean a function $\psi(\cdot) \in$ $\in W^{1,p}_{per}(T)$ such that

$$\left.\begin{array}{l} \int_0^b |\psi'(t)|^{p-2}\psi'(t)y'(t)dt \leq \int_0^b f\left(t, \psi(t), \psi'(t)y(t)\right) dt \\ \text{for all } y \in W^{1,p}_{per}(T) \cap L^p(T)_+. \end{array}\right\}$$

We will assume the existence of an upper and of a lower solution. More specifically we make the following hypothesis:

$\underline{H_0}$: There exist an upper solution φ and a lower solution ψ such that $\psi(t) \leq \varphi(t)$ for all $t \in T$.

Our hypotheses on the right hand side term $f(t,x,y)$ are the following

$\underline{H(f)}$: $f : T \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that

(i) for every $(x,y) \in \mathbf{R} \times \mathbf{R}$, $t \to f(t,x,y)$ is measurable;

(ii) for almost all $t \in T$, $(x,y) \to f(t,x,y)$ is continuous;

(iii) for almost all $t \in T$, all $x \in \left[\psi(t), \varphi(t)\right]$ and all $y \in \mathbf{R}$ we have

$$|f(t,x,y)| \leq a(t) + c|y|^{p-1}$$

where $a \in L^q(T)$ and $c > 0$.

3. Existence of solutions

In this section we prove the existence of a solution in the order interval

$$K = [\psi,\varphi] = \{x \in C(T) : \ \psi(t) \leq x(t) \leq \varphi(t) \text{ for all } t \in T\}.$$

PROPOSITION 1. *If hypotheses H_0 and $H(f)$ hold, then problem* (1) *has at least one solution $x \in K$.*

PROOF. As we already mentioned in the introduction, our proof will combine truncation and penalization techniques, with results from the theory of operators of monotone type.

So we introduce the truncation map $\tau : W^{1,p}(T) \to W^{1,p}(T)$ defined by

$$\tau(x)(t) = \begin{cases} \varphi(t) & \text{if } \ \varphi(t) \leq x(t) \\ x(t) & \text{if } \ \psi(t) \leq x(t) \leq \varphi(t) \\ \psi(t) & \text{if } \ x(t) \leq \psi(t). \end{cases}$$

Note that lemma 7.6, p.145 of GILBARG–TRUDINGER [7], tells us that indeed $\tau(\cdot)$ is $W^{1,p}(T)$-valued. Moreover, it is easy to check that $\tau(\cdot)$ is continuous.

The penalty function $\beta : T \times \mathbf{R} \to \mathbf{R}$ is defined by

$$\beta(t,x) = \begin{cases} (x - \varphi(t))^{p-1} & \text{if } \ \varphi(t) \leq x \\ 0 & \text{if } \ \psi(t) \leq x \leq \varphi(t) \\ -(\psi(t) - x)^{p-1} & \text{if } \ x \leq \psi(t). \end{cases}$$

It is clear that $\beta(\cdot,\cdot)$ is a Caratheodory function (i.e. is measurable in $t \in T$ and continuous in $x \in \mathbf{R}$) and $|\beta(t,x)| \leq a_1(t) + c_1|x|^{p-1}$ a.e. on T with $a_1 \in L^q(T)$, $c_1 > 0$. Moreover, an easy calculation can verify that we have $\int_0^b \beta(t,x(t))x(t)dt \geq \|x\|_p^p - c_2\|x\|_p^{p-1}$ for some $c_2 > 0$ and for all $x \in L^p(T)$.

Using $\tau(\cdot)$ and $\beta(\cdot,\cdot)$ we introduce the following auxiliary periodic problem:

$$\text{(2)} \quad \left\{ \begin{array}{l} -\left(|x'(t)|^{p-2}x'(t)\right)' = f(t,\tau(x)(t),\tau(x)'(t)) - \lambda\beta(t,x(t)) \text{ a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b), \quad \lambda > 0. \end{array} \right\}$$

Let $A : W^{1,p}_{per}(T) \to W^{1,p}_{per}(T)^*$ be the operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2} x'(t) y'(t) dt.$$

Next we establish some useful properties of $A(\,\cdot\,)$.

CLAIM #1. $A(\cdot)$ *is monotone and demicontinuous (hence maximal monotone; see section 2.).*

First we show that $A(\cdot)$ is monotone. So let $x, y \in W^{1,p}_{per}(T)$. We have

$$\langle A(x) - A(y), x - y\rangle =$$

$$= \int_0^b \Big[|x'(t)|^{p-2}x'(t)(x'(t) - y'(t)) - |y'(t)|^{p-2}y'(t)(x'(t) - y'(t))\Big]\, dt \geq$$

$$\geq \int_0^b \Big(|x'(t)|^p - |x'(t)|^{p-1}|y'(t)| - |y'(t)|^{p-1}|x'(t)| + |y'(t)|^p\Big)\, dt \geq$$

$$\geq \|x'\|_p^p - \|x'\|_p^{p-1}\|y'\|_p - \|y'\|_p^{p-1}\|x'\|_p + \|y'\|_p^p =$$

$$= \left(\|x'\|_p^{p-1} - \|y'\|_p^{p-1}\right)\left(\|x'\|_p - \|y'\|_p\right) \geq 0$$

$\Rightarrow$ $A(\cdot)$ is monotone.

Now we will show the demicontinuity of $A(\cdot)$. To this end let $x_n \to x$ in $W^{1,p}_{per}(T)$ as $n \to \infty$. Then for every $y \in W^{1,p}_{per}(T)$, we have

$$|\langle A(x_n) - A(x), y\rangle| = \left|\int_0^b \Big(|x_n'(t)|^{p-2}x_n'(t) - |x'(t)|^{p-2}x'(t)\Big)\, y'(t)dt\right|.$$

Since $x_n \to x$ in $W^{1,p}_{per}(T)$, we have $x_n' \to x'$ in $L^p(T)$ as $n \to \infty$ and so by passing to a subsequence if necessary, we may also assume that $x_n'(t) \to$ $\to x'(t)$ a.e. on T as $n \to \infty$. Using the generalized Lebesgue's convergence theorem (see for example ASH [1], theorem 7.5.2, p. 295), we have that $\left|\int_0^b |x_n'(t)|^{p-2}x_n'(t)y'(t)dt\right| \to \left|\int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt\right|$ as $n \to \infty$. So

$|\langle A(x_n)-A(x),y\rangle| \to 0$ as $n\to\infty$. Since $y\in W^{1,p}_{per}(T)$ was arbitrary, we infer that $A(x_n)\xrightarrow{w} A(x)$ in $W^{1,p}_{per}(T)$ as $n\to\infty$ and so we have proved that $A(\cdot)$ is demicontinuous. Finally recall from section 2, that a monotone, everywhere defined, demicontinuous operator, is maximal monotone.

Let $B: L^p(T)\to L^q(T)$ be the Nemitsky operator corresponding to the penalty function $\beta(t,x)$; i.e. $Bx(\cdot)=\beta(\cdot,x(\cdot))$. It is well-known that $B(\cdot)$ is continuous (Krasnoselskii's theorem; see for example ZEIDLER [12], proposition 26.7, p. 563). Also as a direct consequence of the definition of β we have that $B(\cdot)$ is monotone.

Finally let $F: W^{1,p}(T)\to L^q(T)$ be defined by

$$F(x)(\cdot)=f(\cdot,\tau(x)(\cdot),\tau(x)'(\cdot))$$

Using hypotheses $H(f)$ and the continuity of the truncation map $\tau(\cdot)$, we see that $F(\cdot)$ is continuous.

Now let $R=A+\lambda B-F:\ W^{1,p}_{per}(T)\to W^{1,p}_{per}(T)^*$.

CLAIM #2. *$R(\cdot)$ is pseudomonotone and weakly coercive.*

Since $R(\cdot)$ is bounded, to prove the pseudomonotonicity part of the claim, it suffices to show that $R(\cdot)$ is generalized pseudomonotone (see section 2). To this end let $x_n\xrightarrow{w}x$ in $W^{1,p}_{per}(T)$ and suppose that $\overline{\lim}\langle R(x_n),x_n-x\rangle\le$ ≤ 0. We need to show that $R(x_n)\xrightarrow{w}R(x_n)$ in $W^{1,p}_{per}(T)^*$ and $\langle R(x_n),x_n\rangle\to$ $\to\langle R(x),x\rangle$ as $n\to\infty$. We have

$$\langle R(x_n),x_n-x\rangle=\langle A(x_n)+\lambda B(x_n)-F(x_n),x_n-x\rangle=$$
$$=\langle A(x_n),x_n-x\rangle+\lambda(B(x_n),x_n-x)_{pq}-(F(x_n),x_n-x)_{pq}$$

where by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(T),L^q(T))$. Recall that $W^{1,p}(T)$ embeds compactly in $L^p(T)$ and so $x_n\to x$ in $L^p(T)$ as $n\to\infty$. Therefore $(B(x_n),x_n-x)_{pq}\to 0$ and $(F(x_n),x_n-x)_{pq}\to 0$ as $n\to\infty$. Thus we have

$$\overline{\lim}\,\langle A(x_n),x_n-x\rangle\le 0.$$

But $A(\cdot)$ being maximal monotone (see claim #1), is generalized pseudomonotone (see proposition 2, p. 257 of BROWDER–HESS [2]). Hence we have that

$$A(x_n)\xrightarrow{w}A(x)\text{ in } W^{1,p}_{per}(T)^*\text{ and }\langle A(x_n),x_n\rangle\to\langle A(x),x\rangle\text{ as } n\to\infty.$$

Also note that $\langle A(x_n), x_n\rangle = \|x_n'\|_p^p$ and $\langle A(x), x\rangle = \|x'\|_p^p$ as $n \to \infty$. So $\|x_n'\|_p \to \|x'\|_p$ as $n \to \infty$. We also know that $x_n' \xrightarrow{w} x'$ in $L^p(T)$ as $n \to \infty$. The space $L^p(T)$ being uniformly convex, has the Kadec–Klee property and so $x_n' \to x'$ in $L^p(T)$, hence $x_n \to x$ in $W^{1,p}(T)$ as $n \to \infty$. Thus we have $A(x_n) \xrightarrow{w} A(x)$, $B(x_n) \to B(x)$ and $F(x_n) \to F(x)$ in $W_{per}^{1,p}(T)^*$ as $n \to \infty$ (since $L^q(T)$ is embedded continuously in $W_{per}^{1,p}(T)^*$). So finally $R(x_n) \xrightarrow{w} R(x)$ in $W_{per}^{1,p}(T)$, $\langle R(x_n), x_n\rangle \to \langle R(x), x\rangle$ as $n \to \infty$. This proves that $R(\cdot)$ is pseudomonotone.

Now we will show that $R(\cdot)$ is weakly coercive. We have:

$$\langle R(x), x\rangle \geq \langle A(x), x\rangle + \lambda (B(x), x)_{pq} - \|F(x)\|_q \|x\|_p \geq$$

$$\geq \|x'\|_p^p + \lambda \|x\|_p^p - \lambda c_2 \|x\|_p^{p-1} - \frac{\varepsilon}{q}\|F(x)\|_q^q - \frac{1}{\varepsilon p}\|x\|_p^p \geq \tag{3}$$

$$\geq \|x'\|_p^p + \lambda \|x\|_p^p - \lambda c_2 \|x\|_p^{p-1} - \frac{2^{q-1}\varepsilon}{q}\|a\|_q^q - \frac{2^{q-1}\varepsilon}{q} c_3 \|x'\|_p^p - \frac{1}{\varepsilon p}\|x\|_p^p - \delta$$

for some $c_3, \delta > 0$. Indeed note that

$$\tau(x)'(t) = \begin{cases} \varphi'(t) & \text{if } \varphi(t) \leq x(t) \\ x'(t) & \text{if } \psi(t) \leq x(t) \leq \varphi(t) \\ \psi'(t) & \text{if } x(t) \leq \psi(t) \end{cases}$$

and so $\|\tau(x)'\|_p^p \leq \delta + c_3 \|x'\|_p^p$ for some $c_3, \delta > 0$. Then (3) follows if we use the properties of $B(\cdot)$ and hypothesis $H(f)$(iii). Next we choose $\varepsilon > 0$ such that $\frac{2^{q-1}\varepsilon}{q} c_3 < 1$. Then having fixed $\varepsilon > 0$ this way, we take $\lambda > 0$ large enough so that $\lambda > \frac{1}{\varepsilon p}$. Hence with such choices of λ and ε, from (3) it follows that $R(\cdot)$ is weakly coercive. This proves the claim.

Recall from section 2, that a pseudomonotone, weakly coercive operator is surjective. Thus we can find $x \in W_{per}^{1,p}(T)$ such that $R(x) = 0$.

CLAIM #3. $x \in W_{per}^{1,p}(T)$ *is a solution of the auxiliary problem* (2).

Since $R(x) = 0$, we have

$$\langle A(x), y\rangle = (F(x), y)_{pq} - \lambda (B(x), y)_{pq} \quad \text{for all} \quad y \in W_{per}^{1,p}(T) \Rightarrow$$

$$\int_0^b |x'(t)|^{p-2} x'(t) y'(t)dt = \int_0^b f\left(t, \tau(x)(t), \tau(x)'(t)\right) dt - \lambda \int_0^b \beta(t, x(t)) y(t)dt.$$

Because this last equality is in particular true for every $y \in C_0^\infty(T)$ and $f(\cdot, \tau(x)(\cdot), \tau(x)'(\cdot)) - \lambda\beta(\cdot, x(\cdot)) \in L^q(T)$, it follows that

$$-\left(|x'(t)|^{p-2}x'(t)\right)' = f(t, \tau(x)(t), \tau(x)'(t)) - \lambda\beta(t, x(t))$$

a.e. on T and $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)$. Note that $\eta : \mathbf{R} \to \mathbf{R}$ defined by $\eta(r) = |r|^{p-2}r$ is a strictly increasing continuous function. Hence $\eta^{-1} : \mathbf{R} \to \to \mathbf{R}$ is well-defined and continuous. Because $W^{1,q}(T)$ embeds continuously in $C(T)$, we have that $|x'(\cdot)|^{p-2}x'(\cdot) \in C(T)$. Therefore $t \to x'(t) = = \eta\left(|x'(t)|^{p-2}x'(t)\right)$ belongs in $C(T)$, and so we have that $x \in C^1(T)$. Moreover, from Green's formula, we have

$$-\int_0^b \left(|x'(t)|^{p-2}x'(t)\right)' y(t)dt =$$

$$= -|x'(b)|^{p-2}x'(b)y(b) + |x'(0)|^{p-2}x'(0)y(0) + \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt =$$

$$= -|x'(b)|^{p-2}x'(b)y(b) + |x'(0)|^{p-2}x'(0)y(0) + \int_0^b f(t, \tau(x)(t), \tau(x)'(t))y(t)dt -$$

$$-\lambda \int_0^b \beta(t, x(t))y(t)dt.$$

Since $-\left(|x'(t)|^{p-2}x'(t)\right)' = (f(t, \tau(x)(t), \tau(x)'(t)) - \lambda\beta(t, x(t))$ a.e. we obtain

$$|x'(0)|^{p-2}x'(0)y(0) = |x'(b)|^{p-2}x'(b)y(b).$$

Let $y(\cdot) \equiv 1 \in W^{1,p}_{per}(T)$. Then we have

$$|x'(0)|^{p-2}x'(0) = |x'(b)|^{p-2}x'(b).$$

Using $\eta^{-1}(\cdot)$, we obtain that $x'(0) = x'(b)$. Therefore we have that $x(\cdot)$ is a solution of (2) as claimed.

To finish the proof, it remains to show that $x \in K$.

CLAIM #4. *$x \in K$; i.e. for all $t \in T$, $\psi(t) \le x(t) \le \varphi(t)$.*

Since $\psi \in W^{1,p}_{per}(T)$ is a lower solution of (1), by definition we have

$$\int_0^b |\psi'(t)|^{p-2}\psi'(t)y'(t)dt \le \int_0^b f(t,\psi(t),\psi'(t))y(t)dt$$

for all $y \in W^{1,p}_{per}(T) \cap L^p(T)_+$.

On the other hand since $x(\cdot)$ is a solution of (2), using Green's formula (integration by parts), we have

$$\int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt =$$

$$= \int_0^b f\left(t,\tau(x)(t),\tau(x)'(t)\right) y(t)dt - \lambda \int_0^b \beta(t,x(t))y(t)dt$$

for all $y \in W^{1,p}_{per}(T)$. If we take $y = (\psi - x)_+ \in W^{1,p}_{per}(T) \cap L^p(T)_+$, we obtain

$$\int_0^b \left(|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t)\right)(\psi - x)'_+(t)dt \ge \tag{4}$$

$$\ge \int_0^b \left(f(t,\tau(x)(t),\tau(x)'(t)) - f\left(t,\psi(t),\psi'(t)\right)\right)(\psi - x)_+(t)dt -$$

$$-\lambda \int_0^b \beta(t,x(t))(\psi - x)_+(t)dt.$$

From Gilbarg–Trudinger [7] we know that

$$(\psi - x)'_+(t) = \begin{cases} (\psi - x)'(t) & \text{if } x(t) < \psi(t) \\ 0 & \text{if } \psi(t) \le x(t). \end{cases}$$

So we obtain

$$\int_0^b \left(|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t)\right)(\psi - x)'_+(t)dt = \tag{5}$$

$$= \int\limits_{\{\psi > x\}} \left(|x'(t)|^{p-2} x'(t) - |\psi'(t)|^{p-2} \psi'(t) \right) (\psi' - x')(t) dt \leq 0$$

(recall the fundamental inequality which says that for all $a, b \in \mathbf{R}$ and $p \geq 2$, we have

$$\left(|a|^{p-2} a - |b|^{p-2} b \right) (a - b) \geq 2^{2-p} |a - b|^p).$$

Also note that from the definition of $\tau(\cdot)$, we have

$$-\int_0^b \left(f\left(t, \tau(x)(t), \tau(x)'(t)\right) - f\left(t, \psi(t), \psi'(t)\right) \right) (\psi - x)_+(t) dt = \tag{6}$$

$$= \int\limits_{\{\psi > x\}} \left(f\left(t, \psi(t), \psi'(t)\right) - f\left(t, \psi(t), \psi'(t)\right) \right) (\psi - x)(t) dt = 0.$$

Using (5) and (6) in (4), we obtain

$$\int_0^b \beta(t, x(t)) (\psi - x)_+(t) dt \geq 0$$

$$\Rightarrow - \int\limits_{\{\psi > x\}} (\psi(t) - x(t))^{p-1} (\psi - x)(t) dt =$$

$$= -\int_0^b (\psi - x)_+{}^{p-1}(t) (\psi - x)_+(t) dt \geq 0$$

$$\Rightarrow -\int_0^b (\psi - x)_+{}^p(t) dt \geq 0$$

$$\Rightarrow \int_0^b (\psi - x)_+{}^p(t) dt = 0; \quad \text{i.e. } \psi(t) \leq x(t) \text{ for all } t \in T.$$

Similarly we show that $x(t) \leq \varphi(t)$ for all $t \in T$. Thus $x \in K$ and so $\tau(x) = x$, $\tau(x)' = x'$, $\beta(t, x(t)) = 0$. Since $x(0) = x(b)$, $x'(0) = x'(b)$, we conclude that $x \in K$ is a solution of (1). ∎

4. Existence of extremal solutions

Let S be the set of solutions of (1) in K. In the next proposition we show that S is directed ; i.e. if $x_1, x_2 \in S$, then we can find $x \in S$ such that $(x_1 \vee x_2)(t) = \max\{x_1(t), x_2(t)\} \le x(t)$ for all $t \in T$.

PROPOSITION. *If hypotheses H_0 and $H(f)$ hold, then S is directed.*

PROOF. Let x_1, $x_2 \in S$ and $u = x_1 \vee x_2 \in W^{1,p}_{per}(T)$. We will show that u is a lower solution of (1). Given $\varepsilon > 0$, we introduce the function $\gamma_\varepsilon : \mathbf{R} \to \mathbf{R}$ defined by

$$\gamma_\varepsilon(r) = \begin{cases} 0 & \text{if } \ r \le 0 \\ \frac{r}{\varepsilon} & \text{if } \ 0 \le r \le \varepsilon \\ 1 & \text{if } \ \varepsilon \le r. \end{cases}$$

Evidently $\gamma_\varepsilon(\cdot)$ is Lipschitz continuous and differentiable everywhere except at $r = 0$ and $r = \varepsilon$. Moreover, its derivative is given by

$$\gamma'_\varepsilon(r) = \begin{cases} 0 & \text{if } \ r < 0 \\ \frac{1}{\varepsilon} & \text{if } \ 0 \le r < \varepsilon \\ 0 & \text{if } \ \varepsilon < r. \end{cases}$$

Note that $\gamma_\varepsilon \to \chi_{\{r>0\}}$ pointwise as $\varepsilon \downarrow 0$. Let $k \in C^\infty_{per}(T)$, $k \ge 0$ and define

$$\begin{aligned} \vartheta_1 = (1 - \gamma_\varepsilon(x_2 - x_1))\,k \quad &\text{and} \quad \vartheta_2 = \gamma_\varepsilon(x_2 - x_1)k \\ \Rightarrow \vartheta'_1 = k' - \gamma'_\varepsilon(x_2 - x_1)&(x_2 - x_1)'k - \gamma_\varepsilon(x_2 - x_1)k' \\ \text{and} \quad \vartheta'_2 = \gamma'_\varepsilon(x_2 - x_1)&(x_2 - x_1)'k + \gamma_\varepsilon(x_2 - x_1)k'. \end{aligned}$$

Since $x_1, x_2 \in S$, $\vartheta_1, \vartheta_2 \in C_{per}(T)$ and using Green's formula, we have

$$\int_0^b |x'_1(t)|^{p-2} x'_1(t)\vartheta'_1(t)dt = \int_0^b f\left(t, x_1(t), x'_1(t)\right)\vartheta_1(t)dt \tag{7}$$

and

$$\int_0^b |x'_2(t)|^{p-2} x'_2(t)\vartheta'_2(t)dt = \int_0^b f\left(t, x_2(t), x'_2(t)\right)\vartheta_2(t)dt. \tag{8}$$

Adding (7) and (8) and using the expressions for ϑ_1' and ϑ_2', we obtain

$$\begin{aligned}
&\int_0^b |x_1'(t)|^{p-2}x_1'(t)k'(t)dt-\\
&-\int_0^b |x_1'(t)|^{p-2}\gamma_\varepsilon'\left((x_1-x_2)(t)\right)k(t)x_1'(t)(x_1-x_2)'(t)dt-\\
&-\int_0^b |x_1'(t)|^{p-2}\gamma_\varepsilon\left((x_2-x_1)(t)\right)x_1'(t)k'(t)dt+\\
&+\int_0^b |x_2'(t)|^{p-2}\gamma_\varepsilon\left((x_2-x_1)(t)\right)x_2'(t)k'(t)dt+\\
&+\int_0^b |x_2'(t)|^{p-2}\gamma_\varepsilon'\left((x_2-x_1)(t)\right)x_2'(t)(x_1-x_2)'(t)dt=\\
&=\int_0^b f\left(t,x_1(t),x_1'(t)\right)\vartheta_1(t)dt+\int_0^b f\left(t,x_2(t),x_2'(t)\right)\vartheta_2(t)dt. \qquad (9)
\end{aligned}$$

Denote by ξ_ε the left hand side of (9). Since $\gamma_\varepsilon'\left((x_2-x_1)(t)\right)k(t)\geq 0$ for almost all $t\in T$ and using the inequality $\left(|a|^{p-2}a-|b|^{p-2}b\right)(a-b)\geq$ $\geq 2^{2-p}|a-b|^p$ for all $a,b\in R$ and all $p\geq 2$, we have

$$\begin{aligned}
\xi_\varepsilon \geq &\int_0^b |x_1'(t)|^{p-2}x_2'(t)k'(t)dt-\\
&-\int_0^b \gamma_\varepsilon\left((x_2-x_1)(t)\right)\left(|x_1'(t)|^{p-2}x_1'(t)k'(t)-|x_2'(t)|^{p-2}x_2'(t)k'(t)\right)dt=\rho_\varepsilon\\
&\Rightarrow \int_0^b f\left(t,x_1(t),x_1'(t)\right)\vartheta_1(t)dt+\int_0^b f\left(t,x_2(t),x_2'(t)\right)\vartheta_2(t)dt\geq\rho_\varepsilon.
\end{aligned}$$

Since $\gamma_\varepsilon\left((x_2-x_1)(\cdot)\right)\to\chi_{\{x_2>x_1\}}(\cdot)$ pointwise as $\varepsilon\downarrow 0$, we have that

$$\rho_\varepsilon \to \int\limits_{\{x_1\geq x_2\}} |x_1'(t)|^{p-2}x_1'(t)k'(t)dt + \int\limits_{\{x_2>x_1\}} |x_2'(t)|^{p-2}x_2'(t)k'(t)dt \quad \text{as } \varepsilon\downarrow 0.$$

Also we have

$$\int\limits_0^b f\left(t,x_1(t),x_1'(t)\right)\vartheta_1(t)dt + \int\limits_0^b f\left(t,x_2(t),x_2'(t)\right)\vartheta_2(t)dt \to$$

$$\to \int\limits_0^b f\left(t,u(t),u'(t)\right)k(t)dt \quad \text{as } \varepsilon\downarrow 0.$$

Thus in the limit as $\varepsilon\downarrow 0$, we obtain

$$\int\limits_0^b f(t,u(t),u'(t))k(t)dt \geq \int\limits_{\{x_1\geq x_2\}} |x_1'(t)|^{p-2}x_1'(t)k'(t)dt +$$

$$+\int\limits_{\{x_2>x_1\}} |x_2'(t)|^{p-2}x_2'(t)k'(t)dt = \int\limits_0^b |u'(t)|^{p-2}u'(t)k'(t)dt$$

for all $k\in C^\infty_{per}(T)$, $k\geq 0$. Since such k's are dense in $W^{1,p}_{per}(T)\cap L^p(T)_+$ we deduce that $W^{1,p}_{per}(T)$ is a lower solution of (1). Note that $u\leq\varphi$. From proposition 1 (applied in this case on the pair (u,φ)) we can have a solution x of (1) in $K_1=[u,\varphi]\subseteq K$. Evidently $u\leq x$ and so S is directed. ∎

Now we are ready to state and prove our existence theorem for the extremal solutions of problem (1). So we are going to show that in K, problem (1) has a greatest solution x^* and a least solution x_*; i.e. if $x\in S$, then $x_*\leq x\leq x^*$. The solutions x_*, x^* are called “extremal solutions”.

THEOREM 3. *If hypotheses H_0 and $H(f)$ hold, then problem* (1) *admits extremal solutions in K.*

PROOF. Let C be a chain of S. Let $x=\sup C$. By virtue of corollary 7, p. 336 of DUNFORD–SCHWARTZ [5], we can find an increasing sequence

$\{x_n\}_{n\geq 1} \subseteq C$ such that $x_n \to x$ in $L^p(T)$. Also for every $n \geq 1$ we have via Green's formula

$$\int_0^b |x_n'(t)|^p dt = \int_0^b f\left(t, x_n(t), x_n'(t)\right) x_n(t) dt$$

$$\Rightarrow \; \|x_n'\|_p^p \leq \left(\|a\|_q + c\|x_n'\|_p^{p-1}\right) \|x_n\|_p \leq \left(\|a\|_q + c\|x_n'\|_p^{p-1}\right) M$$

for some $M > 0$. From the above inequality it follows at once that $\{x_n'\}_{n\geq 1} \subseteq$ $\subseteq L^p(T)$ is bounded, hence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$ is bounded. Thus we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T)$ as $n \to \infty$. But $A(x_n) = F(x_n)$, $n \geq 1$ and so

$$\langle A(x_n), x_n - x\rangle = \langle F(x_n), x_n - x\rangle =$$
$$= (F(x_n), x_n - x))_{pq} \leq \|F(x_n)\|_q \|x_n - x\|_p \leq M_1 \|x_n - x\|_p \to 0$$

as $n \to \infty$. So $\overline{\lim} \langle A(x_n), x_n - x\rangle \leq 0$ and as in the proof of proposition 1, from this we can have $x_n \to x$ in $W^{1,p}(T)$ as $n \to \infty$. So in the limit as $n \to \infty$ we have

$$\langle A(x), y\rangle = \langle F(x), y\rangle = (F(x), y)_{pq} \quad \text{for all } y \in W^{1,p}_{per}(T)$$

$$\Rightarrow \int_0^b |x'(t)|^{p-2} x'(t) y'(t) dt = \int_0^b f\left(t, x(t), x'(t)\right) y(t) dt \;\text{ for all } y \in W^{1,p}_{per}(T).$$

Arguing as in the proof of proposition 1, from the above equality, we have

$$\left\{ \begin{array}{l} -\left(|x'(t)|^{p-2} x'(t)\right)' = f\left(t, x(t), x'(t)\right) \quad \text{a.e. on } T \\ x(0) = x(b),\; x'(0) = x'(b), \quad \psi(t) \leq x(t) \leq \varphi(t),\; t \in T. \end{array} \right\}$$

Thus we have proved that every chain in S, has an upper bound in S. Apply Zorn's lemma, to produce a maximal element $x^* \in S$. Since S is directed (proposition 2), x^* is unique and is the greatest element of S in K. Similarly we produce the least element x_* of S in K. ∎

References

[1] R. Ash, *Real Analysis and Probability,* Academic Press, New York, 1971.

[2] F. Browder, P. Hess, Nonlinear mappings of monotone type in Banach spaces, *J. Funct. Anal.,* 11 (1972), 251–294.

[3] C. De Coster, Pairs of positive solutions for one-dimensional p-laplacian, *Nonlinear Anal. T.M.A.*, **23** (1994), 669–681.

[4] C. De Coster, P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results, in *Nonlinear Analysis and BVP's, for ODE's;* ed. by F. Zanolin, Springer, Wien, 1996, 1–78.

[5] N. Dunford, J. Schwartz, *Linear Operators I,* Wiley, New York, 1958.

[6] W. Gao, J. Wang, On a nonlinear second order periodic boundary value problem with Caratheodory functions, *Annales Polonici Math.,* **62** (1995), 283–291.

[7] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, New York, 1977.

[8] J. Nieto, Nonlinear second order periodic boundary value problems with Caratheodory functions, *Appl. Anal.,* **34** (1989), 111–128.

[9] P. Omari, M. Trombetta, Remarks on the lower and upper solutions method for second and third order periodic boundary value problems, *Appl. Math. Comp.,* **50** (1992), 1–21.

[10] N. S. Papageorgiou, F. Papalini, Periodic and boundary value problems for second order differential equations, *Czech. Math. Jour.* – to appear.

[11] M-X. Wang, A. Cabada, J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Caratheodory functions, *Annales Polonici Math.,* **58** (1993), 221–235.

[12] E. Zeidler, *Nonlinear Functional Analysis and Applications II,* Springer-Verlag, New York, 1990.

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THE GENERALIZED CONNECTION IN Osc^3M

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1. Adapted basis in $T(Osc^3M)$ and $T^*(Osc^3M)$

Let $E = Osc^3M$ be a $4n$ dimensional C^∞ manifold. In some local chart (U,φ) some point $u \in E$ has coordinates

$$(x^a, y^{1a}, y^{2a}, y^{3a}) = (y^{0a}, y^{1a}, y^{2a}, y^{3a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, 3.$$

If in some other chart (U',φ') the point $u \in E$ has coordinates $(x^{a'}, y^{1a'}, y^{2a'}, y^{3a'})$, then in $U \cap U'$ the allowable coordinate transformation are given by:

$$(1)\quad \begin{aligned}
&\text{(a)}\quad x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n)\\
&\text{(b)}\quad y^{1a'} = \frac{\partial x^{a'}}{\partial x^a} y^{1a} = \frac{\partial y^{0a'}}{\partial y^{0a}} y^{1a}\\
&\text{(c)}\quad 2y^{2a'} = \frac{\partial y^{1a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{1a'}}{\partial y^{1a}} y^{2a}\\
&\text{(d)}\quad 3y^{3a'} = \frac{\partial y^{2a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{2a'}}{\partial y^{1a}} y^{2a} + 3\frac{\partial y^{2a'}}{\partial y^{2a}} y^{3a}.
\end{aligned}$$

A nice example of the space E can be obtained if the points $(x^a) \in M$ $(dim M = n)$ are considered as the points of the curve $x^a = x^a(t)$ and $y^{\alpha a}$,

$\alpha = 1,2,3$, are defined by

$$y^{1a} = \frac{dx^a}{dt},\ y^{2a} = \frac{1}{2!}\frac{d^2x^a}{dt^2} = \frac{1}{2}\frac{dy^{1a}}{dt},\ y^{3a} = \frac{1}{3!}\frac{d^3x^a}{dt^3} = \frac{1}{3}\frac{dy^{2a}}{dt}.$$

If in $U \cap U'$ the equation

$$x^{a'} = x^{a'}(x^1(t), x^2(t), \dots, (x^n(t))$$

is valid, then it is easy to see that

$$\begin{aligned}
y^{1a'} &= \frac{dx^{a'}}{dt} = y^{1a'}(x^a, y^{1a}),\\
2y^{2a'} &= \frac{dy^{1a'}}{dt} = 2y^{2a'}(x^a, y^{1a}, y^{2a}),\\
3y^{3a'} &= \frac{dy^{2a'}}{dt} = 3y^{3a'}(x^a, y^{1a}, y^{2a}, y^{3a}),
\end{aligned} \tag{2}$$

satisfy (1.1b), (1.1c) and (1.1d) respectively and the explicite form of (1.1) is the following:

$$\begin{aligned}
x^{a'} &= x^{a'}(x^1, x^2, \dots, x^n)\\
y^{1a'} &= \frac{\partial x^{a'}}{\partial x^a} y^{1a},\\
2y^{2a'} &= \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{1b} + 2\frac{\partial x^{a'}}{\partial x^a} y^{2a},\\
6y^{3a'} &= \frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c} y^{1a} y^{1b} y^{1c} + 6\frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{2b} + 6\frac{\partial x^{a'}}{\partial x^a} y^{3a}.
\end{aligned}$$

THEOREM 1.1. *The transformations determined by* (1.1) *form a group.*

Determination of the group of allowable coordinate transformations is the first step to construct some geometry. The second important step is the construction of the adapted basis in $T(E)$, which depends on the choice of the coefficients of the nonlinear connections, here denoted by N and M.

The following abbreviations

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}},\ \alpha = 1,2,3, \text{ and } \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}} \tag{3}$$

will be used. From (1.1) it follows

$$
\begin{aligned}
&\partial_{0a}y^{0a'} = \partial_{1a}y^{1a'} = \partial_{2a}y^{2a'} = \partial_{3a}y^{3a'} = \frac{\partial x^{a'}}{\partial x^a},\\
&\partial_{0a}y^{1a'} = \partial_{1a}y^{2a'} = \partial_{2a}y^{3a'} = \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b}y^{1b},\\
&\partial_{0a}y^{2a'} = \partial_{1a}y^{3a'} = \frac{1}{2}\frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c}y^{1b}y^{1c} + \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b}y^{2b}.
\end{aligned}
\tag{4}
$$

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, \partial_{3a}\} = \{\partial_{\alpha a}\}. \tag{5}$$

The elements of $\bar{B}$ with respect to (1.1) are not transformed as d-tensors. They satisfy the following relations:

$$
\begin{aligned}
\partial_{0a} &= (\partial_{0a}y^{0a'})\partial_{0a'} + (\partial_{0a}y^{1a'})\partial_{1a'} + (\partial_{0a}y^{2a'})\partial_{2a'} + (\partial_{0a}y^{3a'})\partial_{3a'}\\
\partial_{1a} &= (\partial_{1a}y^{1a'})\partial_{1a'} + (\partial_{1a}y^{2a'})\partial_{2a'} + (\partial_{1a}y^{3a'})\partial_{3a'}\\
\partial_{2a} &= (\partial_{2a}y^{2a'})\partial_{2a'} + (\partial_{2a}y^{3a'})\partial_{3a'}\\
\partial_{3a} &= (\partial_{3a}y^{3a'})\partial_{3a'}.
\end{aligned}
\tag{6}
$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dx^a, dy^{1a}, dy^{2a}, dy^{3a}\} = \{dy^{\alpha a}\}. \tag{7}$$

The elements of $\bar{B}^*$ with respect to (1.1) are transformed in the following way (see (1.2)):

(8)

$$
\begin{aligned}
&dx^{a'} = \frac{\partial x^{a'}}{\partial x^a}dx^a \Leftrightarrow dy^{0a'} = (\partial_{0a}y^{0a'})dy^{0a}\\
&dy^{1a'} = (\partial_{0a}y^{1a'})dy^{0a} + (\partial_{1a}y^{1a'})dy^{1a}\\
&dy^{2a'} = (\partial_{0a}y^{2a'})dy^{0a} + (\partial_{1a}y^{2a'})dy^{1a} + (\partial_{2a}y^{2a'})dy^{2a}\\
&dy^{3a'} = (\partial_{0a}y^{3a'})dy^{0a} + (\partial_{1a}y^{3a'})dy^{1a} + (\partial_{2a}y^{3a'})dy^{2a} + (\partial_{3a}y^{3a'})dy^{3a}.
\end{aligned}
$$

The adapted basis B^* of $T^*(E)$ is given by (different from that used in [15]–[17])

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}, \tag{9}$$

where

$$
\begin{aligned}
\delta y^{0a} &= dx^a = dy^{0a}\\
\delta y^{1a} &= dy^{1a} + M^{(1)a}_{\ \ b} dy^{0b}\\
\delta y^{2a} &= dy^{2a} + M^{(1)a}_{\ \ b} dy^{1b} + M^{(2)a}_{\ \ b} dy^{0b}\\
\delta y^{3a} &= dy^{3a} + M^{(1)a}_{\ \ b} dy^{2b} + M^{(2)a}_{\ \ b} dy^{1b} + M^{(3)a}_{\ \ b} dy^{0b}.
\end{aligned}
\tag{10}
$$

THEOREM 1.2. *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a}\delta y^{\alpha a},\ \alpha = 0,1,2,3,$$

are the following equations:

$$
\begin{aligned}
&\text{(a) } M^{(1)a}_{\ \ b}\partial_a x^{a'} = M^{(1)a'}_{\ \ b'}\partial_b x^{b'} + \partial_b y^{1a'}\\
&\text{(b) } M^{(2)a}_{\ \ b}\partial_a x^{a'} = M^{(2)a'}_{\ \ b'}\partial_b x^{b'} + M^{(1)a'}_{\ \ b'}\partial_b y^{1b'} + \partial_b y^{2a'}\\
&\text{(c) } M^{(3)a}_{\ \ b}\partial_a x^{a'} = M^{(3)a'}_{\ \ b'}\partial_b x^{b'} + M^{(2)a'}_{\ \ b'}\partial_b y^{1b'} + M^{(1)a'}_{\ \ b'}\partial_b y^{2b'} + \partial_b y^{3a'}.
\end{aligned}
\tag{11}
$$

From (1.11) and (1.4) it is obvious that (1.11a), (1.11b) and (1.11c) are partial differential equations of second, third and fourth order respectively, so there are infinity functions

$$
\begin{aligned}
M^{(1)a}_{\ \ b} &= M^{(1)a}(y^{0a}, y^{1a}),\\
M^{(2)a}_{\ \ b} &= M^{(2)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}),\\
M^{(3)a}_{\ \ b} &= M^{(3)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}, y^{3a}),
\end{aligned}
$$

which are the solutions of (1.11). From the choise of M depends the adapted basis B^* ((1.9)).

Let us denote the adapted basis of $T(E)$ by B, where

$$B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \delta_{3a}\} = \{\delta_{\alpha a}\},\tag{12}$$

and

$$
\begin{aligned}
\delta_{0a} &= \partial_{0a} - N^{(1)b}_{\ \ a}\partial_{1b} - N^{(2)b}_{\ \ a}\partial_{2b} - N^{(3)b}_{\ \ a}\partial_{3b},\\
\delta_{1a} &= \partial_{1a} - N^{(1)b}_{\ \ a}\partial_{2b} - N^{(2)b}_{\ \ a}\partial_{3b}\\
\delta_{2a} &= \partial_{2a} - N^{(1)b}_{\ \ a}\partial_{3b}\\
\delta_{3a} &= \partial_{3a}.
\end{aligned}
\tag{13}
$$

THEOREM 1.3. *The necessary and sufficient conditions that* B ((1.12)) *be dual to* B^* ((1.9)), *(when* $\bar{B}$ ((1.5)) *is dual to* $\bar{B}^*$ ((1.7)) *i.e.*

$$\left\langle \delta_{\alpha a} \delta y^{\beta b} \right\rangle = \delta_{\alpha}^{\beta} \delta_{a}^{b}$$

are the following relations:

$$(14)\qquad \begin{aligned} N^{(1)b}_{\ \ a} &= M^{(1)b}_{\ \ a} \\ N^{(2)b}_{\ \ a} &= M^{(2)b}_{\ \ a} - N^{(1)c}_{\ \ a} M^{(1)b}_{\ \ c} = M^{(2)b}_{\ \ a} - M^{(1)c}_{\ \ a} M^{(1)b}_{\ \ c} \\ N^{(3)b}_{\ \ a} &= M^{(3)b}_{\ \ a} - N^{(1)c}_{\ \ a} M^{(2)b}_{\ \ c} - N^{(2)c}_{\ \ a} M^{(1)b}_{\ \ c} = \\ &= M^{(3)b}_{\ \ a} - M^{(1)c}_{\ \ a} M^{(2)b}_{\ \ c} - M^{(2)c}_{\ \ a} M^{(1)b}_{\ \ c} + M^{(1)d}_{\ \ a} M^{(1)c}_{\ \ d} M^{(1)b}_{\ \ c}. \end{aligned}$$

From (1.13) and (1.14) it follows

THEOREM 1.4. *The necessary and sufficient conditions that* $\delta_{\alpha a}$ *with respect to* (1.1) *are transformed as* d*-tensors, i.e.*

$$(15)\qquad \delta_{\alpha a'} = \frac{\partial x^{a}}{\partial x^{a'}} \delta_{\alpha a}, \ \alpha = 0, 1, 2, 3,$$

are the following formulae:

$$(16)\qquad \begin{aligned} N^{(1)b'}_{\ \ a'} \partial_a x^{a'} &= N^{(1)b}_{\ \ a} \partial_b x^{b'} - \partial_a y^{1b'} \\ N^{(2)b'}_{\ \ a'} \partial_a x^{a'} &= N^{(2)b}_{\ \ a} \partial_b x^{b'} + N^{(1)b}_{\ \ a} \partial_b y^{1b'} - \partial_a y^{2b'} \\ N^{(3)b'}_{\ \ a'} \partial_a x^{a'} &= N^{(3)b}_{\ \ a} \partial_b x^{b'} + N^{(2)b}_{\ \ a} \partial_b y^{1b'} + N^{(1)b}_{\ \ a} \partial_b y^{2b'} - \partial_a y^{3b'}. \end{aligned}$$

From (1.13) and (1.14) it follows

$$(17)\qquad \begin{aligned} \partial_{3c} &= \delta_{3c} \\ \partial_{2c} &= \delta_{2c} + M^{(1)d}_{\ \ c} \delta_{3d} \\ \partial_{1c} &= \delta_{1c} + M^{(1)d}_{\ \ c} \delta_{2d} + M^{(2)d}_{\ \ c} \delta_{3d} \\ \partial_{0c} &= \delta_{0c} + M^{(1)d}_{\ \ c} \delta_{1d} + M^{(2)d}_{\ \ c} \delta_{2d} + M^{(3)d}_{\ \ c} \delta_{3d}. \end{aligned}$$

2. Decomposition of $T(E)$. Integrability conditions

Let us denote by T_H, T_{V_1}, T_{V_2}, T_{V_3} the subspaces of $T(E)$ spanned by

$$\{\delta_{0a}\},\ \{\delta_{1a}\},\ \{\delta_{2a}\},\ \{\delta_{3a}\}$$

respectively. Then we have

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2} \oplus T_{V_3}.$$

PROPOSITION 2.1. *The horizontal distribution T_H is integrable if all $K_{0a}{}^{\alpha c}{}_{0b}$, $\alpha = 1,2,3$ determined by* (2.2) *and* (2.4) *are equal to zero.*

PROOF. By direct calculation one obtains

$$[\delta_{0a}, \delta_{0b}] = \bar{K}_{0a}{}^{1c}{}_{0b}\partial_{1c} + \bar{K}_{0a}{}^{2c}{}_{0b}\partial_{2c} + \bar{K}_{0a}{}^{3c}{}_{0b}\partial_{3c}, \tag{1}$$

where

$$\bar{K}_{0a}{}^{\alpha c}{}_{0b} = \delta_{0b}N^{(\alpha)c}_{\ a} - \delta_{0a}N^{(\alpha)c}_{\ b},\ \alpha = 1,2,3. \tag{2}$$

The substitution of ∂_{1c}, ∂_{2c}, ∂_{3c} from (1.17) into (2.1) yields

$$[\delta_{0a}, \delta_{0b}] = K_{0a}{}^{1c}{}_{0b}\delta_{1c} + K_{0a}{}^{2c}{}_{0b}\delta_{2c} + K_{0a}{}^{3c}{}_{0b}\delta_{3c}, \tag{3}$$

where

$$\begin{aligned} K_{0a}{}^{1c}{}_{0b} &= \bar{K}_{0a}{}^{1c}{}_{0b} \\ K_{0a}{}^{2c}{}_{0b} &= \bar{K}_{0a}{}^{2c}{}_{0b} + \bar{K}_{0a}{}^{1d}{}_{0b}M^{(1)c}_{\ d} \\ K_{0a}{}^{3c}{}_{0b} &= \bar{K}_{0a}{}^{3c}{}_{0b} + \bar{K}_{0a}{}^{2d}{}_{0b}M^{(1)c}_{\ d} + K_{0a}{}^{1d}{}_{0b}M^{(2)c}_{\ d}. \end{aligned} \tag{4}$$

PROPOSITION 2.2. *The distribution T_{V_1} is integrable if $K_{1a}{}^{\alpha c}{}_{1b}$ $\alpha = 2,3$ determined by* (2.6) *and* (2.8) *are equal to zero.*

PROOF. We have

$$[\delta_{1a}, \delta_{1b}] = \bar{K}_{1a}{}^{2c}{}_{1b}\partial_{2c} + \bar{K}_{1a}{}^{3c}{}_{1b}\partial_{3c}, \tag{5}$$

where

$$\begin{aligned} \bar{K}_{1a}{}^{2c}{}_{1b} &= \delta_{1b}N^{(1)c}_{\ a} - \delta_{1a}N^{(1)c}_{\ b} \\ \bar{K}_{1a}{}^{3c}{}_{1b} &= \delta_{1b}N^{(2)c}_{\ a} - \delta_{1a}N^{(2)c}_{\ b}. \end{aligned} \tag{6}$$

Using (1.17), (2.5) takes the form:

$$[\delta_{1a}, \delta_{1b}] = K_{1a}{}^{2c}{}_{1b}\delta_{2c} + K_{1a}{}^{3c}{}_{1b}\delta_{3c}, \tag{7}$$

where

$$K_{1a}{}^{2c}{}_{1b} = \bar{K}_{1a}{}^{2c}{}_{1b}$$
(8)
$$K_{1a}{}^{3c}{}_{1b} = \bar{K}_{1a}{}^{3c}{}_{1b} + \bar{K}_{1a}{}^{2d}{}_{1b} M^{(1)c}_{\ \ d}.$$

PROPOSITION 2.3. *The distribution T_{V_2} is integrable if $K_{2a}{}^{3c}{}_{2b}$ determined by* (2.9) *and* (2.10) *is equal to zero.*

PROOF. We get

(9)
$$[\delta_{2a}, \delta_{2b}] = K_{2a}{}^{3c}{}_{2b} \delta_{3c},$$

where

(10)
$$K_{2a}{}^{3c}{}_{2b} = (\delta_{2b} N^{(1)c}_{\ \ a} - \delta_{2a} N^{(1)c}_{\ \ b}).$$

PROPOSITION 2.4. *The distribution T_{V_3} is integrable.*

PROOF.

(11)
$$[\delta_{3a}, \delta_{3b}] = 0.$$

PROPOSITION 2.5. *The following formulae are valid:*

(12)
$$[\delta_{0a}, \delta_{1b}] = K_{0a}{}^{1c}{}_{1b} \delta_{1c} + K_{0a}{}^{2c}{}_{1b} \delta_{2c} + K_{0a}{}^{3c}{}_{1b} \delta_{3c},$$

where

$$K_{0a}{}^{1c}{}_{1b} = \bar{K}_{0a}{}^{1c}{}_{1b},$$
(13)
$$K_{0a}{}^{2c}{}_{1b} = \bar{K}_{0a}{}^{2c}{}_{1b} + \bar{K}_{0a}{}^{1d}{}_{1b} M^{(1)c}_{\ \ d},$$
$$K_{0a}{}^{3c}{}_{1b} = \bar{K}_{0a}{}^{3c}{}_{1b} + \bar{K}_{0a}{}^{2d}{}_{1b} M^{(1)c}_{\ \ d} + \bar{K}_{0a}{}^{1d}{}_{1b} M^{(2)c}_{\ \ d},$$

and

$$\bar{K}_{0a}{}^{1c}{}_{1b} = \delta_{1b} N^{(1)c}_{\ \ a}$$
(14)
$$\bar{K}_{0a}{}^{2c}{}_{1b} = \delta_{1b} N^{(2)c}_{\ \ a} - \delta_{0a} N^{(1)c}_{\ \ b}$$
$$\bar{K}_{0a}{}^{3c}{}_{1b} = \delta_{1b} N^{(3)c}_{\ \ a} - \delta_{0a} N^{(2)c}_{\ \ b}.$$

PROPOSITION 2.6. *For $[\delta_{0a}, \delta_{2b}]$ we have*

(15)
$$[\delta_{0a}, \delta_{2b}] = K_{0a}{}^{1c}{}_{2b} \delta_{1c} + K_{0a}{}^{2c}{}_{2b} \delta_{2c} + K_{0a}{}^{3c}{}_{2b} \delta_{3a},$$

where

$$
\begin{aligned}
&K_{0a}{}^{1c}{}_{2b} = \bar{K}_{0a}{}^{1c}{}_{2b}, \\
&K_{0a}{}^{2c}{}_{2b} = \bar{K}_{0a}{}^{2c}{}_{2b} + \bar{K}_{0a}{}^{1d}{}_{2b} M^{(1)c}{}_{d}, \\
&K_{0a}{}^{3c}{}_{2b} = \bar{K}_{0a}{}^{3c}{}_{2b} + \bar{K}_{0a}{}^{2d}{}_{2b} M^{(1)c}{}_{d} + \bar{K}_{0a}{}^{1d}{}_{2b} M^{(2)c}{}_{d},
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
&\bar{K}_{0a}{}^{1c}{}_{2b} = \delta_{2b} N^{(1)c}{}_{a} \\
&\bar{K}_{0a}{}^{2c}{}_{2b} = \delta_{2b} N^{(2)c}{}_{a} \\
&\bar{K}_{0a}{}^{3c}{}_{2b} = \delta_{2b} N^{(3)c}{}_{a} - \delta_{0a} N^{(1)c}{}_{b}.
\end{aligned} \tag{17}
$$

PROPOSITION 2.7. *For* $[\delta_{0a}, \delta_{3b}]$ *we get*

$$
[\delta_{0a}, \delta_{3b}] = K_{0a}{}^{1c}{}_{3b}\delta_{1c} + K_{0a}{}^{2c}{}_{3b}\delta_{2c} + K_{0a}{}^{3c}{}_{3b}\delta_{3c}, \tag{18}
$$

where

$$
\begin{aligned}
&K_{0a}{}^{1c}{}_{3b} = \bar{K}_{0a}{}^{1c}{}_{3b} \\
&K_{0a}{}^{2c}{}_{3b} = \bar{K}_{0a}{}^{2c}{}_{3b} + \bar{K}_{0a}{}^{1d}{}_{3b} M^{(1)c}{}_{d} \\
&K_{0a}{}^{3c}{}_{3b} = \bar{K}_{0a}{}^{3c}{}_{3b} + \bar{K}_{0a}{}^{2d}{}_{3b} M^{(1)c}{}_{d} + \bar{K}_{0a}{}^{1d}{}_{3b} M^{(2)c}{}_{d}
\end{aligned} \tag{19}
$$

and

$$
\bar{K}_{0a}{}^{\alpha c}{}_{3b} = \delta_{3b} N^{(\alpha)c}{}_{a}, \ \alpha = 1, 2, 3. \tag{20}
$$

PROPOSITION 2.8. *For* $[\delta_{1a}, \delta_{2b}]$ *we have*

$$
[\delta_{1a}, \delta_{2b}] = K_{1a}{}^{2c}{}_{2b}\delta_{2c} + K_{1a}{}^{3c}{}_{2b}\delta_{3c}, \tag{21}
$$

where

$$
\begin{aligned}
&K_{1a}{}^{2c}{}_{2b} = \bar{K}_{1a}{}^{2c}{}_{2b} = \delta_{2b} N^{(1)c}{}_{a}, \\
&K_{1a}{}^{3c}{}_{2b} = \bar{K}_{1a}{}^{3c}{}_{2b} + \bar{K}_{1a}{}^{2d}{}_{2b} M^{(1)c}{}_{d}
\end{aligned} \tag{22}
$$

and

$$
\bar{K}_{1a}{}^{3c}{}_{2b} = \delta_{2b} N^{(2)c}{}_{a} - \delta_{1a} N^{(1)c}{}_{b}. \tag{23}
$$

PROPOSITION 2.9. *For* $[\delta_{1a}, \delta_{3b}]$ *we obtain*

$$
[\delta_{1a}, \delta_{3b}] = K_{1a}{}^{2c}{}_{3b}\delta_{2c} + K_{1a}{}^{3c}{}_{3b}\delta_{3c}, \tag{24}
$$

where

(25)
$$K_{1a}{}^{2c}{}_{3b} = \bar{K}_{1a}{}^{2c}{}_{3b}$$
$$K_{1a}{}^{3c}{}_{3b} = \bar{K}_{1a}{}^{3c}{}_{3b} + \bar{K}_{1a}{}^{2d}{}_{3b} M^{(1)c}_{\ \ d}$$

and

(26)
$$\bar{K}_{1a}{}^{2c}{}_{3b} = \delta_{3b} N^{(1)c}_{\ \ a}$$
$$\bar{K}_{1a}{}^{3c}{}_{3b} = \delta_{3b} N^{(2)c}_{\ \ a}.$$

PROPOSITION 2.10. *For* $[\delta_{2a}, \delta_{3b}]$ *we get*

(27)
$$[\delta_{2a}, \delta_{3b}] = K_{2a}{}^{3c}{}_{3b} \delta_{3c},$$

where

(28)
$$K_{2a}{}^{3c}{}_{3b} = \delta_{3b} N^{(1)c}_{\ \ a}.$$

3. Covariant derivatives in $T(E)$

Let $\nabla : T(E) \times T(E) \to T(E)$ be a linear connection, such that

$$\nabla : (X, Y) \to \nabla_X Y \in T(E), \ \forall X, Y \in T(E).$$

The operator ∇ is called a generalized connection. It is called d-connection if $\nabla_X Y$ is in T_H, T_{V_1}, T_{V_2}, T_{V_3} if Y is in T_H, T_{V_1}, T_{V_2}, T_{V_3} respectively.

For the space $Osc^k M$ it has been studied by R. MIRON and GH. ATANASIU in [15], [16].

DEFINITION 3.1. The generalized connection ∇ on $T(E)$ is defined by

(1)
$$\nabla_{\delta_{\alpha a}} \delta_{\beta b} = \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} \delta_{\gamma c}.$$

In (3.1) the summation is going over γ and c.

If Y is any vector field in $T(E)$ and

$$Y = Y^{\beta b} \delta_{\beta b},$$

then

$$\nabla_{\delta_{\alpha a}} Y = \nabla_{\delta_{\alpha a}} (Y^{\beta b} \delta_{\beta b}) = (\delta_{\alpha a} Y^{\beta b}) \delta_{\beta b} + \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} Y^{\beta b} \delta_{\gamma c} =$$
$$= (\delta_{\alpha a} Y^{\beta b} + \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} Y^{\gamma c}) \delta_{\beta b}.$$

Now we define the generalized covariant derivative of vector field Y in the form

$$Y^{\beta b}{}_{|\alpha a} = \delta_{\alpha a} Y^{\beta b} + \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} Y^{\gamma c}. \tag{2}$$

We have

$$\nabla_{\delta_{\alpha a}} Y = (Y^{\beta b}{}_{|\alpha a})\delta_{\beta b}. \tag{3}$$

THEOREM 3.1. *With respect to* (1.1) $Y^{\beta b}{}_{|\alpha a}$ *will be a* d*-tensor field, i.e.*

$$Y^{\beta b'}{}_{|\alpha a'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} Y^{\beta b}{}_{|\alpha a} \tag{4}$$

if all $\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a}$ *are transformed as* d*-tensors, i.e.*

$$\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'} \frac{\partial x^{c'}}{\partial x^c} \tag{5}$$

except $\Gamma_{\beta c}{}^{\beta b}{}_{0a}$ *(no summation over* β, $\beta = 0, 1, 2, 3$*) which have the following transformation law*

$$\Gamma_{\beta c}{}^{\beta b}{}_{0a} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\beta c'}{}^{\beta b'}{}_{\alpha a'} \frac{\partial x^{c'}}{\partial x^c} + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c}, \tag{6}$$

for $\beta = 0, 1, 2, 3$.

PROOF. Starting from (3.4) and using the tensor character of $\delta_{\alpha a'}$ and $Y^{\beta b'}$ we get

$$Y^{\beta b'}{}_{|\alpha a'} = \frac{\partial x^a}{\partial x^{a'}} \left[\frac{\partial x^{b'}}{\partial x^b} \delta_{\alpha a} Y^{\beta b} + Y^{\beta c} \delta_{\alpha a} \frac{\partial x^{b'}}{\partial x^c} \right] + \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'} Y^{\gamma c} \frac{\partial x^{c'}}{\partial x^c} =$$

$$= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} [\delta_{\alpha a} Y^{\beta b} + \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} Y^{\gamma c}].$$

From the above follows:

$$\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'} \frac{\partial x^{c'}}{\partial x^c} + \delta^\beta_\gamma \frac{\partial x^a}{\partial x^{a'}} \delta_{\alpha a} \frac{\partial x^{b'}}{\partial x^c}. \tag{7}$$

If $\beta \neq \gamma$ the last term in (3.7) vanishes.

If $\alpha \neq 0$, then $\delta_{\alpha a} \frac{\partial x^{b'}}{\partial x^c} = 0$ (see (1.13) and (1.1)(a)).

If $\alpha = 0$, then

$$\delta_{0a} \frac{\partial x^{b'}}{\partial x^c} = \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c},$$

which proves (3.6).

4. The torsion tensor of generalized connection

The torsion tensor $T(X, Y)$ is defined as usual by:

(1) $$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

If X and Y expressed in the basis B have the form

(2) $$X = X^{\alpha a} \delta_{\alpha a}, \quad Y = Y^{\beta b} \delta_{\beta b},$$

then using linearity of ∇ and (3.1) we get

(3) $$\begin{aligned} \nabla_X Y &= \nabla_{X^{\alpha a} X^{\alpha a} \delta_{\alpha a}} (Y^{\beta b} \delta_{\beta b}) = \\ &= X^{\alpha a} (\delta_{\alpha a} Y^{\beta b}) \delta_{\beta b} + X^{\alpha a} Y^{\beta b} \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} \delta_{\gamma c}. \end{aligned}$$

Further we have

(4)
$$\begin{aligned} [X, Y] &= [X^{\alpha a} \delta_{\alpha a}, Y^{\beta b} \delta_{\beta b}] = \\ &= X^{\alpha a} (\delta_{\alpha a} Y^{\beta b}) \delta_{\beta b} - Y^{\beta b} (\delta_{\beta b} X^{\alpha a}) \delta_{\alpha a} + X^{\alpha a} Y^{\beta b} [\delta_{\alpha a}, \delta_{\beta b}]. \end{aligned}$$

The substitution of (4.3) and (4.4) into (4.1) gives

THEOREM 4.1. *The torsion tensor of generalized connection has the form*

(5) $$\begin{aligned} T(X, Y) &= X^{\alpha a} Y^{\beta b} [(\Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b}) \delta_{\gamma c} - [\delta_{\alpha a}, \delta_{\beta b}]] = \\ &= T_{\beta b}{}^{\gamma c}{}_{\alpha a} X^{\alpha a} Y^{\beta b} \delta_{\gamma c}, \end{aligned}$$

where

(6) $$T_{\beta b}{}^{\gamma c}{}_{\alpha a} = \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b} - K_{\alpha a}{}^{\gamma c}{}_{\beta b},$$

(7) $$[\delta_{\alpha a}, \delta_{\beta b}] = K_{\alpha a}{}^{\gamma c}{}_{\beta b} \delta_{\gamma c}.$$

From (2.1)–(2.28) follows

$$
\begin{aligned}
X^{\alpha a} Y^{\beta b}[\delta_{\alpha a}, \delta_{\beta b}] = {} & X^{0a} Y^{0b}(K_{0a}{}^{1c}{}_{0b}\delta_{1c} + K_{0a}{}^{2c}{}_{0b}\delta_{2c} + K_{0a}{}^{3c}{}_{0b}\delta_{3c}) + \\
& +(X^{0a} Y^{1b} - X^{1b} Y^{0a})(K_{0a}{}^{1c}{}_{1b}\delta_{1c} + K_{0a}{}^{2c}{}_{1b}\delta_{2c} + K_{0a}{}^{3c}{}_{1b}\delta_{3c}) + \\
& +(X^{0a} Y^{2b} - X^{2b} Y^{0a})(K_{0a}{}^{1c}{}_{2b}\delta_{1c} + K_{0a}{}^{2c}{}_{2b}\delta_{2c} + K_{0a}{}^{3c}{}_{2b}\delta_{3c}) + \\
& +(X^{0a} Y^{3b} - X^{3b} Y^{0a})(K_{0a}{}^{1c}{}_{3b}\delta_{1c} + K_{0a}{}^{2c}{}_{3b}\delta_{2c} + K_{0a}{}^{3c}{}_{3b}\delta_{3c}) + \\
& +X^{1a} Y^{1b}(K_{1a}{}^{2c}{}_{1b}\delta_{2c} + K_{1a}{}^{3c}{}_{1b}\delta_{3c}) + \\
& +(X^{1a} Y^{2b} - X^{2b} Y^{1a})(K_{1a}{}^{2c}{}_{2b}\delta_{2c} + K_{1a}{}^{3c}{}_{2b}\delta_{3c}) + \\
& +(X^{1a} Y^{3b} - X^{3b} Y^{1a})(K_{1a}{}^{2c}{}_{3b}\delta_{2c} + K_{1a}{}^{3c}{}_{3b}\delta_{3c}) + \\
& +X^{2a} Y^{2b}(K_{2a}{}^{3c}{}_{2b}\delta_{3c}) + \\
& +(X^{2a} Y^{3b} - X^{2b} Y^{2a})(K_{2a}{}^{3c}{}_{3b}\delta_{3c}).
\end{aligned}
$$

As in $T_{\beta b}{}^{\gamma c}{}_{\alpha a}$ α, β, γ take values from $\{0, 1, 2, 3\}$, so there are 4^3 different types of torsion coefficients. From (4.6) and (4.8) it is clear that $4^3 - 20$ of them are the difference of connection coefficients, but 20 of them have the additional term $-K_{\alpha a}{}^{\gamma c}{}_{\beta b}$ according to (4.8). If it is supposed that T_H, T_{V_1}, T_{V_2}, T_{V_3} are integrable distributions, then all K's beside $X^{0a} Y^{0b}$, $X^{1a} Y^{1b}$, $X^{2a} Y^{2b}$ are equal to zero.

5. The curvature theory of ∇

The curvature tensor for the generalized connection ∇ is defined as usual

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1}$$

If the notations (4.2) and $Z = Z^{\gamma c}\delta_{\gamma c}$ are used, then

$$\begin{aligned}
&\nabla_X \nabla_Y Z = \nabla_{X^{\alpha a}} X^{\alpha a}\delta_{\alpha a}\nabla_{Y^{\beta b}} Y^{\beta b}\delta_{\beta b} Z^{\gamma c}\delta_{\gamma c} = \\
&\quad = \nabla_{X^{\alpha a}} X^{\alpha a}\delta_{\alpha a}[Y^{\beta b}(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c} + Y^{\beta b}Z^{\gamma c}\Gamma_{\gamma c\ \beta b}^{\ \delta d}\delta_{\delta d}] = \\
&\quad = X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c} + X^{\alpha a}Y^{\beta b}(\delta_{\alpha a}\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c} + \\
&\quad + X^{\alpha a}Y^{\beta b}(\delta_{\beta b}Z^{\gamma c})\Gamma_{\gamma c\ \alpha a}^{\ \delta d}\delta_{\delta d} + X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})Z^{\gamma c}\Gamma_{\gamma c\ \beta b}^{\ \delta d}\delta_{\delta d} + \\
&\quad + X^{\alpha a}Y^{\beta b}(\delta_{\alpha a}Z^{\gamma c})\Gamma_{\gamma c\ \beta b}^{\ \delta d}\delta_{\delta d} + X^{\alpha a}Y^{\beta b}Z^{\gamma c}(\delta_{\alpha a}\Gamma_{\gamma c\ \beta b}^{\ \delta d})\delta_{\delta d} + \\
&\quad + X^{\alpha a}Y^{\beta b}Z^{\gamma c}\Gamma_{\gamma c\ \beta b}^{\ \varepsilon e}\Gamma_{\varepsilon e\ \alpha a}^{\ \delta d}\delta_{\delta d}.
\end{aligned} \tag{2}$$

From (4.4) and (4.7) follows

$$\begin{aligned}
\nabla_{[X\,Y]}Z &= X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c} + X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})Z^{\gamma c}\Gamma_{\gamma c\ \beta b}^{\ \delta d}\delta_{\delta d} - \\
&\quad - Y^{\beta b}(\delta_{\beta b}X^{\alpha a})(\delta_{\alpha a}Z^{\gamma c})\delta_{\gamma c} - Y^{\beta b}(\delta_{\beta b}X^{\alpha a})Z^{\gamma c}\Gamma_{\gamma c\ \alpha a}^{\ \delta d}\delta_{\delta d} + \\
&\quad + X^{\alpha a}Y^{\beta b}[(\delta_{\alpha a}\delta_{\beta b} - \delta_{\beta b}\delta_{\alpha a})Z^{\gamma c}]\delta_{\gamma c} + \\
&\quad + X^{\alpha a}Y^{\beta b}Z^{\gamma c}K_{\alpha a\ \beta b}^{\ \varepsilon e}\Gamma_{\gamma c\ \varepsilon e}^{\ \delta d}\delta_{\delta d}.
\end{aligned} \tag{3}$$

Substituting (5.2) and (5.3) into (5.1) we obtain

THEOREM 5.1. *In Osc^3M the curvature tensor for the generalized connection ∇ has the form:*

$$R(X, Y)Z = R_{\gamma c\ \beta b\alpha a}^{\ \delta d} Z^{\gamma c} Y^{\beta b} X^{\alpha a}\delta_{\delta d}, \tag{4}$$

where

$$R_{\gamma c\ \beta b\,\alpha a}^{\ \delta d} = K_{\gamma c\ \beta b\,\alpha a}^{\ \delta d} - K_{\alpha a\ \beta b}^{\ \varepsilon e}\Gamma_{\gamma c\ \varepsilon e}^{\ \delta d}, \tag{5}$$

$$K_{\gamma c\ \beta b\,\alpha a}^{\ \delta d} = (\delta_{\alpha a}\Gamma_{\gamma c\ \beta b}^{\ \delta d} + \Gamma_{\gamma c\ \beta b}^{\ \varepsilon e}\Gamma_{\varepsilon e\ \alpha a}^{\ \delta d}) - (\alpha a/\beta b). \tag{6}$$

It is clear that in (5.5) are only those $K_{\alpha a\ \beta b}^{\ \varepsilon e}$ are different from zero, which appeare in (4.8). As $\alpha, \beta, \gamma, \delta$ are the elements of the set $\{0, 1, 2, 3\}$, so there exist 4^4 types of curvature tensors.

6. The Ricci equations for ∇

From (3.3) it follows

$$\nabla_Y Z = Y^{\beta b} Z^{\gamma c}{}_{|\beta b} \delta_{\gamma c}, \tag{1}$$

and on the similar way we get

$$\begin{aligned} \nabla_X(\nabla_Y Z) &= X^{\alpha a}(Y^{\beta b} Z^{\gamma c}{}_{|\beta b})_{|\alpha a} \delta_{\gamma c} = \\ &= X^{\alpha a}(Y^{\beta b}{}_{|\alpha a} Z^{\gamma c}{}_{|\beta b} + Y^{\beta b} Z^{\gamma c}{}_{|\beta b|\alpha a})\delta_{\gamma c}. \end{aligned} \tag{2}$$

On the other hand from (3.3) follows

$$\nabla_{[X\,Y]} Z = Z^{\gamma c}{}_{|\delta d}[X, Y]^{\delta d} \delta_{\gamma c} = A + B, \tag{3}$$

where using (3.2) we obtain

$$\begin{aligned} A &= Z^{\gamma c}{}_{|\beta b}(X^{\alpha a}\delta_{\alpha a} Y^{\beta b} - Y^{\alpha a}\delta_{\alpha a} X^{\beta b})\delta_{\gamma c} = \\ &= [X^{\alpha a} Y^{\beta b}{}_{|\alpha a} Z^{\gamma c}{}_{|\beta b} - Y^{\beta b} X^{\alpha a}{}_{|\beta b} Z^{\gamma c}{}_{|\alpha a} - \\ &- (\Gamma_{\beta b}{}^{\delta d}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\delta d}{}_{\beta b}) X^{\alpha a} Y^{\beta b} Z^{\gamma c}{}_{|\delta d}]\delta_{\gamma c} \end{aligned} \tag{4}$$

$$B = K_{\alpha a}{}^{\delta d}{}_{\beta b} X^{\alpha a} Y^{\beta b} Z^{\gamma c}{}_{|\delta d} \delta_{\gamma c}. \tag{5}$$

Substituting (6.5) and (6.4) into (6.3) we get

$$\begin{aligned} \nabla_{[X,Y]} Z &= (X^{\alpha a} Y^{\beta b}{}_{|\alpha a} Z^{\gamma c}{}_{|\beta b} - Y^{\beta b} X^{\alpha a}{}_{|\beta b} Z^{\gamma c}{}_{|\alpha a})\delta_{\gamma c} - \\ &- T_{\beta b}{}^{\delta d}{}_{\alpha a} X^{\alpha a} Y^{\beta b} Z^{\gamma c}{}_{|\delta d} \delta_{\gamma c}. \end{aligned} \tag{6}$$

Substituting (6.2) and (6.6) into (5.1) we get

$$R(X, Y)Z = (Z^{\gamma c}{}_{|\beta b|\alpha a} - Z^{\gamma c}{}_{|\alpha a|\beta b} + T_{\beta b}{}^{\delta d}{}_{\alpha a} Z^{\gamma c}{}_{|\delta d}) X^{\alpha a} Y^{\beta b} \delta_{\gamma c}. \tag{7}$$

From (5.4) and (6.7) it follows

THEOREM 6.1. *The Ricci equations for generalized connection ∇ have the form:*

$$Z^{\gamma c}{}_{|\beta b|\alpha a} - Z^{\gamma c}{}_{|\alpha a|\beta b} + T_{\beta b}{}^{\delta d}{}_{\alpha a} Z^{\gamma c}{}_{|\delta d} = R_{\delta d}{}^{\gamma c}{}_{\beta b \alpha a} Z^{\delta d}. \tag{8}$$

References

[1] ASANOV G.S., *Finsler Geometry, Relativity and Gauge Theories,* D. Reidel Publ. Comp. 1985.

[2] BALAN V., A class of solutions for the generalized gauge Asanov equations, *Studii si Cercetáni Math.,* **45** (1993), 287–292.

[3] BEJANCU A., Fundations of direction-dependent gauge theory, *Seminarul de Mecanico, Univ. Timisoara,* **13** (1988), 1–60.

[4] ČOMIĆ I., The curvature theory of strongly distinguished connection in the recurrent K-Hamilton space, *Indian Journal of Applied Math.,* **23** (1992) 189–202.

[5] ČOMIĆ I., Curvature theory of recurrent Hamilton space with generalized connection, *Analele Stiintifice Univ. Al. J. Cuza din Iasi,* **37**, s.I a. Mat. (1991), 467–476.

[6] ČOMIĆ I., Curvature theory of generalized second order gauge connections, *Publ. Math. Debrecen*, **50** (1997), 97–106.

[7] ČOMIĆ I., Curvature theory of vector bundles and subbundles, *Filomat* (Niš) (1993), 55–66.

[8] ČOMIĆ I., The Ricci and Bianchi identities in the recurrent K-Hamilton spaces, *Proc. of the Third Conf. of Geom.,* Thesaloniki (1991), 137–147.

[9] ČOMIĆ I., KAWAGUCHI H., The curvature theory of dual vector bundles and subbundles, *Tensor, N.S.,* **55** (1994), 20–31.

[10] IKEDA S., On the theory of gravitational field in Finsler space, *Tensor N.S.,* **50** (1991), 256–262.

[11] KAWAGUCHI A., On the Vectors of Higher order and the Extended Affine Connections, *Ann. di Patem. Pura ed Appl.* (IV), **55** (1961), 105–118.

[12] LIBERMANN P. and MARLE CH. M., *Symplectic Geometry and Analytical Mechanics,* D. Reidel Publ. Comp., 1987.

[13] MATSUMOTO M., *Foundations of Finsler Geometry and Special Finsler Spaces,* Kaiseisha Press, Otsu, Japan, 1986.

[14] MIRON R., ANASTASIEI M., *The Geometry of Lagrange Space, Theory and Applications,* Kluwer Academie Publishers, 1993.

[15] MIRON R., ATANASIU GH., Compendium sur les espaces Lagrange d'ordre supérieur, *Seminarul de Mecanica,* **40**, Universitatea din Timisoara, 1994.

[16] MIRON R. and ATANASIU GH., Differential Geometry of the k-Osculator Bundle, *Rev. Roum. Math. Pures et Appl.,* **41** (1996), 205–236.

[17] MIRON R. and ATANASIU GH., Higher Order Lagrange Spaces, *Rev. Roum. Math. Pures et Appl.,* **41** (1996), 251–263.

[18] MIRON R. and KAWAGUCHI T., Lagrangian Geometrical Theories and their Applications to the Physics and Engineering Dynamical Systems, *Tensor Soc.,* (to appear).

[19] MUNTEANU GH., ATANASIU GH., On Miron-connections in Lagrange spaces of second order, *Tensor N.S.,* **50** (1991).

[20] MUNTEANU GH., Metric almost tangent structure of second order, *Bull. Math. Soc. Sci. Mat. Roumanie,* **34** (1990), 49–54.

[21] OPRIS D., Fibres vectorials de Finsler et connexions associes, *The Proc. of Mat. Sem. on Finsler Spaces,* Brasov, (1980), 185–193.

[22] SARDANASHVILY G., ZAKHAROV O., *Gauge Gravitation Theory,* World Scienc. Publishing. Co., 1992.

[23] TRAUTMAN A., *Differential Geometry for Physicists,* Bibliophis Naples, 1984.

[24] YANO K., ISHIHARA S., *Tangent and Cotangent Bundles.,* Marcel Dekker Inc., New York, 1973.

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BOUNDED MEAN CURVATURE ISOMETRIC IMMERSIONS INTO CP^n CONTAINED IN A TUBE AROUND $\mathbb{RP}^n$

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1. Introduction

In the last years, some theorems have been proved stating that a submanifold in a space form with large mean curvature cannot be included into a small geodesic ball (cfr. [Am], [Ha], [HK], [JK], [JX], [Ma]). Recently, F. J. CARRERAS and the authors (cfr. [CGMI]) have studied the analog problem for compact submanifolds inside a ball of a complex space form. In [Gi] and [CGM2] it was considered the problem of the immersibility of a complete Riemannian manifold into a tube of S^n or $\mathbb{CP}^n$ around a totally geodesic S^q or $\mathbb{CP}^q$ respectively, with the mean curvature of the immersion bounded by a number which depends on the radius of the tube, and is related with the mean curvature of the tubular hypersurface. Here we continue this study by considering the same problem for tubes around a totally geodesic $\mathbb{RP}^n$ in $\mathbb{CP}^n$.

Let $\mathbb{CP}^n$ be the complex projective space of real dimension $2n$ and holomorphic sectional curvature $4\lambda > 0$, and let $\mathbb{RP}^n$ be the real projective space of dimension n and sectional curvature λ embedded in $\mathbb{CP}^n$ as a totally geodesic and totally real submanifold. Let M be a complete Riemannian manifold of dimension m, and let $\psi : M \to \mathbb{CP}^n$ be an isometric immersion. Let $r : \mathbb{CP}^n \to \mathbb{R}$ be the distance to the submanifold $\mathbb{RP}^n$ in $\mathbb{CP}^n$, and denote also $r \circ \psi$ by r. Let us denote by ∂_r the gradient of r in $\mathbb{CP}^n$ and by $\partial_r^\top$ the vector field in M defined as the preimage by ψ_* of (the component of) ∂_r (tangent to $\psi(M)$).

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Let $\beta(t)$ be the function

$$\beta(t) = (n-m)\,\mathrm{co}_\lambda(t) + (n-1)\,\mathrm{ta}_\lambda(t) + \mathrm{ta}_{4\lambda}(t),$$

where we use the notation

$$s_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, \qquad c_\lambda(t) = \cos(\sqrt{\lambda}t),$$

$$\mathrm{ta}_\lambda(t) = \sqrt{\lambda}\tan(\sqrt{\lambda}t) \quad \text{and} \quad \mathrm{co}_\lambda(t) = \sqrt{\lambda}\cot(\sqrt{\lambda}t).$$

Let $z^+(\beta) = \inf\{t > 0 \mid \beta(t) = 0\}$. Then $z^+(\beta) < \frac{\pi}{4\sqrt{\lambda}}$.

Given any positive real number ρ such that $\rho < \frac{\pi}{4\sqrt{\lambda}}$, we shall denote by $\mathbb{RP}^n_\rho$ the tube of radius ρ around $\mathbb{RP}^n$ in $\mathbb{CP}^n$ (that is, the set of those points $x \in \mathbb{CP}^n$ such that distance$(\mathbb{RP}^n, x) \le \rho$), and by $\partial\,\mathbb{RP}^n_\rho$ the corresponding tubular hypersurface (the boundary of the tube, which is also the set of the points $x \in \mathbb{CP}^n$ such that distance$(\mathbb{RP}^n, x) = \rho$). It is known that the cut locus of $\mathbb{RP}^n$ in $\mathbb{CP}^n$ is $\mathcal{Q}$, the complex hyperquadric in $\mathbb{CP}^n$. For every $x \in \mathbb{CP}^n - \mathcal{Q}$, let us denote by $p \in \mathbb{RP}^n$ the point such that

$$\text{distance}(p, x) = r(x) = \text{distance}(\mathbb{RP}^n, x),$$

by $\gamma(t)$ the minimal geodesic joining p and x and by τ_t the parallel transport along $\gamma(t)$ from p to x. Then we define the distributions $\mathcal{H}$ and $\mathcal{V}$ by the formulae

$$\tau_t\,\mathbb{RP}^n = \langle J\partial_r\rangle \oplus \mathcal{H}_x \quad \text{and} \quad T_x\,\mathbb{CP}^n = \langle J\partial_r\rangle \oplus \mathcal{H}_x \oplus \mathcal{V}_x \oplus \langle\partial_r\rangle\,.$$

H will denote the mean curvature vector of the immersion ψ.

Now, we state the theorems that we shall prove in this note.

THEOREM 1.1. *Let us assume that M is compact, $m > n$, $\psi(M) \subset \mathbb{RP}^n_\rho$, with $\rho \le z^+(\beta)$, and $m|H| \le |\beta(\rho)|$. Then, $\psi(M) \subset \partial\,\mathbb{RP}^n_\rho$, $mH = \beta(\rho)\partial_r$, $J\partial_r(\psi(x)) \in \psi_{*x}T_xM$, and $\mathcal{H}_{\psi(x)} \subset \psi_{*x}T_xM$ for every $x \in M$.*

Moreover, if $m = 2n-1$, then ψ is an embedding up to a covering map, and M is isometric (up to a Riemannian covering) to $\partial\,\mathbb{RP}^n_\rho$.

Let us remark that the hypothesis $\rho \le z^+(\beta)$ is necessary, because for $\rho > z^+(\beta)$ we have $\partial\,\mathbb{RP}^n_{z^+(\beta)} \subset \mathbb{RP}^n_\rho$, and the mean curvature of $\partial\,\mathbb{RP}^n_{z^+(\beta)}$ is 0.

THEOREM 1.2. *Let us assume that M is complete and non compact, with scalar curvature bounded from below, $m > n$ and $\psi(M) \subset \mathbb{RP}^n_\rho$ with $\rho \le z^+(\beta)$. Then $m\sup|H| \ge |\beta(\rho)|$.*

2. Proof of the theorems

From now on, $\overline{\nabla}$ and ∇ will denote the covariant derivatives in $\mathbb{CP}^n$ and M respectively and Δ will denote the laplacian on M.

If $f : \mathbb{R} \to \mathbb{R}$ is any function, we shall denote by $f(r)$ the function $f \circ r : M \to \mathbb{R}$.

We shall denote by $S(r)$ the $(1,1)$-tensor field on $\mathbb{CP}^n - (\mathcal{Q} \cup \mathbb{RP}^n)$, defined by

$S(r)A = -\overline{\nabla}_A \partial_r$ for any vector $A \in T\,\mathbb{CP}^n$ and any $x \in \mathbb{CP}^n - (\mathcal{Q} \cup \mathbb{RP}^n)$.

It is known (cfr. [CR]) that

$$\begin{aligned} &S(r)J\partial_r = \mathrm{ta}_{4\lambda}(r)J\partial_r, \qquad S(r)\partial_r = 0,\\ (1)\qquad &S(r)X = \mathrm{ta}_\lambda\, X, \quad \text{and} \quad S(r)A = -\,\mathrm{co}_\lambda(r)A \end{aligned}$$

for every $X \in \mathcal{H}_x$, $A \in \mathcal{V}_x$ and every $x \in \mathbb{CP}^n - (\mathcal{Q} \cup \mathbb{RP}^n)$.

The following formula is well known when r is the distance to a point (see, for instance, [CGM1]), and the same computation works here. If $\{e_1, \ldots \ldots, e_m\}$ is an orthonormal basis of the tangent space to M at some point, we have, at this point,

$$(2) \qquad \Delta f(r) = -f''(r)|\partial_r^\top|^2 + f'(r)\left\{\sum_{i=1}^m \langle S(r)e_i, e_i\rangle - m\,\langle H, \partial_r\rangle\right\}.$$

Now, we compute the value of $\langle S(r)e_i, e_i\rangle$. Using (1), and orthonormal basis $\{h_1, \ldots, h_{n-1}\}$ of $\mathcal{H}$ and $\{v_{n+1}, \ldots, v_{2n-1}\}$ of $\mathcal{V}$, we have:

$$\begin{aligned} &\langle S(r)e_i, e_i\rangle = \\ &= \left\langle S(r)\left(\sum_{j=1}^{n-1} \langle e_i, h_j\rangle h_j + \langle e_i, J\partial_r\rangle J\partial_r + \sum_{k=n+1}^{2n-1} \langle e_i, v_k\rangle v_k + \langle e_i, \partial_r\rangle \partial_r\right),\right.\\ &\qquad \left.\sum_{j=1}^{n-1} \langle e_i, h_j\rangle h_j + \langle e_i, J\partial_r\rangle J\partial_r + \sum_{k=n+1}^{2n-1} \langle e_i, v_k\rangle v_k + \langle e_i, \partial_r\rangle \partial_r\right\rangle = \\ &= \sum_{j=1}^{n-1} \langle e_i, h_j\rangle^2 \mathrm{ta}_\lambda(r) + \langle e_i, J\partial_r\rangle^2 \mathrm{ta}_{4\lambda}(r) - \sum_{k=n+1}^{2n-1} \langle e_i, v_k\rangle^2 \mathrm{co}_\lambda(r), \end{aligned}$$

then, having account that $\sum_{i=1}^{m}\left\langle e_i,h_j\right\rangle^2 \leq 1$ with the equality if and only if $h_j\in\psi_{*x}T_xM$ for every j,

$$
\begin{aligned}
\sum_{i=1}^{m}\langle S(r)e_i,e_i\rangle &= \sum_{i=1}^{m}\sum_{j=1}^{n-1}\left\langle e_i,h_j\right\rangle^2\left(\mathrm{ta}_\lambda(r)+\mathrm{co}_\lambda(r)\right)-\\
&\quad -\sum_{i=1}^{m}\left(\sum_{j=1}^{n-1}\left\langle e_i,h_j\right\rangle^2+\sum_{k=n+1}^{2n-1}\langle e_i,v_k\rangle^2\right)\mathrm{co}_\lambda(r)+\sum_{i=1}^{m}\langle e_i,J\partial_r\rangle^2\,\mathrm{ta}_{4\lambda}(r)\leq\\
&\leq (n-1)\left(\mathrm{ta}_\lambda(r)+\mathrm{co}_\lambda(r)\right)-\\
&\quad -\sum_{i=1}^{m}\left(1-\langle e_i,\partial_r\rangle^2-\langle e_i,J\partial_r\rangle^2\right)\mathrm{co}_\lambda(r)+|(J\partial_r)^\top|^2\,\mathrm{ta}_{4\lambda}(r)=\\
&= (n-1)(\mathrm{ta}_\lambda(r)+\mathrm{co}_\lambda(r))-m\,\mathrm{co}_\lambda(r)+\\
&\quad +\left(|\partial_r^\top|^2+|(J\partial_r)^\top|^2\right)\mathrm{co}_\lambda(r)+|(J\partial_r)^\top|^2\,\mathrm{ta}_{4\lambda}(r)=\\
&= (-m+n-1)\,\mathrm{co}_\lambda(r)+(n-1)\,\mathrm{ta}_\lambda(r)+\\
&\quad +\left(|\partial_r^\top|^2+|(J\partial_r)^\top|^2\right)\mathrm{co}_\lambda(r)+|(J\partial_r)^\top|^2\,\mathrm{ta}_{4\lambda}(r).
\end{aligned}
$$

Then, if $f'(r)\geq 0$, by substitution in (2), we get

$$
\begin{aligned}
\Delta f(r) \leq{}& (-f''(r)+\mathrm{co}_\lambda(r)f'(r))|\partial_r^\top|^2+f'(r)(\mathrm{co}_\lambda(r)+\mathrm{ta}_{4\lambda}(r))|(J\partial_r)^\top|^2+\\
&+f'(r)((-m+n-1)\,\mathrm{co}_\lambda(r)+(n-1)\,\mathrm{ta}_\lambda(r)-m\,\langle H,\partial_r\rangle).
\end{aligned}
$$

Now, if we take

$$
f'(t)=s_\lambda(t),\quad\text{then}\quad f(t)=-\frac{c_\lambda(t)}{\lambda}, \tag{3}
$$

we have $-f''(r)+f'(r)\,\mathrm{co}_\lambda(r)=0$, and

$$
\begin{aligned}
\Delta f(r) \leq{}& \left(c_\lambda(r)+s_\lambda(r)\,\mathrm{ta}_{4\lambda}(r)\right)|(J\partial_r)^\top|^2+\\
&+s_\lambda(r)\left((-m+n-1)\,\mathrm{co}_\lambda(r)+(n-1)\,\mathrm{ta}_\lambda(r)-m\,\langle H,\partial_r\rangle\right).
\end{aligned}
$$

Since $|(J\partial_r)^\top|^2 \leq 1$ with the equality if and only if $J\partial_r \in \psi_{*x} T_x M$,

$$\begin{aligned}\Delta f(r) &\leq c_\lambda(r) + \\ &+ s_\lambda(r)\left(\mathrm{ta}_{4\lambda}(r) + (-m+n-1)\,\mathrm{co}_\lambda(r) + (n-1)\,\mathrm{ta}_\lambda(r) - m\,\langle H, \partial_r\rangle\right) = \\ &= s_\lambda(r)\left(\mathrm{ta}_{4\lambda}(r) + (-m+n)\,\mathrm{co}_\lambda(r) + (n-1)\,\mathrm{ta}_\lambda(r) - m\,\langle H, \partial_r\rangle\right) = \\ &= s_\lambda(r)(\beta(r) - m\,\langle H, \partial_r\rangle).\end{aligned}$$

Now, let us study the behaviour of $\beta(t) = (n-m)\,\mathrm{co}_\lambda(t) + (n-1)\,\mathrm{ta}_\lambda(t) + \mathrm{ta}_{4\lambda}(t)$. Since $\lim_{t\to 0} \mathrm{ta}_\mu(t) = 0$, $\lim_{t\to 0+} \mathrm{co}_\lambda(t) = +\infty$ and $m > n$, one gets that

$$\lim_{t\to 0+} \beta(t) = -\infty,$$

moreover

$$\beta' = (m-n)\frac{1}{s_\lambda^2} + \frac{4\lambda}{c_{4\lambda}^2} + (n-1)\frac{\lambda}{c_\lambda^2},$$

then $\beta'(t) > 0$ for $t \in \left]0, z^+(\beta)\right[$ and

$$\beta(t) \text{ is increasing and negative in } \left]0, z^+(\beta)\right[.$$

Then, from this property of β and the fact that $-\langle H, \partial_r\rangle \leq |H|$, with the equality if and only if $H = -|H|\partial_r$, we have

$$\Delta f(r) \leq s_\lambda(r)\left(\beta(\rho) - m\,\langle H, \partial_r\rangle\right) \leq s_\lambda(r)\left(-|\beta(\rho)| + m|H|\right). \tag{4}$$

Now the proof of theorems 1.1 and 1.2 will follow easily from formula (4)

2.1. Proof of Theorem 1.1. If $m|H| \leq |\beta(\rho)|$, it follows from (4) that

$$\Delta f(r) \leq 0, \tag{5}$$

and, if M is compact, by Hopf principle, we have that $\Delta f(r) = 0$, and all the inequalities we have used to get (5) must be equalities, which implies

(i) $r = \rho$ (i.e. $\psi(M) \subset \partial\,\mathbb{RP}^n_\rho$) and $H = -|H|\partial_r$ (i.e. $mH = \beta(\rho)\partial_r$).

(ii) $J\partial_r(\psi(x)) \in \psi_{*x} T_x M$ and

(iii) $\mathcal{H}_{\psi(x)} \subset \psi_{*x} T_m M$.

If $m = 2n-1$, then, for every $x \in M$, ψ_{*x} is bijective and an isometry, then ψ is a local diffeomorphism, then $\psi(M)$ is open in $\partial\,\mathbb{RP}^n_\rho$, and it is compact, then closed, then $\psi(M) = \partial\,\mathbb{RP}^n_\rho$ (because $\partial\,\mathbb{RP}^n_\rho$ is connected). Then $\psi : M \to \partial\,\mathbb{RP}^n_\rho$ is a local isometry and M is compact, then ψ is a covering map.

2.2. Proof of Theorem 1.2. We shall use the following

OMORI'S LEMMA ([Om]). *Let M be a complete Riemannian manifold whose sectional curvature is bounded from below. Let $g : M \to \mathbb{R}$ be a smooth function bounded from above. Then, for every $x \in M$ and every $\epsilon > 0$, there is an $x' \in M$ with the properties*

$$g(x') \geq g(x), \quad \|\operatorname{grad} g\|(x') < \epsilon, \quad \nabla^2 g(X, X) < \epsilon \|X\|^2$$

for every $X \in T_{x'}M$, where ∇ is the Riemannian connection on M.

JORGE–KOUTROUFIOTIS' LEMMA ([JK]) . *Let M and $\overline{M}$ be Riemannian manifolds with $\dim(M) < \dim(\overline{M})$ and assume that the scalar curvature of M is bounded from below. If there is an isometric immersion $\psi : M \to \overline{M}$ satisfying $|H| \leq H_0$ and $\psi(M) \subset B$, with B compact, then the sectional curvature of M is bounded in absolute value.*

If there is some x such that $(\beta(\rho) + m|H|)(x) \geq 0$, then the theorem is obvious. If not, the Jorge–Koutroufiotis Lemma allows us to apply Omori's Lemma to $g = f(r) = -\frac{c_\lambda(r)}{\lambda}$, and we have that for every $x \in M - \mathbb{RP}^n$ and every $\epsilon > 0$, there is an $x' \in M$ such that

$$f(r(x')) \geq f(r(x)) \quad \text{and} \quad \Delta f(r)(x') = -\sum_{i=1}^{m} \nabla^2 f(r)(x')(e_i, e_i) > -\epsilon m.$$

Combining these equations with (4), we get

$$-\frac{c_\lambda}{\lambda}(r(x')) \geq -\frac{c_\lambda}{\lambda}(r(x)),$$

which is equivalent to $r(x') \geq r(x)$ and to

$$s_\lambda(r(x')) \geq s_\lambda(r(x)) = C > 0,$$

and

$$s_\lambda(r(x'))(\beta(\rho) + m|H|)(x') \geq \Delta f(r)(x') > -\epsilon m.$$

From this follows Theorem 1.2, in fact, we have $(\beta(\rho) + m|H|)(x) < 0$ for every $x \in M$, then $s_\lambda(r(x')) > s_\lambda(r(x)) = C > 0$ implies that

$$C(\beta(\rho) + m|H|)(x') \geq s_\lambda(r(x'))(\beta(\rho) + m|H|)(x') > -\epsilon m,$$

and this for every $\epsilon > 0$, then

$$C \sup\{(\beta(\rho) + m|H|)(x); x \in M\} \geq 0 \quad \text{and} \quad \sup\{m|H|(x); x \in M\} \geq |\beta(\rho)|.$$

References

[Am] JU. AMINOV, The exterior diameter of an immersed Riemannian manifold, *Mat. Sb.,* **134.** (1973) 456–460 (Russian); Engl. Transl.: *Math. USSR-Sb.,* **21** (1973), 449–454.

[CGM1] F. J. CARRERAS, F. GIMÉNEZ and V. MIQUEL, Immersions of compact Riemannian manifolds into a ball of a complex space form, *Math. Zeit.,* **225** (1997), 103–113.

[CGM2] F. J. CARRERAS, F. GIMÉNEZ and V. MIQUEL, Bounded mean curvature isometric immersions of a compact Riemannian manifold into a tube, *Glasgow Math. J.*, **40** (1998), 97–107.

[CR] T. E. CECIL and P. J. RYAN, Local sets and real hypersurfaces in Complex Projective Space, *Transactions A. M. S.,* **269** (1982), 481–499.

[Gi] F. GIMÉNEZ, Estimates for the curvature of a submanifold which lies inside a tube, *Journal of Geometry,* **58** (1997), 95–105.

[Ha] TH. HASANIS, Isometric immersions into spheres, *J. Math. Soc. Japan.,* **3** (1981), 551–555.

[HK] TH. HASANIS and D. KOUTROUFIOTIS, Immersions of bounded mean curvature, *Arch. Math.,* **33** (1979), 170–171.

[JK] L. JORGE and D. KOUTROUFIOTIS, An estimate for the curvature of bounded submanifolds, *American J. of Math.,* **103** (1981), 711–725.

[JX] L. JORGE and F. XAVIER, An Inequality between the Exterior-Diameter and the Mean Curvature of Bounded Immerslons, *Math Zeit.,* **178** (1981), 77–82.

[Ma] S. MARKVORSEN, A Sufficient Condition for a Compact Immersion to be Spherical, *Math. Zeit.,* **183** (1983), 407–411.

[Om] H. OMORI, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan,* **19** (1967), 205–214.

ANNALES UNIV. SCI. BUDAPEST., **41** (1998), 63–91

APPLICATIONS OF FRÖLICHER SPACES TO COSMOLOGY

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Introduction

In a previous paper (2) we compared differential spaces, see [10] and [11], with Frölicher spaces. Differential spaces have a significant history in the study of cosmology, see [6], [10] and [5]. Frölicher spaces (otherwise known as smooth spaces) have been studied by Frölicher [3]. Both differential spaces [2] and Frölicher spaces are topological over sets [1] and one can work with them in a way comparable to the way one works with topological spaces. From the fact that Frölicher spaces and differential spaces are topological over sets, both Frölicher spaces and differential spaces have all initial structures, products, equalizers, subobjects, final structures, quotients and coproducts. Every Frölicher space can be viewed as a differential space. More precisely, the category $\mathcal{FRL}$ of Frölicher space is a full subcategory of the category of differential spaces $\mathcal{DSP}$. However, subspaces and products of Frölicher spaces are not the same as subspaces and products of these Frölicher spaces when they are regarded as differential spaces (see [2] for the last two statements).

A category is a collection of mathematical objects (e.g., groups) with associated maps (e.g., homomorphisms). For categories, see [8]. We will attempt to make this paper reasonably independent of category theory.

For easy reference, we now define differential and Frölicher spaces. Let $\mathcal{F}$ be a collection of real valued functions on the set X, $\mathcal{T}$ the initial topology on X defined by these functions and $\mathcal{C}$ be a collection of maps $\mathbf{R} \to X$.

DEFINITION 0.1. The pair $(X, \mathcal{F})$ is a *differential space* with topology $\mathcal{T}$ if

- for each open covering $\{U_i\}_{i\in K}$ of X and function $f : X \to \mathbf{R}$, if $g|_{U_i} = h_i|_{U_i}$ for some $h_i : X \to \mathbf{R} \in \mathcal{F}$ and each $i \in K$, then $g \in \mathcal{F}$.
- if $f_1,\ldots,f_n$ is a collection of functions in $\mathcal{F}$ and $g(x_1,\ldots,x_n)$ is a smooth real-valued function (i.e., infinitely differentiable), then $g \circ (f_1,\ldots,f_n)$ is again in $\mathcal{F}$.

A map $f : (X,\mathcal{F}_X) \to (Y,\mathcal{F}_Y)$ if differential spaces is a map $f : X \to Y$ such that if $f^*\mathcal{Y}_Y$ is the set of all $g \circ f$ with $g \in \mathcal{F}_Y$, then $f^*\mathcal{F}_Y \subset \mathcal{F}_X$. The set of such maps makes differential spaces into a category $\mathcal{DSP}$. A map in $\mathcal{DSP}$ will be called a *differentiable map*. An element of $\mathcal{F}_X$ will be called a *scalar*.

We think of scalars as measuring some feature of X (e.g., temperature or pressure).

The elements of $\mathcal{F}$ determine the differentiable curves $\mathbf{R} \to X$. For a Frölicher space, the differentiable curves $\mathbf{R} \to X$ must determine the scalars on X. More precisely, we define:

DEFINITION 0.2. The triple $(X,\mathcal{F},\mathcal{C})$ is a *Frölicher space* with topology $\mathcal{T}$ if

- if $\mathcal{F} \circ \mathcal{C}$ is the set of all $f \circ g$ such that $f \in \mathcal{F}$ and $g \in \mathcal{C}$ and $\mathcal{M}$ is the sets of smooth maps $\mathbf{R} \to \mathbf{R}$, then $\mathcal{F} \circ \mathcal{C} \subset \mathcal{M}$.
- $\Gamma\mathcal{C} = \{f : A \to \mathbf{R} \,|\, f \circ c$ belongs to $\mathcal{M}$ for all $c \in \mathcal{C}\} = \mathcal{F}$.
- $\Phi\mathcal{F} = \{c : \mathbf{R} \to A \,|\, f \circ c \in \mathcal{M}$ for all $f \in \mathcal{F}\} = \mathcal{C}$.

The elements of $\mathcal{C}$ are often called *contours*.

Again, as shown in [2], if $(X,\mathcal{F},\mathcal{C})$ is a Frölicher space, then $(X,\mathcal{F})$ is a differential space.

DEFINITION 0.3. A map $f : (X,\mathcal{F}_X,\mathcal{C}_X) \to (Y,\mathcal{F}_Y,\mathcal{C}_Y)$ is a map of Frölicher spaces and said to be *smooth* of $f : (X,\mathcal{F}_X) \to (Y,\mathcal{F}_Y)$ is a differentiable map of differential spaces. The category of Frölicher spaces is denoted by $\mathcal{FRL}$.

In this paper, in section 1, we develop a geometric setting for polar coordinates and use the notions developed to introduce a geometrical setting for cosmic strings such as one finds in [5]. After introducing tangent bundles and vector fields for Frölicher spaces, we discuss the Frölicher space structure of the quotient of $\mathbf{R}$ by the equivalence relation identifying -1 with 1 and then examine the Frölicher structure of the the part of $\mathbf{R}^n$ defined by $x_1 \geq 0$ with a view to applications to manifolds with boundaries. We form the quotient L

which identifies points of the form (r,θ) according to the usual conventions of polar coordinates. We would like to identify L with the Frölicher subspace of $\mathbf{R}^3$ defined by $z^2 = x^2 + y^2$, a cone. We let however V (see Example 7) denote the Frölicher space whose underlying set is the subset of $\mathbf{R}^3$ defined by $z^2 = x^2 + y^2$ but with the Frölicher space structure induced from a suitable natural bijection from L to $\{(x,y,z) \mid z^2 = x^2 + y^2\}$. It is the Frölicher space V that we use for examples in our later work. Our approach shows how one can work with quotients even when one can not identify them with a naturally corresponding Frölicher subspace of $\mathbf{R}^n$. The product $\mathbf{R} \times V \times \mathbf{R}$ in $\mathcal{FRL}$ is then the space time $\mathbf{S}$ of a cosmic string. The space time $\mathbf{S}$ is a quotient of $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$. In section 2 we start by showing that our "natural" definition of type $(0,s)$-tensor bundle for a Frölicher space X is smooth isomorphic to the usual $(0,s)$-tensor bundle on X if X is a finite dimensional smooth manifold. We show that the metric $dt^2 + dr^2 + rd\theta + dw^2$ induces a local cone quasi-Riemannian metric, appropriately defined, on the space time $\mathbf{S}$ of a cosmic string. In section 3 we start by characterizing vector fields on V and then define metric connections for arbitrary Frölicher spaces. For the polar Riemannian metric and the Poincare metric on V, the connection coefficients are found. The Riemannian curvature tensor, the Ricci tensor and the scalar curvature are defined for an arbitrary Frölicher space A and zero in these eases. These definitions require a suitable trace function on the set of linear maps $\chi[A] \to \chi[A]$ where $\chi[A]$ is the set of vector fields on A. The Einstein tensor can then be introduced. In section 4 we extend the results of section 3 to the space time $\mathbf{S}$ of a cosmic string and determine geodesics for the Levi–Civita and Poincare metrics on $\mathbf{S}$.

This work shows that one can in a fairly natural way replace the Levi–Civita metric on $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$ by a non-singular metric on the singular (or non-smooth?) Frölicher space $\mathbf{S}$. Since the vector fields on $\mathbf{S}$ vanish at its singularities, the study of connections and their attendant notions become, aside from a change in scalars, the study of connections on an open subset of $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$ for a non-coordinate basis.

1. Tangent bundles: some examples

Let $(X, \mathcal{F}_X)$ be a differential space. See [2] for additional detail and examples related to this section. A *tangent vector* V at $P \in X$ is a derivation

$V : \mathcal{F}_X \to \mathbf{R}$ at P, i.e. a linear map such that

$$V(fg) = f(P)V(g) + g(P)V(f).$$

The set of such vectors form the tangent space TX_P to X at P. Let $c \in \Gamma\mathcal{F}_X$, $f \in \mathcal{F}_X$ suppose that $c(0) = P$. Suppose that V_c is the derivation defined by setting

$$V_c = \lim_{t\to 0} \frac{f \circ c(t) - f \circ c(0)}{t}.$$

Then, TCX_P is the set of all V_c such that $c \in \Gamma\mathcal{F}_X$ and $c(0) = P$. We call TCX_P the *tangent cone* to X at P.

We now use the fact that the category $\mathcal{FRL}$ is Cartesian closed [3]. Let X and Y be Frölicher spaces and $(X, Y) = \{f : X \to Y \mid f \text{ is smooth}\}$. Then, because $\mathcal{FRL}$ is Cartesian closed, (X, Y) has the structure of a Frölicher space in a natural way. In particular,

$\mathcal{F}_X = (X, \mathbf{R})$ will be considered to be a Frölicher space.

DEFINITION 1.1. The sets

$$\mathcal{D}_X = \{D : \mathcal{F}_X \to \mathbf{R} \mid D \text{ is a derivation at some } P \in X\}$$

and $\mathcal{D}_{X,\mathcal{C}}$, which consists of all $V_c \in \mathcal{D}_X$ such that $c \in \mathcal{C}_X$, $c(0) = P$ and $P \in X$ cal be viewed as Frölicher subspaces of $(\mathcal{F}_X, \mathbf{R})$. The *tangent bundle* TX (resp., *tangent cone bundle* TCX) on X is then the Frölicher subspace of $X \times \mathcal{D}_X$ (resp., $X \times \mathcal{D}_{X,\mathcal{C}}$) consisting of all (P, D) such that D is a derivation at P. The projection map $\pi : TX \to X$ (resp., $\Pi : TCX \to X$) is the smooth map sending (P, D) to P. A *vector field* on X is (most properly) a smooth section of Π or (more generally) π. We let $TX_P = \pi^{-1}(P)$ and $TCX_P = \Pi^{-1}(P)$.

We note that if A is a subset of a Frölicher space X, with inclusion i_A: $A \to X$, then A acquires a Frölicher structure by letting $\mathcal{C}_A$ be the set of all $c : \mathbf{R} \to A$ such that $i_A \circ x \in \mathcal{C}_X$.

EXAMPLE 1. In [2] we showed that $T\mathbf{R}^n$ was smooth isomorphic over $\mathbf{R}^n$ too, in effect, the usual tangent bundle by a map $v : T\mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ where if $(P, D) \in TX$ and $D = \sum_{i=1}^{n} d_i \mathbf{e}_i$, using the notation in [9], then $v(P, D) = (P, (d_i))$.

EXAMPLE 2. Let $\sim$ be the smallest equivalence relation on $\mathbf{R}$ for which $-1 \sim 1$. Then, we write $\mathbf{R}/\sim = \mathbf{R}/\{-1, 1\} = \mathcal{D}$ and let $q : \mathbf{R} \to \mathcal{D}$ be the quotient map. Then, f is a scalar on $\mathscr{D}$ if $f \circ q : \mathbf{R} \to \mathbf{R}$ is smooth. One can view $\mathscr{D}$ as a cubic curve defined by $y^2 = x^2(1+x)$ and having one singularity which is a node. In that case, one can write $q(t) = (t^2 - 1, t(t^2 - 1))$ and then f must be smooth on each branch of $\mathscr{D}$. Let $P = q(1) \in \mathscr{D}$. It is clear that $TC\mathscr{D}_P$ consists of two lines meeting at the origin and $T\mathscr{D}_P$ is a two dimensional vector space. Suppose that $\alpha : \mathbf{R} \to \mathbf{R}$ is an increasing smooth (braking) function satisfying $\alpha(t) = 0$ for $|t| \le \frac{1}{8}$ and $\alpha(t) = 1$ for $|t| \ge \frac{1}{2}$. Define a vector field $V_1 : \mathscr{D} \to TC\mathscr{D} \subset T\mathscr{D}$ by setting $V_1(q(t)) = \left(q(t), \alpha(t^2 - 1)\frac{\partial}{\partial t}\right)$. Let ρ_2 denote projection onto the second factor. One has, for a smooth function $g : \mathscr{D} \to \mathbf{R}$,

$$\rho_2 \circ V_1(q(a))(g) = \alpha(a^2 - 1)\frac{\partial g \circ q}{\partial t}\bigg|_{t=a}$$

and $\rho_2 \circ V_1(q(t))$ is 0 in a neighborhood of $a = \pm 1$. The direction in which flow occurs on $\mathscr{D}$ for V_1 is from $t = -\infty$ to ∞. Another vector field $V_2(q(t))$ on $\mathscr{D}$ is given by

$$\rho_2 \circ V_2(q(a))(g) = \begin{cases} \alpha(a^2 - 1)\frac{\partial g \circ q}{\partial t}\Big|_{t=a} & \text{if } a \le 1 \\[2ex] -\alpha(a^2 - 1)\frac{\partial g \circ q}{\partial t}\Big|_{t=a} & \text{if } a \ge 1. \end{cases}$$

The direction in which flow occurs around $\mathscr{D}$ is cyclical going from $-\infty$ to $q(-1)$ to $q(1)$ and then around the path from $q(1)$ to $q(-1)$, repeatedly. We note that any smooth vector field on $\mathscr{D}$ must be 0 at $q(1)$ and hence $q(1)$ is a sink.

EXAMPLE 3. We will provide below in Example 7 a geometrical basis for polar coordinates. As a beginning, let $\approx$, be the smallest equivalence relation on $\mathbf{R}^2$ such that $(x, y) \approx (x', y')$ if and only if $x = x'$ and $y = y' + 2\pi n$ for some $n \in \mathbf{Z}$, the integers. Consider the quotient $\overline{q} : \mathbf{R}^2 \to \mathbf{R}^2/\approx\, = S$. The set S can be identified with the cylinder $K = \{(x, y, z) \mid x^2 + y^2 = 1\}$ by the map $\kappa : S \to K$ sending $q(x, y) \to (\cos(y), \sin(y), x)$. The set S has the initial topology and Frölicher structure determined by the set of all $f : S \to \mathbf{R}$ such that $f \circ q$ is smooth on $\mathbf{R}^2$. The set K is a submanifold of $\mathbf{R}^3$ and one readily sees that K is a homeomorphism. Since $\kappa' : \mathbf{R}^2 \to K$, defined by setting $\kappa'(x, y) = (\cos(y), \sin(y), x)$, is smooth, $\overline{q}$ is a quotient map in $\mathscr{FRL}$ and $\kappa' = k \circ \overline{q}$ is a smooth map. Since κ' and q are locally smooth isomorphisms,

κ^{-1} must be smooth. Thus, $\kappa : S \to K$ is a smooth isomorphism and $\overline{q}$ can be taken to be κ'.

EXAMPLE 4. One would like to handle manifolds with boundary in the context of Frölicher spaces. For this purpose, we consider the set $\mathbf{R}_+\{r \in \in \mathbf{R} \mid r \geq 0\}$ and endow $\mathbf{R}_+$ with its Frölicher subspace structure as a subset of $\mathbf{R}$. The contours $\mathscr{C}_{\mathbf{R}_+}$ on $\mathbf{R}_+$ are those curves $c : \mathbf{R} \to \mathbf{R}_+$ such that, if $i_{\mathbf{R}_+} : \mathbf{R}_+ \to \mathbf{R}$ is the inclusion, then $i_{\mathbf{R}_+} \circ c$ is a smooth curve. Let $\mathscr{F}_+$ consist of functions $f : \mathbf{R}_+ \to \mathbf{R}$ which are right smooth at 0 but otherwise smooth in the usual sense. The following result shows that $\mathscr{F}_{\mathbf{R}_+}$ is as nice as one might expect.

PROPOSITION 1.1. $\mathscr{F}_{\mathbf{R}_+} = \mathscr{F}_+$.

PROOF. It is easy to show first that $\mathscr{F}_+ \circ \mathscr{C}_{\mathbf{R}_+} \subset \mathscr{M}$ and hence that $\mathscr{F}_+ \subset \subset \mathscr{F}_{\mathbf{R}_+}$. The inclusion $\mathscr{F}_{\mathbf{R}_+} \subset \mathscr{F}_+$, and thus the validity of the proposition, then follows as a consequence of the following result to be found (here stated with some simplification) in section 17 of [7]:

PROPOSITION 1.2. (Smooth maps on subsets with collars.) *Let M and E be manifolds of the same dimension and M be a submanifold of E with boundary $\partial M \neq \emptyset$. Then, every smooth map $f : M \to \mathbf{R}$ in the sense of Frölicher spaces extends to an open neighborhood in E of ∂M.*

EXAMPLE 5. We continue with the previous example and show that $\mathbf{R}_+$ can be viewed as $\mathbf{R}$ folded in half in $\mathscr{FRL}$. Let $\mathscr{L}$ be the smallest equivalence relation on $\mathbf{R}$ which identifies any number a with $-a$ and $g_* : \mathbf{R} \to \mathbf{R}/\mathscr{L} = \mathbf{R}$ the quotient in $\mathscr{FRL}$. We prove:

PROPOSITION 1.3. *The quotient $\mathbf{R}/\hbar$ is isomorphic to $(\mathbf{R}_+, \mathscr{F}_{\mathbf{R}_+}, \mathscr{C}_{\mathbf{R}_+})$ in $\mathscr{FRL}$.*

PROOF. Let $\eta : \mathbf{R} \to \mathbf{R}_+$ be a function defined by setting $\eta(x) = x^2$. Clearly, η is smooth. Thus, as q_* is a quotient map, there is a unique map $\nu : \mathbf{R}/\sim \to \mathbf{R}_+$ satisfying $\nu \circ q_* = \eta$. Note that $(q_*, (x)) = x^2$. Clearly, ν is a bijection and one need only show that ν^{-1} is smooth. The smooth maps on $\mathbf{R}/\sim$ are those functions $f : \mathbf{R}/\sim \to \mathbf{R}$ such that $f \circ q$, is smooth on $\mathbf{R}$. As $f \circ q_*$, is a smooth even function, one can apply the result of H. WHITNEY (see [13]) which says that every even function $h : \mathbf{R} \to \mathbf{R}$ can be written $h(t) = g(t^2)$ where $g : \mathbf{R} \to \mathbf{R}$ is smooth. Thus, if $h(t) = f \circ q_*(t)$ and $g\,|_{\mathbf{R}+}$

denotes the restriction of g to $\mathbf{R}_+$, $f \circ \nu^{-1}(t) = f \circ \nu^{-1}(\eta(\sqrt{t})) = f \circ q_*(\sqrt{t}) =$ $= g\ |_{\mathbf{R}_+}(t) \in \mathcal{F}_{\mathbf{R}_+}$ and, applying the fact that $\mathcal{F}_{\mathbf{R}_+} = \mathcal{F}_{\mathbf{R}_+,x} \in \mathcal{C}_{\mathbf{R}_+}$, our proof is complete. ∎

REMARK. For $f \in \mathcal{F}_{\mathbf{R}_+,c} \in \mathcal{C}_{\mathbf{R}_+}$ and with $f(0) = c(0) = 0$, since $f \circ c(t)$ is an even function, $\lim\limits_{t\to 0} \frac{f\circ c(t)}{t} = 0$. Hence, $(TC\mathbf{R}_+)_o = 0$. On the other hand, as $\mathcal{F}_{\mathbf{R}_+} = \mathcal{F}_+$, it follows that $\frac{\partial}{\partial}\Big|_{t=0}$ is a bass for $(T\mathbf{R}_+)_o$.

Let now $\mathbf{R}_+{}^n = \mathbf{R}_+ \times \mathbf{R}^{n-1}$ have the Frölicher subspace structure inherited from $\mathbf{R}^n$. As above, using the fact that the product of a quotient map and identity map is a quotient map in the Cartesian closed category $\mathcal{FRL}$, one can show:

PROPOSITION 1.4. *The set $\mathcal{F}_{\mathbf{R}_+{}^n}$ of scalars on $\mathbf{R}_+{}^n$ is equal to the set $\mathcal{F}_+{}^n$ of maps $f : \mathbf{R}_+{}^n \to \mathbf{R}$ such that f is continuous, f is smooth on the set of points satisfying $x_1 > 0$ and the partials of f, taken from the right (resp., just taken) at a point of a hiperplane $x_1 = 0$ in terms of the first variable (resp., variables other than the first), exist and are continuous on $\mathbf{R}_+{}^n$.*

As for $\mathbf{R}_+$, $\mathbf{R}_+{}^n$ is $\mathbf{R}^n$ folded in two ($n \geq 1$).

EXAMPLE 6. The tangent cone space $(TC\mathbf{R}_+{}^n)_P$ to a point P of $\mathbf{R}_+$ having first coordinate 0 is isomorphic to $\mathbf{R}^{n-1}$. The tangent space $(T\mathbf{R}^n)_P$ for $P \in \mathbf{R}_+{}^n$ is isomorphic to $\mathbf{R}^n$.

EXAMPLE 7. The notation in this example plays an important role in the next section. A set map may define different smooth maps depending on the Frölicher structure of domain and codomain. To avoid any possible ambiguity, we will label the set maps differently corresponding to different domains or codomains.

Let $\simeq$ be the equivalence relation defined on $\mathbf{R}^2$ which is generated by setting $(0,\theta) \simeq (0,\theta')$ for all $\theta,\theta' \in \mathbf{R}$ and $(r,\theta) \simeq (r,\theta+2\pi)$ for all $r,\theta \in \mathbf{R}$. Define a smooth map $\rho : \mathbf{R}^2 \to \mathbf{R}^3$ by setting $\rho(r,\theta) = (r\cos\theta, r\sin\theta, r)$. Notice that $A \simeq B$ implies $\rho(A) = \rho(B)$. Since $\mathcal{FRL}$ has final structures, there is a Frölicher quotient structure on $\mathbf{R}^2/\simeq$ and quotient map $q : \mathbf{R}^2 \to$ $\to \mathbf{R}^2/\simeq$ in $\mathcal{FRL}$ such that, for a unique smooth map j_V, the following diagram commutes:

$$\begin{tikzcd} \mathbf{R}^2 \arrow[r, "q"] \arrow[dr, "\rho"'] & \mathbf{R}^2/\simeq \arrow[d, "j_V"] \\ & \mathbf{R}^3 \end{tikzcd}$$

Fig. 1.

The map j_V is a set bijection onto the cone $V=\{(x,y,z)\mid z^2=x^2+y^2\}$. We will endow V with the quotient structure on $\mathbf{R}^2 \simeq$ that it obtains from the bijection j_V and not necessarily the subspace structure that it inherits from $\mathbf{R}^3$. In this way, the quotient map ρ (unique up to isomorphism) is just the map $q:\mathbf{R}^2 \to V$, equal to ρ as a set map, and the inclusion $j_V : V \to \mathbf{R}^3$ is a smooth map but V is not necessarily a subspace of $\mathbf{R}^3$ under j_V. We will call q the *polar quotient* (map). The map $q:\mathbf{R}^2-\{(r,\theta)\mid r\neq 0\}\to V-\{(0,0,0)\}$ is locally a smooth isomorphism. Of course, although it is not necessary for our arguments, it would be nice to show that V was a subspace of $\mathbf{R}^3$ which we think likely but have found difficult to prove. The cone V with its quotient structure can be viewed as a geometric setting for polar coordinates.

Let $\underline{V}$ denote $\{(x,y,z)\mid x^2+y^2=z^2\}$ with the structure that it has as a Frölicher subspace of $\mathbf{R}^3$ and $i_V : V \to \underline{V}$ denote the identity map. The map $\rho_1 : \mathbf{R}^2 \to \underline{V}$, where $\rho_1(x)=\rho(x)$ for all $x \in \mathbf{R}^2$, is easily seen to be smooth. Thus, suppose $c \in \mathscr{C}_{\mathbf{R}^2}$. As $\rho \circ c : \mathbf{R} \to \mathbf{R}^3$, the composite of two smooth maps, is smooth and $\rho\circ c(t)\in\{(x,y,z)\mid x^2+y^2=z^2\}$ for each $t\in\mathbf{R}$, $\rho_1 \circ c=\rho\circ c \in \mathscr{C}_{\underline{V}}$ and it follows that ρ_1 is smooth. Since q is a quotient map and as a set map agrees with ρ, i_V is the unique smooth map such that $i_V \circ q=\rho_1$.

Most of the important smooth maps that we have defined and will use later appear in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{R}^2 \arrow[r, "q"] \arrow[dr, "\rho_1"] \arrow[ddr, "\rho"'] & V \arrow[d, "i_V"] \\
& \underline{V} \arrow[d, "k_V"] \\
& \mathbf{R}^3
\end{tikzcd}$$

Fig. 2.

where k_V is the smooth inclusion of $\underline{V}$ into $\mathbf{R}^3$ and $(k_V)_* : TC\underline{V} \to TC\mathbf{R}^3 = = t\mathbf{R}^3$ is a one-one fiber preserving map. We also write $j_V = k_V \circ i_V$.

Let K be the cylinder (see Example 3) and $\equiv$ the equivalence relation on K where $(x, y, 0) \equiv (x', y', 0)$ for points (x, y), $(x', y') \in S^1$, the unit circle. Consider the quotient $K/\equiv$ of K by $\equiv$ and the quotient map $\hat{q} : K \to K/\equiv$ in $\mathcal{FRL}$. Let $u^* : K \to \mathbf{R}^3$ be the smooth map $K \to \mathbf{R}^3$ defined by setting $u^*(x, y, z) = (zx, zy, z)$. Since u^* collapses $S^1 \times 0$ to a point, there is thus a unique smooth map $\gamma : L \to \mathbf{R}^3$ which is a bijection onto $\{(x, y, z) \mid z^2 = x^2 + + y^2\}$ making the following diagram commutative.

$$\begin{tikzcd}
K \arrow[r, "\hat{q}"] \arrow[dr, "u^*"'] & \mathbf{R}^2/\equiv \arrow[d, "\gamma"] \\
& \mathbf{R}^3
\end{tikzcd}$$

Fig. 3.

As above, let W be the set $\{(x, y, z) \mid x^2 + y^2 = z^2\}$ but with the quotient structure obtained from $\hat{q}$ via γ and where, as a set map, $\hat{q} = u^*$.

Let κ' be the map defined in Example 3. Using the fact that q is a quotient map in $\mathcal{FRL}$, one obtains a unique smooth bijection $\delta : V \to W$ such that $\hat{q} \circ \kappa = \delta \circ q$. Using the fact that κ' is a local isomorphism, one obtains a unique smooth map $q'' : K \to V$ such that $q'' \circ \kappa' = q$. One checks to see that $P \equiv Q$ implies $q''(P) = q''(Q)$. Using the fact that $\hat{q}$ is a quotient, it follows that there is a unique smooth bijection $\epsilon : W \to V$ such that $e \circ \hat{q} = q''$. One checks that $\delta \circ e \circ \hat{q} \circ \kappa'$ and hence, since $\hat{q} \circ \kappa'$ is surjective, $\delta \circ \epsilon$ is the identity. Similarly, $e \circ \delta \circ q = \epsilon \circ \hat{q} \circ \kappa' = q$ and since q is a surjection, $\epsilon \circ \delta$ is the identity. It follows, as one would expect, that V is isomorphic to W in $\mathcal{FRL}$.

EXAMPLE 8. The representation of physical situations is often very natural using Frölicher structures. Consider a slit in the reflective wall $x_1 = 0$ at $x_2 = 0$ in $\mathbf{R}^2$. Then, the contours (light rays) either

- follow straight lines on either side which when they meet the wall are reflected at equal angle. In particle terms, at a point of reflection, the speed of the contour must be zero.
- are the straight lines which pass through $(0, 0)$ to the other side of the wall.

In this case the scalars need only be smooth along each line and the points of the reflective wall act like points of the boundary of $\mathbf{R_+}^2$. Clearly, there are many useful permutations on the description just given.

2. The singularity of a cosmic string

This section is concerned in part with some of the notions dealt with by GRUSZCZAK, HELLER, SASIN [5] and HELLER, SASIN [4]. Let

$$q^{st} = I_{\mathbf{R}} \times q \times I_{\mathbf{R}} : \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R} \times V \times \mathbf{R}$$

where q is the polar quotient (see Example 7) and $I_{\mathbf{R}}$ is the identity on $\mathbf{R}$. Since the product of a quotient map and an identity map in a Cartesian closed category is a quotient map, q^{st} is a quotient map. The subvariety and Frölicher subspace $\mathbf{S} = \mathbf{R} \times V \times \mathbf{R}$ of $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$ is called in [5] in the case of differential spaces “an ‘extension’ of the space-time manifold of the cosmic string onto its singularity”. The singular locus $sing(\mathbf{S})$ of $\mathbf{S}$ is the set $\mathbf{R} \times \{(0, 0, 0)\} \times \mathbf{R}$. We wish to show that the Levi–Civita metric [5]

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2$$

on $\mathbf{R}^4$ is induced by a metric on $\mathbf{S}$.

DEFINITION 2.1. Let $(\pi^s)^* : \underbrace{TX \oplus TX \oplus \ldots \oplus TX}_{s\text{-times}} \to X$ be the pullback of $\underbrace{\pi \times \pi \times \ldots \times \pi}_{s\text{-times}} : TX \times TX \times \ldots \times TX \to X^s$ along the diagonal map $\Delta : X \to X^s$ which is clearly smooth. The set $\oplus^s TX = \underbrace{TX \oplus TX \oplus \times \oplus TX}_{s\text{-times}}$ is usually called the s-fold *Whitney-sum* of TX.

Let $f : X \to Y$ be an arbitrary smooth map. Then, f induces in turn (using the Cartesian closedness of $\mathcal{FRL}$),

1. a smooth map $f' : (Y, \mathbf{R}) \to (X, \mathbf{R})$ or using alternate notation $f^* : \mathcal{F}_Y \to \to \mathcal{F}_X$,
2. a smooth map $f^{**} : Der(\mathcal{F}_X) \to Der(\mathcal{F}_Y)$ where $Der(\mathcal{F}_X)$ denotes the set of all derivations on the ring $\mathcal{F}_X$ at some point of X,
3. a smooth map
$$F \times f^{**} = f_* : X \times Der(\mathcal{F}_X) \to Y \times Der(\mathcal{F}_Y)$$
and thus a smooth map $f_* : T_X \to T_Y$ such that, for the projections $\pi_X : TX \to X$ and $\pi_Y : TY \to Y$, $\pi_Y \circ f_* = f \circ \pi_X$,
4. a smooth map
$$f^*_{*s} : \mathbf{MULTILIN}(\oplus^* TY; \mathbf{R}) \to \mathbf{MULTILIN}(\oplus^s TX; \mathbf{R})$$
where, for each $P \in X$, $\mathbf{MULTILIN}(\oplus^s TX; \mathbf{R})$ consists of all smooth maps $f : \oplus^s TX \to \mathbf{R}$ such that $f\,|_P : \oplus^s TX_P \to \mathbf{R}$ is a s-multilinear map and $f^*_{*s}(g) = g(\underbrace{f_*, \ldots, f_*}_{s\text{-times}})$. We write **LIN** instead of **MULTILIN** if $s = 1$.

With suitable modification, one obtains corresponding notions on replacing TX by TCX and TY by TCY. Thus, a map $h : TCX \to \mathbf{R}$ is linear if, for each $p \in X$, $h \mid |_P(c_1 T_1 + c_2 T_2) = c_1 h\,|_P\,(T_1) + c_2 h\,|_P\,(T_2)$ whenever $TCP\,|_P$ and $c_1, c_2 \in \mathbf{R}$.

At this stage, our map theoretic discussion, using the Cartesian closedness of $\mathcal{FRL}$, comes screechingly to a halt and, to proceed further, we introduce the tensor bundles $T^0_s X$ of type $(0, s)$ among whose sections, when $s = 2$, are those that one can consider to be quasi-Riemannian metrics.

DEFINITION 2.2. The *tensor bundle* $T^0_s(T)$ of *type* $(0, s)$ is equal the Frölicher space
$$X \times \mathbf{MULTILIN}(\oplus^s TX; \mathbf{R}) / \approx$$

where $(P,f) \approx (Q,g)$ if and only if $P = Q$ and there is a neighborhood U of P such that

$$f\mid_{((\pi^s)^*)^{-1}(U)} = g\mid_{((\pi^s)^*)^{-1}(U)}.$$

The equivalence class of (P,f) will be denoted by $[P,f]$ or $[(P,f)]$. Replacing TX by TCX, one obtains the tensor cone bundles. The projections π_s^0 and Π_s^0 of the tensor and tensor cone bundle are the evident ones. A section s of π_s^0 or Π_s^0 is called a $(0,s)$-*tensor field* if, for each $P \in X$, there is a neighborhood U of P such that $s(Q) = [(Q,K)]$ for some $K \in \in \mathbf{MULTILIN}(\oplus^s TX;\mathbf{R})$ and all $Q \in U$. An 1-*form* is a $(0,1)$-tensor field. a *local cone quasi-Riemannian metric* g is a $(0,2)$-tensor field such that, if g_P is the restriction of g to $(\Pi_2^0)^{-1}(P)$, then

- $g_P(U,V) = g_P(V,U)$.
- The induced preserving map, arising from the Cartesian closedness of $\mathcal{FRL}$ and thus smooth, $g_* : TCX \to T_1^0CV$, where $(g_*)_P(X)(Y) = = g_P(X,Y)$, X, $Y \in \{\pi^{-1}(P)$ and $P \in X$, is a smooth isomorphism.

A *local quasi-Riemannian metric* is defined by taking π_s^0 instead of Π_2^0, TX instead of TCX.

In [2], we have shown, if X is a smooth finite dimensional manifold, that there is a smooth isomorphism between T_1^0X and the usual tensor bundle of type $(0,1)$. We now extend this result.

PROPOSITION 2.1. *Let X be a finite dimensional smooth manifold. Then, the usual tensor bundle of type $(0,s)$ is smooth isometric to T_s^0X.*

PROOF. Let $X = \mathbf{R}^n$ and $\eta : X \times \mathbf{MULTILIN}(\oplus^s TX;\mathbf{R}) \to T_s^0X$ be the quotient map. A map $\tau : \mathbf{R} \to T_s^0X$ is smooth if and only if $f \circ \tau$ is smooth for each smooth map $f : T_s^0X \to \mathbf{R}$. But f is smooth if and only if $f \circ \eta$ is smooth on $X \times \mathbf{MULTILIN}(\oplus^s TX;\mathbf{R})$. Let $[(P,L)]$ be an equivalence class in T^*X and

$$L\mid_{((\pi^s)^*)^{-1}(P)} = \sum_{i_1,i_2,\ldots,i_s}^{n} a_{i_1,i_2,\ldots,i_s,P}\, dx^{i_1}dx^{i_2}\ldots dx^{i_s}\mid_P . \tag{1}$$

There is then a bijection $v : T_s^0X \to \mathbf{R}^n \times \mathbf{R}^{n^s}$ sending $[(P,L)]$ to $(P,(a_{i_1,i_2,\ldots,i_s,P}))$. We wish to show that this bijection is a smooth isomorphism. Let $c : \mathbf{R} \to T_s^0X$ be a contour and write $v(t) = v \circ c(t) = = (P_t,(a_{i_1,i_2,\ldots,i_s,P_t}(t)))$. Using the fact that η is a quotient, one sees that there is a set map $\psi : T^0_s X \to X$ such that $\psi \circ \eta = \lambda_1$ where λ_1 is projection onto the first factor X. As $\lambda_1(P_t, L_t) = P_t$ is smooth in t and $\psi = P_1 \circ v$, where P_1 is the projection of $\mathbf{R}^n \times \mathbf{R}^n$ onto the first of two factors, v is smooth in its first coordinate. Note that this argument shows that π^0_s is smooth.

Define $\theta_{i_1,i_2,\dots,i_s} : X \times \mathbf{MULTILIN}(\oplus^s TX; \mathbf{R}) \to \mathbf{R}$ by setting $\theta_{i_1,i_2,\dots,i_s}(P, L) = a_{i_1,i_2,\dots,i_s,P}$ using (1). Again, as η is a set quotient, there is a map $\rho_{i_1,i_2,\dots,i_s} : T^*X \to \mathbf{R}$ such that $\rho_{i_1,i_2,\dots,i_s} \circ \eta = \theta_{i_1,i_2,\dots,i_s}$. We wish first to show that $\theta_{i_1,i_2,\dots,i_s}$ is smooth. Let $d : \mathbf{R} \to X \times \mathbf{LIN}(TX, \mathbf{R})$ be a smooth map and $d(t) = (P_t, L_t)$. Then, L_t is smooth in t if and only if the associated map $\hat{L} : \mathbf{R} \oplus^s TX \to \mathbf{R}$ defined by $\hat{L}(t, v) = L_t(v)$ is smooth. Identifying $\oplus^s TX$ with $\mathbf{R}^n \times \mathbf{R}^{n^s}$ one can write

$$\hat{L}(t, P_t, w_1, \dots, w_s)) = A(t, P_t) w_1 w_2 \dots w_s$$

where $A(t, P) = (A^{i_1,\dots,i_s}(t, P_t))$ is a $(0, s)$-tensor with entries which are smooth functions of t and P; $w_1, \dots, w_s$ are vectors in TX_{P_t}; and $A(t, P) \cdot w_1 \dots \dots w_s$ is computed using ordinary tensor operations. Let $w_j = \frac{\partial}{\partial x_{i_j}}$. Then, $a_{i_1,\dots,i_s,P_t}(t) = \theta_{i_1,\dots,i_s} \circ c(t) = A(t, P_t) \cdot w_1, \dots, w_s$. Hence, the $\theta_{i_1,\dots,i_s}$ and thus also the $\rho_{i_1,\dots,i_s}$ are smooth. Since $\rho_{i_1,\dots,i_s} = P_{n+(i_1,\dots,i_s)} \circ v$, where $P_{n+(i_1,\dots,i_s)}$ is projection on the $i_1, \dots, i_s$-th coordinate of the second factor of $\mathbf{R}^n \times \mathbf{R}^{ns}$, v must be smooth.

Conversely, suppose that the $a_{i_1,\dots,i_s,P_t}(t)$ for $i_1, \dots, i_s = 1, \dots, n$ and P_t are smooth functions of t. Then, letting

$$\beta(t) = \left(P_t, \sum_{i_1,\dots,i_s=1}^{n} a_{i_1,\dots,i_s,P_t}(t) dx^{i_1} dx^{i_2} \dots dx^{i_s} |_{P_t} \right)$$

and using the identification of $\oplus^s TX$ with $\mathbf{R}^n \times \mathbf{R}^{ns}$, one sees that $\beta : \mathbf{R} \to X \times \mathbf{MULTILIN}(\oplus^s TX; \mathbf{R})$ is smooth and thus $q \circ \beta$ is smooth. Hence, v^{-1} is smooth and the conclusion sought follows.

Patching together local smooth isomorphisms, one extends to arbitrary finite dimensional smooth manifolds. ∎

We return to our smooth map $f : X \to Y$ and let $\chi^0_s[X]$ denote the collection of s-forms on X. We will define the pullback map $(f^0_s)^* : \chi^0_s[Y] \to \chi^0_s[X]$. Suppose that $\alpha : Y \to T^0_s Y$ is a s-form and $P \in Y$. Then, P

has an open neighborhood U_1 such that $\alpha(Q) = [Q, K_1]$ for fixed $K_1 \in$ $\in$ **MULTILIN**$(\oplus^s TY; \mathbf{R})$ and for $Q \in U_1$. We let $(f_s^0)^*\alpha(R) = [R, f_{*s}^* K_1]$ for $R \in f^{-1}(U_1)$. Let $Q \in U_1 \cap U_2$ and $\alpha(S) = [S, K_2]$ for $S \in U_2$ and fixed $K_2 \in$ **MULTILIN**$(\oplus^s TY; \mathbf{R})$. Then, for $R \in f^{-1}(U_1) \cap f^{-1}(U_1)$, $(f_s^0)^*\alpha(R) =$ $= [R, f_{*s}^* K_1]$ and $(f_s^0)^*\alpha(R) = [R, f_{*s}^* K_2]$ in the two representations. Because of the nature of the equivalence relation $\approx$, $K_1 \mid_{((\pi^s)^*)^{-1}(V)} = K_2 \mid_{((\pi^s)^*)^{-1}(V)}$ for some open subset $V \subset U_1 \cap U_2 (\pi : TY \to Y)$. It is easy to see that $f_{*s}^*(K_1 \mid_{((\pi^s)^*)^{-1} f^{-1}(V)} = f_{*s}^*(K_1) \mid_{((\pi^s)^*)^{-1} f^{-1}(V)}$. Thus, $[R, f_{*s}^* K_1] =$ $= [R, f_{*s}^* K_2]$ for $R \in f^{-1}(U_1 \cap U_2)$. It follows that $(f_s^0)^*\alpha$ is a $(0, s)$-form defined on X and smooth since it is smooth on an open covering of X.

Let $\alpha : X \to T_s^X$ be an s-form, $\beta : X \to T_p^X$ be a p-form and $P \in X$. There is an open set U containing P such that $\alpha(Q) = [Q, K]$ and $\beta(Q) =$ $= [Q, L]$ for $K \in$ **MULTILIN**$(\oplus^s TY; \mathbf{R})$, $L \in$ **MULTILIN**$(\oplus^s TY; \mathbf{R})$ and all $Q \in U$. One defines a $p + s$-form $\alpha\beta$ by setting $\alpha\beta(Q) = [Q, KL]$ for $Q \in U$.

With these definitions, it is not difficult to show:

PROPOSITION 2.2. *1. For an s-form α and a p-form β,*

$$(f_{s+p}^0)^*(\alpha\beta) = (f_s^0)^*(\alpha)(f_p^0)^*(\beta).$$

2. For s forms α and β, $(f_s^0)^*(\alpha + \beta) = (f_s^0)^*(\alpha) + (f_s^0)^*(\beta)$.

3. Let $g : Y \to Z$ be a smooth map. Then,

$$((g \circ f)_s^0)^* = (f_s^0)^* \circ (g_s^0)^*.$$

When there is little chance of confusion, we write f^* instead of $(f_s^0)^*$.

We now return to our discussion of cosmic strings. Consider the quotient $\mathbf{S}$ of $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$ described above where $q^{st} : \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R} \to \mathbf{S}$ was the quotient map. We use the notation in Example 7. In order to relate the Levi–Civita metric on $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$ to a metric on $\mathbf{S}$, we for simplicity consider the quotient $q : \mathbf{R}^2 \to V$. Since $q^{st} = I_{\mathbf{R}} \times q \times I_{\mathbf{R}}$, the Levi–Civita metric is pulled back in the first and last coordinate in the evident way (see section 4).

First, we show:

PROPOSITION 2.3. *The tangent come space $TC\underline{V}_{(0,0,0)}$ is equal to*

$$\left\{ a(\rho_1)_* \frac{\partial}{\partial r}\bigg|_{r=0,\theta'} \mid a, \theta' \in \mathbf{R} \right\}$$

and the map $(i_V)_* : TCV_{(0,0,0)} \to TC\underline{V}_{(0,0,0)}$ *is bijective.*

PROOF. Set $B = \left\{ a(\rho_1)_* \frac{\partial}{\partial r}\Big|_{r,0,\theta'} \mid a, \theta' \in \mathbf{R} \right\}$. Let $c : \mathbf{R} \to \underline{V}$ be a contour with $c(0) = (0,0,0)$, $f \in \mathcal{F}_{\underline{V}} \subset \mathcal{F}_V$ and without loss in generality suppose that $f(0,0,0) = 0$. Using the quotient map $\hat{q} : K \to V$ (see Example 7), one sees that f as a function on K has near $z = 0$ the form $k + zH(x, y, z)$ where $(x, y) \in S^1$, $z \in \mathbf{R}$, $k = 0$ as $f(0,0,0) = 0$ and H is a smooth function on K. Thus, near $t = 0$, $f(c(t))$ can be written

$$f(c(t)) = c_3(t)H\left(\frac{c_1(t)}{c_3(t)}, \frac{c_2(t)}{c_3(t)}, c_3(t)\right)$$

if $c_3(t) \neq 0$ and $f(c(t)) = 0$ if $c_3(t) = 0$. If $c_3(t) \neq 0$, one has $\left|\frac{c_1(t)}{c_3(t)}\right|, \left|\frac{c_2(t)}{c_3(t)}\right| \leq 1$ and then, since c_3 is a smooth function $\mathbf{R} \to \mathbf{R}$ and H is smooth, we find

$$\left| c_3(t)G\left(\frac{c_1(t)}{c_3(t)}, \frac{c_2(t)}{c_3(t)}, c_3(t)\right)\right| \leq |c_3(t)|M$$

some constant M and for t near 0. Hence if

$$c_3'(0) = 0, \quad \lim_{t\to 0}\left|\frac{f(c(t))}{t}\right| \leq \lim_{t\to 0}\frac{|c_3(t)|}{|t|}M = 0.$$

If $V_c \in TC\underline{V}_{(0,0,0)}$, then $V_cf = \lim_{t\to 0}\frac{f(c(t))}{t}$. If $c_3'(0) = 0$, as we have just seen, $V_cf = 0$ and hence $V_c \in B$. Thus, one can assume that $c_3'(0) \neq 0$ in which case

$$V_cf = c_3'(0)H\left(\frac{c_1'(0)}{c_3'(0)}, \frac{c_2'(t)}{c_3'(0)}, c_3(0)\right).$$

On the other hand, $(\rho_1)_* \frac{\partial}{\partial r}\Big|_{r=0,\theta'}$ is a derivation on $\mathcal{F}_{\underline{V}}$ where

$$(\rho_1)_* \frac{\partial}{\partial r}\Big|_{r=0,\theta'} f = \frac{\partial}{\partial r}\Big|_{r=0,\theta'} f = G(\cos(\theta'), \sin(\theta'), 0).$$

Since $(c_1(t))^2 + (c_2(t)^2 = (c_3(t))^2$, $2c_1(t)c_1(t)c_1'(t) + 2c_2(t) = 2c_3'(t)c_3(t)$ and, hence, as one can assume that $c_3(t)$ is not zero near $t = 0$ except possibly at 0,

$$\frac{c_1(t)}{c_3(t)}\frac{c_1'(t)}{c_3'(t)} + \frac{c_2(t)}{c_3(t)}\frac{c_2'(t)}{c_3'(t)} = 1$$

near $t=0$ except possibly at 0. Letting $t \to 0$ and using l'Hôspital's rule, one obtains

$$\left(\frac{c_1'(0)}{c_3'(0)}\right)^2 + \left(\frac{c_2'(0)}{c_3'(0)}\right)^2 = 1.$$

This implies that $\left(\frac{c_1'(0)}{c_3'(0)}\right) = \cos(\theta')$ and $\left(\frac{c_3'(0)}{c_1'(0)}\right) = \sin(\theta')$ for some $\theta' \in \mathbf{R}$. Thus,

$$V_c f = c_3'(0)\frac{\partial}{\partial r}\bigg|_{r=0,\theta'} f$$

and then $V_c \in B$. The inclusion $B \subset TC\underline{V}\ |_{(0,0,0)}$ follows from $(\rho_1)_* T\mathbf{R}^2\ |_{r=0,\theta'} = (\rho_1)_* TC\mathbf{R}^*\ |_{r=0,\theta'} \subset TC\underline{V}_{(0,0,0)}$.

Suppose that $(i_V)_* V_c = (i_V)_* V_d$ for $c,\ d \in \mathscr{C}_V$. Then, for all $f \in \mathscr{F}_{\underline{V}}$ and, assuming without loss of generality that $c(0) = d(0) = (0,0,0)$ and $f(0,0,0)=0$, one has

$$\lim_{t\to 0}\frac{f(c(t))}{t} = \lim_{t\to 0}\frac{f(d(t))}{t}.$$

The maps $f_i : \underline{V} \to \mathbf{R}$ defined by setting $f_i(x_1,x_2,x_3)=x_i$ for $i=1$, 2, 3 belong to $\mathscr{F}_{\underline{V}}$. Substituting these functions for f, one finds that $c_i'(0)=d_i'(0)$ for $i=1$, 2, 3.

Suppose that $c_3'(0)=d_3'(0)=0$. It follows from our derivation above that

$$\lim_{t\to 0}\frac{f(c(t))}{t} = \lim_{t\to 0}\frac{f(d(t))}{t} = 0.$$

Suppose that $c_3'(0) = d_3'(0) \neq 0$. Since the $c_i(t)$ are smooth functions with $c_i(0)=0$ for $i=1$, 2, 3, it follows that the functions $u_i(t)=\frac{c_i(t)}{t}$, where $u_i(0) = = \lim_{t\to 0}\frac{c_i(t)}{t}$, are as a consequence of Taylor's theorem smooth. Furthermore, $u_3(0) = c_3'(0) \neq 0$. Thus, for $i=1,\ 2$, $v_i(t) = \frac{c_i(t)}{c_3(t)} = \frac{u_i(t)}{u_3(t)}$ is smooth. Taking limits and taking into account similar formulas for the curve d, one has, using l'Hôspital's rule,

$$\lim_{t\to 0}\frac{f(c(t))}{t} =$$

$$= c_3'(t)H(v_1(0),v_2(0)) + c_3(0)[H_x(v_1(0),v_2(0))v_1'(0) + H_y(v_1(0),v_2(0))v_2'(0)] =$$

$$= c_3'(0)H(v_1(0), v_2(0)) = d_3'(0)H\left(\frac{c_1'(0)}{c_3'(0)}, \frac{c_2'(0)}{c_3'(0)}\right) =$$

$$= d_3'(0)H\left(\frac{d_1'(0)}{d_3'(0)}, \frac{d_2'(0)}{d_3'(0)}\right) = \lim_{t\to 0}\frac{f(d(t))}{t}.$$

Thus, $V_c f = C_d f$ for $f \in \mathcal{F}_V$, $V_c = V_d$ and then, as a consequence, $(i_V)_*$ is one-one. Since $(\rho_1)_* = (i_V \circ q)_* = (i_V)_* \circ (q)_*$ has been shown to be surjective, so is $(i_V)_*$. ∎

Let $dr^2{+}r^2 d\theta^2$ be the Levi–Civita metric restricted to $\mathbf{R}^2$. For the smooth inclusion $j_V : V \to \mathbf{R}^3$ (see Example 7), the vector $\left.\frac{\partial}{\partial r}\right|_{0,\theta}$ is mapped by $\rho * = (j_V \circ q)_*$ to

$$\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

which is a tangent vector in $\mathbf{R}^3$ and tangent to the cone at $(0,0,0)$.

We now consider the 1-form dr on $\mathbf{R}^2$. For a 1-form ω on $\mathbf{R}^3$, let $\omega|_V = = j_V^*\omega$. In particular, one writes $dz|_V = j_V^* dz$. Since $j_V^*\omega X = \omega((j_V)_* X)$ for a vector X on V,

$$dz|_V\left((j_V)_*^{-1}\left(a\cos(\theta')\frac{\partial}{\partial x} + a\sin(\theta')\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\right)\Bigg|_{(0,0,0)} = 1.$$

Under q^* and because of (3) in Proposition 2.2, clearly $dz|_V$ pulls back to dr and then also $dz|_V^2$ pulls back to dr^2.

Next we consider the 1-form $d\theta$ on $\mathbf{R}^2$.The image of $\frac{\partial}{\partial\theta}$ (r arbitrary) under ρ_* is $\psi = (\rho)_*\frac{\partial}{\partial\theta} = -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y}$. The 1-form

$$\psi^* = -\frac{1}{r}\sin(\theta)dx|_V + \frac{\cos(\theta)}{r}dy|_V$$

on $V - \{(0,0,0)\}$ satisfies $\psi^*(\psi|_V) = 1$ where $\psi|_V = (j_V)_*^{-1}(\psi)$. However, as ψ^{-1} is a 1-form on V, for $r = 0\psi^*(\psi|_V)$ must be 0 since ψ is then 0 and ψ^* is linear. Thus, ψ^* is not a (smooth) 1-form. On the other hand,

$$r\psi^* = -\sin(\theta)dx|_V + \cos(\theta)dy|_V = -\frac{y}{z}dx|_V + \frac{x}{z}dy|_V$$

satisfies $r\psi^*(\psi|_V) = r$. Hence, $r\psi^*$ is a (smooth) 1-form. One checks readily that $q^*(r\psi^*) = rd\theta$ and it follows that $q^*(r^2(\psi^*)^2 + dz|_V^2)$ is $dr^2 + r^2 d\theta^2$.

One can also write $r^2(\psi^*)2 + dz|_V^2 = dx|_V^2 + dy|_V^2$. If $\underline{P}_2 : \mathbf{R}^3 \to \mathbf{R}^2$ is smooth projection onto the first 2 coordinates and $P_2 = \underline{P}_2 \circ j_V$, P_2 is also a smooth projection and the variable change from (x,y) to (r,θ) coordinates is given by $q^* \circ P_2^2$ where

$$q^* \circ P_2^*(dx^2 + dy^2) = q^*(dx|_V^2 + dy|_V^2) = dr^2 + r^2 d\theta^2.$$

PROPOSITION 2.4.

- *The metric* $dn^2 = dx|_V^2 + dy|_V^2$ *is a local cone quasi-Riemannian metric on* V.
- *The metric* $dm^2 = dt|_{\mathbf{S}}^2 + dx|_{\mathbf{S}}^2 + dy|_{\mathbf{S}}^2 + dw|_{\mathbf{S}}^2$ *and the Levi–Civita metric* $dy^2 = -dt_{\mathbf{S}}^2 + dx|_{\mathbf{S}}^2 + dy|_{\mathbf{S}}^2 + dw|_{\mathbf{S}}^2$ *are local cone quasi-Riemannian metrics on* $\mathbf{S}$.

PROOF. We show that dn^2 is a local cone quasi-Riemannian metric on V. This proof readily extends to show that dm^2 and du^2 are local cone quasi-Riemannian metrics on $\mathbf{S}$.

The map $(\underline{P}_2)_{*Q} : TC\underline{V}_Q \to TC\mathbf{R}^2_{P_2(Q)}$ is clearly bijective for each $Q \in \underline{V}$. Using Proposition 2.3, it follows that $(P_2)_{*Q} : TCV_Q \to TC\mathbf{R}^2_{P_2Q} =$ $= T\mathbf{R}^2_{P_2(Q)}$ is also bijective for each $Q \in V$. Since $dx^2 + dy^2$ is positive definite on $\mathbf{R}^2$ and $dn^2(X,Y) = (dx^2 + dy^2)(P_{2*}X, P_{2*}Y)$, it follows readily that $g = dn^2$ induces a one-one smooth map $g_* : TCV \to T_1^0 CV$.

We will write down now explicitly the effect of applying g_*. Let $W \in$ $\in TCV$. Then,

$$W = aq_*\frac{\partial}{\partial r} + bq_*\frac{\partial}{\partial\theta} =$$
$$= (a\cos\theta - br\sin\theta)\frac{\partial}{\partial x}\bigg|_V + (a\sin\theta + br\cos\theta)\frac{\partial}{\partial y}\bigg|_V + a\frac{\partial}{\partial z}\bigg|_V.$$

Thus,

$$g_*(W) =$$
$$= (dx|_V^2 + dy|_V^2)(W) = (a\cos\theta - br\sin\theta)dx|_V + (a\sin\theta + br\cos\theta)y|_V.$$

If $L = Cdx|_V + Ddy|_V$, solving $a\cos\theta - br\sin\theta = C$ and $a\sin\theta + br\cos\theta = D$, one finds that

$$W = g_*^{-1}(L) = C\frac{\partial}{\partial x}\bigg|_V + D\frac{\partial}{\partial y}\bigg|_V + (c\cos\theta + D\sin\theta)\frac{\partial}{\partial z}\bigg|_V.$$

Let $x : \mathbf{R} \to T_0^2 CV$ be a smooth curve. Write $c(t) = C(t)dx|_{V,P(t)} + D(t)dy|_{V,P(t)}$ where $P(t) \in V$. The curve $c(t)$ is smooth if, using the Cartesian closedness of $\mathcal{FRL}$, $c(t)(X)$ is a smooth function of t for each $X \in TCV$. Letting $X = \frac{\partial}{\partial x}\Big|_{V,P(t)}$ and $X = \frac{\partial}{\partial y}\Big|_{V,P(t)}$, one sees that C, $D : \mathbf{R} \to \mathbf{R}$ are smooth functions.

Thus, to show that g_*^{-1} is smooth, one need to show that

$$W(t) = g_*^{-1}(c(t)) = \\ = C(t)\frac{\partial}{\partial x}\Big|_{V,P(t)} + D(t)\frac{\partial}{\partial y}\Big|_{V,P(t)} + (C(t)\cos\theta + D(t)\sin\theta)\frac{\partial}{\partial z}\Big|_{V,P(t)}$$

is a smooth function of t. But, using the Cartesian closedness of $\mathcal{FRL}$, $W(t)$ is smooth if $W(t)(f)$ is smooth for smooth functions $f : V \to \mathbf{R}$. Expressing $W(t)$ in polar coordinates, one has

$$W(t) = (C\cos\theta(t) + D\sin\theta(t))q_*\frac{\partial}{\partial r}\Big|_{P(t)} + \frac{(D\cos\theta(t) - C\sin\theta(t))}{r(t)}\frac{\partial}{\partial\theta}\Big|_{P(t)}.$$

Since V has a quotient Frölicher structure given by $q, f \in \mathcal{F}_V$ if and only if f as a function $\mathbf{R}^2 \to \mathbf{R}$ is smooth and can be written $f(r,\theta) = c + rg(r,\theta)$, for r near 0, where g is defined and smooth near $r = 0$ and $f(r,\theta+2\pi) = f(r,\theta)$ for all r, θ. It follows that

$$W(t)(f) = (C\cos\theta(t) + D\sin\theta(t))\frac{\partial rg(r,\theta)}{\partial r}\Big|_{P(t)} + \\ +(D\cos\theta(t) - C\sin\theta(y))\frac{\partial g(r,\theta)}{\partial\theta}\Big|_{P(t)}$$

is a smooth function of t. Thus, g_*^{-1} is smooth. ∎

DEFINITION 2.3. The 2-form $r^2(\psi^*)^2 + dz^2 = dx|_V^2 + dy|_V^2$ on V will be called the *polar Riemannian (cone) metric* on V. Similarly, the 2-form $dt|_S^2 + r^2(\psi^*)^2 + dz|_S^2 + dq|_S^2$ on $\mathbf{S}$ will be called the *polar Riemannian (cone) metric* on $\mathbf{S}$. The metric $-dt|_S^2 + r^2(\psi^*)^2 + dz|_S^2$ has been called the *Levi–Civita (cone) metric* on $\mathbf{S}$.

3. Connections on the Frölicher space cone V

We consider the cone V with the quotient Frölicher structure provided in Example 7 in this section. We extend the results that we obtain in this section to the Frölicher space-time $\mathbf{S}$ of a cosmic string in the next section. A general reference for this section is [9]. As we have shown in the previous section,

$$W_r = q_*\left(\frac{\partial}{\partial r}\right) = \cos(\theta)\frac{\partial}{\partial x}\bigg|_V + \sin(\theta)\frac{\partial}{\partial y}\bigg|_V + \frac{\partial}{\partial z}\bigg|_V$$

and

$$V_\theta = q_*\left(\frac{\partial}{\partial \theta}\right) = -r\sin(\theta)\frac{\partial}{\partial x}\bigg|_V + r\cos(\theta)\frac{\partial}{\partial y}\bigg|_V$$

generate $TV|_{r\cos(\theta),r\sin(\theta),r)}$ for $r\neq 0$ and, for $r = 0$, every vector in $TCV|_{(0,0,0)}$ has the form $a\,W_r$ for suitable $a,\theta\in\mathbf{R}$

PROPOSITION 3.1. *The assignment* $(r\cos(\theta), r\sin(\theta), r)\mapsto V_\theta$ *but not the assignment* $(r\cos(\theta), r\sin(\theta), r)\mapsto W_r$ *defines a smooth vector field on* V.

PROOF. The vector field V_θ defines a smooth vector field on V if and only if $V_\theta f\in\mathcal{F}_V$ for each $f\in\mathcal{F}_V$. But, $f(r,\theta)\in\mathcal{F}_V$ if and only if f as a function $\mathbf{R}^2\to\mathbf{R}$ is smooth and can be written $f(r,\theta)=c+rg(r,\theta)$, for r near 0, where g is defined and smooth near $r=0$ and $f(r,\theta+2\pi)=f(r,\theta)$ for all r, θ. Thus

$$q_*\left(\frac{\partial}{\partial\theta}\right)(f) = \frac{\partial}{\partial\theta}(f) = r\frac{\partial g(r,\theta)}{\partial\theta}$$

near $r=0$ and thus $q_*\left(\frac{\partial}{\partial\theta}\right)(f)\in\mathcal{F}_V$. On the the other hand, near $r=0$,

$$q_*\left(\frac{\partial}{\partial r}\right)(f) = \frac{\partial}{\partial r}(f) = g(r,\theta) + r\frac{\partial g(r,\theta)}{\partial r}$$

and thus $q_*\left(\frac{\partial}{\partial r}\right)(f)$ need not belong to $\mathcal{F}_V$. ∎

Refining the considerations in Proposition 3.1, one sees that the general smooth vector field has the form $\underline{X}=F_\theta(r,\theta)V_\theta+G'_r(r,\theta)W_r$ where $G'_r(r,\theta)$ vanishes at $r=0$, $F_\theta(r,\theta+2\pi)=F_\theta(r,\theta)$ for all r, θ and $G'_r(r,\theta+2\pi)= = G'_r(r,\theta)$ for all r, θ. In particular, if one sets $V_r = rW_r$, V_r is a vector field on V and we see that any smooth vector field on V has $(0,0,0)$ as a stationary point. Thus, $\underline{X}$ can be written $\underline{X}=F_\theta(r,\theta)V_\theta+G_r(r,\theta)V_r$ where $G_r(r,\theta)$ and $F_\theta(r,\theta)$ represent elements of $\mathcal{F}_V$, $F_\theta(r,\theta+2\pi)=F_\theta(r,\theta)$ for all r, θ and $G_r(r,\theta+2\pi)=G_r(r,\theta)$ for all r, θ.

We introduce the general definition of connection, metric connection and then specialize to V in this section and the space time of a cosmic string in the next section.

DEFINITION 3.1. Let A be a smooth space and $\chi[A]$ be the set of smooth vector fields of A. A smooth map $\nabla : \chi[A] \times \chi[A] \to \chi[A]$ is a *covariant derivative* or *connection* if

1. ∇ is bilinear
2. $\nabla_{fX} Y = f \nabla_X Y$ for $f \in \mathcal{F}_A$.
3. $\nabla_X(f Y) = X[f] Y + f \nabla_X Y$ for $f \in \mathcal{F}_A$ (the Leibniz rule).

Let $\chi_s^0[A]$ denote the collection of tensor fields of type $(0,s)$. One defines a smooth map $h_s^0 : \chi_s^0[A] \to$ **MULTILIN**$(\oplus^s TA; \mathbf{R})$ by letting

$$h_s^0(\omega)(X_1(Q), \ldots, X_s(Q)) = \underline{\omega}_P(X_1(Q), \ldots, X_s(Q))$$

for Q in a neighborhood of P, where $\omega(P) = [(P, \underline{\omega}_P)]$ and $\underline{\omega}_P$ defines ω in a neighborhood of P. The relation, where $h = h_2^0$,

(‡) $\nabla_Z(h(g)(X, Y)) = (\nabla_Z h(g))(X, Y) + h(g)(\nabla_Z X, Y) + h(g)(X, \nabla_Z(Y))$

implies that the map $\nabla_Z h(g) : TA \oplus TA \to \mathbf{R}$ is smooth and one defines $\nabla_Z g$ to be the map sending P to $[(P, \nabla_Z h(g))]$.

DEFINITION 3.2. The covariant derivative ∇ is a *metric connection* if and only if $\nabla_X g = 0$ for each $X \in \chi[A]$.

Since any vector field X on V vanishes at $(0,0,0)$, any connection on V can be viewed as a connection on $V - \{(0,0,0)\}$ and, in this way, a connection on $\mathbf{R}^2 - \{(r,\theta) \mid r = 0\}$. Thus, if ∇ is a connection on $\mathbf{R}^2 - \{(r,\theta) \mid r = 0\}$, one defines a connection ∇_* on V using the relation $\nabla_*(q_*(X), q_*(Y)) = \nabla(X, Y)$. For a metric g on V, one defines $\nabla_*(g)$ by setting $\nabla_*(g) = \nabla(q^* g)$ or alternatively, using (‡). Thus, ∇ is a metric connection for $q^* g$ if and only if ∇_* is a metric connection for g. The Levi–Civita or polar connection ∇^L on $\mathbf{R}^2$ is defined to be the unique symmetric metric connection for the metric $dr^2 + r^2 d\theta^2$ on $\mathbf{R}^2$. See [9]. The unique connection on $V - \{(0,0,0)\}$ induced from ∇^L is denoted ∇_*^L. The metric connection

∇^L (see [9]) has been shown to satisfy with respect to the coordinate base $\{e_r, e_\theta\}$ where $e_r = \frac{\partial}{\partial r}$ and $e_\theta = \frac{\partial}{\partial \theta}$:

$$\begin{aligned}
\nabla^L_{e_r} e_r &= e_\gamma \Gamma^\gamma_{rr} = 0 \\
\nabla^L_{e_r} e_\theta &= e_\gamma \Gamma^\gamma_{r\theta} = \frac{1}{r} e_\theta \\
\nabla^L_{e_\theta} e_r &= e_\gamma \Gamma^\gamma_{r} = \frac{1}{r} e_\theta \\
\nabla^L_{e_\theta} e_\theta &= e_\gamma \Gamma^\gamma_{\theta\theta} = -r e_\theta
\end{aligned}$$

where the Γ's are the connection coefficients.

We view V_r and V_θ as providing a non-coordinate basis on V. In terms of V_r and V_θ, the above relations become

$$\begin{aligned}
(\nabla^L_*)_{V_r} V_r &= (\nabla^L_*)_{r W_r} r W_r = r(\nabla^L_*)_{W_r} r W_r = \\
&= r((\nabla^L_*))_{W_r}(r) W_r = r(\nabla^L_*)_{W_r} W_r) = \\
&= r\left(\frac{\partial r}{\partial r} W_r + 0\right) = V_r, \\
(\nabla^L_*)_{V_r} V_\theta &= r(\nabla^L_*)_{W_r} V_\theta = r\frac{1}{r} V_\theta = V_\theta \\
(\nabla^L_*)_{V_\theta} V_r &= V_\theta \\
(\nabla^L_*)_{V_\theta} V_\theta &= -V_r \quad \text{(the } r \text{ disappears).}
\end{aligned}$$

The connection coefficients thus take on a simpler form on the cone V.

PROPOSITION 3.2. *For the geometric setting V for polar coordinates, the non-zero Levi-Civita connection coefficients are*

$$\Gamma^r_{rr} = 1, \quad \Gamma^r_{\theta\theta} = -1, \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1.$$

One defines in the usual way the torsion and Riemannian curvature tensor for a Frölicher space A as follows. Let ∇^A be a connection on A.

DEFINITION 3.3.

1. The smooth map $T^A : \chi[A] \times \chi[A] \to \chi[A]$ defined by setting $T^A(X, Y) = = \nabla^A_X Y - \nabla^A_Y X - [X, Y]$ is called the *torsion tensor* for ∇^A.
2. The smooth map $R^A : \chi[A] \times \chi[A] \times \chi[A] \to \chi[A]$ defined by setting

$$R^A(X, Y, Z) = \nabla^A_X \nabla^A_Y Z - \nabla^A_Y \nabla^A_X Z - \nabla^A_{[X,Y]} Z$$

is called the *Riemannian curvature tensor* for ∇^A.

General results show that as the torsion and Riemannian curvature tensors T and R corresponding to ∇^L are 0, the torsion and Riemannian curvature tensors T_*, and R_*, corresponding to ∇^L_* are trivial. We provide an independent verification of this fact. According to [9], one calculates the curvature tensor in the non-coordinate basis according to the formula

$$R^{\alpha}_{\beta\gamma\delta} = V_\gamma\left[\Gamma^{\alpha}_{\delta\beta}\right] - V_\delta\left[\Gamma^{\alpha}_{\gamma\beta}\right] + \Gamma^{eps}_{\delta\beta}\Gamma^{\alpha}_{\gamma\epsilon} - \Gamma^{\epsilon}_{\gamma\beta}\Gamma^{\alpha}_{\delta\epsilon} - c^{\epsilon}_{\gamma\delta}\Gamma^{\alpha}_{\epsilon\beta}$$

where $[V_\alpha, V_\beta] = c^{\gamma}_{\alpha\beta} V_\gamma$. We show:

PROPOSITION 3.3. *The torsion tensor T_*, and the curvature tensor R_*, of ∇^L_* are trivial.*

PROOF. Here, $\alpha, \beta, \gamma, \delta, \epsilon$, etc. are either r or θ. Clearly, since the $\Gamma^{\alpha}_{\mu\rho}$ are constants, the term $V_\gamma\left[\Gamma^{\alpha}_{\delta\beta}\right] - V_\delta\left[\Gamma^{\alpha}_{\delta\beta}\right]$ in $R^{\alpha}_{\beta\gamma\delta}$ is 0. Also,

$$[V_\theta, V_r] = \left[\frac{\partial}{\partial\theta}, r\frac{\partial}{\partial r}\right] = \frac{\partial}{\partial\theta}\left(r\frac{\partial}{\partial r}\right) - r\frac{\partial}{xr}\frac{\partial}{\partial\theta} = 0.$$

Hence,

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\epsilon}_{\delta\beta}\Gamma^{\alpha}_{\gamma\epsilon} - \Gamma^{\epsilon}_{\gamma\beta}\Gamma^{\alpha}_{\delta\epsilon}.$$

Substituting the relations in Proposition 3.2, a straightforward but tedious calculation shows that all the $R^{\alpha}_{\beta\gamma\delta}$'s are 0.

Proposition 3.2 and the fact that $[V_\theta, V_r] = 0$ imply immediately that $T_* = 0$. ∎

Next, we consider the metric $dp^2 = \psi_*^2 + r^{-2}dz|^2_V$ on V where $q^*(\psi_*^2 + r^{-2}dz|^2_V) = r^{-2}dr^2 + d\theta^2$. For this metric, one shows readily that V_r and V_θ are orthonormal, in the sense that $d^2(V_r, V_\theta) = 0$, $dp^2(V_r, V_r) = dp^2(V_\theta, V_\theta) = 1$ if $r \neq 0$, and $\lim_{r\to 0} dp^2(V_r, V_\theta) = \lim_{r\to 0} dp^2(V_\theta, V_\theta) = 1$. However, since V_r and V_θ are 0 if $r = 0$, dp^2 is only smooth on $V - \{(0,0,0)\}$. Also, $\widehat{dr} = \frac{1}{r}dr$ and $\widehat{d\theta} = d\theta$ are dual vectors in the same way to V_r, and V_θ, respectively, and $\widehat{dr}^2 + \widehat{d\theta}^2 = r^{-2}dr^2 + d\theta^2 = r^{-2}(dr^2 + r^2 d\theta^2)$. The metric $r^{-2}(dr^2 + r^2 d\theta^2)$ on $\mathbf{R}^2 - \{(r,\theta) \mid r \neq 0\}$ is the Poincare metric (in polar coordinates). See [9]. It sometimes is more useful to consider the

Poincare metric than the conformally equivalent metric $dr^2 + r^2 d\theta^2$, $r \neq 0$, and thus, on $V - \{(0,0,0)\}$, the Poincare metric $\psi_*^2 + r^{-2} dz|_V^2$ which is conformally equivalent to the polar (cone) metric on $V - \{(0,0,0)\}$. Simple calculations show that, on $\mathbf{R}^2 - \{(r,\theta) \mid r \neq 0\}$ for the usual coordinate basis, the only non-zero connection coefficient is $\Gamma^r_{rr} = -\frac{1}{r}$ and the curvature tensor is 0. Changing to the non-coordinate basis and calculating as above, one finds:

PROPOSITION 3.4.

- *For the metric* $r^{-2}dr^2 + d\theta^2$ *on* $\mathbf{R}^2 - \{(r,\theta) \mid r \neq 0\}$ *and the non-coordinate basis* $\{V_r, V_\theta\}$, *the connection coefficients are* 0 *and the Riemannian curvature tensor is* 0.
- *For the Poincare metric on* V, *the connection coefficients and the Riemannian curvature tensor are* 0.
- *On the negative side,* $r^{-2}dr^2 + d\theta^2$ *does not provide a duality for* $TCV_{(0,0,0)}$.

In effect, since the connection coefficients are 0 in the Poincare metric, one might then view V_r and V_θ as the partial derivatives on V.

Finally, we consider a metric $d\xi^2$ which is normally viewed as the metric on the sphere but is here viewed as a metric on $\mathbf{R}^2$ and then on V. In the process of developing this example, we introduce some important concepts. The metric $d\xi^2$ is defined by setting $d\xi^2 = dr^2 + \sin(r)^2 d\theta^2$. Clearly, the metric is degenerate if $\sin(r) = 0$ and $q^*(dz|_V^2 + \sin(r)^2 \psi_*^2) = d\xi^2$. The non-zero connection coefficients for $d\xi^2$ and $r \neq 0$, are $\Gamma^r_{\theta\theta} = -\cos(r)\sin(r)$ and $\Gamma^\theta_{r\theta} = \Gamma^{=}_{\theta r} \cot(r)$. Changing to the non-coordinate base consisting of $\widehat{\frac{\partial}{\partial r}} = \frac{\partial}{\partial r}$ and $\widehat{\frac{\partial}{\partial \theta}} = \cos(r)\frac{\partial}{\partial \theta}$, one determines, as in [9]:

PROPOSITION 3.5. *For the metric* $d\xi^2$ *on* $\mathbf{R}^2$ *and, correspondingly, the metric* $dz|_V^2 + \sin(r)^2\psi_*^2$ *on* V, *the non-zero curvature coefficients are* $R^r_{\theta r\theta} = -R^r_{\theta\theta r} = 1$ *and* $R^\theta_{r\theta R} = -R^\theta_{rr\theta} = 1$.

Let M be a finite dimensional smooth manifold, $P \in M$ and R the Riemannian curvature tensor on M. Consider the linear map $L_P : TM_P \to TM_P$ defined by setting, for $Y, Z \in \chi[M]$, $L_P(U) = R(U, Y(P), Z(P))$. In a coordinate basis, the Ricci tensor is defined by the relation

$$\mathcal{RIC}(Y,Z)_p = \left\langle dx^\mu, R(e_\mu, Y(P), Z(P)) \right\rangle .$$

It is easy to see that $\mathcal{RIC}(Y,Z)_P$ is the trace of the map L_P defined above and that its definition doesn't depend on the particular set of coordinates chosen or whether one uses a non-coordinate basis.

In our present example, in the non-coordinate basis,

$$L_P\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & ad-bc\\ -ad+bc & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

if $Y(P)=a\widehat{\frac{\partial}{\partial r}}\Big|_P+b\widehat{\frac{\partial}{\partial\theta}}\Big|_P$ and $Z(P)=\widehat{\frac{\partial}{\partial r}}\Big|_P+d\widehat{\frac{\partial}{\partial\theta}}\Big|_P$.

As the trace of L_P is 0, it thus follows here that $\mathcal{RIC}$ is trivial.

Suppose that, for a Frölicher space A, one has a smooth linear map

$$TR : \mathbf{LIN}(\chi[A],\chi[A]) \to \mathcal{F}_A,$$

called a trace map, which is smooth in $P \in A$. In our present example, $tr(L)(P) = tr(L_P)$ is smooth in P. In many cases, the trace map might be defined in terms of a non-coordinate basis on suitable smooth manifolds, possibly infinite dimensional. Suppose that $\hat{R}$ is the smooth map

$$\hat{R} : \chi[A]\times\chi[A] \to \mathbf{LIN}(\chi[A],\chi[A])$$

induced from the Riemannian curvature tensor R using the Cartesian closedness of $\mathcal{FRL}$. With this motivation, one has:

DEFINITION 3.4. The Ricci tensor $\mathcal{RIC}(X,Y)$ on a Frölicher space A is defined by setting $\mathbf{RIC}(X,Y) = tr\circ\hat{R}(X,Y)$ and is a map $\mathcal{RIC} : \chi[A]\times$ $\times\,\chi[A]\to\mathcal{F}_A$.

To finish this discussion, we will define the scalar curvature and the Einstein tensor for a general Frölicher space. Because of the Cartesian closedness of $\mathcal{FRL}$, the Ricci tensor $\mathcal{RIC}(X,Y)$ induces a smooth linear map $\widehat{\mathcal{RIC}} : \chi[A]\to\mathbf{LIN}(\chi[A],\mathcal{F}_A)$. Let $P,\ Q\in A$.

DEFINITION 3.5. If the smooth map $g_{**} : \chi[A] \to \mathbf{LIN}(\chi[A],\mathcal{F}_A)$ defined by setting $g_{**}(X)(Y) = h(X(Q),Y(Q))$, where $g = [(Q,h)]$ for Q in a neighborhood of P, is a smooth isomorphism, then g is called a *global quasi-Riemannian metric* and the corresponding manifold a *global quasi-Riemannian manifold.*

DEFINITION 3.6. For a global quasi-Riemannian manifold, the scalar curvature $\mathcal{R}$ on a Frölicher space A is defined by setting $\mathcal{R} = TR(g_{**}^{-1}\circ\widehat{\mathcal{RIC}})$.

Thus, to define the Ricci tensor and the scalar curvature using our approach, one needs to have, *a priori*, a suitable trace map for linear endomorphisms of $\chi[A]$ and a global quasi-Riemannian metric. In the above examples, the scalar curvature was always 0. It would be useful to know when a local quasi-Riemannian metric is a global quasi-Riemannian metric.

Using the Cartesian closedness of $\mathcal{FRL}$, the smooth map g_{**} above induces a smooth map $\overline{g}:\chi[A]\times\chi[A]\to\mathcal{F}_A$.

DEFINITION 3.7. The Einstein tensor is the smooth map $\mathcal{RIC}-\overline{g}\mathcal{R}$.

This corresponds to the definition in [12].

4. Connections and geodesics on the space time S of a cosmic string

We consider the examples in section 3 extended to the space time $\mathbf{S} = \mathbf{R}\times V\times\mathbf{R}\subset\mathbf{R}^5$ where, to fix our notation for coordinates $(t,q(r,\theta),w) = (t,x,y,z,w)\in\mathbf{S}$. Recall that $q^{st}:\mathbf{R}\times\mathbf{R}^2\times\mathbf{R}\to\mathbf{S}$ is the quotient map. Let $V_t=(q^{st})_*\frac{\partial}{\partial t}$, $V_r=r(q^{st})_*\frac{\partial}{\partial r}$, $V_\theta=(q^{st})_*\frac{\partial}{\partial\theta}$ and $V_w=(q^{st}_*\frac{\partial}{\partial w}$. Then, as in the previous section the set $\{V_t, V_r, V_\theta, V_w\}$ provides a non-coordinate basis for $\mathbf{S}-(\mathbf{R}\times\{(0,0,0)\}\times\mathbf{R})$ or, correspondingly, $\left\{\frac{\partial}{\partial t}, r\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial w}\right\}$ provides a non-coordinate basis for $\mathbf{R}\times\mathbf{R}^2\times\mathbf{R}-\{(t,r,\theta,w)\mid r=0\}$ (but with scalars arising from the quotient). One can prove, for $(t_0,0,0,w_0)\in sing(\mathbf{S})=\mathbf{R}\times\{(0,0,0)\}\times\mathbf{R}$:

PROPOSITION 4.1. *The tangent cone space* $TC\mathbf{S}_{(t_0,0,0,w_0)}$ *is equal* $\{(a\,V_t+b\,V_r+c\,V_w)|_{t=t_0,r=0,\theta',w=w_0}\mid a,b,c,\theta'\in\mathbf{R}\}$ *and the map* $(I_{\mathbf{R}}\times i_V\times I_{\mathbf{R}})_*$ *from the tangent cone to* $\mathbf{R}\times V\times\mathbf{R}$ *to the tangent cone to* $\mathbf{R}\times V\times\mathbf{R}$ *at* $(t_0,0,0,w_0)$ *is bijective.*

EXAMPLE 9. Let $ds^2=-dt^2+dr^2+r^2d\theta^2+dw^2$ be the Levi–Civita metric on $\mathbf{R}^4$ and

$$\psi^* = -\frac{1}{r}\sin(\theta)dx|_S + \frac{\cos(\theta)}{r}dy|_S,$$

as in section 2. Then, one has

$$(q^{st})^*(-dt|_S^2 + r^2(\psi^*)^2 + dz|_S^2 + dw|_S^2) = ds^2$$

and $-dt|_S^2 + r^2(\psi^*)^2 + dz|_S^2$ is called the *Levi–Civita metric* on $\mathbf{S}$. Extending the Levi–Civita connection ∇^L on $\mathbf{R}\times\mathbf{R}^2\times\mathbf{R}$ to $\mathbf{S}$, as in the previous section, one obtains:

PROPOSITION 4.2. *For the space time* $\mathbf{S}$ *and the given non-coordinate basis, the non-zero Levi–Civita connections are the same as before:*

$$\Gamma^r_{rr} = 1, \quad \Gamma^r_{\theta\theta} = -1, \quad \Gamma^{=}_{\theta r}\Gamma^{\theta}_{r\theta} = 1,$$

The curvature tensor, the Ricci tensor and the scalar curvature vanish.

The geodesic equations (9) are, using s as a parameter:

$$\frac{r''r-(r')^2}{r^2} + \left(\frac{r^2}{r}\right)^2 - (\theta')^2 = 0 \tag{2}$$

$$\theta'' + \left(\frac{r'}{r}\right)\theta' = 0 \tag{3}$$

$$t'', w'' = 0. \tag{4}$$

They are obtained by writing a tangent vector given in terms of r, θ coordinates in terms of V_r and V_θ and then using the connection coefficients associated with V_r and V_θ. Solving equation (4) for θ', one obtains $\theta' =$ $= Ar^{-1}$, with A a constant, which on substituting into equation (3) yields $r" = A$ (assuming $r \neq 0$). Integrating, one obtains $r = As^2+Bs+C$ with B and C constants. Thus,

$$\theta' = \frac{A}{as^2+Bs+C}.$$

PROPOSITION 4.3. *The geodesics on* $\mathbf{S}$ *for the Levi–Civita connection are given, in two cases, as the images of the following paths in* $\mathbf{R}\times\mathbf{R}^2\times\mathbf{R}$ *under* q^{st}*:*

1. *Let* $A = 0$*. Then,* $r = Bs + C$*,* $\theta = D$*,* $t = Es + F$ *and* $w = Gs + H$ *with* E*,* F*,* G *and* H *constants. In this case (extending to* $r = 0$*), one passes through* $sing(\mathbf{S})$ *along a fitted line.*
2. *Suppose that* $A \neq 0$*. Then,* $r = As^2 + Bs + C$ *and*

$$\theta = \int \frac{A}{As^2+Bs+C}ds = F(s)$$

where the inverse of F has period 2π. *In this case, since* θ *is undefined if* $r = 0$, *one cannot pass through* $sing(\mathbf{S})$.

A particular solution to case (2) of the Proposition occurs if $A = 1$, $B = 0$ and $C = 1$ when $r = \tan(\theta)^2 + 1 = \sec^2(\theta)$.

EXAMPLE 10. Call the metric $ds^2 = -dt^2 + r^{-2}dr^2 + d\theta^2 + dw^2$ the space time *polar Poincare metric* on $\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}$. Then,

$$(q^{st})^*(-dt|_S^2 + (\psi^*)^2 + r^{-2}dz|_S^2 + dq|_S^2) = ds^2$$

and the metric $-dt|_S^2 + (\psi^*)^2 + r^{-2}dz|_S^2 + dq|_S^2$ on $\mathbf{S}$ will also be called the polar Poincare metric. Using our earlier results, one obtains:

PROPOSITION 4.4. *For the space time* $\mathbf{S}$, *the connections for the polar Poincare metric and the associated non-coordinate basis vanish.*

A consequence of this proposition is:

PROPOSITION 4.5. *The geodesics on* $\mathbf{S}$ *for the polar Poincare metric are of the form* $r(s) = Ae^{Bs}$, $\theta(s) = C$ *or of the form* $r(s) = A$, $\theta(s) = Ds + C$ *for constants* A, B, C *and* D. *None of the non-trivial geodesics pass through* $sing(\mathbf{S})$.

PROOF. The geodesic equations are: $\frac{r''(r-(r;)^2}{r^2} = 0$ and θ'', t'', $w'' = 0$. Solving these equations, one finds $r(s) = Ae^{Bt}$, $\theta(s) = Ds + C$. If $D \neq 0$, r written as a function of θ must have period 2π, which is impossible unless $B = 0$. Thus, if $D \neq 0$, $r(s) = A$, $\theta(s) = Ds + C$ is a solution and if $D = 0$, $r(s) = Ae^{Bt}$, $\theta(s) = C$ is a solution. ∎

References

[1] G. C. L. BRÜMMER, Topological Categories, *Topology and its applications,* **18** (1984), 27–41.

[2] P. CHERENACK, Frölicher versus Differential Spaces: a Prelude to Cosmology, to appear.

[3] A. FRÖLICHER and A. KRIEGL, *Linear Spaces and Differentiation Theory,* Wiley-Interscience, 1071.

[4] M. HELLER and W. SASIN, Anatomy of the Elementary Quasi-regular Singularity, *Acta Cosmologica,*, **21** (1995), 47–59.

[5] M. HELLER, J. GRUSZCZAK and W. SASIN, Quasiregular singularity of a cosmic string, *Acta Cosmologica*, **18** (1992), 45–55.

[6] M. HELLER, J. GRUSZCZAK and P. MULTARZYNSKI, Physics With and Without the equivalence Principle, *Foundations of Physics,* **19** (1989), 607–618.

[7] A. KRIEGL and P. MICHOR, *The Convenient Setting of Global Analysis,* AMS, 1997.

[8] S. MAC LANE, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[9] M. NAKAHARA, *Geometry, Topology and Physics,* Adam Hilger, 1990.

[10] W, SASIN, Differential Spaces and Singularities in Differential Space Time, *Demonstratio Math.,* **24** (1971), 601–634.

[11] R. SIKORSKI, Differential Modules, *Colloq. Math.,* **24** (1971), 45–79.

[12] R. M. WALD, *General Relativity,* The University of Chicago Press, 1904.

[13] H. WHITNEY, Differential Spaces and Singularities in Differentiable Even Functions, *Duke Math. J.,* **10** (1943), 159–166.

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THE MARCHENKO–PASTUR DISTRIBUTION AS THE LIMIT OF THE EIGENVALUE DENSITY OF SOME SYMMETRIC RANDOM MATRICES

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1. Preliminaries

For a positive number a the corresponding Marchenko–Pastur distribution is

$$\mu_a = \begin{cases} (1-a)\delta_0 + \frac{\sqrt{4a-(x-1-a)^2}}{2\pi x}\chi(x)dx & \text{if } a < 1, \\ \frac{\sqrt{4a-(x-x-1)^2}}{2\pi x}\chi(x)dx & \text{if } a \geq 1, \end{cases}$$

where χ stands for the characteristic function of the interval

$$\left[(1-\sqrt{a})^2, (1+\sqrt{a})^2)\right].$$

This probability distribution appeared in [3] as the limiting mean spectral density of the random matrices

$$B_n(p) = \sum_{i=1}^{p} P_i,$$

where P_i's are independent rank one projections with unitarily invariant distribution and $\frac{p}{n} \to a$. It is clear that for $a < 1$ zero is in the spectrum of $B_n(p)$ with multiplicity $n-p$ yielding the atomic part of the limiting distribution.

The limit theorem we treat concerns the matrix $\frac{TT^t}{n}$, where t stands for the transpose and T is a random matrix with identically distributed independent standardized elements. The Wishart matrix is a particular, but important

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example (see [1], [5]). In [4] an elementary combinatorial proof was given concerning the mean spectral density of the above mentioned matrix. As the result obtained there will serve as a base for this paper, we give it here:

If the entries of the $p(n) \times n$ random matrix $T^{(n)} = \left(\xi_{ij}\right)^{(n)}$ are independent and identically distributed with mean zero and variance 1 and all moments bounded; $\lim_{n\to\infty} \frac{p(n)}{n} = a$ then the mean spectral density of the symmetric $p(n) \times p(n)$ random matrix $\Lambda^{(n)} = \frac{T^{(n)}\left(T^{(n)}\right)^t}{n}$ tends to a variant of the Marchenko–Pastur distribution given by

$$\overline{\mu}_a = \begin{cases} (1-a^{-1})\delta_0 + \frac{\sqrt{4a-(x-1-a)^2}}{2\pi ax}\chi(x)dx & \text{if } 1 < a \\ \frac{\sqrt{4a-(x-x-a)^2}}{2\pi ax}\chi(x)dx & \text{if } 0 < a \leq 1, \end{cases} \tag{1.1}$$

where χ stands for the characteristic function of the interval

$$\left[(1-\sqrt{a})^2, (1+\sqrt{a})^2\right],$$

that is

$$\lim_{n\to\infty} \tau_{p(n)}((\Lambda^{(n)})^k) = m_k(\overline{\mu}_a)$$

for all k, where $m_k(\overline{\mu}_a)$ stands for the kth moment of $\overline{\mu}_a$ and for an $m \times m$ random matrix X

$$\tau_m = \frac{1}{m}\sum_{i=1}^{m} E(X_{ii}).$$

Note that the above variant of the Marchenko–Pastur distribution can be obtained from the original one with the requirement that for its moments $m_n = a\overline{m}_n$ should hold for $n = 2, 3, \ldots$ with $\overline{m}_1 = 1$, where m_n stands for the nth moment of the Marchenko–Pastur distribution, while $\overline{m}_n$ means the nth moment of its variant (of the same parameter a).

The proof is based on the method of moments, which was adapted to random matrices in the classical paper [7], see also [2]. For details see [4], where the moments of the Marchenko–Pastur distribution are also calculated explicitly.

A more general result can be found in [6] concerning sample matrices of independent elements. However, in our special case the proof is simpler (based only on elementary combinatorial methods), partly because the bound for the variance of

$$\frac{1}{n} tr \left(\frac{T^{(n)} \left(T^{(n)}\right)}{n} \right)^k$$

here is of order n while in the general case it was of order n^{-1} (see [6]).

2. Result

THEOREM 2.1. *Assume that the entries of the $p(n) \times n$ random matrix $T^{(n)} = \left(\xi_{ij}^{(n)}\right)$ are independent and identically distributed with mean zero and variance 1 and all moments are bounded. Moreover, assume that $\lim_{n\to\infty} \frac{p(n)}{n} = a$. Then the empirical spectral density of the symmetric $p(n) \times p(n)$ random matrix $\Lambda^{(n)} = \frac{T^{(n)}\left(T^{(n)}\right)^t}{n}$ converges in distribution almost surely to the distribution given by (1.1) as $n \to \infty$, that is*

$$\lim_{n\to\infty} tr_{p(n)}((\Lambda^{(n)})^k) = m_k(\overline{\mu}_a)$$

for all k almost everywhere, where $m_k(\overline{\mu}_a)$ stands for the kth moment of $\overline{\mu}_a$.

PROOF. In many details the proof is similar to that, of Theorem 2.1. of [4].

For the sake of simplicity we write ξ_{ij} in place of $\xi_{ij}^{(n)}$, p for $p(n)$ and Λ in place of $\Lambda^{(n)}$.

We know from [4] that $\tau_{p(n)}(\Lambda^k) = \frac{1}{p(n)} \sum_i E((\Lambda^k))_{ii}) \to m_k(\overline{\mu}_a)$ for all k as $n \to \infty$, so it is sufficient to show that

$$E\left(\sum_{n=1}^{\infty}(tr_{p(n)}((\Lambda^{(n)})^k) - \tau_{p(n)}((\Lambda^{(n)})^k))^2\right) < +\infty$$

for all k. Indeed, this condition implies that

$$\sum_{n=1}^{\infty}(tr_{p(n)}((\Lambda^{(n)})^k) - \tau_{p(n)}((\Lambda^{(n)})^k))^2$$

is finite almost everywhere and therefore $(tr_{p(n)}((\Lambda^{(n)})^k) - \tau_{p(n)}((\Lambda^{(n)})^k))^2$ converges to zero almost surely.

Calculating the expectation value of the expression under the summation we have

$$E((tr_p(\Lambda^k) - \tau_p(\Lambda^k))^2) = E((tr_p(\Lambda^k))^2) - (\tau_p(\Lambda^2))^2$$

As

$E(tr_p(\Lambda^k))^2) =$

$$= \frac{1}{p^2 n^{2k}} \sum_{m_1,m_2,\ldots,m=1}^{p} \sum_{r_1,r_2,\ldots,r_k=1}^{p} \sum_{n_1,n_2,\ldots,n_k=1}^{n} \sum_{s_1,s_2,\ldots,s_k=1}^{n}$$

$E(\xi_{m_1n_1}\xi_{m_2n_1}\xi_{m_2n_2}\xi_{m_2n_2}\ldots\xi_{m_kn_k}\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\xi_{r_2s_2}\ldots\xi_{r_ks_k}\xi_{r_ks_k}\xi_{r_1s_k})$.

and

$(\tau_p(\Lambda^k))^2 =$

$$= \frac{1}{p^2 n^{2k}} \sum_{m_1,m_2,\ldots,m=1}^{p} \sum_{r_1,r_2,\ldots,r_k=1}^{p} \sum_{n_1,n_2,\ldots,n_k=1}^{n} \sum_{s_1,s_2,\ldots,s_k=1}^{n}$$

$$E(\xi_{m_1n_1}\xi_{m_2n_1}\xi_{m_2n_2}\xi_{m_2n_2}\ldots\xi_{m_kn_k}\xi_{m_1n_k})\cdot$$
$$\cdot E(\xi_{r_1s_1}\xi_{r_2s_1}\xi_{r_2s_2}\ldots\xi_{r_ks_k}\xi_{r_ks_k}\xi_{r_1s_k}),$$

we get

$E((tr_p(\Lambda^k) - \tau_p(\Lambda^k))^2) =$

$$= \frac{1}{p^2 n^{2k}} \sum_{m_1,m_2,\ldots,m=1}^{p} \sum_{r_1,r_2,\ldots,r_k=1}^{p} \sum_{n_1,n_2,\ldots,n_k=1}^{n} \sum_{s_1,s_2,\ldots,s_k=1}^{n}$$

$$(E(\xi_{m_1n_1}\xi_{m_2b_1}\ldots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\ldots\xi_{r_1s_k})-$$
$$-E(\xi_{m_1n_1}\xi_{m_2n_2}\ldots\xi_{m_1n_k})\cdot E(\xi_{r_1s_1}\xi_{r_2s_1}\ldots\xi_{r_1s_k})).$$

Let us introduce two numbers, L_1 and L_2 as follows

$$L_1 = \#\{m_1, m_2, \ldots, m_k, r_1, r_2, \ldots, r_k\},\ L_2 = \#\{n_1, n_2, \ldots, n_k, s_1, s_2, \ldots, s_k\}.$$

If we group our terms according to the values of L_1 and L_2, we get

$E((tr_p(\Lambda^k) - \tau_p(\Lambda^k))^2) =$

$$= \frac{1}{p^2 n^{2k}} \sum_{L_1} 1^{2k} \sum_{L_2=1}^{2k} \sum_{(L_1,L_2)} (E(\xi_{m_1n_1}\xi_{m_2n_1}\ldots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\ldots\xi_{r_1s_k})-$$
$$-E(\xi_{m_1n_1}\xi_{m_2n_1}\ldots\xi_{m_1n_k})\cdot E(\xi_{r_1s_2}\xi_{r_2s_1}\ldots\xi_{r_1s_k}),$$

where the last summation is for those terms in which L_1 and L_2 have certain fixed values.

We show that the contribution of those terms for which $L_1 + L_2 \le 2k$ is at most of order n^{-2}.

Due to Hölder inequality

$$\left|E\left(\xi^{(n)}_{m_1n_1}\xi^{(n)}_{m_2n_1}\dots\xi^{(n)}_{m_1n_k}\right)\right| \le E\left(|\xi^{(n)}_{m_1n_1}|^k\right)^{\frac{1}{k}}\dots E\left(|\xi^{(n)}_{m_1n_k}|^k\right)^{\frac{1}{k}} \le C_k(n).$$

(The existence of such $C_k(n)$'s is trivial since the $|\xi_{ij}$ matrix elements are identically distributed and all the moments are bounded.) This estimate implies

$$\frac{1}{p^2n^{2k}}\left|\sum_{(L_1,L_2)}(E(\xi_{m_1n_1}\dots\xi_{r_1s_k}) - E(\xi_{m_1n_1}\dots\xi_{m_1n_k})\cdot E(\xi_{r_1s_1}\dots\xi_{r_1s_k}))\right| \le$$

$$\le \frac{1}{p^2n^{2p}}\sum_{(L_1,L_2)}(|E(\xi_{m_1n_1}\dots\xi_{r_1s_k} - E(\xi_{m_1n_1}\dots\xi_{m_1n_k})\cdot E(\xi_{r_1s_1}\dots\xi_{r_1s_k})|) \le$$

$$\le \frac{1}{p^2n^{2k}}\sum_{(L_1,L_2}(C_{2k}(n) + C_k(n)^2) \le$$

$$\le \frac{1}{p^2n^{2k}}\binom{p}{L_1}L_1^{2k}\binom{n}{L_2}L_2^{2k}(C_{2k}(n) + C_k(n)^2) =$$

$$= \frac{p(p-1)\dots(p-L_1+1)\cdot n(n-1)\dots(n-L_2+1)}{p^2n^{2k}}\cdot$$

$$\cdot\frac{L_1^{2k}\cdot L_2^{2k}}{L_1!\cdot L_2!}\left(C_{2k}(n) + C_k(n)^2\right)$$

and the latest is of order n^{-2} whenever $L_1 + L_2 \le 2k$.

(In the fourth line we used the estimation that the number of sequences corresponding to L_1 and L_2 is not bigger than

$$\binom{p}{L_1}L_1^{2k}\binom{n}{L_2}L_2^{2k},$$

what can be easily checked, keeping in mind that there are p possibilities for each m_i and r_i and n possibilities for each n_i and s_i.) Next we show that if $L_1 + L_2 > 2k + 2$ then

$$\begin{aligned}(2.1)\qquad & E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}) - \\ & - E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k})\cdot E(\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}) = 0.\end{aligned}$$

First we note that

$$E(\xi_{m_1 n_1}\xi_{m_2 n_1}\xi_{m_2 n_2}\xi_{m_3 n_2}\dots\xi_{m_k n_k}\xi_{m_1 n_k}) = 0 \tag{2.2}$$

whenever $l = \#\{m_1, m_2, \dots, m_k\} + \#\{n_1, n_2, \dots, n_k\} > k+1$, as it was shown in [4]. This can be easily proved by induction in k, showing that in every sequence with this property there is at least one ξ_{ij} without repetition which, because of the zero mean and the independence of the ξ_{ij}'s justifies equation (2.2). For further details see the proof of equation (2.4) of [4].

If $L_1 + L_2 > 2k+2$ then $\#\{m_1, m_2, \dots, m_k\} + \#\{n_1, n_2, \dots, n_k\} > k+1$ or $\#\{r_1, r_2, \dots, r_k\} + \#\{s_1, s_2, \dots, s_k\} > k+1$ (at least one of them holds) which according to the above mentioned fact gives that at least one of $E(\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k})$ and $E(\xi_{r_1 s_2}\xi_{r_2 s_1}\dots\xi_{r_1 s_k})$ vanishes yielding

$$E(\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k})\cdot E(\xi_{r_1 s_1}\xi_{r_2 s_1}\dots\xi_{r_1 s_k}) = 0.$$

We show that $E(\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k}\cdot\xi_{r_1 s_1}\xi_{r_2 s_1}\dots\xi_{r_1 s_k}) = 0$ also holds. If in the sequence $\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k}\cdot\xi_{r_1 s_1}\xi_{r_2 s_1}\dots\xi_{r_1 s_k}$ $r_1 = m_1$ then it is of the same type as the one in equation (2.2), with a length of $2k$ and with $l = L >$ $> 2k+2 > 2k+1$, which gives $E(\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k}\cdot\xi_{r_1 s_1}\xi_{r_2 s_1}\dots\xi_{r_1 s_k}) = 0$ on the line of the previous paragraph. If $r_1 \neq m_1$ then this argument cannot be applied directly, but changing the value of every m-type and r-type indices that equals r_1 to the value of m_1 we still get a sequence for which the above argument works since by this procedure we have decreased L exactly by one, so for the length of $2k$ we still have $l = L-1 > 2k+1$, which again gives $E(\xi_{m_1 n_1}\xi_{m_2 n_1}\dots\xi_{m_1 n_k}\cdot\xi_{r_1 s_1}\xi_{r_2 s_1}\dots\xi_{r_1 s_k}) = 0$. (Precisely this we get for the sequence obtained by the changing of certain indices, but as the reason of the vanishing of the expectation value is that in the corresponding sequence there is at least one ξ_{ij} without repetition, one can see that the same holds for the original sequence, too, as by changing the value of some indices to a value that is already present in the sequence we can not produce "lonely" ξ_{ij}'s.)

We have seen that those terms for which $L_1 + L_2 \le 2k$ have a contribution of order n^{-2} or less. On the other hand those terms for which $L_1 + L_2 > 2k+2$ have no contribution at all. Next we show that the terms for which $L_1 + L_2 =$ $= 2k+1$ or $L_1 + L_2 = 2k+2$ have no contribution either. This means that all the terms have a contribution of order n^{-2} (at most), which proves our theorem as $\sum n^{-2} < +\infty$.

We show that if $L_1 + L_2 = 2k+1$ or $L_1 + L_2 = 2k+2$ then for the corresponding terms (2.2) holds.

If in the sequence $\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}$ there is at least one ξ_{ij} without repetition, then because of independence and as $E(\xi_{ij})=0$ both $E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k})$ and $E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k})\cdot$ $\cdot\, E(\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k})$ vanish, giving (2.2).

If there is no "lonely" ξ_{ij} in $\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}$, but every group of ξ_{ij} that equals each other are in the same half of the long sequence (that is there is no $\xi_{m_1n_j}=\xi_{r_ps_q}$ type equality for the sequence) then because of the independence of the ξ_{ij}'s

$$E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k})=$$
$$=E(\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k})\cdot E(\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}),$$

so (2.2) holds again.

The only problem may arise in connection with those sequences in which there is no "lonely" ξ_{ij} and there exist such m_i, n_j, r_p, s_q for which $\xi_{m_in_j}=$ $=\xi_{r_ps_q}$. We show that there are no such sequences if $L_1+L_2\geq 2k+1$.

Let us take an arbitrary sequence $\xi_{m_1n_1}\xi_{m_2n_1}\dots\xi_{m_1n_k}\cdot\xi_{r_1s_1}\xi_{r_2s_1}\dots\xi_{r_1s_k}$ for which $L\geq 2k+1$ and in which there is no "lonely" ξ_{ij}. (If there are no such sequences, then we are ready with the proof.) Let us denote the length of the sequence (i.e. the number of ξ_{ij}'s in it) by H. (So $H=4k$.) As $L>2k+1$ there is at least one x_p (m_p, n_p, r_p or s_p) index which appears only in two ξ_{ij}'s, otherwise each index would appear in at least four ξ_{ij}'s according to the structure of the sequence, but $4L>8k+4$ indices cannot fit in $2H=8!$ sites! As this x_p index appears only in two ξ_{ij}'s and according to our assumption that there is no ξ_{ij} without repetition, this two ξ_{ij}'s form a pair. It is also clear from the structure of the sequences that this two ξ_{ij}'s are neighbors and are in the same half of the long sequence (taking cyclic order in both halves, that is $\xi_{m_1n_1}$ and $\xi_{m_1n_k}$ and also $\xi_{r_1s_1}$ and $\xi_{r_1s_k}$ are considered to be neighbors.)

Let us eliminate this pair from the sequence! By this the value of L decreases by one (x_p disappears, but the other index of the eliminated pair can be found in the neighboring ξ_{ij}'s, too because of the structure of the sequence), while the value of H decreases by two. This way we get a shorter sequence for which $H_1=H-2=4k-2$ and $L_1=L-1\geq 2k$, and which have a similar structure to the original one, though one "half" will be bigger than the other. Indeed, if for example m_p was the index appearing only in two ξ_{ij}'s, than in the subsequence

$$\dots\xi_{m_{p-1}n_{p-1}}\xi_{m_pn_{p-1}}\xi_{m_pn_p}\xi_{m_{p+1}n_1}\dots$$

the second and the third ξ_{ij} form a pair, which means $n_{p-1} = n_p$ and this implies that the structure of the indices in the remaining sequence after eliminating the pair remains the same. If the eliminated pair was $\xi_{m_1 n_1}$ and $\xi_{m_1 n_k}$ (that is the meaning of x_p was m_1), then writing $\xi_{m_2 n_1}$ after $\xi_{m_k n_k}$ (this can be done as the ξ_{ij}'s commute), we get a sequence of the desired structure. Of course the same argument works if x_p is an n_p, r_p or s_p. It is also clear that in the new sequence there is still no "lonely" ξ_{ij}, as the eliminated pair could not have been the member of a triplet since x_p appeared only in them.

In the sequence we obtain by eliminating the corresponding pair there is still at least one x_p (m_p, n_p, r_p or s_p) index which appears only in two ξ_{ij}'s, otherwise each index would appear in at least four ξ_{ij}'s according to the structure of the sequence, but $4L_1 > 8k$ indices cannot fit in $2H_1 = 8k - 4$ sites! This two ξ_{ij}'s form a pair again, and eliminating it we get an even shorter sequence of the same type, for which $L_2 = L - 2$ and $H_2 = H - 4$. The method can be continued until one of the two "halves" becomes the product of two ξ_{ij}'s. Even in this sequence there is at least one x_p (m_p, n_p, r_p or s_p) index which appears only in two ξ_{ij}'s. But if this two ξ_{ij}'s are the last ones in their half, eliminating them L could decrease by two (for example take the sequence $\xi_{11}\xi_{11}\cdot\xi_{r_1 s_1}\xi_{r_2 s_1}\ldots\xi_{r_1 s_k}$, where $r_i \neq 1$ and $s_i \neq 1$ for all $i = 1,\ldots,k$), what would stop our procedure since in that case the key relation $4L_i > 2H_i$ would not hold any more. But if x_p appears (only) in this two ξ_{ij}'s, it means that none of these ξ_{ij}'s can be identical with any ξ_{ij} of the other half. Of course this also holds for the previously eliminated pairs of this half (as each pair had an index appearing only in them), which means that in the original sequence there was $\xi_{m_i n_j}\xi_{r_p s_q}$ type relation, so in this case we are ready with the proof. If x_p is in the other half we can go on with our procedure with no problem until in one of the following steps x_p is in the half containing only two ξ_{ij}'s – when we can use again the above argument, or until the other half reduces to the product of two ξ_{ij}'s, too – when the above argument can be applied to either half in the next step, completing the proof of our statement as well as that of Theorem 2.1.

Note that if ξ_{ij} is complex valued, more precisely $\xi_{ij} = a\eta_{ij}^{(n,re)} + b\eta_{ij}^{(n,im)}$, where $\eta_{ij}^{(n,re)}$ and $\eta_{ij}^{(n,im)}$ form an independent identically distributed standard real family and for the real numbers a and b the relation $a^2 + b^2 = 1$ holds, than for the complex matrix $\Lambda^{(n)} = T^{(n)}\left(T^{(n)}\right)^* / n$ the statement of Theorem

2.1 is still true. The reason for this is the fact that in (2.1) only products $|\xi_{ij}|^2 = \xi_{ij}\overline{\xi}_{ij}$ appear under the expectation value in the contributing terms.

References

[1] T. W. ANDERSON, *An introduction to multivariate statistical analysis,* J. Wiley, New York, 1958.

[2] Z. FÜREDI and J KOMLÓS, The eigenvalues of random symmetric matrices, *Combinatorica,* **1** (1981), 233–241.

[3] V. A. MARCHENKO and L. A. PASTUR, The distribution of eigenvalues in certain sets of random matrices, *Mat. Sb.,* **72** (1967), 507–536.

[4] F. ORAVECZ, D. PETZ, On the eigenvalue distribution of some symmetric random matrices, *Acta Sci. Math.,* **63** (1997), 383–395.

[5] C. R. RAO, *Linear statistical inference and its applications,* J. Wiley, New York, 1973.

[6] K. W. WACHTER, The strong limit of random matrix spectra for sample matrices of independent elements, *Ann. Probab.,* **6** (1978), 1–8.

[7] E. P. WIGNER, On the distribution of the roots of certain symmetric matrices, *Ann. Math.,* **67** (1958), 325–327.

ANNALES UNIV. SCI. BUDAPEST., **41** (1998), 103–115

DIE COMPUTERGESTÜTZTE KLASSIFIZIERUNG DER FLÄCHENEINWICKELUNGEN IN EINEM VIELECK VORGEGEBENER SEITENANZAHL

von

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Einführung

Aus der Flächentopologie ist es wohlbekannt, daß jede geschlossene Fläche (weiterhin nur Fläche genannt) triangulierbar ist. Eine Fläche kann also in einem Vieleck eingewickelt werden. Die Seitenpaare des Vielecks, die dabei identifiziert werden, definieren einen Graph, entlang dem das Vieleck zusammengeklebt wurde, oder umgekehrt, entlang dem die Fläche auseinander geschnitten werden kann, siehe [L-M 1991]. Wie wir wissen, siehe [L-M 1990], kann eine (geschlossene) Fläche F von Geschlecht g stets in einem $2n$-Eck eingewickelt werden, wenn $2n$ eine gerade Zahl ist, die die folgende Bedingung erfüllt:

$$2\alpha g \leq 2n \leq 6\alpha g - 6, \tag{1.1}$$

wobei $n = 0, 1$ zweckmäßig erlaubt sind, und

$$\alpha := \begin{cases} 2 \text{ falls } F \text{ orientierbar ist, und } 0 \leq g; \\ 1 \text{ sonst, dann } 1 \leq g. \end{cases}$$

Die untere Schranke wird dann erreicht, wenn sämtliche Ecken in einem Punkt zusammengeklebt werden. Die obere Schranke wird dann erreicht wenn die Ecken zu

$$x = 2\alpha g - 2 \tag{1.2}$$

verschiedenen Punkten auf F führen. In diesem Fall hat jede Ecke des – durch die Einwicklung – definierten Graphs G den Eckengrad 3.

Ein natürliches Problem ist, zu jeder Fläche F alle möglichen (bisauf kombinatorische Äquivalenz) Einwicklungsvielecke oder Einwicklungsgraphen aufzuzählen. Das war der erste Schritt des Programmes, das in [L-M

1991] aufgestellt wurde. In der vorliegenden Arbeit lösen wir dieses Problem und die besonders wichtige Spezialisierung für die ebenen diskontinuierlichen Gruppen (diese sind zu den Fundamentalgruppen der Flächen isomorph) mit Signature

$$\Gamma = (+ \text{ oder } -,\ g;\ [\ ];\ \{\ \}),\quad g \geq 0 \text{ oder } g \geq 1. \tag{1.3}$$

Das Resultat von [L-J-V 1990] zum Beispiel für die Brezelfläche (d.h. Doppeltorus) mit $\Gamma = (+, 2)$ ist (siehe auch [10]):

(1.4)

8-ecke: 4	10-ecke: 18	12-ecke: 34
14-ecke: 38	16-ecke: 20	18-ecke: 8
insgesammt: 122		

Eine andere sinnvolle Fragestellung ist, für ein gegebene $2n$-Eck alle möglichen Seitenpaarungen und so die möglichen Flächen mit entsprechenden obigen Einwicklungsgraphen zu bestimmen. Mit dieser Frage hatte ich mich in [S 1988] beschäftigt. Das Problem wurde algorithmisch im Sinne [M 1992] gelöst, wie es in Sekt. 2 folgt. Wegen des exponenziell zunehmenden Zeitaufwand konnte ich es damals auf einem Commodore 64 nur bis $2n = 10$ gerechnet werden.

Zur selben Zeit algorithmisierte D. HUSON die von A. DRESS vorgeschlagene und entwickelte Methode der Delaney-Dress-Symbole [D-S 1984], [D-H 1987], [D-H-Z 1992] zunächst für euklidische Pflasterungen. Eine begonnene Zusammenarbeit in [D-H-M 1993] inspirierte weitere Arbeiten wie z.B. [L-M-S 1994]. In [H 1992] ist unter Anderen ein Algorithmus beschreiben, der alle “fundamentalen” Flächen-transitiven Pflasterungen aufzählt, deren N-eckigen Pflastersteine – natürlich gleichzeitig, wegen der Operation der entschprechenden Gruppen – $q(i)$ Ecken vom Grad i haben. Dem Algorithmus gibt man die Typusfunktion.

$$i \to q(i) \quad \text{mit} \quad i \geq 3, \qquad \sum_i q(i) = N = 2n \tag{1.5}$$

beliebig vor, und der Algorithmus generiert dann alle möglichen Typen von Pflasterungen (der euklidische Ebene, der Sphäre oder der hyperbolischen Ebene).

Dieser algorithmus kann auch zu einer Lösung unseres oben erwähnten speziellen Problems in folgenden Formulierung führen:

Man soll alle ebenen Pflasterungen mit den folgenden Eigenschaften bestimmen: Die Steine sind $2n$-Ecke und jede Ecke hat den Grad $2n$. Auf der Pflasterung P operiert eine Homeomorphismengruppe Γ, und zwar auf den Steinen und Ecken von P einfach transitiv.

Zwei Pflasterungen (P_1, Γ_1) und (P_2, Γ_2) heißen *kombinatorisch äquivalent* (*topologisch äquivariant*), wenn eine Homeomorphie $\varphi : P_1 \to P_2$ mit $\Gamma_2 = \varphi \Gamma_1 \varphi^{-1}$ mit existiert. Für die interresanten Fälle $2n \geq 4$ bekommen wir eben die kombinatorische Klassifikation der Flächeneinwickelungen in einem $2n$-Eck, wenn $2n = 2\alpha g$ wie oben angedeutet wurde. $2n = 4$ führt zu den euklidisch-metrischen Realisierungen des Torus (kombinatorisch nur eine) und der Kleinschen Flasche Abb.4.1. (kombinatorisch nur zwei Realisierungen). $2n \geq 6$ führen zu den hyperbolisch-metrischen Realisierungen der nicht orientierbaren Flächen vom Geschlecht $g \geq 3$ und der orientierbaren Flächen mit $g \geq 2$. Kombinatorisch gibt es nur endlich viele Fälle, wie die Tabelle zeigt.

2n	Die Anzahl der Äquivalenzklassen der orientierbaren Seitenpaarungen des $2n$-Eckes
4	2
6	5
8	17
10	79
12	555
14	5284

Tabelle 1.

Unsere Resultate stimmen mit den anderen analogen Fällen aus [L-J-V 1990], [H 1992], [L-M-V 1998] überein, die durch andere Methode gewonnen wurden.

Wir gehen hier ein bischen auch weiter. Wir geben untere und obere Schranken für die auftretenden Anzahlen, die die hohe Komplexität des Problems zeigen. Die Schranken sind grob genug aber hinreichend zu unserem Ziel:

$$\frac{(2n-1)!}{2^{n+1}n!} < o(2n) < \frac{(2n-1)!}{2^{n-1}(n-1)!} \approx \sqrt{2}e^{n \log n - n(1-\log 2)}, \tag{1.6}$$

$$\frac{(2n-1)!}{2n!} < s(2n) < \frac{2(2n-1)!}{(n-1)!} \approx \sqrt{2}e^{n \log n + n(\log 4 - 1)}. \tag{1.7}$$

Hier bezeichnet $o(2n)$ die Anzahl der kombinatorisch nichtäquivalenten orientierbaren Paarungen des $2n$-Eckes, die alle Einwickelungen der orientierbaren Flächen von Geschlecht $o \leq g \leq \frac{n}{2}$ darstellen, $s(2n)$ bezeichnet dieselbe Anzahl für alle (orientierbare und nichtorientierbare) Flächen.

Die Seitenpaarungen eines $2n$-Ecks

Wir betrachten ein $2n$-Eck P $(2n \geq 4)$ und nehmen den üblichen positiven Umlaufsinn (Abb.2.1.). Eine Paarung (Identifizierung) I ordnet jeder gerichteten Seite s_i^{-1} eine andere gerichtete Seite s_i mit einer identifizierenden Homeomorphie s_i zu. Man kann die Paarung I von P wie eine *involutorische Permutation* der zyklisch numerierten Seiten angeben, wo im Index auch die entsprechenden Anfangsecken aufgezeichnet sind. Die Ecke i ist der Schließungspunkt der Seiten $i-1$ und i (mod $2n$).

Zwei Paarungen I_1 und I_2 heißen *kombinatorisch äquivalent*, wenn es eine Automorphie γ von P gibt, sodaß

(2.1) $$I_2 = \gamma I_1 \gamma^{-1}.$$

Da die Automorphismengruppe von P (die Diedergruppe D_{2n} von Ordnung $4n$) aus Ecken-Seiten-Inzidenz treuen Bijektionen bestehen, so dürfen wir sagen: I_1 und I_2 heißen äquivalente Paarungen, wenn sie mit einem Automorphismus von P konjugiert sind. Nun können wir nach den Representanten und den Anzahlen der obigen Äquivalenzklassen fragen. *Wir wollen jedoch jede solche Paarung ausschließen, die zu einer Eckenäquivalenzklasse mit weniger als drei Elemente führt.* In Abb. 2.1. haben wir alle vier nicht-äquivalenten Paarungen dargestellt, die die verschiedenen Einwicklungsgraphen der Brezelfäche mit einer Eckenklasse ergeben (siehe Formel (1.4)). Zum Beispiel ist die zweite Paarung (Abb. 2.1.b) mit der involutorischen Permutation der Seiten (Anfangspunkte in den Indizen) wie folgt beschreiben:

(2.2) $$(1_1, 3_4)(2_2, 5_6)(4_4, 7_8)(6_6, 8_1).$$

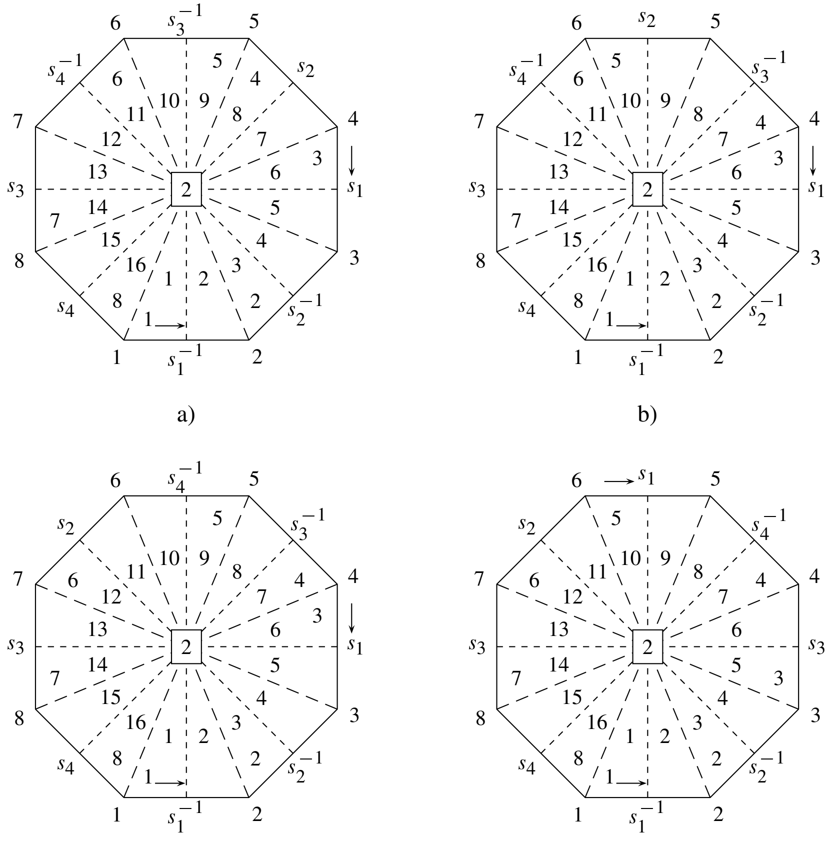
Die Eckpunkte sind dann alle nach

$$1 \xrightarrow{s_1} 4 \xrightarrow{s_3} 8 \xrightarrow{s_4^{-1}} 7 \xrightarrow{s_3^{-1}} 5 \xrightarrow{s_2^{-1}} 3 \xrightarrow{s_1^{-1}} 2 \xrightarrow{s_2} 6 \xrightarrow{s_4} 1$$

äquivalent, das uns eine Darstellung der Fundamentalgruppe

(2.4) $$\Gamma = (s_1, s_2, s_3, s_4, \ -1 = s_4 s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_1)$$

der Brezelfläche liefert.



a) b) c) d)

Abb. 2.1. Alle Seitenpaarungen des 8-ecks P sind dargestellt, die zu den 4 Einwickelungen der Brezelfläche (d.h. Doppeltorus) mit einer Eckenäquivalenzklasse führen. Die Ecke (im Äußeren) und die Seite (im Inneren) sind durch 1, 2, … …, 8 numeriert jede Paarung I ordnet zur positiv-gerichteten Seite s_i^{-1} die gerichtete Seite s_i durch die identifizierende topologische Abbildung s_i (i = 1, 2, 3, 4).

Dieselbe Zuordnung kann auch mit D-Symbole beschrieben werden. Dazu bildet man die baryzentrischen Unterteilung von P auf Dreiecke mit gemeinsamen 2-Zentrum in dem formalen Mittelpunkt von P, mit 1-Zentren in Seitenmittelpunkten und 0-Zentren in Eckpunkten von P. Die gegenüberliegenden (punktierten) 0-Seiten, (gebrochene) 1-Seiten und die 2-Seiten führen zu den σ_0-, σ_1-, bzw. σ_2-Operationen auf der Menge der Dreiecken, d.h. der 16 Elementen der D-Mengen. Die kritische σ_2-Operation ist eben die Paarung, wie in der Formel (2.6) für Abb.2.1.b gegeben ist.

Ebenso können wir im algemeinen Fall für eine Vieleckspaarung die Eckenäquivalenzklassen und die entsprechenden Relationen für die Fundamentalgruppe der dargestellten Fläche aufschreiben. Das Verfahren, wie auch die

Paarungsäquivalenz, ist leicht algorithmisierbar. Man kann sowohl die Orientierbarkeit der Fläche F als auch das Geschleht, nach der Euler-Charakteristik

$$\chi(F) = f_0 - f_1 + f_2 = 2 - \alpha g \tag{2.5}$$

leicht erkennen. Hier, wie üblich, bezeichnet f_i die Anzahl der Äquivalenzklassen des i-dimensionalen Bestandteils des Vielecks P (wobei hier stets $f_2 = 1$), und $\alpha = 2$ für den orientierbaren, $\alpha = 1$ für den nicht-orientierbaren Fall. F ist nicht-orientierbar, wenn P mindestens ein Seitenpaar hat, dessen Richtungen beide im gleichen Sinne mit der Orientierung von P gerichtet sind.

Wir erwähnen, daß die Seitenpaarung in (2.2) auch mit D-Symbolen (zu Ehre von B. N. DELONE (Delaunay), M. S. DELANEY und A. W. M. DRESS) dargestellt werden kann (siehe z.B. in [H 1992]. Dann ist die kritische σ_2-Operation für baryzentrischen Dreiecken 1–16 für Abb.2.1.b wie folgt:

$$\sigma_2 : (1,6)(2,5)(3,10)(4,9)(7,14)(8,13)(11,16)(12,15) \tag{2.6}$$

z.B. $(1,6)(2,5)$ beschreibt das Erzeugende s_1 der Gruppe Γ in (2.4).

Dieselbe Paarung wird in meinem Algorithmus mit der Formel

(2.7) (10)(20)(10)(30)(20)(40)(30)(40) oder kürzer mit 1 2 1 3 2 4 3 4

gekennzeichnet.

3. Der Algorithmus und Beispiele

Im ersten Teil beschäftige ich mich mit orientierbaren Seitenpaarungen des $2n$-Eckes bisauf Automorphien γ (2.1) von der Diedergruppe D_{2n} der Ordnung $4n$.

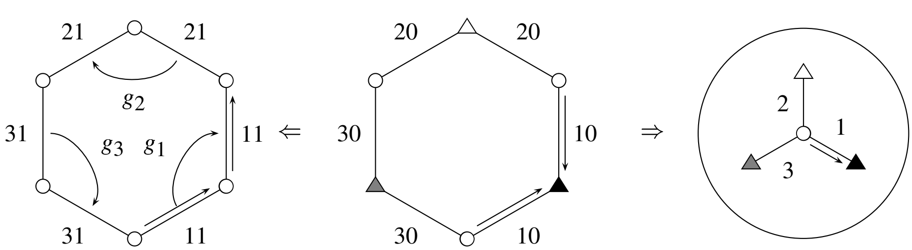


Abb. 3.1.

Im zweiten Teil werden, zu jeder früher gewonnenen Paarung, die möglichen nichtorientierbaren Paarungen mit mindestens einer Zuordnung des Types $\ldots(i1)\ldots(i1)\ldots$ der i-ten Seiten bestimmt, beide nach dem positiven Umlaufsinn des $2n$-Eckes P. Wir zeigen in Abb.3.1 eine solche Paarung

$$(3.1) \qquad (11)(11)(21)(21)(31)(31); \qquad n3,\ (6)$$

des Sechseckes, die zu einer nichtorientierbaren Fläche von Geschlecht 3 führt, denn die 6 Ecken fallen in eine Äquivalenzklasse. Das erscheint auch bei (3.1).

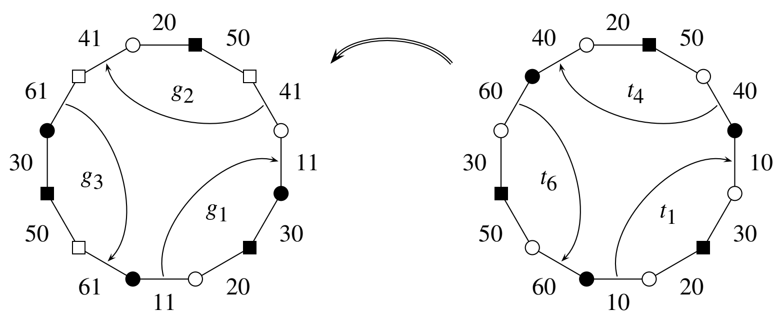


Abb. 3.2.

Die entsprechende orientierbare Paarung würde die Sphäre darstellen, mit trivialen Erzeugenden und Gruppe $\Gamma = 1$. Ein solches Paarung wird im Weiteren ausgeschlossen, denn es gibt eine Eckenklasse, sogar drei solche, zu der weniger als 3 Ecken gehören. Eine solche Einwickelung der Sphäre ist wie trivial (meisterhaft) betrachtet, obwohl sie in die spätere Abschätzungen eingerechnet wird (in der Tabelle 1.). Andere Beispiele stellt Abb.3.2 dar. Die Formel

$$(3.2) \quad (11)(20)(30)(11)(41)(50)(20)(41)(61)(30)(50)(61); \qquad n3,\ (3,3,3,3)$$

schreibt das erste Bild, eine nicht-orientierbare Fläche von Geschlecht 3 (d.h. $n3$) mit 4 Eckenklassen je mit 3 Ecken. Das zweite Bild ist die entsprechende orientierbare Fläche von Geschlecht 2 (d.h. $p2$), kurz

$$(3.3) \qquad 1\ 2\ 3\ 1\ 4\ 5\ 2\ 4\ 6\ 3\ 5\ 6; \qquad p2,\ (3,6,3).$$

Nochmals zeigt die Formel (2.5), daß wir eine Brezelfläche $p2$ mit 3 Eckenklassen $(3,6,3)$ besitzen. In der hyperbolischen Ebene H^2 kann man beide 12-Ecke metrisch realisieren [L-M 1991].

4. Die Resultate, Abschätzungen für nicht äquivalenten Seitenpaarungen eines $2n$-Eckes

Die Tabelle 2 enthält für jede $2n$-Eck $1 \leq n \leq 7$ die Anzahl der entsprechenden Einwickelungen der orientierbaren Flächen pg bzw. der nichtorientierbaren Flächen ng von Geschlecht g.

$2n$	$p1$	$p2$	$p3$	$n2$	$n3$	$n4$	$n5$	$n6$	$n7$
4	1			2					
6	1			2	8				
8		4			22	47			
10		18			24	279	473		
12		34	82		11	682	4928	7192	
14		38	1022			838	20979	110266	144906

Tabelle 2.

Aus der Tabelle der Ergebnisse kann man das „exponenzielle Wachstum” z.B. der Anzahl $s(n)$ der nichtäquivalenten Seitenpaarungen des $2n$-Eckes auslesen.

(1) Wir stellen zunächst elementare Schranke für die orientierbare Seitenpaarungen des $2n$-Eckes. Wir legen das $2n$-Eck mit Ecken- und Seitennumerierung nach dem positiven Umlaufsinn fest (siehe Abb.2.1). Zur positiv-gerichteten Seite 1 ordnen wir eine andere Seite nach der entgegengesetzte Richtung. Das kann man auf $2n-1$ verschiedenen Weisen machen. Dann kommt die nächste freie positive Seite mit einer in Gegenrichtung zugeordneten neuen Seite. Nach dem ersten festen Paar kann man das zweite Paar auf $2n-3$ Weisen auswählen, und so weiter. Endlich können wir

$$(2n-1)(2n-3)\ldots\cdot 3\cdot 1 = \frac{(2n-1)!}{2^{n-1}(n-1)!} \tag{4.1}$$

orientierbare Paarungen im allgemeinen Sinne konstruieren. Unter den Eckenäquivalenzklassen können nämlich auch ein- oder zweielementige Klassen vorkommen, wie in Abb.3.1 oder in Abb.4.1. In (4.1) haben wir eine obere Schranke für die Anzahl $o(2n)$ der Äquivalenzklassen der orientierbaren Seitenpaarungen, denn jede feste Paarung kann ihre Äquivalenten bei den Diedersymmetrien des am Anfang festen $2n$-Eckes

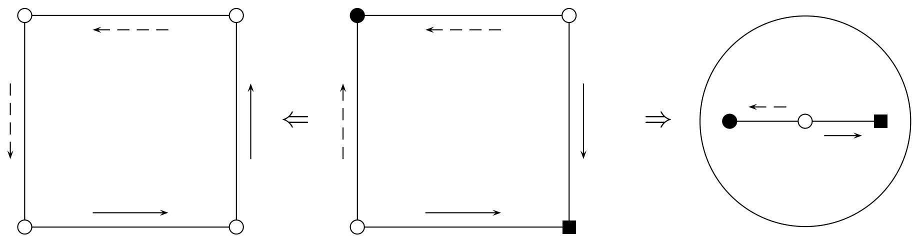


Abb. 4.1.

haben. Diese Symmetriegruppe ist die Diedergruppe der Ordnung $4n$, wie die Presentation

(4.2) $$D_{2n} = (\sigma_0, c - 1 = \sigma_0^2 = c^{2n} = \sigma_0 c \sigma_0 c)$$

zeigt. Hier ist

(4.3) $$c = (1, 2, 3, \ldots, (2n))$$

Die zyklische Eckenpermutation der Ordnung $2n$, und

(4.4) $$\sigma_0 = (1, 2n)(2, 2n - 1) \ldots (n, n + 1)$$

ist eine Spiegelung mit zweielementigen Eckenzyklen. Eine andere Spiegelung ist

(4.5) $$\sigma_1 = \sigma_0 c = (1)(2, 2n) \ldots (n, n + 2)(n + 1).$$

Eine feste Paarung kann bei der obigen Aufzählung höchstens $4n$ Äquivalenten haben. Durch $4n$ dividiert in (4.1), kriegt man eine untere Schranke für $o(2n)$:

(4.6) $$\frac{(2n - 1)!}{2^{n+1} \cdot n!} < o(2n) < \frac{(2n - 1)!}{2^{n-1}(n - 1)!},$$

denn einige Paarungen transformieren in sich bei gewissen Untergruppen von D_{2n}. Zum Beispiel

(4.7) $$1\ 2\ 3\ \ldots\ n\ 1\ 2\ 3\ \ldots\ n$$

transformiert in sich unter der vollen Gruppe D_{2n}, denn jedes Element von D_{2n} läßt die Paarung in (4.7) invariant, wir können nämlich von der Numerierung der Paare absehen.

(2) Für die $s(2n)$ Äquivalenzklassen der sämtlichen Paarungen können wir einen analogen Gedankengang verfolgen. Zum festen $2n$-Eck können wir das erste Paar auf $2 \cdot (2n - 1)$ Weisen auswählen, weil die Richtung auf der ersten Bildseiten mit dem Umlaufsinn entweder entgegengesetzt

oder gleichgesetzt sein kann. Diese 2 Möglichkeiten kommen für jedes Seitenpaar vor. So gewinnen wir die Schranke

$$\frac{(2n-1)!}{2\cdot n!} < s(2n) < \frac{2\cdot(2n-1)!}{(n-1)!}. \tag{4.8}$$

Die Formel von Stirling

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi\cdot n} \tag{4.9}$$

ermöglicht uns die Schranke bequemer abzuschätzen ($e = 2{,}71828\ldots$ ist die Eulersche Zahl).

Zum Beispiel aus (4.6) haben wir

$$\frac{(2n-1)!}{2^{n+1}\cdot n!} \approx \frac{\left(\frac{2n-1}{e}\right)^{2n-1}\cdot\sqrt{2\pi(2n-1)}\cdot\left(\frac{n}{e}\right)^{2n-1}}{2^{n+1}\left(\frac{n}{e}\right)^n\sqrt{2\pi\cdot n}\cdot\left(\frac{n}{e}\right)^{2n-1}} = \tag{4.10}$$

$$= \frac{\left(2-\frac{1}{n}\right)^{\frac{1}{2}}}{2^{n+1}}\left(2-\frac{1}{n}\right)^{2n-1}\cdot\left(\frac{n}{e}\right)^{n-1} =$$

$$= 2^{n-\frac{3}{2}}\cdot\left(1-\frac{1}{2n}\right)^{2n-\frac{1}{2}}\cdot\left(\frac{n}{e}\right)^{n-1} \approx \frac{1}{\sqrt{8}}2^n\cdot e^{-1}\cdot\left(\frac{n}{e}\right)^{n-1} =$$

$$= \frac{1}{\sqrt{8}}e^{n\log 2+(\log n-1)(n-1)-1} = \frac{1}{\sqrt{8}}e^{n\log n+n(\log 2-1)-\log n}.$$

So gewinnen wir die folgenden Schranke:

$$\frac{1}{\sqrt{8}}e^{n\log n-n(1-\log 2)-\log n} < o(2n) < \sqrt{2}\cdot e^{n\log n-n(1-\log 2)}, \tag{4.11}$$

$$\frac{1}{\sqrt{8}}e^{n\log n+n(\log 4-1)-\log n} < s(2n) > \sqrt{2}e^{n\log n+n(\log 4-1)}, \tag{4.12}$$

wie in den Formeln (1.6–7) in der Einführung. Aus unseren Abschätzungen ergeben sich z.B.

$$\begin{array}{rcccl} 4826 & < & o(14) & < & 135135 \\ 617760 & < & s(14) & < & 17297280 \\ 63344 & < & o(16) & < & 2027025 \\ 16216200 & < & s(16) & < & 518918400 \end{array} \tag{4.13}$$

nach den Formeln (4.6) und (4.8), und

$$\begin{array}{ccccc} 4855 & < & o(14) & < & 135941 \\ 621444 & < & s(14) & < & 17400425 \\ 63675 & < & o(16) & < & 2037600 \\ 16300802 & < & s(16) & < & 521625686 \end{array} \tag{4.14}$$

nach den Formeln (4.11) und (4.12).

5. Schlußbemerkungen

Unsere Formel (2.1) ermöglicht uns natürlich die genauen Werte von $o(2n)$ oder $s(2n)$ für kleineres n direkt zu rechnen.

Dazu kann man die Elemente von D_{2n} wie Eckenpermutationen in Zyklendarstellung typisieren, wie G. PÓLYA das im Begriff des zyklischen Indexes gemacht hatte. Der zyklische Index von D_{2n} ist ein formales Polynom von den Veränderlichen x_1, x_2, …, x_{2n}, abhängig von Zyklenlängen der Elemente:

$$I(x_1, x_2, \ldots, x_{2n}) = \frac{1}{4n}\left[x_1^{2n} + nx_1^2x_2^{n-1} + (n+1)x_2^n + \ldots + \varphi(2n)x_{2n}\right] \tag{5.1}$$

Hier x_1^{2n} entspricht der Identität

$$(1)(2)\ldots(2n), \tag{5.2}$$

$n \cdot x_1^2x_2^{n-1}$ entspricht den n Elementen der Form (4.5).

Die n Spiegelungen der Form (4.4) und die zentrale Punktspiegelung

$$(1, n+1)(2, n+2)\ldots(n, 2n) \tag{5.3}$$

entsprechen dem Glied $(n+1)x_2^n$.

Das Glied $\varphi(2n)x_{2n}$ entspricht den zyklischen Permutationen der Form

$$\big(1, 2, \ldots, (2n)\big). \tag{5.4}$$

Hier bezeichnet $\varphi(2n)$ die Eulersche Funktion, d.h. die Anzahl der Teilerfremden zu $2n$.

Eine Seitenpaarung I in (2.1) soll dann so angegeben werden, daß die Wirkung der Diedergruppe D_{2n} verfolgt werden kann.

Kritisch werden diejenigen Paarungen, die unter gewissen nicht-trivialen Untergruppen von D_{2n} invariant sind.

Zu einer "besseren" unteren Schranke für $o(n)$ geben wir eine andere Interpretation. Zu der Menge der Seiten $\{1, 2, \ldots, 2n\}$ ordnen wir die Elemente der Menge $\{r_1, r_2, \ldots, r_n\}$, so daß jeder Wert zweimal genutzt wird. Das machen wir bisauf D_{2n}-Äquivalenz. Wie es bekannt ist, hat man die Substitution

$$x_i = r_1^i + r_2^i + \ldots + r_n^i \quad \text{in} \quad I(x_1, x_2, \ldots, x_{2n}) \tag{5.5}$$

durchzuführen und den Koeffizient des Gliedes

$$A_{22\ldots 2} r_1^2 r_2^2 \ldots r_n^2 \tag{5.6}$$

zu bestimmen. Dazu braucht man nur die ersten drei Glieder von $I(x_1, x_2, \ldots \ldots, x_{2n})$ in (5.1). So kriegt man mit einfacher Rechnung

$$A_{22\ldots 2} = \frac{(2n-1)!}{2^{n+1}} + \frac{1}{4} n! + \frac{1}{4n}(n+1)!. \tag{5.7}$$

Das wäre eine grobe obere Abschätzung für die orientierbaren Seitenpaarungen. Jede Permutation der Werte r_1, r_2, $\ldots$, r_n führt aber zur äquivalenten Paarungen, die in $A_{22\ldots 2}$ höchtens $n!$-mal gezählt werden. So bekommen wir die gewünschte untere Schranke

$$\frac{(2n-1)!}{n! \cdot 2^{n+1}} + \frac{1}{4} + \frac{n+1}{4n} < o(n), \tag{5.8}$$

die nur formal besser als die in (4.6).

Literatur

[D-S 1984] A. DRESS-R. SCHARLAU, Zur Klassifikation äquivarianter Pflasterungen, *Mitteilungen aus dem Math. Seminar Giessen*, **164.** Coxeter-Festschrift (1984).

[D-H 1987] A. DRESS–D. HUSON, On tilings of the plane, *Geometriae Dedicata,* **24** (1987), 295–310.

[D-H-Z 1992] O. DELGADO FRIEDRICHS–D. HUSON–E. ZAMORZAEVA, The classification of 2-isohedral tilings of the plane, *Geometriae Dedicata,* **42** (1992), 43–117.

[D-H-M 1993] A. DRESS–D. HUSON–E. MOLNÁR, The classification of the face-transitive periodic three-dimensional tilings, *Acta Crystallographica,* **A 49** (1993), 806–817.

[H 1992] D. HUSON, The generation and classification of tile-k-transitive tilings of the euclidean plane, the sphere and the hyperbolic plane, *Geometriae Dedicata,* **47** (1993), 269–296.

[L-M 1990] Z. LUČIĆ–E. MOLNÁR, Combinatorial classification of fundamental domains of finite area for planar discontinuous groups, *Archiv Math.,* **54** (1990), 511–520.

[L-M 1991] Z. LUČIĆ–E. MOLNÁR, Fundamental domains for planar discontinuous groups and uniform tilings, *Geometriae Dedicata,* **40** (1991), 125–143.

[L-J-V 1990] Z. LUČIĆ–P. JANIČIĆ–N. VASILJEVIĆ, Manuscript 1990.

[L-M-S 1994] Z. LUČIĆ–E. MOLNÁR–M. STOJANOVIĆ, The 14 infinite series of isotoxal tilings in planes of constant curvature, *Periodica Math. Hung.,* **29** (1994), 177–195.

[L-M-V 1998] Z. LUČIĆ–E. MOLNÁR–N. VASILJEVIĆ, Combinatorial structure of fundamental polygons of finite area for plane discontinuous groups, Manuscript 1998.

[M 1992] E. MOLNÁR, Polyhedron complexes with simply transitive group actions and their metric realizations, *Acta Math. Hung.,* **59** (1992), 175–216.

[S 1989] E. STETTNER, Felületek számítógépes osztályozása (Über die Klassifizierung der Flächen mit Computer), Eötvös Loránd Tudományegyetem TTK Mat. Szakm. Csop. (Eötvös L. Univ. Budapest, Math. Fachmethod. Gruppe), 1989.

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CURVES IN PSEUDO-GALILEAN GEOMETRY

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1. Pseudo-Galilean space

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature (0,0,+,-), explained in [4]).The absolute of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$ where ω is the ideal (absolute) plane, f line in ω and I is the fixed hyperbolic involution of the points of f. In appropriate affine coordinates the group

$$B_6 : \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ d & \operatorname{ch}\varphi & \operatorname{sh}\varphi \\ e & \operatorname{sh}\varphi & \operatorname{ch}\varphi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

of pseudo-Galilean proper motions will preserve the absolute . Let the group

$$\bar{B}_6 := \left\langle B_6, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle \tag{2}$$

be called the motion group of the pseudo-Galilean space G_3^1. The motion group $\bar{B}_6$ leaves invariant the absolute figure and defines the other invariants of this geometry.

Now in affine coordinates the group $\bar{B}_6$ acts as follows

$$\bar{B}_6 : \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ d & \eta\operatorname{ch}\varphi & \eta\operatorname{sh}\varphi \\ e & \eta\operatorname{sh}\varphi & \eta\operatorname{ch}\varphi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3}$$

where η is +1 or -1.

There will be six classes of points on which B_6 acts transitively:

1. the proper points $(1 : x : y : z) \sim (x, y, z)$;

2. the non-absolute ideal points $(0:1:y:z)$ spanned by unit vectors $(1,y,z)$;
3. the spacelike absolute points, which can be written (by the projective sign freedom) into the form $(0:0:\mathrm{ch}\varphi:\mathrm{sh}\varphi)$;
4. the timelike absolute points $(0:0:\mathrm{sh}\varphi:\mathrm{ch}\varphi)$;
5. one lightlike absolute point $(0:0:1:1)$;
6. the other lightlike absolute point $(0:0:1:-1)$.

Distance between two proper points $P_i(x_i,y_i,z_i)$, $i=1,2$; $P_1 \neq P_2$ is

$$d(P_1,P_2) = \begin{cases} |x_2-x_1|, & x_1 \neq x_2 \\ \sqrt{\left|(y_2-y_1)^2-(z_2-z_1)^2\right|}, & x_1 = x_2. \end{cases}$$

A vector (i.e. a proper point pair class) of G_3^1 represents an ideal point of ω.

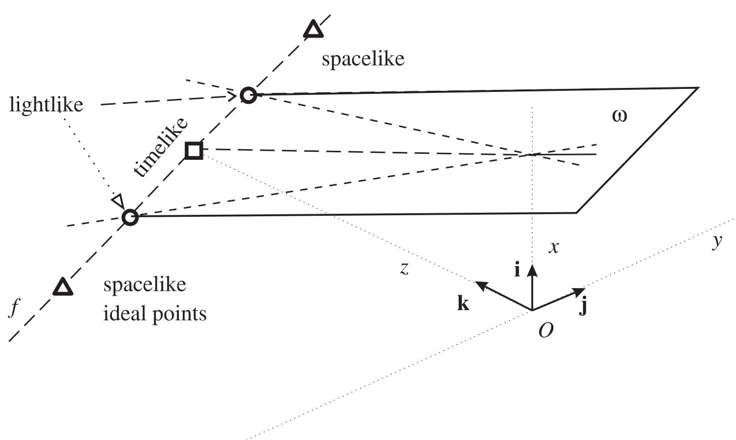

Fig. 1.

According to the group $\bar{B}_6$ there are non-isotropic and isotropic vectors. A vector $\mathbf{x}(x,y,z)$ is non-isotropic if $x \neq 0$. All unit non-isotropic vectors have $(1,y,z)$ form. For isotropic vectors $x=0$ holds. Again, there are four types of isotropic vectors: spacelike $(y^2-z^2>0)$, timelike $(y^2-z^2<0)$ and two types of lightlike $(y=\pm z)$ vectors. A non-lightlike isotropic vector is unit vector if $y^2-z^2=\pm 1$ (Figure 1).

The scalar product between two vectors $\mathbf{a}(a_1,a_2,a_3), \mathbf{b}(b_1,b_2,b_3)$ is

$$s(\mathbf{a},\mathbf{b}) = \begin{cases} a_1b_1 & \text{if } a_1 \neq 0 \vee b_1 \neq 0 \\ a_2b_2 - a_3b_3 & \text{if } a_1 = 0 \wedge b_1 = 0. \end{cases}$$

The scalar product of two vectors in G_3^1 is an invariant under the group B_6 (and under the group $\bar{B}_6$ as well).

A trihedron $(T_0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with a proper origin $T_0(x_0, y_0, z_0) \sim (1 : x_0 : y_0 : z_0)$, is orthonormal in pseudo-Galilean sense iff the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ have the following form: $\mathbf{e}_1 = (1, y_1, z_1)$, $\mathbf{e}_2 = (0, y_2, z_2)$, $\mathbf{e}_3 = (0, \varepsilon z_2, \varepsilon y_2)$, with $y_2^2 - z_2^2 = \delta$, where each of ε, δ is +1, or -1.

An above trihedron $(T_0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called positively oriented if for its vectors $\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$, i.e. $y_2^2 - z_2^2 = \varepsilon$ stand. This concept will be invariant under B_6 (and under $\bar{B}_6$ as well). The reason is, B_6 (and $\bar{B}_6$) keeps spacelike and timelike vectors, respectively and $\det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is invariant under B_6 (and under $\bar{B}_6$ as well).

THEOREM 1.1. *For any two equally oriented orthonormal trihedra in pseudo-Galilean space G_3^1 there is a unique $\bar{B}_6$-motion which transforms one trihedron into the other one.*

PROOF. Let $\{T_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{T_0', \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ be equally oriented orthonormal trihedra, given by the proper points $T_0(x_0, y_0, z_0)$ and $T_0'(x_0', y_0', z_0')$, by the vectors $\mathbf{e}_i = (x_i, y_i, z_i)$ and $\mathbf{e}_i' = (x_i', y_i', z_i')$, $i = 1, 2, 3$. That is the followings hold: $x_1 = x_1' = 1, x_2 = x_3 = x_2' = x_3' = 0$, and $(y_3, z_3) = (\varepsilon y_2, \varepsilon z_2)$, $(y_3', z_3') = (\varepsilon y_2', \varepsilon z_2')$ with $y_2^2 - z_2^2 = y_2'^2 - z_2'^2 = \delta$ where each of ε, δ is +1 or -1.

As consequence is $\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \det(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3') = \varepsilon$. Then we get unique real parameters $\varphi, \eta; d, e; a, b, c$ for $\bar{B}_6$-motion as follows step by step.

By $\mathbf{e}_2 = (0, y_2, z_2) \mapsto \mathbf{e}_2' = (0, y_2', z_2')$ we uniquely get φ and η from

$$\begin{aligned} y_2' &= \eta(\operatorname{ch}\varphi\, y_2 + \operatorname{sh}\varphi\, z_2) \\ z_2' &= \eta(\operatorname{sh}\varphi\, y_2 + \operatorname{ch}\varphi\, z_2) \end{aligned} \tag{6}$$

and $\mathbf{e}_3 \mapsto \mathbf{e}_3'$ holds by equal orientation, too.

Now, by $\mathbf{e}_1 = (1, y_1, z_1) \mapsto \mathbf{e}_1' = (1, y_1', z_1')$ we uniquely get d and e from

$$\begin{aligned} y_1' &= d + \eta(\operatorname{ch}\varphi\, y_1 + \operatorname{sh}\varphi\, z_1) \\ z_1' &= e + \eta(\operatorname{sh}\varphi\, y_1 + \operatorname{ch}\varphi\, z_1). \end{aligned} \tag{7}$$

Finally, we get a, b, c from $T_0 \mapsto T_0'$ by substitution into (3). ∎

The angle measure between two non-isotropic vectors $\mathbf{a}(1, a_2, a_3)$, $\mathbf{b}(1, b_2, b_3)$ is defined as the length of their difference vector

$$m(\mathbf{a}, \mathbf{b}) = \sqrt{\left|(b_2 - a_2)^2 - (b_3 - a_3)^2\right|}. \tag{8}$$

The planes, different from ω, are called proper planes. We differ two classes of proper planes in pseudo-Galilean space G_3^1: the proper planes which intersect f and those which contain f. In the absolute plane and in the proper planes which contain f, pseudo-Euclidean geometry holds. There are also three subclasses of the proper planes which intersect f: lightlike, spacelike and timelike planes, depending on the point of f polar to the plane considered. In these planes the isotropic plane geometry holds. From the projective point of view, the types of planes depend on the polarity of the signature $(0, 0, +, -)$. A plane $(u_0 : u_1 : u_2 : u_3)$ has its pole $(0 : 0 : u_2 : -u_3)$. A plane which is not incident to its pole can be timelike or spacelike iff $u_2^2 - u_3^2$ is less or bigger than 0, respectively. A proper plane incident to its pole i.e. $u_2^2 - u_3^2 = 0$, is lightlike. A plane $x = const$ i.e. $(u_0 : u_1 : 0 : 0)$ has no pole (since $(0 : 0 : 0 : 0)$ is not a point), and the induced geometry in it is pseudo-Euclidean plane geometry of the projective signature $(0, +, -)$.

Let α be a spacelike or timelike plane and $F(\alpha)$ the intersection of the absolute line f and α. The point $F(\alpha)$ is called the absolute point of α. Now, let $F^{\perp}(\alpha) = I(F(\alpha))$ be the point on f orthogonal to $F(\alpha)$ according to the hyperbolic involution I.

All isotropic lines through $F^{\perp}(\alpha)$ are called pseudo-Galilean normals of the plane α. $F^{\perp}(\alpha)$ is said to be the pole of α.

More about pseudo-Galilean space, in general, is described in [2]. The theory of curves in Galilean space (of the projective signature $(0, 0, +, +)$) is presented in [6].

2. Spatial curves in G_3^1

Let c be a spatial curve given first by

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \tag{9}$$

where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and t run through a real interval.

DEFINITION 2.1. A curve c given by (9) is admissible if

$$\dot{x}(t) \neq 0. \tag{10}$$

Then the curve c can be given by

$$\mathbf{r}(x) = (x, y(x), z(x)) \tag{11}$$

and we assume in addition that

$$y''^2 - z''^2 \neq 0. \tag{12}$$

From now on, we will denote the derivation by x by upper prime $'$.

DEFINITION 2.2. For an admissible curve given by (9) the parameter of arc length is defined by

$$ds = |\dot{x}(t)dt| = |dx|. \tag{13}$$

For simplicity we assume $ds = dx$ and $s = x$ as the arc length of the curve c.

The vector $\mathbf{t}(x) = \mathbf{r}'(x)$ is called the tangential unit vector of an admissible curve c in a point $P(x)$. Further, we define the so called osculating plane of c spanned by the vectors $\mathbf{r}'(x)$ and $\mathbf{r}''(x)$ in the same point. The absolute point of the osculating plane is

$$H(0 : 0 : y''(x) : z''(x)). \tag{14}$$

We have assumed in (12) that H is not lightlike. H is a point at infinity of a line which direction vector is $\mathbf{r}''(x)$. Then the unit vector

$$\mathbf{n}(x) = \frac{\mathbf{r}''(x)}{\sqrt{\left|y''^2(x) - z''^2(x)\right|}} \tag{15}$$

is called the principal normal vector of the curve c in the point P.

Now the vector

$$\mathbf{b}(x) = \frac{(0, \varepsilon z''(x), \varepsilon y''(x))}{\sqrt{\left|y''^2(x) - z''^2(x)\right|}} \tag{16}$$

is orthogonal in pseudo-Galilean sense to the osculating plane and we call it the binormal vector of the given curve in the point P. Here $\varepsilon = +1$ or -1 is chosen by the criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$. That means

$$\left|y''^2 - z''^2\right| = \varepsilon(y''^2 - z''^2). \tag{17}$$

By the above construction the following can be summarized.

DEFINITION 2.3. In each point of an admissible curve in $\mathbf{G}_3^1$ the associated orthonormal (in pseudo-Galilean sense) trihedron $\{\mathbf{t}(x), \mathbf{n}(x), \mathbf{b}(x)\}$ can be defined. This trihedron is called pseudo-Galilean Frenet trihedron (Figure 2).

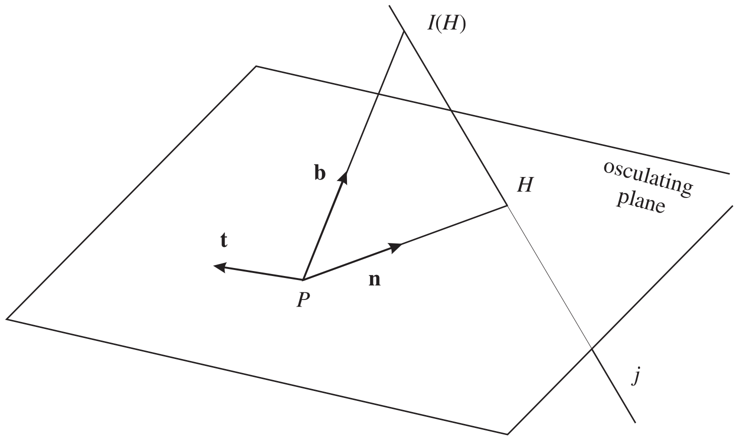

Fig. 2.

If a curve is parametrized by the arc length i. e. given by (11), then the tangent vector is non-isotropic and has the form of

$$\mathbf{t}(x) = \mathbf{r}'(x) = (1, y'(x), z'(x)). \tag{18}$$

Now we have

$$\mathbf{t}'(x) = \mathbf{r}''(x) = (0, y''(x), z''(x)). \tag{19}$$

According to the classical analogy we write (15) in the form

$$\mathbf{r}''(x) = \kappa(x)\mathbf{n}(x), \tag{20}$$

and so the curvature of an admissible curve c can be defined as follows

$$\kappa(x) = \sqrt{\left|y''^2(x) - z''^2(x)\right|}. \tag{21}$$

DEFINITION 2.4. The curve c given by (11) is timelike (t-curve for short), or c is spacelike (s-curve) if $\mathbf{n}(x)$ is a spacelike or a timelike vector, respectively.

It is easy to prove the next theorem by simple derivations, if we take into consideration our ε-convention at (16) and (17) as it follows also later on, from (31) up to (34).

THEOREM 2.5. *For the pseudo-Galilean Frenet trihedron of an admissible curve c given by* (11) *the following derivative Frenet formulas are true.*

$$\begin{aligned} \mathbf{t}'(x) &= \kappa(x)\mathbf{n}(x), \\ \mathbf{n}'(x) &= \tau(x)\mathbf{b}(x), \\ \mathbf{b}'(x) &= \tau(x)\mathbf{n}(x), \end{aligned} \tag{22}$$

where $\kappa(x)$ is the pseudo-Galilean curvature given by (21) *and $\tau(x)$ is the pseudo-Galilean torsion of c defined by*

$$\tau(x) = \frac{y''z''' - y'''z''}{\kappa^2(x)}. \tag{23}$$

The formula (23) can be written as

$$\tau(x) = \frac{\det(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\kappa^2(x)}. \tag{24}$$

Further, we obtain

$$\mathbf{r}'''(x) = \kappa'(x)\mathbf{n}(x) + \kappa(x)\tau(x)\mathbf{b}(x). \tag{25}$$

The same relation holds in Galilean space as it is shown in [6] as well as in double isotropic space that is proven in [1], but it is not true in simply isotropic space (see [8]).

Let us notice that the curvature κ and the torsion τ associated to a given curve are invariants under the pseudo-Galilean motions.

In G_3^1 the functions κ and τ have the analogous geometrical meaning as in Euclidean space. We set $(x, y(x), z(x))$ and $(x+\triangle x, y(x+\triangle x), z(x+\triangle x))$ to be two "infinitesimally near", briefly, neighbour points on curve c. According to (8) the angle between the tangents in these points we compute as follows

$$\Delta\varphi = \sqrt{\left|(y'(x) - y'(x+\Delta x))^2 - (z'(x) - z'(x+\Delta x))^2\right|}, \tag{26}$$

and then

$$\Delta\varphi = \sqrt{\left|y''^2(x) - z''^2(x)\right|}\Delta x + \dots \quad . \tag{27}$$

Now we have

$$\kappa(x) = \lim_{\Delta x \to 0}\left|\frac{\Delta\varphi}{\Delta x}\right| \left(= \left|\frac{d\varphi}{dx}\right|\right). \tag{28}$$

In contrary to the geometrical interpretation of the curvature in Euclidean and Galilean space, in pseudo-Galilean case κ measures the absolute value of the change of the angle between tangents in neighbour points.

Similarly, for the torsion we set $\Delta\psi$ the signed angle between the osculating planes or the binormals in the points $(x, y(x), z(x))$ and $(x+\Delta x, y(x+ +\Delta x), z(x+\Delta x))$ from the tangent view of c, and obtain, not detailed,

$$\tau(x) = \lim_{\Delta x \to 0}\left(\frac{\Delta\psi}{\Delta x}\right)\left(= \frac{d\psi}{dx}\right). \tag{29}$$

Direction cone (ruled surface whose generators are the tangents of a given curve and its directrix is the curve itself) of an admissible curve c intersects the ideal plane ω in the plane curve c_ω given by

$$c_\omega : (0 : 1 : y'(x) : z'(x)). \tag{30}$$

The curve c_ω is not an admissible curve, because its first derivative $(0 : 0 : y''(x) : z''(x))$ is isotropic. First, the tangent vector $\bar{\mathbf{t}}(x)$ of c_ω is defined as the isotropic unit vector

$$\bar{\mathbf{t}}(x) = \left(0 : 0 : \frac{y''(x)}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))}} : \frac{z''(x)}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))}}\right).$$

Then

$$\frac{d}{ds}\bar{\mathbf{t}} = k(s)\bar{\mathbf{n}}(s) \tag{31}$$

defines the signed curvature $k(s)$, s denotes now the arc length of the curve c_ω in the pseudo-Euclidean plane ω. We require

$$\det(\bar{\mathbf{t}}, \bar{\mathbf{n}}) = \begin{vmatrix} y''(x) & \varepsilon z''(x) \\ z''(x) & \varepsilon y''(x) \end{vmatrix} \frac{1}{\kappa^2(x)} = 1$$

as usual. We get

$$\frac{d}{dx}\bar{\mathbf{t}} = \left(0 : 0 : \frac{\varepsilon z''(x)(y''(x)z'''(x) - z''(x)y'''(x))}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))^3}} : \right.$$
$$\left. : \frac{\varepsilon y''(x)(y''(x)z'''(x) - z''(x)y'''(x))}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))^3}}\right)$$

and

$$\frac{d}{ds} = \frac{1}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))}}\frac{d}{dx}. \tag{32}$$

Obviously

$$\bar{\mathbf{n}}(x) = \left(0 : 0 : \frac{\varepsilon z''(x)}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))}} : \frac{\varepsilon y''(x)}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))}}\right), \tag{33}$$

i.e. the normal vector of c_ω is just the binormal vector of c, and we get from (17)

$$k(x) = \frac{y''(x)z'''(x) - z''(x)y'''(x)}{\sqrt{\varepsilon(y''^2(x) - z''^2(x))^3}}. \tag{34}$$

Immediately we obtain

$$k(x) = \frac{\tau(x)}{\kappa(x)}. \tag{35}$$

The function $k(x)$ is called conical curvature of the curve c in a point $P(x)$.

Using Taylor series of the equation of a spatial curve c ($\mathbf{r} = \mathbf{r}(x)$) in a neighbourhood of the origin P ($x = 0$) we have

$$\mathbf{r} = \mathbf{r}'x + \frac{1}{2}\mathbf{r}''x^2 + \frac{1}{6}\mathbf{r}'''x^3 + o(x^4). \tag{36}$$

Now we assume that the vectors of the associated trihedron coincide with the new coordinate axes for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, e.g. $\mathbf{t} = (1, 0, 0)$, $\mathbf{n} = (0, 1, 0)$, $\mathbf{b} = (0, 0, 1)$. From the statement $\mathbf{r}' = \mathbf{t}$ and equations (20) and (25) the so called canonical expression of a curve follows. So we get

$$\begin{aligned} \tilde{x}(x) &= x \\ \tilde{y}(x) &= \frac{\kappa}{2}x^2 + \frac{\kappa'}{6}x^3 \\ \tilde{z}(x) &= \frac{\kappa\tau}{6}x^3. \end{aligned} \tag{37}$$

Immediately from (37) we can prove the following proposition that is true also in Euclidean space as it is shown e.g. in [3].

PROPOSITION 2.6. *If in a point P of a curve c the curvature and the torsion differ from zero the following statements are true.*

a) *Orthogonal projection of the curve c on the osculating plane in a point P is regular curve and* $\mathbf{t}$ *is the tangent vector in this point.*

b) *Parallel projection from the ideal point of* $\mathbf{t}$ *of the curve c onto the normal plane in a point P has a singularity of the first order and* $\mathbf{n}$ *is the tangent vector in this point.*

c) *Orthogonal projection of the curve* c *on the rectification plane in a point* P *has an inflection point and* $\mathbf{t}$ *is the tangent of the projection of* c *in point* P.

Considering the last equation in (37) we could describe the usual geometrical meaning of the sign of the torsion τ.

3. The fundamental theorem of the pseudo-Galilean curves

The fundamental theorem of the pseudo-Galilean theory of curves differs crucially from the analogous theorem in Euclidean, isotropic or Galilean space. Actually, the uniqueness in this theorem is not fulfilled and the reason for this is the existence of pseudo-Euclidean planes in pseudo-Galilean space. As it is well known in pseudo-Euclidean plane geometry, the uniqueness in the fundamental theorem of plane curves does not hold yet.

THEOREM 3.1. *Let* $\kappa = \kappa(x)$ *and* $\tau = \tau(x)$ *be given functions so that* $0 < \kappa(x) \in C^1$, $0 \neq \tau(x) \in C$. *There are two admissible curves (one timelike and one spacelike) so that the following statements are true:*

(1) c *passes through a given point;*

(2) *at this point the Frenet trihedron of* c *coincides with a given orthonormal positively oriented trihedron;*

(3) c *can be represented as a vector function* $\mathbf{r}(x) \in C^3$ *with arc length parameter* x;

(4) $\kappa(x)$ *and* $\tau(x)$ *are the curvature and torsion of* c, *respectively.*

PROOF. First, if we use (32) we compute the parameter of arc length s of c_ω as follows

$$s(x) = \int_0^x \kappa(v)dv. \tag{38}$$

So x can be expressed from (38) as a function of s and from (35) we have the conical curvature $k(s) = \frac{\tau(x(s))}{\kappa(x(s))}$ as a function of s. Now the Frenet formula (31) for $\bar{\mathbf{t}}(s) = (0:0:\varepsilon\bar{y}(s):\varepsilon\bar{z}(s))$ and $\bar{\mathbf{n}}(s) = (0:0:\bar{z}(s):\bar{y}(s))$ with

$$\varepsilon\left(\bar{y}^2(s) - \bar{z}^2(s)\right) = 1 \tag{39}$$

provides us

$$\frac{d}{ds}\bar{y}(s) = k(s)\varepsilon\bar{z}(s), \quad \frac{d}{ds}\bar{z}(s) = k(s)\varepsilon\bar{y}(s), \tag{40}$$

with $\varepsilon = +1$ or -1, which is a differential equation system for $\bar{\mathbf{t}}(s)$ of the curve c_ω. Further, from the condition (39) we have

$$\bar{z}^2(s) = \bar{y}^2(s) - \varepsilon. \tag{41}$$

This means that there are two different equations for $\bar{y}(s)$ (and for $\bar{z}(s)$ as well) and each differential equation has the unique solution which satisfies the initial conditions from the Theorem (condition (2)). Finally, the original curve c can be determined by

$$y''(x) = \kappa(x)\bar{y}(x), \quad z''(x) = \kappa(x)\bar{z}(x), \tag{42}$$

where the initial values of c are fixed in the conditions (1) and (2) of the Theorem.

It is obvious from (41) that one obtained curve is spacelike and the other is timelike. ∎

Let us prove some other aspects of the fundamental theorem of curves in pseudo-Galilean space.

The fact that the notions of s-curve and t-curve are invariants under the pseudo-Galilean motions could be formulated in the next lemma.

LEMMA 3.2. *The s-curves (t-curves) in pseudo-Galilean space, under the proper pseudo-Galilean motions B_6, are transformed into s-curves (t-curves).*

Using the Lemma 3.2 and Theorem 1.1, we can prove the following theorem as it is done for E^3 e.g. in [5].

THEOREM 3.3. *Two admissible s-curves (t-curves) in G_3^1 are $\bar{B}_6$-equivalent if and only if they have the same natural equations for $\kappa(x)$ and $\tau(x)$.*

References

[1] H. BRAUNER, Geometrie des zweifach isotropen Raumes II, *J.Reine. Angew. Math.* **226** (1967), 132–158.

[2] B. DIVJAK, Geometrija pseudogalilejevih prostora, Ph. D. thesis, PMF Zagreb, 1997.

[3] E. KRUPPA, Zur Differentialgeometrie der Strahlflächen und Raumkurven, *Sb. Akad. Wiss. Wien IIa*, **157** (1949), 143–176.

[4] E. MOLNÁR, The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beitr. Algebra Geom.,* **38** (1997), 261–288.

[5] P. K. RAŠEVSKI, *Kurs differencialnoi geometrii,* The. teoret. Lit., Moskva, 1956.

[6] O. RÖSCHEL, Die Geometrie des Galileischen Raumes, Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben, 1984.

[7] B. A. ROZENFELD–I. M. YAGLOM–E. U. YASINSKAYA, Projective metrics (in Russian), *Uspehi Mat. Nauk,* **19** (1964), 51–113.

[8] H. SACHS, *Isotrope Geometrie des Raumes,* Vieweg, Braunschweig/Wiesbaden, 1990.

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INSTABILITY OF DIFFERENCE EQUATIONS

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1. Introduction

Let us consider are nonautonomous system of difference equations

$$y(n+1) = A(n)y(n), \qquad n \in \mathbf{N} = \{0,1,2,3,\dots\}, \tag{1}$$

for which all matrices $A(n)$ are invertible. The fundamental matrix Ψ of this system is defined by

$$\Psi(n) = \prod_{s=0}^{n-1} A(s) = A(n-1)\dots A(1)A(0), \qquad \prod_{s=0}^{-1} A(s) = I,$$

where I denotes the identity matrix. This paper concerns the unstable properties of the null solution of the nonautonomous difference equation

$$x(n+1) = A(n)x(n) + f(n,x(n)), \qquad f(n,0) = 0, \tag{2}$$

where f is defined on $\mathbf{N} \times \{x : |x| < H\}$, $H \in (0,\infty]$. This problem has been investigated from the beginning of this century by PERRON [13] and LI [10].

THEOREM A. [8] *Assume that $f(n,x)$ is continuous in the variable x. Moreover, uniformly with respect to $n \in \mathbf{N}$, let us assume that*

$$\lim_{|x|\to 0} \frac{f(n,x)}{|x|} = 0.$$

If $A(n) = A =$ constant and the matrix A has at least one eigenvalue satisfying $|\mu| > 1$, then this solution $x = 0$ of equation (2) *is unstable.*

A more general result on instability was given by COPPEL for ordinary differential equations [4], which discrete version can be found in [1]:

THEOREM B. [1] *Assume that* $f(n,x)$ *is continuous in the variable* x *and*

$$|f(n,y)| \leq \gamma |y|, \qquad \gamma = constant. \tag{3}$$

Moreover, assume that P*, a projection matrix,* $P \neq I$*, satisfies*

$$\sum_{s=n_0}^{n-1} |\Psi(n) P \Psi^{-1}(s+1)| + \sum_{s=n}^{\infty} |\Psi(n)(I-P)\Psi^{-1}(s+1)| \leq K. \tag{4}$$

where K *is a constant. If* $K\gamma < 1$*, then the null solution of* (2) *is unstable.*

Despite the importance of Theorems A and B, the instability of a large class of systems cannot be described by these theorems. The aim of this paper is to provide a method of investigation of the unstable properties of the system (2) relying on the dichotomic properties of the nonautonomous system (1). This methodology was proposed, for ordinary differential equations in [11]. We will obtain not only the discrete version of the results exposed in [2], [3], [7], [8], [11], but we will communicate two results on the instability of System (2) (Theorem 1 and Theorem 2 of our text) that cannot be obtained from Theorems A and B.

Our main hypotheses are the following:

(L) *For some positive* ρ*, the function* $f(n,x)$ *appearing in system* (2) *satisfies the following Lipschitz condition*

$$|f(n,x) - f(n,y)| \leq \gamma(n)|x-y|, \qquad |x| \leq \rho, \qquad |y| \leq \rho.$$

(D) *System* (1) *has an ordinary dichotomy* [1]*. By this we mean the existence of a projection matrix* P *such that*

$$\begin{aligned} |\Psi(n) P \Psi^{-1}(m)| &\leq K, \qquad n \geq m \geq 0, \\ |\Psi(n)(I-P)\Psi^{-1}(m)| &\leq K, \qquad m \geq n \geq 0, \end{aligned} \tag{5}$$

where K *is a constant.*

2. Preliminaries

In what follows, the sequences $\{y(n,n_0,\xi)\}$, $\{x(n,n_0,\xi)\}$ respectively stand for the solutions of system (1) and (2) with initial condition ξ at the initial time n_0. V denotes the space $\mathbf{R}^r$ or $\mathbf{C}^r$ with a fixed norm $|\cdot|$. For a sequence $x : \mathbf{N} \to V$, we will denote $|x|_\infty = \sup\{|x(n)| : n \in \mathbf{N}\}$. The space of all sequences $x : \mathbf{N} \to V$ such that $|x|_\infty < \infty$ will be denoted by ℓ^∞. The sequential space of all summable sequences will be denoted by ℓ^1, with norm $|x|_1 = \sum_{n=0}^{\infty} |\gamma(n)|$. In the sequel we shall consider the closed ball $B_\infty[0,\rho] = \{x \in \ell^\infty : |x| \le \rho\}$.

DEFINITION 1. The solution $x = 0$ of equation (1) is unstable if there exists $\varepsilon > 0$ and $n_0 \in \mathbf{N}$ such that for all $\delta > 0$, there exists a $\xi \in V$, $|\xi| < \delta$ and an $n_\delta \ge 0$ such that

$$|x(n_\delta, n_0, \xi)| \ge \varepsilon.$$

DEFINITION 2. The solution $x = 0$ of equation (1) is asymptotically unstable, if for all $n_0 \in \mathbf{N}$ and all $\delta > 0$, there exists a $\xi \in V$, $|\xi| < \delta$, such that

$$\lim_{n\to\infty} \sup |x(n, n_0, \xi)| > 0.$$

In what follows we will use the following subspaces of initial conditions:

$$V_1 = \{\xi \in V : \Psi(n)\xi \in \ell^\infty\}, \qquad V_0 = \{\xi \in V_1 : \lim_{n\to\infty} x(n,0,\xi) = 0\}.$$

We will require the following property of ordinary dichotomies [12], [5].

THEOREM C. *Let us assume that system* (1) *has an ordinary dichotomy, then system* (1) *has an ordinary dichotomy with projection Q iff*

$$V_0 \subset Q[V] \subset V_1.$$

3. Instability

THEOREM 1. *Let us assume that equation* (1) *possesses am ordinary dichotomy and an unbounded solution. If the function* f *satisfies the condition (L), and* $\{\gamma(n)\}$ *is summable, then the null solution of equation* (2) *is unstable.*

PROOF. Let us assume that the null solution of equation (2) is stable. Then for $\varepsilon = \rho$ and $n_0 \in \mathbf{N}$ there exists a $\delta > 0$ such between

$$|x_0| < \delta \quad \text{implies} \quad |x(n, n_0, x_0)| < \rho, \quad \forall\, n \geq n_0.$$

We will show that this is impossible. Let us assume that

$$K \sum_{s=n_0}^{\infty} \gamma(s) < 1 \tag{6}$$

and let us consider the operator

$$\begin{aligned} \mathcal{U}(x)(n) = {} & \sum_{s=n_0}^{n-1} \Psi(n) P \Psi^{-1}(s+1) f(s, x(s)) \\ & - \sum_{s=n}^{\infty} \Psi(n)(I - P)\Psi^{-1}(s+1) f(s, x(s)), \end{aligned} \tag{7}$$

for which

$$|\mathcal{U}(x)(n)| \leq K \sum_{s=n_0}^{\infty} \gamma(s)|x(s)| \leq K \sum_{s=n_0}^{\infty} \gamma(s)|x|_{\infty}. \tag{8}$$

This condition (6) implies

$$\mathcal{U} : B_{\infty}[0, \rho] \to B_{\infty}[0, \rho].$$

Moreover

$$\begin{aligned} |\mathcal{U}(x)(n) - \mathcal{U}(y)(n)| & \leq K \left(\sum_{s=n_0}^{n-1} + \sum_{s=n}^{\infty} \right) \gamma(s)|x(s) - y(s)| \\ & \leq K \sum_{s=n_0}^{\infty} \gamma(s)|x - y|_{\infty}. \end{aligned} \tag{9}$$

Let us consider the sequence $\{y(n)\}$ defined by

$$y(n) = x(n, n_0, x_0) - \mathcal{U}(x(\,\cdot\,, n_0, x_0))(n), \qquad |x_0| < \delta.$$

It is easy to see that the sequence $\{y(n)\}_{n=n_0}^{\infty}$ is a bounded solution of the equation (1). Hence $y(n_0) \in \Psi(n_0)[V_1]$. From Theorem C, we may assume and do that $\Psi(n_0)[V_1] = \Psi(n_0)[V]$. Let n_0 be chosen with the properties

(10) $$x_0 \in \Psi(n_0)(I - P)\Psi^{-1}(n_0)[V], \qquad x_0 \neq 0, \quad \text{and} \quad |x_0| < \delta.$$

From the definition of sequence $\{y(n)\}_{n=n_0}^{\infty}$ we obtain

$$y(n_0) = x_0 = \Psi(n_0)(I - P\Psi^{-1}(n_0)\sum_{s=n_0}^{\infty}\Psi(n_0)\Psi^{-1}(s+1)f(s, x(s, n_0, x_0)).$$

This implies $y(n_0) \in \Psi(n_0)(I - P)\Psi^{-1}(n_0)[V]$ and therefore $y(n_0) = 0$. Consequently the sequence $\{x(n, n_0, x_0)\}$ satisfies the equation

$$x(\,\cdot\,, n_0, x_0) = \mathcal{U}(x(\,\cdot\,, n_0, x_0)).$$

Thus, any solution $x(\cdot, n_0, x_0)$, where x_0 satisfies (10), is a fixed point of the dichotomic operator $\mathcal{U}$. But from (8) and (9), we see that operator $\mathcal{U}$, is a contraction acting from $B_\infty[0,\rho]$ to $B_\infty[0,\rho]$. Moreover $\mathcal{U}(0) = 0$, therefore $x(m, m_0, x_0) = 0$ for all $n \in \mathbf{N}$, giving the contradiction $x_0 = x(n_0, n_0, x_0) \neq 0$. ∎

Let us assume a more general hypothesis than (L).

(L') *For any* $\rho > 0$ *there exists a sequence* $\{\gamma(n,\rho)\}$ *such that*

$$|f(n,x) - f(n,y)| \le \gamma(x,\rho)|x - y|, \quad |x|, \quad |y| \le \rho.$$

The proof of the preceding theorem suggests the following result.

THEOREM 2. *Let us assume that equation* (1) *possesses an ordinary dichotomy with projection* $P = 0$ *and* f *satisfy (L'). If for any* $\rho > 0$ *the sequence* $\{\gamma(n,\rho)\}$ *is summable and all nontrivial solutions of equation* (1) *are unbounded, then all nontrivial solutions of equation* (2) *are unbounded.*

PROOF. Let us assume that equation (2) has a bounded and nontrivial solution $\{\tilde{x}(n)\}$, $|\tilde{x}(n)| \le M$, $\forall\, n \in \mathbf{N}$. Let us consider a fixed n_0 such that

(11) $$K\sum_{s=n_0}^{\infty}\gamma(s, M) < 1.$$

From (11), repeating the same reasoning of the proof of Theorem 1, we obtain that $\mathcal{U} : B_\infty[0, M] \to B_\infty[0, M]$ and the operator U is a contraction. The sequence defined by

$$y(n) = \tilde{x}(n) - \mathcal{U}(\tilde{x})(n)$$

is a bounded solution of equation (1). Therefore $\tilde{x} = \mathcal{U}(\tilde{x})$ implying $\tilde{x} = 0$, but this contradicts $\tilde{x} = 0$. ∎

4. Asymptotic instability

By a slight modification of the proof of Theorem 1 we obtain

THEOREM 3. *If* $V_0 \neq V_1$ *and equation* (1) *has an ordinary dichotomy, then under condition (L), where* γ *is summable, the null solution of equation* (2) *is asymptotically unstable.*

PROOF. Let us consider $\xi \in \Psi(n_0)[V_1] \setminus \Psi(n_0)[V_0]$. Then

$$\limsup_{n\to\infty} |y(n, n_0, \xi)| > 0. \tag{12}$$

From the definition of space V_1, there exists a constant N such that

$$|y(n, n_0, \xi)| \leq N|\xi|, \qquad \forall \xi \in \Psi(n_0)[V_1], \qquad \forall n \geq n_0. \tag{13}$$

Following the proof of Theorem 1, we may prove that, under condition (6), the integral equation

$$x(n) = y(n, n_0, \xi) + \mathcal{U}(x)(n) \tag{14}$$

has a unique bounded solution in $B[0,\rho]$ satisfying equation (2). From (14) it follows

$$|x(n)| \leq |y(\,\cdot\,, n_0, \xi)|_\infty |K|\gamma|_1|x|_\infty, \qquad n \geq n_0.$$

Thus

$$|x|_\infty \leq \frac{|y(\,\cdot\,, n_0, \xi)|_\infty}{1 - K|\gamma|_1}. \tag{15}$$

This inequality and (13) imply

$$|x(n_0)| \leq \frac{N|\xi|}{1 - K|\gamma|_1}.$$

In other words, the initial condition $|x(n_0)|$ is small according to the smallness of ξ. From Theorem C, the projection of the ordinary dichotomy (5) can be chosen with the asymptotic property

$$\lim_{n\to\infty} \Psi(n)P = 0.$$

Let $m \in \mathbf{N}$, $m > n_0$. From (15) we have the following estimates

$$\left|\sum_{s=n_0}^{n-1} \Psi(n)P\Psi^{-1}(s+1)f(s, x(s))\right| \leq$$

$$\leq \left|\Psi(n)P\sum_{s=n_0}^{m} \Psi^{-1}(s+1)f(s, x(s))\right| + K\frac{|y(\,\cdot\,, n_0, .\xi)|_\infty}{1 - K|\gamma|_1}\sum_{s=m+1}^{\infty} \gamma(s).$$

and

$$\left|\sum_{s=n}^{\infty}\Psi(n)(I-P)\Psi^{-1}(s+1)f(s,x(s))\right| \le K\frac{|y(\,\cdot\,,n_0,\xi)|_\infty}{1-K|\gamma|_1}\sum_{s=n}^{\infty}\gamma(s).$$

From these estimates, it follows

$$\limsup_{n\to\infty}\mathcal{U}(y(n))=0$$

and therefore

$$\limsup_{n\to\infty}|x(n)| = \limsup_{n\to\infty}|y(n,n_0,\xi)|>0. \blacksquare$$

5. Applications

5.1. Example 1

Consider the difference systems

$$x(n+1)=Ax(n)+f(n,x(n)),\qquad f(n,0)=0,$$

where f satisfies the condition (L) with γ a summable sequence. Let us assume that the eigenvalues of matrix A satisfy $|\mu|\le 1$ and the eigenvalues μ such that $|\mu|=1$ at least one is not Jordan simple. From Theorem 1 it follows that $x=0$ is unstable. If all eigenvalues satisfying $|\mu|=1$ are Jordan simple, then this system is asymptotically unstable. This analysis cannot be explained neither by the Theorem A nor by the Theorem B.

5.2 Example 2

Let us consider the following second order difference equations

$$x(n+2)-g(n)x(n)=f(n,x(n),x(n+1)), \tag{16}$$

where $\{g(n)\}$ is a sequence of real numbers, $g(n)\neq 0,\ \forall n, f$ is a real function defined $\mathbf{N}\times\mathbf{R}\times\mathbf{R}$, $f(\,\cdot\,,0,0)=0$. Let us solve the homogeneous equation

$$y(n+2)-g(n)y(n)=0. \tag{17}$$

For, let u and v be the respective solutions of the first order scalar equations

$$\begin{aligned} u(n+1)&=\quad g(2n)u(n) && v(0)=1, && n\in\mathbf{N},\\ v(n+1)&=g(2n+1)u(n) && v(0)=1, && n\in\mathbf{N}. \end{aligned}$$

Therefore

$$u(n)=\prod_{k=0}^{n-1} g(2k), \quad \text{and} \quad v(n)=\prod_{k=0}^{n-1} g(2k+1).$$

A fundamental set of solutions of equation (17) is given by

$$y_1(n)=\begin{cases} u(n/2), & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} \qquad y_2(n)=\begin{cases} 0, & n \text{ is even} \\ v\left(\frac{n-1}{2}\right), & n \text{ is odd.} \end{cases}$$

Consequently, the fundamental matrix of equation (17) is

$$\Phi(n)=\begin{pmatrix} y_1(n) & y_2(n) \\ y_1(n+1) & y_2(n+1) \end{pmatrix}.$$

We will establish conditions under which system (17) yields an ordinary dichotomy. Previously, we shall obtain a formula to calculate $\Phi^{-1}(n)$. The sequence $\{(\Phi^{-1})^T(n)\}$ (the transpose of matrix $\Phi^{-1}(n)$) satisfies the equation

$$W(n+1)=\begin{pmatrix} 0 & 1 \\ 1/g(n) & 0 \end{pmatrix} W(n), \qquad W(0)=I.$$

It is easy to see that the solution of this problem is the sequence

$$W(n)=\begin{pmatrix} w_1(n) & w_2(n) \\ w_1(n+1) & w_2(n+1) \end{pmatrix},$$

where

$$w_1(n)=\begin{cases} \frac{1}{u(n/2)}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}, \qquad w_2(n)=\begin{cases} 0, & n \text{ is even} \\ \frac{1}{v\left(\frac{n-1}{2}\right)}, & n \text{ is odd} \end{cases}.$$

Since $W^T(n)=\Phi^{-1}(n)$ then

$$\Phi^{-1}(n)=\begin{pmatrix} w_1(n) & w_1(n+1) \\ w_2(n) & w_2(n+1) \end{pmatrix}.$$

Let us define the two dimensional projection $P=\operatorname{diag}\{1,0\}$, then

$$\Phi(n)P\Phi^{-1}(m)=\begin{pmatrix} y_1(n)w_1(m) & y_1(n)w_1(m+1) \\ y_1(n+1)w_1(m) & y_1(n+1)w_1(m+1) \end{pmatrix}$$

and

$$\Phi(n)(I-P)\Phi^{-1}(m)=\begin{pmatrix} y_2(n)w_2(m) & y_2(n)w_2(m+1) \\ y_2(n+1)w_2(m) & y_2(n+1)w_2(m+1) \end{pmatrix}.$$

The number $|\Phi(n)P\Phi^{-1}(m)|$ is equal to

- $|y_1(n)w_1(m)| = \prod_{k=m}^{n-1} |g(2k)|$, for n even, m even,
- $|y_1(n)w_1(m+1)| = \frac{1}{|g(2m)|} \prod_{k=m}^{n-1} |g(2k)|$, for n even, m odd,
- $|y_1(n+1)w_1(m)| = |g(2n)| \prod_{k=m}^{n-1} |g(2k)|$, n odd, m even,
- $|y_1(n+1)w_1(m+1)| = \frac{|g(2n)|}{|g(2m)|} \prod_{k=m}^{n-1} |g(2k)|$, for n odd, m odd.

Therefore $|\Phi(n)P\Phi^{-1}(m)|$ is bounded for $n \geq m$ if

(18) $$\frac{1}{|g(2m)|} \prod_{k=m}^{n-1} |g(2k)| \leq M, \qquad n \geq m,\ M = \text{constant}.$$

By similar calculations, $|\Phi(n)(I-P)\Phi^{-1}(m)|$ is bounded, for $m \geq n$, if

(19) $$\frac{1}{|g(2n+1)|} \prod_{k=n}^{m-1} |g(2k+1)| \geq M^{-1}, \qquad m \geq n,\ M = \text{constant}.$$

From Theorem 1, if the function $f(n,x,y)$ is Lipschitz continuous

$|f(n,x_1,y_1)-f(n,x_2,y_2)| \leq \gamma(n)(|x_1-x_2|+|y_1-y_2|)$. $\gamma \in \ell^1$, $|x_i|$, $|y_i| \leq \rho$,

for some positive ρ, conditions (18) and (19) are fulfilled and

$$\lim_{n\to\infty} \prod_{k=0}^{n-1} |g(2k+1)| = \infty,$$

then the null solution of equation (16) is unstable. On the other hand, if

$$\limsup_{n\to\infty} \prod_{k=0}^{n-1} |g(2k)| > 0,$$

and conditions (18) and (19) are satisfied, then from Theorem 2 follows the asymptotic instability of the null solution of equation (17). In the above calculations we could consider the projection matrix $P = \operatorname{diag}\{0,1\}$ instead of $P = \operatorname{diag}\{1,0\}$ to obtain similar results of instability of equation (16).

5.3. Example 3

Let us consider the equation

$$(20)\qquad \begin{cases} x_1(n+1) = 4x_2(n) + f_1(n)2x_2(n)x_1^2(n), \\ x_2(n+1) = \dfrac{1}{2}x_1(n) + f_2(n)x_1(n)x_2^2(n). \end{cases}$$

For $f_1(n) = -1 = -f_2(n)$, this example was considered in [6] (Example 4.35), where it was proven the instability of the null solution by means of the Liapounov function $V(x_1,x_2) = x_1^2 + 16x_2^2$. Under these conditions on coefficients f_1 and f_2, the instability of the null solution can be obtained from Theorem B. If these coefficients are unbounded sequences, then a straightforward application of Theorem B is not possible.

In order to study this example we introduce the change of variables

$$(21)\qquad x_1(n) = r^n y_1(n), \qquad x_2(n) = \rho^n y_2(n).$$

We obtain the system

$$(22)\qquad \begin{cases} y_1(n+1) = \dfrac{4}{r}\left(\dfrac{\rho}{r}\right)^n y_2(n) + r^{-1}(\rho r)^n f_1(n) y_2(n) y_1^2(n), \\ y_2(n+1) = \dfrac{1}{2\rho}\left(\dfrac{r}{\rho}\right)^n y_1(n) + \rho^{-1}(r\rho)^n f_2(n) y_1(n) y_2^2(n). \end{cases}$$

The linear component of this system is defined by the equations

$$(23)\qquad z_1(n+1) = \frac{4}{r}\left(\frac{\rho}{r}\right)^n z_2(n), \qquad z_2(n+1) = \frac{1}{2\rho}\left(\frac{r}{\rho}\right)^n z_1(n).$$

This system is equivalent to the pair of second order equations

$$z_1(n+2) = \frac{2}{r^2} z_1(n), \qquad z_2(n+2) = \frac{2}{\rho^2} = \frac{2}{\rho^2} z_2(n).$$

If we choose r and ρ such that $1 < r < \sqrt{2}$, $0 < \rho < 1$, $r\rho < 1$, then all solutions of system (23) are unbounded. For these r and ρ, if the sequences $\{f_i(n)(r\rho)^n\}$, $i = 1, 2$, are summable, then according to Theorem 2, all the solutions of system (22) are unbounded. But, under such conditions, the change of variables (21) implies that all nontrivial solutions of equation (20) starting from the manifold $y_2 = 0$ are unbounded.

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References

[1] AGARWAL R. P., *Difference Equation and Inequalities (Theory, Methods and applications),* Pure and Applied Mathematics, Marcel Dekker, New York (1992).

[2] AVIS R., NAULIN R., Asymptotic instability of nonlinear differential equations, *Electronic Journal of Differential Equations,* **16** (1997), 1–7.

[3] BOWNDS J. M., Stability implications on the asymptotic between of second order differential equations, *Proc. Amer. Math. Soc.,* **39** (1973), 169–172.

[4] COPPEL W. A., On the stability of ordinary differential equations, *J. London Math. Soc.,* **39** (1969), 255–260.

[5] COPPEL W. A., *Dichotomies in Stability Theory,* Lecture Notes 629, Springer, Berlin (1978).

[6] ELAYDI S. E., *An Introduction to Difference Equations,* Springer Verlag, Berlin (1996).

[7] HATVANI L., PINTÉR L., On perturbation of unstable second order linear differential equations, *Proc. Amer. Soc.,* **61** (1976), 36–38.

[8] LA SALLE L. P., *The Stability and Control of Discrete Processes,* Springer Verlag, Berlin (1990).

[9] LEZAMA L., NAULIN R., Inestabilidad por primera aproximación, to appear in *Revista Saber,* Universidad de Oriente, Venezuela (1997).

[10] LI TA, Die Stabilitätsfrage bei Differenzengleichungen, *Math. Zeitschr.,* **32** (1930), 99–141.

[11] NAULIN R., Instability of nonautonomous differential systems, to appear in *Differential Equations and Dynamical Systems* (1997).

[12] NAULIN R., PINTO M., Projections for dichotomies in linear differential equations (submitted) (1997).

[13] PERRON O, Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen, *J. Reine und Angew. Math.,* **161** (1929), 41–64.

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ABOUT THE ERDŐS PAIRS

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Introduction

A few years ago, PAUL ERDŐS posed a problem about the partitions of a sequence of integers in the following terms.

Let us consider two natural integers $m < n$, and a partition into two parts A_1 and A_2 of the sequence $[m,n[= \{m, m+1, \ldots, n-1\}$ of all the integers from m to $n-1$. We shall call it an *Erdős partition* if there are, in at least one part, some (distinct) elements whose sum is equal to n. If *every* partition of $[m,n[$ is an Erdős partion, (m,n) an *Erdős pair*.

If (m,n) is an Erdős pair, then (m',n) for $m' < m$, and also (km,kn) for every positive integer k, are clearly Erdős pairs (it is enough to consider induced partitions). On the other hand, $(1,n)$ is an Erdős pair if (and only if) $n \geq 12$ (the remarks of section 1 below enable us to check this easily). So we have, for fixed $n \geq 12$, a maximal value $l_2(n)$ for m such that (m,n) is an Erdős pair, then a number $l_2 = \liminf(l_2(n)/n)$ when n tends to infinity. The problem posed by PAUL ERDŐS in 1992 was to determine the number l_2. In fact, it is quite easy to convince oneself that $l_2 \leq 1/4$. B. BOLLOBÁS and J. JIN [B-J] proved that $l_2 = 1/4$, giving the value of $l_2(n)$ for n large enough (however without precising a bound).

The first aim of the present note is to give a simpler an more "visual" proof of these results.[1]Moreover, the method which is used here allows us

[1] Had I known about the paper of B. BOLLOBÁS et G. JIN, I would certainly not have started this work; so, many thanks to my colleague D. DUMONT for having told me about the Erdős problem before, and about the reference [B-J] afterwards!

to determine easily the exact bound beyond which the general expression of $l_2(n)$ is valid.

Some sections (in particular lemma 1 and its corollary 1) are presented so that they can be applied to the more general case of partitions into p parts of $[m,n[$. For $l_p(n)$ and its inferior limit l_p defined in the obvious way, the conjecture is that $l_p = 1/2p$ (see [B-J]).

The second aim of this note is to present (in section 4) a new construction which gives support to this conjecture and allows us to improve the related results of [B-J]: we show that $l_p(n) < n/2p$ for every n, hence $l_p \le 1/2p$.

The subject discussed here is linked to the general theme of the Ramsey-type problems (see [G-R-S]) in additive number theory. Among the classical results to which it is related, let us mention those of DAVENPORT [D], ERDŐS and HEILBRONN [E-H]. More recent results have been established by BOLLOBÁS, ERDŐS and JIN ([B-E-J], [B-J 2]).

I am very grateful to VILMOS KOMORNIK for his encouragement and his interest in this work.

1. A few simple remarks

Let us search for a 2-partition (two parts only) $[m,n[= A_1 \cup A_2$ which is not of Erdős. Hence we exclude the partitions for which some part certainly contains elements whose sum is n. It is natural to exclude first those for which some A_i contains *two* elements whose sum is n. We can easily "see" these partitions: let us display the numbers $m, \ldots, n-1$ so that two numbers whose sum is n face each other:

if $n = 2p+1$:
$$\begin{array}{ccccccccc} & & m & m+1 & \ldots & k & \ldots & p-1 & p \\ n-1 & \ldots & n-m & n-m-1 & \ldots & n-k & \ldots & p+2 & p+1 \end{array}$$
if $n = 2p+2$:
$$\begin{array}{cccccccccc} & & m & m+1 & \ldots & k & \ldots & p-1 & p & p+1 \\ n-1 & \ldots & n-m & n-m-1 & \ldots & n-k & \ldots & p+3 & p+2 & \end{array}$$

Thus, the partitions first excluded are those for which some complete column is contained in a part A_i. Notice that the elements in the incomplete columns on the left cannot occur in a sum equal to n, hence no matter the parts to which they belong.

The display chosen above for the numbers $[m,n[$ also enables us to locate qualitatively three elements or more whose sum is n: we can see immediately that at most one of these elements is in the second row, and that this element must be on the right of the others:

* * *
*

When the 2-partition is not of Erdős, if two numbers u and v are in the same part A_i and if $n-u-v$ is distinct from u and v, then $n-u-v$ is not in A_i. In particular, if $u+v$ is in a column which is complete and distinct from those of u and v, then $u+v$ is also in A_i: otherwise, the element $n-u-v$ which faces it would be in A_i, and this is impossible.

These remarks allow us to prove easily that, for example, $(1,n)$ is an Erdős pair if (and only if) $n \geq 12$; or that $(2,n)$ is an Erdős pair for $n=15$ or $n \geq 17$.

2. Flips and pincers

DEFINITION. Given a partition of $[m,n[$, let us call a *flip* a sequence of two consecutive numbers which do not belong to the same part.

If the partition is not of Erdős, we know that the sequence $[p,p+1,p+2]$ necessarily contains a flip, since p faces either $p+1$ or $p+2$ (according to the parity of n), and two numbers with sum n do not belong to the same part.

This remark can be generalized in the following way.

LEMMA 1. *For $j=2, 3, 4, \ldots$ with $j<n/m$, there is, surrounding n/j, a sequence of $j+1$ consecutive integers which contains j elements whose sum is n. With the convention that the smallest number of the sequence must figure in that sum, this smallest number is equal to the integer part of $n/j-(j-1)/2$.*

PROOF. Denote by q the integer part of n/j (resp. $n/j+1/2$) if j is odd (resp. even). The sequence of length $j+1$ which is centered in $q+1/2$ (resp. q) contains j integers whose sum is n; the element which must be omitted in the sum is determined by the class of n mod j. Let us specify that the last element (i.e. the $(j+1)^{\text{th}}$) of the sequence has to be omitted when n (resp. $n+j/2$) is a multiple of j, with j odd (resp. even): thus, in that case, we have j *consecutive* integers whose sum is n.

NOTATION. Let us denote by S_j the sequence of length $j+1$ or j (according to the case) which has been described in the proof of lemma 1: it is the shortest sequence which contains j elements with sum n.

EXAMPLE. $(j=2)$ $S_2=[p,p+1,p+2]$ if n is even, and $S_2=[p,p+1]$ if n is odd.

COROLLARY 1. *If* $[m,n[$ *is equipped with a partition which is not of Erdős, then in each sequence* S_j *contained in* $[m,n[$ *there is at least one flip.*

Let us notice that asymptotically, i.e. for n large enough with the ratio n/m fixed, these sequences are contained in $[m,n[$ and are disjoint.

DEFINITION. Given a partition of $[m,n[$, let us say that four elements $x_1 < x_2 < y_2 < y_1$ form *pincers* if:

i) x_i and y_i belong to a same part A_i (with distinct A_1 and A_2)

ii) $x_1 + y_1 = x_2 + y_2 \leq n - m$

iii) the column of $x_i + y_i$ is distinct from those of x_1, y_1, x_2, y_2.

3. "Pincers extraction" of Erdős 2-partitions

In this section we shall only consider *2-partitions* (partitions into two parts only) of $[m,n[$. In this case, it is enough to find two "well placed" flips in order to make sure that the partition is of Erdős. Moreover, the sequences which contain a flip near to n/j (cf. lemma 1) can be shortened.

LEMMA 2. *Given a* 2*-partition which is not of Erdős, if two numbers* x *and* y *are in a same part* A_i *and if* $x+y$ *is in a column which is complete and distinct from those of* x *and* y*, then* $x+y$ *is also in* A_i.

PROOF. The part A_i does not contain the element $n-x-y$, thus it contains $x+y$.

LEMMA 3. *If a* 2*-partition contains pincers, then it is of Erdős.*

PROOF. Consider a set of pincers (x_1,x_2,y_2,y_1). The element $n-(x_i+y_i)$ forms, either with x_1 and y_1, or with x_2 and y_2, a triple whose sum is n in its part. By condition iii), these three elements are distinct.

The following task is to obtain effectively pincers from adequate flips. In fact, two flips which are respectively situated near $n/3$ and $n/4$ will suffice (for n large enough: $n \geq 56$), essentially because "$1/3 < 1/4 + 1/3 < 2/3$": as the elements x_1, x_2, y_2, y_1 apt to form pincers are near $n/4$ or $n/3$, the sum $x_1 + y_1 = x_2 + y_2$ will certainly be in a *distinct* column (cf. condition iii)), in fact in a column which is further right, as $x_i + y_i$ will be between $n/3$ and $2n/3$.

For the 2-partitions, the following lemma gives a sequence which is shorter than S_4, so it will allow us to situate more precisely a flip near $n/4$.

NOTATION. Denote by V the integer part of $(n-3)/4$.

LEMMA 4. *Assume that* $n \geq 11$. *Given a 2-partition of* $[V, n[$, *if the sequence* $[V, V+1, V+2, V+3]$ *is contained in a part, then the partition is of Erdős. In the case* $n \neq 2 \mod 4$, *it is enough for* $[V, V+1, V+2]$ *to be contained in a part.*

PROOF. In fact, V is the integer part of $(p-1)/2$ (let us recall that p, which is the smallest element of S_2, is itself the integer part of $(n-1)/2$). Hence, following lemma 1, $[V, V+1, V+2]$ contains two elements whose sum is p. As V must be one of these two elements (either $p = V + (V+1)$, or $p = V + (V+2)$), we can also find in $[V, V+1, V+2]$ two elements whose sum is $p+1$.

Suppose now that the partition is not of Erdős, the described sequence being contained in a part. If n is odd, we have $p+1 = n-p$; as p and $p+1$ are in the same part by lemma 2, we arrive to a contradiction. If n is even, we have $p+2 = n-p$; as $p+2$ is equal either to $(V+1)+(V+2)$ or to $(V+1)+(V+3)$ (according to the parity of p), lemma 2 leads once again to a contradiction.

The bound $n \geq 11$ traduces the condition that the column of p must be disjoint from the considered sequence, in order to apply lemma 2.

REMARK 1. As it is formulated, this proof may also be viewed as the starting point of a recursive process which could be useful for the study of the 3-partitions (of course lemma 2 must be replaced by the corresponding lemma).

REMARK 2. In case $n \neq 2$ mod 4, the sequence $[V, V+1, V+2]$ of lemma 4 corresponds to S_4 deprived of its first and last elements: in this case, we see that the integer part of $(n-6)/4$ is equal to $V-1$. But in case $n = 2$ mod 4, lemma 4 does not improve the result of lemma 1: the integer parts of $(n-6)/4$ and $(n-3)/4$ are equal, thus $[V, V+1, V+2, V+3] = S_4$ in this case.

NOTATION. Denote by S_4' the sequence of length 3 or 4 (according to the case) which has been described in lemma 4.

DEFINITION. A flip $[x, x+1]$ is said to be of *type* 12 if x is an element of the part A_1 and. $x+1$ of A_2. More generally, the *type* of a given sequence is the corresponding sequence of the indices of the parts which contain the given elements.

The following lemma may be seen as a complement to lemma 4. Picking out a few other cases which force the 2-partition to be of Erdős, it will allow us to precise the values of n such that the pair (V, n) is of Erdős.

LEMMA 5. *Consider the sequence of the first four or five (according to the case) elements of* $[V,n[$, *with* $n > 18$. *If its type falls within one of the following cases, then the 2-partition is of Erdős:*

for $n = 0 \mod 4$, *type* 11#1

for $n = 1 \mod 4$, *type* 11#1, 1#11 *or* 11##1

for $n = 2 \mod 4$, *type* 11##1 *or* 1#1#1

where # *is a generic character for* 1 *or* 2.

PROOF. Similar to the proof of lemma 4. Apply lemma 2, after having noticed that two elements which face each other (either p and $n-p$, or $p-1$ and $n-p+1$, according to the case) can be obtained as sums of elements belonging to the same part.

THEOREM. *For* $n \geq 56$, *every* 2-*partition of* $[m,n[$ *with* $m \leq V$ *is of Erdős. (*$n = 55$, $m = V$*) There is a 2 partition of* $[13,55[$ *which is not of Erdős. For every* n, *there is a* 2-*partition of* $[V+1,n[$ *which is not of Erdős.*

PROOF OF THE THEOREM. Let us suppose that the partition induced on $[V,n[$ is not of Erdős. By lemma 4 and corollary 1, there is at least one flip in each of the sequences S_4' and S_3. We shall distinguish two cases, depending on their types.

Let U be the smallest element of S_3. Recall that V is the integer part of $(n-3)/4$, and that U is the integer part of $n/3-1$ (lemma 1).

CASE 1. *The sequences* S_4' *and* S_3 *contain respectively flips with different types.*

First, let us make sure that S_4' and S_3 are disjoint (provided that $n \geq 42$). If $n \neq 2 \mod 4$, we see immediately that $V+3 \leq U$. If $n = 2 \mod 4$, then $V = (n-6)/4$, hence $V+4 \leq U$.

Hence, the two flips provide four elements $x_1 < x_2 < y_2 < y_1$ which apply for forming pincers. It only remains to verify condition iii), in order for $n - (x_i + y_i)$ to be distinct from x_1, x_2, y_2, y_1. In fact, we shall prove that $x_i + y_i$ is too small for its column to reach y_1 (equivalently, $x_i + y_i$ is on the right of y_1), which is expressed by the inequality: $x_1 + 2 \cdot y_1 < n$. The most "unfavourable" situation is the one in which x_1 and y_1 are as large as possible, the flips being situated at the ends of the sequences: let us say S_4' of type 1112 or 112, together with S_3 of type 2221 or 221, according to the class of $n \mod 12$. The pincers condition iii) is expressed by an inequation involving V, U and n. Now, the expressions of V and U as functions of n

provide a bound which is a sufficient condition: $n \geq 47$ is valid no matter the class of $n \mod 12$. Let us summarize this little study in the following table:

type of S_4'	type of S_3	pincers condition iii)	sufficient condition obtained	
1112	221 ($n=0 \mod 3$)	$V+2U+6<n$	$\frac{(n-3)}{4}+\frac{2n}{3}+4<n$:	$n>39$
112	221 ($n=0 \mod 3$)	$V+2U+5<n$	$\frac{(n-3)}{4}+\frac{2n}{3}+4<n$:	$n>39$
112	2221 ($n \neq 0 \mod 3$)	$V+2U+7<n$	$\frac{(n-3)}{4}+\frac{2(n-1)}{3}+5<n$:	$n>43$
1112 ($n=2 \mod 4$)	2221 ($n \neq 0 \mod 3$)	$V+2U+8<n$	$\frac{(n-6)}{4}+\frac{2(n-1)}{3}+6<n$:	$n>46$

CASE 2. *The sequences S_4' and S_3 do not contain flips of distinct types.*

In this case, there is one (unique) flip in each sequence; let us say, of type 12. The fact that the two sequences are disjoint does no longer ensure the existence of pincers: it is enough to concatenate S_4' of type 1112 with S_3 of type 1222 in order to be convinced of that.

First, notice that S_4' and S_3 are not contiguous as soon as $n \geq 54$ (we have $V+4 \leq U$, or $V+5 \leq U$, according to the length of S_4').

i) If there is an element of type 2 between the two flips, then the flip of S_4' gives us the first two elements of pincers whose fourth element is, at most, equal to U: therefore, the pincers condition iii) is evidently fulfilled.
Here are some examples: [1112]2[1222], [1122][1222], [112]221...[1112].

ii) If there is no element of type 2 between the two flips and if S_4' and S_3 are not contiguous, then the configuration is of type [112]1...[1...2]. Four elements are likely to form pincers (of type 2,1,1,2, now). The most "unfavourable" situation is the one in which the flip of S_3 is at the end of the sequence: the extremal elements of the pincers are the last elements of S_4' and S_3. It is enough to adapt the study which has been made in case 1. It may be noticed that the configuration [1112]1... (for $n=2 \mod 4$) has not even to be studied anymore: it is of Erdős by lemma 5.

type of S_4'	type of S_3	pincers condition iii)	sufficient condition obtained	
112	112 ($n=0 \mod 3$)	$V+2U+6<n$	$\frac{(n-3)}{4}+\frac{2n}{3}+4<n$:	$n>39$
112	1112 ($n \neq 0 \mod 3$)	$V+2U+8<n$	$\frac{(n-3)}{4}+\frac{2(n-1)}{3}+6<n$:	$n>55$

Finally, taking into account all the preceding considerations, we obtain the bound $n \geq 56$, which is valid in any case.

The previous analysis may also be exploited in order to construct a non-Erdős partition of $[13,55[$. As a consequence of both case 2 above and lemma 2, the partition is (and must be) of the following type:

		13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
type:		1	1	2	1	1	1	1	2	2	2	2	2	2	2	1
	...	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28
type:	...	2	2	1	2	2	2	2	1	1	1	1	1	1	1	2

It is easy to give a 2-partition of $[V+1,n[$ which is not of Erdős. But a more general construction will be presented in the next section.

4. Non-Erdős p-partitions

The following construction allows us to obtain very simply, for every integer n and whatever the number of parts p, a non-Erdős p-partition of $[m,n[$, provided that $m \geq n/2p$.

Notice that the example for $p = 3$ and n divisible by 60 of [B-J] is a product of a different process (besides it has to be slightly modified in order to be conclusive).

We begin with the study of p-partitions of an interval $[m/n,1[$ of *real numbers*.

PROPOSITION. *There exists a non-Erdős p-partition of the real interval $[1/2p,1[$: none of the p parts contain a finite subfamily whose sum is equal to* 1.

PROOF. The parts $A_1, \ldots, A_p$ are constructed in order for the interior of each of these parts to be stable by addition. Moreover, the parts to which the boundary points will be attributed must be chosen adequately. The choice which is made below will allow to improve (very slightly) the bound m when the proposition will be applied to the p-partitions of the integers $[m,n[$.

Let us define the following intervals, where i $(1 \leq i \leq p)$ is a counter for the parts, and k $(k \geq 0)$ is a counter for the connected components of each part:

– for $k = 0$,

$$J_i^0 = \begin{cases} \left[\frac{1}{2p}, \frac{1}{2p-1}\right], & \text{if } i = 1 \\ \left]\frac{1}{2p-i+1}, \frac{1}{2p-i}\right], & \text{if } 2 \leq i \leq p \end{cases}$$

– for $k \geq 1$ with $2^k \leq 2p - i$,

$$J_i^k = \begin{cases} \left]\frac{2^k}{2p-i+1}, \frac{2^k}{2p-i}\right], & \text{if } \frac{2^k}{2p-i} \leq \frac{1}{2} \\ \left]\frac{2^k}{2p-i+1}, \frac{2^k}{2p-i}\right[, & \text{if } \frac{2^k}{2p-i+1} = \frac{1}{2} \\ \left[\frac{2^k}{2p-i+1}, \frac{2^k}{2p-i}\right[, & \text{if } \frac{2^k}{2p-i+1} > \frac{1}{2} \end{cases}.$$

Let A_i be the union of the J_i^k $(k \geq 0)$. In short, if we leave aside the boundary points, a partition of $]1/2p, 1/p[$ is extended homothetically to $]1/2p, 1[$.

An elementary checking may (and has to) be made: $1/2$ is a boundary-point.

i) There are a value $k = K$, and a value $i = I$, such that

$$\frac{2^K}{2p - I + 1} = \frac{1}{2}.$$

Indeed, let us write p to the base 2: $p = b_K b_{K-1} \ldots b_1 b_0$ (where $b_K = 1$).

We have: $2^K \leq p \leq 1{+}2{+}\ldots{+}2^K = 2^{K+1}-1$, equivalently: $p < 2^{K+1} \leq 2p$, from where we obtain the desired value of i (with $1 \leq i \leq p$).

Of course, 1 is then a boundary-point too, and the partition is well-defined.

ii) Clearly, for elements taken in J_i^0 and not all of them identical, the sum is different from 1.

iii) From i), ii) and from the choice which has been made at the boundary-points of the intervals J_i^k, we deduce that the sum of elements of A_i, all of them being distinct, is different from 1. i The only case where the checking is not trivial is the one with all the elements taking the form

$$\frac{2^k}{2p - i}$$

(i.e. ends of intervals). Denoting by k_i the largest integer such that

$$\frac{2^{k_i}}{2p - i} \leq \frac{1}{2}$$

(in fact, $k_i = K$ or $k_i = K - 1$), the sum of the given elements is at most

$$\frac{1}{2p - i}\left(2^{k_i+1} - 1\right) \leq \frac{1}{2p - i}(2p - i - 1) < 1.$$

Notice that the case where all the elements are at the beginning of intervals is immediate ($\frac{1}{2p} + \frac{2^k}{2p} \neq 1$ as $1 + 2^k$ is odd).

As an evident corollary of proposition 1, we obtain

PROPOSITION 2. *If $m/n \geq 1/2p$, then there exists a non-Erdős partition into p parts of the interval of natural numbers $[m, n[$.*

For some values of $n \mod 2p$, this bound obtained for m can be slightly improved (one unit less), as a result of the discretisation: among the reals, the only elements which are to be taken into account are the multiples of $1/n$, and they have to be situated in relation to $1/2p$.

For $p = 2$ for example:

– if $n = 0 \mod 4$ or $n = 3 \mod 4$, the bound which has been given by proposition 2 is optimal: the condition $m \geq n/4$ is equivalent to $m \geq V + 1$ (cf. §3.).

– if $n = 1 \mod 4$, the smallest element of the interval J_1^0 is $(n+3)/4$, that is $V+2$ in the considered case; it is straightforward to check that the partition remains non-Erdős if the preceding integer $(n-1)/4$ is joined to A_1.

– if $n = 2 \mod 4$, the smallest element of the interval J_1^0 is $V + 2$ too. As before, the preceding integer, $(n-2)/4$, can be joined to A_1.

Hence we have, for every value n, a non-Erdős 2-partition of $[V + 1, n[$.

References

[B-J] B. BOLLOBÁS, G. JIN, A partition problem of Erdős for integers, Combinatorics, Paul Erdős is Eighty (vol.l), Keszthely 1993, pp. 79–89, Bolyai Math. Studies, Budapest 1993

[B-J 2] B. BOLLOBÁS, G. JIN, Partition problems in additive number theory, *J. Number Theory,* **56** (1996), 167–187.

[B-E-J] B. BOLLOBÁS, P ERDŐS, G. JIN, Ramsey problems in additive number theory, *Acta Arith.,* **64** (1993), 341–355.

[E-H] P. ERDŐS, H. HEILBRONN, On the addition of residue classes mod p, *Acta Arith.,* **9** (1964), 149–159.

[D] H. DAVENPORT, On the addition of residue classes, *Journ. London Math. Soc.,* **10** (1935), 30–32.

[G-R-S] R. GRAHAM, B. ROTHSCHILD, J. SPENCER, *Ramsey Theory,* 2nd edition, Wiley-Interscience 1990.

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ON CERTAIN QUASI-EINSTEIN SEMISYMMETRIC HYPERSURFACES

By

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1. Introduction

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ and let ∇ be its Levi–Civita connection. A manifold (M, g) is said to be an *Einstein manifold* if the following condition is fulfilled on M: $S = \frac{\kappa}{n} g$, where S and κ denote the Ricci tensor and the scalar curvature of (M, g), respectively. According to [11], a manifold (M, g) is called Ricci-simple if $\operatorname{rank}(S) \leq 1$ holds at every point $x \in M$. Conformally flat Ricci-simple manifolds were investigated in [2] and [11]. Einstein manifolds as well as Ricci-simple manifolds form subclasses of the class of quasi-Einstein manifolds. A manifold (M, g) is called a quasi-Einstein manifold if at every point $x \in M$ its Ricci tensor is decomposed in two parts: a metrical part and a part of rank at most one, i.e. if $S = \alpha g + \beta w \otimes w$ holds at x, where $w \in T^*_x(M)$ and $\alpha, \beta \in \mathbb{R}$. A manifold (M, g) is said to be semisymmetric ([23], [25]) if $R \cdot R = 0$ holds on M. Every semisymmetric manifold is *Ricci-semisymmetric* ($R \cdot S = 0$). The converse statement is not true. Every semi-Riemannian semisymmetric manifold (M, g) satisfies at every point the following condition ([15]):

$(*)$ the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent.

This condition is equivalent to $R \cdot C = LQ(S, C)$ on the set $U = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, for a certain function L on U. There exist non-semisymmetric manifolds satisfying $(*)$. Manifolds satisfying $(*)$ were recently investigated in: [8], [15], [17], [18] and [19].

We prove that some hypersurfaces fulfill $(*)$. Namely, the main result of this paper (see Theorem 5.1) states that if M is a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$ with signature $(n+1-s, s)$, $n \geq 4$, satisfying at every point $x \in M$ the following relation

$$S = \frac{\kappa}{n-1} g + \beta w \otimes w, \qquad w \in T_x^*(M), \qquad \alpha, \beta \in \mathbb{R}, \tag{1}$$

then $(*)$ holds at every point of M. Precisely, the following condition is fulfilled on M

$$R \cdot C = Q(S, C). \tag{2}$$

Hypersurfaces of $\mathbb{E}_s^{n+1}$, $n \geq$ satisfying $(*)$ will be studied in a subsequent paper

In Section 2 we give definitions of basical tensors used in the paper. In this section we give also a presentation of certain classes of manifolds of pseudosymmetry type. In Section 3 we prove that if (M, g) is a Ricci-semisymmetric manifold satisfying (1) and

$$R \cdot R = Q(S, R) \tag{3}$$

at every point $x \in M$ then (2) holds on M (see Theorem 3.1). In Section 4 we give an example of a semisymmetric warped product manifold satisfying assumptions of Theorem 3.1. In Section 5 we present results on quasi-Einstein Ricci-semisymmetric as well as semisymmetric hypersurfaces. Since every hypersurface of $\mathbb{E}_s^{n+1}$ satisfies (3), Theorem 5.1 is an immediate consequence of Theorem 3.1. Theorem 5.2 shows that certain quasi-Einstein semisymmetric hypersurfaces are Ricci-simple semisymmetric manifolds. Finally, in Theorem 5.3 a curvature characterization of Ricci-simple semisymmetric manifolds is given. In this section we give also an example of a semisymmetric hypersurface satisfying assumptions of theorems 5.1–5.3.

2. Certain pseudosymmetry type manifolds

We define on a semi-Riemannian manifold (M, g) the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right),$$

respectively, where A is a $(0,2)$-tensor on M, $X, Y, Z \in \Xi(M), \Xi(M)$ being the Lie algebra of vector fields of M. The Ricci operator $\mathcal{S}$ is defined by $S(X, Y) = g(X, \mathcal{S} Y)$, where S is the Ricci tensor and κ the scalar curvature of (M, g), respectively. We define the tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M, g), by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively. For $(0,2)$-tensors A and B we define its Kulkarni–Nomizu product $A \wedge B$ by

$$(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) - \\ - A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).$$

We note that the Weyl tensor C can be presented in the following form

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \tag{4}$$

For a $(0,2)$-tensor A we define the $(0,4)$-tensor $\overline{A}$ by $\overline{A} = \frac{1}{2} A \wedge A$. Thus we have

$$\overline{A}(X_1, X_2, X_3, X_4) = A(X_1, X_4)A(X_2, X_3) - A(X_1, X_3)A(X_2, X_4).$$

For a $(0,k)$-tensor T, $k \geq 1$ and a symmetric $(0,2)$-tensor A we define the $(0,k)$-tensor $A \cdot T$ and the $(0,k+2)$-tensors $R \cdot T$ and $Q(A, T)$ by

$$(A \cdot T)(X_1, \ldots, X_k) = -T(\mathcal{A}X_1, X_2, \ldots, X_k) - \ldots - T(X_1, X_2, \ldots, \mathcal{A}X_k),$$
$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_k) = \\ = -T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),$$
$$Q(A, T)(X_1, \ldots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k) = \\ = -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),$$

where $\mathcal{A}$ is the corresponding to A endomorphism of $\Xi(M)$ defined by $g(\mathcal{A}X, Y) = A(X, Y)$. Putting in the above formulas $T = R$, $T = S$ or $T = C$, $A = g$ or $A = S$, we obtain the tensors: $R \cdot R$, $R \cdot S$, $R \cdot C$, $C \cdot S$, $Q(g, R)$, $Q(g, S)$, $Q(g, C)$, $Q(S, R)$, $Q(S, C)$, $S \cdot R$ and $S \cdot C$.

As a proper generalization of locally symmetric spaces ($\nabla R = 0$) semisymmetric manifolds were studied by many authors. We mention that complete semisymmetric hypersurfaces of Euclidean spaces as well as semisymmetric Lorentzian hypersurfaces of Minkowski spaces were classified in [24] and in [26], respectively. The profound investigation of several properties of semisymmetric manifolds, gave rise to their next generalization: the pseudosymmetric manifolds. A manifold (M, g) is said to be pseudosymmetric ([13], [27]) if at every point of M the following condition is satisfied:

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This condition is equivalent to $R \cdot R = L_R Q(g, R)$ on the set $U_R = \left\{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\right\}$, where L_R is a certain function on U_R. Evidently, every semisymmetric manifold is pseudosymmetric. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric and a review of results on pseudosymmetric manifolds is given in [13] (see also [27]). It is known that a 3-dimensional semi-Riemannian manifold is pseudosymmetric if and only if it is quasi-Einstein ([21], Theorem 1). We mention also that Theorem 2 of [14] provides a classification of all pseudosymmetric 3-dimensional pseudosymmetric totally real submanifolds of S^6.

It is easy to see that if $(*)_1$ holds on a manifold (M, g), then at every point of M the following condition is satisfied:

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The converse statement is not true (see, e.g. [13]). A manifold (M, g) is called Ricci-pseudosymmetric if at every point of M the condition $(*)_2$ is fulfilled. If a manifold (M, g) is Ricci-pseudosymmetric then

$$R \cdot S = L_S Q(g, S), \tag{5}$$

holds on the set $U_S = \left\{x \in M \mid S \neq \frac{\kappa}{n} g \text{ at } x\right\}$, where L_S is a certain function on U_S.

As it was shown in [20](Proposition 3.1), at every point of a hypersurface M of a semi-Riemannian space of constant curvature $N^{n+1}(c)$ the following condition is fulfilled:

$(*)_3$ the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

More precisely,

$$R \cdot R - Q(S, R) = -\frac{(n-2)\tilde{\kappa}}{n(n+1)} Q(g, C) \tag{6}$$

holds on M, where $\tilde{\kappa}$ is the scalar curvature of the ambient space. It is clear that if the ambient space is a semi-Euclidean space $\mathbb{E}^{n+1}_s$ then (6) reduces to (3). Note also that every pseudosymmetric Einstein manifold realizes $(*)_3$. Pseudosymmetric manifolds satisfying $(*)_3$ were investigated in [16].

Semi-Riemannian manifolds satisfying $(*)_1$–$(*)_3$, $(*)$ or other conditions of this kind (see [13] and [27]) are called manifolds of *pseudosymmetry type*. Hypersurfaces satisfying such conditions were studied among others in: [1], [2], [7], [9], [20] and [22].

3. Ricci-semisymmetric manifolds

Using the definitions presented in Section 2 we can prove that if A and B are symmetric $(0,2)$-tensor fields on a semi-Riemannian manifold (M,g), $n \geq 3$, then we have on M: $Q(A, A \wedge B) = -Q(B, \overline{A})$. In particular, from this it follows that (cf. [2], Lemma 3.1): $Q(g, g \wedge S) = -Q(S, G)$, $Q(S, g \wedge S) = -Q(g, \overline{S})$ and

$$Q(S,C) = Q(S,R) + \frac{1}{n-2} Q(g,\overline{S}) + \frac{\kappa}{(n-2)(n-1)} Q(S,G). \tag{7}$$

PROPOSITION 3.1. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold.*

(i) *If* (5) *is satisfied at x then we have at this point*

$$R \cdot C = R \cdot R + \frac{1}{n-2} L_S Q(S,G). \tag{8}$$

(ii) *If* (1) *is satisfied at x then we have at this point*

$$\overline{S} = \left(\frac{\kappa}{n-1}\right)^2 G + \frac{\kappa}{n-1} g \wedge (\beta w \otimes w), \tag{9}$$

$$Q(S,C) = Q(S,R). \tag{10}$$

(iii) *If* (1), (5) *and*

$$R \cdot R = Q(S,R) + LQ(g,C), \qquad L \in \mathbb{R}, \tag{11}$$

are satisfied at x then the following relation is fulfilled at x

$$R \cdot C = Q(S,C) + LQ(g,R) + \frac{1}{n-2}(L + L_S) Q(S,G). \tag{12}$$

(iv) *If $(*)_2$ is satisfied at x then we have at this point*

$$R(\mathcal{S}X, Y, W, Z) + R(\mathcal{S}W, Y, Z, X) + R(\mathcal{S}Z, Y, Z, W) = 0. \tag{13}$$

PROOF. (i) The tensor $R \cdot C$, by (4), takes the following form

$$R \cdot C = R \cdot \left(R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G\right) = R \cdot R - \frac{1}{n-2} R \cdot (g \wedge S).$$

But this, by making use of (5), turns into (8).

(ii) (9) is an immediate consequence of (1). From (9) we obtain

$$Q(g,\overline{S}) = \frac{\kappa}{n-1}Q(g, g\wedge(\beta w\otimes w)) = \tag{14}$$

$$= -\frac{\kappa}{n-1}Q((\beta w\otimes w), G) = -\frac{\kappa}{n-1}Q(S,G). \tag{15}$$

But this reduces (7) to (10).

(iii) Using the following identity ([7], Remark 2.1)

$$Q(g,C) = Q(g,R) + \frac{1}{n-2}Q(S,G), \tag{16}$$

(8), (10) and (11), we obtain our assertion

$$\begin{aligned} R\cdot C &= R\cdot R + \frac{1}{n-2}L_S Q(S,G) = \\ &= Q(S,R) + LQ(g,C) + \frac{1}{n-2}L_S Q(S,G) = \\ &= Q(S,C) + LQ(g,C) + \frac{1}{n-2}L_S Q(S,G) = \\ &= Q(S,C) + LQ(g,R) + \frac{1}{n-2}(L+L_S)Q(S,G). \end{aligned}$$

(iv) Evidently, if $x\in M-U_S$ then (13) holds at x. If $x\in U_S$ then (5) implies

$$\begin{aligned} &R(\mathcal{S}X,Y,W,Z) + R(X,\mathcal{S}Y,W,Z) = L_S(g(X,W)S(Y,Z)+ \\ &\qquad +g(Y,W)S(X,Z) - g(X,Z)S(Y,W) - g(Y,Z)S(X,W)). \end{aligned}$$

Summing this cyclically in X, W, Z we obtain (13).

As an immediate consequence of the above results we have the following

THEOREM 3.1. *If* (M,g), $n \geq 4$, *is a Ricci-semisymmetric semi-Riemannian manifold fulfilling* (1) *at every point* $x\in M$ *then the conditions:* $R\cdot R = Q(S,R)$ *and* $R\cdot C = Q(S,C)$ *are equivalent on* M.

COROLLARY 3.1. *If* (M,g), $n\geq 4$, *is a Ricci-semisymmetric Ricci-simple semi-Riemannian manifold with vanishing scalar curvature then the conditions:* $R\cdot R = Q(S,R)$ *and* $R\cdot C = Q(S,C)$ *are equivalent on* M.

THEOREM 3.2. *If* $M,g)$, $n\geq 4$, *is a semi-Riemannian manifold fulfilling* (1) *then*

$$Q(S,R) = 0 \tag{17}$$

and

(18) $$Q(S,C)=0$$

are equivalent on M. *Moreover, if at a point* $x \in M$ *the tensors* S *and* C *are non-zero and* (17) *or* (18) *holds at this point then the scalar curvature* κ *vanishes at* x.

PROOF. The first part of our assertion is an immediate consequence of Proposition 3.1 (ii). Let at a point $x \in M$ the tensors S and C be non-zero and let (17) be satisfied at x. This, in view of Theorem 4.1(i) of [5], implies $R \cdot R = 0$, whence

(19) $$R \cdot C = 0.$$

From (18), in view of Theorem 3.1 of [15] it follows that $R \cdot R = \frac{\kappa}{n-1} Q(g,R)$, whence $R \cdot C = \frac{\kappa}{n-1} Q(g,C)$. This, by (19), and the assumption $C \neq 0$ implies $\kappa = 0$. Our theorem is thus proved.

4. Warped products

Let (M,g) be a semi-Riemannian manifold covered by a system of charts $\{W; x^k\}$. We denote by g_{ij}, R_{hijk}, S_{ij}, $S_i^j = g^{ij} S_{ik}$, $G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ and

(20) $$C_{hijk} = $$
$$= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) + \frac{\kappa}{(n-2)(n-1)} G_{hijk},$$

the local components of the metric tensor g, the Riemann–Christoffel curvature tensor R, the Ricci tensor S, the Ricci operator $\mathcal{S}$, the tensor G and the Weyl tensor C, respectively. Let now $(\overline{M}, \overline{g})$ and $(\tilde{N}, \tilde{g})$, $\dim \overline{M} = p$, $\dim \tilde{N} = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\tilde{V}; y^\alpha\}$, respectively. Let F be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times_F \tilde{N}$ of $(\overline{M}, \overline{g})$ and $(\tilde{N}, \tilde{g})$ is the product manifold $\overline{M} \times \tilde{N}$ with the metric $g = \overline{g} \times_F \tilde{g}$ defined by

$$\overline{g} \times_F \tilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \tilde{g},$$

where $\pi_1 : \overline{M} \times \tilde{N} \to \overline{M}$ and $\pi_2 : \overline{M} \times \tilde{N} \to \tilde{N}$ are the natural projections on $\overline{M}$ and $\tilde{N}$, respectively. Let $\{\overline{U} \times \tilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \tilde{N}$. The local components of the metric $g = \overline{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F \tilde{g}_{\alpha\beta}$

if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a,b,c,\ldots \in \{1,\ldots,p\}$, $\alpha,\beta,\gamma,\ldots \in \{p+1,\ldots,n\}$ and $h,i,j,k,l,m \in \{1,2,\ldots,n\}$. We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\tilde{g}$). The local components Γ^h_{ij} of the Levi–Civita connection ∇ of $\overline{M} \times_F \tilde{N}$ are the following:

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \qquad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \qquad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\overline{g}^{ab} F_b \tilde{g}_{\alpha\beta},$$

$$\Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta, \quad \Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}. \tag{21}$$

The local components

$$R_{hijk} = g_{hl} R^l_{ijk} = g_{hl}(\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj}), \qquad \partial_k = \frac{\partial}{\partial x^k},$$

of the Riemann–Christoffel curvature tensor R and the local components S_{ij} of the Ricci tensor S of the warped product $\overline{M} \times_F \tilde{N}$ which may not vanish identically are the following:

$$R_{abcd} = \overline{R}_{abcd}, \; R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \tilde{g}_{\alpha\beta}, \; R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\beta} - \frac{\Delta_1 F}{4} \tilde{G}_{\alpha\beta\gamma\delta}, \tag{22}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \; S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - \frac{1}{2}\left(tr(T) + \frac{n-p-1}{2F}\Delta_1 F \right) \tilde{g}_{\alpha\beta}, \tag{23}$$

$$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \; tr(T) = \overline{g}^{ab} T_{ab}, \; \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b, \tag{24}$$

and T is the $(0,2)$-tensor with the local components T_{ab}. The scalar curvature κ of $\overline{M} \times_F \tilde{N}$ satisfies the following relation

$$\kappa = \overline{\kappa} + \frac{\tilde{\kappa}}{F} - \frac{n-p}{F}\left(tr(T) + \frac{n-p-1}{4F}\Delta_1 F \right). \tag{25}$$

EXAMPLE 4.1. Let $(\tilde{N}, \tilde{g})$, be a 1-dimensional Riemannian manifold. Let $\overline{M}$ be a non-empty open connected subset of $\mathbb{R}^p$, $p = n-1 \geq 3$, equipped with the standart metric $\overline{g}$, $\overline{g}_{ab} = \varepsilon_a \delta_{ab}$, $\varepsilon_a = \pm 1$. We put $F = F(x^1, \ldots, x^p) = = k \exp(\xi_a x^a)$, where $\xi_1, \ldots, \xi_p$ and k are constants such that $\xi_1^2 + \ldots \xi_p^2 > 0$, $\overline{g}^{ab}\xi_a\xi_b = 0$ and $k > 0$. We consider the warped product $\overline{M} \times_F \tilde{N}$. Now (22)–(25) turn into

$$R_{abcd} = 0, \qquad R_{nabn} = -\frac{F}{4}\xi_a\xi_b\tilde{g}_{nn},$$

$$T_{ab} = \frac{F}{2}\xi_a\xi_b, \qquad tr(T) = 0, \qquad \Delta_1 F = 0,$$

$$S_{ab} = -\frac{1}{4}\xi_a\xi_b, \qquad S_{nn} = 0, \qquad S^2 = 0, \qquad \kappa = 0, \tag{26}$$

respectively. In [6] (Example 5.2) it was shown that the manifold $\overline{M} \times {}_F\tilde{N}$ satisfies (3). Furthermore, in virtue of (26), Lemma 4 of [12] implies $R\cdot R = 0$, whence $R\cdot S = 0$. Thus we see that the manifold $\overline{M}\times {}_F\tilde{N}$ realizes assumptions of Corollary 3.1. In addition (26) implies $S \cdot R = 0$. Further, using (20), (26), and (12)–(16) of [12], we state that the local components of the Weyl tensor C_{rstu} of the manifold $\overline{M} \times {}_F\tilde{N}$ are the following

$$C_{abcd} = \frac{1}{4(n-2)}(g_{ad}\xi_b\xi_c + g_{bc}\xi_a\xi_d - g_{ac}\xi_b\xi_d - g_{bd}\xi_a\xi_c),$$

$$C_{annd} = -\frac{n-3}{4(n-2)}\xi_a\xi_d g_{nn}. \tag{27}$$

We note that $\overline{g}^{pq} S_{rp} C_{qstu} = 0$. Therefore the tensors $S \cdot C = 0$ the $C \cdot S = 0$ must vanish.

5. Pseudosymmetric hypersurfaces

Let M, $n = \dim M \geq 3$, be a connected hypersurface immersed isometrically in a semi-Riemannian manifold $(N, \tilde{g})$. We denote by g the metric tensor of M, induced from the metric tensor $\tilde{g}$. Further, we denote by $\tilde{\nabla}$ and ∇ the Levi–Civita connections corresponding to the metric tensors $\tilde{g}$ and g, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \tilde{g}(\xi, \xi) = \pm 1$. We can present the *Gauss* formula and the Weingarten formula of M in N in the following form: $\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$, and $\tilde{\nabla}_X \xi = -\mathcal{A}(X)$, respectively, where X, Y are vector fields tangent to M, H is the *second fundamental* tensor of M in N, $\mathcal{A}$ is the shape operator of M in N and $g(\mathcal{A}(X), Y) = \tilde{\nabla}_X \xi = -\mathcal{A}(X)$. Furthermore, for $k > 1$ we also have that $H^k(X, Y) = g(\mathcal{A}^k(X), Y)$ and $tr(H^k) = tr(\mathcal{A}^k)$. We assume that the ambient space is a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 3$. The curvature tensor R of M, by the Gauss equation of M in N, satisfies $R = \frac{\varepsilon}{2} H \wedge H$. Let the equations $x^r = x^r(y^h)$ be the local parametric expression of M in $(N, \tilde{g})$, where y^r and x^r are the local coordinates of M and N, respectively, and $h, i, j, k, l, m \in \{1, \ldots, n\}$ and $r, s, t, u \in \{1, \ldots, n+1\}$. Thus we have

$$R_{hijk} = \varepsilon\left(H_{hk}H_{ij} - H_{hj}H_{ik}\right), \qquad \varepsilon = \pm 1, \tag{28}$$

where R_{hijk} and H_{hk} are the local components of the tensors R and H, respectively.

We present now applications of Theorem 3.1 to hypersurfaces of spaces of constant curvature. Evidently, we replace (11) by (6). We have

PROPOSITION 5.1. *Let M be a hypersurface of a space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If* (1) *and* (5) *are satisfied at a point $x \in M$ then the relation*

$$(29)\quad R\cdot C = Q(S,C) - \frac{(n-2)}{n(n+1)}Q(g,R) + \frac{1}{n-2}\left(L_S - \frac{(n-2)\tilde{\kappa}}{n(n+1)}\right)Q(S,G),$$

holds at x. In addition, if at a point x $\mathcal{A}^2$ is not a linear combination of $\mathcal{A}$ and the identity transformation Id at x then (29) *turns into*

$$(30)\qquad R\cdot C = Q(S,C) - \frac{(n-2)\tilde{\kappa}}{n(n+1)}Q(g,R) - \frac{(n-3)\tilde{\kappa}}{(n-2)n(n+1)}Q(S,G).$$

PROOF. (29) is an immediate consequence of an application of (6) to Proposition 3.1. Since $\mathcal{A}^2$ is not a linear combination of $\mathcal{A}$ and the identity transformation Id at x, in view of Propositions 3.1 and 3.2 of [7], we have at x: $R\cdot S = \frac{\tilde{\kappa}}{n(n+1)}Q(g,S)$, i.e. $L_S = \frac{\tilde{\kappa}}{n(n+1)}$. Applying the last relation in (29) we obtain (30), which completes the proof.

In particular, if $\tilde{\kappa} = 0$, Proposition 5.1 implies the following

THEOREM 5.1. *Let M be a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$. If* (1) *is fulfilled on M then $R\cdot C = Q(S,C)$ holds on M.*

We note that the above theorem follows also from Theorem 3.1. Furthermore, Corollary 3.1 as well as Theorem 5.1 imply

COROLLARY 5.1 *If M is a Ricci-semisymmetric Ricci-simple hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, with vanishing scalar curvature, then $R\cdot C = Q(S,C)$ holds on M.*

As an immediate consequence of Theorem 3.2 and (3) we have

PROPOSITION 5.2. *Let M be a semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying* (1) *at every point. If the tensors C and $S - \frac{\kappa}{n-1}g$ are non-zero at every point of a dense subset of M then the hypersurface M is a Ricci-simple manifold satisfying $R\cdot C = Q(S,C) = 0$.*

EXAMPLE 5.1. Let $\overline{M} \times {}_F\tilde{N}$ be a warped product defined in Example 4.1. Since (3) holds on $\overline{M} \times {}_F\tilde{N}$, this manifold can not be realized as a hypersurface of a semi-Riemannian space of constant curvature $N^{n+1}(c)$, with non-zero curvature c. However, the manifold $\overline{M} \times {}_F \tilde{N}$ can be realized as a hypersurface of a semi-Euclidean space. Let H be the $(0,2)$-tensor on $\overline{M} \times {}_F \tilde{N}$, with the local components H_{ij}, defined by $H_{ab} = -\frac{\varepsilon}{4l}\sqrt{F}\xi_a\xi_b$, $H_{an} = 0$, $H_{nn} = l\sqrt{F}\tilde{g}_{nn}$, where $\varepsilon = \pm 1$ and l is a positive constant. The tensor H fulfills (28) and $H^3 = tr(H)H^2$. Furthermore, by making use of (21), we can check that H is a Codazzi tensor. Thus we see that the manifold $\overline{M} \times {}_F \tilde{N}$ can be realized as a hypersurface immersed isometrically in a semi-Euclidean space $\mathbb{E}^{n+1}_s$. Such hypersurface satisfies assumptions of Theorem 5.2.

REMARK 5.1. As it was shown in previous section from (26) it follows that $S \cdot R = 0$ holds on $\overline{M} \times {}_F \tilde{N}$. Thus we have an example of a hypersurface satisfying assumptions of Theorem 3.1 of [1].

In [4] Riemannian hypersurfaces of Euclidean spaces satisfying the conditions $R \cdot C = 0$ or $C \cdot R = 0$ were investigated. With this subject is related the following result.

THEOREM 5.3. *Let M be a hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 4$, with vanishing scalar curvature.*

(i) *If M is a Ricci-simple semisymmetric manifold then $R \cdot C = 0$ and $C \cdot R = 0$ hold on M.*

(ii) *If $R \cdot C = 0$ and $C \cdot R = 0$ are satisfied on M then $M \cap U_C$ is a Ricci-simple semisymmetric manifold.*

PROOF. The following identities are fulfilled on every semi-Riemannian manifold

$$(C \cdot R)_{hijklm} = -\frac{1}{n-2}(g_{hl} S^s_m R_{sijk} - g_{hm} S^s_l R_{sijk} - g_{il} S^s_m R_{shjk} +$$

$$+|g_{im} S^s_l R_{shjk} + g_{jl} + S^s_m R_{skhi} - g_{jm} S^s_l R_{skhi} - g_{kl} S^s_m R_{sjhi} + g_{km} S^s_l R_{sjhi}) +$$

$$+(R \cdot R)_{hijklm} - \frac{1}{n-2} Q(S,R)_{hijklm} + \frac{\kappa}{(n-2)(n-1)} Q(g,R)_{hijklm}, \tag{31}$$

$$g^{hm} Q(S,R)_{hijklm} = \kappa R_{lijk} + S_{lk} S_{ij} - S_{ik} S_{jl} - S^s_i R_{sljk} +$$

$$+ S^s_l R_{sijk} + S^s_j R_{sikl} + S^s_k R_{silj}. \tag{32}$$

(i) First of all, semisymmetry of M implies: $R \cdot C = 0$ and $Q(S,R) = 0$. The last relation, together with the relations $\kappa = 0$ and $\operatorname{rank} S \leq 1$, reduces

(32) to

$$S_i^s R_{sljk} = S_l^s R_{sijk} + S_j^s R_{sikl} + S_k^s R_{silj},$$

whence, by Proposition 3.1(iv), we obtain $S_i^s R_{sljk} = 0$. Now (31) gives $C \cdot R = 0$.

(ii) From Theorem 3.1 of [3] it follows that $R \cdot R = 0$ holds on U_C. Now (32) turns into $S_i^s R_{sljk} = S_{lk} S_{ij} - S_{ik} S_{jl}$. Thus (31) implies $Q(g, S \wedge S) = 0$, whence, in view of Lemma 1.1 of [10], it follows that

$$S \wedge S = \lambda G, \qquad \lambda \in \mathbb{R}, \tag{33}$$

holds at every point $x \in U_C$. Further, (33) gives $Q(S, S \wedge S) = \lambda Q(S, G)$ and, in a consequence, $\lambda Q(S, G) = 0$. Evidently, if λ vanishes at a point $x \in U_C$, then (33) implies $\operatorname{rank} S \leq 1$. If $\lambda \neq 0$ at x, then the last equality gives $Q(S, G) = 0$, whence $S = \frac{\kappa}{n} g$, and $S = 0$. Our theorem is thus proved.

COROLLARY 5.2. *Let M be a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$, with vanishing scalar curvature. Then $M \cap U_C$ is a Ricci-simple semisymmetric manifold of and only if $R \cdot C = 0$ and $C \cdot R = 0$ hold on $M \cap U_C$.*

The hypersurface defined in Example 5.1 satisfies assumptions of Corollary 5.2.

References

[1] K. ARSLAN, R DESZCZ and R. EZENTAŞ, On a certain subclass of hypersurfaces in semi-Euclidean spaces, *Soochow J. Math.,* in print.

[2] K. ARSLAN, R. DESZCZ, R. EZENTAS and M. HOTLOŚ, On a certain subclass of conformally flat manifolds, *Bull. Inst. Math. Acad. Sinica,* **26** (1998), 283–299.

[3] K. ARSLAN, R. DESZCZ and S. YAPRAK, On Weyl pseudosymmetric hypersurfaces, *Colloq. Math.,* **72** (1997), 353–361.

[4] D. E. BLAIR, P. VERHEYEN and L. VERSTRAELEN, Hypersurfaces satisfaisant a $R \cdot C = 0$ ou $C \cdot R = 0$, *C. R. Acad. Bulgare Sci.,* **37** (1984), 459–462.

[5] F. DEFEVER and R. DESZCZ, On semi-Riemannian manifolds satisfying the condition $R \cdot R = Q(S, R)$, in: *Geometry and Topology of Submanifolds,* III, World Sci. Publishing, River Edge, NJ, 1990, 108–130.

[6] F. DEFEVER and R. DESZCZ, On warped product manifolds satisfying a certain curvature condition, *Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Natur.,* **69** (1991), 213–236.

[7] F. DEFEVER, R. DESZCZ, P. DHOOGHE, L. VERSTRAELEN and Ş. YAPRAK, On Ricci-pseudosymmetric hypersurfaces in spaces of constant curvature, *Results in Math.,* **27** (1995), 227–236.

[8] F. DEFEVER, R. DESZCZ, M. HOTLOŚ, M. KUCHARSKI and Z. ŞENTÜRK, Generalisations of Robertson–Walker spaces, *Dept. Math., Agricultural Univ. Wrocław,* Preprint No. 61, 1998.

[9] F. DEFEVER, R. DESZCZ, and L. VERSTRAELEN, On semisymmetric para-Kähler manifolds, *Acta Math. Hungarica,* **74** (1997), 7–17.

[10] J. DEPREZ, R. DESZCZ and L. VERSTRAELEN, Examples of pseudosymmetric conformally flat warped products, *Chinese J. Math.,* **17** (1989), 51–65.

[11] J. DEPREZ, W. ROTER and L. VERSTRAELEN, Conditions on the projective curvature tensor of conformally flat Riemannian manifolds, *Kyungpook Math. J.,* **29** (1989), 153–165.

[12] R. DESZCZ, On four-dimensional Riemannian warped product manifolds satisfying certain pseudosymmetry curvature conditions, *Colloq. Math.,* **62** (1991), 103–120.

[13] R. DESZCZ, On pseudosymmetric spaces, *Bull. Soc. Math. Belg.,* **44** (1992), Sér. A, 1–34.

[14] R. DESZCZ, F. DILLEN, L. VRANCKEN and L. VERSTRAELEN, Quasi-Einstein totally real submanifolds of $S^6(1)$, *Tôhoku Math. J.,* in print. Preprint No. 47, 1997.

[15] R. DESZCZ and M. HOTLOŚ, On a certain extension of the class of semisymmetric manifolds, *Publ. Inst. Math.* (Beograd) (N.S), **63(77)** (1998), 115–130.

[16] R. DESZCZ and M. HOTLOŚ, On a certain subclass of pseudosymmetric manifolds, *Publ. Math. Debrecen,* **53** (1998), 29–48.

[17] R. DESZCZ, M. HOTLOŚ and Z. ŞENTÜRK, On the equivalence of the Ricci-pseudosymmetry, *Colloq. Math.,* **79** (1999), 211–227.

[18] R. DESZCZ, M. HOTLOŚ and Z. ŞENTÜRK, On a certain application of the Patterson's curvature identity, *Dept. Math., Agricultural Univ. Wrocław,* Preprint No. 55, 1997.

[19] R. DESZCZ and M. KUCHARSKI, On a certain curvature property of generalized Robertson–Walker spacetimes, *Tsukuba J. Math.,* in print.

[20] R. DESZCZ and L. VERSTRAELEN, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: *Geometry and Topology of Submanifolds,* III, World Sci. Publishing, River Edge, NJ. 1991, 131–147.

[21] R. DESZCZ, L. VERSTRAELEN and S. S. YAPRAK, Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor, *Chinese J. Math.,* **22** (1994), 139–157.

[22] R. DESZCZ, L. VERSTRAELEN and S. YAPRAK, On 2-quasi-umbilical hypersurfaces in conformally flat spaces, *Acta Math. Hungarica,* **78** (1998), 45–57.

[23] Z. I. SZABÓ, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, I. The local version, *J. Differential Geom.,* **17** (1982), 531–582.

[24] Z. I. SZABÓ, Classification and construction of complete hypersurfaces satisfying $R(X, Y) \cdot R = 0$, *Acta Sci. Math.,* **47** (1984), 321–348.

[25] Z. I. SZABÓ, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, II. Global verseion, *Geom. Dedicata,* **19** (1985), 65–108.

[26] I. VAN DE WOESTIJNE and L. VERSTRAELEN, Semi-symmetric Lorentzian hypersurfaces, *Tôhoku Math. J.,* **39** (1987), 81–88.

[27] L. VERSTRAELEN, Comments on pseudosymmetry in the sense of Ryszard Deszcz, in: *Geometry and Topology of Submanifolds,* VI. World Sci. Publishing, River Edge, NJ, 1994, 199–209.

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SUM FUNCTIONS OF CERTAIN GENERALIZED DIVISORS

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and let $A(n)$ be a subset of the positive divisors of n for each $n \in \mathbb{N}$. The A-convolution of the arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d). \tag{1}$$

W.NARKIEWICZ [Nar63] defined the A-convolution (1) to be regular if

(a) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power $p^a (a \geq 1)$.

It can be proved, see [Nar63], that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbb{N}, (m,n) = 1$,

(ii) for every prime power $p^a (a \geq 1)$ there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$.

For example, the Dirichlet convolution D, where $D(n)$ is the set of all positive divisors of n, and the unitary convolution U, where $U(n)$ is the set of all unitary divisors of n (i.e. divisors d of n with $(d, n/d) = 1$), are regular.

If A is a regular convolution and $k \in \mathbb{N}$ let $(a,b)_{A,k}$ denote the greatest k-th power divisor of a which belongs to $A(b)$.

In [T95], [TH96], [T97-i] we introduced the notion of cross-convolution as a special case of Narkiewicz's regular convolution as follows. We say that A is a cross-convolution if for every prime p we have either $A(p^a) = = \{1,p,p^2,...,p^a\} \equiv D(p^a)$ for every $a \in \mathbb{N}$ or $A(p^a) = \{1,p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let P and Q be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution U.

Furthermore, let $(P) = \{1\} \cup \{n \in \mathbb{N} :$ each prime factor of n belongs to $P\}$, $(Q) = \{1\} \cup \{n \in \mathbf{N} :$ each prime factor of n belongs to $Q\}$. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$.

Now suppose A is a regular convolution, S is an arbitrary subset of $\mathbb{N}$ and $k,u \in \mathbb{N}$. For $d,n \in \mathbb{N}$ we say that d is a semi-(S,A,k,u) divisor of n if $d|n$ and $((d,(n/d)^u)_{A,k})^{1/k} \in S$.

It is clear that a semi-$(\mathbb{N},A,k,u)$ divisor is a divisor in the usual sense for every A and for every $k,u \in \mathbb{N}$. A semi-$(\{1\},D,k,1)$ divisor is the same as a semi-$(Q_k,D,1,1)$ divisor, where Q_k is the set of k-free integers (i. e. integers not divisible by the k-th power of any integer > 1) and this is the notion of the k-ary divisor studied by J. CHIDAMBARASWAMY [Chi70], D. SURYANARAYANA [Sur71], D.SURYANARAYANA and V. SIVA RAMA PRASAD [SurSiv73]. If in addition $k = 1$, then we reobtain the concept of the unitary divisor (or block divisor) introduced by R.VAIDYANATHASWAMY [V31] and investigated by E. COHEN [Co60], [Co61] and others.

Furthermore, a semi-$(\{1\},U,k,1)$ divisor is the same as a semi-$(Q_k^*,U,1,1)$ divisor, where Q_k^* is the set of unitarily k-free integers (i. e. integers not divisible unitarily by the k-th power of any integer > 1) and this is the notion of the semi-k-ary divisor discussed by D.SURYANARAYANA and V. SIVA RAMA PRASAD [SurSiv73]. If in addition $k = 1$, then we reobtain the concept of the semi-unitary divisor introduced by J. CHIDAMBARASWAMY [Chi67]. Among the many other interesting particular cases we mention here only the following one. Let P be an arbitrary subset of the set of prime numbers: $P \subseteq \mathbb{P}$ and $S = (P)$ defined as above, $A = D, u = 1$, then we reobtain the notion of $P-k$-divisor investigated by us in [T].

It may be noted that if d is a semi-(S,A,k,u) divisor of n and $A \neq D$, then n/d (the complementary divisor to d of n) need not be a semi-(S,A,k,u) divisor of n.

Let $\sigma^{(s)}_{(S,A,k,u),r}(n)$ and $\sigma^{(cs)}_{(S,A,k,u),r}(n)$ denote the sum of r-th powers of the semi-(S,A,k,u) divisors of n and the sum of r-th powers of the complementary semi-(S,A,k,u) divisors of n, respectively.

In this paper we deduce an asymptotic formula for the function $\sigma^{(s)}_{(S,A,k,u),r}(n)$ in case of an arbitrary regular convolution A, an arbitrary subset S, for $k,u \in \mathbb{N}$ and for $r \geq u$. We also obtain an asymptotic formula for the function $\sigma^{(cs)}_{(S,A,k,u),r}(n)$ if A is a cross-convolution, $k \in \mathbb{N}, r \geq u$ and, for the sake of simplicity, if S is a multiplicative subset (i. e. its characteristic function is multiplicative) and $u = 1$.

In fact we obtain slightly more general results regarding the convolutions

$$G(n) = \sum_{\substack{de=n \\ ((d,e^u)_{A,k})^{1/k} \in S}} f(d)g(e) \tag{2}$$

and

$$H(n) = \sum_{\substack{de=n \\ ((d,e^u)_{A,k})^{1/k} \in S}} h(d)f(e) \tag{3}$$

where $f(n) = n^r, r \geq u$, g and h are bounded arithmetical functions, assuming the above hypothesis.

Our results generalize and unify the corresponding known results concerning divisors in the usual sense, unitary-divisors, k-ary divisors, semi-unitary divisors and semi-k-ary divisors.

The method we use is elementary, it applies certain familiar estimates regarding the sums $\sum_{n \leq x} n^s$ and $\sum_{n \leq x} \tau(n) n^s$, cf. [T97-i]. We point out that using the following result of A. Walfisz [W63], p. 99,

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2 x^2}{12} + O(x \log^{2/3} x), \tag{4}$$

which is the best known result concerning the sum of divisors function, some error terms of our formulae can be improved. In this way we also improve some known error terms regarding the special cases mentioned above.

2. Asymptotic formulae for $G(n)$

If A is a regular convolution and $S \subseteq \mathbb{N}$ define the generalized Möbius function $\mu_{S,A}$ by $\mu_{S,A} *_A I = \rho_S$, where $I(n) = 1, n \in \mathbb{N}$ and ρ_S is the characteristic function of S. Hence, by Möbius inversion, one has $\mu_{S,A} = \rho_S *_A \mu_A$.

REMARK 1. For every regular convolution A, for every subset S and $n \in \mathbb{N}$ we have $|\mu_{S,A}(n)| \le \tau(n)$, the number of divisors of n. Moreover, if S is multiplicative, i. e. ρ_S is multiplicative, then $|\mu_{S,A}(n)| \le 1$, see [TH96], Lemmas 1 and 2.

For $k \in \mathbb{N}$ let $A_k(n) = \{d \in \mathbb{N} : d^k \in A(n^k)\}$. It is known that the A_k convolution is regular whenever the A convolution is regular.

We need the following generalization of the Euler ϕ-function. For a regular convolution A, for $S \subseteq \mathbb{N}$ and $k, u, n \in \mathbb{N}$ let $\phi_{S,A,k,u}(n)$ denote the number of integers m (mod n^u) such that $((m, n^u)_{A,k})^{1/k} \in S$. For $u = k$ the function $\phi_{S,A,k,k}(n) \equiv \phi_{S,A,k}(n)$ was introduced by P. HAUKKANEN [H88]. Note that $\phi_{\{1\},D,1}(n) \equiv \phi(n)$ is the Euler function. For other special cases we refer to [H88], [T97-i] and [SurSiv73].

We investigate the following Legendre-type function:

$$\phi_{S,A,k,u}(n,x,r) = \sum_{\substack{m \le x \\ ((m,n^u)_{A,k})^{1/k} \in S}} m^r.$$

LEMMA 1. (cf. [SurSiv73], Lemma 3.1) *If A is a regular convolution, $S \subseteq \mathbb{N}$, $k, u, n \in \mathbb{N}, r, x \ge 1$ are real numbers, then*

$$\phi_{S,A,k,u}(n,x,r) = \sum_{d^k \in A(n^u)} d^{kr} \mu_{S,A_k}(d) \sum_{e \le x/d^k} e^r.$$

If in addition $r \ge 0$, then

$$\phi_{S,A,k,u}(n,x,r) = \frac{\phi_{S,A,k,u}(n) x^{r+1}}{n^u (r+1)} + O(x^r f_{S,k}(n^u)),$$

where $f_{S,k}(n) = \tau_k(n) \equiv \sum_{d^k | n} 1$ (S multiplicative), $J_k(n) \equiv \sum_{d^k | n} \tau(d)$ (S not multiplicative).

PROOF. Using that $d^k \in A((a,b)_{A,k})$ if and only if $d^k | a$ and $d^k \in A(b)$, see [Sit78], Theorem 4.2, we have

$\phi_{S,A,k,u}(n,x,r) =$

$$= \sum_{m \le x} m^r \rho_S(((m,n^u)_{A,k})^{1/k}) = \sum_{m \le x} m^r \sum_{d \in A_k((m,n^u)_{A,k})^{1/k}} \mu_{S,A_k}(d) =$$

$$= \sum_{m \le x} m^r \sum_{\substack{d^k | m \\ d^k \in A(n^u)}} \mu_{S,A_k}(d) = \sum_{d^k \in A(n^u)} d^{kr} \mu_{S,A_k}(d) \sum_{e \le x/d^k} e^r,$$

where $m = d^k e$. Note that for $r = 0$ and $x = n^u$ we get

$$\phi_{S,A,k,u}(n) = n^u \sum_{d^k \in A(n^u)} \frac{\mu_{S,A_k}(d)}{d^k}. \tag{5}$$

Furthermore, if $r \ge 0$ we use the estimate $\sum_{n \le x} n^r = \frac{x^{r+1}}{r+1} + O(x^r)$ and obtain

$$\phi_{S,A,k,u}(n,x,r) = \sum_{d^k \in A(n^u)} d^{kr} \mu_{S,A_k}(d) \left(\frac{x^{r+1}}{(r+1)d^{k(r+1)}} + O\left(\frac{x^r}{d^{kr}}\right)\right) =$$

$$= \frac{x^{r+1}}{r+1} \sum_{d^k \in A(n^u)} \frac{\mu_{S,A_k}(d)}{d^k} + O\left(x^r \sum_{d^k \in A(n^u)} |\mu_{S,A_k}(d)| \right) =$$

$$= \frac{x^{r+1}}{r+1} \cdot \frac{\phi_{S,A,k,u}(n)}{n^u} + O(x^r f_{S,k}(n^u)),$$

by (5) and by Remark 1. ∎

LEMMA 2. *If A is a regular convolution, $S \subseteq \mathbb{N}$, $k, u \in \mathbb{N}, r > 0$ and g is a bounded function, then the series*

$$\sum_{n=1}^{\infty} \frac{g(n)\phi_{S,A,k,u}(n)}{n^{u+r+1}}$$

is absolutely convergent. Let $\alpha^{(g)}_{(S,A,k,u),r}$ denote its sum. If in addition S and g are multiplicative and $u = k$, then

$$\alpha^{(g)}_{(S,A,k,k),r} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{g(p^m)}{p^{m(r+1)}} \left(1 + \sum_{i=1}^{m/t} \frac{\rho_S(p^{it}) - \rho_S(p^{(i-1)t})}{p^{itk}} \right)\right),$$

where $t = t_{A_k}(p^m)$.

PROOF. By its definition $\phi_{S,A,k,u}(n) \le n^u$ for every $n \in \mathbb{N}$, hence the general term of the series is $O(1/n^{r+1})$, where $r > 0$ and the absolute convergence of the series follows at once.

If S and g are multiplicative, then $\phi_{S,A,k,k}$ is also multiplicative, see (5), therefore the series can be expanded into an infinite product of Euler type and we use that

$$\mu_{S,A_k}(p^m) = \rho_S(p^m) - \rho_S(p^{m-t})$$

for every prime power p^m. ∎

REMARK 2. If in addition $S = \{1\}$, then

$$\alpha^{(g)}_{(S,A,k,k),r} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{g(p^m)}{p^{m(r+1)}} \left(1 - \frac{1}{p^{tk}}\right)\right).$$

LEMMA 3. (cf. [SurSiv73], Lemma 3.6) *If $k, u \in \mathbb{N}$ and $r \ge u$, then*

$$\sum_{n\le x} \frac{\tau_k(n^u)}{n^r} = \begin{cases} O(1), & r > u, \\ O(\log x), & r = u,\ k > 1 \\ O(\log^2 x), & r = u,\ k = 1, \end{cases} \tag{6}$$

$$\sum_{n\le x} \frac{J_k(n)}{n^r} = \begin{cases} O(1), & r > u, \\ O(\log x), & r = u,\ k > 1 \\ O(\log^3 x), & r = u,\ k = 1. \end{cases} \tag{7}$$

PROOF. We have

$$\sum_{n\le x} \frac{\tau_k(n^u)}{n^r} = \sum_{n\le x} \frac{1}{n^r} \sum_{d^k \mid n^u} 1 = \sum_{d^k e = n^u \le x^u} \frac{1}{(d^k e)^{r/u}} =$$

$$= \sum_{d \le x^{u/k}} \frac{1}{d^{kr/u}} \sum_{e \le x^u/d^k} \frac{1}{e^{r/u}},$$

which is for $r > u$:

$$\sum_{d \le x^{u/k}} \frac{1}{d^{kr/u}} O(1) = O\left(\sum_{d \le x^{u/k}} \frac{1}{d^{kr/u}}\right) = O(1);$$

and for $r = u$ it is

$$\sum_{d \le x^{u/k}} \frac{1}{d^k} O(\log \frac{x^u}{d^k}) = O\left(\log x \sum_{d \le x^{u/k}} \frac{1}{d^k}\right) = \begin{cases} O(\log x), & k > 1, \\ O(\log^2 x), & k = 1. \end{cases}$$

The second estimate can be obtained similarly. ∎

LEMMA 4. *For $n, m \in \mathbb{N}$ let*

$$\sigma'(n,m) \equiv \sum_{\substack{de=n \\ (e,m)=1}} d.$$

Then

$$\sum_{n\le x} \sigma'(n,m) = \frac{\pi^2}{12} \cdot \frac{\phi_2(m)}{m^2} x^2 + O\left(\frac{\psi(m)}{m} x \log^{2/3} x\right),$$

where ϕ_2 and ψ are the Jordan function of order 2 given by $\phi_2(n) = n^2 \prod_{p|n}\left(1-\frac{1}{p^2}\right)$ and the Dedekind function defined by $\psi(n) = n \prod_{p|n}(1+\frac{1}{p})$, respectively.

PROOF. Let μ denote the Möbius function. We have

$$\sum_{n\le x} \sigma'(n,m) = \sum_{\substack{de=n\le x \\ (e,m)=1}} d = \sum_{de\le x} d \sum_{a|(e,m)} \mu(a) = \sum_{a|m} \mu(a) \sum_{dab\le x} d =$$

$$= \sum_{a|m} \mu(a) \sum_{db\le x/a} d = \sum_{a|m} \mu(a) \sum_{n\le x/a} \sigma(n)$$

where $e = ab$. Now using Walfisz' result (4) we obtain

$$\sum_{n\le x} \sigma'(n,m) = \sum_{a|m} \mu(a) \left(\frac{\pi^2}{12} \cdot \frac{x^2}{a^2} + O\left(\frac{x}{a} \log^{2/3} \frac{x}{a}\right)\right) =$$

$$= \frac{\pi^2}{12} x^2 \sum_{a|m} \frac{\mu(a)}{a^2} + O\left(x \log^{2/3} x \sum_{a|m} \frac{\mu^2(a)}{a}\right),$$

which gives the desired result. ∎

Now we are ready to prove the following

THEOREM 1. *If A is a regular convolution, S is a subset of $\mathbb{N}$, $k, u \in \mathbb{N}$, $f(n) = n^r$, $r \ge u$ and g is a bounded function, then for the function $G(n)$ given by (2) we have*

$$\sum_{n\le x} G(n) = \alpha^{(g)}_{(S,A,k,u),r} \frac{x^{r+1}}{r+1} + O(x^r \log^q x),$$

where $\alpha^{(g)}_{(S,A,k,u),r}$ *is given in Lemma 2 and* $q = 0$ *(*$r > u$*),* 1 *(*$r = u, k > 1$*),* 2 *(*$r = u, k = 1$ *and* S *multiplicative),* 3 *(*$r = u, k = 1$ *and* S *not multiplicative).*

If A *is a cross-convolution,* $S \subseteq \mathbb{N}$*,* $r = u = 1, k \in \mathbb{N}$ *and* $g(n) = 1, n \in \mathbb{N}$*, then the exponents of the* log *factors can be improved with* $1/3$*, i. e. we have* $q = 2/3$ *(*$k > 1$*),* $5/3$ *(*$k = 1$ *and* S *multiplicative),* $8/3$ *(*$k = 1$ *and* S *not multiplicative).*

PROOF. We have, using Lemma 1,

$$\sum_{n \le x} G(n) = \sum_{\substack{de = n \le x \\ ((d,e^u)_{A,k})^{1/k} \in S}} g(e) d^r = \sum_{e \le x} g(e) \sum_{\substack{d \le x/e \\ ((d,e^u)_{A,k})^{1/k} \in S}} d^r =$$

$$= \sum_{e \le x} g(e) \phi_{S,A,k,u}(e, x/e, r) =$$

$$= \sum_{e \le x} g(e) \left(\frac{\phi_{S,A,k,u}(e)}{e^u (r+1)} \left(\frac{x}{e} \right)^{r+1} + O\left(\left(\frac{x}{e} \right)^r f_{S,k}(e^u) \right) \right) =$$

$$= \frac{x^{r+1}}{r+1} \sum_{e \le x} \frac{g(e) \phi_{S,A,k,u}(e)}{e^{u+r+1}} + O\left(x^r \sum_{e \le x} \frac{f_{S,k}(e^u)}{e^r} \right).$$

Now using Lemma 2 we get

$$\sum_{n \le x} G(n) = \alpha^{(g)}_{(S,A,k,u),r} \frac{x^{r+1}}{r+1} + O\left(x^{r+1} \sum_{e > x} \frac{1}{e^{r+1}} \right) + O\left(x^r \sum_{e \le x} \frac{f_{S,k}(e^u)}{e^r} \right).$$

Here the first O-term is $O(x^{r+1} x^{-r}) = O(x)$ and for the second O-term we apply Lemma 3.

Now suppose that A is a cross-convolution (note that in this case $A_k = A$ for every $k \in \mathbb{N}$), $S \subseteq \mathbb{N}$, $r = u = 1$ and $g(n) = 1, n \in \mathbb{N}$. We use the following treatment:

$$\sigma^{(s)}_{(S,A,k,1),1}(n) = \sum_{de = n} d \rho_S(((d,e)_{A,k})^{1/k}) = \sum_{de = n} d \sum_{a^k \in A((d,e)_{A,k})} \mu_{S,A_k}(a) =$$

$$= \sum_{de = n} d \sum_{\substack{a^k \mid d \\ a^k \in A(e)}} \mu_{S,A_k}(a) = \sum_{\substack{a^{2k} bc = n \\ a^k \in A(e)}} a^k b \mu_{S,A}(a) =$$

$$= \sum_{a^{2k}j=n} a^k \mu_{S,A}(a) \sum_{\substack{bc=j \\ (c,a_Q)=1}} b = \sum_{a^{2k}j=n} a^k \mu_{S,A}(a) \sigma'(j, a_Q),$$

where $d = a^k b, e = a^k c$. Now using Lemma 4 we obtain

$$\sum_{n \le x} \sigma^{(s)}_{(S,A,k,1),1}(n) = \sum_{a^{2k} j \le x} a^k \mu_{S,A}(a) \sigma'(j, a_Q) =$$

$$= \sum_{a \le \sqrt[2k]{x}} a^k \mu_{S,A}(a) \sum_{j \le x/a^{2k}} \sigma'(j, a_Q) =$$

$$= \sum_{a \le \sqrt[2k]{x}} a^k \mu_{S,A}(a) \Bigg(\frac{\pi^2}{12} \cdot \frac{\phi_2(a_Q)}{a_Q^2} \left(\frac{x}{a^{2k}} \right)^2 +$$

$$+ O\left(\frac{\psi(a_Q)}{a_Q} \frac{x}{a^{2k}} \log^{2/3} \left(\frac{x}{a^{2k}} \right) \right) \Bigg) =$$

$$= \frac{\pi^2}{12} x^2 \sum_{a \le \sqrt[2k]{x}} \frac{\mu_{S,A}(a) \phi_2(a_Q)}{a_Q^2 a^{3k}} + O\left(x \log^{2/3} x \sum_{a \le \sqrt[2k]{x}} \frac{|\mu_{S,A}(a)| \psi(a_Q)}{a^k a_Q} \right) =$$

$$= \frac{\pi^2}{12} x^2 \sum_{a=1}^{\infty} \frac{\mu_{S,A}(a) \phi_2(a_Q)}{a_Q^2 a^{3k}} + O\left(x^2 \sum_{a > \sqrt[2k]{x}} \frac{|\mu_{S,A}(a)|}{a^{3k}} \right) +$$

$$+ O\left(x \log^{2/3} x \sum_{a \le \sqrt[2k]{x}} \frac{|\mu_{S,A}(a)| \psi(a)}{a^{k+1}} \right).$$

Here the first O-term is

$$O\left(x^2 \sum_{a > \sqrt[2k]{x}} \frac{\tau(a)}{a^{3k}} \right) = O\left(x^2 \cdot x^{1/2k - 3/2} \log x \right) =$$

$$= O\left(x^{1/2 + 1/2k} \log x \right) = O(\log x).$$

The second O-term is for S multiplicative

$$O\left(x\log^{2/3}x\sum_{a\le\sqrt[2k]{x}}\frac{\psi(a)}{a^{k+1}}\right)=\left(x\log^{2/3}x\sum_{a\le\sqrt[2k]{x}}\frac{\sigma(a)}{a^{k+1}}\right)=$$

$$=\begin{cases}O(\log^{2/3}x), & k>1,\\ O(\log^{5/3}x), & k=1;\end{cases}$$

and for S not multiplicative it is

$$O\left(x\log^{2/3}x\sum_{a\le\sqrt[2k]{x}}\frac{\tau(a)\sigma(a)}{a^{k+1}}\right)=\begin{cases}O(\log^{2/3}x), & k>1,\\ O(\log^{8/3}x), & k=1,\end{cases}$$

see [T-ix], Lemma 3, and the proof is complete. ∎

COROLLARY 1. *If A is a regular convolution, S is a subset of $\mathbb{N}$, $k,u\in\mathbb{N}$ and $r\ge u$, then we have*

$$\sum_{n\le x}\sigma^{(s)}_{(S,A,k,u),r}(n)=\alpha_{(S,A,k,u),r}\frac{x^{r+1}}{r+1}+O(x^r\log^q x),$$

where $\alpha_{(S,A,k,u),r}\equiv\alpha^{(I)}_{(S,A,k,u),r}$ and q is given in Theorem 1.

If A is a cross-convolution, $S\subseteq\mathbb{N}$, $r=u=1$ and $k\in\mathbb{N}$, then q can be improved with $1/3$.

In case of k-ary divisors (take $S=\{1\},A=D,u=1$) and semi-k-ary divisors (take $S=\{1\},A=U,u=1$) we reobtain formulae (4.3) and (4.4) given in [SurSiv73] with better error terms. For the unitary divisor sum function the same improved error term is due in [SSur73], see also [T97-i], Theorem 12 and [T], Corollary 3.3.

2. Asymptotic formulae for $H(n)$

For the simplicity we consider in what follows S multiplicative and $u=1$.

LEMMA 5. (cf. [SurSiv73], Lemma 3.4) *If A is a cross-convolution, $S\subseteq\subseteq\mathbb{N}$ is multiplicative, $k,n\in\mathbb{N},r,x\in\mathbb{R},x\ge 1,r\ge 0$, then*

$$\sum_{\substack{m\le x\\ ((n,m)_{A,k})^{1/k}\in S}}m^r=\frac{\psi_{S,A,k}(n)x^{r+1}}{n(r+1)}+O(x^r f_{A,k}(n)),$$

where

$$\psi_{S,A,k}(n) = n \sum_{d^k|n} \frac{\mu_{S,A}(d)\phi(d)}{d^k d_Q},$$

and $f_{A,k}(n) = \tau_k(n) \equiv \sum_{d^k|n} 1$ *(Q finite set),* $J_k(n) \equiv \sum_{d^k|n} \tau(d)$ *(Q infinite set).*

PROOF. Similarly to Lemma 1,

$$\sum_{\substack{m\le x\\ ((n,m)_{A,k})^{1/k}\in S}} m^r = \sum_{m\le x} m^r \rho_S(((n,m)_{A,k})^{1/k}) = \sum_{m\le x} m^r \sum_{\substack{d^k|n\\ d^k\in A(m)}} \mu_{S,A_k}(d) =$$

$$= \sum_{d^k|n} d^{kr}\mu_{S,A}(d) \sum_{\substack{e\le x/d^k\\ (e,d)\in(P)}} e^r.$$

Now using the estimate

$$\sum_{\substack{n\le x\\ (n,a)\in(P)}} n^r = \frac{\phi(a_Q)x^{r+1}}{a_Q(r+1)} + O(x^r\eta_Q(a))$$

valid for every $r \ge 0$ and $a \in \mathbb{N}$, where $\eta_Q(a) = 1$ (Q finite set), $\tau(a)$ (Q infinite set), cf. [T97-i], Lemma 7, we get

$$\sum_{\substack{m\le x\\ ((n,m)_{A,k})^{1/k}\in S}} m^r = \sum_{d^k|n} d^{kr}\mu_{S,A}(d) \left(\frac{x^{r+1}\phi(d_Q)}{(r+1)d^{k(r+1)}d_Q} + O\left(\eta_Q(d)\frac{x^r}{d^{kr}}\right)\right) =$$

$$= \frac{x^{r+1}}{r+1}\sum_{d^k|n} \frac{\mu_{S,A}(d)\phi(d_Q)}{d^k d_Q} + O\left(x^r \sum_{d^k\in A(n)} |\mu_{S,A}(d)|\eta_Q(d)\right),$$

and we use Remark 1. ∎

REMARK 3. $|\psi_{S,A,k}(n)| \le \sigma(n)$ for every $n \in \mathbb{N}$. Indeed,

$$|\psi_{S,A,k}(n)| \le n \sum_{d^k|n} \frac{|\mu_{S,A}(d)|\phi(d_Q)}{d^k d_Q} \le n \sum_{d^k|n} \frac{1}{d^k} \le n \sum_{d|n} \frac{1}{d} = \sigma(n).$$

LEMMA 6. *If A is a cross-convolution, S is multiplicative, $k \in \mathbb{N}, r > 0$ and h is a bounded function, then the series*

$$\sum_{n=1}^{\infty} \frac{h(n)\psi_{S,A,k}(n)}{n^{r+2}}$$

is absolutely convergent. Let $\beta^{(h)}_{(S,A,k),r}$ denote its sum.

PROOF. The absolute convergence follows at once by Remark 3. ∎

The proof of the subsequent result is similar to the proof of Lemma 4, see also [SitSur81], Lemma 2.1.

LEMMA 7. *For $n, m \in \mathbb{N}$ let*

$$\sigma(n,m) \equiv \sum_{\substack{de=n \\ (d,m)=1}} d.$$

Then

$$\sum_{n \le x} \sigma(n,m) = \frac{\pi^2}{12} \cdot \frac{\phi(m)}{m} x^2 + O\left(\tau(m) x \log^{2/3} x\right).$$

THEOREM 2. *If A is a cross-convolution, S is a multiplicative subset of $\mathbb{N}$, $k \in \mathbb{N}, u = 1, f(n) = n^r, r \ge 1$ and h is a bounded function, then for the function $H(n)$ given by (3) we have*

$$\sum_{n \le x} H(n) = \beta^{(h)}_{(S,A,k),r} \frac{x^{r+1}}{r+1} + O(x^r \log^q x),$$

where $\beta^{(h)}_{S,A,k,r}$ is given in Lemma 6 and $q = 0$ ($r > 1$), 1 ($r = 1, k > 1$), 2 ($r = k = 1$ and Q finite set), 3 ($r = k = 1$ and Q infinite set).

If in addition $r = 1$ and $h(n) = 1, n \in \mathbb{N}$, then q can be improved with $1/3$, i. e. we have $q = 2/3$ ($k > 1$), $5/3$ ($k = 1$ and Q finite), $8/3$ ($k = 1$ and Q infinite).

PROOF. We have, using Lemma 5,

$$\sum_{n \le x} H(n) = \sum_{\substack{de=n \le x \\ ((d,e)_{A,k})^{1/k} \in S}} h(d) e^r = \sum_{d \le x} h(d) \sum_{\substack{e \le x/d \\ ((d,e)_{A,k})^{1/k} \in S}} e^r =$$

$$= \sum_{d \le x} h(d) \left(\frac{\psi_{S,A,k}(d)}{d(r+1)} \left(\frac{x}{d}\right)^{r+1} + O\left(\left(\frac{x}{d}\right)^r f_{A,k}(d) \right) \right) =$$

$$= \frac{x^{r+1}}{r+1} \sum_{d \le x} \frac{h(d)\psi_{S,A,k}(d)}{d^{r+2}} + O\left(x^r \sum_{d \le x} \frac{f_{A,k}(d)}{d^r} \right).$$

Now using Lemma 6 we get

$$\sum_{n \le x} H(n) = \beta^{(h)}_{(S,A,k),r} \frac{x^{r+1}}{r+1} + O\left(x^{r+1} \sum_{d > x} \frac{\sigma(d)}{d^{r+2}} \right) + O\left(x^r \sum_{d \le x} \frac{f_{A,k}(d)}{d^r} \right).$$

Here the first O-term is $O(x^{r+1}x^{-r}) = O(x)$, see [T-ix], Lemma 3/a, and for the second O-term we apply Lemma 3 with $u = 1$.

Now suppose that $r = 1$ and $h(n) = 1, n \in \mathbb{N}$. We have

$$\sigma^{(cs)}_{(S,A,k,1),1}(n) = \sum_{de=n} e\rho_S(((d,e)_{A,k})^{1/k}) =$$

$$= \sum_{de=n} e \sum_{\substack{a^k | d \\ a^k \in A(e)}} \mu_{S,A}(a) = \sum_{\substack{a^{2k} bc=n \\ a^k \in A(e)}} a^k c \mu_{S,A}(a) =$$

$$= \sum_{a^{2k} j=n} a^k \mu_{S,A}(a) \sum_{\substack{bc=j \\ (c,a_Q)=1}} c = \sum_{a^{2k} j=n} a^k \mu_{S,A}(a) \sigma(j, a_Q),$$

where $d=a^k b, e=a^k c$. Now using Lemma 7 we obtain

$$\sum_{n\le x}\sigma^{(cs)}_{(S,A,k,1),1}(n)=\sum_{a^{2k}j\le x}a^k\mu_{S,A}(a)\sigma(j,a_Q)=$$

$$=\sum_{a\le \sqrt[2k]{x}}a^k\mu_{S,A}(a)\sum_{j\le x/a^{2k}}\sigma(j,a_Q)=$$

$$=\sum_{a\le \sqrt[2k]{x}}a^k\mu_{S,A}(a)\left(\frac{\pi^2}{12}\cdot\frac{\phi(a_Q)x^2}{a^{4k}a_Q}+O\left(\tau(a_Q)\frac{x}{a^{2k}}\log^{2/3}\left(\frac{x}{a^{2k}}\right)\right)\right)=$$

$$=\frac{\pi^2}{12}x^2\sum_{a\le \sqrt[2k]{x}}\frac{\mu_{S,A}(a)\phi(a_Q)}{a_Q a^{3k}}+O\left(x\log^{2/3}x\sum_{a\le \sqrt[2k]{x}}\frac{|\mu_{S,A}(a)|\tau(a_Q)}{a^k}\right)=$$

$$=\frac{\pi^2}{12}x^2\sum_{a=1}^{\infty}\frac{\mu_{S,A}(a)\phi(a_Q)}{a_Q a^{3k}}+O\left(x^2\sum_{a> \sqrt[2k]{x}}\frac{|\mu_{S,A}(a)|}{a^{3k}}\right)+$$

$$+O\left(x\log^{2/3}x\sum_{a\le \sqrt[2k]{x}}\frac{\tau(a_Q)}{a^k}\right).$$

Here the first O-term is $O(\log x)$ and the second O-term is for Q finite

$$O\left(x\log^{2/3}x\sum_{a\le \sqrt[2k]{x}}\frac{1}{a^k}\right)=\begin{cases}O(x\log^{2/3}x), & k>1,\\ O(\log^{5/3}x), & k=1;\end{cases}$$

and for Q infinite it is

$$O\left(x\log^{2/3}x\sum_{a\le \sqrt[2k]{x}}\frac{\tau(a)}{a^{k+1}}\right)=\begin{cases}O(\log^{2/3}x), & k>1,\\ O(\log^{8/3}x), & k=1,\end{cases}$$

and the proof is complete. ∎

COROLLARY 2. *If A is a cross-convolution, S is multiplicative, $k\in\mathbb{N}$ and $r\ge 1$ then we have*

$$\sum_{n\le x}\sigma^{(cs)}_{(S,A,k,1),r}(n)=\beta_{(S,A,k),r}\frac{x^{r+1}}{r+1}+O(x^r\log^q x),$$

where $\beta_{(S,A,k),r} \equiv \beta^{(I)}_{(S,A,k),r}$ *and* q *is given in Theorem 2.*

If $A = D$ (in particular for k-ary divisors) Theorem 2 reduces to Theorem 1 and for semi-k-ary divisors (take $S = \{1\}, A = U, u = 1$) we reobtain formula (4.12) given in [SurSiv73] with better error term.

References

[Chi67] J. CHIDAMBARASWAMY, Sum functions of unitary and semi-unitary divisors, *J. Indian Math. Soc.,* **31** (1967), 117–126.

[Chi70] J. CHIDAMBARASWAMY, The K-unitary convolution of certain arithmetical functions, *Publ. Math. Debrecen,* **17** (1970), 67–74.

[Co60] E.COHEN, Arithmetical functions associated with the unitary divisors of an integer, *Math. Z.,* **74** (1960), 66–80.

[Co61] E.COHEN, Unitary products of arithmetical functions, *Acta Arith.,* **7** (1961), 29–38.

[H88] P. HAUKKANEN, Some generalized totient functions, *Math. Student,* **56** (1988), 65–74.

[Nar63] W. NARKIEWICZ, On a class of arithmetical convolutions, *Colloq. Math.,* **10** (1963), 81–94.

[SSur73] R. SITA RAMA CHANDRA RAO, D. SURYANARAYANA, On $\sum_{n\leq x}\sigma^*(n)$ and $\sum_{n\leq x}\phi^*(n)$, *Proc. Amer. Math. Soc.,* **41** (1973), 61–66.

[Sit78] V. SITA RAMAIAH, Arithmetical sums in regular convolutions, *J. Reine Angew. Math.,* **303/304** (1978), 265–283.

[SitSur81] V. SITA RAMAIAH, D. SURYANARAYANA, An order result involving the σ function, *Indian J. Pure Appl. Math.,* **12** (1981), 1192–1201.

[Sur71] D. SURYANARAYANA, Some theorems concerning the k-ary divisors of an integer, *Math. Student,* **39** (1971), 384–394.

[SurSiv73] D. SURYANARAYANA, V. SIVA RAMA PRASAD, Sum functions of k-ary and semi-k-ary divisors, *J. Austr. Math. Soc.,* **15** (1973), 148–162.

[T95] L. TÓTH, Contributions to the theory of arithmetical functions defined by regular convolutions, (in Rumanian), thesis, "Babeş-Bolyai" University, Cluj-Napoca, 1995.

[TH96] L. TÓTH, P. HAUKKANEN, A generalization of Euler's ϕ-function with respect to a set of polynomials, *Ann. Univ. Sci. Budapest. Rolando Eötvös, Sect. Math.,* **39** (1996), 69–83.

[T97-i] L. TÓTH, Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions, *Publ. Math. Debrecen,* **50** (1997), 159–176.

[T-ix] L. TÓTH, Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, IX. On the product of certain fuctions, submitted.

[T] L. TÓTH, The number and the sum of $P-k$-ary divisors of m which are prime to n, submitted.

[V31] R.VAIDYANATHASWAMY, The theory of multiplicative arithmetical functions, *Trans. Amer. Math. Soc.,* **33** (1931), 579–662.

[W63] A. WALFISZ, Weylsche Exponentialsummen in der neueren Zahlentheorie, *Mathematische Forschungsberichte,* XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

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FINITE ELEMENT ANALYSIS FOR THE HEAT CONDUCTION EQUATION WITH THE THIRD BOUNDARY CONDITION

By

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1. Introduction

The initial-boundary value problem of the parabolic type serves as a mathematical model in many practical applications. Particularly, the parabolic problem with a mixed boundary condition has a great importance in the description of various physical phenomena, see [1], [7], [17], [18].

The numerical solution of such a problem is presented in [4] and [5], where the convergence is established for the one-dimensional diffusion equation. The one-dimensional nonlinear problem is considered in [17]. The three-dimensional problem with the homogeneous Dirichlet boundary condition is analysed in [15]. Also some comparison between the finite element method and the finite volume method is presented in [14] for the problem with the mixed homogeneous boundary condition.

In our work we consider the numerical solution of the nonstationary heat conduction problem in $\mathbb{R}^d$ with a uniformly elliptic part and a mixed nonhomogeneous boundary condition. We estimate the convergence of the semidiscrete solutions in the finite element subspaces to the weak solution. Further, we also analyse the convergence of the fully discretized solutions.

In the paper we show that the method of elliptic projection, introduced in [14] and [15], can be successfully applied to the problem with third boundary

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condition. In addition, we prove the applicability of a general class of fully discretized schemes, considered in [10] and [14] only for some special case for the problem with first boundary condition.

The paper is organized in the following manner. In Section 2 we give all necessary mathematical facts and define the weak formulation of the problem, for the details we refer also to [2], [8], [11] and [13]. Section 3 is devoted to the semidiscretization in the standard finite element subspaces. The basic idea is to use the elliptic projection of the exact solution as a comparison function, (cf. [15], [16]). In Section 4 we consider the numerical solution of the semidiscrete problem via the θ-method (see [12]). We prove the second order of the convergence for the Crank–Nicolson–Galerkin scheme, while that is of the first order for the other schemes.

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2. Mathematical background and notations

We consider the partial differential equation of the parabolic type

$$\frac{\partial u}{\partial t} - \operatorname{div}(\mathscr{A} \operatorname{grad} u) = f \quad \text{in } (0, T) \times \Omega, \tag{2.1}$$

with a mixed nonhomogeneous boundary condition

$$\alpha u + n^T \mathscr{A} \operatorname{grad} u = g \quad \text{on } (0, T) \times \partial\Omega, \tag{2.2}$$

where $t \in (0, T)$, $T > 0$, matrix $\mathscr{A} = \mathscr{A}(x) = (a_{ij}(x))_{i,j=1}^d$, $x \in \Omega$, $\alpha = \alpha(s) \geq 0$, $s \in \partial\Omega$, and n is the outward unit normal to Ω.

The initial condition is imposed:

$$u(x, 0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

In this paper we shall employ the following standard notations:

Ω	a polyhedral domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$,
$(\cdot, \cdot)$	a scalar product in $L^2(\Omega)$,
$\langle \cdot, \cdot \rangle$	a scalar product in $L^2(\partial\Omega)$,
$\lVert \cdot \rVert_s = (\sum_{\lvert m\rvert \leq s} \int_\Omega \lvert D^m v\rvert^2 dx)^{1/2}$	the standard norm of function $v \in H^s(\Omega)$,

$\lvert \cdot \rvert_s = (\sum\limits_{\lvert m\rvert = s} \int\limits_{\Omega} \lvert D^m v\rvert^2 dx)^{1/2}$	the standard seminorm of function $v \in H^s(\Omega)$,
$\lVert \cdot \rVert \equiv \lVert \cdot \rVert_0$	for simplicity,
$\lVert \cdot \rVert_{0,\partial\Omega}$	a norm in space $L^2(\partial\Omega)$,
V_h	a finite element subspace of $H^1(\Omega)$,
$C([0,T], L^2(\cdot))$	the space of continuous mappings from $[0,T]$ into $L^2(\cdot)$,
$C^k([0,T], H^s(\Omega))$	the space of the k-times continuously differentiable mappings from $(0,T)$ into $H^s(\Omega)$,
$H^1((0,T), H^1(\Omega))$	the space of the measurable mappings from $(0,T)$ into $H^1(\Omega)$ which belong to $H^1((0,T))$,
$H^1((0,T), V_h)$	the space of the measurable mappings from $(0,T)$ into V_h which belong to $H^1((0,T))$,
$P_r(K)$	a space of polynomials of the r-th degree over domain K,
C_i	the generic constant.

The matrix $\mathscr{A}$ is supposed to be symmetric and uniformly positive definite:

$$C_0 \eta^T \eta \le \eta^T \mathscr{A}(x) \eta \quad \forall \eta \in \mathbb{R}^d \quad \forall x \in \overline{\Omega}. \tag{2.4}$$

The given mappings a_{ij} and α are assumed to be bounded measurable functions:

$$\operatorname{ess\,sup}_{x,i,j} \lvert a_{ij}(x)\rvert \le C_1, \quad \operatorname{ess\,sup}_s \lvert \alpha(s)\rvert \le C_1. \tag{2.5}$$

If the problem (2.1)–(2.3) has the classical solution $u = u(t,x)$, then it satisfies the relation

$$(u'(t), v) + a(u,v) = F(t;v) \quad \forall v \in H^1(\Omega), \tag{2.6}$$

where $u' = \partial u/\partial t$ and a is a symmetric positive definite bilinear form defined as follows

$$a(v,w) = (\mathscr{A} \operatorname{grad} v, \operatorname{grad} w) + \langle \alpha v, w\rangle, \tag{2.7}$$

and

$$F(t;v) = (f(\cdot\,,t), v) + \langle g(\cdot\,,t), v\rangle. \tag{2.8}$$

For the functions $v \in H^1(\Omega)$ we recall the Friedrichs' inequality

$$\lVert v\rVert_1 \le C_2(\lvert v\rvert_1^2 + \lVert v\rVert_{0,\partial\Omega}^2)^{1/2} \tag{2.9}$$

and the trace theorem

$$\|v\|_{0,\partial\Omega} \leq C_3 \|v\|_{1,\Omega}, \tag{2.10}$$

respectively. Using (2.9) and (2.10) we may easily prove that

$$a(v,v) \geq C_4 \|v\|_1^2, \tag{2.11}$$

and

$$|a(u,v)| \leq C_5 \|u\|_1 \|v\|_1. \tag{2.12}$$

In order to formulate a weak formulation of the problem (2.1)–(2.3) we further assume that $a_{ij} \in L^\infty(\Omega)$, $\alpha \in L^\infty(\partial\Omega)$, $u^0 \in H^1(\Omega) \cap C(\overline{\Omega})$, $f \in$ $\in C([0,T], L^2(\Omega))$ and $g \in C([0,T], L^2(\partial\Omega))$, then the weak formulation reads: find $u \in H^1((0,T), H^1(\Omega))$ (*weak solution*) which satisfies (2.6) for a.e. fixed $t \in (0,T)$ and (2.3) for a.e. $x \in \Omega$.

REMARK 2.1. For the existence and uniqueness of the weak solution u under the above conditions we refer, e.g., to [9, Ch.3] and [11, Ch.11].

3. Semidiscrete Galerkin approximations

Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$, consisting of elements K_i and having the standard properties (see [3]). We are assumed to have the finite-dimensional subspace

$$V_h = \{v_h \in H^1(\Omega) \mid v_h|_K \in P_r(K) \quad \forall K \in \mathcal{T}_h\} \tag{3.1}$$

such that

$$\inf_{v_h \in V_h} \{\|v - v_h\| + h\|\operatorname{grad}(v - v_h)\|\} \leq C_6 h^s \|v\|_s, \quad 1 \leq s \leq r, \tag{3.2}$$

for $v \in H^s(\Omega)$ (the standard finite element spaces made e.g. of V_h). Let v^j, $j = 1, \ldots, N$, be the basis functions in the space V_h.

First we present the semidiscrete Galerkin approximation of equation (2.1), consisting of finding $u_h \in H^1((0,T), V_h)$ which satisfies

$$(u_h'(t), v_h) + a(u_h, v_h) = F(t; v_h) \quad \forall v_h \in V_h \tag{3.3}$$

for a.e. $t \in (0,T)$. Thus we have performed a discretization only in the space variables. The initial condition (2.3) can be approximated as follows

$$(u_h(0), v_h) = (u^0, v_h) \quad \forall v_h \in V_h. \tag{3.4}$$

Looking for u_h in the form

$$u_h(t,x) = \sum_{j=1}^{N} z_j(t) v^j(x), \quad t \in (0,T),\ x \in \Omega \tag{3.5}$$

and, using (2.6), we get a system of ordinary differential equations of the first order

$$\sum_{j=1}^{N} (v^j, v^i) z_j'(t) + \sum_{j=1}^{N} a(v^j, v^i) z_j(t) = F(t; v^i), \quad i = 1, \dots, N, \tag{3.6}$$

for the unknown functions $z_1, \dots, z_N$, respectively. On the basis of (3.4) the corresponding initial conditions at the point $t = 0$ can be determined via the relations

$$\sum_{j=1}^{N} (v^j, v^i) z_j(0) = (v^0, u^i), \quad i = 1, \dots, N. \tag{3.7}$$

REMARK 3.1. Using the notations

$$z(t) = (z_1(t), \dots, z_N(t))^T, \qquad M = ((v^i, v^j))_{i,j=1}^N,$$

$$A = (a(v^i, v^j))_{i,j=1}^N, \qquad \mathcal{F}(t) = (F(t; v^1), \dots, F(t; v^N))^T$$

and

$$z^0 = ((v^1, u^0), \dots, (v^N, u^0))^T,$$

we may rewrite the Cauchy problem (3.6)–(3.7) in the following matrix form

$$Mz'(t) + Az(t) = \mathcal{F}(t), \quad t \in (0,T), \tag{3.8}$$

$$Mz(0) = z^0. \tag{3.9}$$

Since the matrix M is symmetric and positive definite, the following corollary holds.

COROLLARY 3.2. *The problem* (2.1)–(2.3) *has a unique semidiscrete solution in $H^1((0,T), V_h)$ in the sense of* (3.3)–(3.4).

In order to derive the rate of convergence of the semidiscrete solution we shall employ the idea of the elliptic projection of $w \in H^1(\Omega)$ with respect to the scalar product $a(\cdot,\cdot)$, being described in the following way: we seek an element $w_h \in V_h$ such that the relation

$$a(w_h, \chi_h) = a(w, \chi_h) \tag{3.10}$$

holds for all $\chi_h \in V_h$. Since this problem has always a unique solution, then the definition of the so-called elliptic projection, introduced by

$$P : H^1(\Omega) \to V_h, \; Pw = w_h,$$

is correct. Consequently, we have the relation

$$a(Pw, \chi_h) = a(w, \chi_h) \quad \forall \chi_h \in V_h. \tag{3.11}$$

LEMMA 3.3. *For the projection P we have the estimate*

$$\|Pv - v\| + h\|\operatorname{grad}(Pv - v)\| \le C_7 h^s \|v\|_s \quad \textit{for all } v \in H^s(\Omega),\; 1 \le s \le r. \tag{3.12}$$

PROOF. First we consider the second term on the left hand side of (3.12). In view of (2.11), (3.11) with $\chi_h = Pv - v_h$, and (2.12) we have the inequalities

$$\|Pv - v\|_1^2 \le \frac{1}{C_4} a(Pv - v, Pv - v) = \frac{1}{C_4} a(Pv - v, v_h - v) \le$$
$$\le \frac{C_5}{C_4}\|Pv - v\|_1 \|v_h - v\|_1.$$

Thus the estimate

$$\|Pv - v\|_1 \le \frac{C_5}{C_4}\|v_h - v\|_1 \tag{3.13}$$

holds. Taking into account (3.2) and (3.13) we obtain

$$\|\operatorname{grad}(Pv - v)\| \le C_8 h^{s-1}\|v\|_s. \tag{3.14}$$

To get an estimate for the first term on the left hand side of (3.12) we consider the following auxiliary problem in Ω

$$-\operatorname{div}(\mathscr{A} \operatorname{grad} \psi) = \varphi \tag{3.15}$$

with homogeneous boundary condition

$$\alpha\psi + n^T \mathscr{A} \operatorname{grad} \psi = 0, \tag{3.16}$$

where $\varphi \in L_2(\Omega)$ will be defined later. Multiplying (3.15) by the term $Pv - v$ and using (3.2) we get

$$(Pv - v, \varphi) = -(Pv - v, \operatorname{div}(\mathscr{A} \operatorname{grad} \psi)) =$$
$$= (\operatorname{grad}(Pv - v), \mathscr{A} \operatorname{grad} \psi) - \langle Pv - v, n^T \mathscr{A} \operatorname{grad} \psi\rangle =$$
$$= (\operatorname{grad}(Pv - v), \mathscr{A} \operatorname{grad} \psi) + \langle \alpha(Pv - v), \psi\rangle =$$
$$= a(Pv - v, \psi) = a(Pv - v, \psi - P\psi) \le$$
$$\le C_5\|Pv - v\|_1 \|\psi - P\psi\|_1 \le C_9 h^s \|v\|_s \|\psi\|_2.$$

Due to the regularity in $H^2(\Omega)$ for the problem (3.15)–(3.16) we have a priori estimate $\|\psi\|_2 \leq C_{10}\|\varphi\|$. Consequently, taking $\varphi = Pv - v$ in the above chain of inequalities, we obtain

$$\|Pv - v\| \leq C_{11}h^s\|v\|_s. \tag{3.17}$$

Clearly, the estimates (3.14) and (3.17) together result in the statement of the lemma. ∎

THEOREM 3.4. *Let u and u_h be the solutions of* (2.6) *and* (3.3), *respectively. Then for $t \geq 0$ the estimate*

$$\|u(t) - u_h(t)\| \leq \|u_h(0) - u^0\| + C_{12}h^r\left(\|u^0\|_r + \int_0^t \|u'\|_r\,ds\right) \tag{3.18}$$

holds.

PROOF. We compare the solution of the semidiscrete problem to the elliptic projection. First we note that

$$u_h - u = (u_h - Pu) + (Pu - u) \equiv \sigma + \varrho. \tag{3.19}$$

In view of Lemma 3.3 we observe

$$\|\varrho(t)\| \leq C_7h^r\|u(t)\|_r \leq C_7h^r(\|u^0\|_r + \int_0^t \|u'\|_r\,ds). \tag{3.20}$$

Further from the definition of σ and (3.11) we get

$$\begin{aligned}(\sigma', v_h) + a(\sigma, v_h) &= (u_{h,t}, v_h) + a(u_h, v_h) - (Pu', v_h) - a(Pu, v_h) = \\ &= (f, v_h) + \langle g, v_h\rangle - (Pu', v_h) - a(u, v_h) = (u' - Pu', v_h).\end{aligned}$$

Consequently,

$$(\sigma', v_h) + a(\sigma, v_h) = -(\varrho', v_h), \quad \forall v_h \in V_h. \tag{3.21}$$

Choosing in (3.21) $v_h = \sigma$ we get

$$(\sigma', \sigma) + a(\sigma, \sigma) = -(\varrho', \sigma), \tag{3.22}$$

Due to the relations $0 < C_4\|\operatorname{grad}\sigma\|^2 \leq C_4\|\sigma\|_1^2 \leq a(\sigma,\sigma)$ and the Cauchy–Schwarz inequality, the relation (3.22) implies

$$(\sigma', \sigma) \leq \|\varrho'\| \cdot \|\sigma\|.$$

Since $(\sigma', \sigma) = \frac{1}{2}\frac{d}{dt}(\sigma, \sigma)$ we obtain

$$\frac{d}{dt}\|\sigma\| \leq \|\varrho'\|.$$

Integrating the above inequality on the interval $[0,t]$ we get

$$\|\sigma(t)\| \le \|\sigma(0)\| + \int_0^t \|\varrho'\|\, ds. \tag{3.23}$$

We observe

$$\begin{aligned} \|\sigma(0)\| &= \|u_h(0) - Pu^0\| \le \|u_h(0) - u^0\| + \|Pu^0 - u^0\| \le \\ &\le \|u_h(0) - u^0\| + C_{13}h^r\|u^0\|_r. \end{aligned} \tag{3.24}$$

Further, taking into account the obvious inequality

$$\|\varrho'\| = \|Pu' - u'\| \le C_{14}h^r\|u'\|_r,$$

(3.20), (3.23) and (3.24) result together in the proof. ∎

REMARK 3.5 Theorem 3.4 shows that the semidiscretization in V_h results in an optimal order of approximation in $L^2(\Omega)$.

4. Discretization in time

In this section we consider the fully discretized problem, obtained via the θ-method (see [12]) for the numerical solution of the semidiscrete (Cauchy) problem (3.3).

We introduce some notations: let τ be the time step, U^n be the approximation in V_h of $u(t)$ at $t_n = n\tau$, $n = 1, 2, \ldots$, $t_0 \equiv 0$, and the finite difference operator $\bar{\partial}_t$ be defined as follows

$$\bar{\partial}_t U^n = \frac{1}{\tau}(U^n - U^{n-1}). \tag{4.1}$$

We also assume that $\theta \in [0.5, 1]$ is any fixed parameter and $t_{n,\theta} = t_{n-1} + \theta\tau$. Then the θ-method, applied to (3.3), yields the relations

$$(\bar{\partial}_t U^n, v_h) + a(\theta U^n + (1-\theta)U^{n-1}, v_h) = F(t_{n,\theta}, v_h) \quad \forall v_h \in V_h,\ n = 1, 2, \ldots, \tag{4.2}$$

where

$$U^0 = u_h(0). \tag{4.3}$$

Obviously, this defines U^n implicitly by means of U^{n-1}.

REMARK 4.1 Our goal is to obtain unconditionally stable methods (see [12]), i.e., we restrict ourselves only to the case $\theta \in [0.5, 1]$.

Further we assume that $u \in C^3((0,T), H^r(\Omega))$. In order to estimate the global error $U^n - u(t_n)$ we shall split the error into two parts

$$U^n - u(t_n) = (U^n - Pu(t_n)) + (Pu(t_n) - u(t_n)) \equiv \sigma^n + \varrho^n. \tag{4.4}$$

Due to Lemma 3.3, we have the estimate

$$\|\varrho^n\| \le C_7 h^r \|u(t_n)\|_r \le C_7 h^r \left(\|u^0\|_r + \int_0^{t_n} \|u'(s)\|_r \, ds \right). \tag{4.5}$$

Consider the term σ^n. Let L denote the elliptic part of the parabolic operator, i.e.,

$$Lu = \operatorname{div}(\mathscr{A}(x) \operatorname{grad} u), \tag{4.6}$$

defined on the set of functions satisfying (2.2).

Then for any $v \in H^1(\Omega)$ we have

$$(-Lu, v) = a(u,v) - \langle g, v \rangle. \tag{4.7}$$

Using (4.2), (3.11), (2.6) and (4.7), respectively, we obtain

$$\begin{aligned}
&(\bar\partial_t \sigma^n, v_h) + a(\theta\sigma^n + (1-\theta)\sigma^{n-1}, v_h) = \\
&= (\bar\partial_t U^n, v_h) - (\bar\partial_t Pu(t_n), v_h) + a(\theta U^n + (1-\theta)U^{n-1}, v_h) - \\
&\quad - a(\theta Pu(t_n), v_h) - a((1-\theta)Pu(t_{n-1}), v_h) = \\
&= F(t_{n,\theta}, v_h) - (\bar\partial_t Pu(t_n), v_h) - \theta a(u(t_n), v_h) - (1-\theta) a(u(t_{n-1}), v_h) = \\
&= (u'(t_{n,\theta}), v_h) + a(u(t_{n,\theta}), v_h) - (\bar\partial_t Pu(t_n), v_h) - \\
&\quad - \theta a(u(t_n), v_h) - (1-\theta) a(u(t_{n-1}), v_h) = \\
&= (u'(t_{n,\theta}), v_h) - (Lu(t_{n,\theta}), v_h) + \langle g, v_h \rangle - (\bar\partial_t Pu(t_n), v_h) + \\
&\quad + \theta[(Lu(t_n), v_h) - \langle g, v_h \rangle] + (1-\theta)[(Lu(t_{n-1}), v_h) - \langle g, v_h \rangle] = \\
&= (u'(t_{n,\theta}), v_h) - (\bar\partial_t Pu(t_n), v_h) - (L(u(t_{n,\theta}) - \\
&\qquad\qquad - \theta u(t_n) - (1-\theta) u(t_{n-1})), v_h).
\end{aligned}$$

Consequently, using the notation

$$\begin{aligned}
\omega^n &= [(P - I)\bar\partial_t u(t_n)] + [\bar\partial_t u(t_n) - u'(t_{n,\theta})] \\
&+ [L(u(t_{n,\theta}) - \theta u(t_n) - (1-\theta) u(t_{n-1}))] \equiv \omega_1^n + \omega_2^n + \omega_3^n,
\end{aligned} \tag{4.8}$$

we get the relation

$$(\bar\partial_t \sigma^n, v_h) + a(\theta\sigma^n + (1-\theta)\sigma^{n-1}, v_h) = -(\omega^n, v_h) \quad \forall v_h \in V_h. \tag{4.9}$$

Now, choosing $v_h = \theta\sigma^n + (1-\theta)\sigma^{n-1}$ in (4.9), and employing the nonnegativity of the bilinear form a, we obtain

$$(\bar{\partial}_t \sigma^n, \theta\sigma^n + (1-\theta)\sigma^{n-1}) \le \|\omega^n\|(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|). \tag{4.10}$$

First consider the Crank–Nicolson scheme, i.e., $\theta = 0.5$. Then, (4.10) implies

$$\|\sigma^n\|^2 - \|\sigma^{n-1}\|^2 \le \tau\|\omega^n\|(\|\sigma^n\| + \|\sigma^{n-1}\|),$$

i.e.,

$$\|\sigma^n\| \le \|\sigma^{n-1}\| + \tau\|\omega^n\|.$$

Consequently, we have the estimate

$$\|\sigma^n\| \le \|\sigma^0\| + \tau\sum_{j=1}^{n}(\|\omega_1^j\| + \|\omega_2^j\| + \|\omega_3^j\|). \tag{4.11}$$

Obviously, as before,

$$\|\sigma^0\| = \|u_h(0) - Pu(0)\| \le \|u_h(0) - u^0\| + C_2 h^r \|u^0\|_r. \tag{4.12}$$

Note that

$$\omega_1^j = (P-I)\frac{1}{\tau}\int_{t_{j-1}}^{t_j} u'(s)\,ds = \frac{1}{\tau}\int_{t_{j-1}}^{t_j}(P-I)u'(s)\,ds,$$

where I is the identity operator. Hence we obtain

$$\tau\sum_{j=1}^{n}\|\omega_1^j\| \le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} C_{15}h^r\|u'(s)\|_r\,ds = C_{15}h^r\int_0^{t_n}\|u'(s)\|_r\,ds. \tag{4.13}$$

Further, using a simple integral equality we get

$$\|\omega_2^j\| = \|\bar{\partial}_t u(t_j) - u'(t_{j,\frac{1}{2}})\| = \frac{1}{2\tau}\left\|\int_{t_{j-1}}^{t_{j,\frac{1}{2}}}(s-t_{j-1})^2 u'''(s)\,ds + \right.$$
$$\left. + \int_{t_{j,\frac{1}{2}}}^{t_j}(s-t_j)^2 u'''(s)\,ds\right\| \le C_{16}\tau\int_{t_{j-1}}^{t_j}\|u'''(s)\|\,ds. \tag{4.14}$$

Similarly, using the Taylor expansion, we obtain

$$\|\omega_3^j\| = \left\|L(u(t_{j,\frac{1}{2}}) - \frac{1}{2}u(t_j) - \frac{1}{2}u(t_{j-1})\right\| \le C_{17}\tau\int_{t_{j-1}}^{t_j}\|Lu''(s)\|\,ds. \tag{4.15}$$

Finally, substituting (4.12), (4.13), (4.14) and (4.15) into (4.11), we can summarize our results as follows.

THEOREM 4.2. *For the global error of the fully discretized Crank–Nicolson–Galerkin finite element method the following estimate holds:*

$$\begin{aligned}\|U^n - u(t_n)\| \le \|u_h(0) - u^0\| + C_{18}h^r(\|u^0\|_r + \int_0^{t_n} \|u'(s)\|_r\, ds)+ \\ + C_{19}\tau^2 \int_0^{t_n} (\|u'''(s)\|_r + \|Lu''(s)\|_r)\, ds.\end{aligned} \tag{4.16}$$

Consider now the case $\theta \in (0.5, 1]$. For the left hand side of (4.10) we have the identity

$$(\bar{\partial}_t\sigma^n, \theta\sigma^n + (1-\theta)\sigma^{n-1}) = \frac{1}{\tau}[\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)(\sigma^{n-1}, \sigma^n)].$$

Using the Cauchy–Schwarz inequality in the above equality we may rewrite (4.10) in the following form:

$$\begin{aligned}\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)\|\sigma^{n-1}\|\|\sigma^n\| \le \\ \le \tau\|\omega^n\|(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|).\end{aligned} \tag{4.17}$$

Further we observe that

$$\begin{aligned}\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)\|\sigma^{n-1}\|\|\sigma^n\| = \\ = (\|\sigma^n\| - \|\sigma^{n-1}\|)(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|).\end{aligned}$$

Therefore (4.17) results in the relation

$$\|\sigma^n\| \le \|\sigma^{n-1}\| + \tau\|\omega^n\|.$$

Consequently we can apply the proof of Theorem 4.2. Clearly, the estimates (4.12), (4.13) and (4.14) hold. For the term ω_3^j we have the bound

$$\|\omega_3^j\| \le \int_{t_{j-1}}^{t_j} \|Lu'(s)\|\, ds. \tag{4.18}$$

Using all these estimates we can summarize our result as follows.

THEOREM 4.3. *For the global error of the fully discretized Galerkin method with $\theta \in (0.5, 1]$ the estimate*

(4.19)
$$\|U^n - u(t_n)\| \le C_{20}\|u_h(0) - u^0\| + C_{21}\, h^r \left(\|u^0\|_r + \int_0^{t_n} \|u'(s)\|_r \, ds \right) +$$
$$+ C_{22}\, \tau^2 \int_0^{t_n} \|u'''(s)\|_r \, ds + \tau \int_0^{t_n} \|Lu'(s)\|_r \, ds$$

holds.

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References

[1] AKRIVIS, G. and DOUGALIS, V., On a conservative, high-order accurate finite element scheme for the "parabolic" equation, *Comput. Acoustics,* **1** (1989), 17–26.

[2] AXELSSON, O. and BARKER, V. A., *Finite element solution of boundary value problems. Theory and computation,* Academic Press, New York, 1984.

[3] CIARLET, P. G., *The finite element method for elliptic problems,* North-Holland, Amsterdam, 1978.

[4] DOUGLAS, J. and DUPONT, T., Galerkin methods for parabolic equations, *SIAM J. Numer. Anal.,* **7** (1970), 575–626.

[5] FIX, G. and NASSIF, N., On finite element approximation in time dependent problems, *Numer. Math.,* **19** (1972), 127–135.

[6] JOHNSON, C., *Numerical solutions of partial differential equations by the finite element method,* Cambridge University Press, Cambridge, 1988.

[7] KŘÍŽEK, M. and NEITTAANMÄKI, P., *Mathematical and numerical modelling in electrical engineering: theory and applications,* Kluwer Academic Publishers, 1996.

[8] LADYZENSKAJA, O. A., SOLONNIKOV, V. A. and URAL'CEVA, N. N., *Linear and quasilinear equations of parabolic type,* American Mathematical Society, Providence, R.I., 1967.

[9] LIONS, J. L. and MAGENES, E., *Problèmes aux limites non homogènes et applications,* Dunod, Paris, 1968.

[10] NEITTAANMÄKI, P. and TIBA, D., *Optimal control of nonlinear parabolic systems. Theory, algorithms, and applications,* Marcel Dekker, New York, 1994.

[11] REKTORYS, K., *The method of discretization in time and partial differential equations,* D. Reidel, Dordrecht, 1982.

[12] SAMARSKII, A. A., *Theory of difference schemes,* Nauka, Moscow, 1977, (in Russian).

[13] SIMON, L. AND BADERKO, E. A., *Theory of linear partial differential equations of second order,* Tankönyvkiadó, Budapest, 1983, (in Hungarian).

[14] STEINBACH, J., Comparison of finite element and finite volumes schemes for variational inequalities, *East-West J. Numer. Math.,* **4** (1996), 207–235.

[15] THOMÉE, V., *Galerkin finite element methods for parabolic problems,* Springer, Berlin, 1997.

[16] WHEELER, M. F., A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations, *SIAM J. Numer. Anal.,* **10** (1973), 723–759.

[17] WHITE, R. E., *An introduction to the finite element method with applications to nonlinear problems,* J. Wiley & Sons, New York, 1985.

[18] ZHU, J., WU, H. and WANG, J., A mixed method for the mixed initial–boundary value problems of equations of semiconductor devices, *SIAM J. Numer. Anal.,* **31** (1994), 731–744.

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AVERAGING OPERATORS WITH SUPPORTS IN FIBERS

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Introduction.

All spaces considered here are compact (Hausdorff) spaces. We recall that an averaging operator for a continuous onto map $\phi : S \to T$ is a continuous linear operator $u : C(S) \to C(T)$ such that $u(f \circ \phi) = f$ for all $f \in C(T)$. In [4], WOLFE poses the following problem:

PROBLEM 9. *Let T be the quotient space S obtained by identifying corresponding points in two disjoint homeomorphic closed subsets. Let $\phi : S \to$ $\to T$ be the quotient map. Give necessary and suffiecient conditions for ϕ to have an averaging operator.*

It is well known (see Example 1 below) that the existence of an extension operator from one of the closed subsets to S is a sufficient condition. By ([2], Prop. 4.1), the image of a continuous function by a norm-one averaging operator can be calculated on each point $t \in T$ as an integral (an average) of f over the fiber $\phi^{-1}(t)$ with respect to certain probability measure. However, this is no longer true for averaging operators with higher norm. Here we prove that if we restrict ourselves to certain class of averaging operators that satisfy a weak form of this property the existence of an extension operator is also necessary when the spaces are zero-dimensional.

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Averaging operators with supports in fibers

Let S be space. We call $C(S)$ the Banach space of all real-valued continuous functions on S (endowed with the supremum norm) and $M(S)$ its dual space, identified with the set of all regular signed measures on S. We consider in this space the weak*-topology, i.e., a net $\{\mu_\alpha\} \subset M(S)$ converges to a measure μ if and only if

$$\lim_\alpha \int f\, d\mu_\alpha = \int f\, d\mu,$$

for all $f \in C(S)$.

For a linear operator $u : C(S) \to C(T)$, its integral representation is the map $\mu : T \to M(S)$ given by $\mu_t = u^*(\delta_t)$, where $u^* : M(T) \to M(S)$ is the dual map $x \mapsto u \circ x$ and δ_t is the Dirac measure with support $\{t\}$ (see [1, Theorem 4.11]). This map is continuous and completely determines u. In fact we have $u(f)(t) = \int f \mu_t$, for all $f \in C(S)$. Also, $\|u\| = \sup\limits_{t \in T} |\mu_t|(S)$.

Let $\phi : S \to T$ be a continuous onto map. The fiber of a point $t \in T$ (with respect to ϕ) is the set $\phi^{-1}(t)$. The point t is plural if its fiber has more than one point.

An averaging operator for ϕ (in the sense given in the introduction) satisfies the following property: $\mu_t(\phi^{-1}(B)) = \delta_t(B)$ for all Borel subset $B \subset T$ and $t \in T$ (see [1, Lemma 6.1]).

Recall that the support of a measure $\mu \in M(S)$ is given by

$$\operatorname{supp} \mu = S \setminus \bigcup \{A \subset S \mid A \text{ is open and } |\mu|(A) = 0\}.$$

DEFINITION 1. We say that an averaging operator u for a continuous onto map $\phi : S \to T$ has *supports in fibers* if $\operatorname{supp} \mu_t \subset \phi^{-1}(t)$ for all t in the closure of the set of all plural points of ϕ.

By [2] Prop. 4.1 we have that all regular (i.e. norm-one) averaging operators have supports in fibers. We will need the following example. Recall that if A is a closed subset of a space S, an extension operator from $C(A)$ into $C(S)$ (or, with an abuse of language, from A to S) is a continuous linear operator $E : C(A) \to C(S)$ such that $E(f)|_A = f$, for each $f \in C(A)$.

EXAMPLE 1. Let $\phi : S \to T$ be the quotient map obtained by pointwise identification of two disjoint closed subsets of S through a homeomorphism

$h : A_1 \to A_2$. Suppose that there exist an extension operator $E : C(A_1) \to$ $\to C(S)$. Let $g \in C(S)$ be any function such that $g(A_1) = 1$, $g(A_2) = 0$ and $0 \leq g \leq 1$. Then the function

$$u(f)(\phi(s)) = f(s) + g(s)E(f|_{A_2} \circ h - f|_{A_1})(s)$$

defines an averaging operator for ϕ. For a plural point $t = \phi(s_1) = \phi(s_2)$ (with $s_i \in A_i$), the associated measure is $\mu_t = \delta_{s_2}$. Hence u has supports in fibers. ∎

It is easy to see that from any averaging operator we can construct another one without supports in fibers (except in trivial cases) but, on the other hand, many maps having an averaging operator admit also an averaging operator with supports in fibers. We give an example of a map that does not admit any averaging operator with supports in fibers. In order to show this we prove the following Lemma:

LEMMA 2. *Let $\phi : S \to T$ be a continuous onto map. Let A be the closure of the set of plural points of ϕ. Suppose that $[hi$ has an averaging operator with supports on fibers. Then every pair of open subsets U, V in A such that $\overline{\phi^{-1}(U)} \cap \overline{\phi^{-1}} = \emptyset$ satisfies $\overline{U} \cap \overline{V} = \emptyset$.*

PROOF. Suppose that there is a point $p \in \overline{U} \cap \overline{V}$. Let $\{t_\alpha\} \subset U$ be a net converging to p. Then for each a α $\operatorname{supp}\mu_{t_\alpha} \subset \phi^{-1}(t_\alpha) \subset \phi^{-1}(U)$ and it is easy to show that this implies $\operatorname{supp}\mu_p \subset \overline{\phi^{-1}(U)}$. By the same argument, we have that $\operatorname{supp}\mu_p \subset \phi^{-1}(V)$, and this is a contradiction. ∎

EXAMPLE 2. Consider the standard quotient map $\phi : [0,1]^2 \to T$ from the unit square onto the torus T (which identifies corresponding points in opposite edges). It is easy to see that ϕ does not satisfy the property of the previous lemma: Take $p = \phi(0,0) = \phi(0,1)$, $U = \phi(]0,1/3[)$ and $V = \phi(]2/3,1[)$. Hence ϕ does not have an averaging operator with supports in fibers. On the other hand, you can express ϕ as the composition of two identification maps (from the square to a cylinder and from the cylinder to the torus) and each one has an averaging operator by Example 1. The composition of these operators is an averaging operator for ϕ.

The result

In order to prove our main result we need the following lemma due to Wolfe [3] Lemma 2.2:

LEMMA 3. *Let* $\phi : S \to T$ *be a continuous onto map. Let* $J \subset T$ *be the closure of the set of all plural points of* ϕ. *Let* $J_1 = \phi^{-1}(J)$. *Let* $\phi_1 : J_1 \to J$ *be the restriction of* ϕ. *Then there exists an averaging* u *operator for* ϕ *if and only if there exists an averaging operator* u_1 *for* ϕ_1 *and a continuous linear operator* $E : C(J_1) \to C(S)$ *which extends the functions in the kernel of* u_1.

The proof of this Lemma shows that the operator u_1 is given by $u_1(f) = = u(\tilde{f})_A$, where f is any continuous extension of f to S.

Consider now a quotient map $\phi : S \to T$ obtained from pointwise identification of two disjoint closed homeomorphic subsets A_1, A_2 of a zero-dimensional space S. By Example 1 we know that the existence of an extension operator from one A_i to S implies the existence of an averaging operator for ϕ. We are going to show that the converse is essentially true, but it can not be stated as simply as one could think at first glance:

EXAMPLE 3. Take $S = \beta\mathbb{N}_1 \oplus (\beta\mathbb{N}_2 \setminus \mathbb{N}_2)$, where $\beta\mathbb{N}_i$ is the Stone–Čech compactification of the discrete countable space $\mathbb{N}_i$. Let $A_i = \beta\mathbb{N}_i \setminus \mathbb{N}_i$. Phillips theorem implies that there is not any extension operator from A_1 to S. Obviously, such an operator does exists from A_2 to S (in [4, § 5, Ex. 2] another example of such situation is given in which the sets A_i are nowhere dense in S). Change now S by the topological sum of two disjoint copies of S. So we have four pairwise disjoint closed subsets A_{11}, A_{12}, A_{21}, A_{22}, and only the first and the second of them have an extension operator to S. If we call $A_1 = A_{11} \cup A_{12}$ and $A_2 = A_{21} \cap A_{22}$, then it is easy to see that there exists an extension operator from A_1 to S, and so the quotient map identifying A_1 with A_2 has an averaging operator (with supports in fibers). However, we can also call $B_1 = A_{11} \cup A_{22}$ and $B_2 = A_{21} \cup A_{12}$ and the quotient map identifying these sets is exactly the same than before, and so it has an averaging operator, but now neither B_1 nor B_2 has an extension operator.

In view of this Example, the right statement of our theorem happens to be the following:

THEOREM 4. *Let* S *be a compact zero-dimensional space and let* A_1, A_2 *be two disjoint closed nowhere dense subsets of* S. *Let* $h : A_1 \to A_2$ *be a homeormorphsm. Let* $f : S \to T$ *be the quotient map obtained by identifying the points* x *and* $h(x)$ *to one single point* $\phi(x) \in T$. *Let* $A \subset T$ *be the image of the sets* A_i.

There exists an averaging operatot for ϕ with supports in fibers if and only if there exists a partition $A_1 = A_{11} \cup A_{12}$ such that calling $A_{2j} = h(A_{1j})$, $B_1 = A_{11} \cup B_{22}$, and $B_2 = A_{12} \cup B_{21}$, then the sets B_1 thes B_2 provide the same identification map and such that there exists an extension operator $E : C(B_1) \to C(S)$.

PROOF. One implication is Example 1. Suppose that ϕ has an averaging operator with supports in fibers. For each $t \in A$ and each $i = 1,2$, let $t_i \in A_i$ be such that $\phi(t_i) = t$. By hypothesis we have that $\operatorname{supp}\mu_t \subset \{t_1, t_2\}$, where μ is the integral representation of the averaging operator of ϕ. So

$$\mu_t = \alpha_{t_1}\delta_{t_1} + \alpha_{t_2}\delta_{t_2}, \tag{1}$$

for certain real numbers α_{t_i}.

Let C_1, C_2 be clopen subsets of S such that $A_i \subset C_i$ for $i = 1,2$.

Clearly, $\alpha_{t_i} = \mu_t(\chi_{C_i})$, where χ_{C_i} is the characteristic function of C_i. This proves that the functions $t \mapsto \alpha_{t_i}$ are continuous for each i.

Since $1 = \mu_t(\phi^{-1}(t)) = \alpha_{t_1} + \alpha_{t_2}$, there exists an index, for instance 2, such that $\alpha_{t_2} \geq 1/2$. By continuity, there exists a clopen neighborhood W of t in A such that for all $t' \in W$ we have $\alpha_{t'_2} \geq 1/2$. For each $i = 1,2$, let $W_i = \phi^{-1}(W) \cap A_i$.

Define an extension operator $E : C(W_1) \to C(A_1 \cup A_2)$ given by

$$E(f)(x) = \begin{cases} f(x) & \text{if } x \in W_1 \\ -\frac{\alpha_{\phi(x)1}}{\alpha_{\phi(x)2}} f(h^{-1}(x)) & \text{if } x \in W_2 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, E is a continuous linear operator. In fact, $\|E\| \leq 2\|u\|$.

Let $u_1 : C(A_1 \cup A_2) \to C(A)$ be the restriction of the given averaging operator, i.e., the operator defined by $u_1(f) = u(\tilde{f})|_A$, where $\tilde{f}$ is any extension of f to S.

It is easy to see that the measures μ_t associated to μ_1 are the restrictions of those associated to u, and since supports are contained in fibers, formula (1) remains valid for u_1.

By Lemma 3 there is a linear operator $E_1 : C(A_1 \cup A_2) \to C(S)$ which extends the functions in the kernel of u_1. Let us see that each $E(f)$ is in this kernel, and so the composition of E followed by E_1 is an extension operator from $C(W_1)$ into $C(S)$.

Take $f \in C(W_1)$. If $t' \in W$, then

$$u_1(E_1(f))(t') = \mu_{t'}(E_1(f)) = \alpha_{t'_1} E_1(f)(x_{t'_1}) + \alpha_{t'_2} E_1(f)(x_{t'_2}) =$$
$$= \alpha_{t'_1} E_i(f)(x_{t'_1}) + \alpha_{t'_2} E_1(f)(x_{t'_2}) =$$
$$= \alpha_{t'_1} f(x_{t'_1}) + (\alpha_{t'_2}(-\alpha_{t'_1})/\alpha_{t'_2}) f(x_{t'_1}) = 0.$$

If $t' \in A \setminus W$ we also have $u_1(E(f))(t') = \mu_{t'}(E(f)) = 0$ by definition of E.

So we have proved that every $t \in A$ has a clopen neighborhood W_t in A such that if $W_{it} = f^{-1}(W_t) \cap A_i$, there exists an extension operator $E_t : C(W_{it}) \to C(S)$ for some index i (not necessarily equal to 1 for every t).

Obviously, if W_t has this property, also does every clopen neighborhood of t contained in W_t, and if $t' \in W_t$, then $W_{t'} = W_t$ is a clopen neighborhood of t' with the same property.

Take a finite subcover from $\{W_t\}$ and remove common (clopen) intersections. So we get a finite partition $A = \bigcup_{j=1}^{m} W_j$ in clopen subsets satisfying previous conditions. Now we can take A_{11} as the join of all $f^{-1}(W_j) \cap A_1$ for which there exists an extension operator to S, $A_{12} = A_1 \setminus A_{11}$, let also A_22 be the analogous to A_{11} in A_2 and $A_{21} = A_2 \setminus A_{22}$. Our construction guarantees that these sets are those we were looking for.

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References

[1] W. G. BADE, The Banach space $C(S)$, Lecture Notes Series, **26** (1971), Aarthus Universitet.

[2] A. PE⁄LCZYŃSKI, *Linear Extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions,* Dissertationes Math. Rozprawy Mat. **58** (1968).

[3] J. WOLFE, Injective Banach Spaces of type $C(T)$, *Israel J. Math.,* **18** (1974), 133–140.

[4] J. WOLFE, Injective Banach Spaces of Continuous Functions, *Trans. Amer. Math. Soc.,* **235** (1978), 115–139.

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