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GENERALIZED SEPARABILITY IN VECTOR-VALUED FUNCTION SPACES

By

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1. Introduction

Let $C(X,E)$ the vector space of all continuous E-valued functions on X, and let $C_b(X,E)$ $(C_0(X,E))$ denote the subspace of $C(X,E)$ consisting of those functions which are bounded (vanish at infinity). When E is the real or complex field, these spaces are denoted by $C(X)$, $C_b(X)$, and $C_0(X)$. Let $\beta_0, u(k)$ denote the substrict, uniform (compact-open) topologies on $C_b(X,E)$ $(C(X,E))$. The fundamental result on the separability of $(C_b(X), \|\cdot\|)$ was obtained by M. and S. KREIN [13] in 1940. Since then several authors have studied the separability of scalar- or vector-valued functions under various topologies; see, e.g. [2], [4], [6], [8], [11], [14], [18], l9]. In [4], GULICK and SCHMETS considered the notion of seminorm separability and obtained its characterizations for $(C_b(X),\beta_0)$ and $(C(X),k)$. This result was later extended in [10] to the case of E-valued functions where E is a locally convex space.

In this paper we introduce a more general notion of separability, namely, the neighbourhood separability, and study it for the spaces $(C_b(X,E),\beta_0)$, $(C(X,E),k)$, $(C_b(X,E),u)$, and $(C_0(X,E),u)$ without assuming the local convexity of E. We also give some examples to show that our results are applicable in certain non-locally convex situations: Our arguments are similar to those used by the author in [9], [11], where E is any TVS. Here we consider a generalized separability but need the additional hypothesis of semi-convexity on E.

2. Preliminaries

Throughout this paper, unless stated otherwise, X will denote a completely regular Hausdorff space and E a Hausdorff TVS with a base $\mathcal{W}$ of neighbourhoods of 0. We shall denote by $C(X) \otimes E$ the vector space spanned by the set of all functions of the form $g \otimes a$, where $g \in C(X)$, $a \in E$, and $(g \otimes a)(x) = g(x)a$ $(x \in X)$. The substrict topology β_0 [1], [7] on $C_b(X, E)$ is the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$U(\phi, W) = \{f \in C_b(X, E) : \phi(x)f(x) \in W \text{ for all } x \in X\},$$

where $\phi \in B_0(X)$, the set of all bounded functions on X vanishing at infinity, and $W \in \mathcal{W}$. On $C_b(X)$, the β_0-topology is determined by the family $\{\|\phi\cdot : \phi \in B_0(X)\}$ of seminorms, where $\|\phi f\| = \sup\{|\phi(x)f(x)| : x \in X\}$ $(f \in C_b(X))$. Note that $k \leq \beta_0 \leq u$ on $C_b(X, E)$ with k or β_0 equal to u iff X is compact.

A locally convex space L is called *seminorm separable* [3], [4] if, for every continuous seminorm p on L, (L, p) is separable. This notion of separability may be formulated in the general setting as follows. Let L be a TVS, and let V be a neighbourhood of 0 in L. A subset H of L is said to be V*-dense* in L if, for any $z \in L$ and $\delta > 0$, there exists an element $y \in H$ such that $y - z \in \delta V$. L is called *neighbourhood separable* if, for every neighbourhood V of 0, there exists a countable V-dense subset of L. Clearly, separability implies neighbourhood separability. If L is locally convex, then it is neighbourhood separable iff it is seminorm separable.

E is called a *semi-convex* space [5] if it has a base $\mathcal{W}$ of neighbourhoods of 0 consisting of semi-convex sets i.e. the sets W which satisfy $W + W \subseteq$ $\subseteq tW$ for some $t > 0$. The class of semi-convex spaces includes all locally convex and all locally bounded spaces.

3. Main results

THEOREM 3.1. *Let E be a semi-convex space with nontrivial dual E'. Then the following statements are equivalent.*

(a) $(C_b(X) \otimes E, \beta_0)$ *is neighbourhood separable.*

(b) $(C(X) \otimes E, k)$ *is neighbourhood separable.*

(c) *Every compact subset of X is metrizable and E is neighbourhood separable.*

PROOF. (a) $\Rightarrow$ (b). This follows from the facts that $k \leq \beta_0$ on $C_b(X, E)$ and $C_b(X) \otimes E$ is dense in $(C(X) \otimes E, k)$.

(b) $\Rightarrow$ (c). Suppose $(C(X) \otimes E, k)$ is neighbourhood separable, and let K be a compact subset of X. Choose $\psi \in E'$ and $c \in E$ with $\psi(c) = 1$. There exists a balanced $W \in \mathcal{W}$ such that $|\psi(a)| < 1$ for all $a \in W$. Let $\{h_n\}$ be a $U(\chi_K, W)$-dense subset of $C(X) \otimes E$. Then $\{\psi \circ h_n\}$ is dense in $(C(X), \|\chi_K \cdot \|)$ and so, by ([4], Theorem 2b), K is metrizable. Now, let $V \in \mathcal{W}$. For any fixed $z \in X$, choose $\{f\}$ as a $U(\chi_{\{z\}}, V)$-dense subset of $C(X) \otimes E$. Then $\{f_n(z)\}$ is V-dense in E.

(c) $\Rightarrow$ (a). Let $\phi \in B(X)$ with $0 \leq \phi \leq 1$, and let $W \in \mathcal{W}$ be balanced and semi-convex. By ([4], Theorem 3b) $(C_b(X), \|\phi \cdot \|)$ is separable and so it has a countable dense subset $\{g_m\}$, say. Choose $\{a_n\}$ as a W-dense subset of E. Let H be the countable subspace generated by $\{g_m \otimes a_n : m, n = 1, 2, \ldots\}$ over rationals. Then H is $U(\phi, W)$-dense in $C_b(X) \otimes E$, as follows. Let $f = \sum_{i=1}^{q} f_i \otimes b_i$ ($f_i \in C_b(X)$, $b_i \in E$) be in $C_b(X) \otimes E$ and $\delta > 0$. Choose $r > 0$ such that each $b_i \in rW$. Let $s = \max\{\|f_i\| : 1 \leq i \leq q\}$. Since W is semi-convex, we can choose $t > r(s+1)$ such that $W + W + \ldots + W$ ($2q$-terms) $\subseteq tW$. For each $i = 1, 2, \ldots, q$, there exist $g_{m_i} \in \{g_m\}$ and $a_{n_i} \in \{a_n\}$ such that $\|\phi(g_{m_i} - f_i)\| < \delta/t^2$ and $a_{n_i} - b_i \in (\delta/t^2)W$. Let $g = \sum_{i=1}^{q} g_{m_i} \otimes a_{n_i}$. Then $g \in H$ and it is easily verified that $g - f \in \delta\, U(\phi, W)$. This completes the proof.

THEOREM 3.2. *Let E be a semi-convex space with non-trivial dual. Then*

(i) *$(C_b(X) \otimes E, u)$ is neighbourhood separable iff X is a compact metric space and E is neighbourhood separable.*

(ii) *Suppose X is locally compact. Then $(C(X) \otimes E, u)$ is neighbourhood separable iff X is a σ-compact metric space and E is neighbourhood separable.*

PROOF. (i) In this case $(C_b(X), u)$ is seminorm separable $\Leftrightarrow$ it is separable $\Leftrightarrow$ X is a compact metric space [13]. The proof now follows just as in Theorem 3.1.

(ii) If X is locally compact, $(C_0(X), u)$ is seminorm separable $\Leftrightarrow$ it is separable $\Leftrightarrow$ X is a σ-compact metric space [14]. Again the proof follows just as in Theorem 3.1.

REMARK 3.3. If X has finite covering dimension or E is locally convex or E has the approximation property or E is an F-space with a basis, then $C_b(X)\otimes E$ is β_0-dense in $C_b(X, E)$ and $C(X)\otimes E$ is k-dense in $C(X, E)$ (see [7], [9], [16]). Hence, under these assumptions, Theorems 3.1 and 3. 2 hold with $C_b(X)\otimes E$, $C(X)\otimes E$, and $C_0(X)\otimes E$ replaced by $C_b(X, E)$, $C(X, E)$, and $C_0(X, E)$, respectively. It is not known whether or not the above 'density' results hold for E a locally bounded space. However, we are able to establish the following theorem.

THEOREM 3.4. *Let X be any Hausdorff space and E a locally bounded space. Then $(C_b(X, E), \beta_0)$ is neighbourhood separable iff $(C(X, E), k)$ is so.*

PROOF. Suppose $(C(X, E), k)$ is neighbourhood separable. Let $\phi \in B_0(X)$ with $0 \le \phi \le 1$ and $W \in \mathscr{W}$. Let V be a bounded neighbourhood of 0 in E. By KLEE ([12], Theorems 4 and 5), there exists a closed 'shrinkable' neighbourhood S of 0 in E with $S \subseteq V$ and such that the Minkowski functional ρ of S is continuous. Then, for any $r > 0$, the function $h_r : E \to rS$ defined by

$$h_r(a) \begin{cases} a & \text{if } a \in rS \\ (r/\rho(a))a & \text{if } a \in E \setminus rS \end{cases}$$

is continuous. Choose $t \ge 1$ such that $V + V \subseteq tS$ and $V + V \subseteq tW$. For each $m = 1, 2, \ldots$, there exists a compact set $K_m \subseteq X$ such that $\phi(x) < 1/tm^2$ for $x \in X \setminus K_m$. Corresponding to each K_m, choose $\{f_{mn} : n = 1, 2, \ldots\}$ as a $U(\chi_K, V)$-dense subset of $C(X, E)$. We now show that$\{h_n \circ f_{mn} : m, n = 1, 2, \ldots\}$ is $U(\phi, W)$-dense in $C_b(X, E)$. Let $f \in C_b(X, E)$ and $0 < \delta < 1$. Choose integers $M \ge 1/\delta$ and $N \ge 1$ such that $f(X) \subset \frac{M\delta}{t} V$ and $(f_{MN} - f)(K_M) \subseteq \frac{\delta}{t} V$. Let $x \in X$. If $x \in K_M$, then $f_{MN}(x) \in MS$ and so

$$\phi(x)(h_M \circ f_{MN}(x) - f(x)) = \phi(x)(f_{MN}(x)f(x)) \in \delta W.$$

If $x \in X \setminus K_M$, then

$$\phi(x)(h_M \circ f_{MN}(x) - f(x)) = \\ = \begin{cases} \phi(x)(f_{MN}(x) - f(x)) & \text{if } f_{MN}(x) \in MS \\ \phi(x)\left[\frac{M}{\rho(f_{MN}(x))} f_{MN}(x) - f(x)\right] & \text{if } f_{MN}(x) \in E \setminus MS. \end{cases}$$

Thus $h_M \circ f_{MN} - f \in U(\phi, W)$, as required.

The converse follows from the fact that $C_b(X, E)$ is dense in $(C(X, E), k)$ ([9], Theorem 3.2).

Finally we give some examples concerning the above results and the Remark 3.3.

4. Examples

For $0 < p < 1$, let $\ell_p = \ell_p(N)$ denote the usual space of scalar sequences, and let $h_p = h_p(D)$ denote the Hardy space of certain harmonic functions on the unit disc D of the complex plane (see [15]). ℓ_p and h_d are not locally convex but are locally bounded and their duals separate points; ℓ_p is separable while h_p is not since it contains a copy of ℓ_∞ ([15], Theorem 3.5). Let Ω be the first uncountable ordinal and $\Gamma < \Omega$ a limit ordinal.

Let $X = [0, \Omega)$, $X_1 = [0, \Omega]$, $Y = [0, \Gamma)$, and $Y_1 = [0, \Gamma]$, each endowed with the order topology ([17], p.68). Then

1. $(C_b(X, E), \beta_0)$ and $(C(X, E), k)$ are neighbourhood separable for $E = \ell_p$, but not for $E = h_p$. Here every compact subset of X is metrizable although X is not.

2. $(C_b(Y_1, \ell_p), u)$ and $(C_0(Y, \ell_p), u)$ are (neighbourhood) separable, but $(C(X_1, \ell_p), u)$ and $(C_0(X, \ell_p), u)$ are not since X_1 is not metrizable and X is not a σ-compact metric space.

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SEMIGROUPS OVER WHICH NO AUTOMATON HAS PROPER ESSENTIAL CONGRUENCES

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1. Introduction

Let S be a semigroup and A an S-automaton (throughout the paper, automata are understood as right automata). A congruence σ on A is said to be essential if for every congruence $\alpha \neq \iota$ (the identity relation) on A we have $\alpha \cap \sigma \neq \iota$. It follows from the definitions that an S-automaton B is an essential extension of an S-automaton A if and only if the Rees congruence of B defined by A is essential. BERTHIAUME [1] proved that an S-automaton is injective if and only if it has no essential extensions. Hence all S-automata are injective iff no S-automaton has proper essential Rees congruences. Monoids with this property, called completely injective monoids, have been considered by several authors, see for example [2], [3]. In the present paper we restrict our consideration to the case where the Rees congruences are replaced by all congruences, and prove that, for a semigroup S, no S-automaton has proper essential right congruences if and only if S is a finite chain considered as a semilattice.

2. Results

Two distinguished congruences are present in every S-automaton M : ι, the identity relation, and ν, the universal relation. They are defined respectively by

$$a\iota b \Longleftrightarrow a = b \text{ and } a\nu b \Longleftrightarrow a, b \in M.$$

Congruences are ordered in the same way as equivalence relations are. We say that a congruence σ is proper if $\sigma \neq \nu$.

Let $\mathbf{D}$ be the class of all semigroups S such that no S-automaton has proper essential congruences. Let $S \in \mathbf{D}$ and denote by S^1 the semigroup S with identity adjoined (even if S itself has an identity). By the assumption, S^1 has no proper essential S-congruences and then, clearly, it has no proper essential S^1-congruences either, so $S^1 \in \mathbf{D}$. By [4] Theorem 1, every right ideal of S^1 is generated by an idempotent. In particular, S contains a left identity and every right ideal of S has an idempotent generator. Furthermore, by [4] Theorem 2, the set of right ideals of S is linearly ordered by inclusion. Suppose $J_1 \subseteq J_2 \ldots \subseteq J_n \subseteq \ldots$ is an ascending chain of right ideals, then $\cup J_n$ is also a right ideal, hence it is generated by an idempotent e, but $e \in J_k$ for some k, and then $J_k = J_{k+1} = \ldots$. This shows that S satisfies the ascending chain condition on right ideals.

LEMMA A. *Suppose J, K are ideals of a semigroup $S \in \mathbf{D}$ such that K is strictly contained in J. Then $J \setminus K$ is a subsemigroup of S.*

PROOF. Suppose for some a and b in J we have $ab \in K$. We show either $a \in K$ or $b \in K$.

We know there exists an idempotent f such that $aS^1 = fS^1$, which implies $f = as$ for some s in S^1. Since right ideals are linearly ordered, either $sS^1 \subseteq bS^1$ or $bS^1 \subseteq sS^1$. In the first case $s = bd$ for some d in S^1. Therefore, $f = as = abd \in K$, which implies that $a \in K$. In the second case $b = su$ for some u in S^1, so $fu = asu = ab$. If $f \in K$ then again $a \in K$. Suppose $f \notin K$. We claim that in this case $fu \in K$ implies $u \in K$. To prove this, let $T(f, K) = \{v \in S^1 : fv \in K\}$. Clearly, $T(f, K)$ is a right ideal of S, therefore there exists an idempotent e in S such that $T(f, K) = eS$. It follows easily that either $fe = e$ or $ef = f$. Let v be in $T(f, K)$. Then $v = ev$ and if $fe = e$, then $fev = ev$ implies $fv = v$ and this implies in turn $v \in K$. If, on the other hand, $ef = f$, then in view of $ef \in eS = T(f, K)$ we have $f = fef \in K$ contrary to our assumption, and this completes the proof of the claim. So in case $a \notin K$ we get $u \in K$, and since $b = su$, therefore $b \in K$. ∎

LEMMA B. *If S is a semigroup in $\mathbf{D}$, then S has a zero element.*

PROOF. Let S^0 denote the semigroup $S \cup 0$ (S with a zero element 0 adjoined). Clearly, S^0 is an S-automaton. Since $S \in \mathbf{D}$, S is an injective S-automaton and hence there exists an S-homomorphism $f : S^0 \to S$ which extends the identity map, 1_S, on S. For each $a \in S$, we have $f(0)a = f(0a) = f(0)$. Thus $f(0)$ is a left zero element of S, call it z. Let s be an arbitrary element of S. As $\{z\}$ and $\{sz\}$ are right ideals, we have $z = sz$, i.e., z is a zero element. ∎

Now we are ready to prove the characterization given in the introduction.

THEOREM. *S is a semigroup for which no S-automaton has proper essential congruences if and only if S is a finite set of idempotents $\{e_1, \dots, e_n\}$ with $e_i e_j = e_k$ where $k = \min\{i, j\}$, $e_i, e_j \in S$.*

PROOF. Let $I_n = \{e_1 = 0, e_2, e_3, \dots, e_n\}$ be a two-sided ideal of S, $S \in \mathbf{D}$, such that (i) for $i \leq j$, $e_i e_j = e_j e_i = e_i$ and (ii) for every $e_i \in I_n$ and $s \in S \setminus I_n$, $se_i = e_i s = e_i$. We prove that if $I_n \neq S$, then we can find an ideal I_{n+1} with $n + 1$ elements containing I_n and satisfying properties similar to (i) and (ii) for I_{n+1}. By Lemma A, $S \setminus I_n$ is a subsemigroup of S. Define

$$(a, b) \in \rho \iff (a, b \in I_n \text{ or } a, b \in S \setminus I_n).$$

This ρ is a proper right congruence and is not essential. Therefore there exists a non-identity right congruence σ on S such that $\sigma \cap \rho = \iota$, that is, there exists $(a, b) \in \sigma$ such that $(a, b) \notin \rho$. So either $a \in I_n$ and $b \in S \setminus I_n$ or the other way round. Without loss of generality we assume the former case. Then $b \in S \setminus I_n$ implies that there is an idempotent $f \in S \setminus I_n$ with $bS^1 = fS^1$. Therefore, $f = bs$ for some $s \in S \setminus I_n$ (note that I_n is a two-sided ideal) and we have

$$(a, b) \in \sigma \implies (a, b)s = (as, bs) = (as, f) \in \sigma.$$

But $as = a$ by part (ii) of the hypothesis, so $(a, f) \in \sigma$. Note that a cannot be zero, because otherwise the zero equivalence class of σ which is a right ideal has to contain I_n and this contradicts $\sigma \cap \rho = \iota$. The only choice for a is e_n. Now consider the right ideal generated by f, say J. This is the ideal that we are looking for. Clearly $I_n \subset J$. Claim: $J = \{e_1 = 0, e_2, e_3, \dots, e_n, f\} = I_n \cup \{f\}$. Suppose there is $j \in J$ such that $j \neq e_i$ for $1 \leq i < n$, i.e., $j \notin I_n$. We will show that $j = f$. Since $fS^1 = J$, $j = ft$ and $t \notin I_n$ (since $j \notin I_n$). Now $(a, f) \in \sigma$, $(a, j) = (at, ft) \in \sigma$, hence $(j, f) \in \sigma$. Also note that $(j, f) \in \rho$, therefore $(j, f) \in \sigma \cap \rho = \iota$, i.e., $j = f$. This completes the proof of the claim.

Take now any $s \in S \setminus J$. Then $fs \in J \setminus I_n$, so $fs = f$. Furthermore, since $sf \notin I_n$ and the right ideals $fS = J$ and $sfS = \{sf\} \cup I_n$ are comparable, $sf = f$ must hold as well, and $I_{n+1} = J$ is a two-sided ideal with the required properties.

Starting with $I_1 = \{0\}$, which exists by Lemma B, we get now an ascending chain of ideals I_i of S:

$$I_1 = \{0\} \subset I_2 \subset I_3 \subset \dots$$

As S satisfies the ascending chain condition on right ideals, we must have $S = I_m$ for some m. This proves the necessity of the condition.

As for the converse, notice that the one-element semigroup I_1 clearly belongs to **D**, and that $I_n = I^1_{n-1}$ for $n \geq 2$; now the statement follows from our very first observation stating that $S \in \mathbf{D}$ implies $S^1 \in \mathbf{D}$ for any semigroup S. ∎

REMARK. A careful analysis of the above considerations yields the following.

PROPOSITION. *If S is a semigroup with zero and a left identity element, then S has no proper essential right congruences if and only if S is a finite set of idempotents $\{e_1, \ldots, e_n\}$ satisfying $e_i e_j = e_k$ with $k = \min\{i, j\}$.* ∎

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STABILITY OF LURIE-TYPE EVOLUTION EQUATIONS WITH MULTIPLE NON-LINEARITIES IN HILBERT SPACES*

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1. Introduction

In paper [1], the author have investigated the problem of Lurie abstract system

$$(1.1)\qquad \begin{cases} \dot{x}(t) = Ax + b\mu \\ \dot{\mu} = \phi(\sigma) \\ \sigma = (c,x) - \rho\mu, \end{cases}$$

where the operator A is either bounded, or is assumed to generate a C_0 strongly continuous group $T(t)$, $t \in (-\infty, +\infty) = E$, on a real space X, with inner product $(\,\cdot\,,\,\cdot\,)$ and norm $|\,\cdot\,|$; $b, c \in X$, $\mu, \rho \in E$, and $\phi : E \to E$ is a continuous uniformly Lipschitzian nonlinear function, which satisfies the property

$$(1.2)\qquad \sigma\phi(\sigma) > 0, \qquad \sigma \neq 0 \quad (\text{so that } \phi(0) = 0).$$
$$|\phi(\sigma)| < k(|\sigma|) \text{ for } \sigma \in E,$$

for some monotonically nondecreasing function $k(s)$, $s \in E^+ = [0, \infty)$. and system

$$(1.3)\qquad \begin{cases} \dot{x}(t) = f(x) + b\mu \\ \dot{\mu} = \phi(\sigma) \\ \sigma = (c,x) - \rho\mu, \end{cases}$$

where b, c, μ and x are as those of (1.1), $f : X \to X$ is a continuous, Frechet differentiable function which Frechet derivative at x given by $A(x)$.

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$\phi : E \to E$ is continuous, uniformly Lipschitzian and $-f$ is a monotonic function, that is, there exists a constant M such that

$$(f(u) - f(v), u - v) \leq M|u - v|^2 \qquad u, v \in X.$$

which are motivated by some special cases arising in ordinary differential equations, neutral functional differential equations, and equations arising in reactor dynamics. In finite dimensional ordinary differential equations, A is a stable $n \times n$ constant matrix, μ is a scalar actuator which controls the system where the equation of the actuator is given by $\dot{\mu} = \phi(\sigma)$. Here the scalar σ represents an error function which is given by $\sigma = c^T x - \rho\mu$, where $c \in E^*$, and $\rho \in (-\infty, +\infty)$. The function $\phi(\sigma)$ denotes the speed nf the response of the actuator to the error signal. This situation has had a long history and was extensive in paper [1] is to show that the simple easily verifiable criteria given by LA SALLE and LEFSCHETZ [3] for the finite dimensional case, holds analogously in infinite dimensions. In this paper, further, using a more general Lyapunov functional, we generalize the results of [1] to the Lurie-type evolution equations with multiple non-linearities in Hilbert spaces.

2. Main results

We consider the system

$$\begin{cases} \dot{x}(t) = Ax + \sum_{i=1}^{m} b_i \mu_i \\ \dot{\mu}_i = \phi_i(\sigma_i) \\ \sigma_i = (c_i, x) - \rho_i \mu_i, \qquad (i = 1, 2, ..., m) \end{cases} \tag{2.1}$$

where the operator A is either bounded, or is assumed to generate a C_0 strongly continuous group $T(t)$, $t \in (-\infty, +\infty) = E$, on a real space X, with inner product $(\cdot, \cdot)$ and norm $|\cdot|$; $b_i, c_i \in X$, $\mu_i, \rho_i \in E$, and $\phi_i : E \to E$ is a continuous uniformly Lipschitzian nonlinear function which satisfies the property

$$\begin{aligned} &\sigma_i \phi_i(\sigma_i) > 0, \qquad \sigma_i \neq 0 \qquad (\text{so that } \phi_i(0) = 0), \\ &|\phi_i(\sigma_i)| \leq k(|\sigma_i|) \qquad \text{for} \qquad \sigma_i \in E, \end{aligned} \tag{2.2}$$

$i = 1, 2, \ldots, m$, for some monotonically nondecreasing function $k(s)$, $s \in E^+$ $+ = [0, \infty)$.

We assume that the linear plant equation

(2.3) $$\dot{x} = Ax$$

is exponentially stable, that is, there are constants $M \geq 1$, $\alpha > 0$ such that

(2.4) $$|T(t)|_{L(x)} \leq Me^{-\alpha t}, \qquad t \geq 0$$

where $L(X)$ is the Banach space of bounded linear operators from X to X. We note that because of condition (2.4) on the system (2.3) we are guaranteed by PAO [1, theorem 2.1] and [1, theorem 2.2] that there is unique symmetric positive definite bounded operator P on X such that

(2.5) $$(PAx, x) + (x, PAx) = -(x, x)$$

where $T(t)$ is a strongly continuous group; and such that

(2.6) $$(A^* Px, x) + (PAx, x)G \leq -\lambda|x|^2$$

for any λ with $0 < \lambda < 1$, if $T(t)$ is a strongly continuous semigroup and A is bounded. A similar result is given by WALKER [2, theorem 9.7] when A satisfies

(2.7)
$$(x, (A - wI)x)_X \leq 0 \qquad \text{for real} \qquad w \in E \qquad \text{for all} \qquad x \in \text{domain of } A$$

It is clear that (2.1) can be viewed in the Hilbert space $H = X \times E$, with inner product $(\cdot, \cdot)$ defined by

$$((x_1, r_1), (x_2, r_2))_H = (x_1, x_2) + r_1 r_2.$$

The condition (2.2) assures us that bounded solutions of (2.1) are global in E [3, p.28], and precompact [3; p.29].

With these basic assumptions, we can now state the problem of Lurie as follows: find conditions on b_i, c_i, ρ_i, μ_i and A, $i = 1, 2, \ldots, m$ such that for any ϕ_i, the zero solution of the system (2.1) is uniformly asymptotically stable in the large. The stability concept is similar to those in [9, p.168].

THEOREM 2.1. *Suppose the origin is the only singular point of* (2.1). *Let P be the unique symmetric positive bounded operator on H given by* (2.6).

(i) *Suppose λ in* (2.6) *satisfies*

(2.8) $$\alpha_i \lambda_i \rho_i > |Pb_i + \alpha_i c_i/2|^2, \qquad i = 1, 2, \ldots, m$$

$\sum_{i=1}^{m} \lambda_i = \lambda$, *and* $\lambda_i > 0$, $\alpha_i > 0$, $i = 1, 2, \ldots, m$.

(ii) *Assume that*

$$\int_0^{\sigma_i} \phi_i(s)ds \to \infty \qquad as \qquad \sigma \to \infty. \tag{2.9}$$

Then the trivial solution of (2.1) *is uniformly asymptotically stable in the large.*

PROOF. Since (2.2) is uniformly exponentially stable, there exists unique symmetric positive definite bounded operator on H such that

$$A^* Px, x) + (PAx, x) \leq -\lambda |x|^2 \tag{2.6}$$

for some λ with $0 < \lambda \leq 1$, we now see P to define the functional on H, namely,

$$V(x,\mu) = \left(Ax + \sum_{i=1}^{m} b_i \mu_i, P\left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) \right) \tag{2.10}$$

Let

$$U(x,\mu) = \sum_{i=1}^{m} \int_0^{\sigma_i} \alpha_i \phi(s) ds, \qquad \sigma_i = (c_i, x) - \rho_i \mu_i$$

and set

$$W = V + U \tag{2.11}$$

Note that $W(x,\mu) = 0$ if and only if $x = 0$, $\mu_i = 0$. Indeed $W = 0$ if and only if $Ax + \sum_{i=1}^{m} b_i \mu_i = 0$ and $\phi(\sigma) = 0$.

Since 0 is the only singular point of (1.1), this last equality holds if and only if $x = 0$ and $\mu_i = 0$. Since $\phi_1(\sigma) = 0$ if and only if $\sigma_i = 0$ we have that $\phi_i(\sigma_i) = 0$ if and only if $(c_i, x) - \phi_i(\sigma_i)\rho_i = 0$. We have prove that $Ax + \sum_{i=1}^{m} b_i \mu_i = 0$ and $(c_i, x) - \phi_i(\sigma_i)\rho_i = 0$ if and only if $x = 0$, $\mu_i = 0$, $i = 1, 2, \ldots, m$. Hence, W is positive definite. Since P is a symmetric and positive definite if and only if

$$\delta |x|^2 \leq |(x, Px)| \leq l |x|^2$$

where $\|P\| \le l$, we have the estimate

$$
\delta \left| Ax + \sum_{i=1}^{m} b_i \mu_i \right|^2 + \sum_{i=1}^{m} \int_0^{\sigma_i} \alpha_i \phi(s) ds \le W(x,\mu) \le
$$

(2.12)

$$
\le l \left| Ax + \sum_{i=1}^{m} b_i \mu_i \right|^2 + \sum_{i=1}^{m} \int_0^{\sigma_i} \alpha_i \phi(s) ds.
$$

We now find the derivative of V along solutions of (1.1):

$$
\dot{V} = \left(\frac{d}{dt} \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right), P \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) \right) +
$$

$$
+ \left(Ax + \sum_{i=1}^{m} b_i \mu_i, P \frac{d}{dt} \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) \right)
$$

then,

$$
\frac{d}{dt} \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) = A\dot{x} + \sum_{i=1}^{m} b_i \dot{\mu}_i = A \left[Ax + \sum_{i=1}^{m} b_i \mu_i \right] + \sum_{i=1}^{m} b_i \phi_i(\sigma_i),
$$

so that

$$
\dot{V} = \left(A \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) + \sum_{i=1}^{m} b_i \phi_i(\sigma_i), P \left(Ax + \sum_{j=1}^{m} b_j \mu_j \right) \right) +
$$

$$
+ \left(Ax + \sum_{j=1}^{m} b_j \mu_j, P \left(A \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right) + \sum_{i=1}^{m} b_i \phi_i(\sigma_i) \right) \right) =
$$

(2.13)

$$
= \left(Ax + \sum_{i=1}^{m} b_i \mu_i, PA \left(Ax + \sum_{j=1}^{m} b_j \mu_j \right) \right) +
$$

$$
+ \left(PA \left(Ax + \sum_{j=1}^{m} b_j \mu_j \right), Ax + \sum_{i=1}^{m} b_i \mu_i \right) +
$$

$$
+ \left(Ax + \sum_{i=1}^{m} b_i \mu_i, P \sum_{i=1}^{m} b_j \phi_j(\sigma_j) \right) + P \left(\sum_{j=1}^{m} b_j \phi_j(\sigma_j), Ax + \sum_{i=1}^{m} b_i \mu_i \right).
$$

And

$$\dot{U} = \sum_{i=1}^{m} \alpha_i \phi_i(\sigma_i)\dot{\sigma}_i = \sum_{i=1}^{m} \alpha_i \left[\left(c_i, Ax + \sum_{j=1}^{m} b_j \mu_j \right) - \rho_i \phi_i(\sigma_i) \right] =$$

$$(2.14) \qquad = \sum_{i=1}^{m} \alpha_i \left(Ax + \sum_{j=1}^{m} b_j \mu_j, c_i \right) \phi_i(\sigma_i) - \sum_{i=1}^{m} \alpha_i \rho_i \phi_i^2(\sigma_i).$$

Hence, using (2.13), (2.14), (2.5)

$$\dot{W} = \left(A^* P \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right), Ax + \sum_{j=1}^{m} b_j \mu_j \right) +$$

$$+ \left(PA \left(Ax + \sum_{i=1}^{m} b_i \mu_i \right), Ax + \sum_{j=1}^{m} b_j \mu_j \right) +$$

$$+ \left(Ax + \sum_{i=1}^{m} b_i \mu_i, P \sum_{j=1}^{m} b_j \phi_j(\sigma_j) \right) + P \left(\sum_{j=1}^{m} b_j \phi_j(\sigma_j), Ax + \sum_{i=1}^{m} b_i \mu_i \right) +$$

$$+ \sum_{i=1}^{m} \left(Ax + \sum_{j=1}^{m} b_j \mu_j, \alpha_i c_i \right) \phi_i(\sigma_i) - \sum_{i=1}^{m} \alpha_i \rho_i \phi_i^2(\sigma_i) \leq$$

$$\leq \lambda \left| Ax + \sum_{j=1}^{m} b_j \mu_j \right|^2 - \sum_{i=1}^{m} \alpha_i \rho_i \phi_i^2(\sigma_i) +$$

$$+2 \left(Ax + \sum_{j=1}^{m} b_j \mu_j, \sum_{i=1}^{m} Pb_i + \alpha_i c_i / 2 \right) \phi_i(\sigma_i).$$

Hence

$$\dot{W} \leq -\sum_{i=1}^{m} \lambda_i \left| Ax + \sum_{j=1}^{m} b_j \mu_j \right|^2 - \sum_{i=1}^{m} \alpha_i \rho_i \phi_i^2(\sigma_i) +$$

$$(2.15) \qquad + 2 \left(\left| Ax + \sum_{j=1}^{m} b_j \mu_j \right| \cdot \left| \sum_{j=1}^{m} Pb_j + \alpha_j c_j / 2 \right| \right) \phi_i(\sigma_i)$$

where $\sum_{i=1}^{m} \lambda_i = \lambda$, and $\lambda_i > 0$, $i = 1, 2, \dots, m$.

The conditions for this quadratic form in $Ax + \sum_{i=1}^{m} b_i \mu_i$ and $\phi_i(\sigma_i)$ given in (2.15) to be negative definite is

$$\alpha_i \lambda_i \rho_i > |Pb_i + \alpha_i c_i/2|^2, \qquad i = 1, 2, \dots, m.$$

Since these inequalities are valid,

$$W \leq -k \left[\left| Ax + \sum_{i=1}^{m} b_i \mu_i \right|^2 + \sum_{i=1}^{m} \rho_i \phi_i^2(\sigma_i) \right] \leq 0 \tag{2.16}$$

for some sufficient small k.

$$\dot{W} = 0 \quad \text{only if} \quad \left| Ax + \sum_{i=1}^{m} b_i \mu_i \right| = 0 \quad \text{and} \quad |\phi_i(\sigma_i)| = 0.$$

This is true only if $x = 0$, $\mu_i = 0$. Therefore

$$\dot{W} = 0 \qquad \text{only if} \qquad x = 0,\ \mu_i = 0.$$

Hence,

$$S = (x, \mu) \in H : \ \dot{W}(x, \mu) = 0 = (0, 0)$$

where $\mu = \text{col}\,(\mu_1, \mu_2, \dots, \mu_m)$. By (5.16)

$$W(x(t), \mu(t)) < W(x(0), \mu(0)), \qquad t \geq 0.$$

By (2.12) every solution $(x, \mu) \in H$ of (2.1) is bounded. Invoking [6, Th 4.1] orbits of (1.1) are precompact sets of H. The invariance principle of Hale [8, p.50] yields that every solution $(x(t), \mu(t)) \to 0$ as $t \to \infty$. This concludes the proof. ∎

3. Nonlinear system

In this section, we consider the system

$$\begin{cases} \dot{x}(t) = f(x) + \sum_{i=1}^{m} b_i \mu_i \\ \dot{\mu}_i = \phi_i(\sigma_i) \\ \sigma_i = (x_i, x) - \rho_i \mu_i, \end{cases} \tag{3.1}$$

where b_i, c_i, μ_i, ρ_i are as before and $f : X \to X$ is a continuous, Frechet differentiable function which Frechet derivative at x given by $A(x)$. To ensure that solutions of (3.1) exist in the future we assume, for example that $\phi : E \to E$ is continuous, uniformly Lipschitzian and that $-f$ is a monotonic function, that is, there exists a constant M such that

$$(f(u) = f(v), u - v) \leq M|u - v|^2, \qquad u, v \in X.$$

We now introduce a basic stability comparison theorem for the system

$$\dot{x} = l(t, x) \tag{3.2}$$

where $l : E^+ \times X \to X$ is continuous, which is adapted from [1,p.982].

THEOREM 3.1. *Assume the following*

(i) $V \in C(E^+ \times X, E^+)$ *and for* $(t, x_1), (t, x_2) \in X^+ \times X$

$$|V(t_1, x) - V(t, x_2)| \leq L(t)|x_1 - x_2| \tag{3.3}$$

where $L(t) > 0$ *and continuous on* E^+.

(ii) *There exists a function* $g \in C(E^+ \times E^+, E)$ *such that for each* $(t, x) \in$
$\in E^+ \times X$

(3.4)

$$D_+ V(t, x) = \lim_{k^+ \to 0} \sup h^{-1}[V(t + h, x + hl(t, x) - V(t, x))] \leq g(t, V(t, x)).$$

(iii) *For each* $(t_0, r_0) \in E^+ \times E^*$ *the maximal solution* $r(t, t_0, r_0)$ *of the scalar initial value problem*

$$\dot{r} = g(t, r), \qquad r(t_0) = r_0 \tag{3.5}$$

exists in future.

$f(t, 0) = 0$, $g(t, 0) = 0$, *and* $V(t, 0) = 0$, $t \in E^+$.

There exist functions $a, b : E^+ \to E^+$ *such that* $b(r)$, $a(r)$ *are increasing in* r *and*

$$b(\|x\|) \leq V(t, x) \leq a(\|x\|) \qquad \text{for } (t, x) \in E^+ \times X. \tag{3.6}$$

Then, if the trivial solution of (3.5) *is uniformly asymptotically stable in the large then so is that of* (3.2).

THEOREM 3.2. *Assume that in* (3.1) $f(0) = 0$, $\phi(0) = 0$ *and that* $A(x)$ *is the Frechet derivative of* $f(x)$ *at* x. *Suppose*

(i) *there is a symmetric positive definite operator* P *such that*

$$((PA(x) + A^*(x)P)y, y) \leq -\lambda|y|^2 \tag{3.7}$$

for all x and y in X and some $\lambda > 0$, where A^* is the adjoint of A;

(ii) $\phi_i(s)\,\mathrm{sgn}\ s > 0$, $\phi_i(s)\,\mathrm{sgn}\ s \to \infty$ *as* $|s| \to \infty$; $\dot{\phi}_i(s) > 1/2\lambda_i$ *for some* $\lambda_1 \geq |P|$;

(iii) $\left| f(x) + \sum_{i=1}^{m} b_i\mu_i \right| \to \infty$ *as* $|x| + \sum_{i=1}^{m} |\mu_i| \to \infty$;

(iv) $\alpha_i\lambda_i\rho_i > |Pb_i + \alpha_i c_i/2|^2$, $i = 1, 2, \ldots, m$.

Then, the origin of (3.1) *is uniformly asymptotically stable in the large.*

PROOF. Let $V : H \to E$ be defined by

$$V = W + U \tag{3.8}$$

where

$$W = \left(f(x) + \sum_{i=1}^{m} b_i\mu_i,\ P\left(f(x) + \sum_{j=1}^{m} b_j\mu_i \right) \right) \tag{3.9}$$

$$U = \sum_{i=1}^{m} \int_0^{\sigma_i} \alpha_i\phi_i(s)ds. \tag{3.10}$$

Since P is positive definite and symmetric, there exist positive constants γ_1, γ_2, such that

$$\gamma_2 \left| f(x) + \sum_{i=1}^{m} b_i\mu_i \right|^2 \leq \gamma_1 \left| f(x) + \sum_{i=1}^{m} b_i\mu_i \right|^2 \tag{3.11}$$

where $\gamma_1 \geq \lambda_{\max}(P)$, hence

$$\begin{aligned} &\gamma_2 \left| f(x) + \sum_{i=1}^{m} b_i\mu_i \right|^2 + \sum_{i=1}^{m} \int_0^{\sigma_i} \alpha_i\phi(s)ds \leq V(x,\mu) \leq \\ &\leq \left| \gamma_1 f(x) + \sum_{i=1}^{m} b_i\mu_i \right|^2 + \sum_{i=1}^{m} \sigma_0^{\sigma_i}\alpha_i\phi(s)ds. \end{aligned} \tag{3.12}$$

By the hypothesis (ii), $\int_0^{\sigma_i} \phi_i(s)ds \to \infty$ as $\sigma_i \to \infty$ so that by the (iii) of theorem 3.1 is fulfilled in view of (3.12).

Obviously, V is differentiable.

Calculating the time derivative of W along solutions of (3.1), we obtain

$$\dot{W} = \left(\frac{d}{dt}\left(f(x) + \sum_{i=1}^{m} b_i\mu_i\right), P\left(f(x) + \sum_{j=1}^{m} b_j\mu_j\right)\right) + \\ + \left(f(x) + \sum_{i=1}^{m} b_i\mu_i, P\frac{d}{dt}\left(f(x) + \sum_{j=1}^{m} b_j\mu_j\right)\right),$$

also,

$$\frac{d}{dt}\left(f(x) + \sum_{i=1}^{m} b_i\mu_i\right) = A(x)\dot{x} + \sum_{i=1}^{m} b_i\dot{\mu}_i = \\ = A(s)\left[f(x) + \sum_{i=1}^{m} b_i\mu_i\right] + \sum_{i=1}^{m} b_i\phi_i(\sigma_i).$$

Thus, we obtain

$$\dot{W} = \left(A(x)\left(f(x) + \sum_{i=1}^{m} b_i\mu_i\right) + \sum_{i=1}^{m} b_i\phi_i(\sigma_i), P\left(f(x) + \sum_{j=1}^{m} b_j\mu_j\right)\right) + \\ + \left(f(x) + \sum_{j=1}^{m} b_j\mu_j, P\left(A(x)\left(f(x) + \sum_{i=1}^{m} b_i\mu_i\right) + \sum_{i=1}^{m} b_i\phi_i(\sigma_i)\right)\right) = \\ = \left(f(x) + \sum_{i=1}^{m} b_i\mu_i, PA(x)\left(f(x) + \sum_{j=1}^{m} b_j\mu_j\right)\right) + \\ + A^*(x)P\left(f(x) + \sum_{j=1}^{m} b_j\mu_j\right), \\ f(x) + \sum_{i=1}^{m} b_i\mu_i 2\left(P\sum_{i=1}^{m} b_i\phi_i(\sigma_i), f(x) + \sum_{j=1}^{m} b_j\mu_j\right).$$

Because of hypothesis (3.7), we deduce from the above that

$$\dot{W} \leq -\lambda\left|f(x) + \sum_{i=1}^{m} b_i\mu_i\right|^2 + 2\left(P\sum_{i=1}^{m} b_i\phi_i(\sigma_i), f(x) + \sum_{j=1}^{m} b_j\mu_j\right). \tag{3.13}$$

Also, the derivative of U along solutions of (3.1) is

$$\dot{U} = \alpha_i \sum_{i=1}^{m} \alpha_i \phi_i(\sigma_i)\dot{\sigma}_i =$$

$$= \sum_{i=1}^{m} \alpha_i \left[\left(C_i, f(x) + \sum_{j=1}^{m} b_j \mu_j \right) - \rho_i \phi_i(\sigma_i) \right] \phi_i(\sigma_i) = \tag{3.14}$$

$$= \sum_{i=1}^{m} \alpha_i \left(f(x) + \sum_{j=1}^{m} b_j \mu_j, c_i \right) \phi_i(\sigma_i) - \sum_{i=1}^{m} \alpha_i \rho_i \phi^2(\sigma_i).$$

Hence, using (3.13), (3.14), we have

$$\dot{V} \le -\lambda \left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right|^2 - \sum_{i=1}^{m} \alpha_i \rho_i \phi_i^2(\sigma_i) +$$

$$+ 2 \left(\sum_{i=1}^{m} \phi_i(\sigma_i)(Pb_i + c_i/2), f(x) + \sum_{j=1}^{m} b_j \mu_j \right).$$

Hence

$$\dot{V} \le -\sum_{i=1}^{m} \lambda_i \left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right|^2 - \sum_{i=1}^{m} \rho_i \phi_i^2(\sigma_i) +$$

$$+ 2 \left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right| \cdot \left| \sum_{j=1}^{m} Pb_j + c_j/2 \right| \cdot |\phi_i(\sigma_i) + \tag{3.15}$$

where $\sum_{i=1}^{m} \lambda_i = \lambda$, and $\lambda_i > 0$, $i = 1, 2, \ldots, m$.

The right-hand side of (3.15) is a quadratic form in

$$\left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right| \quad \text{and} \quad |\phi_i(\sigma_i)|, \quad i = 1, 2, \ldots, m.$$

It is clearly negative definite since

$$\alpha_i \lambda_i \rho_i > |Pb_i + \alpha_i c_i/2|^2, \qquad i = 1, 2, \ldots, m.$$

$$\dot{W} \le -\gamma_3 \left[\left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right|^2 + \sum_{i=1}^{m} \rho_i \phi_i^2(\sigma_i) \right]. \tag{3.16}$$

We now use the inequality

$$V(x,\mu) - \left(\sum_{i=1}^{m} \int_0^{\sigma_i} \phi_i(s)ds\right) /\gamma_1 \leq \left| f(x) + \sum_{i=1}^{m} b_i \mu_i \right|^2 ,$$

From (3.12), we deduce that

$$\dot{V} \leq -\gamma_3 V/\gamma_1 - 1/\gamma_1 \left\{ \sum_{i=1}^{m} \int_0^{\sigma_1} \phi_i(s)ds + \sum_{i=1}^{m} \phi_1^2(\sigma_1) \right\} .$$

Clearly

$$\sum_{i=1}^{m} \phi_i^2(\sigma_i) = 2 \int_0^{\sigma_i} \phi_i(s)\phi_i'(s)ds.$$

Since $\phi_i^2(0) = 0$, therefore

$$\frac{1}{\gamma_1} \sum_{i=1}^{m} \int_0^{\sigma_i} \phi_i(s)ds - \sum_{i=1}^{m} \phi_i^2(\sigma_i) = \sum_{i=1}^{m} \int_0^{\sigma_i} \left[\frac{1}{\gamma_1} - 2\phi_i^2(s)\right] \phi_i(s)ds \leq 0 \tag{3.17}$$

since $\phi_i(s)\,\mathrm{sgn}\; s > 0$ and $\frac{1}{2}\gamma_1 - \phi_i' \leq 0$ by (ii).

With the above remarks we obtain the inequality

$$\dot{V}(x,\mu) \leq \frac{\gamma_3}{\gamma_1} B(x,\mu). \tag{3.18}$$

Thus the comparison equation (3.5) take the form

$$\dot{r}(t) = \frac{\gamma_3}{\gamma_1} r(t), \quad r(0) = r_0. \tag{3.19}$$

It is easy to observe that the solution

$$r(t) = r_0 e^{-\gamma_3/\gamma_1 (t-t_0)}, \qquad t \geq t_0$$

of (3.19) is uniformly asymptotically stable. Theorem 3.1 of [9] yield the uniform asymptotic stability of the trivial solution of (3.1). This concludes the proof. ∎

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CONTRA-SUPER-CONTINUOUS FUNCTIONS

By

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1. Introduction

In 1996, DONTCHEV [3] introduced and investigated the class of contra-continuous functions. He showed that contra-continuity is weaker than strong continuity [11] and stronger than LC-continuity [6]. He also introduced a stronger form of S-closedness [20] called strongly S-closedness and showed that the contra-continuous images of strongly S-closed spaces are compact [3, Theorem 3.8]. Quite recently, DONTCHEV and NOIRI [4] among others introduced the class of RC-continuous functions. They showed that this class lies between the class of contra-continuous functions and B-continuous functions [21].

In this paper, we further investigate the class of RC-continuous functions. We obtain some decomposition theorems of RC-continuity. We also introduce and investigate a new generalized form of continuity called contra-super-continuity. It turns out that contra-super-continuity lies strictly between RC-continuity and contra-continuity.

2. Preliminaries

In this paper, spaces X and Y are always assumed to be topological spaces. Let A be a subset of X. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{Int}\,(\mathrm{Cl}\,(A))$ (resp. $A = \mathrm{Cl}\,(\mathrm{Int}\,(A))$), where $\mathrm{Cl}\,(A)$ (resp. $\mathrm{Int}\,(A)$) denotes the closure (resp. interior) of A. The family of all regular open (resp. regular closed) subsets of X will be denoted by $\mathrm{RO}\,(X)$ (resp. $\mathrm{RC}\,(X)$). The following notions are due to VELIČKO [22]: A point x

in X is said to be δ-*cluster* point of A if $A \cap U \neq \emptyset$ for every $U \in \mathrm{RO}(X)$ containing x. The set of all δ-cluster points of A is called the δ-*closure* of A and is denoted by $\mathrm{Cl}_{\delta}(A)$. If $A = \mathrm{Cl}_{\delta}(A)$ then A is called δ-*closed*. The complement of a δ-closed set is called δ-*open*.

A subset A is called *preopen* [13] (resp. *semi-open* [10], β-*open* [1]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of a preopen set is called *preclosed*. The intersection of all preclosed sets containing a subset A is called the *preclosure* [5] of A and is denoted by $\mathrm{pCl}(A)$. The family of all open (resp. closed, preopen, semi-open, β-open) subsets of X will be denoted by $\mathrm{O}(X)$ (resp. $\mathrm{C}(X)$, $\mathrm{PO}(X)$, $\mathrm{SO}(X)$, $\beta(X)$). We set $\mathrm{O}(X,x) = \{U \in \mathrm{O}(X) \mid x \in U\}$ for $x \in X$. We define similarly $\mathrm{C}(X,x)$, $\mathrm{PO}(X,x)$, $\mathrm{SO}(X,x)$, $\beta(X,x)$ and $\mathrm{RO}(X,x)$.

DEFINITION 2.1. A function $f : X \to Y$ is called *precontinuous* [13] (resp. *semi-continuous* [10], β-*continuous* [1]) if for each $x \in X$ and each $V \in \mathrm{O}(X,f(x))$, there exists $U \in \mathrm{PO}(X,x)$ (resp. $U \in \mathrm{SO}(X,x)$, $U \in$ $\in \beta(X,x)$) such that $f(U) \subset V$.

DEFINITION 2.2. A function $f : X \to Y$ is called *contra-continuous* [3] (resp. *contra-precontinuous* [7]) if $f^{-1}(U)$ is closed (resp. preclosed) in X for each open set U of Y.

REMARK 2.1. Every contra-continuous function is contra-precontinuous but the converse is not true as the following example shows.

EXAMPLE 2.1. Let $X = \{a,b\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. The identity function $f : (X,\tau) \to (X,\sigma)$ is contra-precontinuous but not contra-continuous.

3. RC-continuous functions

DEFINITION 3.1. A function $f : X \to Y$ is called RC-continuous [4] if $f^{-1}(U)$ is regular closed in X for each open set U of Y.

As a decomposition theorem of RC-continuity, DONTCHEV and NOIRI [4, Theorem 3.11] proved that a function is RC-continuous if and only if it is β-continuous and contra-continuous. We need the following lemma to obtain further decomposition theorems of RC-continuity.

LEMMA 3.1. *The following properties are equivalent for a subset A of a space X:*

(1) $A \in \mathrm{RC}(X)$*;*

(2) *A is semi-open and closed;*

(3) *A is semi-open and preclosed.*

THEOREM 3.1. *The following properties are equivalent for a function $f : X \to Y$:*

(1) *f is* RC *-continuous;*

(2) *f is semi-continuous and contra-continuous;*

(3) *f is semi-continuous and contra-precontinuous.*

PROOF. The proof is obvious by Lemma 3.1.

Recall that a space is called *extremally disconnected* if the closure of every open set in it is open.

A function $f : X \to Y$ is said to be *perfectly continuous* [17] if the preimage $f^{-1}(V)$ of every open set V of Y is clopen in X.

THEOREM 3.2. *If $f : X \to Y$ is* RC *-continuous and X is extremally disconnected, then f is perfectly continuous.*

PROOF. Let V be any open set of Y. Since f is RC -continuous, $f^{-1}(V)$ is a regular closed set of X. Since X is extremally disconnected and Cl (Int $(f^{-1}(V))) = f^{-1}(V)$, $f^{-1}(V)$ is open in X and hence f is perfectly continuous.

Recall that a space X is said to be *preconnected* [18] if X cannot be expressed as the union of two nonempty preopen sets.

THEOREM 3.3. *If $f : X \to Y$ is a contra-precontinuous surjection and X is preconnected, then Y is connected.*

PROOF. Suppose that Y is not connected. There exist nonempty open sets V_1 and V_2 of Y such that $V_1 \cup V_2 = Y$ and $V_1 \cap V_2 = \emptyset$. Therefore, V_1 and V_2 are closed and open in Y and hence $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are preopen in X since f is contra-precontinuous. Moreover, we have $f^{-1}(V_1) \cup f^{-1}(V_2) =$ $= X$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Therefore, X is not preconnected.

COROLLARY 3.1. *If $f : X \to Y$ is an* RC *-continuous surjection and X is preconnected, then Y is connected.*

4. Contra-super-continuous functions

In this section, we introduce a new type of continuity called contra-super-continuity which is weaker than RC-continuity and stronger than contra-continuity.

DEFINITION 4.1. A function $f : X \to Y$ is called contra-super-continuous if for each $x \in X$ and each $V \in \mathrm{C}\,(Y, f(x))$, there exists $U \in \mathrm{RO}\,(X, x)$ such that $f(U) \subset V$.

REMARK 4.1. We have the following implications:

RC-continuity $\Longrightarrow$ contra-super-continuity $\Longrightarrow$ contra-continuity.

But the converses need not be true as the following two examples show.

EXAMPLE 4.1. The *digital line* or the so-called *Khalimsky line* [8], [9] is the set of all integers $\mathbb{Z}$, equipped with the topology κ, generated by subbase $\mathfrak{T}_\kappa = \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$. Let $(\mathbb{Z}, \kappa)$ be the digital line and $f : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ be a function defined as follows: $f(x) = 0$ if x is odd; $f(x) = 1$ if x is even. Then, f is contra-super-continuous. For any closed set F of $(\mathbb{Z}, \kappa)$, we have $f^{-1}(F) = \mathbb{Z}$ if $0, 1 \in F$; $f^{-1}(F)$ is all odd numbers if $0 \in F$ and $1 \notin F$ and hence $f^{-1}(F)$ is δ-open in $(\mathbb{Z}, \kappa)$. Since $f^{-1}(0)$ is dense in $(\mathbb{Z}, \kappa)$, $f^{-1}(0)$ is not regular open and f is not RC-continuous.

EXAMPLE 4.2. Let $X = \{a, b\}$ be the Sierpiński space by setting $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, X\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is contra-continuous [3, Example 2.5] but not contra-super-continuous.

DEFINITION 4.2. Let A be a subset of a space (X, τ). The set $\cap\{U \in \tau \,|\, A \subset U\}$ is called the *kernel* of A [14] and is denoted by ker(A). In [12], the kernel of A is called the *Λ-set*.

LEMMA 4.1. *The following properties hold for subsets A, B of a space X:*

(1) $x \in \ker(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any* $F \in \mathrm{C}\,(X, x)$.

(2) $A \subset \ker(A)$ *and* $A = \ker(A)$ *if A is open in X.*

(3) *If* $A \subset B$, *then* $\ker(A) \subset \ker(B)$.

PROOF. Obvious. (2) and (3) are stated in [12, p. 140].

THEOREM 4.1. *The following are equivalent for a function $f : X \to Y$:*

(1) f *is contra-super-continuous;*

(2) *For each* $x \in X$ *and each* $V \in C(Y, f(x))$, *there exists* $U \in O(X, x)$ *such that* $f(\mathrm{Int}(\mathrm{Cl}(U))) \subset V$;

(3) *The preimage of any closed set of* Y *is* δ*-open in* X;

(4) *The preimage of any open set of* Y *is* δ*-closed in* X;

(5) $f(\mathrm{Cl}_\delta(A)) \subset \ker(f(A))$ *for every subset* A *of* X;

(6) $\mathrm{Cl}_\delta(f^{-1}(B)) \subset f^{-1}(\ker(B))$ *for every subset* B *of* Y.

PROOF. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (3) $\Longleftrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (5): Let A be any subset of X. Suppose that $y \notin \ker(f(A))$. Then by Lemma 4.1 there exists $F \in C(Y, y)$ such that $F \cap f(A) = \emptyset$; hence $f^{-1}(F) \cap \mathrm{Cl}_\delta(A) = \emptyset$. Therefore, we have $F \cap f(\mathrm{Cl}_\delta(A)) = \emptyset$ and $y \notin f(\mathrm{Cl}_\delta(A))$. Consequently, we obtain $f(\mathrm{Cl}_\delta(A)) \subset \ker(f(A))$.

(5) $\Longrightarrow$ (6): Let B be any subset of Y. By (5) and Lemma 4.1, we have

$$f(\mathrm{Cl}_\delta(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$$

and $\mathrm{Cl}_\delta(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(6) $\Longrightarrow$ (4): Let V be any open set of Y. Then, by Lemma 4.1 we have

$$\mathrm{Cl}_\delta(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$$

This shows that $f^{-1}(V)$ is δ-closed in X.

(4) $\Longrightarrow$ (1): Let $x \in X$ and $V \in C(Y, f(x))$. Since (3) and (4) are equivalent, $f^{-1}(V)$ is δ-open in X and $x \in f^{-1}(V)$. Hence there exists $U \in RO(X, x)$ such that $U \subset f^{-1}(V)$. Therefore, we obtain $f(U) \subset V$.

COROLLARY 4.1. *A function* $f : X \to Y$ *is contra-super-continuous if and only if* $f : X_s \to Y$ *is contra-continuous, where* X_s *denotes the semi-regularization of* X.

Recall that a function $f : X \to Y$ is called super-continuous [15] (resp. strongly θ-continuous [16]) if for each $x \in X$ and each $V \in O(Y, f(x))$, there exists $U \in O(X, x)$ such that $f(\mathrm{Int}(\mathrm{Cl}(U))) \subset V$ (resp. $f(\mathrm{Cl}(U)) \subset V$).

THEOREM 4.2. *If a function* $f : X \to Y$ *is contra-super-continuous and* Y *is regular, then* f *is super-continuous and strongly* θ*-continuous.*

PROOF. Let $x \in X$ and $V \in O(Y, f(x))$. By the regularity of Y, there exists $V_0 \in O(Y, f(x))$ such that $\mathrm{Cl}(V_0) \subset V$. By Theorem 4.1, there exists

$U \in O(X,x)$ such that $f(\text{Int}\ (\text{Cl}\ (U))) \subset \text{Cl}\ (V_0)$. Next, we shall show that $f(\text{Cl}\ (U)) \subset V$. Suppose that $y \notin V$. Since $f^{-1}(Y - V)$ is δ-open and $U \cap f^{-1}(Y - V) = \emptyset$, we have $\text{Cl}\ (U) \cap f^{-1}(Y - V) = \emptyset$ and hence $f(\text{Cl}\ (U)) \cap (Y - V) = \emptyset$. This shows that $f(\text{Cl}\ (U)) \subset V$. Therefore, f is strongly θ-continuous.

5. Contra-super-closed graphs

Recall that for a function $f : X \to Y$ the subset $\{(x,f(x)) \mid x \in X\} \subset \subset X \times Y$ is called the *graph* of f and is denoted by $G(f)$.

DEFINITION 5.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be *contra-super-closed* if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in \text{RO}\ (X,x)$ and $V \in C\ (Y,y) such that (U \times V) \cap G(f) = \emptyset$.

LEMMA 5.1. *The graph of $f : X \to Y$ is contra-super-closed in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in \text{RO}\ (X,x)$ and $V \in C\ (Y,y) such that f(U) \cap V = \emptyset$.*

DONTCHEV [3] introduced and investigated the notion of strong S-closedness. By definition, a space X is called *strongly S-closed* if every closed cover of X has a finite subcover.

A subset A of a space X is *strongly S-closed* if the subspace A is strongly S-closed.

THEOREM 5.1. *If $f : X \to Y$ has a contra-super-closed graph, then the inverse image of a strongly S-closed set K of Y is δ-closed in X.*

PROOF. Let K be a strongly S-closed set of Y and $x \notin f^{-1}(K)$. For each $k \in K$, $(x,k) \notin G(f)$ and by Lemma 5.1 there exist $U_k \in \text{RO}\ (X,x)$ and $V_k \in C\ (Y,k)$ such that $f(U_k) \cap V_k = \emptyset$. Since $\{K \cap V_k | k \in K\}$ is a closed cover of the subspace K, there exists a finite subset K_0 of K such that $K \subset \cup\{V_k \mid k \in K_0\}$. Set $U = \cap\{U_k \mid k \in K_0\}$, then $U \in \text{RO}\ (X,x)$ and $f(U) \cap K = \emptyset$. Therefore, we have $U \cap f^{-1}(K) = \emptyset$ and $f^{-1}(K)$ is δ-closed in X.

THEOREM 5.2. *If $f : X \to Y$ has a contra-super-closed graph and Y is strongly S-closed, then f is contra-super-continuous.*

PROOF. Let V be any open set of Y. Since Y is strongly S-closed, V is a strongly S-closed set of Y and by Theorem 5.1 $f^{-1}(V)$ is δ-closed in X. Therefore, by Theorem 4.1 f is contra-super-continuous.

COROLLARY 5.1. *Let Y be a strongly S-closed and Urysohn space. Then, the following properties are equivalent for a function $f : X \to Y$*

(1) *$G(f)$ is contra-super-closed;*

(2) *$f^{-1}(K)$ is δ-closed in X for every strongly S-closed set K of Y;*

(3) *f is contra-super-continuous.*

PROOF. (3) $\implies$ (1): Let $(x, y) \in (X \times Y) - G(f)$. Then $f(x) \neq y$ and there exist open sets V and W of Y such that $f(x) \in V$, $y \in W$ and $\mathrm{Cl}\,(V) \cap \cap \mathrm{Cl}\,(W) = \emptyset$. By (3), there exists $U \in \mathrm{RO}\,(X, x)$ such that $f(U) \subset \mathrm{Cl}\,(V)$; hence $f(U) \cap \mathrm{Cl}\,(W) = \emptyset$. By Lemma 5.1, $G(f)$ is contra-super-closed. Recall that a subset A of a space X is said to be N-closed relative to X [2] if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of A by open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{\mathrm{Int}\,(CI(V_\alpha)) \mid \alpha \in \nabla_0\}$. A space X is said to be *nearly compact* [19] if the set X is N-closed relative to X.

THEOREM 5.3. *If $f : X \to Y$ is contra-super-continuous and K is N-closed relative to X, then $f(K)$ is strongly S-closed in Y.*

PROOF. Let K be N-closed relative to X and $\{F_\alpha \mid \alpha \in \nabla\}$ be a closed cover of the subspace $f(K)$. For each $\alpha \in \nabla$, there exists a closed set C_α of Y such that $F_\alpha = f(K) \cap C_\alpha$. For each $x \in K$, there exist $\alpha(x) \in \nabla$ and $U_x \in \mathrm{O}(X, x)$ such that $f(\mathrm{Int}\,(\mathrm{Cl}\,(U_x))) \subset C_{\alpha(x)}$. Since $\{U_x \mid x \in \in K\}$ is a cover of K by open sets of X, there exists a finite subset K_0 of K such that $K \subset \cup\{\mathrm{Int}\,(\mathrm{Cl}\,(U_x)) \mid x \in K_0\}$. Therefore, we obtain $f(K) \subset \cup\{f(\mathrm{Int}\,(\mathrm{Cl}\,(U_x))) \mid x \in K_0\} \subset \cup\{C_{\alpha(x)} \mid x \in K_0\}$ and hence $f(K) = \cup\{F_{\alpha(x)} \mid x \in K_0\}$. This shows that $f(K)$ is strongly S-closed in Y.

COROLLARY 5.2. *If $f : X \to Y$ is a contra-super-continuous surjection and X is nearly compact, then Y is strongly S-closed.*

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ON TWO QUESTIONS OF F. SZÁSZ

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0. In this paper we characterize the rings R in which the ideal nR is a ring direct summand of R for every integer n. This answers one question posed by F. SZÁSZ (see [1, Problem 79]). We also give a characterization of rings in which every element is a left multiplier (see [1, Problem 82]). F. SZÁSZ was the first who has characterized rings whose all elements are left multipliers [2]. Our result describes such rings more precisely.

Throughout this paper p is a prime. We assume that all rings are associative. For a ring R, R^+ denotes the additive group of R, $|x|$ the order of an element $x \in R$ in R^+, $\mathcal{F}(R) = \{r \in R \mid nr = 0$ for some nonzero integer $n\}$, $\exp(R^+)$ the exponent of R^+, and $\langle X \rangle$ the subgroup of R^+ generated by a subset X.

In the sequel we will use the following notations: $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{Z}_{p^n}$ the ring of integers modulo a prime power p^n. All other notations and terminology can be found, for example, in [3], [4].

1. THEOREM 1. *Let R be an associative ring. Then the ideal nR is a ring direct summand of R for every integer n if and only if R is one of the following types:*

(i) $R = (0)$.

(ii) *R is a $\mathbb{Q}$-algebra.*

(iii) *$R = \sum_p {}^{\oplus} R_p$ is a ring direct sum of R_p, where R_p^+ is the p-component of R^+ and every ring R_p satisfies one of the following conditions:*

1) *$R_p^2 = (0)$ and R_p^+ is a divisible group;*

2) $R_p = A \oplus B$ *is a ring direct sum, where* $A^2 = (0)$, A^+ *is a divisible group, and* B^+ *is a group of exponent* p;

3) R_p *is a ring of characteristic* p.

(iv) $R = \mathcal{F}(R) \oplus K$ *is a ring direct sum with* $\mathcal{F}(R)$ *of type* (iii) *and* K *of type* (ii).

The proof of the "if" part of Theorem 1 is obvious. We prove the "only if" part. Therefore let R be a ring in which the ideal nR is a ring direct summand of R for every integer n.

LEMMA 1.1. *If the additive group* R^+ *is torsion-free then* R *is a* $\mathbb{Q}$-*algebra.*

PROOF. If R^+ is not divisible then nR is a proper ideal of R for some integer n and therefore

$$R = nR \oplus S$$

is a ring direct sum with a nontrivial ideal S. Then

$$nS < nR \cap S$$

and consequently $nS = (0)$, a contradiction. Hence R^+ is a divisible group. The lemma is proved.

LEMMA 1.2. *If* R^+ *is a* p-*group then* R *satisfies one of the following conditions:* (1) $R^2 = (0)$ *and* R^+ *is a divisible group.*

(2) $R = A \oplus B$ *is a ring direct sum, where* $A^2 = (0)$, A^+ *is a divisible group, and* B^+ *is a group of exponent* p.

(3) R *is a ring of characteristic* p.

PROOF. If the additive group R^+ is divisible then obviously $R^2 = (0)$. Suppose that R^+ is not divisible and pR is a proper ideal of R for some prime p. Then

$$R = pR \oplus S$$

is a ring direct sum, where S is a nontrivial ideal of R with $pS = (0)$. It is easy to see that $(pR)^+$ is a divisible group and consequently $(pR)^2 = (0)$. The lemma is proved.

COROLLARY 1.3. *If R^+ is torsion then R is a ring of type* (iii).

LEMMA 1.4. *If the additive group R^+ is mixed then R is a ring of type* (iv).

PROOF. (a) Suppose that R^+ is a divisible group. Then the torsion part $\mathcal{F}(R)$ is also divisible and by Theorem 21.3 of [3]

$$R^+ = \mathcal{F}(R) \oplus K$$

is a ring direct sum, where K is a nontrivial divisible torsion-free group. By lemma 1.2

$$\mathcal{F}(R)^2 = (0).$$

Further, clearly that

$$\mathcal{F}(R)K = K\mathcal{F}(R) = (0)$$

and consequently R is of type (iv).

(b) Now suppose that R^+ is a non-divisible group and therefore

$$R = nR \oplus A$$

is a ring direct sum for some integer n and some nontrivial ideal A. By Lemma 9.4 of [3]

$$\mathcal{F}(R)^+ = n\mathcal{F}(R) \oplus (A \cap \mathcal{F}(R))$$

is a group direct sum. This means that $\mathcal{F}(R)$ satisfies the conditions (1), (2) or (3) of Lemma 1.2. Finally, it is easy to see that

$$(R) \oplus K$$

is a ring direct sum, where K is a nontrivial $\mathbb{Q}$-algebra. The lemma is proved. ∎

Lemmas 1.1, 1.4 and Corollary 1.3 give the "only if" part of Theorem 1.

2. Following [1, Problem 82] an element a of a ring R is called a left multiplier if there exists an integer n such that

$$(a + n)R = (0). \tag{1}$$

The following theorem rectifies Szász' result [2].

THEOREM 2. *Let R be an associative ring. Then every element of R is a left multiplier if and only if R is one of the following types:*

(α) $R^2 = (0)$.

(β) $R \cong \mathbb{Z}_{p^n}$.

(γ) *$R \cong m\mathbb{Z}$ ($m \in \mathbb{Z}$).*

(δ) *there exists an element e in R such that $R = A + e \cdot \mathbb{Z}$, where A is the left annihilator of R, $ex = mx$ for all elements x in R, and $R/A \cong m\mathbb{Z}$ (m is a nontrivial integer).*

(ϵ) *there exists an element $e \in R$ such that $R = A + e \cdot \mathbb{Z}_{p^n}$, where A is the left annihilator of R and $ex = x$ for all elements x in R.*

(ζ) *$R = \sum_{i=1}^{m} {}^{\oplus} R_i$ is a ring direct sum, where R_i^+ is the p_i-component of R^+, $\exp(R_i^+) = p_i^{m_i}$, $R_s = A_s + e_s \cdot \mathbb{Z}_{p_s m_s}$, A_s is the left annihilator of R_s, $R_j = e_j \cdot \mathbb{Z}_{p_j m_j}$, $R_t^2 = (0)$, $e_h x = -n_h x$ for all elements x in R and, furthermore, $n_h \equiv -1 \pmod{p_h^{m_h}}$ and $n_h \equiv 0 \pmod{exp(R_l^+)}$ for the integers $l \in \{1, \ldots, m\} \setminus \{h\}$ ($1 \le k \le n \le m$; $s = 1, \ldots, k$; $j = k+1, \ldots, n$; $t = n+1, \ldots, m$; $h = 1, \ldots, n$).*

Only the “only if” part requires proof. In what follows R will be a ring in which every element is a left multiplier.

LEMMA 2.1. *If the additive group R^+ is torsion-free then R is a ring of type (α), (γ) or (δ).*

PROOF. Firstly, suppose that R is a nontrivial ring without zero-divisors. Let a and b be any nontrivial elements of R. Then there are integers n and m such that

$$ab = -nb, \qquad ba = -ma$$

and therefore

$$(ab - ba)a = -nba + ma^2 = -nma + mna = 0.$$

By our hypothesis, this yields that

$$-nb = ab = ba = -ma.$$

We conclude that R^+ is an abelian group of rank 1 (see [4, chapter XIII]). By Theorem 121.1 of [4] there is an isomorphism

$$f : R \to m\mathbb{Z}(q_j^{-1}; j \in J),$$

where m is a nontrivial integer, $\{q_j \mid j \in J\}$ is a set of primes such that q_j and m are relatively prime. Suppose that J is a nonempty set and $q \in \{q_j \mid j \in J\}$. Then there exists an element r of R such that

$$f(qr) = m.$$

Since

$$rx = -sx$$

for some integer s and for all x in R, we have

$$mf(x) = f(qr)f(x) = f(qrx) = qf(rx) = -qsf(x)$$

and consequently

$$m = -qs,$$

a contradiction. Hence the set J is empty and

$$R \cong m\mathbb{Z}.$$

Now suppose that

$$ar = 0$$

for some nontrivial elements a and r of R. Then

$$arb = 0$$

for every b in R. Since

$$rb = -sb$$

for some integer s, we have that

$$aR = (0).$$

Denote $\{a \in R \mid aR = (0)\}$ by A. Assume that $A \neq R$. Then $A^2 = (0)$ and as proved above there is an isomorphism

$$g : R/A \to m\mathbb{Z}$$

for some nontrivial integer m. Let e be an element of R such that

$$g(e + A) = m.$$

Then

$$A \cap e \cdot \mathbb{Z} = (0)$$

and

$$R = A + e \cdot \mathbb{Z}.$$

By our hypothesis, there is an integer l such that

$$ex = -lx$$

for all x in R. Then

$$m^2 = (g(e + A))^2 = g((e + A)^2) = -lg(e + A) = -lm$$

and consequently

$$m = -l.$$

The lemma is proved. ∎

LEMMA 2.2. *The additive group R^+ is either torsion-free or torsion.*

PROOF. Suppose that R^+ is mixed. Let a be a nontrivial element of finite order and b be a nontrivial element of infinite order in R^+. Since there is an integer n with condition (1), we have that

$$nb \in \mathcal{F}(R),$$

a contradiction. The lemma is proved. ∎

LEMMA 2.3. *If R^+ is a p-group then it is a group of finite exponent.*

PROOF. Let a be a nontrivial element of R which satisfies condition (1). Suppose that R^+ is a group of infinite exponent. Then there is an element c of R^+ such that

$$n|a| < |c|.$$

Put $d = |a|c$. Then

$$0 = |a|ac = ad = -nd \neq 0,$$

a contradiction. The lemma is proved. ∎

LEMMA 2.4. *If R^+ is a p-group then R is a ring of one of the types: (α), (β) or (ϵ).*

PROOF. Suppose that R^2 is nontrivial and take any nontrivial elements $a, b \in R$. Then by the hypothesis there are integers m and n such that

$$at = -nt, \qquad bt = -mt$$

for all t in R. Hence

$$(ab - ba)r = a(br) - b(ar) = mnr - nmr = 0 \tag{2}$$

for every element r of R. Denote $\{a \in R \mid aR = (0)\}$ by A.

(a) Suppose that $A = (0)$. Then R is a commutative ring and (2) yields that

$$ma = nb.$$

Further, by Lemma 2.3 R^+ is a group of exponent p^k for some positive integer k. Suppose that $|a| = p^k$ and $|b| = p^l$, where $1 \leq l \leq k$. If $ma = nb = 0$ then

$bt = -mt = 0$ for every $t \in R$, and so $b \in A$, a contradiction. Hence nb is nontrivial and there exists the integers n_0 and x such that

$$n = n_0 p^x,$$

where $0 \leq x < l$, n_0 and p are relatively prime. It is easy to see that there is an integer m_0 such that

$$m = m_0 p^{k-l+x},$$

where m_0 and p are relatively prime. Since

$$(n_0 b - m_0 p^{k-l} a)t = (-n_0 m + m_0 p^{k-l} n)t = 0$$

for every $t \in R$, our hypothesis yields that

$$n_0 b - m_0 p^{k-l} a = 0.$$

This yields that the subgroup

$$\langle p^{k-l} a, b \rangle = \langle p^{k-l} a \rangle = \langle b \rangle$$

is cyclic and consequently $\Omega_l(R^+) = \{x \in R \mid |x| \leq p^l\}$ is a locally cyclic group. Hence $\Omega_l(R^+) \cong \mathbb{Z}_{p^l}$ and therefore

$$R \cong \mathbb{Z}_{n^k}.$$

(b) Now suppose that A is nontrivial. Then as stated above

$$R/A \cong \mathbb{Z}_{p^k}$$

for a positive integer k. Let u be an inverse image in R of the identity element of R/A. Then

$$u^2 - u = d$$

for some element d of A and consequently

$$ud = du = 0.$$

Denote $u + d$ by e. Then

$$e^2 = (u + d)^2 = u^2 = u + d = e$$

and e is a nontrivial idempotent of R. Since there exists $s \in \mathbb{Z}$ such that

$$ex = -sx \tag{3}$$

for all x in R, we have that

$$s \equiv -1 \pmod{|e|}. \tag{4}$$

Assume that

$$|e| < |a| = \exp(R^{+}).$$

Then

$$0 = |e|ea = -s|e|a$$

and consequently

$$s \equiv 0 \pmod{p},$$

a contradiction. Thus

$$|e| = exp(R^{+})$$

and in view of (3) and (4)

$$ex = x.$$

Therefore

$$d = ed = (u + d)d = ud = 0.$$

Hence R is a noncommutative ring of exponent p^k and

$$R = A + e \cdot \mathbb{Z}_{p^k}.$$

The lemma is proved. ∎

REMARK 2.5. Let $R = A \oplus B$ be a ring direct sum, where $B \cong \mathbb{Z}_{p^n}$ and A is a nontrivial ring such that A^{+} is a p-group. Then R contains an element which is not a left multiplier. Indeed, if e is the identity element of B then there exists an integer m such that

$$ex = mx$$

for all x in R and, in particular,

$$ma = ea = 0 \tag{5}$$

for every nontrivial element a of A. On other hand, $e^2 = e$ and so m and p are relatively prime, contradicting (5).

COROLLARY 2.6. *If R^{+} is torsion then R is a ring of one of the types: (α), (β), (ϵ) or (ζ).*

PROOF. Suppose that R^2 is nontrivial and R^{+} is a non-primary group. Then

$$R^{+} = \sum_{p\in\pi}{}^{\oplus} R_p$$

is a group direct sum, where π is some set of primes, R_p is the p-component of R^+ (see [3, Theorem 8.4]). It is easy to see that there is a prime p such that R_p^2 is nontrivial and by Lemma 2.4 $\exp(R^+) = p^n$ for some integer n.

Let a_p be an element of R_p such that $a_p R_p \neq (0)$. By the hypothesis, there is an integer n_p with

$$a_p x = -n_p x$$

for all x in R. If $q \in \pi \setminus \{p\}$ and $x \in R_q$ then

$$n_p \equiv 0 \pmod{\exp(R_q^+)}.$$

From this it follows that $\pi = \{p_1, \dots, p_m\}$ is finite. Hence

$$R = R_1 \oplus \dots \oplus R_k \oplus R_{k+1} \oplus \dots \oplus R_n \oplus \dots \oplus R_m$$

is a ring direct sum, where R_i^+ is the p_i-component of R^+, $\exp(R_i^+) = p_i^{m_i}$, $R_s = A_s + e_s \cdot \mathbb{Z}_{p_s} m_s$, $A_s R_s = (0)$, $R_j = e_j \cdot \mathbb{Z}_{p_j} m_j$, $R_t^2 = (0)$, $e_h x = -n_h x$ for all x in R, and, furthermore, $n_h \equiv -1 \pmod{p_h^{m_h}}$ and $n_h \equiv 0 \pmod{\exp(R_r^+)}$ for all integers $r \in \{1, \dots, m\} \setminus \{h\}$ $(1 \leq k \leq n \leq m;\ i = 1, \dots, m;\ s = 1, \dots, k;\ j = k+1, \dots, n;\ t = n+1, \dots, m;\ h = 1, \dots, n)$. This completes the proof.

Now Theorem 2 follows from Lemmas 2.1, 2.2, 2.4 and Corollary 2.6.

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ON WEIGHTED $(0,2)$-INTERPOLATION ON INFINITE INTERVAL $(-\infty,+\infty)$

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1. Introduction

E. EGERVÁRY and P. TURÁN discovered the (0,2)-interpolation in order to get approximate solutions of the differential equation:

$$y'' + fy = 0 \tag{1.1}$$

Their results have been studied by several mathematicians. These interpolation polynomials usually cannot be determined uniquely. Moreover, their explicit forms are very complicated, which have been obtained first by K. K. MATHUR and A. SHARMA [8] taking the nodes as the roots of the n^{th} Hermite polynomial in the case of n even. They have also proved that there exist infinitely many solutions when n is odd. To avoid these difficulties P. TURÁN suggested to study the problem of:

Weighted $(0,2)$-interpolation. Let (a,b) be a finite or infinite interval such that

$$-\infty < a < x_{n,n} < \ldots < x_{1,n} < b < \infty \qquad (n \in \mathbf{N}) \tag{1.2}$$

and $w \in C^2(a,b)$ be a weight function. How can a polynomial R_n of lowest possible degree satisfying the conditions:

$$R_n(x_{i,n}) = y_{i,n}, \qquad (wR_n)''(x_{i,n}) = y''_{i,n}, \qquad i = 1(1)n \tag{1.3}$$

be determined where $y_{i,n}$ and $y''_{i,n}$ are arbitrary real numbers? J. BALÁZS [1] was the first to settle this problem on the roots of n^{th} Ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$ with weight function $w(x) = (1-x^2)^{\alpha+1/2}$, $(x \in [-1,-1])$. He proved that generally there does not exit any polynomial of degree $\leq 2n-1$ satisfying the conditions (1.3). Taking an additional condition:

$$R_n(0)=\sum_{i=1}^{n} y_{i,n} l_{i,n}^2(0) \tag{1.4}$$

where 0 is not a nodal point belonging to (1.2), he showed that there does exist a unique polynomial of degree $\leq 2n$ (n even) and also proved a convergence theorem. If n is odd, then uniqueness ie not true.

L. SZILI [12] studied the analogous problem on nodes as the zeros of the n^{th} Hermite polynomial $H_n(x)$, with weight function $w(x)=e^{-x^2/2}$. For arbitrary n consider the polynomials

$$\overline{A}_{i,n}(x)=\frac{l_{i,n}^2(x)}{2}+(n+1-x_{i,n})\frac{H_n(x)}{H_n'(x_{i,n})}\infty_0^x l_{i,n}(t)dt+ \tag{1.5}$$

$$+\frac{H_n'(x)}{2H_n'(x_{i,n})}l_{i,n}(x)-x\frac{H_n(x)}{H_n'(x_{i,n})}l_{i,n}(x)-\frac{H_n(x)}{2H_n'(x_{i,n})}l_{i,n}'(0), \qquad i=1(1)n$$

$$\overline{B}_{i,n}(x)=\frac{e^{x_{i,n}^2/2}H_n(x)}{2H_n'(x_{i,n})}\int_0^x l_{i,n}(t)dt, \qquad i=1(1)n. \tag{1.6}$$

For n even SZILI [12] established that

$$\overline{R}_n(x)=\sum_{i=1}^{n} y_{i,n}\overline{A}_{i,n}(x)+\sum_{i=1}^{n} y_{i,n}''\overline{B}_{i,n}(x) \tag{1.7}$$

is the uniquely determined polynomial of degree $\leq 2n$ satisfying the condition (1.3) and (1.4). Furthermore he proved that if the function $f:\mathbf{R}\to\mathbf{R}$, is continuously differentiable such that

$$\lim_{|x|\to\infty} x^{2r}w(x)f(x)=0, \qquad r=1,2,\ldots \quad \text{and} \quad \lim_{|x|\to\infty} w(x)f'(x)=0 \tag{1.8}$$

then for $\overline{R}_n$ given by (1.7), together with

$$y_{i,n}=f(x_{i,n}),\; y_{i,n}''=O\left(\sqrt{n}e^{\beta x_{i,n}^2}\omega\left(f',\frac{1}{n}\right)\right) \tag{1.9}$$

$$i=1(1)n; \quad n=2,4,\ldots, \quad 0\leq\beta\leq\frac{1}{2}$$

and

$$\overline{R}_n(0) = \sum_{i=1}^{n} f(x_{i,n}) l_{i,n}^2(0) \tag{1.10}$$

the following estimates holds:

$$e^{-yx^2} |f(x) - \overline{R}_n(f, x)| = O\left(\omega\left(f', \frac{1}{\sqrt{n}}\right) \log n\right), \qquad x \in \mathbf{R} \tag{1.11}$$

where $y > 1$, O does not depend on n and x and $\omega(f, \delta)$ is the Freud modulus of continuity.

Later I. JOÓ [7] sharpened this result and showed that

$$e^{x^2} \left|f(x) - \overline{R}(f, x)\right| = O\left(\omega\left(f', \frac{1}{\sqrt{n}}\right)\right) + O\left(\frac{1}{\sqrt{n}}\right), \tag{1.12}$$

holds for all $x \in R$. He improved (1.11) by eliminating factor $\log n$, taking $\beta = \frac{1}{2}$ in (1.9) and replacing $\gamma > 1$ by $\gamma = 1$.

In SZILI'S, BALÁZS'S and many other papers on weighted (0, 2)-interpolation end PÁL type interpolation processes, results have been obtained under the special condition (1.4), which looks to be artificial. Also, in almost every lacunary interpolation formula, which satisfy the conditions (1.3) and (1.4), it has been proved that for odd n the interpolation polynomial of degree $\leq 2n$ either does not exist or if it exists there are infinitely many. In this connection, we raise the following:

PROBLEM. For each positive integer n does there exist a unique weighted (0, 2)-interpolatory polynomial R_n, of degree $\leq 2n$ satisfying the condition (1.3) and

$$\begin{gathered} R_n(0) = y_{0,n}, \qquad \text{if } n \text{ is even} \\ \text{or} \\ R_n'(0) = y_{0,n}', \qquad \text{if } n \text{ is odd.} \end{gathered} \tag{1.13}$$

If it exists what will be its explicit form and does it converge?

REMARK. The referee points out that J. BALÁZS [3] and L. SZILI [13] have also studied. analogous problems for weighted (0, 2)-interpolation and T. F. XIE [14], L. G. PÁL [9] and Z. F. SEBESTYÉN [10] have investigated such type of modification in Pál interpolation.

In this paper, we answer this problem in affirmative, taking the roots of the n^{th} Hermite polynomial $H_n(x)$ as nodes.

In section 2, we give some preliminaries and state new results in section 3. The estimates of the fundamental polynomials and the convergence theorems have been proved in sections 4 and 5 respectively.

2. Preliminaries

Let $H_n(x)$ be the n^{th} Hermite polynomial with usual normalization

$$\int_{-\infty}^{+\infty} H_n(t)H_m(t)e^{-t^2}dt = \sqrt{\pi}2^n n!\delta_{n,m}, \qquad (n,m \in \mathbf{N}) \tag{2.1}$$

which satisfies the differential equation:

$$\begin{cases} H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \\ H_n'(x) = 2nH_{n-1}(x). \end{cases} \tag{2.2}$$

It is well known that the roots $x_{i,n}$ of $H_n(x)$ satisfy the following relations:
(2.3)

$$\begin{cases} -\infty < x_{n,n} < \ldots < x_{\frac{n}{2}+1,n} < 0 < x_{\frac{n}{2},n} < \ldots < x_{1,n} < +\infty & (n = 2m) \\ -\infty < x_{n,n} < \ldots < x_{\frac{n-1}{2},n} = 0 < \ldots < x_{1,n} < +\infty & (n = 2m+1) \\ x_{i,n} = -x_{n-i+1,n} & \left(i = 1, 2, \ldots, \frac{n}{2}\right) \end{cases}$$

Let $l_{i,n}$, denote the Lagrange fundamental polynomial corresponding to the nodal points $x_{i,n}$, then

$$l_{i,n}(x) = \frac{H_n(x)}{H_n'(x_{i,n})(x - x_{i,n})}, \qquad i = 1(1)n \tag{2.4}$$

leading to

$$l_{i,n}'(x_{i,n}) = x_{i,n}. \tag{2.5}$$

For the roots of $H_n(x)$, we have

$$x \sim \frac{i^2}{n}, \qquad i = 1(1)n \tag{2.6}$$

$$H_n(x) = O(1)n^{-1/4}\sqrt{2^n n!}\left(1 + \sqrt[3]{|x|}\right)e^{x^2/2}, \qquad x \in \mathbf{R} \tag{2.7}$$

$$|H_n'(x_{i,n})| \geq c2^{n+1}\left(\frac{n}{2}\right)!e^{\delta x_{i,n}^2}, \qquad 0 < \delta < 1 \tag{2.8}$$

$$\sum_{i=1}^{n} \frac{e^{\delta^2 x_{i,n}^2}}{H_n'(x_{i,n})^2} = O\left(\frac{1}{2^{n+1}n!}\right), \qquad 0<\delta<\delta^*<1 \tag{2.9}$$

$$\sum_{i=1}^{n} e^{x_{i,n}^2} l_{i,n}^2(x) = O\left(e^{x^2}\right) \tag{2.10}$$

$$|H_n(0)| = \frac{n!}{\left(\frac{n}{2}\right)!} \qquad \text{for even } n \tag{2.11}$$

$$\frac{2^n\left(\left(\frac{n}{2}\right)!\right)^2}{(n+1)!} \sim n^{-1/2}. \tag{2.12}$$

The above results have been taken from a paper of L. SZILI [12], we shall also require the following estimates given by I. JOÓ [7].

$$\sum_{i=1}^{n} w^{x_{i,n}^2/2} |\overline{A}_{i,n}(x)| = O\left(\sqrt{n}e^{x^2}\right) \tag{2.13}$$

and

$$\sum_{i=1}^{n} e^{x_{i,n}^2/2} |\overline{B}_{i,n}(x)| = O\left(\frac{e^{x^2}}{\sqrt{n}}\right), \tag{2.14}$$

where $\overline{A}_{i,n}(x)$ and $\overline{B}_{i,n}(x)$ are given by (1.5) and (1.6) respectively.

We shall use the following notations in the sequel.

$$x_i = x_{i,n}, \qquad l_i = l_{i,n} \qquad \overline{A}_i = \overline{A}_{i,l}, \qquad \overline{B}_i = \overline{B}_{i,n}.$$

3. New results

THEOREM 1. *If the nodal points are the roots of the n^{th} Hermite polynomial $H_n(x)$ and the weight function is $w(x) = e^{-x^2/2}$ ($x \in \mathbf{R}$), then there exists a unique polynomial R_n of degree $\le 2n$, satisfying the conditions* (1.3) *and* (1.13) *for n odd.*

THEOREM 2. *Let*

$$A_i(x) = l_i^2(x) - \frac{H_n(x)}{H_n'(x)}\left[\int_0^x \frac{l_i'(t) - x_i l_i(t)}{(t-x_i)}dt - \frac{(1-x_i^2)}{2}\int_0^x l_i(t)dt\right], \tag{3.1}$$

$$i = 1(1)n$$

$$(3.2)\qquad B_i(x) = \frac{e^{x_i^2/2} H_n(x)}{2H_n'(x_i)} \int_0^1 l_i(t)dt, \qquad i = 1(1)n$$

and

$$(3.3)\qquad C_0(x) = \frac{H_n(x)}{H_n'(0)} \qquad (x_{\frac{n+1}{2},n} = 0).$$

Then

$$(3.4)\qquad R_n(x) = \sum_{i=1}^{n} y_i A_i(x) + \sum_{i=1}^{n} y_i'' B_i(x) + y_0' C_0(x)$$

is the uniquely determined polynomial of degree $\leq 2n$ *satisfying the conditions (1.3) and (1.13), if* n *is odd.*

THEOREM 3. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuously differentiable function such that*

$$(3.5)\qquad \lim_{|x|\to\infty} x^{2r} f(x) e^{-x^2/2} = 0, \quad r = 0, 1, 2, \ldots \text{ and } \lim_{|x|\to\infty} f'(x) e^{-x^2/2} = 0.$$

Further, let

$$(3.6)\qquad y_1 = f(x_i), \qquad y_i'' = O\left(e^{x_i^2/2} \omega\left(f', \frac{1}{\sqrt{n}}\right)\right), \qquad i = 1(1)n \quad \text{and} \quad y_0' = f'(0),$$

then the interpolatory polynomial $R_n(f, x)$ $(n = 3, 5, \ldots)$ *given by (3.4) satisfies the estimate:*

$$(3.7)\qquad e^{-x^2} |f(x) - R_n(f, x)| = O(1)\omega\left(f', \frac{1}{\sqrt{n}}\right), \qquad x \in \mathbf{R}$$

where O *does not depend on* n *and* x *and* $\omega(f', \delta)$ *is the Freud modulus of continuity of* f'.

In the case of n *even, analogously to Theorem 1, there does exist a weighted* $(0, 2)$*-interpolatory polynomial* R_n^* *of degree* $\leq 2n$, *satisfying the conditions (1.3) and (1.13). Further, let*

$$(3.8)\qquad A_i^*(x) =$$

$$= \begin{cases} \frac{H_n(x)}{H_n(0)}, & \text{for } i = 0 \ (x_0 = 0) \\ \frac{x l_i^2(x)}{x_i} - \frac{H_n(x)}{H_n'(x_i)} \left[\frac{1}{x_i} \int_0^x \frac{t\{l_i'(t) - x_i l_i(t)\}}{(t - x_i)} dt + \frac{(1+x_i^2)}{2} \int_0^x l_i(t) dt \right], & i = 1(1)n \end{cases}$$

and

$$B_i^*(x) = \frac{e^{x_i^2/2} H_n(x)}{H_n'(x_i)} \int_0^x l_i(t) dt, \qquad i = 1(1)n. \tag{3.9}$$

Then

$$R_n^*(x) = \sum_{i=0}^{n} y_i^* A_i^*(x) + \sum_{i=1}^{n} y_i''^* B_i^*(x) \tag{3.10}$$

is the uniquely determined polynomial of degree $\leq 2n$ *satisfying the conditions (1.3) and (1.13) for* n *even.*

THEOREM 4. *Let a function* $f : \mathbf{R} \to \mathbf{R}$ *be continuously differentiable, satisfying the requirements* (3.5) *and, suppose the numbers* y_i^* *and* $y_i''^*$ *are such that*

$$y_i^* = f(x_i), \qquad i = 0(1)n$$

$$y_i''^* = O\left(e^{x_i^2/2} \omega\left(f', \frac{1}{\sqrt{n}}\right)\right), \qquad i = 1(1)n.$$

Then for interpolatory polynomial R_n^* *given by* (3.10)*, we have the estimate:*

$$e^{-x^2} |f(x) - R_n^*(f,x)| = O(1)\omega\left(f', \frac{1}{\sqrt{n}}\right), \qquad x \in \mathbf{R},\ n = 2,\ 4,\ \ldots$$

where O *does not depend on* n *and* x. $\omega(f',\delta)$ *is the Freud modulus of continuity of* f'.

We shall prove only our main Theorems 3 and 4, because the proof of the other theorems are quite similar to that of the theorems in [1].

4. Basic estimates with respect to the fundamental polynomials (n odd)

LEMMA 1. *$A_i(x)$ given in (3.1) can be represented in a convenient form as: For $x_i \neq 0$*

$$A_i(x) = \overline{A}_i(x) - \frac{(1-x_i^2)H_n(x)}{2H_n'(x_i)} \int_0^x l_i(t)dt, \tag{4.1}$$

where $\overline{A}_i(x)$ is given by (1.5).

For $x_i = 0$

$$A_i(x) = \frac{l_i^2(x)}{2} + \frac{(1-x_i^2+2n)H_n(x)}{2H_n'(x_i)} \int_0^x l_i(t)dt + \tag{4.2}$$

$$+\frac{H_n'(x)}{2H_n'(x_i)} l_i(x) - x\frac{H_n(x)}{H_n(x_i)} l_i(x).$$

PROOF. Let us consider the case $x_i > 0$ and assume that $x < 0$. Since $A_i(x)$ is a polynomial, (4.1) is enough to be proved for such x. The case for $x_i < 0$ can be treated similarly. By (2.2) and (2.4), it follows that
(4.3)

$$x l_i(x) - l_i'(x) = \frac{x - x_i}{2}\left[l_i''(x) - 2xl_i'(x) + 2nl_i(x)\right] \qquad x \in \mathbf{R},\ i = 1(1)n.$$

By (3.1) and (4.3), we get

$$A_i(x) = l_i^2(x) - \frac{(1+x_i^2)H_n(x)}{2H_n'(x_i)} \int_0^x l_i(t)dt +$$

$$+\frac{H_n(x)}{2H_n'(x_i)} \int_0^x \left[l_i''(t) - 2tl_i'(t) + 2nl_i(t)\right] dt =$$

$$= \frac{l_i^2(x)}{2} + \frac{(1-x_i^2+2n)H_n(x)}{2H_n'(x_i)} \int_i^x l_i(t)dt +$$

$$+\frac{H_n'(x)}{2H_n'(x_i)} l_i(x) - x\frac{H_n(x)}{H_n'(x_i)} l_i(x) - \frac{H_n(x)}{2H_n'(x_i)} l_i'(0) =$$

$$= \overline{A}_i(x) - \frac{(1-x_i^2)H_n(x)}{2H_n'(x_i)} \int_0^x l_i(t)dt.$$

For $x_i = 0$, (3.1) reduces to
(4.4)

$$A_i(x) = l_i^2(x) + \frac{(1-x_i^2)H_n(x)}{2H_n'(x_i)} \int_i^x l_i(t)dt - \frac{H_n(x)}{H_n'(x_i)} \int_0^x \frac{tH_n(t)-H_n(t)}{t^3}dt.$$

Integrating the last term by parts and using

$$\lim_{t\to 0} \frac{tH_n'(t)-H_n(t)}{t^2} = \lim_{t\to 0} \frac{H_n''(t)}{2} = 0 \tag{4.5}$$

we get (4.2), which completes the proof of the lemma. ∎

LEMMA 2. *Let n be odd, then*

$$\sum_{i=1}^n e^{x_i^2/2}|A_i(x)| = O\left(e^{x^2}\sqrt{n}\right) \tag{4.6}$$

$$\sum_{i=1}^n e^{x_i^2/2}|B_i(x)| = O\left(\frac{e^{x^2}}{\sqrt{n}}\right) \tag{4.7}$$

and

$$|C_0(x)| = O\left(\frac{e^{x^2}}{\sqrt{n}}\right). \tag{4.8}$$

PROOF. We remark that (2.7)–(2.10), (2.13) and (2.14) are also valid for n odd, then we have

$$\sum_{i=1}^n w^{x_i^2/2} \frac{|(1-x_i^2)|\;|H_n(x)|}{2|H_n'(x_i)|} \left|\int_0^x l_i(t)dt\right| = O(1)\frac{e^{x^2}}{\sqrt{n}} \tag{4.9}$$

(see [7], lemma 4). By lemma 1
(4.10)

$$\sum_{i=1}^n e^{x_i^2/2}|A_i(x)| = \sum_{i=1}^n e^{x_i^2/2}|\overline{A}_i(x)| + \sum_{i=1}^n e^{x_i^2/2} \frac{|(1-x_i^2)|\;|H_n(x)|}{2|H_n'(x_i)|} \left|\int_0^x l_i(t)dt\right|,$$

which owing to (2.13) and (4.9), proves (4.6).

By (3.2) and (1.6), we have

$$\sum_{i=1}^{n} e^{x_i^2/2}|B_i(x)| = \sum_{i=1}^{n} e^{x_i^2/2}|\overline{B}_i(x)|, \tag{4.11}$$

which proves (4.7) owing to (2.14).

From (3.3) using (2.7), (2.8) and (2.12), we get (4.8), which completes the proof of the lemma. ∎

LEMMA 3 ([5] Theorem 4 and [4], Theorem 1). *If* $f \in C'(\mathbf{R})$,

$$\lim_{x\to\pm\infty} x^{2r} f(x)w(x) = 0, \quad r = 0, 1, \dots \quad \textit{and} \quad \lim_{x\to\pm\infty} f'(x)w(x) = 0,$$

then there exists a polynomial $p_n(x)$, *of degree* $\leq n$, *such that for* $x \in \mathbf{R}$,

$$w(x)|f(x) - p_n(x)| = O(1)\frac{1}{\sqrt{n}}\omega\left(f, \frac{1}{\sqrt{n}}\right) \tag{4.12}$$

$$w(x)|f'(x) - p_n'(x)| = O(1)\omega\left(f', \frac{1}{\sqrt{n}}\right). \tag{4.13}$$

Furthermore ([12], Lemma 4), we have for $x \in \mathbf{R}$

$$w(x)|p_n(x)| = O(1) \tag{4.14}$$

$$w(x)|p_n'(x)| = O(1) \tag{4.15}$$

and

$$w(x)|p_n''(x)| = O(1)\sqrt{n}\,\omega\left(f', \frac{1}{\sqrt{n}}\right) \quad \textit{for } |x| < \sqrt{2n+1}. \tag{4.16}$$

PROOF OF THEOREM 3. Let n be odd. From the uniqueness of the polynomial $R_n(x)$ in (3.4), it follows that every polynomial Q_n of degree $\leq 2n$ satisfies the relation:

$$Q_n(x) = \sum_{i=1}^{n} Q_n(x_i)A_i(x) + \sum_{i=1}^{n}(wQ_n)''(x_i)B_i(x) + Q_n'(0)C_0(x). \tag{4.17}$$

Let p_n be a polynomial of degree $\leq 2n$ satisfying Lemma 3, then we have

$$e^{-x^2}|f(x) - R_n(x) \leq e^{-x^2}|f(x) - p_n(x)| + e^{-x^2}|p_n(x) - R_n(x)| = \tag{4.18}$$

$$= O(1)\left[e^{-x^2/2}e^{-x^2/2}|f(x) - p_n(x)| + e^{-x^2}\left|\sum_{i=1}^{n}(f_n(x_i) - p_n(x_i))A_i(x)\right| + \right.$$

$$\left. + e^{-x^2}\left|\sum_{i=1}^{n}\{(wp_n)''(x_i) - y_i''\}B_i(x)\right| + e^{-x^2}|f'(0) - p_n'(0)|\;|C_o(x)|\right].$$

Using Lemmas 2 and 3, we have

$$e^{-x^2}|f(x) = R_n(x)| = \tag{4.19}$$

$$= O(1)\left[\omega\left(f',\frac{1}{\sqrt{n}}\right) + e^{-x^2}\sum_{i=1}^{n}|y_i''B_i(x)| + e^{-x^2}\sum_{i=1}^{n}w(x_i)|p_n''(x_i)B_i(x)| + \right.$$

$$\left. + e^{-x^2}\sum_{i=1}^{n}|w'(x_i)|p_n'(x_i)B_i(x)| + e^{-x^2}\sum_{i=1}^{n}|w''(x_i)||p_n(x_i)B_i(x)|\right].$$

By Lemma 2 and 3, we have

$$e^{-x^2}\sum_{i=1}^{n}|w''(x_i)||p_n(x_i)B_i(x)| = O(1)\frac{1}{\sqrt{n}} \tag{4.20}$$

$$e^{-x^2}\sum_{i=1}^{n}|w'(x_i)||p_n'(x_i)B_i(x)| = O(1)\frac{1}{\sqrt{n}} \tag{4.21}$$

and

$$e^{-x^2}\sum_{i=1}^{n}w(x_i)|p_n''(x_i)B_i(x)| = O(1)\omega\left(f',\frac{1}{\sqrt{n}}\right). \tag{4.22}$$

By (3.7), Lemma 2, 3 and (4.19)–(4.22), the theorem follows.

5. Basic estimates with respect to the fundamental polynomials (n even)

LEMMA 4. *For n even, $A_i^*(x)$ given by* (3.8), *can be written in a convenient form as:*

$$A_i^*(x) = \overline{A}_i(x) - \frac{(1-x_i^2)H_n(x)}{2H_n'(x_i)}\int_0^x l_i(t)dt + \frac{H_n(x)l_i(0)}{x_iH_n'(x_i)}, \quad i = 1(1)n \tag{5.1}$$

where $\overline{A}_i(x)$ is given by (1.5).

The lemma follows exactly on the same steps as given in Lemma 1, so we omit its proof.

LEMMA 5. *For n even*

$$\sum_{i=0}^{n}e^{x_i^2/2}|A_i^*(x)| = O(1)\left(\sqrt{n}e^{x^2}\right) \tag{5.2}$$

and

$$\sum_{i=1}^{n} e^{x_i^2/2}|B_i(x)| = O(1)\left(\frac{e^{x^2}}{\sqrt{n}}\right), \tag{5.3}$$

where $A_i^*(x)$ *and* $B_i^*(x)$ *are given by* (5.1) *and* (3.9) *respectively.*

PROOF. When $i = 0$, $x_0 = 0$, then we have

$$e^{x_i^2/2}|A_0^*(x)| = |A_0^*(x)| = \frac{|H_n(x)|}{|H_n(0)|} = O(1)e^{x^2}. \tag{5.4}$$

Since $|1 - x_i^2| \le e^{x_i^2}$, from (2.14), we have

$$\sum_{i=1}^{n} \frac{|(1 - x_i^2)|e^{x_i^2/2}|H_n(x)|}{2|H_n'(x_i)} \left|\int_0^x l_i(t)dt\right| = \sum_{i=1}^{n} e^{x_i^2}|\overline{B}_i(x)| = O(1)\left(\frac{e^{x^2}}{\sqrt{n}}\right). \tag{5.5}$$

Also

$$\sum_{i=1}^{n} e^{x_i^2/2}\frac{|H_n(x)||H_n(0)|}{x_i^2 H_n'(x_i)^2} = \sum_{0<x_i\le 1} e^{x_i^2/2}\frac{|H_n(x)||H_n(0)|}{x_i^2 H_n'(x_i)^2} +$$

$$+\sum_{x_i>1} e^{x_i^2/2}\frac{|H_n(x)||H_n(0)|}{x_i^2 H_n'(x_i)^2} \equiv I_1 + I_2. \tag{5.6}$$

Thus by (2.7) and (2.9), we have

$$I_2 \le |H_n(x)||H_n(0)|\sum_{i=1}^{n} \frac{e^{x_i^2/2}}{H_n'(x_i)^2} = O(1)\left(\frac{e^{x^2/2}}{\sqrt{n}}\right). \tag{5.7}$$

Also by (2.6)–(2.8) and (2.12), we have

$$I_1 \le |H_n(x)||H_n(0)|\sum_{i=1}^{n} \frac{e^{x_i^2/2}}{x_i^2 H_n'(x_i)^2} = \tag{5.8}$$

$$O(1)\frac{n^{-1/2}2^n n! e^{x^2/2}}{2^{2n}\left[\left(\frac{n}{2}\right)!\right]^2} n\log n = O(1)e^{x^2}\log n.$$

From (5.6)–(5.8), we get

$$\sum_{i=1}^{n} e^{x_i^2/2}\frac{|H_n(x)||H_n(0)|}{x_i^2 H_n'(x_i)^2} = O(1)e^{x^2}\log n. \tag{5.9}$$

Thus by (5.4), (5.5) and (5.9), (5.2) follows.

Also, by (3.9), (1.6) and (2.14), (5.3) follows. Hence the lemma is proved. ∎

PROOF OF THEOREM 4. Following the same steps as in the proof of Theorem 3, the Theorem follows. We omit details. ∎

REMARKS. An analogous problem can be raised on the zeros of ultraspherical polynomial as nodes which we shall take up elsewhere.

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QUASIMINIMAL REPRODUCING QUADRATIC SPLINE INTERPOLATION

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1. Introduction

Let $\Omega_n = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be an equidistant partition of the real interval $[a,b]$ for a natural n with $h = \frac{b-a}{n}$. The function $s : [a,b] \mapsto \mathbf{R}$ is a quadratic spline on Ω_n — in short: $s \in S_2(\Omega_n)$ —, if $s \in C^1[a,b]$ and $s|_{[x_{k-1},x_k]} \in \mathbf{P}_2$, $k = 1,\ldots,n$, i.e. the restriction of s to any subinterval is an at most second degree polynomial. Further, $s \in S_2(\Omega_n)$ is an interpolant for a given function $f : [a,b] \mapsto \mathbf{R}$, if $s(x_i) = f(x_i)$, $i = 0,\ldots,n$.

These interpolatory conditions do not determine s uniquely: it remains a free parameter. Usually the additional (initial) condition $s'(x_0) = f'(x_0)$ is required, if $f'(x_0)$ exists [8], or one interpolates at other points than the knots $(x_i)_{i=0}^n$ ([3], p. 255) and/or periodicity of f is assumed [6], [7].

The aim of the paper is to specify the free parameter in a natural manner, using some guiding principles, cf. (3) and (4). To this we shall need the B-spline basis $(B_{2,i})_{i\in\mathbf{Z}}$, for which the sequence $(x_i)_0^n$ has to be extended by "virtual" knots $x_i = a + ih$ for $i < 0$ and $i > n$. Then, for any integer i,

$$B_{2,i}(x) = \frac{1}{2h^2}\begin{cases} (x - x_i)^2, & x \in [x_i, x_{i+1}],\\ h^2 + 2h(x - x_{i+1}) - 2(x - x_{i+1})^2, & x \in [x_{i+1}, x_{i+2}],\\ (x - x_{i+3})^2, & x \in [x_{i+2}, x_{i+3}],\\ 0 & \text{else.} \end{cases}$$

The nontrivial function values and derivatives at the knots are computed to be $B_{2,i}(x_{i+1}) = B_{2,i}(x_{i+2}) = \frac{1}{2}$, and $B'_{2,i}(x_{i+1}) = -B'_{2,i}(x_{i+2}) = \frac{1}{h}$. Since

$\dim(S_2(\Omega_n)) = n + 2$, any $s \in S_2(\Omega_n)$ can be uniquely written as

$$s = \sum_{j=-2}^{n-1} c_j B_{2,j},$$

and a quadratic spline interpolant for f satisfies the linear equations

$$s(x_i) = \sum_{j=-2}^{n-1} c_j B_{2,j}(x_i) = \frac{1}{2}c_{i-2} + \frac{1}{2}c_{i-1} = f(x_i), \qquad i = 0,\ldots,n. \tag{1}$$

Introducing the upper bidiagonal $(n+1) \times (n+2)$ matrix

$$M = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix},$$

the column vector $c = (c_i)_{i=-2}^{n-1}$ of unknown coefficients, and the column vector $\hat{f} = (f(x_i))_{i=0}^{n}$ of function values, (1) assumes

$$Mc = \hat{f}. \tag{2}$$

PROBLEM SETTING. Given Ω_n and f, find a solution c of (2) such that

$$f \in \mathbf{P}_2 \Rightarrow s \equiv f, \tag{3}$$

i.e. the method is reproducing, and

$$Md = \hat{f} \Rightarrow \|d\| \geq \|c\|, \tag{4}$$

i.e. $\|c\|^2 = \sum\limits_{i=-2}^{n-1} c_i^2$ is minimal.

Unfortunately, the above requirements are conflicting.

EXAMPLE 1. Let $\Omega_2 = \{-1, 0, 1\}$ and $f(x) = x(x+1)/2$. Then the minimal solution of (2) is given by $c_{\min} = \frac{1}{2}(1, -1, 1, 3)^\top$, giving

$$s_{\min}(x) = \begin{cases} x(x+1), & x \in [-1, 0], \\ x, & x \in [0, 1]. \end{cases}$$

On the other hand, the spline with coefficients $c_{\text{rep}} = \frac{1}{4}(1, -1, 1, 7)^\top$ is reproducing ($s_{\text{rep}} = f$), however, $\|c_{\text{rep}}\|^2 = 3.25 > 3 = \|c_{\min}\|^2$. ∎

Hence we have to weaken the requirement 'minimality' (the reproducing property being more essential). For this aim, we re-scale the four B-splines with smallest and largest indices by introducing

$$\tilde{B}_{2,-2} = \frac{1}{\alpha_n} B_{2,-2}, \qquad \tilde{B}_{2,-1} = \frac{1}{\beta_n} B_{2,-1},$$

$$\tilde{B}_{2,n-2} = \frac{1}{\beta_n} B_{2,n-2}, \qquad \tilde{B}_{2,n-1} = \frac{1}{\alpha_n} B_{2,n-1}.$$

(Take into account their smaller support for explanation.)

Then requirement (4) becomes the condition

$$(4') \qquad \alpha_n^2 c_{-2}^2 + \beta_n^2 c_{-1}^2 + c_0^2 + \ldots + c_{n-3}^2 + \beta_n^2 c_{n-2}^2 + \alpha_n^2 c_{n-1}^2 \to \min$$

of quasiminimality.

Consequently, in the final problem setting we have to determine the conditions for α_n, β_n to yield a reproducing (see (3)) and quasiminimal (see (4′)) solution of (2). The calculations result in the following.

If n is odd, one obtains the equations

$$(5) \qquad 2\alpha_n^2 + 1 = 2\beta_n^2$$

and

$$(6) \qquad 2n(n+1)\alpha_n^2 + (n-1)^2 = 2n(n-1)\beta_n^2$$

with the unique solution

$$\alpha_n^2 = \frac{n-1}{4n}, \qquad \beta_n^2 = \frac{3n-1}{4n}.$$

If n is even, we get only one equation,

$$(7) \qquad 2(n+1)\alpha_n^2 + n - 2 = 2(n-1)\beta_n^2,$$

the one parameter-solutions of which are

$$\alpha_n^2 = \frac{n+t}{4(n+1)}, \qquad \beta_n^2 = \frac{3n+t-4}{4(n-1)}, \qquad t \in \mathbf{R},$$

cf. (9) below as a special case for $t = 1$.

The following tableau gives the relations necessary to reproduce the powers 1, id and id^2 in the quasiminimal way (4′). (The missing number indicates that, for n even, $f = 1$ is reproduced automatically.)

f	n even	n odd
1	–	(5)
id	(7)	(5)
id^2	(7)	(6)

A remark from linear algebra makes the method more convenient to use.

OBSERVATION. Let A be a $k \times (k+1)$ full rank matrix, b a column vector of length k, and r be a row vector of length $k+1$. If r is orthogonal to the rows of A, then the minimal solution of the system $Ax = b$ coincides with the unique solution of

$$\begin{pmatrix} A \\ r \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

a linear system with a *square* invertible $(k+1) \times (k+1)$ matrix.

Applying this for $A = M$, $b = \hat{f}$, $x = c$, the corresponding row vector r is easily calculated to be $(-1, 1, -1, \ldots)$. However, the presence of α_n and β_n changes the situation, and the final form of the quasiminimal reproducing quadratic interpolating spline (cf. (2), (3), (4′)) is that of the system of linear equations

$$\begin{pmatrix} M \\ r \end{pmatrix} c = \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix}, \tag{8}$$

with

$$r = (-\alpha_n^2, \beta_n^2, -1, 1, \ldots, (-1)^n, (-1)^{n+1}\beta_n^2, (-1)^{n+2}\alpha_n^2).$$

Here the row vector r can be replaced by its any nonzero scalar multiple, since the corresponding linear equation is homogeneous. For instance, in the next section we will investigate the case

$$n \text{ even}, \quad \alpha_n^2 = 1/4, \; \beta_n^2 = 3/4, \tag{9}$$

and, we multiply the corresponding row by 2 such that all diagonal elements of the matrix are the same.

As an illustration, see (8), (9) for $n = 4$:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 3 & -4 & 4 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ 0 \end{pmatrix}.$$

It is a critical point to determine the derivatives of the spline at knots. The simple form of the last row with parameters (9) enables us to give explicit formulas for them. Let $s'(:) = (s'(x_i))_{i=0}^n$ be a column vector, and define the matrix M_1 by

$$s'(x_i) = \sum_{j=-2}^{n-1} c_j B'_{2,j}(x_i), \qquad i = 0, \ldots, n \Leftrightarrow s'(:) = \frac{1}{h} M_1 c.$$

In view of (8) it holds that

$$s'(:) = \frac{1}{h} M_1 \begin{pmatrix} M \\ r \end{pmatrix}^{-1} \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix},$$

giving a connection between the function values and the spline derivatives. Define $\tilde{Q}$ by dropping the last column of the matrix above, i.e. let

$$M_1 \begin{pmatrix} M \\ r \end{pmatrix}^{-1} = (\tilde{Q}, last),$$

with 'last' being indifferent. Then $\tilde{Q}$ is a square matrix, and so is $Q = n\tilde{Q}$, which will prove to be integer. With these matrices we obviously have

$$s'(:) = \frac{1}{h} \tilde{Q} \hat{f} = \frac{1}{b-a} Q \tilde{f}. \tag{10}$$

To sum up, the quite regular structure of Q in this formula enables us to give an algebraic treatment, in contrast to "midpoint interpolation", (interpolation at the averages of adjacent knots), see e.g. the same title of section 2 in [7]: "*Estimation* of the Inverse of the Coefficient Matrix".

REMARK. It is worth calling the attention of the reader, that in the next section B-splines $B_{2,i}$ no more occur. The very simi1ar notation B_{2i} will stand for the Bernoulli numbers with subscript $2i$.

2. Results

In this section we give several formulas for the derivatives of the spline calculated by (8), (9). Note that the knowledge of the value $s'(x_i)$ for some i ($i = 1, \ldots, n-1$) suffices to construct the spline on the interval $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, hence first we examine the matrix Q. Notice that row and column one are provided by subscript zero.

LEMMA 1. *The matrix* $Q = (q_{i,j})_{i,j=0}^{n}$ *defined above is an* $(n+1)$*-th order skew centrosymmetric matrix, i.e.* $q_{i,j} = -q_{n-i,n-j}$, $i, j = 0, \ldots, n$. *Its elements are:*

$$q_{0,0} = 1 - 2n,\ q_{i,i} = 2(2i-n),\ 1 \le i \le n-1,\ q_{n,n} = 2n-1,$$

$$q_{i,0} = (-1)^{i},\ 1 \le i \le n,\ q_{i,n} = (-1)^{i+1},\ 0 \le i \le n-1,$$

$$q_{i,j} = 4j(-1)^{i+j},\ 0 < j < i \le n,\ q_{i,j} = 4(n-j)(-1)^{i+j+1},\ 0 \le i < j < n.$$

Moreover, Q *transforms the 'power vectors' into their 'derivatives':*

$$Q\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\quad Q\begin{pmatrix}0\\1\\\vdots\\n\end{pmatrix} = n\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix},\quad Q\begin{pmatrix}0^2\\1^2\\\vdots\\n^2\end{pmatrix} = 2n\begin{pmatrix}0\\1\\\vdots\\n\end{pmatrix},$$

and

$$(b-a)s'(x_i) = \sum_{j=0}^{n} q_{i,j} f(x_j), \quad i = 0, 1, \ldots, n.$$

PROOF. To calculate the entries, write matrix $\binom{M}{r}$ as a sum of an upper bidiagonal and a strictly lower triangular matrix. Then, inverting this upper bidiagonal (Toeplitz) matrix is easy, while the other term is a dyad (a rank one matrix), hence the Sherman–Morrison formula [4] can be used.

The 3 so-called 'power' relations are equivalent with the reproducing property, while the last equality follows from the definition of Q. ∎

In case of knots with odd subscript, a more concrete form of the derivative will be given.

THEOREM 1. *Let* $f \in C^4[a,b]$, *and* Ω_n *be an equidistant partition for* $[a,b]$ *with* n *even. Then for the spline defined by (8), (9) we have*

$$(b-a)s'(x_{2i+1}) =$$

$$= \frac{n}{2}[(f(x_{2i+2}) - f(x_{2i})] - \sum_{j=1}^{i} j\,\Delta^4 f_{2j-2} + \sum_{j=i+1}^{n/2-1} \left(\frac{n}{2} - j\right)\Delta^4 f_{2j-2},$$

where Δ^4 *stands for the fourth order difference:*

$$\Delta^4 f_{2j-2} = f(x_{2j+2}) - 4f(x_{2j+1}) + 6f(x_{2j}) - 4f(x_{2j-1}) + f(x_{2j-2}).$$

Further,

$$s'(x_{2i+1}) - f'(x_{2i+1}) =$$
$$\frac{f^{(3)}(\xi)}{6}h^2 - \frac{i(i+1)f^{(4)}(\xi_-)}{2(b-a)}h^4 + \frac{\left(\frac{n}{2}-i\right)\left(\frac{n}{2}-i-1\right)f^{(4)}(\xi_+)}{2(b-a)}h^4,$$

where $\xi, \xi_-, \xi_+ \in [a,b]$.

PROOF. To the first equality write the i-th row of Q $(0 \le i \le n)$ as a linear combination of the vectors $(0,\dots,0,1,-4,6,-4,1,0,\dots,0)$ and a single vector of the form $(0,\dots,0,-1,0,1,0,\dots,0)$.

The second equality follows from $f(x+h)-f(x-h) = 2hf'(x)+\frac{h^3}{3}f^{(3)}(\xi)$, $\Delta^4 f_{2j-2} = h^4 f^{(4)}(\tilde{\xi})$, $(\xi, \tilde{\xi} \in [a,b])$, and the Darboux property of $f^{(4)}$.

EXAMPLE 2. If $n = 4$, we have

$$Q = \begin{pmatrix} -7 & 12 & -8 & 4 & -1 \\ -1 & -4 & 8 & -4 & 1 \\ 1 & -4 & 0 & 4 & -1 \\ -1 & 4 & -8 & 4 & 1 \\ 1 & -4 & 8 & -12 & 7 \end{pmatrix}.$$

For the second row, e.g., we obtain the decomposition

$$(-1,-4,8,-4,1) = 2(-1,0,1,0,0) + (1,-4,6,-4,1),$$

the negative sum in Theorem 1 is now vacuous. If $f = \mathrm{id}^3$, we get

$$s'(x_{2i+1}) = f'(x_{2i+1}) + h^2 = 3x_{2i+1}^2 + h^2, \qquad i = 0,\dots,n-1,$$

which yields an interesting interpretation: when applying the method for id^3, the resulting second degree spline coincides on $[x_{2i}, x_{2i+1}]$ with the Lagrange interpolation polynomial based on the abscissas x_{2i}, x_{2i+1}, x_{2i+2}!

COROLLARY. *We have* $s'(x_{2i+1}) - f'(x_{2i+1}) = O(h^2)$, *hence calculating* s *on the adjoining subintervals gives the best possible error* $\|e\| \equiv \|s-f\|_{C[a,b]} = O(h^3)$.

This follows from the theorem. See also the remark in [6]: “For quadratic spline interpolation, the assertion $\|e\| = O(h^3)$ cannot be improved”.

REMARK. It is usual to give an expansion for the error $s'(x_i) - f'(x_i)$, cf. e.g. the article [8], see also remark one in the last section. In what follows we give an algebraic treatment, that is we obtain formulas for polynomials.

LEMMA 2. *There are numbers* $\{\lambda_{i,n}^{(p)}\}_{p\geq 1}$ *such that for any polynomial* f *the derivatives of the spline* (3), (4) *can be written as*

$$s'(x_i) = \frac{1}{n}\sum_{p\geq 1}\lambda_{i,n}^{(p)}\frac{f^{(p)}(x_i)}{p!}h^{p-1}, \qquad 0\leq i\leq n.$$

PROOF. Applying Lemma 1, and a Taylor expansion, one obtains

$$s'(x_i) = \frac{1}{b-a}\sum_{j=0}^{n} q_{i,j} f(x_i+(j-i)h) = \frac{1}{nh}\sum_{p\geq 0}\sum_{j=0}^{n} q_{i,j}(j-i)^p\frac{f^{(p)}(x_i)}{p!}h^p,$$

showing that

$$\lambda_{i,n}^{(p)} = \sum_{j=0}^{n} q_{i,j}(j-i)^p$$

meets the requirements. ∎

REMARK. This enables us to calculate the first several coefficients:

$$\lambda_{i,n}^{(1)} = 1, \quad \lambda_{i,n}^{(2)} = 0, \quad \lambda_{i,n}^{(3)} = -\frac{1}{2}\left(1+3(-1)^i\right), \quad \lambda_{i,n}^{(4)} = 3(-1)^{i+1}(n-2i),$$

$$\lambda_{i,n}^{(5)} = 1+5(-1)^{i+1}\left(n^2-1-3i(n-i)\right),$$

$$\lambda_{i,n}^{(6)} = \frac{15}{4}(-1)^{i+1}(n-2i)\left((n-2i)^2+n^2-4\right).$$

However, subsequently we will need a formula for $\lambda_{i,n}^{(p)}$ with *at least* one i but for *all* $p\geq 1$. To this aim we will make use of the Bernoulli polynomials $B_i(x)$ and Bernoulli numbers $B_i\equiv B_i(0)$, $i=1,2,\ldots$, the first several of them are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.$$

Denote for brevity $B_i^*(x) = B_i(x) - B_i(0)$, $i=1,2,\ldots$ Then the following can be proved.

LEMMA 3. *For* n *even, and* p *positive integer we have*

$$\lambda_{0,n}^{(p)} = \frac{4}{p+1}\left[B_{p+1}^*(n+1) - 2^{p+1}B_{p+1}^*((n+2)/2)\right] +$$

$$+\frac{4}{(p+2)n}\left[2^{p+2}B_{p+2}^*((n+2)/2) - B_{p+2}^*(n+1)\right] - n^{p-1}.$$

PROOF. By virtue of Lemma 1 and Lemma 2, we have to prove

$$n\lambda_{0,n}^{(p)} =$$
$$= 4\left(-1^{p+1} + 2^{p+1} - 3^{p+1} + \ldots + n^{p+1}\right) - 4n\left(-1^p + 2^p - 3^p + \ldots + n^p\right) - n^p.$$

Since sums of this kind can be expressed by Bernoulli polynomials as

$$-1^p + 2^p - \ldots + (2k)^p = \frac{1}{p+1}\left(2^{p+1}B_{p+1}^*(k+1) - B_{p+1}^*(2k+1)\right),$$

cf. e.g. [2], applying the well known identities

$$B_m(x) = \sum_{k=0}^{m}\binom{m}{k}B_k x^{m-k}, \qquad B_m(0) = B_m, \qquad m \geq 1$$

yields the result. ∎

Our last preparing lemma gives an identity for the divided difference of arbitrary polynomials.

LEMMA 4. *For a polynomial f and distinct x, y real numbers we have*

$$\frac{f(x) - f(y)}{x - y} = \sum_{k \geq 2} 2(2^k - 1)B_k \frac{f^{(k-1)}(x) + f^{(k-1)}(y)}{k!}(x - y)^{k-2}. \tag{11}$$

PROOF. It can be assumed that $y = 0$ and $f(x) = x^p$. Then

$$f^{(k-1)}(x) = \frac{p!}{(p-k+1)!}x^{p-k+1}$$

is zero at $x = 0$, if $p - k + 1 > 0$, and nonzero, if $k = p + 1$. However, this occurs only for p odd, because Bernoulli numbers of odd index are equal to zero (except $B_1 = -\frac{1}{2}$). Hence these cases must be handled separately.

If p is even, comparing the coefficients of x^{p-1} gives

$$1 = \sum_{k=2}^{p+1}(2^k - 1)B_k \frac{p!}{(p-k+1)!k!} = \frac{2}{p+1}\sum_{k=2}^{p+1}(2^k - 1)B_k\binom{p+1}{k},$$

which is equivalent to

$$\sum_{k=1}^{p+1}(2^k - 1)B_k\binom{p+1}{k} = 0, \qquad \text{i.e.} \quad B_{p+1}(1/2) = B_{p+1},$$

a standard formula for Bernoulli polynomials.

If p is odd, we have to prove

$$1 = \frac{2}{p+1}\sum_{k=2}^{p+1}(2^k - 1)B_k\binom{p+1}{k}(1+\delta_{k,p+1}),$$

(δ being the Kronecker symbol) or, equivalently,

$$B_{p+1}(1/2) = (2^{-p} - 1)B_{p+1},$$

which can be derived by using the generator function.

THEOREM 2. *Let s be the reproducing quasiminimal (cf.* (3), (4′)*) quadratic spline interpolant given by* (8), (9) *for a polynomial f, where $a = x_0 < < x_1 < \ldots < x_n = b$ is an equidistant partition of $[a,b]$ with n even. Then for any i, $i = 0, 1, \ldots, n$ it holds that*

$$s'(x_i) = f'(x_i) + \sum_{j\geq 4} 4(2^j - 1)B_j\frac{h^j}{j}\left(\frac{f^{(j-1)}(x_i)}{(j-1)!} + (-1)^i\frac{f^{(j-2)}[a,b]}{(j-2)!}\right),$$

where $\{B_j\}$ are the Bernoulli numbers, and $g[\,\cdot\,,\,\cdot\,]$ stands for the (first order) divided difierence of the function g.

PROOF. By introducing the quantities $\{\alpha_i\}_{i=0}^n$ and β as

$$\alpha_i = \sum_{j\geq 2} 4(2^j - 1)B_j h^j\frac{f^{(j-1)}(x_i)}{j!}, \qquad \beta = \sum_{j\geq 4} 4(2^j - 1)B_j h^j\frac{f^{(j-2)}[a,b]}{j(j-2)!},$$

the statement assumes

$$s'(x_i) = \alpha_i + (-1)^i\beta, \qquad 0 \leq i \leq n.$$

Instead of these we first concentrate on the sums

$$s'(x_i) + s'(x_{i+1}) = \alpha_i + \alpha_{i+1}, \qquad i = 0, 1, \ldots, n-1$$

of “adjoining” equalities, because they do not contain β. Using the elementary identity

$$s'(x_i) + s'(x_{i+1}) = 2s[x_i, x_{i+1}],$$

characteristic for quadratic polynomials, and the interpolatory conditions in the form $s[x_i, x_{i+1}] = f[x_i, x_{i+1}]$, Lemma 4 yie1ds the above β-free equa1ities, hence it remains to prove only one of the original relations.

For a given index i $(0 \leq i \leq n)$ we expand the numerator of $f^{(j)}[a,b]$ as

$$f^{(j)}(b) - f^{(j)}(a) = f^{(j)}(x_i + (n-i)h) - f^{(j)}(x_i - ih) =$$
$$= \sum_{k\geq 1} \frac{f^{(j+k)}(x_i)}{k!} \left[(n-i)^k - (-i)^k\right] h^k.$$

Choose now $i = 0$, and rewrite the divided differences in β according to

$$f^{(j)}[a,b] = \frac{f^{(j)}(b) - f^{(j)}(a)}{b-a} = \sum_{k\geq 1} \frac{f^{(j+k)}(x_0)}{k!} n^{k-1} h^{k-1}.$$

Comparing the coefficients of h^{p-1} with those in Lemma 2, we see that

$$\lambda_{0,n}^{(p)} = \frac{4(2^{p+1}-1)B_{p+1}}{p+1} + 4\sum_{k=2}^{p} \frac{(2^{k+1}-1)B_{k+1}}{k+1} \binom{p}{k-1} n^{p-k}$$

is to be proven. However, the representation given by Lemma 3 yields an alternative formula for the $\lambda_{0,n}^{(p)}$-s, hence a manipulation using only Bernoulli polynomials completes the proof. ∎

3. Concluding remarks

1. For $(b-a)$-periodic functions the sum β disappears and we get

$$s'(x_i) = \alpha_i = f'(x_i) - \frac{h^2}{12} f^{(3)}(x_i) + \frac{h^4}{120} f^{(5)}(x_i) - \frac{31}{20160} h^6 f^{(7)}(x_i) + \ldots$$

This sum is also found in [8], (3.7) (with the coefficient of h^6 being erroneous), although there the initial condition $s'(x_0) = f'(x_0)$ (in that notation: $s_0' = y_0'$) was used, and no further restrictions were assumed.

To clear up the matter, we introduce the notation s_{init} and s_{qmin} for the splines investigated there and here. If $f = \text{id}^3$, $x_0 = 0$, $i = 0$, $h = x_1$, then for the first subinterval $[x_0, x_1]$ we have $s_{\text{init}}'(x) = hx^2$, giving $s_{\text{init}}'(0) = 0$, while the formula above would yield $\alpha_0 = f'(0) - h^2 f^{(3)}(0) = -h^2/2 \neq 0$.

We conclude that, although the calculus of finite difference operators is efficient, it results in an approximative equality, which must yet be completed. So, for the initial value problem investigated in [8], the precise form of the derivative is $s_{\text{init}}'(x_i) = \alpha_i + (-1)^i [f'(x_0) - \alpha_0]$, $i = 0, \ldots, n$.

2. As a consequence of Lemma 4, we get a quadratura formula, exact for all polynomials and giving the integral by derivatives. To get it, multiply (11) by $(x - y)$, and assume that f is a primitive function of some g. The Newton–Leibniz formula yields

$$\int_a^b g = \sum_{j=0}^{\infty} 2(2^{j+2} - 1)B_{j+2}\frac{g^{(j)}(a) + g^{(j)}(b)}{(j+2)!}(b-a)^{j+1}. \tag{12}$$

Compare now this sum by the special case of the famous Euler–McLaurin formula [5]:

$$\int_a^b g = \frac{g(b)+g(a)}{2!}(b-a) - 2\frac{g'(b)-g'(a)}{4!}(b-a)^2 + \frac{g^{(3)}(b)-g^{(3)}(a)}{6!}(b-a)^4 -$$

$$= \frac{4}{3}\frac{g^{(5)}(b)-g^{(5)}(a)}{8!}(b-a)^6 + \frac{21}{8}\frac{g^{(7)}(b)-g^{(7)}(a)}{10!}(b-a)^8 - \dots$$

To see the details, we display also the first several terms of (12):

$$\int_a^b g = \frac{g(a)+g(b)}{2!}(b-a) - \frac{g''(a)+g''(b)}{4!}(b-a)^3 + 3\frac{g^{(4)}(a)+g^{(4)}(b)}{6!}(b-a)^5 -$$

$$-17\frac{g^{(6)}(a)+g^{(6)}(b)}{8!}(b-a)^7 + 155\frac{g^{(8)}(a)+g^{(8)}(b)}{10!}(b-a)^9 - \dots$$

Observe that both can be considered as a continuation of the trapezoidal formula. Further, all the coefficients $1, -1, 3, -17, 155$ in (12) are integers. This is not by chance: their moduli are the so-called Genocchi-numbers (see [2], Ex. 6.24), which are known to be odd integers.

3. Although so far we restricted ourselves mainly to polynomials, it is not difficult to state convergence results for entire functions using the asymptotic expansion $B_j \approx 2(-1)^{(j/2-1)} j!/(2\pi)^j$ (j large, even) for Bernoulli numbers (see [1], p. 389). Consider e.g. the typical sum

$$\sum_{j \geq 0} c^j B_j M_j h^j / j!,$$

where c is a positive constant (here $c = 1$ or $c = 2$) and $M_j = \|f^{(j)}\|_{C[a,b]}$. For functions with $M^* \equiv \limsup_{j \geq 1} M_j^{1/j} < \infty$ the Cauchy criterion yields

$h < \frac{2\pi}{cM^*}$, which means that the series above is absolutely convergent for small enough h, i.e. for large enough n.

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COMPLEXES AND COMPONENTS

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What is a complex in the topology? There are several answers to this question. In general a complex in a topological space $(X, \mathcal{T})$ is a system $\mathcal{M}$ of subsets of X satisfying some special requirements.

First observe that for an arbitrary system $\mathcal{M}$ of subsets of X the topological space $(X, \mathcal{T})$ induces a topology $\mathcal{T}'$ on $\mathcal{M}$.

Indeed, for $A \in \mathcal{M}$ let

(1) $$\mathrm{St}_A = \{B \in \mathcal{M};\ A \subset \overline{B}\},$$

($\overline{B}$ is the closure of B in $(X, \mathcal{T})$, cf. [1] p.29.). We then clearly have

(2) $$A \in \mathrm{St}_A,$$

(3) $$B \in \mathrm{St}_a \Rightarrow \mathrm{St}_B \subset \mathrm{St}_A\,.$$

Let $\mathcal{T}'$ be that topology on $\mathcal{M}$, where for each $A \in \mathcal{M}$ the singleton $\{\mathrm{St}_A\}$ is a base for $(\mathcal{M}, \mathcal{T}')$ at the element A (cf. [1] p.28.). According to (2) and (3) that topology $\mathcal{T}'$ is well defined (cf. [1] Proposition 1.2.3 p.39.).

Now we can raise the question: under which conditions is the connectedness in $(X, \mathcal{T})$ of the set $\cup\mathcal{M}$, i.e. the connectedness of the subspace of $(X, \mathcal{T})$ induced by the set $\cup\mathcal{M}$ equivalent to the connectedness of the topological space $(\mathcal{M}, \mathcal{T}')$ or even more, when is the system of components of $\cup\mathcal{M}$ — denoted by $\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M})$ — the same as the system $\{\cup\mathcal{C};\ C \in \mathrm{Comp}_{\mathcal{T}'}\mathcal{M}\}$, where $\mathrm{Comp}_{\mathcal{T}'}\mathcal{M}$ is the system of components of the topological space $(\mathcal{M}, \mathcal{T}')$.

We call the system of sets $\mathcal{M}$ a *regular complex* in $(X, \mathcal{T})$ if

(A) $$\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M}) = \{\cup\mathcal{C};\ \mathcal{C} \in \mathrm{Comp}_{\mathcal{T}'}\mathcal{M}\}.$$

We now have the following

THEOREM. *Equality* (A) *holds if the following conditions are fulfilled.*

(i) *Each member of* $\mathcal{M}$ *is a nonempty connected set in* $(X,\mathcal{T})$.

(ii) $\mathcal{M}$ *is locally finite in* $(X,\mathcal{T})$.

(iii) *For* A, $B \in \mathcal{M}$ *the relation* $A \cap \overline{B} \neq \emptyset$ *implies* $A \subset \overline{B}$.

Before the proof of this theorem first observe that no one of these three conditions can be omitted.

Indeed, if $\emptyset$ is the only element of $\mathcal{M}$ then conditions (ii) and (iii) are fulfilled while $\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M}) = \emptyset \neq \{\emptyset\} = \mathrm{Comp}_{\mathcal{T}'}\mathcal{M}$.

Likewise, conditions (ii) and (iii) are fulfilled whenever $\mathcal{M}$ is a singleton and the only element of $\mathcal{M}$ is nonconnected. In this case $\mathrm{Comp}_{\mathcal{T}'}(\cup M)$ has at least two elements while $\mathrm{Comp}_{\mathcal{T}'}\mathcal{M}$ is a singleton.

Now let X be the set $\mathbf{R}$ of the real numbers and let $\mathcal{T}$ be the natural topology on $\mathbf{R}$. Let $\mathcal{M}$ be the family of all singletons of $\mathbf{R}$. Then conditions (i) and (iii) are fulfilled while $\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M})$ is a singleton and $\mathrm{Comp}_{\mathcal{T}'}\mathcal{M}$ is an infinite set.

On the other hand if $X = \mathbf{R}$ and $\mathcal{T}$ are the same as before, $A = \{y \in \in \mathbf{R};\ y \leq 0\}$, $B = \{y \in \mathbf{R};\ y \geq 0\}$ and $\mathcal{M} = \{A, B\}$ then conditions (i) and (ii) are fulfilled while $\mathrm{Comp}_{\mathcal{T}}(\cup M)$ is a singleton and $\mathrm{Comp}_{\mathcal{T}'}\mathcal{M}$ consists of two elements.

We now prepare the proof of the theorem. First we prove a lemma.

LEMMA 1. *Let* $\mathrm{Comp}_{\mathcal{T}^*} Y$ *be the system of components of a topological space* $(Y,\mathcal{T}^*)$. *Let* $\mathcal{K}$ *be the decomposition of* Y *into nonempty pairwise disjoint open and connected sets, i.e.*

$\cup\mathcal{K} = Y$,

K_1, $K_2 \in \mathcal{K}$ *and* $K_1 \cap K_2 \neq \emptyset \to K_1 = K_2$,

and

$K \in \mathcal{K} \Rightarrow K$ *is nonempty open and connected in* $(Y,\mathcal{T}^*)$.

Then $\mathcal{K}$ *is the system of components of* $(Y,\mathcal{T}^*)$, *i.e.* $\mathcal{K} = \mathrm{Comp}_{\mathcal{T}^*} Y$.

Indeed, for each $K \in \mathcal{K}$ we have

$$K = Y \setminus (\cup\{K';\ K' \in \mathcal{K} \setminus \{K\}\}).$$

Thus K is closed in $(Y,\mathcal{T}^*)$.

Let $K \in \mathcal{K}$ and $q \in K$. Since $K \neq \emptyset$ there is such a q. Let Q be the component of q in $(Y,\mathcal{T}^*)$ (cf. [1] p.438). Since K is connected in $(Y,\mathcal{T}^*)$ and $q \in K$ it follows $K \subset Q$. Thus K is a nonempty open and closed susbset of the connected set Q. Accordingly $K = Q$ (cf. [1] 6.1.1.(i) and 6.1.1.(ii) p.432). K is a component of $(Y, \mathcal{T}^*)$, $K \in \mathrm{Comp}_{\mathcal{T}^*}\ Y$ and that means

(4) $$\mathcal{K} \subset \mathrm{Comp}_{\mathcal{T}^*}\ Y.$$

Now since $\mathrm{Comp}_{\mathcal{T}^*}\ Y$ is a decomposition of Y into nonempty pairwise disjoint sets and $\cup\mathcal{K} = \cup\,\mathrm{Comp}_{\mathcal{T}^*}\ Y = Y$ (4) implies

$$\mathcal{K} = \mathrm{Comp}_{\mathcal{T}^*}\ Y$$

as required. ∎

Unless stated to the contrary in the remainder let $(X, \mathcal{T})$ be a topological space and $\mathcal{M}$ a system of subsets of X satisfying conditions (i), (ii) and (iii). Let $\mathcal{T}'$ be the topology on $\mathcal{M}$, where for each $A \in \mathcal{M}$ the singleton $\{\mathrm{St}_A\}$ (see (1)) is a base for $(\mathcal{M}, \mathcal{T}')$ at the element A.

First observe that for A, $B \in \mathcal{M}$ according to (iii)

(5) $$A \cap B \neq \emptyset \Rightarrow A \subset \overline{B}.$$

Also, for A, $B \in \mathcal{M}$

(6) $$A \in \overline{\{B\}}^{\mathcal{T}'} \Leftrightarrow A \subset \overline{B},$$

where $\overline{\{B\}}^{\mathcal{T}'}$ is the closure of the singleton $\{B\}$ in the space $(\mathcal{M}, \mathcal{T}')$.

Indeed, $A \in \overline{\{B\}}^{\mathcal{T}'}$ means that each element of the base for $(M, \mathcal{T}')$ at A intersects $\{B\}$. Accordingly by (1)

$$A \in \overline{\{B\}}^{\mathcal{T}'} \Leftrightarrow \mathrm{St}_A \cap \{B\} \neq \emptyset \Leftrightarrow B \in \mathrm{St}_A \Leftrightarrow A \subset \overline{B}$$

as required.

Observe also that for A, $B \in \mathcal{M}$

(7) $$A \subset \overline{B} \Rightarrow \{A, B\} \text{ is connected in } (\mathcal{M}, \mathcal{T}')$$

and

(8) $$A \subset \overline{B} \Rightarrow A \cup B \text{ is connected in } (X, \mathcal{T}).$$

Indeed, by (6) $A \subset \overline{B}$ implies $A \in \overline{\{B\}}^{\mathcal{T}'}$ and thus $\{B\} \subset \{A, B\} \subset \overline{\{B\}}^{\mathcal{T}'}$. However the singleton $\{B\}$ is connected and thus each set containing $\{B\}$ and contained in $\overline{\{B\}}^{\mathcal{T}'}$ is connected (see [1] 6.1.11. p.435). Accordingly $\{A, B\}$ is connected in $(\mathcal{M}, \mathcal{T}')$ as required.

On the other hand by $A \subset \overline{B}$ we have $B \subset A \cup B \subset \overline{B}$. However B is connected in $(X, \mathcal{T})$ by (i) and thus the set $A \cup B$ containing B and contained in $\overline{B}$ is connected in $(X, \mathcal{T})$.

Consider now a finite sequence

$$\gamma = \langle A_1, \ldots, A_k \rangle \tag{9}$$

in $\mathcal{M}$. That means γ is a map from an initial segment $\{1, \ldots, k\}$ of the set of positive integers into $\mathcal{M}$ and for $i = 1, \ldots, k$ $\gamma(i)$ is A_i. Clearly

$$\operatorname{im}\gamma = \{A_1, \ldots, A_k\} \tag{10}$$

and

$$\cup \operatorname{im}\gamma = A_1 \cup \ldots \cup A_k. \tag{11}$$

The sequence γ is said to be an *$\mathcal{M}$-chain* if either $k = 1$ or $k > 1$ and for each $i \in \{1, \ldots, k-1\}$ one of the following conditions holds

(a) $A_i \subset \overline{A_{i+1}}$,

(b) $A_{i+1} \subset \overline{A_i}$.

k itself is called the *combinatoric length* of γ.

PROPOSITION 2. *Let γ be an $\mathcal{M}$-chain. Then* $\operatorname{im}\gamma$ *is a connected set in* $(\mathcal{M}, \mathcal{T}')$.

We argue by induction with respect to the combinatoric length k of γ.

If $k = 1$ then $\operatorname{im}\gamma$ is a singleton and thus it is connected.

Now suppose that $k \geq 2$ and the assertion is true for each $\mathcal{M}$-chain γ' with the combinatoric length $k-1$. Let $\gamma = \langle A_1, \ldots, A_k \rangle$ be an $\mathcal{M}$-chain with the combinatoric length k and let $\gamma' = \langle A_1, \ldots, A_{k-1} \rangle$. Then clearly

$$\begin{aligned} \operatorname{im}\gamma &= \{A_1, \ldots, A_k\} = \\ &= \{A_1, \ldots, A_{k-1}\} \cup \{A_{k-1}, A_k\} = \\ &= \operatorname{im}\gamma' \cup \{A_{k-1}, A_k\}. \end{aligned} \tag{12}$$

By the induction hypothesis $\operatorname{im}\gamma'$ is a connected set in $(\mathcal{M}, \mathcal{T}')$ and by $A_{k-1} \in$ $\in \operatorname{im}\gamma' \cap \{A_{k-1}, A_k\}$ we have

$$\operatorname{im}\gamma' \cap \{A_{k-1}, A_k\} \neq \emptyset. \tag{13}$$

However by (a), (b) and (7) $\{A_{k-1}, A_k\}$ is a connected set in $(\mathcal{M}, \mathcal{T}')$. Thus by (12) and (13) $\operatorname{im}\gamma$ is connected in $(\mathcal{M}, \mathcal{T}')$ (cf. [1] 6.1.10. p. 435) as required. ∎

Now let $\gamma = \langle A_1, \ldots, A_k \rangle$ be an $\mathcal{M}$-chain. We then say that γ *connects* A_1 and A_k. We clearly have

(14) $$A_1, A_k \in \operatorname{im}\gamma = \{A_1, \ldots, A_k\}.$$

Let A, $A' \in \mathcal{M}$. A and A' are said to be $\mathcal{M}$-*equivalent* – and we use the notation $A \underset{\mathcal{M}}{\sim} A'$ – if there is an $\mathcal{M}$-chain connecting A and A'. $\underset{\mathcal{M}}{\sim}$ is clearly an equivalence relation on the system of sets $\mathcal{M}$. Let $E(\mathcal{M})$ be the family of the equivalence classes of the relation $\underset{\mathcal{M}}{\sim}$ (cf. [1] pp. 15–16). $E(\mathcal{M})$ is a family of nonempty pairwise disjoint subsystems of $\mathcal{M}$, where

(15) $$\cup E(\mathcal{M}) = \mathcal{M}$$

and for each A, $B \in \mathcal{M}$ we hive

(16) $$A \underset{\mathcal{M}}{\sim} B \Leftrightarrow \exists\ \mathcal{C} \in E(\mathcal{M}):\ A, B \in \mathcal{C}.$$

Observe that $\mathcal{C} \in E(\mathcal{M})$, $A \in \mathcal{C}$ and $B \underset{\mathcal{M}}{\sim} A$ clearly imply $B \in \mathcal{C}$, i.e.

(17) $$\mathcal{C} \in E(\mathcal{M}),\ A \in \mathcal{C},\ A \underset{\mathcal{M}}{\sim} B \Rightarrow B \in \mathcal{C}.$$

Observe also that for $\mathcal{C} \in E(\mathcal{M})$, $A \in \mathcal{C}$, $B \in \mathcal{M}$ the relation $A \cap \overline{B} \neq \emptyset$ implies $B \in \mathcal{C}$, i.e.

(18) $$\mathcal{C} \in E(\mathcal{M}),\ A \in \mathcal{C},\ B \in \mathcal{M},\ A \cap \overline{B} \neq \emptyset \Rightarrow B \in \mathcal{C}.$$

Indeed, if $A \cap \overline{B} \neq \emptyset$ then by (iii) we have $A \subset \overline{B}$ and thus $\langle A, B \rangle$ is an $\mathcal{M}$-chain (see (a)) connecting A and B. Accordingly we have $A \underset{\mathcal{M}}{\sim} B$ and thus by (17) it follows (18).

PROPOSITION 3. *Let $\mathcal{C} \in E(\mathcal{M})$ and $A \in \mathcal{C}$. Let $\gamma = \langle A_1, \ldots, A_k \rangle$ be an $\mathcal{M}$-chain with $A_1 = A$. Then* $\operatorname{im}\gamma \subset \mathcal{C}$.

Indeed, let $i \in \{1, \ldots, k\}$ and $\gamma_i = \langle A_1, \ldots, A_i \rangle$. γ_i is clearly an $\mathcal{M}$-chain connecting $A_1 = A$ and A_i. Thus we have $A \underset{\mathcal{M}}{\sim} A_i$ and since $A \in \mathcal{C}$ according to (17) we may conclude $A_i \in \mathcal{C}$. Consequently $\operatorname{im}\gamma \subset \mathcal{C}$ as required. ∎

PROPOSITION 4. *Let $\mathcal{C} \in E(\mathcal{M})$. Then $\mathcal{C}$ is connected in* $(\mathcal{M}, \mathcal{T}')$.

Indeed, let $A \in \mathcal{C}$. Since $\mathcal{C} \neq \emptyset$ there is such an A. For each $B \in \mathcal{C}$ let γ_B be an $\mathcal{M}$-chain connecting A and B. By (14) we have A, $B \in \operatorname{im}\gamma_B$ and by Proposition 3 we obtain $\operatorname{im}\gamma_B \subset \mathcal{C}$. Thus

(19) $$\mathcal{C} = \cup\{\operatorname{im}\gamma_B;\ B \in \mathcal{C}\}$$

and since for each $B \in \mathcal{C}$ we have $A \in \operatorname{im}\gamma_B$ it follows

$$\cap\{\operatorname{im}\gamma_B;\ B \in \mathcal{C}\} \neq \emptyset.$$

Hence by (19) and Proposition 2 $\mathcal{C}$ is connected in $(\mathcal{M}, \mathcal{T}')$ as required (cf. [1] 6.1.10. p.435). ∎

PROPOSITION 5. *For each $\mathcal{C} \in E(\mathcal{M})$ $\mathcal{C}$ is open in $(\mathcal{M}, \mathcal{T}')$.*

Indeed, let $A \in \mathcal{C}$ and $B \in \mathrm{St}_A$. Then $B \in \mathcal{M}$, $A \subset \overline{B}$ (see (1)) and since $A \neq \emptyset$ (see (i)) we have $A \cap \overline{B} \neq \emptyset$. Accordingly by (18) we obtain $B \in \mathcal{C}$. Hence $\mathrm{St}_A \subset \mathcal{C}$ and thus by (2) we have

$$\mathcal{C} = \cup\{\mathrm{St}_A;\ A \in \mathcal{C}\},$$

where $\{\mathrm{St}_A;\ A \in \mathcal{C}\}$ is a subfamily of $\cup\{\{\mathrm{St}_A\};\ A \in \mathcal{M}\}$. Consequently $\mathcal{C}$ is open in $(\mathcal{M}, \mathcal{T}')$ as required (cf. [1] 1.2.3. p.39.). ∎

Now by Propositions 4, 5 and Lemma 1 we have

$$E(\mathcal{M}) = \mathrm{Comp}_{\mathcal{T}'}\mathcal{M}. \tag{B}$$

Hence to prove the equality (A) we need only to show that

$$\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M}) = \{\cup\mathcal{C};\ \mathcal{C} \in E(\mathcal{M})\}. \tag{C}$$

PROPOSITION 6. *Let γ be an $\mathcal{M}$-chain. Then $\cup\operatorname{im}\gamma$ is a connected set in $(X, \mathcal{T})$.*

We argue by induction with respect to the combinatoric length k of γ.

If $k = 1$ then $\gamma = \langle A_1 \rangle$ and $\cup\operatorname{im}\gamma = A_1$ is connected in (X, T) by (i).

Now suppose that $k \geq 2$ and that the assertion is true for each $\mathcal{M}$-chain γ' with the combinatoric length $k-1$. Let $\gamma = \langle A_1 \ldots, A_k \rangle$ be an $\mathcal{M}$-chain with the combinatoric length k and let $\gamma' = \langle A_1, \ldots, A_{k-1} \rangle$. Then clearly

$$\begin{aligned} \cup\operatorname{im}\gamma &= A_1 \cup \ldots \cup A_k = \\ &= (A_1 \cup \ldots \cup A_{k-1}) \cup (A_{k-1} \cup A_k) = \\ &= (\cup\operatorname{im}\gamma') \cup (A_{k-1} \cup A_k). \end{aligned} \tag{20}$$

By the induction hypothesis $\cup\operatorname{im}\gamma'$ is a connected set in $(X, \mathcal{T})$, moreover by $A_{k-1} \subset (\cup\operatorname{im}\gamma') \cap (A_{k-1} \cup A_k)$ and by $A_{k-1} \neq \emptyset$ (see (i)) we have

$$\left(\cup\operatorname{im}\gamma'\right) \cap (A_{k-1} \cup A_k) \neq \emptyset. \tag{21}$$

However by (a), (b) and (8) $A_{k-1} \cup A_k$ is connected in $(X, \mathcal{T})$ and thus by (20) and (21) $\cup\operatorname{im}\gamma$ is connected in $(X, \mathcal{T})$ as required (cf. [1] 6.1.10. p.435). ∎

PROPOSITION 7. *Let* $\mathscr{C} \in E(\mathscr{M})$ *and let* $\gamma = \langle A_1, \ldots, A_k \rangle$ *be an* $\mathscr{M}$*-chain with* $A_1 \in \mathscr{C}$. *Then* $(\cup \operatorname{im} \gamma) \subset (\cup \mathscr{C})$.

Indeed, by Proposition 3 we have $\operatorname{im} \gamma \subset \mathscr{C}$ and thus $(\cup \operatorname{im} \gamma) \subset (\cup \mathscr{C})$ as required. ∎

PROPOSITION 8. *Let* $\mathscr{C} \in E(\mathscr{M})$. *Then* $\cup \mathscr{C}$ *is a nonempty connected set in* $(X, \mathscr{T})$.

Indeed, let $A \in \mathscr{C}$ and $y \in A$. Since $A \neq \emptyset$ (see (i)) there is such a y. Hence by $y \in (\cup \mathscr{C})$ we have

$$\cup \mathscr{C} \neq \emptyset. \tag{22}$$

Let $p \in \cup \mathscr{C}$. Choose a B_p from $\mathscr{C}$ with $p \in B_p$. By (16) we obtain $A \underset{\mathscr{M}}{\sim} B_p$ and thus there is an $\mathscr{M}$-chain γ_p which connects A and B_p. By A, $B_p \in \operatorname{im} \gamma_p$ (see (14)) we then have

$$y, p \in (\cup \operatorname{im} \gamma_p). \tag{23}$$

Hence by Proposition 7 we obtain

$$\cup \mathscr{C} = \cup \{\cup \operatorname{im} \gamma_p;\ p \in (\cup \mathscr{C})\}. \tag{24}$$

On the other hand (23) implies

$$\cap \{\cup \operatorname{im} \gamma_p;\ p \in (\cup \mathscr{C})\} \neq \emptyset.$$

Accordingly by (22), (24) and Proposition 6 we may conclude that $\cup \mathscr{C}$ is a nonempty connected set in $(X, \mathscr{T})$ (see [1] 6.1.10. p. 435) as required. ∎

PROPOSITION 9. *For* $\mathscr{C}, \mathscr{D} \in E(\mathscr{M})$

$$(\cup \mathscr{C}) \cap (\cup \mathscr{D}) \neq \emptyset \Rightarrow \mathscr{C} = \mathscr{D}.$$

Indeed, let $y \in A \cap B$, where $A \in \mathscr{C}$ and $B \in \mathscr{D}$. Then $A \cap \overline{B} \neq \emptyset$ and thus by (18) $B \in \mathscr{C}$. Hence

$$\mathscr{C} \cap \mathscr{D} \neq \emptyset. \tag{25}$$

Since $\mathscr{C}$ and $\mathscr{D}$ are equivalence classes of the equivalence relation $\underset{\mathscr{M}}{\sim}$ (25) implies $\mathscr{C} = \mathscr{D}$ as required. ∎

LEMMA 10. *According to Propositions 9 and 8* $\{\cup \mathscr{C};\ \mathscr{C} \in E(\mathscr{M})\}$ *is a system of pairwise disjoint nonempty sets with*

$$\cup \{\cup \mathscr{C};\ \mathscr{C} \in E(\mathscr{M})\} = \cup(\cup E(\mathscr{M})) = \cup \mathscr{M} \tag{26}$$

(cf. (15)). ∎

In the sequel let $\mathcal{T}_1$ be the subspace topology of $\cup\mathcal{M}$ induced by the topology $\mathcal{T}$. Then clearly

$$\mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M}) = \mathrm{Comp}_{\mathcal{T}_1}(\cup\mathcal{M}). \tag{27}$$

PROPOSITION 11. *Let $\mathcal{C} \in E(\mathcal{M})$. Then $\cup\mathcal{C}$ is open in $(\cup\mathcal{M}, \mathcal{T}_1)$.*

Indeed, let $p \in (\cup\mathcal{C})$. Let A be a member of $\mathcal{C}$ with $p \in A$. Let U be an open neighbourhood of p meeting only a finite number of members of $\mathcal{M}$ (see (ii)). Thus

$$\mathcal{M}' = \{B \in \mathcal{M};\ U \cap B \neq \emptyset\} \tag{28}$$

is a finite system of sets. Let

$$\mathcal{N} = \{B \in \mathcal{M}';\ p \notin \overline{B}\}. \tag{29}$$

Then $\mathcal{N} \subset \mathcal{M}'$ consequently $\mathcal{N}$ is finite as well. Hence

$$V = X \setminus \cup\{\overline{B};\ B \in \mathcal{N}\} = X \setminus \overline{\cup\mathcal{N}} \tag{30}$$

is an open set in $(X, \mathcal{T})$ containing p. Accordingly $U \cap V$ is an open neighbourhood of p moreover by (28), (29) and (30) for each $B' \in \mathcal{M}$ with $U \cap V \cap B' \neq \emptyset$ we have $B' \in \mathcal{M}' \setminus \mathcal{N}$ and thus $p \in \overline{B'}$. Now let $q \in U \cap V \cap (\cup\mathcal{M})$ and choose $B' \in \mathcal{M}$ with $q \in B'$. Then $U \cap V \cap B' \neq \emptyset$ consequently $p \in \overline{B'}$ and thus $A \cap \overline{B'} \neq \emptyset$. Hence by (18) we have $B' \in \mathcal{C}$ which implies $q \in (\cup\mathcal{C})$. Accordingly

$$U \cap V \cap (\cup\mathcal{M}) \subset (\cup\mathcal{C}),$$

where $U \cap V \cap (\cup\mathcal{M})$ is open in $(\cup\mathcal{M}, \mathcal{T}_1)$ and $p \in U \cap V \cap (\cup\mathcal{M})$. Consequently $\cup\mathcal{C}$ is the union of open sets in $(\cup\mathcal{M}, \mathcal{T}_1)$. $\cup\mathcal{C}$ is open in $(\cup\mathcal{M}, \mathcal{T}_1)$ as required. ∎

According to Propositions 11, 8 and Lemmas 10 and 1 we now have

$$\mathrm{Comp}_{\mathcal{T}_1}(\cup\mathcal{M}) = \mathrm{Comp}_{\mathcal{T}}(\cup\mathcal{M}) = \{\cup\mathcal{C};\ \mathcal{C} \in E(\mathcal{M})\}$$

(see also (27)). Equality (C) and thus equality (A) holds.

The proof of the theorem is complete. ∎

We now show two examples of regular complexes.

I. SIMPLICIAL COMPLEXES. Let n be a positive integer and let $(X, \mathcal{T})$ be the euclidean n-space with the natural topology. Hence $X = \mathbf{R}^n$. For any simplex s in $\mathbf{R}^n$ let the relative interior $\mathrm{int}_r\, s$ be the interior of s in its supporting plane. Let $\mathcal{M}$ be a finite system of pairwise disjoint subsets of $\mathbf{R}^n$, where each member of $\mathcal{M}$ is the relative interior of a simplex and where for each member of $\mathcal{M}$ of the form $\mathrm{int}_r\, s$ and for each face s' of s the relative interior of s' belongs to $\mathcal{M}$. Such systems are said to be *simplicial complexes*.

If $\mathcal{M}$ is a simplicial complex, then it clearly satisfies conditions (i) and (ii). It also satisfies (iii).

Indeed, let $A = \mathrm{int}_r\, s$, $B = \mathrm{int}_r\, s' \in \mathcal{M}$ with $A \cap \overline{B} \neq \emptyset$. Let $q \in A \cap \overline{B}$. Then by $q \in \overline{B} = s'$ there is a proper or nonproper face s'' of s' with $q \in \mathrm{int}_r\, s''$. By the assumption we have $\mathrm{int}_r\, s'' \in \mathcal{M}$ and $A \cap \mathrm{int}_r\, s'' \neq \emptyset$. Since the members of $\mathcal{M}$ are pairwise disjoint it follows

$$A = \mathrm{int}_r\, s'' \subset s'' \subset s' = \overline{\mathrm{int}_r\, s'} = \overline{B}$$

as required.

By the theorem above $\mathcal{M}$ is a regular complex. Since each subsystem $\mathcal{M}'$ of $\mathcal{M}$ satisfies conditions (i), (ii) and (iii) it follows that each subsystem of a simplicial complex is a regular complex.

II. Planar graphs. Let $(X, \mathcal{T})$ be a Hausdorff-space. For each simple arc v in $(X, \mathcal{T})$ let the kernel of v – denoted by $\ker v$ – be the set of the separating points of v, i.e. v itself without its two endpoints. Let $\mathcal{M}$ be a finite system of pairwise disjoint subsets of X, where each member of $\mathcal{M}$ is either a singleton or the kernel of a simple arc. Suppose that for each member of $\mathcal{M}$ of the form $\ker v$ and for each endpoint p of v the singleton $\{p\}$ belongs to $\mathcal{M}$. Then $\mathcal{M}$ clearly satisfies conditions (i) and (ii). It also satisfies condition (iii).

Indeed, let A, $B \in \mathcal{M}$ with

$$A \cap \overline{B} \neq \emptyset. \tag{31}$$

If B is s singleton then $\overline{B} = B$ and thus (31) implies

$$A \cap B \neq \emptyset. \tag{32}$$

However the members of $\mathcal{M}$ are pairwise disjoint, consequently by (32) we have $A = B$ and thus $A \subset \overline{B}$.

Now suppose that (31) holds and A is a singleton. Then clearly $A \subset \overline{B}$.

$A = B$ implies also $A \subset \overline{B}$.

Hence we need only to consider the case $A = \ker v$, $B = \ker v'$,

$$A \neq B \tag{33}$$

and

(34) $$A \cap \overline{B} \neq \emptyset.$$

Now (33) implies $A \cap B = \emptyset$. Hence by $\overline{B} = v'$ from (34) follows the existence of an endpoint p of v' with $p \in A = \ker v$. However by assumption we then have $\{p\} \in \mathcal{M}$. Moreover

(35) $$\{p\} \cap A \neq \emptyset.$$

Since the members of $\mathcal{M}$ are pairwise disjoint by (35) we obtain $A = \ker v =$ $= \{p\}$. But A is an infinite set, it cannot be a singleton. A contradiction arises, this case $A = \ker v$, $B = \ker v'$, $A \neq B$, $A \cap \overline{B} \neq \emptyset$ cannot occur.

Thus $\mathcal{M}$ satisfies condition (iii), $\mathcal{M}$ is a regular complex.

In the case if $(X, \mathcal{T})$ is the plane with the usual topology, $\mathcal{M}$ is said to be a *planar graph*.

Reference

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ON A SQUARE FUNCTION WITH RESPECT TO VILENKIN SYSTEM

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1. Introduction

The so-called Sunouchi operator S is well-known in the Walsh–Fourier analysis. S was introduced and firstly investigated by SUNOUCHI [12], [13], showing among others that S characterizes the L^p spaces for $p > 1$. This characterization fails to hold for $p = 1$, namely S doesn't form a bounded map from L^1 into L^1. However, in SIMON [7] we showed that $S : H^1 \to L^1$ is bounded, where H^1 is the dyadic Hardy space. Furthermore, a conjecture was formulated, whether H^1 can be characterized by S. The positive answer due to DALY and PHILLIPS [1], that is, the H^1-norm of a function f with mean value zero is equivalent to the L^1-norm of Sf. For the H^p $(0 < p \leq 1)$ version of Simon's and Daly and Phillips's results see SIMON [8], [11] and DALY and PHILLIPS [2].

In the so-called bounded-case the Vilenkin analogue of (H^1, L^1)-result was given by GÁT [3] (see also [4]). Moreover, he proved that if the Vilenkin group has an unbounded structure and H^1 is defined by means of the usual maximal function then S isn't bounded. Furthermore, if a modified H^1 space is considered (this martingale Hardy space was introduced by SIMON [6]), then a necessary and sufficient condition can be given for the Vilenkin group that $S : H^1 \to L^1$ to be bounded. All Vilenkin groups with bounded structure and also certain groups without this boundedness property staisfy Gát's condition.

Thus in the bounded case the (H^1, L^1)-boundedness of S remains true also for Vilenkin systems. We extended the Gát's result to H^p spaces in SIMON [11], showing the (H^p, L^p)-boundedness of S for all $0 < p \leq 1$.

Moreover, the equivalence $\|f\|_{H^p} \sim \|Sf\|_p$ $(0 < p \leq 1)$ was proved for f with mean value zero.

In this note we give a generalization of the Sunouchi operator and prove that in the bounded case the (H^p, L^p) $(0 < p \leq 1)$ result remains true. We will also investigate the role of the boundedness structure of the Vilenkin group.

2. Preliminaries and notations

In this section we introduce the most important definitions and notations and formulate some known results with respect to the Vilenkin system, which play a basic role in the further investigations. For details see VILENKIN [14] and the book SCHIPP–WADE–SIMON and Pál [5].

To the definition of the Vilenkin system let $m = (m_0, m_1, \dots, m_k, \dots)$ be given as a sequence of natural numbers with terms $m_k \geq 2$ $(k \in \mathbf{N} := \{0, 1, \dots \dots\})$. For all $k \in \mathbf{N}$ denote by Z_{m_k} the m_kth discrete cyclic group, where Z_{m_k} is represented by $\{0, 1, \dots, m_k - 1\}$. The so-called Vilenkin group G_m is the complete direct product of Z_{m_k}'s. The group G_m is a compact Abelian group with Haar measure 1, its elements are of the form $(x_0, x_1, \dots, x_k, \dots)$, where $x_k \in Z_{m_k}$ $(k \in \mathbf{N})$. The topology of the group G_m is completely determined by the sets

$$I_n := I_n(0) := \{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_j = 0 \qquad (j = 0, \dots, n-1)\}$$

$(0 \neq n \in \mathbf{N},\ I_0 := G_m)$. The Haar measure of I_n, is M_n, where the generalized powers M_n, $(n \in \mathbf{N})$ are defined in the following way: $M_0 := 1$, $M_n := = \prod_{j=0}^{n-1} m_j$ $(0 < n \in \mathbf{N})$. The symbol L^p $(0 < p \leq \infty)$ will denote the usual Lebesgue space of complex-valued functions f defined on G_m with the norm (or quasinorm) $\|f\|_p := (\int |f|^p)^{1/p}$ $(p < \infty)$, $\|f\|_\infty := \operatorname{ess\,sup} |f|$.

To the description of the characters of G_m let

$$r_n(x) := \exp \frac{2\pi i x_n}{m_n}$$

$(n \in \mathbf{N},\ x = (x_0, x_1, \dots) \in G_m,\ i := \sqrt{-1})$ and

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k M_k$ ($n_k \in Z_{m_k}$ ($k \in \mathbf{N}$)). Then the character system of G_m, (this is called Vilenkin system) is none another than $\{\Psi_n : n \in \mathbf{N}\}$. In the special case $m_n = 2$ ($n \in \mathbf{N}$) we get the Walsh–Paley system.

If $f \in L^1$ then $\hat{f}(k) := \int_{G_m} f\overline{\Psi}_k$ ($k \in \mathbf{N}$) is the usual Fourier coefficient of f. Let $S_n f$ ($n \in \mathbf{N}$) be the n-th partial sum of f, i.e.

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \cdot \psi_k .$$

Furthermore, let

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (0 < n \in \mathbf{N})$$

be the n-th Fejér's mean of f.

If $f \in L^1$ then the (martingale) maximal function of f is given by

$$f^*(x) = \sup_n |(S_{M_n} f)(x)| = \sup_n M_n \left| \int_{I_n(x)} f \right| \qquad (x \in G_m),$$

where $I_n(x)$ is the coset of I_n by x. In this connection we recall a good property of the Dirichlet kernels $D_{M_n} := \sum_{k=0}^{M_n - 1} \Psi_k$ ($n \in \mathbf{N}$), namely

(1) $$D_{M_n}(x) = \begin{cases} M_n & (x \in I_n) \\ 0 & (x \in G_m \setminus I_n). \end{cases}$$

It is known (see e.g. WEISZ [17]) that the maximal operator $L^p \ni f \to f^*$ ($1 < p < \infty$) is L^p-bounded, that is, $\|f^*\|_p \le C_p \|f\|_p$.

(From now on c_p, C_p, C will denote positive constants depending at most on p, not always the same at different occurrences.)

Define the (martingale) Hardy space H^p for $0 < p < \infty$ as the space of $f \in L^1$ for which

$$\|f\|_{H^p} := \|f^*\|_p < \infty.$$

Then $\|f\|_{H_p}$ is equivalent to $\|Qf\|_p$, i.e.

$$c_p \|Qf\|_p \le \|f\|_{H^p} \le C_p \|Qf\|_p \qquad (0 < p < \infty),$$

where Qf is the quadratic variation of f:

$$Qf := \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} |S_{M_{n+1}}f - S_{M_n}f|^2 \right)^{1/2}.$$

Moreover, for $1 < p < \infty$ we get the equivalence

$$c_p \|f\|_p \le \|f\|_{H^p} \le C_p \|f\|_p.$$

Let qf be the conditional quadratic variation of f defined by

$$qf := \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} S_{M_n}(|S_{M_{n+1}}f - S_{M_n}f|^2) \right)^{1/2}.$$

It is not hard to see that

$$qf := \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} \sum_{l=1}^{m_n-1} |S_{(l+1)M_n}f - S_{l M_n}f|^2 \right)^{1/2} =$$

$$= \left(|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} \sum_{l=1}^{m_n-1} \left| \sum_{k=l M_n}^{(l+1)M_n-1} \hat{f}(k)\Psi_k \right|^2 \right)^{1/2}.$$

Furthermore, in the case $0 < p \le 2$ we have $\|Qf\|_p \le C_p \|qf\|_p$, while for $2 \le p < \infty$ the converse inequality $\|qf\|_p \le C_p \|Qf\|_p$ holds (see WEISZ [17]). Moreover, if m is bounded then $(S_{M_n}f,\ n \in \mathbf{N})$ is a so-called regular martingale and so $\|Qf\|_p$ is equivalent to $\|qf\|_p$ for all $0 < p < \infty$. That is, $\sup_n m_n < \infty$ implies

$$c_p \|qf\|_p \le \|Qf\|_p \le C_p \|qf\|_p \qquad (0 < p < \infty). \tag{2}$$

A simple example shows that in (2) the boundedness of m is essential. Indeed, for $n \in \mathbf{N}$ let f_n be defined by $f_n := D_{M_{n+1}} - D_{M_n}$. Then by (1)

$$\|Qf_n\|_p = \|f_n\|_p = \left(\frac{(M_{n+1} - M_n)^p}{m_{n+1}} + M_n^p \left(\frac{1}{M_n} - \frac{1}{M_{n+1}} \right) \right)^{1/p} =$$

$$= \left((m_n - 1)^p \frac{M_n^{p-1}}{m_n} + (m_n - 1) M_n^{p-1} \frac{1}{m_n} \right)^{1/p} =$$

$$= \frac{M_n^{1-1/p}}{m_n^{1/p}} \left((m_n - 1)^p + m_n - 1 \right)^{1/p}.$$

On the other hand

$$qf_n = \left(\sum_{l=1}^{m_n-1} \left| \sum_{k=lM_n}^{(l+1)M_n-1} \Psi_k \right|^2 \right)^{1/2} = \left(\sum_{l=1}^{m_n-1} D_{M_n}^2 \right)^{1/2} = \sqrt{m_n - 1} D_{M_n},$$

which implies $\|qf_n\|_p = \sqrt{m_n - 1} M_n^{1-1/p}$. We get by a simple calculation that in the case $\sup_n m_n = \infty$ the estimations $\|Qf_n\|_p \leq C_p \|qf_n\|_p$ $(2 < p < \infty)$ and $\|qf_n\|_p \leq C_p \|Qf_n\|_p$ $(0 < p < 2)$ cannot be true for all $n \in \mathbf{N}$, resp.

3. Results

The purpose of this note is to investigate the operator T given by

$$Tf := \left(\sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} |S_{jM_n} f - \sigma_{jM_n} f|^2 \right)^{1/2} \qquad (f \in L^1).$$

This is a modification of the Vilenkin analogue of the Sunouchi operator S introduced and firstly investigated by GÁT [3] as

$$Sf := \left(\sum_{n=0}^{\infty} |S_{M_n} f - \sigma_{M_n} f|^2 \right)^{1/2} \qquad (f \in L^1).$$

It is clear that $Sf \leq Tf$ $(f \in L^1)$. We proved in SIMON [11] that $S : H^p \to L^p$ $(0 < p \leq 1)$ is bounded assumed $\sup_n m_n < \infty$. This is the extension of Gát's result for $p = 1$. Moreover, $\|Sf\|_p$ is equivalent to $\|f\|_{H^p}$ for $f \in H^p$ with mean value zero. (For the Walsh case, i.e. when $m_n = 2$ $(n \in \mathbf{N})$ see SIMON [8] and DALY and PHILLIPS [2].) The first statement of this work is the next

THEOREM. *Let m be bounded and $0 < p \leq 1$. Then $T : H^p \to L^p$ is bounded. Moreover, there exist positive constants c_p, C_p depending only on p such that for all $f \in H^p$ with $\hat{f}(0) = 0$ we have*

$$c_p \|f\|_{H^p} \leq \|Tf\|_p \leq C_p \|f\|_{H^p}.$$

PROOF. Let $f \in L^1$ and write Tf in the following form:

$$Tf = \left(\sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} \left| \sum_{k=0}^{jM_n-1} \frac{k}{jM_n} \hat{f}(k)\Psi_k \right|^2 \right)^{1/2} = \left(\sum_{n=0}^{\infty} \left(\left| \sum_{k=0}^{M_n-1} \frac{k}{M_n} \hat{f}(k)\Psi_k \right|^2 + \right.\right.$$

$$\left.\left. + \sum_{j=2}^{m_n-1} \left(\left| \frac{1}{jM_n} \sum_{k=0}^{M_n-1} k\hat{f}(k)\Psi_k + \frac{1}{jM_n} \sum_{k=M_n}^{jM_n-1} k\hat{f}(k)\Psi_k \right|^2 \right) \right) \right)^{1/2} \leq$$

$$\leq \left(\sum_{n=0}^{\infty} \left(1 + 2 \sum_{j=2}^{m_n-1} \frac{1}{j^2} \right) \left| \frac{1}{M_n} \sum_{k=0}^{M_n-1} k\hat{f}(k)\Psi_k \right|^2 \right)^{1/2} +$$

$$+ \sqrt{2} \left(\sum_{n=0}^{\infty} \sum_{j=2}^{m_n-1} \left| \frac{1}{jM_n} \sum_{k=M_n}^{jM_n-1} k\hat{f}(k)\Psi_k \right|^2 \right)^{1/2} \leq C(Sf + Rf), \tag{3}$$

where

$$Rf := \left(\sum_{n=0}^{\infty} \sum_{j=2}^{m_n-1} \left| \frac{1}{jM_n} \sum_{k=M_n}^{jM_n-1} k\hat{f}(k)\Psi_k \right|^2 \right)^{1/2} =$$

$$= \left(\sum_{n=0}^{\infty} \sum_{j=2}^{m_n-1} \left| \frac{1}{j} \sum_{l=1}^{j-1} \sum_{k=lM_n}^{(l+1)M_n-1} \frac{k}{M_n} \hat{f}(k)\Psi_k \right|^2 \right)^{1/2}.$$

Introduce the multiplier M by

$$Mf := \sum_{n=0}^{\infty} \sum_{i=M_n}^{M_{n+1}-1} \frac{i}{M_n} \hat{f}(i)\Psi_i .$$

Furthermore, define the mapping τ on the set of the complex-valued sequences as follows: if $a = (a_{nj}, n \in \mathbf{N}, j = 1, \ldots, m_n - 1)$ is such a sequence then let the sequence $\tau(a) := b = (b_{nj}, n \in \mathbf{N}, j = 1, \ldots, m_n - 1)$ be given by

$$b_{nj} := \frac{1}{j} \sum_{l=1}^{j-1} a_{nl} .$$

It is not hard to see that τ is ℓ_2-bounded when $\sup_n m_n < \infty$:

$$\|\tau(a)\|_{\ell_2}^2 \leq \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} |b_{nj}|^2 \leq \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{j^2} \left(\sum_{l=1}^{j} |a_{nl}| \right)^2 \leq$$

$$\leq \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{j} \sum_{l=1}^{j} |a_{nl}|^2 \leq \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} \frac{1}{j} \sum_{l=1}^{m_n-1} |a_{nl}|^2 \leq$$

$$\leq C \sum_{n=0}^{\infty} (\log m_n) \sum_{l=1}^{m_n-1} |a_{nl}|^2 \leq C \sum_{n=0}^{\infty} \sum_{l=1}^{m_n-1} |a_{nl}|^2,$$

that is, $\|\tau a\|_{\ell_2} \leq C \cdot \|a\|_{\ell_2}$. Hence assumed the boundedness of m we get

$$Rf \leq \left\|\tau \left((S_{(l+1)M_n}(Mf) - S_{lM_n}(Mf),\ n \in \mathbf{N},\ l = 1, \ldots, m_n - 1) \right) \right\|_{\ell_2} \leq$$

$$\leq C \|(S_{(l+1)M_n}(Mf) - S_{lM_n}(Mf),\ n \in \mathbf{N},\ l = 1, \ldots, m_n - 1)\|_{\ell_2} =$$

$$= C \left(\sum_{n=0}^{\infty} \sum_{l=1}^{m_n-1} |S_{(l+1)M_n}(Mf) - S_{lM_n}(Mf)|^2 \right)^{1/2} = Cq(Mf).$$

Therefore the next inequalities hold for all $0 < p < \infty$:

$$\|Rf\|_p \leq C_p \|q(Mf)\|_p \leq C_p \|Q(Mf)\|_p \leq C_p \|Mf\|_{H_p}. \tag{4}$$

In SIMON [11] we showed the (H^p, H^p)-boundedness of the multiplier M, i.e. the inequality $\|Mf\|_{H^p} \leq C_p \|f\|_{H^p}$ $(f \in H^p)$ for $0 < p \leq 1$. Thus $\|Rf\|_p \leq C_p \|f\|_{H^p}$ $(0 < p \leq 1)$ is also true from which $\|Tf\|_p \leq C_p \|f\|_{H^p}$ $(0 < p \leq 1)$ follows by the previous considerations.

Moreover, $Sf \leq Tf$ yields $\|Sf\|_p \leq \|Tf\|_p$ $(p > 0)$. If $0 < p \leq 1$, $f \in H^p$ and $\hat{f}(0) = 0$ then (see Simon [11]) $\|f\|_{H^p} \leq C_p \|Sf\|_p$. Hence we get $\|f\|_{H^p} \leq C_p \|Tf\|_p$. This proves Theorem 1. ∎

Especially, the equivalence $\|Sf\|_p \sim \|Tf\|_p$ $(f \in H^p, f(0) = 0)$ follows for all $0 < p \leq 1$ provided the boundedness of m.

The multiplier M is obviously (L^2, L^2)-bounded, i.e. $\|Mf\|_2 \leq C_2 \|f\|_2$ $(f \in L^2)$. From this it follows by interpolation (see WEISZ [15]) that M is of weak type $(1,1)$ and (L^p, L^p)-bounded for all $1 < p < \infty$. This implies by (4) the boundedness of $R : L^p \to L^p$ $(1 < p < \infty)$ which leads to weak type $(1,1)$ of R by interpolation. We remark that $Sf \leq CQ(Mf)$ (see SIMON [11]) thus our previous assertions hold also for S instead of R. In other words we get by (3) the following

COROLLARY. *Assume the boundedness of* m. *Then* T *is of weak type* $(1,1)$. *Moreover, for all* $1 < p < \infty$ *there exits a constant* C_p *depending only on* p *such that* $\|Tf\|_p \le \|f\|_p$ $(f \in L^p)$.

Finally we show that for unbounded m the equivalence $\|Tf\|_p \sim \|Sf\|_p$ $(0 < p \le 1)$ does not hold in general. Indeed, for $N \in \mathbf{N}$ let f_N be defined by

$$f_N := \sum_{l=1}^{m_N-1} r_N^l = \sum_{l=1}^{m_N-1} \Psi_{lM_N}.$$

Then

$$Tf_N \ge \left(\sum_{j=2}^{m_N-1} \left|S_{jM_N} f_N - \sigma_{jM_N} f_N\right|^2\right)^{1/2} =$$

$$= \left(\sum_{j=2}^{m_N-1} \left|\frac{1}{jM_N} \sum_{i=0}^{jM_N-1} i\hat{f}_N(i)\Psi_i\right|^2\right)^{1/2} = \left(\sum_{j=2}^{m_N-1} \frac{1}{j^2} \left|\sum_{l=1}^{j-1} l r_N^l\right|^2\right)^{1/2}.$$

If $A_t := \{x \in G_m : x_N = t\}$ $(t = 0, \ldots, m_N - 1)$ then the measure of A_t is equal to $1/m_N$ and $r_N(x) = 1$ $(x \in A_0)$. Therefore

$$\|Tf_N\|_p^p \ge \int_{A_0} |Tf_N|^p = \frac{1}{m_N} \left(\sum_{j=2}^{m_N-1} \frac{1}{j^2} \left(\sum_{l=1}^{j-1} l\right)^2\right)^{p/2} \ge$$

$$\ge C_p \frac{1}{m_N} \left(\sum_{j=2}^{m_N-1} j^2\right)^{p/2} \ge C_p \frac{1}{m_N} m_N^{3p/2},$$

i.e. $\|Tf_N\|_p \ge C_p m_N^{3/2-1/p}$. On the other hand

$$Sf_N = \left(\sum_{n=0}^{\infty} \left|\frac{1}{M_n} \sum_{k=0}^{M_n-1} k\hat{f}\Psi_k\right|^2\right)^{1/2} =$$

$$= \left(\left| \frac{1}{M_{N+1}} \sum_{l=1}^{m_N-1} l M_N r_N^l \right|^2 + \sum_{n=N+2}^{\infty} \left| \frac{1}{M_n} \sum_{l=1}^{m_N-1} l M_N r_N^l \right|^2 \right)^{1/2} =$$

$$= \frac{1}{M_{N+1}} \left| \sum_{l=1}^{m_N-1} l M_N r_N^l \right| \left(1 + \sum_{m=N+2}^{\infty} \left(\frac{M_{N+1}}{M_n} \right)^2 \right)^{1/2} \leq C \frac{1}{m_N} \left| \sum_{l=1}^{m_N-1} l r_N^l \right| .$$

From this estimation we get

$$\|Sf_N\|_p^p = \sum_{t=0}^{m_N-1} \int_{A_t} |Sf_N|^p \leq C_p \frac{1}{m_N^p} \sum_{t=0}^{m_N-1} \frac{1}{m_N} \left| \sum_{l=1}^{m_N-1} l \exp \frac{2\pi i l t}{m_N} \right|^p ,$$

where

$$\left| \sum_{l=0}^{m_N-1} l \exp \frac{2\pi i l t}{m_N} \right| = \begin{cases} \sum_{l=1}^{m_N-1} l = \frac{m_N(m_N-1)}{2} & (t = 0) \\ \frac{m_N}{\left| \exp \frac{2\pi i t}{m_N} - 1 \right|} & (t = 1, \ldots, m_N - 1). \end{cases}$$

Hence it follows for $\|Sf_N\|_p^p$ that

$$\|Sf_N\|_p^p \leq C_p m_N^{-p-1} \left(\frac{m_N^p (m_N - 1)^p}{2^p} + \sum_{t=1}^{m_N-1} \frac{m_N^p}{2^p \sin^p \frac{\pi t}{m_N}} \right) \leq$$

$$\leq C_p m_N^{-p-1} \left(m_N^{2p} + \sum_{t=1}^{m_N-1} \frac{m_N^{2p}}{t^p} \right) \leq C_p m_N^{p-1} \begin{cases} \log m_N & (p = 1) \\ m_N^{1-p} & (0 < p < 1). \end{cases}$$

Therefore we have $\|Sf_N\|_1 \leq C_1 \log m_N$ and $\|Sf_N\|_p \leq C_p$ $(0 < p < 1)$, resp. Assumed $\|Tf_N\|_p \leq C_p \|Sf_N\|_p$ $(0 < p \leq 1)$ we get $\sqrt{m_N} \leq C_1 \log m_N$ $(N \in \in \mathbf{N})$ which does not hold when m is unbounded. Analogously $m_N^{3/2-1/p} \leq \leq C_p$ $(N \in \mathbf{N})$ would be true for $0 < p < 1$, which fails to hold if $2/3 < p < 1$ and $\sup_m m_n = \infty$.

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PARTIAL RAPID STABILIZATION OF LINEAR DISTRIBUTED SYSTEMS

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1. Controllable states

Consider the problem

$$x' = Ax + Bu, \qquad x(0) = x_0, \tag{1.1}$$

where A is a densely defined, closed linear operator in some Hilbert space H and B is a bounded linear operator of another Hilbert space G into $D(A^*)'$. (Here and in the sequel the adjoint is denoted by a star $*$ and the dual by a prime $'$, so that $D(A^*)'$ is the dual space of the domain of the adjoint of A.)

Fix a positive number T. We recall that a state $x_0 \in H$ is called controllable (in time T) if there is a *control function* $u \in L^2(0, T; G)$ such that the solution of (1.1) satisfies $x(T) = 0$.

In order to characterize the controllable states, it is useful to introduce the *dual problem*

$$\varphi' = -A^*\varphi, \qquad \varphi(0) = \varphi_0, \qquad \psi = B^*\varphi. \tag{1.2}$$

The function ψ is called an *observation.*

The problems (1.1) and (1.2) are well posed if the following three hypotheses are satisfied (see [7] for details):

- (H1) the operator A^* generates a group e^{sA^*} in H';

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- (H2) we have $D(A^*) \subset D(B^*)$, and there exist two numbers $\lambda \in \mathbb{C}$ and $c \in \mathbb{R}$ such that

$$\|B^*\varphi_0\|_{G'} \le c\|(A+\lambda I)^*\varphi_0\|_{H'} \quad \text{for all} \quad \varphi_0 \in D(A^*);$$

- (H3) there exists a number $c' \in \mathbb{R}$ such that

$$\|\psi\|_{L^2(0,T;G)} \le c'\|\varphi_0\|_{H'} \quad \text{for all} \quad \varphi_0 \in D(A^*).$$

In order to simplify notations we identify G with its dual G' in the sequel.

Under these assumptions the formula

$$\langle \Lambda\varphi_0, \psi_0\rangle_{H,H'} := \int_0^T (B^* e^{-sA^*}\varphi_0, B^* e^{-sA^*}\psi_0)_G ds, \qquad \varphi_0, \psi_0 \in H' \tag{1.3}$$

defines a bounded, nonnegative selfadjoint operator Λ of H' into H. If the operators A and B are bounded (for example in the finite-dimensional case), then Λ can be defined equivalently by the simpler formula

$$\Lambda := \int_0^T e^{-sA} B B^* e^{-sA^*} ds. \tag{1.4}$$

We shall also use this formula in the general case as a mere convenient abbreviation for (1.3).

Next, we assume the following

- (H4) there exists a number c such that

$$\inf_{\psi_0 \in N(\Lambda)} \|\varphi_0 + \psi_0\|_{H'}^2 \le c \int_0^T \|B^* e^{-sA^*}\varphi_0\|_G^2$$

for all $\varphi_0 \in D(A^*)$.

If Λ is one-to-one, then this hypothesis reduces to the usual *inverse inequalities* in the terminology of the Hilbert Uniqueness Method, see [9] or [10]. An easy generalization of the results in [10], pp. 95–101 (see also [5]), leads to the

LEMMA 1. *Assume* (H1)–(H4). *Then Λ has a closed range $R(\Lambda)$. Furthermore, the quotient map Λ with respect to the kernel $N(\Lambda)$ of Λ is an isomorphism of $H'/N(\Lambda)$ onto $R(\Lambda)$.*

2. Rapid partial stabilization

We continue the study of problem (1.1). We seek feedback controls of the form $u = Fx$ which stabilize rapidly the controllable component of the solutions.

More generally, the following extension of the results in [7] allows us to construct feedbacks which stabilize only some of the first modes of the solutions. This is typical in experiments. It would be interesting to compare the results obtained by using the feedback constructed below with those obtained by BOURQUIN et al. [3], [4] by using global feedbacks.

We assume throughout this section that the assumptions (H1)–(H4) are satisfied. Fix a positive number ω, set $T_\omega = T + (2\omega)^{-1}$, and introduce the function e_ω defined by the formula

$$e_\omega(s) = \begin{cases} e^{-2\omega s} & \text{if } 0 \le s \le T, \\ 2\omega e^{-2\omega T}(T_\omega - s) & \text{if } T \le s \le T_\omega. \end{cases}$$

Note that

$$e_\omega(0) = 1,\ e_\omega(T_\omega) = 0,\ \text{and } e_\omega' + 2\omega e_\omega \le 0 \text{ everywhere.} \tag{2.1}$$

Similarly to the definition of the operator Λ in the preceding section, the formula

$$\langle \Lambda_\omega \varphi_0, \psi_0 \rangle_{H,H'} := \int_0^{T_\omega} e_\omega(s)(B^* e^{-sA*}\varphi_0, B^* e^{-sA^*}\psi_0)_G ds, \quad \varphi_0, \psi_0 \in H' \tag{2.2}$$

defines a bounded, nonnegative selfadjoint operator Λ_ω of H' into H. Formally, this is equivalent to the formula

$$\Lambda_\omega := \int_0^{T_\omega} e_\omega(s) e^{-sA} B B^* e^{-sA^*} ds, \tag{2.3}$$

the latter being justified if both A and B are bounded operators.

One can readily verify that lemma 1 remains valid for Λ_ω instead of Λ: the quotient map $\tilde{\Lambda}_\omega$ of Λ_ω with respect to its kernel $N(\Lambda_\omega)$ is an isomorphism of $H'/R(\Lambda_\omega)^\perp$ onto the closed subspace $R(\Lambda_\omega)$.

Consider a bounded linear map C of H into $R(\Lambda_\omega)$. Then its adjoint C^* vanishes on $R(\Lambda_\omega)^\perp$. Indeed, if $\varphi_0 \in H'$ is orthogonal to $R(\Lambda_\omega)$, then

$$\langle C^*\varphi_0, x_0\rangle = \langle \varphi_0, Cx_0\rangle = 0$$

(because $Cx_0 \in R(\Lambda_\omega)$) for all $x_0 \in H$ and hence $C^*\varphi_0 = 0$. Therefore the formula $C^*(\tilde{\Lambda}_\omega)^{-1}C$ defines a bounded linear operator of H into H'.

We are going to study the effect of the feedback control

$$u := -B^*C^*(\tilde{\Lambda}_\omega)^{-1}Cx$$

in problem (1.1). So let us consider the system

$$x' = Ax - BB^*C^*(\tilde{\Lambda}_\omega)^{-1}Cx, \qquad x(0) = x_0. \tag{2.4}$$

THEOREM 1. *Assume* (H1)–(H4)*. Then the problem* (2.4) *is well posed in H. Furthermore, if $AC = CA$, then there exists a constant M such that*

$$\|Cx(t)\|_H \le M\|x_0\|_H e^{-\omega t}. \tag{2.5}$$

for all $t \ge 0$ and for all $x_0 \in H$.

Rougly speaking, the (controllable) component of the solutions belonging to the range of C tends to zero rapidly.

PROOF. The well posedness of the problem (2.4) follows from a theorem of FLANDOLI [6]. In order to simplify notations we give a formal proof of (2.5) by using the formula (2.3). However, this proof is entirely correct if A and B are bounded operators, and it can be rewritten as a correct proof in the general case exactly as it was done in [7] for the case where C is the identity operator.

Set $P = \tilde{\Lambda}_\omega$ for brevity. If x solves (2.4), then a simple computation, using also the commutativity relation $AC = CA$, leads to the following identity:

$$\frac{d}{dt}\left\langle P^{-1}Cx, Cx\right\rangle = \left\langle p^{-1}Cx,\ (AP + PA^* - 2CBB^*C^*)P^{-1}Cx\right\rangle. \tag{2.6}$$

Let us show that

$$AP + PA^* - 2CBB^*C^* < -2\omega P. \tag{2.7}$$

For this evaluate the integral

$$\int_0^{T_\omega} \frac{d}{ds}(e_\omega(s)Ce^{-sA}ABB^*e^{-sA^*}C^*)ds$$

in two different ways. First, applying the Newton–Leibniz formula and then using the equalities $e_\omega(0) = 1$ and $e_\omega(T^\omega) = 0$, this integral is equal to $-CBB^*C^*$. Next, differentiating first the expression inside the integral by using the Leibniz rule and then integrating the resulting three terms separately, we obtain the expression

$$\int_0^{T_\omega} e_\omega'(s)Ce^{-sA}BB^*e^{-sA^*}C^*ds - AP - PA^*.$$

Thanks to (2.1) the first term is less than or equal to $-2\omega P$. Hence

$$-CBB^*C^* \leq -2\omega P - AP - PA^*.$$

Since $CBB^*C^* \geq 0$, this implies (2.7).

Using (2.7) we deduce from (2.6) the inequality

$$\frac{d}{dt}\left\langle P^{-1}Cx, Cx\right\rangle \leq -2\omega\left\langle P^{-1}Cx, Cx\right\rangle.$$

Integrating we obtain

$$\left\langle P^{-1}Cx(t), Cx(t)\right\rangle \leq -2\omega\left\langle P^{-1}Cx_0, Cx_0\right\rangle e^{-2\omega t} \tag{2.8}$$

for all $t \geq 0$.

Now observe that there exist two positive constants c_1 and c_2 such that

$$c_1\|Cx_1\|_H^2 \leq \left\langle P^{-1}Cx_1, Cx_1\right\rangle \leq c_2\|Cx_1\|_H^2$$

for all $x_1 \in H$. Indeed, this follows from the fact that $\left\langle P^{-1}y, y\right\rangle$ is a continuous symmetric and coercive bilinear form on the range $R(C)$ of C by hypotheses (H3) and (H4).

Using these inequalities we deduce from (2.8) that

$$c_1\|Cx(t)\|_H^2 \leq c_2\|Cx_0\|_H^2 e^{-2\omega t}$$

for all $t \geq 0$. This implies (2.5) with $M = \sqrt{c_2/c_1}$. ∎

3. An application

Consider the wave equation with Dirichlet boundary control:

$$
(3.1) \qquad \begin{cases} y'' - \Delta y = 0 \text{ in } \Omega \times \mathbb{R}, \\ y = u \text{ on } \Gamma \times \mathcal{R}, \\ y(0) = y_0 \text{ and } y'(0) = y_1 \text{ in } \Omega, \end{cases}
$$

where Ω is a bounded open set in $\mathbb{R}^N$ having a boundary Γ of class C^2. If we rewrite in the form

$$
x' = Ax + Bu, \qquad x(0) = x_0,
$$

then its dual problem

$$
\varphi' = -A^*\varphi, \quad \varphi(0) = \varphi_0, \quad \psi = B^*\varphi
$$

takes the following form (cf. [7], section 4):

$$
(3.2) \qquad \begin{cases} \xi'' - \Delta\xi = 0 \text{ in } \Omega \times \mathbb{R}, \\ \xi = 0 \text{ on } \Gamma \times \mathbb{R}, \\ \xi(0) = \xi_0 \text{ and } \xi'(0) = \xi_1 \text{ in } \Omega, \\ \psi = \partial_\nu \xi, \end{cases}
$$

where ∂_ν denotes the normal derivative with respect to the outward unit normal vector ν to the boundary.

For the dual problem H', A^* and B^* are given by the following formulae:

$$
\begin{gathered} H' = H_0^1(\Omega) \times L^2(\Omega), \\ D(A^*) = D(B^*) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \\ A^*(\eta_0, \eta_1) = (-\eta_1, -\Delta\eta_0), \\ B^*(\eta_0, \eta_1) = \partial_\nu \eta_0. \end{gathered}
$$

Let us observe that H' has an orthogonal basis consisting of eigenfunctions of A^*. Indeed, fix an orthogonal basis (z_n) of $L^2(\Omega)$ formed by eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$:

$$
\begin{gathered} -\Delta z_n = \lambda_n z_n \quad \text{in} \quad \Omega, \\ z_n = 0 \quad \text{on} \quad \Gamma, \\ 0 < \lambda_1 \le \lambda_2 \le \dots, \\ \lambda_n \to +\infty. \end{gathered}
$$

Now set

$$\omega_{2n-1} = i\sqrt{\lambda_n}, \qquad \Phi_{2n-1} = (z_n, -\omega_{2n-1} z_n)$$

and

$$\omega_{2n} = -i\sqrt{\lambda_n}, \qquad \Phi_{2n} = (z_n, -\omega_{2n} z_n)$$

for $n = 1, 2, \ldots$ Then one can readily verify that (Φ_n) is an orthogonal basis in H', formed by eigenfunctions of A^*:

$$A^*\Phi_n = \omega_n \Phi_n, \qquad n = 1, 2, \ldots$$

Fix an integer $M \geq 1$ and denote by C^* the orthogonal projection of H' onto the linear subspace spanned by the first M eigenfunctions $\Phi_1, \ldots, \Phi_M$. We have clearly $A^*C^* = C^*A^*$, so that $CA = AC$. We may therefore apply theorem 1.

Let us identify the operators A and C. It is defined in the dual space $H = H''$ of H'. If we identify $L^2(\Omega)$ with its dual, then $H = H^{-1}(\Omega) \times L^2(\Omega)$ and the sequence (Φ_n) is also an orthogonal basis in H. Furthermore, we have

$$\langle \Phi_n, \Phi_m \rangle_{H,H'} = 0$$

whenever $n \neq m$. Let us normalize the eigenfunctions Φ_n such that

$$\langle \Phi_n, \Phi_n \rangle_{H,H'} = 1$$

for all n. Now, given n arbitrarily, we have

$$\langle C\Phi_n, \Phi_m \rangle_{H,H'} = \langle \Phi_n, C^*\Phi_m \rangle_{H,H'} = \langle \Phi_n, \Phi_m \rangle_{H,H'} = \delta_{nm}$$

if $m \leq M$, and

$$\langle C\Phi_n, \Phi_m \rangle_{H,H'} = \langle \Phi_n, C^*\Phi_m \rangle_{H,H'} = \langle \Phi_n, 0 \rangle_{H,H'} = 0$$

if $m > M$. This means that C is the orthogonal projection of H onto the linear subspace spanned by $\Phi_1, \ldots, \Phi_M$.

Hence theorem 1 provides a feedback which rapidly stabilizes the first M components of the solution (with respect to the basis (Φ_n)).

REMARK. The same method can be applied for three other systems studied in sections 5, 6 and 7 of [7] and to Maxwell's equations by using the results in [8].

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ON THE NUMBER OF ZEROS OF NONOSCILLATORY SOLUTIONS TO SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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0. Preliminaries and new results

We consider the second order half-linear differential equation

$$(\mathrm{A}_\lambda) \qquad (|y'|^{\alpha-1}y')' + \lambda q(t)|y|^{\alpha-1}y = 0, \qquad t \geq a, \quad a \in \mathbb{R},$$

which can be rewritten as

$$(|y'|^{\alpha} \operatorname{sgn} \, y')' + \lambda q(t)|y|^{\alpha} \operatorname{sgn} \, y = 0, \qquad t \geq a,$$

or in a shorter form

$$(y'^{\alpha *})' + \lambda q(t) y^{\alpha *} = 0,$$

where $u^{\alpha *}$ denotes the function $|u|^{\alpha} \operatorname{sgn} u$.

Concerning (A_λ) we make the following hypotheses:

(H)
- (a) $\alpha > 0$ is a constant;
- (b) $\lambda > 0$ is a parameter;
- (c) $q : [a, \infty) \to [0, \infty)$ is a piecewise continuous function, and $q(t) \not\equiv 0$ on any interval $[T, \infty)$, $T \geq a$.

By a solution $y(t)$ of (A_λ) it is meant a function $y : [a, \infty) \to \mathbb{R}$ which is continuously differentiable on $[a, \infty)$ together with $|y'|^{\alpha-1}y'$ and satisfies (A_λ) at every point of $[a, \infty)$ where $q(t)$ is continuous. Particularly, the function $y(t) \equiv 0$ as the trivial solution will be excluded from our investigations

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because here we are interested in the question that how many zeros a solution may have on $[a,\infty)$, and this question has no meaning in the case of the trivial solution.

It is known that the zeros of a nontrivial solution are not accumulating at any finite point from $[a,\infty)$ (see [5]). A nontrivial solution of (A_λ) is said to be oscillatory if it has a sequence of zeros clustering at $t = \infty$ and nonoscillatory otherwise. A nonoscillatory solution has at most a finite number of zeros and it is eventually positive or negative.

In the limit case $\lambda = 0$ the functions of the form $k_0 + k_1 t$, where k_0 and k_1 are constants, are the only solutions of (A_0). Clearly, they are all nonoscillatory.

If $\alpha = 1$, then the differential equation (A_λ) is a well-known linear second order differential equation

$$(L_\lambda) \qquad y'' + \lambda q(t) y = 0,$$

thus (A_λ) can be considered as a natural generalization of the linear differential equations to some nonlinear differential equations.

Qualitative properties of solutions to half-linear equations of the form (A_λ) were studied first by MIRZOV [13] and ELBERT [3]. Further analysis on (A_λ) was made by several authors including DEL PINO et al. [2], ELBERT [4], [5], HOSHINO et al. [8], KUSANO et al. [9]–[11] and LI and YEH [12].

Their study shows a surprizing similarity between the qualitative properties of the solutions of (A_λ) and those of (L_λ). For example, the Sturmian theory of linear second order differential equations can be extended almost literatim and verbatim to half-linear differential equations of the form (A_λ) (see [13] or [3]). By this theory, the interlacing property of the zeros of solutions holds for (A_λ), too. Hence, for each fixed $\lambda > 0$, all nontrivial solutions of (A_λ) are either oscillatory, in which case the differential equation (A_λ) is called oscillatory, or else nonoscillatory and (A_λ) is called nonoscillatory. We are interested in the situation when (A_λ) is nonoscillatory for every $\lambda > 0$ and we are concerned with the question: Is it possible to count the number of zeros of (nonoscillatory) solutions of (A_λ)? Here we give a partial answer to this question by showing that considerably precise information about the number of zeros can be drawn for some particular types of solutions of (A_λ). Actually, we take up two types of solutions $\{y_0(t;\lambda)\}$ and $\{y_1(t;\lambda)\}$ such that

$$(0.1) \qquad \lim_{t\to\infty} y_0(t;\lambda) = k_0 \quad \text{for some constant } k_0 \neq 0$$

and

$$\lim_{t\to\infty} [y_1(t;\lambda) - k_1(t-a)] = 0 \quad \text{for some constant } k_1 \neq 0. \tag{0.2}$$

The solutions $y_0(t;\lambda)$ and $y_1(t;\lambda)$, if they exist, are referred to as a *subdominant solution* and a *dominant solution* of (A_λ), respectively. The subdominant and dominant solutions are essentially unique in the sense that if $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ denote the particular solutions of (A_λ) with the properties that

$$\lim_{t\to\infty} Y_0(t;\lambda) = 1 \tag{0.3}$$

and

$$\lim_{t\to\infty} [Y_1(t;\lambda) - (t-a)] = 0, \tag{0.4}$$

then the solutions $y_0(t;\lambda)$ and $y_1(t;\lambda)$ satisfying (0.1) and (0.2) are constant multiples of $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$, respectively, that is, $y_0(t;\lambda) = k_0\, Y_0(t;\lambda)$ and $y_1(t;\lambda) = k_1\, Y_1(t;\lambda)$ on $[a,\infty)$.

According to Lemma 1 from [6], any ultimately positive solution $y(t;\lambda)$ of (A_λ) has one of the following asymptotic behavior:

$$\begin{aligned}
&\text{(i)} \quad \lim_{t\to\infty} |y'(t;\lambda)|^{\alpha-1} y'(t;\lambda) = \text{const} > 0;\\
&\text{(ii)} \quad \lim_{t\to\infty} |y'(t;\lambda)|^{\alpha-1} y'(t;\lambda) = 0, \quad \lim_{t\to\infty} y(t;\lambda) = \infty;\\
&\text{(iii)} \quad \lim_{t\to\infty} |y'(t;\lambda)|^{\alpha-1} y'(t;\lambda) = 0, \quad \lim_{t\to\infty} y(t;\lambda) = \text{const} > 0.
\end{aligned} \tag{0.5}$$

According to this result, our dominant solution $Y_1(t;\lambda)$ and subdominant solution $Y_0(t;\lambda)$ are the extremal cases subject to (i) and (iii) in (0.5), respectively.

Then the question to be answered will be that how the number of zeros of $Y_0(t;\lambda)$ or $Y_1(t;\lambda)$ changes as the value of λ varies from zero to infinity. Now our answer to the question concerning the number of zeros of the solutions (A_λ) is formulated in the following theorems.

THEOREM 0.1. *Suppose*

$$\int^{\infty} \left(\int_t^{\infty} q(s)\,ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{0.6}$$

Then, for every $\lambda > 0$, (A_λ) *has a unique solution* $Y_0(t;\lambda)$ *satisfying* (0.3), *and there exists a sequence* $\{\lambda_n^{(0)}\}_{n=0}^{\infty}$ *of positive parameters with the properties that*

(i) $0 = \lambda_0^{(0)} < \lambda_1^{(0)} < \cdots < \lambda_n^{(0)} < \cdots, \quad \lim_{n\to\infty} \lambda_n^{(0)} = \infty$;

(ii) *for* $\lambda \in (\lambda_{n-1}^{(0)}, \lambda_n^{(0)})$, $n = 1, 2, \cdots$, $Y_0(t;\lambda)$ *has exactly* $n-1$ *zeros in* (a,∞) *and* $Y_0(a;\lambda) \neq 0$;

(iii) *for* $\lambda = \lambda_n^{(0)}$, $n = 1, 2, \cdots$, $Y_0(t;\lambda)$ *has exactly* $n-1$ *zeros in* (a,∞) *and* $Y_0(a;\lambda) = 0$.

THEOREM 0.2. *Let the sequence* $\{\lambda_n^{(0)}\}_{n=0}^{\infty}$ *be defined as in Theorem* 0.1. *Then the number of zeros of any (nontrivial) solution* $y(t;\lambda)$ *on* $[a,\infty)$ *can be:*

(i) *exactly* n *if* $\lambda = \lambda_n^{(0)}$ $(n = 1, 2, \cdots)$;

(ii) *either* $n-1$ *or* n *if* $\lambda_{n-1}^{(0)} < \lambda < \lambda_n^{(0)}$, *and both cases occur.*

THEOREM 0.3. *Suppose*

$$\int^{\infty} t^{\alpha+1} q(t)\,dt < \infty. \tag{0.7}$$

Then, for every $\lambda > 0$, (A_λ) *has a unique solution* $Y_1(t;\lambda)$ *satisfying* (0.4), *and there exists a sequence* $\{\lambda_n^{(1)}\}_{n=0}^{\infty}$ *of positive parameters with the properties that*

(i) $0 = \lambda_0^{(1)} < \lambda_1^{(1)} < \cdots < \lambda_n^{(1)} < \cdots, \quad \lim_{n\to\infty} \lambda_n^{(1)} = \infty$;

(ii) *for* $\lambda \in (\lambda_{n-1}^{(1)}, \lambda_n^{(1)})$, $n = 1, 2, \cdots$, $Y_1(t,\lambda) = 0$ *has exactly* n *zeros in* (a,∞) *and* $Y_1(a;\lambda) \neq 0$;

(iii) *for* $\lambda = \lambda_n^{(1)}$, $n = 1, 2, \cdots$, $Y_1(t;\lambda)$ *has exactly* n *zeros in* (a,∞) *and* $Y_1(a;\lambda) = 0$,

(iv) *the parameters* $\{\lambda_n^{(0)}\}$ *and* $\{\lambda_n^{(1)}\}$ *have the interlacing property* $0 = \lambda_0^{(1)} = \lambda_0^{(0)} < \lambda_1^{(1)} < \lambda_1^{(0)} < \cdots < \lambda_n^{(1)} < \lambda_n^{(0)} < \cdots$.

Concerning the conditions (0.6) and (0.7), we observe that (0.7) implies (0.6) because $q(t) \geq 0$ and

$$\int_t^{\infty} q(s)\,ds \leq \frac{1}{t^{\alpha+1}} \int_t^{\infty} s^{\alpha+1} q(s)\,ds \leq \frac{1}{t^{\alpha+1}} \int_a^{\infty} s^{\alpha+1} q(s)\,ds = \frac{K}{t^{\alpha+1}},$$

and

$$\int_a^\infty \left(\int_t^\infty q(s)\,ds \right)^{\frac{1}{\alpha}} \le \int_a^\infty \left(\frac{K}{t^{\alpha+1}} \right)^{\frac{1}{\alpha}} dt < \infty.$$

The proofs of the above theorems will be given in Section 2, while the preparation for those proofs will be made in Section 1. In Section 3 we extend these theorems to more general equations of the form

$$(\mathrm{B}_\lambda) \qquad (r(t)|y'|^{\alpha-1}y')' + \lambda q(t)|y|^{\alpha-1}y = 0, \qquad t \ge a,$$

where $r : [a, \infty) \to (0, \infty)$ is a continuous function, and we apply the extended results to the qualitative study of partial differential equations involving the p-Laplace operator of the type

$$(\mathrm{C}_\lambda) \qquad \operatorname{div}(|Du|^{p-2}Du) + \lambda c(|x|)|u|^{p-2}u = 0, \qquad x \in E_a,$$

where $p > 1$ is a constant, Du denotes the gradient of u in $\mathbb{R}^N$, $N \ge 2$, E_a is the exterior of a ball of radius $a > 0$ centered at the origin, and $c : [a, \infty) \to (0, \infty)$ is a continuous function.

In a recent paper [7] the notion of principal solutions known in the theory of nonoscillatory second order linear differential equations has been extended to nonoscillatory general half-linear equations (to more general than the ones of (A_λ)). Roughly speaking, a principal solution $y^*(t;\lambda)$ of (A_λ), which is determined up to a constant factor, is the "smallest" possible solution among the solutions of (A_λ). The principal solution $y^*(t;\lambda)$ enjoys several extremal properties (see [7]). We mention only one here. Suppose $y^*(t;\lambda)$ has zeros at $t = t_k$, $k = 1, 2, \ldots, m$ (and no zero any more). Then every other solution (linearly independent of $y^*(t;\lambda)$) has one zero in (t_1, t_2), (t_2, t_3), $\ldots$, (t_m, ∞). This fact will be recalled later as the extremal property of the zeros of $y^*(t;\lambda)$. Also it will be shown that the notion of the principal solutions and the notion of the subdominant solutions coincide for (A_λ).

In the proof of Theorems 0.1 and 0.2 extensive use is made of the generalized sine function introduced in [3]. The *generalized sine function* $S = S(\tau)$ is defined as the solution of the half-linear equation

$$(0.8) \qquad (|\dot{S}|^{\alpha-1}\dot{S})^{\cdot} + \alpha|S|^{\alpha-1}S = 0 \quad \left(\cdot = \frac{d}{d\tau} \right)$$

subject to the initial conditions $S(0) = 0$, $\dot{S}(0) = 1$. The generalized sine function $S(\tau)$ enjoys several properties remarkably resembling the trigonometric sine function $\sin\tau$. First of all it is periodic with period $2\pi_\alpha$, where

$$\pi_\alpha = \frac{2\pi}{\alpha + 1} \Big/ \sin\frac{\pi}{\alpha + 1}, \tag{0.9}$$

i.e. $S(\tau + 2\pi_\alpha) = S(\tau)$. Even more, $S(\tau + \pi_\alpha) = -S(\tau)$ for all τ and $S(\tau)$ is an odd function having zeros at $\tau = j\pi_\alpha$, $j \in \mathbb{Z}$. The derivative $\dot{S}(\tau)$ has zeros only at $\frac{1}{2}\pi_\alpha + j\pi_\alpha$, $j \in \mathbb{Z}$. Furthermore, the generalized Pythagorean theorem holds for $S(\tau)$:

$$|S(\tau)|^{\alpha+1} + |\dot{S}(\tau)|^{\alpha+1} = 1 \qquad \text{for all} \quad \tau. \tag{0.10}$$

The generalized tangent function $T(\tau)$ is defined by

$$T(\tau) = \frac{S(\tau)}{\dot{S}(\tau)} \qquad \text{for} \quad \tau \neq \frac{\pi_\alpha}{2} + m\pi_\alpha, \quad m \in \mathbb{Z}. \tag{0.11}$$

The generalized tangent function $T(\tau)$ is periodic with period π_α and satisfies

$$\dot{T} = 1 + |T|^{\alpha+1} > 0, \tag{0.12}$$

so that $T(\tau)$ is strictly increasing on the subintervals where it is continuous.

In [3] the generalized sine function is used to generalize in a natural way the notion of the Prüfer transformation, known for Sturm-Liouville equations, to half-linear equations of the type (A_λ). In the proof of Theorems 0.1 and 0.2 a crucial role will be played by this generalized Prüfer transformation, which consists in associating with a nontrivial solution $y(t;\lambda)$ of (A_λ) the generalized polar functions $\varphi(t;\lambda)$ and $\rho(t;\lambda)$ defined by

$$y(t;\lambda) = \rho(t;\lambda)S(\varphi(t;\lambda)), \qquad y'(t;\lambda) = \rho(t;\lambda)\dot{S}(\varphi(t;\lambda)) \tag{0.13}$$

where by (0.10)

$$\rho(t;\lambda) = \left(|y(t;\lambda)|^{\alpha+1} + |y'(t;\lambda)|^{\alpha+1}\right)^{\frac{1}{\alpha+1}}. \tag{0.14}$$

As in the classical Prüfer transformation, it can be shown that $\varphi(t;\lambda)$ and $\rho(t;\lambda)$ are continuously differentiable functions of t, and $(\varphi,\rho) = (\varphi(t;\lambda),\rho(t;\lambda))$ is a solution of the system of differential equations

$$\begin{aligned} \varphi' &= |\dot{S}(\varphi)|^{\alpha+1} + \frac{\lambda q(t)}{\alpha}|S(\varphi)|^{\alpha+1}, \\ \rho' &= \rho\left(1 - \frac{\lambda q(t)}{\alpha}\right)\dot{S}(\varphi)|S(\varphi)|^{\alpha-1}S(\varphi). \end{aligned} \tag{0.15}$$

Since the right hand side of the first equation in (0.15) is a Lipschitz function with respect to φ and the second equation is linear with respect to ρ, there exists, for every φ_0 and $\rho_0 > 0$, a unique solution $(\varphi(t;\lambda), \rho(t;\lambda))$ of (0.15) on $[a,\infty)$ such that $\varphi(t_0;\lambda) = \varphi_0$ and $\rho(t_0;\rho) = \rho_0$, where $t_0 \in [a,\infty)$ is any given initial point.

Let us observe that the function $\varphi(t;\lambda)$ can be found independently of $\rho(t;\lambda)$ and that if $\varphi(t;\lambda)$ is already known, then the function $\rho(t;\lambda)$ can be determined as

$$\rho(t;\lambda) = \rho_0 \exp\left[\int_{t_0}^{t} \left(1 - \frac{\lambda q(s)}{\alpha}\right) \dot{S}(\varphi(s;\lambda))|S(\varphi(s;\lambda))|^{\alpha-1} S(\varphi(s;\lambda))ds\right].$$

Hence $\rho(t;\lambda) > 0$ in $[a,\infty)$. On the other hand, if $(\varphi(t;\lambda), \rho(t;\lambda))$ is a solution of (0.15), then $(\varphi(t;\lambda) \pm \pi_\alpha, \rho(t;\lambda))$ is also a solution, which corresponds to the fact that if $y(t;\lambda)$ is a solution of (A_λ), then so is the function $-y(t;\lambda)$.

1. Some lemmas

We begin by formulating a fundamental lemma on the existence, uniquess and continuous dependence on parameters for solutions of the half-linear equation (A_λ).

LEMMA 1.1. *Let $t_0 \in [a,\infty)$ be fixed and let ξ and η be any given real constants. Then, for every $\lambda > 0$, (A_λ) has a unique solution $y(t) = y(t;t_0,\xi,\eta,\lambda)$ defined on $[a,\infty)$ and satisfying the initial condition*

$$y(t_0) = \xi, \qquad y'(t_0) = \eta.$$

This solution is a continuous function of $(t,t_0,\xi,\eta,\lambda) \in [a,\infty) \times [a,\infty) \times \mathbb{R} \times \mathbb{R} \times (0,\infty)$.

To prove this lemma it suffices to apply to (A_λ) a standard continuous dependence result (e.g. [1], pp. 18–19) in the theory of ordinary differential equations after observing that ([13], [3]) the initial value problem in question has a unique solution existing on $[a,\infty)$.

Crucial to our study of zeros of nonoscillatory solutions of (A_λ) is the following comparison theorem for the half-linear equations

$$(|y'|^{\alpha-1}y')' + q(t)|y|^{\alpha-1}y = 0, \tag{1.1}$$

$$(|z'|^{\alpha-1}z')' + Q(t)|z|^{\alpha-1}z = 0, \tag{1.2}$$

where $\alpha > 0$ is a constant, and $q(t)$ and $Q(t)$ are continuous functions on some interval $J \subset \mathbb{R}$.

LEMMA 1.2. *Suppose that* $Q(t) \geq q(t)$ *on* J. *Suppose that* (1.1) *has a nontrivial solution* $y(t)$ *with two zeros* t_1, t_2, *in* J, $t_1 < t_2$. *Then, any nontrivial solution* $z(t)$ *of* (1.2) *has a zero in* (t_1, t_2) *unless it is a constant multiple of* $y(t)$.

This theorem proven in [3, 13] was the prototype of the extensions of Sturmian theorems concerning the linear differential equations $(p(t)y')' + {}+ q(t)y = 0$ to half-linear differential equations of the form $(p(t)|y'|^{\alpha-1}y')' + {}+ q(t)|y|^{\alpha-1}y = 0$.

An important consequence of Lemma 1.2 is that if $y_1(t)$ and $y_2(t)$ are nontrivial, linearly independent solutions of (1.1), then the zeros of $y_1(t)$ separate and are separated by those of $y_2(t)$. Applying this result to (A_λ), we see that, for every fixed $\lambda > 0$, the solutions of (A_λ) are either all oscillatory ((A_λ) is oscillatory) or else all nonoscillatory ((A_λ) is nonoscillatory). Here we recall a result from [10].

LEMMA 1.3. *Differential equation* (A_λ) *is nonoscillatory for every* $\lambda > 0$ *if and only if* $q(t)$ *is integrable on* $[a, \infty)$ *and*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = 0. \tag{1.3}$$

This lemma is a direct generalization of a well-known result of NEHARI [14] for (L_λ).

Now it seems to be too ambitious for us to make an attempt to calculate the number of zeros of solutions of (A_λ) under the sole hypothesis (1.3) because the set of q's subject to (1.3) is too large. Instead of this, we imposed the more stringent condition (0.6) or (0.7) in the introduction to determine two classes of solutions of (A_λ) namely the subdominant solutions $\{y_0(t;\lambda)\}_{\lambda>0}$ and the dominant solutions $\{y_1(t;\lambda)\}_{\lambda>0}$. Here we are going to carry out an effective analysis concerning the dependence of the number of zeros upon the parameter λ. The subdominant and dominant solutions satisfying (0.1) and (0.2), respectively, are two extreme classes of solutions of (A_λ) on which our subsequent considerations are focused. Sharp conditions for the existence of such solutions are available.

LEMMA 1.4. (i) *There exist subdominant solutions of* (A_λ) *for every* $\lambda \geq 0$ *if and only if* (0.6) *holds. The subdominant solutions of* (A_λ) *are essentially unique, actually they are constant multiples of* $Y_0(t;\lambda)$ *specified in* (0.3) *which is the unique solution of the integral equation*

$$Y_0(t;\lambda) = 1 - \int_t^\infty \left(\int_s^\infty \lambda q(r)(Y_0(r;\lambda))^{\alpha *} dr \right)^{\frac{1}{\alpha} *} ds, \qquad t \geq a. \tag{1.4}$$

(ii) *There exist dominant solutions of* (A_λ) *for every* $\lambda \geq 0$ *if and only if* (0.7) *holds. These dominant solutions of* (A_λ) *are essentially unique, they are constant multiples of* $Y_1(t;\lambda)$ *specified in* (0.4) *which is the unique solution of the integral equation*

$$Y_1(t;\lambda) = t - a - \int_t^\infty \left[\left(1 + \int_s^\infty \lambda q(r)(Y_1(r;\lambda))^{\alpha *} dr \right)^{\frac{1}{\alpha} *} - 1 \right] ds, \ t \geq a. \tag{1.5}$$

PROOF. We begin by deriving the integral equations characterizing the subdominant and dominant solutions of (A_λ) for any $\lambda > 0$, fixed.

Let $y_0(t;\lambda)$ be a subdominant solution of (A_λ) subject to (0.1) with $k_0 > 0$. Then there exists $t_0 \geq a$ such that $y_0(t;\lambda) > 0$ for $t \geq t_0$. Then $y_0(t;\lambda) > 0$ is concave on $[t_0, \infty)$ and by (0.1) it is easy to show that

$$y_0'(t;\lambda) > 0 \qquad \text{for} \quad t \geq t_0 \quad \text{and} \quad \lim_{t \to \infty} y_0'(t;\lambda) = 0. \tag{1.6}$$

Integrating the equation (A_λ) with $y = y_0(t;\lambda)$ twice from t to ∞ and using (0.1) and (1.6), we have the integral equation

$$y_0(t;\lambda) = k_0 - \int_t^\infty \left(\int_s^\infty \lambda q(r)(y_0(r;\lambda))^{\alpha} dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geq t_0. \tag{1.7}$$

Conversely, if $y_0(t;\lambda)$ is positive, continuous on $[t_0, \infty)$ and satisfies (1.7), then differentiation of (1.7) shows that it is a subdominant solution of (A_λ) on $[t_0, \infty)$ satisfying (0.1).

Similarly, let $y_1(t;\lambda)$ be a dominant solution of (A_λ), subject to (0.2) with $k_1 > 0$. Suppose that $y_1(t;\lambda) > 0$ for $t \geq t_0$. We find that

$$y_1'(t;\lambda) > 0 \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \lim_{t \to \infty} y_1'(t;\lambda) = k_1 \tag{1.8}$$

then integrating (A_λ) with $y = y_1(t;\lambda)$ from t to ∞ and using the limit value of $y_1'(t;\lambda)$ from (1.8), we obtain

$$y_1'(t;\lambda) = \left(k_1^\alpha + \int_t^\infty \lambda q(s)(y_1(s;\lambda))^\alpha\,ds\right)^{\frac{1}{\alpha}}, \qquad t \geq t_0,$$

or

(1.9)

$$[y_1(t;\lambda) - k_1(t-a)]' = \left(k_1^\alpha + \int_t^\infty \lambda q(s)(y_1(s;\lambda))^\alpha\,ds\right)^{\frac{1}{\alpha}} - k_1, \qquad t \geq t_0.$$

Integration of (1.9) with the use of (0.2) yields the integral equation for $y_1(t;\lambda)$:

(1.10)

$$y_1(t;\lambda) = k_1(t-a) - \int_t^\infty \left[\left(k_1^\alpha + \int_s^\infty \lambda q(r)(y_1(r;\lambda))^\alpha\,dr\right)^{\frac{1}{\alpha}} - k_1\right] ds, \ t \geq t_0.$$

Conversely, if $y_1(t;\lambda)$ is a positive, continuous function solving (1.10), then $y_1(t;\lambda)$ is a dominant solution of (A_λ) satisfying (0.2).

Now we prove the "only if" parts of our lemma. Assume that (A_λ) has a subdominant solution $y_0(t;\lambda)$ and a dominant solution $y_1(t;\lambda)$ of (A_λ), both of which are positive for $t \geq t_0$. Then, as shown by the above arguments, the multiple integrals on the right hand side of (1.7) and (1.10) are convergent. As for $y_0(t;\lambda)$, since $y_0(t;\lambda) \geq y_0(t_0;\lambda)$ for $t \geq t_0$, we see that

$$\lambda^{\frac{1}{\alpha}} y_0(t_0;\lambda) \int_{t_0}^\infty \left(\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} dt \leq \int_{t_0}^\infty \left(\int_t^\infty \lambda q(s)(y_0(s;\lambda))^\alpha\,ds\right)^{\frac{1}{\alpha}} dt < \infty,$$

which implies the truth of (0.6). As for $y_1(t;\lambda)$, take $t_1 \geq t_0$ so large that $y_1(t;\lambda) \geq k_1 t/2$ for $t \geq t_1$ and

$$\int_{t_1}^\infty \lambda q(s)(y_1(s;\lambda))^\alpha\,ds \leq \frac{1}{2}k_1^\alpha. \tag{1.11}$$

Using (1.11) and the inequality

$$u^{\frac{1}{\alpha}} - v^{\frac{1}{\alpha}} \geq \min\left\{1, \frac{1}{\alpha}\right\} \min\left\{u^{\frac{1}{\alpha}-1}, v^{\frac{1}{\alpha}-1}\right\} (u - v)$$

valid for $u \geq v > 0$, we find that there exists a constant $c > 0$ depending only on α and k_1 such that

$$\left(k_1^{\alpha} + \int_t^{\infty} \lambda q(s)(y_1(s;\lambda))^{\alpha} ds\right)^{\frac{1}{\alpha}} - k_1 \geq c\lambda \int_t^{\infty} q(s)(y_1(s;\lambda))^{\alpha} ds, \qquad t \geq t_1.$$

Then the integrability of the left hand side of the above implies that

$$\infty > \int_{t_1}^{\infty} \int_t^{\infty} q(s)(y_1(s;\lambda))^{\alpha} ds dt \geq \left(\frac{k_1}{2}\right)^{\alpha} \int_{t_1}^{\infty} \int_t^{\infty} q(s)s^{\alpha} ds dt =$$

$$= \left(\frac{k_1}{2}\right)^{\alpha} \int_{t_1}^{\infty} s^{\alpha}(s - t_1)q(s)ds,$$

which is clearly equivalent to (0.7).

Let us turn to the proof of the "if" parts of the lemma. Suppose that (0.6) holds. Let $\lambda > 0$ be fixed arbitrarily. Define $K_0(\alpha)$ by

$$K_0(\alpha) = \begin{cases} \alpha 2^{\alpha-1} & \text{for} \quad \alpha \geq 1 \\ \frac{1}{\alpha} 2^{1-\alpha} & \text{for} \quad 0 < \alpha < 1 \end{cases}$$

and choose $t_0 \geq a$ so large that

$$K_0(\alpha) \int_{t_0}^{\infty} \left(\int_t^{\infty} \lambda q(s)ds\right)^{\frac{1}{\alpha}} dt \leq \frac{1}{2}. \tag{1.12}$$

Let $\mathcal{B}$ denote the Banach space of all bounded continuous functions $y(t)$ on $[t_0, \infty)$ with the norm $\|\cdot\|$ given by

$$\|y\| = \sup_{t \geq t_0} |y(t)|. \tag{1.13}$$

For any fixed constant $k_0 > 0$ define the subset $\mathcal{Y}_0$ of $\mathcal{B}$ by

$$\mathcal{Y}_0 = \left\{y \in \mathcal{B} : \quad \frac{1}{2}k_0 \leq y(t) \leq k_0, \quad t \geq t_0\right\}$$

and the operator $\mathcal{F}_0 : \mathcal{Y}_0 \to \mathcal{B}$ by

$$(\mathcal{F}_0 y)(t) = k_0 - \int_t^\infty \left(\int_s^\infty \lambda q(r)(y(r))^\alpha dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geq t_0. \tag{1.14}$$

It is clear that $y \in \mathcal{Y}_0$ implies $\mathcal{F}_0 y \in \mathcal{Y}_0$. We want to show that the operator $\mathcal{F}_0$ is a contraction on $\mathcal{Y}_0$. The inequality

$$|u^\nu - v^\nu| \leq \max\{1, \nu\} \max\left\{u^{\nu-1}, v^{\nu-1}\right\} |u - v| \tag{1.15}$$

holding for positive u, v and ν is used for this purpose. For any two elements y and z of $\mathcal{Y}_0$ we have, by using (1.15) twice,

$$\left| \left(\int_s^\infty \lambda q(r)(y(r))^\alpha dr \right)^{\frac{1}{\alpha}} - \left(\int_s^\infty \lambda q(r)(z(r))^\alpha dr \right)^{\frac{1}{\alpha}} \right| \leq \max\left\{1, \frac{1}{\alpha}\right\}$$

$$\max\left\{ \left(\int_s^\infty \lambda q(r)(y(r))^\alpha dr \right)^{\frac{1}{\alpha}-1}, \left(\int_s^\infty \lambda q(r)(z(r))^\alpha dr \right)^{\frac{1}{\alpha}-1} \right\}$$

$$\int_s^\infty \lambda q(r) \max\{1, \alpha\} \max\left\{(y(r))^{\alpha-1}, (z(r))^{\alpha-1}\right\} |y(r) - z(r)| dr, \qquad s \geq t_0,$$

which implies that

$$|(\mathcal{F}_0 y)(t) - (\mathcal{F}_0 z)(t)| \leq$$

$$\leq K_0(\alpha) \int_t^\infty \left(\int_s^\infty \lambda q(r) dr \right)^{\frac{1}{\alpha}-1} \left(\int_s^\infty \lambda q(r) |y(r) - z(r)| dr \right) ds$$

for $t \geq t_0$. In view of (1.13) and (1.12) we see that

$$\|\mathcal{F}_0 y - \mathcal{F}_0 z\| \leq K_0(\alpha) \|y - z\| \int_{t_0}^\infty \left(\int_s^\infty \lambda q(r) dr \right)^{\frac{1}{\alpha}} ds \leq \frac{1}{2} \|y - z\|,$$

demonstrating that $\mathcal{F}_0$ is a contraction mapping defined on $\mathcal{Y}_0$.

Consequently, $\mathcal{F}_0$ has a unique fixed point in $\mathcal{Y}_0$. Denote this fixed point by $y_0 = y_0(t;\lambda)$, then the definition (1.14) of $\mathcal{F}_0$ shows that $y_0(t;\lambda)$ is a solution of the integral equation (1.7) on $[t_0,\infty)$ satisfying (1.6). If we continue $y_0(t;\lambda)$ to the left of t_0 as a solution of (A_λ) (which is possible by Lemma 1.1), we obtain a subdominant solution of (A_λ) defined on the entire interval $[a,\infty)$.

Suppose next that (0.7) holds. Let $\lambda > 0$ be fixed arbitrarily and let $K_1(\alpha)$ be defined by

$$K_1(\alpha) = \begin{cases} \alpha & \text{for} \quad \alpha \geq 1 \\ \frac{1}{\alpha} 2^{\frac{1}{\alpha}-\alpha} & \text{for} \quad 0 < \alpha < 1. \end{cases}$$

Choose $t_0 > a$ so large that

(1.16)

$$K_1(\alpha)\lambda \int_{t_0}^{\infty} (t-a)^{\alpha} q(t)dt \leq \frac{1}{2}, \qquad K_1(\alpha)\lambda \int_{t_0}^{\infty} (t-a)^{\alpha+1} q(t)dt \leq \frac{1}{2}(t_0 - a).$$

For any given constant $k_1 > 0$ define $\mathcal{Y}_1$ to be the set of all continuous functions $y(t)$ on $[t_0,\infty)$ such that

$$\frac{1}{2}k_1(t-a) \leq y(t) \leq k_1(t-a) \qquad \text{for} \quad t \geq t_0 \quad \text{and} \quad \lim_{t\to\infty}[y(t)-k_1(t-a)] = 0.$$

Clearly, $\mathcal{Y}_1$ is a complete metric space with the distance function

$$d(y,z) = \sup_{t\geq t_0} |y(t) - z(t)|, \qquad y,z \in \mathcal{Y}_1.$$

Consider the mapping $\mathcal{F}_1 : \mathcal{Y}_1 \to C[t_0,\infty)$ defined by

(1.17)

$$(\mathcal{F}_1 y)(t) = k_1(t-a) - \int_t^{\infty} \left[\left(k_1^{\alpha} + \int_s^{\infty} \lambda q(r)(y(r))^{\alpha} dr \right)^{\frac{1}{\alpha}} - k_1 \right] ds, \qquad t \geq t_0.$$

It can be shown that $\mathcal{F}_1$ is a contraction on $\mathcal{Y}_1$. In fact, $y \in \mathcal{Y}_1$ implies, with the use of (1.15) and (1.16), that

$$0 < k_1(t-a) - (\mathcal{F}_1 y)(t) \leq \max\left\{1, \frac{1}{\alpha}\right\} \cdot$$

$$\cdot \int_t^\infty \max\left\{k_1^{1-\alpha},\ \left(k_1^\alpha + \int_s^\infty \lambda q(r)(y(r))^\alpha dr\right)^{\frac{1}{\alpha}-1}\right\} \int_s^\infty \lambda q(r)(y(r))^\alpha dr ds$$

$$\le k_1 K_1(\alpha) \int_t^\infty \left(\int_s^\infty \lambda q(r)(r-a)^\alpha dr\right) ds$$

$$\le k_1 K_1(\alpha) \lambda \int_{t_0}^\infty (r-a)^{\alpha+1} q(r) dr \le \frac{1}{2} k_1 (t-a)$$

for $t \ge t_0$, and $\lim_{t\to\infty} [(\mathcal{F}y)(t) - k_1(t-a)] = 0$. This shows that $\mathcal{F}_1$ maps $\mathcal{Y}_1$ into itself. Furthermore, using (1.15) twice, we obtain for $y, z \in \mathcal{Y}_1$

$$|(\mathcal{F}_1 y)(t) - (\mathcal{F}_1 z)(t)| \le \max\left\{1, \frac{1}{\alpha}\right\} \max\{1, \alpha\}$$

$$\int_t^\infty \left[\max\left\{\left(k_1^\alpha + \int_s^\infty \lambda q(r)(y(r))^\alpha dr\right)^{\frac{1}{\alpha}-1}, \left(k_1^\alpha + \int_s^\infty \lambda q(r)(z(r))^\alpha dr\right)^{\frac{1}{\alpha}-1}\right\}\right.$$

$$\left.\int_s^\infty \lambda q(r) \max\left\{(y(r))^{\alpha-1},\ (z(r))^{\alpha-1}\right\} |y(r) - z(r)| dr\right] ds \le$$

$$\le K_1(\alpha) \int_{t_0}^\infty \left(\int_s^\infty \lambda q(r)(r-a)^{\alpha-1} |y(r) - z(r)| dr\right) ds, \qquad t \ge t_0,$$

from which it follows that

$$d(\mathcal{F}_1 y, \mathcal{F}_1 z) \le K_1(\alpha) d(y,z) \int_{t_0}^\infty \left(\int_s^\infty \lambda q(r)(r-a)^{\alpha-1} dr\right) ds \le$$

$$\le K_1(\alpha) d(y,z) \int_{t_0}^\infty \lambda (r-a)^\alpha q(r) dr \le \frac{1}{2} d(y,z)$$

for any $y, z \in \mathcal{Y}_1$. The contraction mapping principle then ensures the existence of a unique element $y_1 = y_1(t;\lambda)$ in $\mathcal{Y}_1$ such that $y_1 = \mathcal{F}_1 y_1$, which, by the definition (1.17) of $\mathscr{F}_1$, is equivalent to the integral equation (1.10). Therefore, $y_1(t;\lambda)$ is a positive solution on $[t_0,\infty)$ satisfying (1.7), and the desired dominant solution of (A_λ) on $[a,\infty)$ is obtained via continuation of $y_1(t;\lambda)$ to the left of t_0.

The subdominant solution $y_0(t;\lambda)$ [respectively dominant solution $y_1(t;\lambda)$] constructed above is uniquely determined by its "terminal" value k_0 [respectively k_1]. This fact, combined with the observation that any constant multiple of a solution of (A_λ) is also a solution, shows that any two subdominant [respectively dominant] solutions of (A_λ) differ only by a constant mutiplicative factor. The subdominant and dominant solutions of (A_λ), if exist, are essentially unique in this sense, and so it suffices for us to examine the solutions $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ which satisfy (0.3) and (0.4), respectively. This completes the proof of Lemma 1.4.

REMARK 1.1. By (0.5), a principal solution $y^*(t;\lambda)$ must have the property $\lim_{t\to\infty} y^*(t;\lambda) = C > 0$ because it can be compared with the solution $Y_0(t;\lambda)$. Hence $y^*(t;\lambda)$ is a subdominant solution. By the uniqueness, proved in Lemma 1.4, we have $y^*(t;\lambda) = CY_0(t;\lambda)$. Consequently, the notion of principal solutions and the notion of subdominant solutions coincide if (0.6) holds.

2. Proof of main theorems

This section is devoted to the proof of Theorems 0.1, 0.2, 0.3, respectively, stated in the introduction, which are concerned with the subdominant solution $Y_0(t;\lambda)$ of (A_λ) satisfying (0.3) and the dominant solution $Y_1(t;\lambda)$ of (A_λ) satisfying (0.4).

In the preceding section, the existence of $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ for each $\lambda > 0$ is guaranteed by the conditions (0.6) and (0.7), respectively. First we show the continuous dependence of $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ upon the parameter $\lambda > 0$.

LEMMA 2.1. *For each fixed $t \geq a$, the functions $Y_0(t;\lambda)$, $Y_0'(t;\lambda)$, $Y_1(t;\lambda)$ and $Y_1'(t;\lambda)$ are continuous functions of λ for $\lambda > 0$.*

PROOF. (i) Let a positive constant Λ_0 be fixed and choose $t_0 \geq a$ so large that

$$K_0(\alpha)\int_{t_0}^{\infty}\left(\int_t^{\infty}\Lambda_0 q(s)ds\right)^{\frac{1}{\alpha}}dt \leq \frac{1}{2},$$

where $K_0(\alpha)$ is as in (1.12). Then from the proof of Lemma 1.4 we see that, for any $\lambda \in (0, \Lambda_0]$, $Y_0(t;\lambda)$ is the solution of the integral equation (1.4) and satisfies $\frac{1}{2} \leq Y_0(t;\lambda) \leq 1$ for $t \geq t_0$. For $0 < \lambda < \lambda' \leq \Lambda_0$, we compute

$$Y_0(t;\lambda) - Y_0(t;\lambda') = I_1 + I_2, \tag{2.1}$$

where

$$I_1 = \left(\lambda'^{\frac{1}{\alpha}} - \lambda^{\frac{1}{\alpha}}\right)\int_t^{\infty}\left(\int_s^{\infty} q(r)(Y_0(r;\lambda'))^{\alpha}dr\right)^{\frac{1}{\alpha}}ds$$

and

$$I_2 = \lambda^{\frac{1}{\alpha}}\int_t^{\infty}\left[\left(\int_s^{\infty} q(r)(Y_0(r;\lambda'))^{\alpha}dr\right)^{\frac{1}{\alpha}} - \left(\int_s^{\infty} q(r)(Y_0(r;\lambda))^{\alpha}dr\right)^{\frac{1}{\alpha}}\right]ds$$

for $t \geq t_0$. It is clear that

$$0 < I_1 \leq \left(\lambda'^{\frac{1}{\alpha}} - \lambda^{\frac{1}{\alpha}}\right)\int_{t_0}^{\infty}\left(\int_s^{\infty} q(r)dr\right)^{\frac{1}{\alpha}}ds \leq c_0\left(\lambda'^{\frac{1}{\alpha}} - \lambda^{\frac{1}{\alpha}}\right), \tag{2.2}$$

where $c_0 = 1/2K_0(\alpha)\Lambda_0^{\frac{1}{\alpha}}$. Applying the inequality (1.15) twice, we obtain for $t \geq t_0$

(2.3)

$$|I_2| \leq \lambda^{\frac{1}{\alpha}}K_0(\alpha)\int_t^{\infty}\left(\int_s^{\infty} q(r)dr\right)^{\frac{1}{\alpha}-1}\left(\int_s^{\infty} q(r)|Y_0(r;\lambda') - Y_0(r;\lambda)|dr\right)ds.$$

Since $\lim_{t\to\infty}[Y_0(t;\lambda') - Y_0(t;\lambda)] = 0$, we can define the function $u : [t_0, \infty) \to [0, \infty)$ by

$$u(t) = \sup_{r\geq t}|Y_0(r;\lambda') - Y_0(r;\lambda)| = \max_{r\geq t}|Y_0(r;\lambda') - Y_0(r;\lambda)|, \quad t \geq t_0.$$

Note that $u(t)$ is continuous and nonincreasing on $[t_0, \infty)$. Using (2.2) and (2.3), we find in (2.1) that

$$u(t) \le c_0 \left(\lambda'^{\frac{1}{\alpha}} - \lambda^{\frac{1}{\alpha}} \right) + \lambda^{\frac{1}{\alpha}} K_0(\alpha) \int_t^\infty \left(\int_s^\infty q(r) dr \right)^{\frac{1}{\alpha}} u(s) ds, \qquad t \ge t_0.$$

By the well-known Gronwall inequality (on unbounded interval) it follows

$$|Y_0(t;\lambda) - Y_0(t;\lambda')| \le u(t) \le$$

$$\le c_0 \left(\lambda'^{\frac{1}{\alpha}} - \lambda^{\frac{1}{\alpha}} \right) \exp \left[\lambda^{\frac{1}{\alpha}} K_0(\alpha) \int_t^\infty \left(\int_s^\infty q(r) dr \right)^{\frac{1}{\alpha}} ds \right], \qquad t \ge t_0,$$

for $0 < \lambda < \lambda' \le \Lambda_0$. This implies the continuity of $Y_0(t;\lambda)$ for each fixed $t \in [t_0, \infty)$ as a function of λ in $(0, \Lambda_0]$. The continuity of $Y_0'(t;\lambda)$ with respect to $\lambda \in (0, \Lambda_0]$ for fixed $t \in [t_0, \infty)$ is a direct consequence of the integral equation

$$Y_0'(t;\lambda) = \left(\int_t^\infty \lambda q(s) (Y_0(s;\lambda))^\alpha ds \right)^{\frac{1}{\alpha}}, \qquad t \ge t_0.$$

Application of Lemma 1.1 then shows that $Y_0(t;\lambda)$ and $Y_0'(t;\lambda)$ are continuous functions of $\lambda \in (0, \Lambda_0]$ for each fixed t in the interval $[a, t_0]$. Since $\Lambda_0 > 0$ is arbitrary, we conclude that, for each fixed $t \in [a, \infty)$, $Y_0(t;\lambda)$ and $Y_0'(t;\lambda)$ are continuous with respect to $\lambda \in (0, \infty)$.

(ii) Let $\Lambda_1 > 0$ be fixed arbitrarily and choose $t_0 > a$ so large that

$$K_1(\alpha) \int_{t_0}^\infty (t-a)^\alpha \Lambda_1 q(t) dt \le \frac{1}{2}, \qquad K_1(\alpha) \int_{t_0}^\infty (t-a)^{\alpha+1} \Lambda_1 q(t) dt \le \frac{1}{2} t_0,$$

where $K_1(\alpha)$ is the same as in (1.16). Then, similarly, $Y_1(t;\lambda)$ is the unique solution of the integral equation (1.5) and it satisfies $\frac{1}{2}(t-a) \le Y_1(t;\lambda) \le (t-a)$, $t \ge t_0$, for any $\lambda \in (0, \Lambda_1]$. Now, by (1.5) we have for $0 < \lambda < \lambda' \le \Lambda_1$

$$Y_1(t;\lambda) - Y_1(t;\lambda') = J_1 + J_2, \tag{2.4}$$

where

$J_1 =$

$$
= \int_t^\infty \left[\left(1 + \lambda' \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}} - \left(1 + \lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}} \right] ds,
$$

$J_2 =$

$$
= \int_t^\infty \left[\left(1 + \lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}} - \left(1 + \lambda \int_s^\infty q(r)(Y_1(r;\lambda))^\alpha dr \right)^{\frac{1}{\alpha}} \right] ds.
$$

Making use of (1.15), we obtain for the integrand of J_1

$$
\left| \left(1 + \lambda' \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}} - \left(1 + \lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}} \right| \le
$$

$$
\le \max\left\{1, \frac{1}{\alpha}\right\} \cdot \max\left\{ \left(1 + \lambda' \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}-1}, \right.
$$

$$
\left. \left(1 + \lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \right)^{\frac{1}{\alpha}-1} \right\}
$$

$$
(\lambda' - \lambda) \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr \le K_1(\alpha)(\lambda' - \lambda) \int_s^\infty q(r)(r-a)^\alpha dr,
$$

hence

$$
|J_1| \le K_1(\alpha)(\lambda' - \lambda) \int_t^\infty \int_s^\infty (r-a)^\alpha q(r) dr ds \le \tag{2.5}
$$

$$
\le K_1(\alpha)(\lambda' - \lambda) \int_{t_0}^\infty (s-a)^{\alpha+1} q(s) ds \le c_1(\lambda' - \lambda),
$$

where $c_1 = t_0/2K_1(\alpha)\Lambda_1$. To estimate J_2, we first note that

$$\left| \left(1+\lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr\right)^{\frac{1}{\alpha}} - \left(1+\lambda \int_s^\infty q(r)(Y_1(r;\lambda))^\alpha dr\right)^{\frac{1}{\alpha}} \right| \le$$

$$\le \max\left\{1, \frac{1}{\alpha}\right\} \max\left\{ \left(1+\lambda \int_s^\infty q(r)(Y_1(r;\lambda'))^\alpha dr\right)^{\frac{1}{\alpha}-1}, \right.$$

$$\left. \left(1+\lambda \int_s^\infty q(r)(Y_1(r;\lambda))^\alpha dr\right)^{\frac{1}{\alpha}-1} \right\}$$

$$\lambda \int_s^\infty q(r) \max\{1,\alpha\} \max\left\{(Y_1(r;\lambda))^{\alpha-1},\ (Y_1(r;\lambda'))^{\alpha-1}\right\} \cdot$$

$$\cdot |Y_1(r;\lambda) - Y_2(r;\lambda')| dr \le$$

$$\le K_1(\alpha)\lambda \int_s^\infty q(r)(r-a)^{\alpha-1} |Y_1(r;\lambda) - Y_1(r;\lambda')| dr, \qquad s \ge t_0.$$

Since $\lim_{t\to\infty} \left[Y_1(t;\lambda) - Y_1(t;\lambda')\right] = 0$, we can define the continuous nonincreasing function $u : [t_0,\infty) \to [0,\infty)$ by

$$u(t) = \sup_{r\ge t} |Y_1(r;\lambda) - Y_1(r;\lambda')| = \max_{r\ge t} |Y_1(r;\lambda) - Y_1(r;\lambda')|, \qquad t \ge t_0,$$

then we have in the above inequality

$$|J_2| \le K_1(\alpha)\lambda \int_t^\infty \left(\int_s^\infty q(r)(r-a)^{\alpha-1} dr\right) u(s) ds, \qquad t \ge t_0. \tag{2.6}$$

Combining (2.4) with (2.5) and (2.6), we get

$$u(t) \le c_1(\lambda' - \lambda) + \lambda K_1(\alpha) \int_t^\infty \left(\int_s^\infty q(r)(r-a)^{\alpha-1} dr\right) u(s) ds, \qquad t \ge t_0.$$

By Gronwall inequality we conclude that

$$|Y_1(t;\lambda) - Y_1(t;\lambda')| \le c_1(\lambda' - \lambda)\exp\left(\lambda K_1(\alpha)\int_t^\infty (r-a)^\alpha q(r)dr\right), \quad t \ge t_0,$$

which establishes the continuity of $Y_1(t;\lambda)$ with respect to $\lambda \in (0, \Lambda_1]$ for each fixed $t \in [t_0, \infty)$. The continuity of $Y_1'(t;\lambda)$ follows again from the integral representation

$$Y_1'(t;\lambda) = \left(1 + \int_t^\infty \lambda q(s)(Y_1(s;\lambda))^\alpha ds\right)^{\frac{1}{\alpha}}, \qquad t \ge t_0.$$

On the finite interval $[a, t_0]$ the continuity of $Y_1(t;\lambda)$ and $Y_1'(t;\lambda)$ with respect to $\lambda \in (0, \Lambda_1]$ is guaranteed by Lemma 1.1. Since Λ_1 can be chosen arbitrarily large, the proof of Lemma 2.1 is complete. ∎

Let us be concerned with the number of zeros of the subdominant and dominant solutions $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ of (A_λ).

LEMMA 2.2. (i) *The subdominant solution* $Y_0(t;\lambda)$ *has no zeros in* $[a, \infty)$ *for* $0 < \lambda < \lambda_0$ *provided* λ_0 *is sufficiently small, while the dominant solution* $Y_1(t;\lambda)$ *has exactly one zero in* $[a, \infty)$ *for these* λ*'s.*

(ii) *The number of zeros of* $Y_0(t;\lambda)$ *and* $Y_1(t;\lambda)$ *in* (a, ∞) *can be made as large as possible if* $\lambda > 0$ *is taken sufficiently large.*

PROOF. The first statement of (i) is almost clear from the proof of Lemma 1.4. In fact, if $\lambda_0 > 0$ is chosen small enough so that

$$K_0(\alpha)\int_a^\infty \left(\int_t^\infty \lambda_0 q(s)ds\right)^{\frac{1}{\alpha}} dt \le \frac{1}{2}$$

and the integral operator $\mathcal{F}_0$ is defined by

$$(\mathcal{F}_0 y)(t) = 1 - \int_t^\infty \left(\int_s^\infty \lambda q(r)(y(r))^\alpha\right)^{\frac{1}{\alpha}} ds, \qquad t \ge a,$$

then it can be verified that, for any $\lambda \in (0, \lambda_0)$, $\mathcal{F}_0$ is a contraction on the set

$$\mathcal{Y}_0 = \left\{y \in C[a, \infty): \quad \frac{1}{2} \le y(t) \le 1 \quad \text{for} \quad t \ge a\right\},$$

so that $\mathcal{F}_0$ has a fixed point in $\mathcal{Y}_0$, which gives rise to the desired positive subdominant solution $Y_0(t;\lambda)$ of (A_λ) on $[a,\infty)$ for $\lambda \in (0,\lambda_0)$.

Concerning the dominant solution $Y_1(t;\lambda)$ for $0 < \lambda < \lambda_0$ it is clear — due to Lemma 1.2 — that $Y_1(t;\lambda)$ may have at most one zero. Hence we have to exclude the possibility of the case when $Y_1(t;\lambda)$ has no zeros. Indeed, this would imply $Y_1(t;\lambda) > 0$ for all $t > a$. Hence by (1.5) we would have

$$Y_1(a;\lambda) = -\int_a^\infty \left[\left(1 + \int_s^\infty \lambda q(r)(Y_1(r;\lambda))^{\alpha *} dr \right)^{\frac{1}{\alpha}*} - 1 \right] ds < 0.$$

Therefore $Y_1(t;\lambda)$ must be negative in some (right) neighborhood of $t = a$, which proves the fact that $Y_1(t;\lambda)$ has exactly one zero on $[a,\infty)$.

The second statement (ii) follows again from Lemma 1.2. Consider the constant coefficient equation

$$(|y'|^{\alpha-1}y')' + \alpha\mu^{\alpha+1}|y|^{\alpha-1}y = 0, \qquad \mu > 0, \tag{2.7}$$

which has a solution $S(\mu t)$, where $S(\tau)$ is the generalized sine function, defined as a particular solution to (0.8). Hence $S(\mu t)$ has zeros at $t = j\pi_\alpha/\mu$, $j \in \mathbb{Z}$. Let the interval $[t',t''] \subset [a,\infty)$ be chosen in the way that $q(t) > 0$ for all $t \in [t',t'']$. Let $k \in \mathbb{N}$ be any given integer and take $\mu > 0$ large enough so that $S(\mu t)$ has at least $k+1$ zeros in $[t',t'']$. Then, comparing the differential equations (2.7) and (A_λ) with $\lambda > 0$, $\lambda \min_{[t',t'']} q(t) \geq \alpha\mu^{\alpha+1}$, we conclude by Lemma 2.2 that, for such values of λ, all solutions of (A_λ) have at least k zeros in $[t',t'']$, hence in $[a,\infty)$. This shows that the statement (ii) is true and the proof of Lemma 2.2 is completed.

PROOF OF THEOREM 0.1. Since (0.6) holds, there exists a unique subdominant solution $Y_0(t;\lambda)$ of (A_λ) satisfying (0.3) for every $\lambda > 0$. Let $(\varphi_0(t;\lambda),\rho_0(t;\lambda))$ be the polar functions associated with $Y_0(t;\lambda)$. We note that $\varphi_0(t;\lambda)$ is a solution of the first differential equation in (0.15) and the right hand side of this differential equation is nonnegative, hence $\varphi_0(t;\lambda)$ is a nondecreasing function of t for each fixed $\lambda > 0$. From (0.3) and (0.14) we can see that

$$\lim_{t\to\infty} \rho_0(t;\lambda) = 1, \quad \lim_{t\to\infty} \dot{S}(\varphi_0(t;\lambda)) = 0$$

hence

$$\lim_{t\to\infty} \varphi_0(t;\lambda) = \frac{\pi_\alpha}{2} + l\pi_\alpha \quad \text{for some } l \in \mathbb{Z}.$$

Without loss of generality we may suppose that $l = 0$ and we have

(2.8) $$\lim_{t\to\infty} \varphi_0(t;\lambda) = \frac{\pi_\alpha}{2}.$$

Now we claim that the inequality

(2.9) $$\varphi_0(t;\lambda) < \frac{\pi_\alpha}{2} \quad \text{for } a \le t < \infty,\ \lambda > 0$$

holds. This can be seen indirect way. Since $\varphi_0(t;\lambda)$ is nondecreasing, the relation $\varphi_0(t';\lambda) = \pi_\alpha/2$ would imply the equality $\varphi_0(t;\lambda) \equiv \pi_\alpha/2$ for all $t' \le t < \infty$, consequently $\varphi_0'(t;\lambda) \equiv 0$ for $t \ge t'$. From the differential equation of φ in (0.15) we would have $q(t) \equiv 0$ for $t \ge t'$. But this possibility is excluded by (c) in (H), which proves (2.9).

By Lemma 2.2 it happens that $Y_0(t;\lambda) = \rho_0(t;\lambda)S(\varphi_0(t;\lambda)) > 0$ if λ is small, hence the inequality $0 < \varphi_0(t;\lambda) < \pi_a/2$ must hold for all $t \in$ $\in [a,\infty)$. Suppose that $Y_0(t;\lambda)$ has one or several zeros. Since $Y_0(t;\lambda)$ is nonoscillatory, there exist only finitely many zeros:

$$a \le t_1 < t_2 < \cdots < t_m < \infty.$$

It is clear that $S(\varphi_0(t;\lambda)) \neq 0$ between the consecutive zeros t_k, t_{k+1} of $Y_0(t;\lambda)$. So, we have

$$\varphi_0(t_{k+1};\lambda) - \varphi_0(t_k;\lambda) = \pi_\alpha,$$

and consequently

$$\varphi_0(t_k;\lambda) = (k-m)\pi_\alpha, \qquad k = 1, 2, \cdots, m.$$

We list the basic properties of $\varphi_0(t;\lambda)$:

(a) $\varphi_0(t;\lambda)$ is continuous in $\lambda \in (0,\infty)$ for each fixed $t \ge a$;

(b) $\varphi_0(t;\lambda)$ is a strictly decreasing function of $\lambda > 0$ for each fixed $t \ge a$;

(c) $\lim_{\lambda\to 0+} \varphi_0(a;\lambda) = \frac{\pi_\alpha}{2}$;

(d) $\lim_{\lambda\to\infty} \varphi_0(a;\lambda) = -\infty$.

The property (a) is an immediate consequence of Lemma 2.1 combined with the definition of $\varphi_0(t;\lambda)$.

The property (b) follows from the results on the principal solutions of (A_λ) obtained in the paper [7], but we present here another proof of it without referring to the properties of the principal solutions. Assume that (b) is not true. Then there exist λ, λ' with $\lambda > \lambda' > 0$ and $b \in [a,\infty)$ such that

$\varphi_0(b;\lambda) \geq \varphi_0(b;\lambda')$. Then, comparing the first order differential equation (0.15) for $\varphi_0(t;\lambda)$ and $\varphi_0(t;\lambda')$, we have

$$\varphi_0(t;\lambda) \geq \varphi_0(t;\lambda') \qquad \text{for} \quad t \in [b,\infty),$$

and there exists $c \in [b,\infty)$ (depending on $q(t)$) such that

$$\varphi_0(t;\lambda) \geq \varphi_0(t;\lambda') > 0 \qquad \text{for} \quad t \in (c,\infty),$$

where the lower bound 0 is ensured by (2.8). Thus we have $Y_0(t;\lambda) > 0$, $Y_0(t;\lambda') > 0$ for $t \geq c$. By the definition of $T(\varphi)$ in (0.11) and $\varphi_0(t;\lambda)$, $\varphi_0(t;\lambda')$ in (0.13), we have

$$\frac{1}{T(\varphi_0(t;\lambda))} = \frac{Y_0'(t;\lambda)}{Y_0(t;\lambda)} \leq \frac{Y_0'(t;\lambda')}{Y_0(t;\lambda')} = \frac{1}{T(\varphi_0(t;\lambda'))} \tag{2.10}$$

for all sufficiently large t, i.e. $t \geq c$. Integrating (2.10) from t to τ, $t \leq \tau$, we get

$$\log Y_0(\tau;\lambda) - \log Y_0(t;\lambda) \leq \log Y_0(\tau;\lambda') - \log Y_0(t;\lambda'),$$

for which, in the limit as $\tau \to \infty$, it follows by (0.3) that

$$\log Y_0(t;\lambda) \geq \log Y_0(t;\lambda') \quad \text{or} \quad Y_0(t;\lambda) \geq Y_0(t;\lambda') \qquad \text{for all} \quad t \geq c.$$

Then we obtain from (1.4)

$$Y_0'(t;\lambda) = \left(\int_t^\infty \lambda q(s)(Y_0(s;\lambda))^\alpha\, ds\right)^{\frac{1}{\alpha}} \geq$$

$$\geq \left(\int_t^\infty \lambda' q(s)(Y_0(s;\lambda'))^\alpha\, ds\right)^{\frac{1}{\alpha}} = Y_0'(t;\lambda')$$

for $t \geq c$. Define $V(t) = Y_0(t;\lambda) - Y_0(t;\lambda')$. Then, $V(t) \geq 0$ and $V'(t) \geq 0$ for $t \geq c$. Since $V(t) \to 0$ as $t \to \infty$, we conclude that $V(t) \equiv 0$, that is, $Y_0(t;\lambda) \equiv Y_0(t;\lambda')$ for $t \geq c$. Then by (1.4) we have

$$V(t) = (\lambda^{\frac{1}{\alpha}} - \lambda'^{\frac{1}{\alpha}}) \int_t^\infty \left(\int_s^\infty q(r)(Y_0(r;\lambda))^\alpha\, dr\right)^{\frac{1}{\alpha}} ds = 0, \quad t \geq c,$$

which contradicts the fact that $\lambda' \neq \lambda$ and $q(t)$ is nonnegative and not identically zero on $[c,\infty)$ according to (c) in (H), and the property (b) is justified.

The statement (c) follows from the fact that $\lim_{\lambda\to 0+} Y_0(a;\lambda) = 1$ and $\lim_{\lambda\to 0+} Y_0'(a;\lambda) = 0$.

Finally, to verify (d) it suffices to combine (ii) of Lemma 2.2 with the observation that $Y_0(t;\lambda)$ has $k \in \mathbb{N}$ zeros in $[a,\infty)$ if and only if $-k\pi_\alpha < < \varphi_0(a;\lambda) \le -(k-1)\pi_\alpha$.

Let us now consider $\varphi_0(a;\lambda)$ as a function of $\lambda > 0$. Since it is continuous by (a), strictly decreasing by (b), positive for small $\lambda > 0$ by (c) and tending to $-\infty$ as $\lambda \to \infty$, for each $n \in \mathbb{N}$ there exists a unique number $\lambda_n^{(0)} > 0$ such that $\varphi_0(a;\lambda_n^{(0)}) = -(n-1)\pi_\alpha$. Put $\lambda_0^{(0)} = 0$. Then, as it is easily seen, the sequence of parameters $\{\lambda_n^{(0)}\}_{n=0}^{\infty}$ has the properties (i), (ii), (iii) stated in Theorem 0.1 which completes the proof.

Proof of Theorem 0.2. (i) Suppose that $Y_0(t;\lambda_n^{(0)})$ has zeros at $t_1 = = a < t_2 < \cdots < t_n < \infty$. By the exremal property of the zeros of principal solutions every other, linearly independent solution $y(t;\lambda_n^{(0)})$ has one zero in (t_1,t_2), (t_2,t_3), $\ldots$, (t_n,∞), i.e. $y(t;\lambda_n^{(0)})$ has exactly n zeros in $[a,\infty)$.

(ii) Now we distinguish two cases: (a) $Y_0(t;\lambda)$ has no zeros in $[a,\infty)$ (i.e. $n = 1$); and (b) $Y_0(t;\lambda)$ has $n-1$ zeros in $[a,\infty)$, where $n \ge 2$, by (ii) in Theorem 0.1. In case (a) it is clear that any solution $y(t;\lambda)$ may have at most one zero. In case (b) let $(a <)\ t_1 < t_2 < \cdots < t_{n-1}\ (< \infty)$ be $n-1$ zeros of $Y_0(t;\lambda)$. Then we see by the extremal property of the zeros of principal solutions that any solution $y(t;\lambda)$ has one zero in (t_1,t_2), (t_2,t_3), $\ldots$, (t_{n-1},∞), which means that $y(t;\lambda)$ has at least $n-1$ zeros. Some solutions may have another zero in $[a,t_1)$, too, i.e. they have n zeros which completes the proof of Theorem 0.2. ∎

Proof of Theorem 0.3. By (0.7) the differential equation (A_λ) possesses a unique dominant solution $Y_1(t;\lambda)$ for every $\lambda > 0$, satisfying (0.4). Let $(\varphi_1(t;\lambda), \rho_1(t;\lambda))$ denote the polar functions associated with $Y_1(t;\lambda)$. Since $\lim_{t\to\infty} [Y_1(t;\lambda)/t - 1] = \lim_{t\to\infty} [Y_1'(t;\lambda) - 1] = 0$, we find from (0.13) that

$$\lim_{t\to\infty} \frac{\rho_1(t;\lambda)}{t} = 1, \quad \lim_{t\to\infty} S(\varphi_1(t;\lambda)) = 1, \quad \lim_{t\to\infty} \dot{S}(\varphi_1(t;\lambda)) = 0.$$

Hence we can fix the function $\varphi_1(t;\lambda)$ by setting its terminal value

$$\lim_{t\to\infty} \varphi_1(t;\lambda) = \frac{\pi_\alpha}{2}.$$

As in the case of $\varphi_0(t;\lambda)$, it is easy to verify that

$$\varphi_1(t;\lambda) < \frac{\pi_\alpha}{2} \quad \text{for } a \le t < \infty,\ \lambda > 0,$$

and $\varphi_1(t;\lambda)$ has the following basic properties:

(a) $\varphi_1(t;\lambda)$ is continuous in $\lambda \in (0,\infty)$ for each fixed $t \ge a$;

(b) $\varphi_1(t;\lambda)$ is a strictly decreasing function of $\lambda > 0$ for each fixed $t \ge a$;

(c) $\lim\limits_{\lambda\to 0+} \varphi_1(a;\lambda) = 0$;

(d) $\lim\limits_{\lambda\to\ \infty} \varphi_1(a;\lambda) = -\infty$.

The property (a) above is evident.

The proof of property (b) is very similar to the one of the property (b) of $\varphi_0(t;\lambda)$ in Theorem 0.1, thus here we shall indicate only the steps where some modification takes place. The proof is indirect and one could suppose the existence of $c \in [a,\infty)$ and λ, λ', ($\lambda > \lambda' > 0$) such that

$$\left(\frac{\pi_\alpha}{2} >\right) \varphi_1(t;\lambda) \ge \varphi_1(t;\lambda') > 0 \qquad \text{for} \quad t \in (c,\infty),$$

which implies

$$\frac{Y_1'(t;\lambda)}{Y_1(t;\lambda)} \le \frac{Y_1'(t;\lambda')}{Y_1(t;\lambda')} \quad \text{for } t \ge c. \tag{2.11}$$

Integration of (2.11) on $[t,\tau]$ gives

$$\log Y_1(\tau;\lambda) - \log Y_1(\tau,\lambda') \le \log Y_1(t;\lambda) - \log Y_1(t;\lambda')$$

for $\tau \ge t \ge c$. Since the left hand side of this inequality can be written as

$$\log \frac{Y_1(\tau;\lambda)}{\tau} - \log \frac{Y_1(\tau,\lambda')}{\tau}$$

which tends to 0 if τ tends to ∞ due to (0.4). Therefore

$$\log Y_1(t;\lambda) - \log Y_1(t;\lambda') \ge 0 \quad \text{or} \quad Y_1(t;\lambda) \ge Y_1(t;\lambda') \qquad \text{for} \quad t \ge c.$$

By making use of the integral equation (1.5), we obtain $Y_1'(t;\lambda) \ge Y_1'(t;\lambda')$ for $t \ge c$ and the function $V(t) = Y_1(t;\lambda) - Y_1(t;\lambda')$ is identically zero on $[c,\infty)$, which is again a contradiction. This completes the proof of (b).

Property (c) follows from the fact (see Lemma 2.1) that $\lim\limits_{\lambda\to 0+} Y_1(t;\lambda) =$ $= t - a$, hence

$$\lim_{\lambda\to 0+} \varphi_1(a;\lambda) = 0.$$

Property (d) can be justified in the similar way as in Theorem 0.1.

The properties (a)–(d) of $\varphi_1(t;\lambda)$ enable us to find, for each $n \in \mathbb{N}$, a unique number $\lambda_n^{(1)} > 0$ such that $\varphi_1(a;\lambda_n^{(1)}) = -n\pi_\alpha$. Then, it is readily verified that the sequence of parameters $\{\lambda_n^{(1)}\}_{n=0}^{\infty}$ with $\lambda_0^{(1)} = 0$ has the properties (i), (ii) and (iii) in Theorem 0.2.

To show the property (iv) let us consider the dominant solution $Y_1(t;\lambda_n^{(1)})$ which has n zeros in (a,∞) and also a zero at $t = a$. Hence the zeros of $Y_1(t;\lambda_n^{(1)})$ are: a, t_1, ..., t_n. Now the corresponding subdominant/principal solution $Y_0(t;\lambda_n^{(1)})$ has one zero in (a,t_1), (t_1,t_2), ..., (t_{n-1},t_n), and by the extremal property of the zeros of principal solutions, it has no zero in $[t_n,\infty)$. Consequently, $Y_0(t;\lambda_n^{(1)})$ has exactly n zeros in (a,∞) and no zero at $t = a$. By Theorem 0.1 we have in this case the only possibility $\lambda_{n-1}^{(0)} < \lambda_n^{(1)} < \lambda_n^{(0)}$, which completes the proof of Theorem 0.3.

REMARK 2.1. We would like to call the reader's attention to an inherent property of the subdominant solution $Y_0(t;\lambda)$. Let $\tilde{a} \in (a,\infty)$ and consider differential equation (A_λ) on $[\tilde{a},\infty)$. Then on $[\tilde{a},\infty)$ we can define the subdominant solution $\tilde{Y}_0(t;\lambda)$. By the uniqueness stated in Theorem 0.1 we have $\tilde{Y}_0(t;\lambda) \equiv Y_0(t;\lambda)$ for $t \geq \tilde{a}$. But concerning $Y_1(t;\lambda)$ this is not the case. Clearly, on $[\tilde{a},\infty)$ we do have a unique dominant solution $\tilde{Y}_1(t;\lambda)$ but there is no guarantee that it will coincide with $Y_1(t;\lambda)$ on $[\tilde{a},\infty)$.

To achieve uniqueness for the dominant solutions in this sense, we should modify the definition. Such a definition could be made by the requirement

$$\lim_{t\to\infty} \left[Y_1(t;\lambda) - t \right] = 0$$

instead of (0.4). However, using this modified definition, we must face different behavior of $Y_1(t;\lambda)$ at $t = a$. We have chosen (0.4) as our definition because the treatment seemed to be simpler in this case.

3. Extension and application

A) *Extension.* We show that Theorems 0.1, 0.2 and 0.3 can be extended to more general equation (B_λ) where the function $r : [a,\infty) \to (0,\infty)$ is continuous and satisfies

$$\int_a^\infty (r(t))^{-\frac{1}{\alpha}} dt = \infty. \tag{3.1}$$

Define the function $R(t)$ by

$$R(t) = \int_a^t (r(\sigma))^{-\frac{1}{\alpha}} d\sigma, \quad t \geq a, \tag{3.2}$$

and perform the change of variables given by

$$s = R(t), \qquad \eta(s) = y(t).$$

Then (B_λ) is transformed into the equation

$$(|\dot\eta|^{\alpha-1}\dot\eta)^{\cdot} + \lambda Q(s)|\eta|^{\alpha-1}\eta = 0, \qquad s \geq 0, \tag{3.3}$$

which is of the same type as (A_λ), where $Q(s) = (r(t))^{\frac{1}{\alpha}} q(t)$ and a dot denotes differentiation with respect to s. This equation possesses a unique subdominant solution $\eta_0(s;\lambda)$ satisfying $\lim_{s\to\infty} [\eta_0(s;\lambda) - 1] = 0$ if and only if

$$\int_0^\infty \left(\int_s^\infty Q(\sigma) d\sigma\right)^{\frac{1}{\alpha}} ds < \infty, \tag{3.4}$$

and a unique dominant solution $\eta_1(s;\lambda)$ satisfying $\lim_{s\to\infty} [\eta_1(s;\lambda) - s] = 0$ if and only if

$$\int_0^\infty s^{\alpha+1} Q(s)\, ds < \infty. \tag{3.5}$$

The conditions (3.4) and (3.5) are equivalent, respectively, to

$$\int_a^\infty \left[(r(t))^{-1} \int_t^\infty q(s) ds\right]^{\frac{1}{\alpha}} dt < \infty \tag{3.6}$$

and

$$\int_a^\infty (R(t))^{\alpha+1} q(t) dt < \infty. \tag{3.7}$$

The solutions $Y_0(t;\lambda)$ and $Y_1(t;\lambda)$ of (B_λ) defined by $Y_0(t;\lambda) = \eta_0(s;\lambda)$ and $Y_1(t;\lambda) = \eta_1(s;\lambda)$ satisfy

(3.8) $$\lim_{t\to\infty} [\,Y_0(t;\lambda) - 1] = 0$$

and

(3.9) $$\lim_{t\to\infty} [\,Y_1(t;\lambda) - R(t)] = 0,$$

and are referred to as a subdominant solution and a dominant solution of (B_λ), respectively.

It is possible to count the number of zeros of these two types of solutions of (B_λ) in the sense of the following theorems which follow immediately from Theorems 0.1 and 0.3 applied to the equation (3.3).

THEOREM 3.1. *Suppose that* (3.1) *and* (3.6) *hold. Then, for every* $\lambda > 0$, (B_λ) *has a unique solution* $Y_0(t;\lambda)$ *satisfying* (3.8) *and there exists a sequence of positive parameters* $\{\lambda_n^{(0)}\}_{n=0}^\infty$ *with the properties that* (i), (ii), (iii) *of Theorem* 0.1 *hold.*

THEOREM 3.2. *Suppose that* (3.1) *and* (3.7) *hold. Then, for every* $\lambda > 0$, (B_λ) *has a unique solution* $Y_1(t;\lambda)$ *satisfying* (3.9) *and there exists a sequence of positive parameters* $\{\lambda_n^{(1)}\}_{n=0}^\infty$ *with the properties that* (i), (ii), (iii), (iv) *of Theorem* 0.3 *hold.*

B) *Application.* Let us consider the partial differential equation (C_λ) where $p > 1$ is a constant, $\lambda > 0$ is a parameter, Du denotes the gradient vector of $u = u(x)$ in $\mathbb{R}^N$, $N \geq 2$, E_a is the complement in $\mathbb{R}^N$ of the ball of radius $a > 0$ centered at the origin, and $c : [a,\infty) \to (0,\infty)$ is a continuous function.

We are interested in the radial solutions of (C_λ) satisfying the boundary condition

(D) $$u(x) = 0, \qquad x \in \partial E_a = \{x \in \mathbb{R}^N : \ |x| = a\}.$$

Observe that a radial function $u = y(|x|)$ is a solution of the exterior Dirichlet problem (C_λ)–(D) if and only if $y = y(t;\lambda)$ satisfies the ordinary differential equation

(3.10) $$(t^{N-1}|y'|^{p-2}y')' + \lambda t^{N-1}c(t)|y|^{p-2}y = 0, \qquad t \geq a,$$

which is a special case of (B_λ) with $\alpha = p-1$, $r(t) = t^{N-1}$ and $q(t) = t^{N-1}c(t)$, and the initial condition $y(a;\lambda) = 0$. Then condition (3.1) is satisfied for this $r(t)$ if and only if $p \geq N$, in which case the function $R(t)$ given by (3.2) becomes

(3.11)
$$R(t) = \frac{p-1}{p-N}\left(t^{\frac{p-N}{p-1}} - a^{\frac{p-N}{p-1}}\right) \text{ for } p > N, \quad R(t) = \log\frac{t}{a} \text{ for } p = N.$$

Now the condition (3.6) reduces to

$$\int^{\infty}\left(t^{1-N}\int_t^{\infty} s^{N-1}c(s)ds\right)^{\frac{1}{p-1}} dt < \infty, \tag{3.12}$$

which ensures the existence of a subdominant solution $Y_0(t;\lambda)$ of (3.10) satisfying

$$\lim_{t\to\infty}[Y_0(t;\lambda) - 1] = 0.$$

The condition (3.7) is equivalent to

(3.13)
$$\int^{\infty} t^{p-\frac{N-1}{p-1}} c(t)dt < \infty \text{ for } p > N, \quad \int^{\infty} t^{N-1}(\log t)^p c(t)dt < \infty \text{ for } p = N,$$

which ensures the existence of a dominant solution $Y_1(t;\lambda)$ of (3.10) satisfying

$$\lim_{t\to\infty}[Y_1(t;\lambda) - R(t)] = 0.$$

We define the radial solutions $U_0(x;\lambda)$ and $U_1(x;\lambda)$ by

$$U_0(x;\lambda) = Y_0(|x|;\lambda), \qquad U_1(x;\lambda) = Y_1(|x|;\lambda), \qquad x \in E_a,$$

and raise the question as to for what values of $\lambda > 0$, $U_0(x;\lambda)$ and $U_1(x;\lambda)$ are solutions of the problem (C_λ)–(D). An answer to this question follows immediately from Theorems 3.1, 3.2 applied to (3.10).

THEOREM 3.3. *Suppose that* $p \geq N$ *and* (3.12) *holds. Then, there exists a sequence of parameters* $\{\lambda_n^{(0)}\}_{n=1}^{\infty}$ *with the properties that*

(i) $0 < \lambda_1^{(0)} < \lambda_2^{(0)} < \cdots < \lambda_n^{(0)} < \cdots, \quad \lim_{n\to\infty}\lambda_n^{(0)} = \infty$;

(ii) $U_0(x;\lambda_n^{(0)})$ *is a radial solution of* (C_λ)–(D) *satisfying*
$$\lim_{|x|\to\infty}[U_0(x;\lambda_n^{(0)}) - 1] = 0;$$

(iii) $U_0(x;\lambda_n^{(0)})$ *has exactly* $n-1$ *spherical nodes in the interior of* E_a.

THEOREM 3.4. *Suppose that* $p \geq N$ *and* (3.13) *holds. Suppose in addition that* $a > 0$. *Then, there exists a sequence of parameters* $\{\lambda_n^{(1)}\}_{n=1}^{\infty}$ *with the properties that*

(i) $0 < \lambda_1^{(1)} < \lambda_2^{(1)} < \cdots < \lambda_n^{(1)} < \cdots, \quad \lim_{n\to\infty} \lambda_n^{(1)} = \infty$;

(ii) $U_1(x;\lambda_n^{(1)})$ *is a radial solution of* (C_λ)–(D) *satisfying* $\lim_{|x|\to\infty} [U_1(x;\lambda_n^{(1)}) - R(|x|)] = 0$;

(iii) $U_1(x;\lambda_n^{(1)})$ *has exactly* n *spherical nodes in the interior of* E_a.

REMARK 3.2. It remains to consider the equation (B_λ) in the case where $r(t)$ satisfies

$$\int_a^{\infty} (r(t))^{-\frac{1}{\alpha}}\, dt < \infty. \tag{3.14}$$

This case requires an independent analysis, since no change of variables is known which reduces (B_λ) subject to (3.14) to an equation of the type (A_λ). Any result for such an equation (B_λ) would automatically yield a corresponding result regarding radially symmetric solutions to the elliptic boundary value problem (C_λ)–(D) in the case where p and N satisfy $p < N$.

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ASYMPTOTIC APPROXIMATION BY QUADRATIC SPLINE CURVES

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1. Introduction and statement of result

Let C be a convex curve in the Euclidean plane $\mathbb{E}^2$ and let $\mathcal{P}_n(C)$ be the set of convex polygons with at most n vertices all lying on C and with the same endpoints as C. If the distance $\delta(C, P_n)$ of C and $P_n \in \mathcal{P}_n(C)$ is measured by the area of the region between C and P_n, then the problem of asymptotic best approximation is to study the behavior of

$$\delta(C, \mathcal{P}_n) = \inf\{\delta(C, P_n) \ : \ P_n \in \mathcal{P}_n(C)\}$$

as $n \to \infty$. For a C of differentiability class $\mathcal{C}^2$ with positive curvature function $\kappa(t)$, the following asymptotic formula was given by L. FEJES TÓTH [4], [5]

$$\delta(C, \mathcal{P}_n) \sim \frac{1}{12}\left(\int_0^l \kappa^{1/3}(t)dt\right)^3 \frac{1}{n^2} \quad \text{as} \quad n \to \infty, \tag{1}$$

where t is the arc length and l the length of C. A complete proof of this result is due to D. E. MCCLURE and R. A. VITALE [12]. Formula (1) holds also for general convex curves without any smoothness assumption, if κ is now the generalized curvature (see [10]) and these polytopal approximation problems are also solved for approximation of smooth convex surfaces in d-dimensional space (see P. M. GRUBER [7]). For more information we refer to the surveys [6], [8].

Here we consider the problem of approximating a planar curve C by quadratic spline curves and derive an asymptotic formula in this case. Let Q_n

be the set of quadratic spline curves with n knots, i.e. of curves consisting of n pieces of parabolas with common tangents at their common points. These curves are tangent continuous, i.e. their tangents (but not necessarily their tangent vectors) vary continuously. For a given planar curve C, let $\mathcal{Q}_n(C)$ be the set of $Q_n \in \mathcal{Q}_n$ with the same endpoints as C, with their knots lying on C, and with common tangents with C at their knots. As in the case of polygons, we measure the distance of C and Q_n by the area of the region between C and Q_n and study

$$\delta(C,\mathcal{Q}_n) = \inf\{\delta(C,Q_n) \ : \ Q_n \in \mathcal{Q}_n(C)\} \tag{2}$$

as $n \to \infty$.

Like the determination of the asymptotic behavior of $\delta(C,P_n)$, this problem of spline approximation is equi-affine invariant, i.e. invariant with respect to area preserving affine transformations. Therefore the asymptotic formula can be described with the help of notions from affine differential geometry (see the next section for definitions). Denote by $\lambda = \lambda(C)$ the affine length of C and by $k(s)$ the affine curvature of C given as a function of the affine arc length s, $0 \le s \le \lambda$.

THEOREM. *Let C be a curve in $\mathbb{E}^2$ of differentiability class $\mathcal{C}^4$ with positive affine curvature (or with negative affine curvature). Then*

$$\delta(C,\mathcal{Q}_n) \sim \frac{1}{240}\left(\int_0^{\lambda} |k^{1/5}(s)|ds\right)^5 \frac{1}{n^4} \tag{3}$$

as $n \to \infty$.

The restriction to curves with positive (or negative) affine curvature makes it possible to use in the proof arguments similar to that of the proof of the asymptotic formula (1). To determine just the order of approximation of this problem of spline approximation, it is also possible to use arguments from related approximation schemes (see [14], [2]).

For the problem of approximation of functions of one variable by spline functions asymptotic formulae were derived by D. D. PENCE and P. W. SMITH [13]. For polygons the problem of area approximation of a convex curve is equivalent to the problem of L^1-approximation of a function by linear splines. But quadratic spline functions have to be a class $\mathcal{C}^1$, whereas quadratic spline curves only have to be tangent continuous. The order of approximation is also different for these problems. For quadratic spline functions with n knots the order of approximation is $1/n^3$, whereas for quadratic spline curves with n knots we have $1/n^4$.

2. Tools from affine differential geometry

For the following notions from affine differential geometry, see the monographs by W. BLASCHKE [1] or K. LEICHTWEISS [9]. A recent survey on planar affine differential geometry is also contained in [3].

Let C be a planar curve of class $\mathscr{C}^2$ with positive curvature. The affine arc length is given by

$$s(t) = \int_0^t \kappa^{1/3}(\tau)d\tau, \qquad 0 < t < l, \tag{4}$$

where t is the ordinary arc length, $\kappa(t)$ the curvature and l is the length of C. The affine length λ of C is

$$\lambda = \int_0^l \kappa^{1/3}(\tau)d\tau.$$

In the following, let $x : [0,\lambda] \to \mathbb{E}^2$ be an affine arclength parametrization of C. The affine curvature of C at the point $x(s)$ is then given by

$$k(s) = |x''(s), x'''(s)|, \tag{5}$$

where $'$ denotes differentiation with respect to affine arc length and $|\cdot,\cdot|$ stands for determinant. The affine curvature $k(s)$ for $0 \le s \le \lambda$ determines a curve up to an area-preserving affine transformation. The curves with $k(s) = 0$ are parabolas. For a curve C with $k(s) > 0$ (or $k(s) < 0$) for $0 \le s \le \lambda$, the quadratic arc, i.e. a piece of a parabola, with endpoints on C and common tangents with C at these points lies completely on one side of C.

3. Proof of the Theorem

We only give the proof for $k > 0$, the proof for $k < 0$ is analogue. We need the following lemma.

LEMMA. *For* $0 \le s_1 \le s_2 \le \lambda$, *let* $G(s_1, s_2)$ *be the area of the moon-shaped piece between* C *and the quadratic curve with endpoints* $x(s_1)$ *and* $x(s_2)$. *Then*

$$G(s_1, s_2) = \frac{1}{2}\left(k(s_1)\frac{(s_2 - s_1)^5}{5!} + o\left((s_2 - s_1)^5\right)\right)$$

uniformly for all $0 \le s_1 \le s_2 \le \lambda$ *as* $(s_2 - s_1) \to 0$.

PROOF. We first calculate the area $F(s_1, s_2)$ of the piece between the curve C and the line segment joining $x(s_1)$ and $x(s_2)$. We have

$$F(s_1, s_2) = \frac{1}{2}\int_{s_1}^{s_2} |x(s) - x(s_1), x'(s)| ds$$

and applying Taylor's formula yields

$$F(s_1, s_2) = \frac{1}{2}\left(\frac{(s_2 - s_1)^3}{3!} - k(s_1)\frac{(s_2 - s_1)^5}{5!} + o\left((s_2 - s_1)^5\right)\right) \tag{6}$$

uniformly for all $0 \le s_1 \le s_2 \le \lambda$ as $(s_2 - s_1) \to 0$ (cf. [11], Lemma 1).

Second, we calculate the area $H(s_1, s_2)$ of the triangle formed by the line segment joining $x(s_1)$ and $x(s_2)$ and the tangents at $x(s_1)$ and $x(s_2)$. We have

$$H(s_1, s_2) = \frac{1}{2}\frac{|x'(s_1), x(s_2) - x(s_1)|\;|x(s_2) - x(s_1), x'(s_2)|}{|x'(s_1), x'(s_2)|}$$

and by Taylor's formula and some calculation we see that

$$H(s_1, s_2) = \frac{1}{2}\left(\frac{(s_2 - s_1)^3}{4} + o\left((s_2 - s_1)^5\right)\right) \tag{7}$$

uniformly for all $0 \le s_1 \le s_2 \le \lambda$ as $(s_2 - s_1) \to 0$. Since $k(s) > 0$ implies that the quadratic arc with endpoints at $x(s_1)$ and $x(s_2)$ lies completely on one side of C, we have

$$G(s_1, s_2) = \frac{2}{3}H(s_1, s_2) - F(s_1, s_2),$$

which together with (6) and (7) implies the lemma. ∎

First we introduce a new parameter r by

$$r(s) = \int_0^s k^{1/5}(\sigma) d\sigma,$$

which is possible, since $k(s) > 0$. Define

$$\rho = \int_0^\lambda k^{1/5}(\sigma)d\sigma.$$

By the mean value theorem of integral calculus there is a σ, $s_1 < \sigma < s_2$, such that

$$r(s_2) - r(s_1) = k^{1/5}(\sigma)(s_2 - s_1)$$

holds. Setting $r_1 = r(s_1)$ and $r_2 = r(s_2)$, we therefore have

$$(r_2 - r_1)^5 = k(s_1)(s_2 - s_1)^5 + o\left((s_2 - s_1)^5\right) \tag{8}$$

uniformly for all $0 \leq s_1 < s_2 \leq \lambda$ as $(s_2 - s_1) \to 0$. Let $K(r_1, r_2)$ be the area of the moon-shaped piece between C and the quadratic arc with knots at $x(r_1)$ and $x(r_2)$. Then by Lemma 3 and (8) we obtain that

$$K(r_1, r_2) = \frac{1}{240}(r_2 - r_1)^5 + o\left((r_2 - r_1)^5\right) \tag{9}$$

uniformly for all $0 \leq r_1 \leq r_2 \leq \rho$ as $(r_2 - r_1) \to 0$.

We take $Q_{n+1} \in \mathcal{Q}_{n+1}(C)$ with knots at $r_{ni} = \frac{i}{n}\rho$ for $0 \leq i \leq n$. Then by (9)

$$\delta(C, \mathcal{Q}_{n+1}) \leq \delta(C, Q_{n+1}) = \sum_{i=1}^{n} K(r_{n,i-1}, r_{ni}) = \frac{1}{240}\frac{\rho^5}{n^4} + o\left(\frac{1}{n^4}\right)$$

as $n \to \infty$ and consequently

$$\limsup_{n\to\infty}(n+1)^4\delta(C, \mathcal{Q}_{n+1}) \leq \frac{1}{240}\rho^5. \tag{10}$$

In order to show the opposite inequality, let $Q_{n+1} \in \mathcal{Q}_{n+1}(C)$ be a sequence of best approximating quadratic spline curves, i.e. for Q_{n+1} the infimum in (2) is attained. Such sequences exist, since $\delta(C, \mathcal{Q}_{n+1})$ depends continuously on its knots. Let $x(r_{ni})$ for $i = 0, \ldots, n$ be the knots of Q_{n+1} and set $\rho_{ni} = r_{ni} - r_{n,i-1}$. Since $\lim_{n\to\infty} \delta(C, \mathcal{Q}_{n+1}) \to 0$, $\max_{i=1,\ldots,n} \rho_{ni} \to 0$ as $n \to \infty$. Choose $\varepsilon > 0$. (9) shows that there is an integer n_0 such that for $n \geq n_0$

$$\left| K(r_{n,i-1}, r_{ni})^{1/5} - \left(\frac{1}{240}\right)^{1/5} \rho_{ni} \right| < \varepsilon\rho_{ni} \tag{11}$$

for $i = 1, \ldots, n$. Rewriting the mean value inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})^{1/5}\right)^5 \le \frac{1}{n}\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})$$

in the form

$$\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})^{1/5} \le n^{4/5}\left(\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})\right)$$

and combining this with (11), we see that

$$\begin{aligned}\left(\frac{1}{240}\right)^{1/5}\rho = \left(\frac{1}{240}\right)^{1/5}\sum_{i=1}^{n}\rho_{ni} < \sum_{i=1}^{n}\left(K(r_{n,i-1}, r_{ni})^{1/5} + \varepsilon\rho_{ni}\right) \le \\ \le n^{4/5}\left(\sum_{i=1}^{n} K(r_{n,i-1}, r_{ni})\right)^{1/5} + \varepsilon\rho = \\ = n^{4/5}\delta(C, \mathcal{Q}_{n+1})^{1/5} + \varepsilon\rho.\end{aligned}$$

Therefore

$$\left(\frac{1}{240}\right)^{1/5}\rho \le \liminf_{n\to\infty} n^{4/5}\delta(C, \mathcal{Q}_{n+1})^{1/5} + \varepsilon\rho.$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$\frac{1}{240}\rho^5 \le \liminf_{n\to\infty}(n+1)^4\delta(C, \mathcal{Q}_{n+1}).$$

Combined with (10) this concludes the proof of the theorem.

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PARADOXICAL SETS UNDER $\mathrm{SL}_2[\mathbf{R}]$

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Let $\mathrm{SL}_2[\mathbf{R}]$ denote the group of 2×2 real matrices with determinant 1, and let $X = \mathbf{R}^2 \setminus \{(0,0)\}$. It is well-known that the action of $\mathrm{SL}_2[\mathbf{R}]$ on the set X is locally commutative and that X is paradoxical under $\mathrm{SL}_2[\mathbf{R}]$ (see [5], pp. 39 and 97). S. WAGON asked if the interior of the unit square is paradoxical under $\mathrm{SL}_2[\mathbf{R}]$, or even under $\mathrm{SL}_2[\mathbf{Z}]$ ([5], Question 7.4, p. 101). JAN MYCIELSKI proved that the answer to the second question is negative. Moreover, as he proved in Theorem 2 of [3], no bounded set with nonempty interior can be paradoxical under $\mathrm{SL}_2[\mathbf{Z}]$. Mycielski also proved that the interior of the unit square is paradoxical under $\mathrm{SL}_2[\mathbf{R}]$, subject to the following conjecture (C): there is a free group F acting on the set $D = \{x \in \mathbf{R}^2 : 0 < |x| < 1\}$ without nontrivial fixed points such that each transformation of F is the union of finitely many elements of $\mathrm{SL}_2[\mathbf{R}]$, restricted to subsets of D.

In fact, MYCIELSKI proves that conjecture (C) implies the following more general result: if A and B are bounded subsets of X with nonempty interior, and either they contain triangles with one vertex at $(0,0)$ or their distance from the origin is positive, then A and B are $\mathrm{SL}_2[\mathbf{R}]$-equidecomposable.

In this note we prove this statement without using conjecture (C). The question, whether conjecture (C) is true or not, remains open.

THEOREM 1. *Let $\mathcal{A}$ denote the system of all bounded subsets of $\mathbf{R}^2$ with nonempty interior and with positive distance from the origin. Then each element of $\mathcal{A}$ is paradoxical under* $\mathrm{SL}_2[\mathbf{R}]$, *and any two elements of $\mathcal{A}$ are* $\mathrm{SL}_2[\mathbf{R}]$*-equidecomposable.*

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If a bounded set $A \subset X$ contains a triangle with one vertex at $(0,0)$ then there are rotations $\rho_1, \dots, \rho_n$ about the origin such that $\rho_1(A) \cup \dots \cup \rho_n(A)$ contains a set of the form $\{x \in \mathbf{R}^2 : 0 < |x| < r\}$. The fact that all these sets are paradoxical and are equidecomposable to each other, is a consequence of the next result.

THEOREM 2. *Let $\mathcal{B}$ denote the system of those bounded sets $A \subset \mathbf{R}^2 \setminus \{(0,0)\}$ for which there are linear transformations $\alpha_1, \dots, \alpha_n \in \mathrm{SL}_2[\mathbf{R}]$ such that $\bigcup_{i=1}^{n} \alpha_i(A) \cup \{(0,0)\}$ contains a neighbourhood of the origin. Then each element of $\mathcal{B}$ is paradoxical under $\mathrm{SL}_2[\mathbf{R}]$, and any two elements of $\mathcal{B}$ are $\mathrm{SL}_2[\mathbf{R}]$-equidecomposable.*

LEMMA 1. *Let the real numbers a_k, b_k, c_k, d_k $(k \in I)$ be algebraically independent over the rationals, and suppose that the numbers $D_k = a_k d_k - b_k c_k$ are positive for every $k \in I$. Then the linear transformations*

$$\alpha_k = \begin{bmatrix} a_k/\sqrt{D_k} & b_k/\sqrt{D_k} \\ c_k/\sqrt{D_k} & d_k/\sqrt{D_k} \end{bmatrix}$$

$(k \in I)$ generate a free subgroup of $\mathrm{SL}_2[\mathbf{R}]$.

PROOF. It is well-known that $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{ax+b}{cx+d}$ defines a homomorphism of $\mathrm{SL}_2[\mathbf{R}]$ into the group of linear fractional transformations. By a theorem of JOHN VON NEUMANN [4], the transformations $\beta_k = (a_k x + b_k)/(c_k x + d_k)$ $(k \in I)$ generate a free group (see also [1], Theorem 3.1). Since $\phi(\alpha_k) = \beta_k$ for every $k \in I$, the statement of the Lemma follows. ∎

The following lemma is a special case of Theorem 3 of [2].

LEMMA 2. *Let G be a locally commutative group acting on a set X and suppose that G is freely generated by the transformations $f_1, \dots, f_n$. Let A and B be subsets of X such that*

(i) *for every $x \in A$ there are indices $1 \le i, j \le n$, $i \ne j$ such that $f_i(x) \in B$, $f_j(x) \in B$;*

and

(ii) *for every $y \in B$ there are indices $1 \le i, j \le n$, $i \ne j$ and points $x_i, x_j \in A$ such that $f_i(x_i) = f_j(x_j) = y$.*

Then A and B are G-equidecomposable.

PROOF OF THEOREM 1. First we note that the action of $SL_2[\mathbf{R}]$ on X is transitive. Indeed, if $x, y \in X$ are given, then a rotation maps x into $|x|$, a linear map of the form $\begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix}$ maps $|x|$ into $|y|$, and another rotation maps $|y|$ into y. Then the composition of these maps belongs to $SL_2[\mathbf{R}]$, and maps x into y.

Let $A, B \in \mathscr{A}$ be fixed. Then every $x \in X$ can be mapped, by a linear transformation $\alpha_x = \begin{bmatrix} u^x_{11} & u^x_{12} \\ u^x_{21} & u^x_{22} \end{bmatrix} \in SL_2[\mathbf{R}]$ into int B. By the continuity of α_x, there is an $\varepsilon_x > 0$ such that whenever $|z - x| < \varepsilon_x$ and $|a_{ij} - u^x_{ij}| < \varepsilon_x$ for every $i, j = 1, 2$, then $\alpha(z) \in B$, where $\alpha = \begin{bmatrix} a_{11}/\sqrt{D} & a_{12}/\sqrt{D} \\ a_{21}/\sqrt{D} & a_{22}/\sqrt{D} \end{bmatrix}$ and $D = a_{11}a_{22} - a_{12}a_{21}$. Since cl A (the closure of A) is compact, there is a finite set $U \subset$ cl A such that the open balls $B(x, \varepsilon_x)$ $(x \in U)$ cover cl A. In the same way we find another finite set $V \subset B$ and positive numbers η_y for every $y \in B$ with the following properties: the open balls $B(y, \eta_y)$ $(y \in V)$ cover cl B, and for every $y \in V$ there are numbers v^y_{ij} $(i, j = 1, 2)$ such that whenever $|w - y| < \eta_y$ and $|c_{ij} - v^y_{ij}| < \eta_y$ for every $i, j = 1, 2$, then $\gamma(w) \in A$, where $\gamma = \begin{bmatrix} c_{11}/\sqrt{D} & c_{12}/\sqrt{D} \\ c_{21}/\sqrt{D} & c_{22}/\sqrt{D} \end{bmatrix}$ and $D = c_{11}c_{22} - c_{12}c_{21}$.

Now we choose real numbers a^x_{ij}, b^x_{ij}, c^y_{ij}, d^y_{ij} $(i, j = 1, 2;\ x \in U,\ y \in V)$ such that they are algebraically independent over the rationals and satisfy the following inequalities: $|a^x_{ij} - u^x_{ij}| < \varepsilon_x$, $|b^x_{ij} - u^x_{ij}| < \varepsilon_x$ for every $i, j = 1, 2$ and $x \in U$, and $|c^y_{ij} - v^y_{ij}| < \eta_y$, $|d^y_{ij} - v^y_{ij}| < \eta_y$ for every $i, j = 1, 2$ and $y \in V$. Let $D^x_a = a^x_{11}a^x_{22} - a^x_{12}a^x_{21}$, $D^x_b = b^x_{11}b^x_{22} - b^x_{12}b^x_{21}$, $D^y_c = c^y_{11}c^y_{22} - c^y_{12}c^y_{21}$, $D^y_d = d^y_{11}d^y_{22} - d^y_{12}d^y_{21}$ for every $x \in U$ and $y \in V$. We put $\alpha^x = \begin{bmatrix} a^x_{11}/\sqrt{D^x_a} & a^x_{12}/\sqrt{D^x_a} \\ a^x_{21}/\sqrt{D^x_a} & a^x_{22}/\sqrt{D^x_a} \end{bmatrix}$, and define β^x, γ^y, δ^y analogously. By Lemma 1, the transformations α^x, β^x $(x \in U)$ and γ^y, δ^y $(y \in V)$ generate a free group $G \subset SL_2[\mathbf{R}]$. Since the action of $SL_2[\mathbf{R}]$ is locally commutative on X, so is the action of G. Now it follows from the construction that for every $z \in A$ there is an $x \in U$ such that $\alpha^x(z) \in B$ and $\beta^x(z) \in B$, and for every $w \in B$ there is a $y \in V$ and there are points $z_1, z_2 \in A$

such that $(\gamma^y)^{-1}(z_1) = (\delta^y)^{-1}(z_2) = w$. Then, by Lemma 2, A and B are G-equidecomposable. Since $G \subset \mathrm{SL}_2[\mathbf{R}]$, it follows that any two elements of $\mathscr{A}$ are $\mathrm{SL}_2[\mathbf{R}]$-equidecomposable. Now every element $A \in \mathscr{A}$ contains two disjoint subsets also belonging to $\mathscr{A}$, and thus A is paradoxical. ∎

PROOF OF THEOREM 2. Let $A \in \mathscr{B}$ be fixed, and suppose that A is not paradoxical. Then, by Tarski's theorem ([5], Corollary 9.2, p. 128), there is a finitely additive and $\mathrm{SL}_2[\mathbf{R}]$-invariant measure defined on all subsets of X such that $\mu(A) = 1$. Let B_r denote the punctured disk $\{x \in X : 0 < < |x| \leq r\}$. By the definition of the class $\mathscr{B}$, there are linear transformations $\alpha_1, \ldots, \alpha_n \in \mathrm{SL}_2[\mathbf{R}]$ and there is an $r > 0$ such that $B_r \subset \bigcup_{i=1}^{n} \alpha_i(A)$. Thus we have $m = \mu(B_r) < \infty$. By Theorem 1, every element of the class $\mathscr{A}$ is equidecomposable to a subset of B_r and, consequently, has finite μ-measure. Since all these sets, by the same theorem, are paradoxical, they must be of μ-measure zero. In particular, all annuli of the form $\{x \in X : a < |x| \leq b\}$ $(0 < a < b)$ are of measure zero. As μ is nonnegative, it follows that all bounded sets having positive distance from the origin are of measure zero. Then $\mu(H \cap B_s) = \mu(H)$ for every bounded set $H \subset X$ and for every $s > 0$, since $\mu(H \setminus B_s) = 0$. Therefore $\mu(B_s) = \mu(B_r) = m$ for every s. Since $A \subset B_s$ if s is large enough, we have $m > 0$, and thus $0 < m < \infty$.

Now we define $\nu(H) = \mu(H \cap B_r)$ for every $H \subset X$. Then ν is a finitely additive measure defined on all subsets of X such that $\nu(X) = m \in (0, \infty)$. Let $H \subset X$ and $\alpha \in \mathrm{SL}_2[\mathbf{R}]$ be arbitrary; we prove $\nu(\alpha(H)) = \nu(H)$. Let $t > 0$ be so small that $B_t \subset \alpha^{-1}(B_r)$. Then, using the facts that μ is $\mathrm{SL}_2[\mathbf{R}]$-invariant and $\alpha^{-1}(B_r)$ is bounded, we obtain

$$\begin{aligned}\nu(\alpha(H)) &= \mu(\alpha(H) \cap B_r) = \mu(H \cap \alpha^{-1}(B_r)) = \mu(H \cap \alpha^{-1}(B_r) \cap B_t) = \\ &= \mu(H \cap B_t) = \mu(H \cap B_t \cap B_r) = \mu(H \cap B_r) = \\ &= \nu(H),\end{aligned}$$

as we stated. We proved that ν is a finitely additive measure on all subsets of X, and it is invariant under $\mathrm{SL}_2[\mathbf{R}]$. Since $0 < \nu(X) < \infty$, this contradicts the fact that X is paradoxical under $\mathrm{SL}_2[\mathbf{R}]$. This contradiction proves that A is paradoxical for every $A \in \mathscr{B}$.

Now we show that any two elements of $\mathscr{B}$ are $\mathrm{SL}_2[\mathbf{R}]$-equidecomposable. Let S denote the type semigroup of the action of $\mathrm{SL}_2[\mathbf{R}]$ on X, and let $[H] \in S$ be the type of the set $H \subset X$ (see [5], Chapter 8). If $0 < r < s$ then, by Theorem 1, $B_s \setminus B_r$ is equidecomposable to a subset of B_r and thus $[B_r] \leq [B_s] \leq 2[B_r]$. Now B_r, being an element of $\mathscr{B}$, is paradoxical. Therefore $[B_r] \leq [B_s] \leq 2[B_r] = [B_r]$, and thus $[B_r] = [B_s]$ for every $r, s > 0$.

If $A \in \mathscr{B}$ then, by the definition of the class $\mathscr{B}$, we have $[B_r] \leq n \cdot [A]$, where n is a positive integer and r is small enough. On the other hand, if s is large enough then $A \subset B_s$ and thus $[A] \leq [B_s]$. Therefore $n \cdot [B_s] = n \cdot [B_r] = [B_r] \leq n \cdot [A] \leq n \cdot [B_s]$, which implies $n \cdot [A] = n \cdot [B_s]$, and, by cancellation ([5], Theorem 8.7), $[A] = [B_s]$. Thus the sets A and B_s are $SL_2[\mathbf{R}]$-equidecomposable for every $A \in \mathscr{B}$ and for every $s > 0$, proving that any two elements of $\mathscr{B}$ are $SL_2[\mathbf{R}]$-equidecomposable. ∎

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RÉPARTITION DES VALEURS DE LA FONCTION φ D'EULER ET DE LA FONCTION SOMME DES DIVISEURS SUR LES ENTIERS SANS GRAND FACTEUR PREMIER

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1. Introduction

Soient m, n des entiers stictement positifs, φ la fonction d'Euler

$$\varphi(n) := \#\{1 \leq m \leq n; (m,n) = 1\},$$

σ la fonction somme des diviseurs

$$\sigma(n) := \sum_{d \mid n} d,$$

et soit $P(n)$ le plus grand facteur premier divisant n.

Pour x, y deux réels strictement positifs et f une fonction multiplicative, notons

$$N(x,y) := \sum_{\substack{\varphi(n)\leq x\\ P(n)\leq y}} 1, \qquad S(x,y) := \sum_{\substack{\sigma(n)\leq x\\ P(n)\leq y}} 1, \qquad \psi_f(x,y) := \sum_{\substack{n\leq x\\ P(n)\leq y}} f(n),$$

$$F(x,y) := \sum_{\substack{f(n)\leq x\\ P(n)\leq y}} 1 \quad \text{et} \quad F(x) := \sum_{f(n)\leq x} 1.$$

La résolution de certains problèmes en théorie des nombres nécessite la connaissance du comportement de $\psi_f(x,y)$. En particulier $\psi(x,y)$ correspondant au cas $f(n) = 1$ intervient dans la théorie du crible et a fait l'objet de nombreux travaux (cf. par exemple [2], [3], [4], [5], [8]); HILDEBRAND [3], utilisant l'identité

$$(1.1) \qquad (\log x)\psi(x,y) - \int_1^x \frac{\psi(t,y)}{t}dt = \sum_{\substack{p^r\leq x\\ p\leq y}} \psi\left(\frac{x}{p^r}, y\right)\log p,$$

et une itération sur $u := \frac{\log x}{\log y}$, a démontré que

$$\psi(x,y) = x\rho(u)\left\{1 + O_\varepsilon\left(\frac{u\log(u+1)}{\log x}\right)\right\}, \tag{1.2}$$

uniformément dans le domaine

$$(H_\epsilon) \qquad x \geq 3, \qquad \exp(\log\log x)^{\frac{5}{3}+\epsilon} \leq y \leq x,$$

oú ρ est la fonction de Dickman définie comme l'unique solution continue sur $[0,\infty[$ du système

$$\begin{cases} \rho(u) = 1 & 0 \leq u \leq 1, \\ -u\rho'(u) = \rho(u-1) & u \geq 1. \end{cases}$$

Le but de ce travail est d'établir des résultats analogues à (1.2) en particulier pour $N(x,y)$ et $S(x,y)$; cela résultera d'un résultat plus général concernant $F(x,y)$ pour la classe de fonctions $\mathscr{E}$ suivante.

DÉFINITION 1. Etant données deux constantes positives B et H, on note $\mathscr{E} := \mathscr{E}(B,H)$ la classe de fonctions multiplicatives vérifiant:

i) Pour $k \geq 1$ il existe un polynôme Q_k unitaire, de degré k et à coefficients dans $[-1, B]$ tel que:

$$f(p^k) = Q_k(p) \qquad (p \text{ premier}).$$

ii) Pour tout nombre premier p

$$1 \leq f(p^j) \leq f(p^k) \qquad (1 \leq j \leq k).$$

iii) $f(n) \gg n(\log\log(n))^{-H} \qquad (n \geq 3)$.

Il résulte de i) que, pour tout $f \in \mathscr{E}$ et tout nombre premier p, on a

$$p^k - \frac{p^k-1}{p-1} \leq f(p^k) \leq p^k + B\frac{p^k-1}{p-1} \qquad (k \geq 1)$$

ce qui entraine que

$$1 \leq f(p^k) \leq p^k(1+B) \tag{1.3}$$

et

$$\frac{1}{2}p^k \leq f(p^k) \qquad (p \geq 3). \tag{1.4}$$

Remarquons que si $x \leq Q_1(y)$ et $f \in \mathscr{E}$ alors, tout diviseur premier d'un entier n, vérifiant $f(n) \leq x$, est plus petit que y. En effet, si p est un tel diviseur, d'après ii), on a $Q_1(p)) = f(p) \le f(n) \le x \le Q_1(y)$, ce qui implique, compte tenu de la croissance de la fonction Q_1, que $p \le y$ et par suite, on a

$$F(x,y) = F(x) \qquad (x \le Q_1(y)).$$

La quantité $F(x)$ pour $f \in \mathcal{E}$, fut étudiée par SMATI [9], il a montré par une méthode élémentaire que

$$F(x) = A(f)x\left\{1 + O_c\left(\exp -c\sqrt{\log x \log\log x}\right)\right\} \qquad (x \to \infty), \tag{1.5}$$

oú c est une constante arbitraire $< \frac{1}{\sqrt{2}}$ et

$$A(f) := \prod_{p\ge 2}\left(1 - p^{-1}\right)\left(1 + \sum_{k\ge 1}(Q_k(p))^{-1}\right),$$

améliorant ainsi un résultat de IVIĆ [6].

Notre résultat principal est:

THÉORÈME 1. *Pour toute fonction f de la classe $\mathcal{E}$ et tout $\varepsilon > 0$ il existe un réel $x_0 > 0$, tel que l'on a*

$$F(x,y) = A(f)x\rho(v)\left\{1 + O_\varepsilon\left(\frac{v\log(v+1)}{\log x}\right)\right\}.$$

uniformément pour $x \ge x_0$ et $1 \le v \le \frac{\log x}{\log\log x^{\frac{5}{3}+\varepsilon}}$ avec

$$v := \frac{\log x}{\log(Q_1(y))}.$$

Nous en déduisons les deux corollaires suivants

COROLLAIRE 1. *Pour tout $\varepsilon > 0$ il existe un réel $x_0 > 0$, tel que l'on a*

$$N(x,y) = Ax\rho(v)\left\{1 + O_\varepsilon\left(\frac{v\log(v+1)}{\log x}\right)\right\},$$

uniformément pour $x \ge x_0$ et $\exp(\log\log x)^{\frac{5}{3}+\varepsilon} \le y \le x+1$, avec $v = \frac{\log x}{\log(y-1)}$ et $A = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$.

Ceci améliore un résultat antérieur de A. SMATI et J. WU [10].

COROLLAIRE 2. *Pour tout* $\varepsilon > 0$ *il existe un réel* $x_0 > 0$, *tel que l'on a*

$$S(x,y) = A'x\rho(v)\left\{1 + O_\varepsilon\left(\frac{v\log(v+1)}{\log x}\right)\right\},$$

uniformément pour $x \geq x_0$ *et* $\exp(\log\log x)^{\frac{5}{3}+\varepsilon} \leq y \leq x-1$, *avec* $v = \frac{\log x}{\log(y+1)}$ *et* $A' = \prod_{p\geq 2}\left(1-p^{-1}\right)\left(1+\sum_{k=1}^{\infty}\frac{p-1}{p^{k+1}-1}\right)$.

Un résultat similaire à notre théorème 1 vient d'être obtenu indépendamment par A. SMATI et J. WU [11], par des méthodes différentes. Leur méthode est analytique et utilise une technique récente de M. BALAZARD et G. TENENBAUM [2], fondée sur des estimations de sommes d'exponentielles dûes à KARATSUBA [7].

Notre méthode s'inspire de celle de A. HILDEBRAND [3] qui utilise l'équation fonctionnelle (1.1). En remarquant q'une inégalité fonctionnelle suffit, nous établirons

$$(\log x)F(x,y) - \int_1^x \frac{F(t,y)}{t}dt \leq \sum_{\substack{f(p^k)\leq x\\ p\leq y}} F\left(\frac{x}{f(p^k)},y\right)\log(f(p^k)), \tag{1.6}$$

ce qui nous permettra d'obtenir le théorème par une itération sur le paramètre $v = \frac{\log x}{\log(Q_1(y))}$.

Certaines de nos lemmes, en apparence analogues aux lemmes de A. HILDEBRAND [3], nécessitent cependant une démonstration spécifique.

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2. Lemmes

Dans ce paragraphe, nous regroupons des lemmes techniques qui sont utiles par la suite. Commençons tout d'abord par donner des estimations de certaines sommes faisant intervenir $f(p)$, quand f est dans $\mathcal{E}$.

LEMME 1. *Pour toute fonction f de la classe $\mathscr{E}$, il existe une constante C, telle que pour tout réel $x \geq 3$ et tout $\varepsilon > 0$ on a*

$$\sum_{f(p)\leq x} \frac{1}{f(p)} = \log\log x + C + O\left(\frac{1}{\log x}\right), \tag{2.1}$$

$$\sum_{f(p)\leq x} \log(f(p)) = x\left(1 + O\varepsilon\left(\exp -(\log x)^{\frac{3}{5}-\varepsilon}\right)\right), \tag{2.2}$$

$$\pi_f(x) := \sum_{f(p)\leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \tag{2.3}$$

et les constantes implicites intervenant dans les termes de reste dépendent au plus de B.

Ce lemme est une conséquence facile du théorème des nombres premiers, compte tenu de l'identité

$$f(p) = p + b \qquad (f \in \mathscr{E},\ b \in [-1, B]).$$

Le lemme suivant est dû à A. HILDEBRAND.

LEMME 2. *(*Lemme 1 de HILDEBRAND [3].*) La fonction ρ de Dickman est décroissante pour $v \geq 0$ et vérifie*

$$\rho(v) = 1 - \log v \qquad (1 \leq v \leq 2), \tag{2.4}$$

$$v\rho(v) = \int_{v-1}^{v} \rho(t)dt \qquad (v \geq 1), \tag{2.5}$$

$$0 < \rho(v) \leq 1 \qquad (v \geq 0), \tag{2.6}$$

$$\frac{\rho'(v)}{\rho(v)} \ll \log(v\log^2(v)) \qquad (v \geq e^4) \tag{2.7}$$

$$\frac{\rho(v-t)}{\rho(v)} \ll (v\log^2(v+1))t, \tag{2.8}$$

uniformément pour $1 \leq v$ et $0 \leq t \leq v$.

LEMME 3. *Uniformément pour $y \geq 3$ et $1 \leq v \leq \sqrt{Q_1(y)}$, on a:*

$$\int_0^v \rho(v-t)Q_1(y)^{-t}dt \ll \frac{\rho(v)}{\log Q_1(y)}. \tag{2.9}$$

PREUVE. En appliquant (2.8) et tenant compte de l'hypothèse $1 \leq v \leq$ $\leq \sqrt{Q_1(y)}$, nous obtenons

$$\frac{1}{\rho(v)} \int_0^v \rho(v-t) Q_1(y) - t\,dt \ll \int_0^v \left(\frac{v \log^2(v+1)}{Q_1(y)} \right)^t dt \leq$$

$$\leq \int_0^{\sqrt{Q_1(y)}} \left(\frac{\log^2(Q_1(y)+1)}{\sqrt{Q_1(y)}} \right)^t dt \leq \frac{1}{\lg Q_1(y)},$$

oú nous avons utilisé les inégalités suivantes

$$Q_1(y) \geq y - 1 \geq 2 \qquad (y \geq 3)$$

$$\frac{\log^2(u+1)}{u} < 1 \qquad (u > 1).$$

LEMME 4. *Pour toute fonction f de la classe $\mathcal{E}$ et uniformément pour $y \geq 3$ et $1 \leq v \leq Q_1(y)^{1/4}$ on a:*

$$\frac{1}{\rho(v)} \sum_{\substack{f(p^k \leq Q_1(y)^v \\ p \leq y,\ k \geq 2}} \frac{\log(f(p^k))}{f(p^k)} \rho \left(v - \frac{\log f(p^k)}{\log(Q_1(y))} \right) \ll 1. \tag{2.10}$$

PREUVE. La majoration (2.8) entraine que

$$\frac{1}{\rho(v)} \sum_{\substack{f(p^k) \leq Q_1(y)^v \\ p \leq y,\ k \geq 2}} \frac{\log f(p^k)}{f(p^k)} \rho \left(v - \frac{\log f(p^k)}{\log Q_1(y)} \right) \ll \sum_{\substack{f(p^k) \leq Q_1(y)^v \\ p \leq y,\ k \geq 2}} \frac{\log f(p^k)}{f(p^k)^{1-\beta}},$$

avec, $\beta := \frac{\log(v \log^2(v+1)}{\log Q_1(y)}$.

Choisissons $y_0 \geq 3$ de façon que

$$\beta \leq 1/3, \qquad \text{pour} \quad y \geq y_0 \quad \text{et} \quad v < Q_1(y)^{\frac{1}{4}}$$

Il s'ensuit que

$$\sum_{\substack{f(p^k) \leq Q_1(y)^v \\ p \leq y,\ k \geq 2}} \frac{\log f(p^k)}{f(p^k)^{1-\beta}} \leq \sum_{\substack{f(p^k) \leq Q_1(y)^v \\ p \leq y,\ k \geq 2}} \frac{\log f(p^k)}{f(p^k)^{2/3}} \leq$$

$$\leq \sum_{\substack{f\leq (p^k)\leq Q_1(y)^v\\ 3\leq p\leq y,\ k\geq 2}} \frac{\log f(p^k)}{f(p^k)^{2/3}} + \sum_{\substack{f(2^k\leq Q_1(y)^v\\ k\geq 2}} \frac{\log f(2^k)}{f(2^k)^{2/3}} =$$

$$= \qquad S_1 \qquad + \qquad S_2.$$

Majorons S_1, compte tenu de (1.3) et (1.4), par

$$\ll \sum_{3\leq p\leq y} \log p \sum_{\substack{p^k\leq 2x\\ k\geq 2}} \frac{k}{\left(p^{2/3}\right)^k} \ll \sum_{3\leq p\leq y} \frac{\log p}{p^{4/3}} \sum_{m\geq 0} \frac{m+2}{\left(p^{2/3}\right)^m}.$$

La série $\sum_{m\geq 0} \frac{m+2}{t^m}$ est uniformément convergente pour $0 < t < \frac{1}{2}$, ce qui entraine que

$$S_1 \ll \sum_{3\leq p\leq y} \frac{\log p}{p^{4/3}} \ll 1.$$

Puisque $f \in \mathscr{E}$, il existe un entier $k_0 \geq 2$, tel que

$$f(2^k) \gg \frac{2^k}{(\log\log 2^k)^H} \geq 2^{k-1} \qquad (k \geq k_0).$$

S_2 est alors majorée par

$$\ll \sum_{k=2}^{k_0} \frac{k\log 2 + \log(1+B)}{f\left(2^k\right)^{2/3}} + \sum_{k\geq k_0} \frac{k\log 2 + \log(1+B)}{2^{k-1}} \ll 1,$$

ce qui termine la preuve du lemme pour $y \geq y_0$. Dans le cas $3 \leq y \leq y_0$, le résultat cherché est trivial puisque le membre de gauche de (2.10) est une somme finie.

LEMME 5. *Pour toute fonction f de la classe $\mathscr{E}$ et tout ε strictement positif fixé, on a uniformément pour $y \geq y_0(\varepsilon)$, $v \geq 1$ et $0 \leq \theta \leq 1$*

(2.11)

$$\sum_{f(p)\leq Q_1(y)^\theta} \frac{\log f(p)}{f(p)} \rho\left(v - \frac{\log f(p)}{\log Q_1(y)}\right) = \log Q_1(y) \int_{v-\theta}^{v} \rho(t)dt + R(v,y),$$

oú

$$R(v,y) \ll_\varepsilon \rho(v)\left\{1 + v\log^2(v+1)\exp(-\log^{3/5-\epsilon}((Q_1(y))\right\}.$$

PREUVE. Le membre de gauche est égal à l'integrale de Stieltjes suivante:

$$\sum_{f(p)\leq Q_1(y)^\theta} \frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right) = \int_1^{Q_1(y)^\theta} \frac{\rho\left(v-\frac{\log s}{\log Q_1(y)}\right)}{s} d(m_1(s)),$$

oú $m_1(s) := \sum_{f(p)\leq s} \log f(p)$; par une intégration par parties, on a

$$\sum_{f(p)\leq Q_1(y)^\theta} \frac{\log p}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right) =$$

$$= m_1(Q_1(y)^\theta)\frac{\rho(v-\theta)}{Q_1(y)^\theta} - \int_1^{Q_1(y)^\theta} m_1(s)\frac{d}{ds}\left(\frac{\rho\left(v-\frac{\log s}{\log Q_1(y)}\right)}{s}\right) ds.$$

Il résulte de (2.2), compte tenu du changement de variables $s = Q_1(y)^t$, que

$$\sum_{f(p)\leq Q_1(y)^\theta} \frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right) = P+R,$$

avec

$$\begin{aligned} P &= \rho(v-\theta) + \int_0^\theta \{\rho'(v-t) + (\log Q_1(y))\rho(v-t)\}dt \\ &= (\log Q_1(y)) \int_{v-\theta}^{v} \rho(t)dt + \rho(v), \end{aligned}$$

et

$$\begin{aligned} R \ll_\varepsilon{} & \rho(v-\theta)\exp\left(-(\theta\log Q_1(y))^{\frac{3}{5}-\varepsilon}\right) + \\ & + \int_0^\theta \{|\rho'(v-t)| + \log(Q_1(y))\rho(v-t)\}\exp\left(-t(\log Q_1(y))^{\frac{3}{5}-\varepsilon}\right) dt. \end{aligned}$$

Posons

$$U := \log(v\log^2(v+1)) \qquad \text{et} \qquad V := \log Q_1(y),$$

les majorations (2.7) et (2.8) entrainent que

$$R \ll_{\varepsilon} \rho(v)\left(\exp(\theta U - (\theta V)^{\frac{3}{5}-\varepsilon}) + (U+V)\int_0^1 \exp(tU - (tV)^{\frac{3}{5}-\varepsilon})dt\right).$$

Le premier terme du membre de droite est

$$\leq \rho(v)\sup(1; \exp(U - V^{\frac{3}{5}-\varepsilon})),$$

ce qui est admissible.

Pour le second terme, nous distinguerons deux cas.

Si $U \leq \frac{1}{2}V^{\frac{3}{5}-\varepsilon}$, majorons le terme en question par

$$\leq (U+V)\int_0^1 \exp\left(-\frac{1}{2}(tV)^{\frac{3}{5}-\varepsilon}\right)dt \leq \frac{U+V}{V}\int_0^{V^{\frac{3}{5}-\varepsilon}} \exp\left(-\frac{1}{2}t\right)^{\frac{3}{5}-\varepsilon} dt \ll 1.$$

Si $U > \frac{1}{2}V^{\frac{3}{5}-\varepsilon}$, nous pouvons écrire

$$\int_0^1 \exp\left(tU - (tV)^{\frac{3}{5}-\varepsilon}\right)dt \leq$$

$$\leq \int_0^{\frac{1}{2}} \exp(tU)dt + \int_{\frac{1}{2}}^1 \exp\left(tU - \left(\frac{V}{2}\right)^{\frac{3}{5}-\varepsilon}\right)dt \ll$$

$$\ll \frac{1}{U}\exp\left(\frac{U}{2}\right) + \frac{1}{U}\exp\left(U - \left(\frac{V}{2}\right)^{\frac{3}{5}-\varepsilon}\right) \ll$$

$$\ll \frac{1}{U}\exp\left(U - \frac{1}{4}V^{\frac{3}{5}-\varepsilon}\right) + \frac{1}{U}\exp\left(U - \left(\frac{V}{2}\right)^{\frac{3}{5}-\varepsilon}\right) \ll_{\varepsilon}$$

$$\ll_{\varepsilon} \frac{1}{U+V}\exp\left(U - (V)^{\frac{3}{5}-\varepsilon}\right),$$

ce qui achève la preuve du lemme 5.

3. Démonstration du théorème

Commençons tout d'abord par traiter le cas $1 \le v \le 2$.

PROPOSITION 1. *Pour toute fonction f de la classe $\mathcal{E}$, il existe un réel x_0 positif, tel que pour $x \ge x_0$ et $1 \le v \le 2$ on a*

$$F(x, y) = A(f)x\rho(v)\left(1 + O\left(\frac{1}{\log Q_1(y)}\right)\right). \tag{3.1}$$

PREUVE. Nous utiliserons l'égalité

$$F(x) = A(f)x\left\{1 + O\left(\exp -c\sqrt{\log x}\right)\right\} \qquad (x \ge 1), \tag{3.2}$$

qui est une forme faible de (1.5).

Remarquons dans ce cas que, si $f(n) \le x$, il existe au plus un nombre premier p vérifiant

$$p > y, \qquad \exists k \ge 1 \quad \text{tel que} \quad p^k \parallel n. \tag{$*$}$$

En effet, si q est un autre premier vérifiant $(*)$, ii) entraine que

$$f(p)f(q) = f(pq) \le f(n) \qquad (f \in \mathcal{E}).$$

Le membre de gauche est strictement supérieur à $(Q_1(y))^2$, celui de droite est inférieur à x, ce qui est impossible puisque $x \le (Q_1(y))^2$. Ecrivons alors $F(x, y)$ sous la forme

$$F(x, y) = \sum_{\substack{f(n)\le x \\ P(n)\le y}} 1 = \sum_{f(n)\le x} 1 - \sum_{\substack{f(n)\le x \\ \exists p>y,\, k\ge 1 \text{ et } p^k \parallel n}} 1,$$

Il suit, en regroupant la deuxième somme suivant l'unique nombre premier vérifiant $(*)$, que

$$F(x, y) = F(x) - \sum_{y<p} \sum_{\substack{f(n)\le x \\ p^k \parallel n,\ k\ge 1}} 1 = F(x) - \sum_{\substack{y<p \\ f(p^k)\le x}} \sum_{\substack{f(m)\le \frac{x}{f(p^k)} \\ (p,m)=1}} 1.$$

Les conditions $f(m) \le \frac{x}{f(p^k)}$ et $p > y$ entrainent que $(p, m) = 1$, en effet, si q est un nombre premier divisant m, alors, compte tenu de ii),

$$Q_1(q) = f(q) \le f(m) \le \frac{x}{f(p^k)} \le \frac{x}{f(p)} \le \frac{x}{Q_1(y)} \le Q_1(y).$$

Par suite, tout diviseur premier de m est plus petit que y et donc $(p,m)=1$. Il en découle que

$$F(x,y) = F(x) - \sum_{\substack{y<p \\ f(p^k)\le x}} \sum_{f(m)\le \frac{x}{f(p^k)}} 1 =$$

$$= F(x) - \sum_{\substack{y<p \\ f(p)\le x}} F\left(\frac{x}{f(p)}\right) - \sum_{\substack{y<p \\ f(p^k)\le x,\ k\ge 2}} F\left(\frac{x}{f(p^k)}\right),$$

$$F(x,y) = F(x) \quad - \quad S_1 \quad - \quad S_2 \tag{3.3}$$

Il résulte de (1.3), compte tenu de l'inégalité $F(x) \ll x$ valable pour $x>0$, que

$$S_2 \ll \sum_{\substack{y<p \\ f(p^k)\le x,\ k\ge 2}} \frac{x}{f(p^k)} \ll x \sum_{y<p} \sum_{k\ge 2} \frac{1}{p^k} \ll \frac{x}{y}, \tag{3.4}$$

ce qui est admissible.

Grâce à (3.2) nous écrivons

$$S_1 = A(f)x \sum_{\substack{y<p \\ f(p)\le x}} \frac{1}{f(p)} + O\left(x \sum_{\substack{y<p \\ f(p)\le x}} \frac{\exp -c\sqrt{\log\left(\frac{x}{f(p)}\right)}}{f(p)} \right)$$

$$= \qquad P_1 \qquad + \qquad R_1.$$

la fonction Q_1, qui est égale à f sur l'ensemble des nombres premiers, est injective et croissante, ce qui implique que le terme principal de S_1 est égal à

$$P_1 = A(f)x \sum_{Q_1(y)<f(p)\le x} \frac{1}{f(p)};$$

il s' ensuit, en vertu de (2.1), que

$$P_1 = A(f)x \left[\log v + O\left(\frac{1}{\log(Q_1(y))}\right)\right].$$

Par ailleurs

$$R_1 \ll x \sum_{Q_1(y)<f(p)\le x} \frac{\exp -c\sqrt{\log\left(\frac{x}{f(p)}\right)}}{f(p)} \ll x \int_{Q_1(y)}^{x} \frac{\exp -c\sqrt{\log \frac{x}{t}}}{t} d\pi_f(t),$$

une double intégration par parties et l'égalité (2.3) fournissent

$$R_1 \ll$$

$$\ll \frac{x}{\log Q_1(y)} \left(1 + \int_{Q_1(y)}^{x} \frac{1}{2\sqrt{\log \frac{x}{t}}} \frac{\exp -c\sqrt{\log \frac{x}{t}}}{t \log t} dt + \int_{Q_1(y)}^{x} \frac{\exp -c\sqrt{\log \frac{x}{t}}}{t} dt \right).$$

Il en résulte, compte tenu du changement de variable $s = \sqrt{\log \frac{x}{t}}$, que

$$R_1 \ll \frac{x}{\log Q_1(y)} \left(1 + \int_0^{\sqrt{\log \frac{x}{Q_1(y)}}} \frac{\exp(-cs)}{(\log x - s^2)} ds + \int_0^{\sqrt{\log \frac{x}{Q_1(y)}}} 2s \exp(-cs) ds \right).$$

Les deux intégrales sont convergentes et donc $R_1 = O\left(\frac{x}{\log(Q_1(x)}\right)$; par suite

$$S_1 = A(f)x \left[\log v + O\left(\frac{1}{\log Q_1(y)} \right) \right]. \tag{3.5}$$

La proposition découle alors de (3.2), (3.3), (3.4) et (3.5).

Preuve de l'inégalité (1.7)

Considérons la quantité

$$\sum_{\substack{f(n) \le x \\ P(n) \le y}} \log f(n) \qquad (f \in \mathcal{E}),$$

qu'on interprète de deux manières différentes: d'une part en l'écrivant sous la forme d'une intégrale de Stieltjes et en intégrant par partie,

$$\sum_{\substack{f(n) \le x \\ P(n) \le y}} \log f(n) = \int_1^x (\log t) dF(t, y) = (\log x) F(x, y) - \int_1^x \frac{F(t, y)}{t} dt,$$

d'autre part en remplaçant $\log f(n)$ par $\sum\limits_{p^l \,\|\, n} \log f(p^k)$ et en permutant les sommes,

$$\sum_{\substack{f(n) \le x \\ P(n) \le y}} \log f(n) = \sum_{\substack{f(n) \le x \\ P(n) \le y}} \sum_{p^k \,\|\, n} \log f(p^k) =$$

$$= \sum_{\substack{p\le y\\ f(p^k)\le x}} \log f(p^k) \sum_{\substack{f(m)\le \frac{x}{f(p^k)}\\ P(m)\le y,\ (m,p)=1}} 1 \le \sum_{\substack{p\le y\\ f(p^k)\le x}} \log f(p^k) \sum_{\substack{f(m)\le \frac{x}{f(p^k)}\\ P(m)\le y}} 1.$$

Ces deux expressions regroupées donnent l'inégalité souhaitée.

Fin de la démonstration du théorème

Pour $v \ge 1$, posons

$$F(x,y) = A(f)x\rho(v)(1+\Delta(v,y)) \tag{$**$}$$

et notons

$$\Delta^*(v,y) := \sup_{\frac12\le v'\le v} |\Delta(v',y)|,$$

$$\Delta^{**}(v,y) := \sup_{0<v'\le v} |\Delta(v',y)|.$$

Compte tenu de (3.2), il existe $x_0 > 0$ tel que

$$|\Delta(v,y)| \le 1 \qquad (x > x_0 \quad \text{et} \quad 0 < v \le 1),$$

ce qui nous permet d'écrire

$$\Delta^{**}(v,y) \le 1 + \Delta^*(v,y) \qquad (x \ge x_0). \tag{3.6}$$

Soit $v_t := \frac{\log t}{\log x}$; il résulte de $(**)$ que

$$F(t,y) = A(f)t\rho(v)(1+\Delta(v_t,y)) \qquad (1 \le t \le x).$$

L'identité précédente et l'inégalité (1.7) fournissent

$$A(f)x \log x\rho(v)(1+\Delta(v,y)) \le$$

$$\le A(f)\int_1^x \rho\left(\frac{\log t}{\log Q_1(y)}\right)\left(1+\Delta\left(\frac{\log t}{\log Q_1(y)},y\right)\right)dt+$$

$$+A(f)x\sum_{f(p)\le Q_1(y)} \frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\left(1+\Delta\left(v-\frac{\log f(p)}{\log Q_1(y)},y\right)\right)+$$

$$+A(f)x\sum_{\substack{f(p^k)\le x\\ p\le y,\ k\ge 2}} \frac{\log f(p^k)}{f(p^k)}\rho\left(v-\frac{\log f(p^k)}{\log Q_1(y)}\right)\left(1+\Delta\left(v-\frac{\log f(p^k)}{\log Q_1(y)},y\right)\right),$$

ce qui donne, après simplification,

$$1+\Delta(v,y) \le \frac{1}{x\rho(v)\log x}\int_1^x \rho\left(\frac{\log t}{\log Q_1(y)}\right)\left(1+\Delta\left(\frac{\log t}{\log Q_1(y)},y\right)\right)dt+$$

$$+\frac{1}{\rho(v)\log x}\sum_{f(p)\le Q_1(y)}\frac{\log f(p)}{f(p)}\cdot$$

$$\cdot\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\left(1+\Delta\left(v-\frac{\log f(p)}{\log Q_1(y)},y\right)\right)+$$

$$+\frac{1}{\rho(v)\log x}\sum_{\substack{f(p^k)\le x\\ p\le y,\ k\ge 2}}\frac{\log f(p^k)}{f(p^k)}\cdot$$

$$\cdot\rho\left(v-\frac{\log f(p^k)}{\log Q_1(y)}\right)\left(1+\Delta\left(v-\frac{\log f(p^k)}{\log Q_1(y)},y\right)\right).$$

Soit

$$1+\Delta(v,y)\le A_1+A_2+A_3,$$

ou encore

$$|\Delta(v,y)|\le |A_1|+|A_2-1|+|A_3|. \tag{3.7}$$

Il découle de (2.9), compte tenu du changement de variable $\frac{\log t}{\log Q_1(y)} = v - t'$, que

$$|A_1| \ll \frac{(1+\Delta^{**}(v,y))}{v\log Q_1(y)}. \tag{3.8}$$

En vertu de (2.10), nous pouvons majorer $|A_3|$ par

$$|A_3| \ll \frac{1+\Delta^{**}(v,y)}{v\log Q_1(y)}. \tag{3.9}$$

Pour $|A_2-1|$, nous introduisons la fonction

$$\alpha(v)=\frac{1}{v\rho(v)}\int_{v-\frac{1}{2}}^{v}\rho(s)ds;$$

alors, par (2.5),

$$\frac{1}{v\rho(v)}\int_{v-1}^{v-\frac{1}{2}}\rho(s)ds = (1-\alpha(v))$$

et par suite

$$|A_2 - 1| =$$

$$= \left| \frac{1}{\rho(v)\log x}\sum_{f(p)\le\sqrt{Q_1(y)}}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\Delta\left(v-\frac{\log f(p)}{\log Q_1(y)},y\right)+\right.$$

$$+\frac{1}{\rho(v)\log x}\sum_{\substack{\sqrt{Q_1(y)}<\\<f(p)\le Q_1(y)}}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\Delta\left(v-\frac{\log f(p)}{\log Q_1(y)},y\right)+$$

$$+\frac{1}{\rho(v)\log x}\sum_{f(p)\le\sqrt{Q_1(y)}}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)-\alpha(v)+$$

$$\left.+\frac{1}{\rho(v)\log x}\sum_{\sqrt{Q_1(y)}<f(p)\le Q_1(y)}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)-(1-\alpha(v))\right|.$$

Ou encore

$$|A_2 - 1| \le \frac{1}{\rho(v)\log x}\sum_{f(p)\le\sqrt{Q_1(y)}}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\Delta^*(v,y)+$$

$$+\frac{1}{\rho(v)\log x}\sum_{\sqrt{Q_1(y)}<f(p)\le Q_1(y)}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)\Delta^*\left(v-\frac{1}{2},y\right)+$$

$$+\left|\frac{1}{\rho(v)\log x}\sum_{f(p)\le\sqrt{Q_1(y)}}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)-\alpha(v)\right|+$$

$$+\left|\frac{1}{\rho(v)\log x}\sum_{\sqrt{Q_1(y)}<f(p)\le Q_1(y)}\frac{\log f(p)}{f(p)}\rho\left(v-\frac{\log f(p)}{\log Q_1(y)}\right)-(1-\alpha(v))\right|.$$

Il résulte des applications du Lemme 5 dans les deux cas $\theta = \frac{1}{2}$ et $\theta = 1$, pour $y \geq y_0$ et1 $\leq v \leq \dfrac{\log x}{\log\log x^{\frac{5}{3}+\varepsilon}}$, que

$$|A_2 - 1| \leq \alpha(v)\Delta^*(v,y) + (1-\alpha(v)\Delta^*\left(v - \frac{1}{2}, y\right) + O_\varepsilon\left(\frac{1+\Delta^{**}(v,y)}{v\log Q_1(y)}\right)$$

ce qui donne, en tenant compte de (3.7), (3.8) et (3.9),

$$|\Delta(v,y)| \leq \alpha(v)\Delta^*(v,y) + (1-\alpha(v))\Delta^*\left(v - \frac{1}{2}, y\right) + O_\varepsilon\left(\frac{1+\Delta^{**}(v,y)}{v\log Q_1(y)}\right).$$

II vient par (3.6) que

$$(3.10)\ |\Delta(v,y)| \leq \alpha(v)\Delta^*(v,y) + (1-\alpha(v))\Delta^*\left(v - \frac{1}{2}, y\right) + O_\varepsilon\left(\frac{1+\Delta^{*}(v,y)}{v\log Q_1(y)}\right) \quad (x \geq x_0).$$

Par ailleurs, on a

$$\frac{1}{2}\left(\Delta^*(v,y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right) - \left(\alpha(v)\Delta^*(v,y) + (1-\alpha(v))\Delta^*\left(v - \frac{1}{2}, y\right)\right) =$$
$$= \left(\frac{1}{2} - \alpha(v)\right)\left(\Delta^*(v,y) - \Delta^*\left(v - \frac{1}{2}, y\right)\right).$$

La croissance de la fonction ρ sur $]0,\infty]$ et l'égalité (2.2) impliquent, que

$$2\alpha(v) = \frac{2}{v\rho(v)}\int_{v-\frac{1}{2}}^{v}\rho(s)ds \leq \frac{1}{v\rho(v)}\int_{v-1}^{v-\frac{1}{2}}\rho(s)ds + \frac{1}{v\rho(v)}\int_{v-\frac{1}{2}}^{v}\rho(s)ds = 1$$

et donc $\alpha(v) \leq \frac{1}{2}$; cela entraine que

$$\alpha(v)\Delta^*(v,y) + (1-\alpha(v\Delta^*\left(v - \frac{1}{2}y\right) \leq \frac{1}{2}\left(\Delta^*(v,y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right).$$

Ainsi nous obtenons, compte tenu de l'inégalité (3.10),

$$|\Delta(v,y)| \leq \frac{1}{2}\left(\Delta^*(v,y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right) + O_\varepsilon\left(\frac{1+\Delta^{*}(v,y)}{v\log Q_1(y)}\right).$$

Prenons pour le moment un réel v', $\frac{1}{2} \leq v' \leq v$.

Si $v - \frac{1}{2} \leq v' \leq v$, la majoration

$$\frac{1}{v'} \ll \frac{1}{v}$$

et la croissance en t de la fonction $\Delta^*(t, y)$ impliquent que

$$|\Delta(v', y)| \leq \frac{1}{2}\left(\Delta^*(v, y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right) + O_\varepsilon\left(\frac{1 + \Delta^*(v, y)}{v \log Q_1(y)}\right). \tag{3.11}$$

Si $\frac{1}{2} \leq v' \leq v - \frac{1}{2}$, nous avons les inégalités suivantes

$$|\Delta(v', y)| \leq \Delta^*(v', y) \leq \Delta^*\left(v - \frac{1}{2}, y\right) \leq \frac{1}{2}\left(\Delta^*(v, y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right).$$

Il s'ensuit que l'inégalité (3.11) est vérifiée pour tout $\frac{1}{2} \leq v' \leq v$.

Ainsi

$$\Delta^*(v, y) \leq \frac{1}{2}\left(\Delta^*(v, y) + \Delta^*\left(v - \frac{1}{2}, y\right)\right) + O_\varepsilon\left(\frac{1 + \Delta^*(v, y)}{v \log Q_1(y)}\right),$$

ou encore

$$\Delta^*(v, y) \leq \Delta^*\left(v - \frac{1}{2}, y\right) + O_\varepsilon\left(\frac{1 + \Delta^*(v, y)}{v \log Q_1(y)}\right).$$

Une itération de cette inégalité jusqu'à v_0, $\frac{3}{2} \leq v_0 \leq 2$, donne

$$\Delta^*(v, y) \leq \Delta^*(v_0, y) + O_\varepsilon\left\{\frac{\log v}{\log Q_1(y)}(1 + \Delta^*(v, y))\right\}.$$

Il en découle, compte tenu de (3.1), que

$$\Delta^*(v, y) \ll \left(1 + \Delta^*(v, y)\right)\frac{\log(v + 1)}{\log Q_1(y)} \qquad (x \geq x_0)$$

$$\Delta^* \ll \frac{\log(v + 1)}{\log Q_1(y)} \qquad (x \geq x_0),$$

ce qui achève la démonstration du théorème.

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NONLINEAR ELLIPTIC EQUATIONS WITH DISCONTINUOUS NONLINEARITIES

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1. Introduction

In this paper we study quasilinear elliptic problems. Using techniques from nonsmooth critical point theory, as this was formulated by CHANG [7], we prove two existence theorems under different nonresonance conditions.

Elliptic problems with discontinuities have been studied in the past, primarily for semilinear problems. The methods employed for the analysis of such problems vary. They can be traced in the works of AMBROSETTI–BADIALE [1] (which is based on Clarke's dual variational principle), CHANG [7] (which uses nonsmooth critical point theory), RAUCH [12] (which is based on truncation and approximation techniques) and STUART [13] (which employs the method of upper and lower solutions). Nonlinear problems were considered only recently by BOUCHERIF–SLIMANI [5] and DRABEK [9], who deal with one-dimensional problems (i.e. $N = 1$). In Boucherif–Slimani the discontinuous forcing term has only one jump discontinuity while in Drabek the forcing term is a Caratheodory function. Higher dimensional problems (i.e. $N > 1$) were considered by ARCOYA–CALAHORRANO [3], BOCCARDO–DRABEK–GIACHETTI–KUCERA [4], BOUGOUIMA [6] and COSTA–MAGALHAES [8].

2. Mathematical preliminaries

Let $Z \subset \mathbf{R}^N$ be a bounded domain with a C^1-boundary Γ. The problem under consideration is the following:

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$$(1)\qquad \left\{\begin{array}{ll} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = f(z,x(z)) & \text{in} \quad Z \\ x|_\Gamma = 0, 2 \le p < \infty. & \end{array}\right\}.$$

Here $f : Z \times \mathbf{R} \to \mathbf{R}$ is in general discontinuous and so problem (1) need not to have a solution. In order to be able to develop a reasonable existence theory, we need to pass to a multivalued variant of (1) by, roughly speaking, filling in the gaps at the discontinuity points of $f(z, \cdot)$. For this purpose we define the following two functions

$$f_0(z,r) = \varliminf_{r'\to r} f(z,r') = \lim_{\delta\downarrow 0}\ ess\inf_{|r'-r|<\delta} f(z,r')$$

and

$$f_1(z,r) = \varlimsup_{r'\to r} f(z,r') = \lim_{\delta\downarrow 0}\ ess\sup_{|r'-r|<\delta} f(z,r')$$

Using these two functions we define the multifunction $\hat{f}(z,r) = [f_0(z,r), f_1(z,r)]$ and instead of (1), we consider the following multivalued variant of it:

$$(2)\qquad \left\{\begin{array}{ll} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \hat{f}(z,x(z)) & \text{in} \quad Z \\ x|_\Gamma = 0, 2 \le p < \infty. & \end{array}\right\}$$

It is this problem that we shall investigate in this paper.

DEFINITION. By a solution of (2), we mean a function $x \in W_0^{1,p}(Z)$ for which there exists a function $g \in L^q(Z)\left(\frac{1}{p}+\frac{1}{q}=1\right)$ such that $g(z) \in \hat{f}(z,x(z))$ a.e. on Z and

$$\left\{\begin{array}{ll} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) = g(z) & \text{a.e. on} \quad Z \\ x|_\Gamma = 0. & \end{array}\right\}$$

In ARCOYA–CALAHORRANO [3], $f(z, \cdot)$ has only one jump discontinuity at $x = 0$ and the authors give conditions under which a solution of (2) is in fact a solution of (1). Note that if $f(z,x)$ is a Caratheodory function (i.e. $z \to f(z,x)$ is measurable and $x \to f(z,x)$ is continuous) then $\hat{f}(z,x) = \{f(z,x)\}$ and so problems (1) and (2) coincide.

As we already mentioned in the introduction, our approach will be variational and will use the nonsmooth critical point theory for locally Lipschitz functionals developed by CHANG [7]. So for the convenience of the reader, we will recall here the main aspects of this theory, which we will need in the sequel.

Let X be a Banach space and X^* its topological dual. A function $f : X \to \mathbf{R}$ is said to be locally Lipschitz, if for every $x \in X$, there exists a neighbourhood U of x and a constant k depending on U such that $|f(y) - - f(z)| \leq k\|y - z\|$ for all $y, z \in U$. Recall that a proper, convex and lower semicontinuous function $g : X \to \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain, $\operatorname{dom} g = \{x \in X : g(x) < +\infty\}$. Given a direction $y \in X$, we can define a generalized directional derivative of $f(\cdot)$ at x in the direction y, by

$$f^0(x; y) = \overline{\lim_{\substack{x' \to x \\ \lambda \downarrow 0}}} \frac{f(x' + \lambda y) - f(x')}{\lambda}.$$

The function $f^0(x; \cdot)$ is sublinear and continuous. So it is the support function of the convex set $\partial f(x)$ defined by

$$\partial f(x) = \{x^* \in X^* : (x^*, y) \leq f^0(x; y) \quad \text{for all} \quad y \in X\}.$$

We call $\partial f(x)$ the "subdifferential of $f(\cdot)$ at x". Evidently for every $x \in X$, $\partial f(x)$ is a nonempty, convex and w^*-compact set. Moreover, if $f(\cdot)$ is also convex, then $\partial f(\cdot)$ coincides with the subdifferential in the sense of convex analysis and $f^0(x; \cdot) = f'(x; \cdot)$ where $f'(x; \cdot)$ is the directional derivative of $f(\cdot)$ at x. If $f_1, f_2 : X \to \mathbf{R}$ are both locally Lipschitz functions then $\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + f_2(x)$. Also for every $\lambda \in \mathbf{R}$, $\partial(\lambda f)(x) = \lambda \partial f(x)$. If $f : X \to \mathbf{R}$ is continuously Gateaux differentiable, the $\partial f(x) = \{Df(x)\}$. Finally if $x \in X$ is a local minimum of $f(\cdot)$, then $0 \in \partial f(x)$.

In the variational approach we look for the critical points of an appropriately defined "energy" functional. Here due to the presence of discontinuities, the energy functional is not a C^1 but only locally Lipschitz. So the notion of a "critical point" is defined as follows:

DEFINITION. Let $f : X \to \mathbf{R}$ be locally Lipschitz. A point $x_0 \in X$ is said to be a "critical point" of $f(\cdot)$, if $0 \in \partial f(x_0)$.

A basic tool in the variational theory is a compactness type condition on the functional $f(\cdot)$, known as the "Palais–Smale condition" ((PS)-condition). In the present context of locally Lipschitz functions, this condition takes the following form:

DEFINITION. A locally Lipschitz functional $f : X \to \mathbf{R}$ satisfies the "(PS)-condition", if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ along which $\{f(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) = \min[\|v\|_{X^*} : v \in \partial f(x_n)] \overset{n\to\infty}{\longrightarrow} 0$, possesses a convergent subsequence.

Using this notion we have the following nonsmooth version of the well-known "Mountain Pass Theorem" of AMBROSETTI–RABINOWITZ [2] (see CHANG [7], theorem 3.4).

THEOREM 1. *If X is a reflexive Banach space, $f : X \to \mathrm{R}$ is a locally Lipschitz functional which satisfies the (PS)-condition and also satisfies the following conditions:*

(i) $f(0) = 0$ *and there exist* $\rho, \xi > 0$ *with* $f|_{\partial B_\rho} \geq \xi$, *(*$B_\rho = \{x \in X : \|x\| < \rho\}$ *and* $\partial B_\rho = \{x \in X : \|x\| = \rho\}$*);*

(ii) *there exists* $y \in X$ *with* $\|y\| > \rho$ *such that* $f(y) \leq 0$,

then $f(\cdot)$ *possesses a critical point* $x \in X$ *with critical value* $f(x) \geq \xi$.

In the sequel we will use the first eigenvalue of the following quasilinear eigenvalue problem:

$$\left\{ \begin{array}{ll} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) & \text{in} \quad Z \\ x|_\Gamma = 0. & \end{array} \right\}. \tag{3}$$

From LINDQVIST [11] we know that there exists the first eigenvalue $\lambda_1 > 0$ of (3), which is simple and isolated. A corresponding eigenfunction $u_1 \in W_0^{1,p}(Z) \cap L^\infty(Z)$ can be choosen so that $u_1(z) > 0$ a.e. on Z. Also the first eigenvalue is the minimum of the Rayleigh quotient:

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z),\ x \neq 0\right], \qquad \lambda_1 = \frac{\|Du_1\|_p^p}{\|u_1\|_p^p}.$$

3. Existence theorems

In this section we prove two existence theorems for problem (2). The first existence theorem is based on a nonresonance condition in which we have crossing of the first eigenvalue $\lambda_1 > 0$. More precisely our hypotheses on $f(z,x)$ are the following:

H(f)$_1$: $f : Z \times \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that

(i) $f_0(z,r)$, $f_1(z,r)$ are finite and N-measurable (i.e. for all $x : Z \to \mathbf{R}$ measurable, $z \to f_0(z,x(z))$ and $z \to f_1(z,x(z))$ are both measurable);

(ii) there exist $\alpha, \beta \in L^\infty(Z)$ such that $\lim_{r\to-\infty} \frac{f(z,r)}{|r|^{p-2}r} = \alpha(z)$ and $\lim_{r\to\infty} \frac{f(z,r)}{|r|^{p-2}r} = \beta(z)$ uniformly for almost all $z \in Z$ and with $ess \sup_{z\in Z} \beta(z) < \lambda_1 < ess \inf_{z\in Z} \alpha(z)$;

(iii) $\lim_{|r|\to 0} \frac{|f(z,r)|}{|r|^{p_1-1}} < +\infty$ uniformly for almost all $z \in Z$. with $p < p_1 \leq \frac{Np}{N-p}$ if $p < N$, $p < p_1 < \infty$ if $p \geq N$ and for every $M > 0$ there exists $\gamma_M \in L^\infty(Z)$ such that $|f(z,r)| \leq \gamma_M(z)$ a.e. on Z, for all $|r| \leq M$.

REMARK. From hypotheses $H(f)_1$(ii) and (iii) it easily follows that $f(z,r)$ satisfies the following growth condition: $|f(z,r)| \leq \alpha_1(z) + c_1|r|^{p-1}$ a.e. on Z, with $\alpha_1 \in L^q(Z)$, $c_1 > 0$.

In what follows let $F(z,x) = \int_0^x f(z,r)dr$ (the potential function corresponding to f). Then let $G : W_0^{1,p}(Z) \to \mathbf{R}$ be defined by $G(x) = \int_Z F(z,x(z))dz$. From CHANG [7] we know that $G(\cdot)$ is locally Lipschitz. Then let $R : W_0^{1,p}(Z) \to \mathbf{R}$ be defined by $R(x) = \frac{1}{p}\|Dx\|_p^p - G(x)$. Evidently $R(\cdot)$ is locally Lipschitz.

PROPOSITION 2. *If hypotheses* $H(f)_1$ *hold, then* $R(\cdot)$ *satisfies the (PS)-condition.*

PROOF. Let $\{x_n\}_{n\geq1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $\sup_{n\geq1} |R(x_n)| < \infty$ and $m(x_n) = \inf[\|x^*\| : x^* \in \partial R(x_n)] \overset{n\to\infty}{\longrightarrow} 0$. We will show that $\{x_n\}_{n\geq1}$ has a strongly convergent subsequence. To this end let $x_n^* \in \partial R(x_n)$ be such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. The existence of these elements follows from the weak compactness of $\partial R(x_n) \subseteq W^{-1,q}(Z)$ and the weak lower semicontinuity of the norm functional. Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbf{R}^N} dz \quad \text{for all} \quad x, y \in W_0^{1,p}(Z).$$

It is easy to see that $A(\cdot)$ is monotone and demicontinuous (hence maximal monotone). Also if $J : W_0^{1,p} \to \mathbf{R}$ is defined by $J(x) = \frac{1}{p}\|Dx\|_p^p$, then

$J'(x) = A(x)$. Hence $\partial R(x_n) \subseteq A(x_n) - \partial G(x_n)$ and so $x_n^* = A(x_n) - g_n$ with $g_n \in \partial G(x_n)$, $n \geq 1$ and $\|A(x_n) - g_n\|_* \overset{n\to\infty}{\longrightarrow} 0$.

Now suppose that $\{x_n\}_{n\geq 1}$ is not bounded is $W_0^{1,p}(Z)$. Hence by passing to a subsequence if necessary, we may assume that $\|x_n\|_{1,p} \overset{n\to\infty}{\longrightarrow} \infty$. Let $y_n = \frac{x_n}{\|x_n\|_{1,p}}$, $n \geq 1$. Evidently $\|y_n\|_{1,p} = 1$. Thus by passing to a subsequence if necessary and using the fact that $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$, we may assume that $y_n \overset{w}{\rightarrow} y$ in $W_0^{1,p}(Z)$, $y_n \overset{n\to\infty}{\longrightarrow} y$ in $L^p(Z)$, $y_n(z) \overset{n\to\infty}{\longrightarrow} y(z)$ a.e. on Z. Vitali's theorem will be applied.

We will show that

$$\frac{f_0(z, x_n(z))}{\|x_n\|_{1,p}^{p-1}} \longrightarrow \beta(z)|y^+(z)|^{p-2}y^+(z) - \alpha(z)|y^-(z)|^{p-2}y^-(z) \text{ a.e. on } Z \tag{4}$$

and

$$\frac{f_1(z, x_n(z))}{\|x_n\|_{1,p}^{p-1}} \longrightarrow \beta(z)|y^+(z)|^{p-2}y^+(z) - \alpha(z)|y^-(z)|^{p-2}y^-(z) \text{ a.e. on } Z. \tag{5}$$

To prove (4) and (5) we proceed as follows. Let $Z_+ = \{z \in Z : y(z) > 0\}$, $Z_- = \{z \in Z : y(z) < 0\}$ and $Z_0 = \{z \in Z : y(z) = 0\}$. For almost all $z \in Z_+$ there exists $|v_n| \leq 1$ such that

$$|f_0(z, x_n(z)) - f(z, x_n(z) + v_n)| \leq \frac{1}{n}.$$

Note that for almost all $z \in Z_+$, $y_n(z) = \frac{x_n(z)}{\|x_n\|_{1,p}} \to y(z) > 0$ and so we find that $x_n(z) \overset{n\to\infty}{\longrightarrow} +\infty$. We have

$$\left|\frac{f_0(z, x_n(z))}{\|x_n\|_{1,p}^{p-1}} - \beta(z)|y^+(z)|^{p-2}y^+(z)\right| \leq$$

$$\leq \left|\frac{f(z, x_n(z) + v_n)}{\|x_n\|_{1,p}^{p-1}} - \beta(z)|y^+(z)|^{p-2}y^+(z)\right| + \frac{1}{n}\frac{1}{\|x_n\|_{1,p}^{p-1}} \leq$$

$$\leq \left|\frac{f(z, x_n(z) + v_n)}{\|x_n\|_{1,p}^{p-1}} - \beta(z)|y_n(z)|^{p-2}y_n(z)\right| +$$

$$+|\beta(z)|\left||y_n(z)|^{p-2}y_n(z)-|y^+(z)|^{p-2}y^+(z)\right|+\frac{1}{n\|x_n\|_{1,p}^{p-1}} \quad \text{a.e. on } Z_+$$

Since $x_n(z)\underset{n\to\infty}{\longrightarrow}+\infty$, a.e. on Z_+, by virtue of hypothesis $\mathrm{H(f)}_1$(ii), we have

$$\left|\frac{f(z,x_n(z)+v_n)}{\|x_n\|_{1,p}^{p-1}}-\beta(z)|y_n(z)|^{p-2}y_n(z)\right|\overset{n\to\infty}{\longrightarrow}0 \text{ a.e. on } Z_+.$$

Also $|\beta(z)|\left||y_n(z)|^{p-2}y_n(z)-|y^+(z)|^{p-2}y^+(z)\right|\overset{n\to\infty}{\longrightarrow}0$ a.e. on Z_+ and $\frac{1}{n\|x\|_{1,p}^{p-1}}\overset{n\to\infty}{\longrightarrow}0$. Hence we have that

$$\frac{f_0(z,x_n(z))}{\|x_n\|_{1,p}^{p-1}}\overset{n\to\infty}{\longrightarrow}\beta(z)|y^+(z)|^{p-2}y^+(z) \text{ a.e. in } Z_+.$$

With a similar argument we can show that

$$\frac{f_0(z,x_n(z))}{\|x_n\|_{1,p}^{p-1}}\overset{n\to\infty}{\longrightarrow}-\alpha(z)|y^-(z)|^{p-2}y^-(z) \text{ a.e. on } Z_0.$$

Replacing f_0 by f_1 in the above arguments we can also check that

$$\frac{f_1(z,x_n(z))}{\|x_n\|_{1,p}^{p-1}}\overset{n\to\infty}{\longrightarrow}\beta(z)|y^+(z)|^{p-2}y^+(z) \text{ a.e. on } Z_+.$$

and

$$\frac{f_1(z,x_n(z))}{\|x_n\|_{1,p}^{p-1}}\overset{n\to\infty}{\longrightarrow}-\alpha(z)|y^-(z)|^{p-2}y^-(z) \text{ a.e. in } Z_-.$$

$$y^-(x)=-y(x) \qquad \text{if} \qquad y(x)<0.$$

So we have proved that (4) and (5) hold on $Z_+\cup Z_-$. Finally since $\frac{x_n(z)}{\|x_n\|_{1,p}}=y_n(z)\to 0$ for almost all $z\in Z_0$ from the growth property of f (see the remark following hypotheses $\mathrm{H(f)}_1$), we have

$$\left|\frac{f(z,x_n(z))}{\|x_n\|_{1,p}^{p-1}}\right|\le\frac{1}{\|x_n\|_{1,p}^{p-1}}(\alpha_1(z)+c_1|x_n(z)|^{p-1})\to 0 \text{ a.e. on } Z_0 \ \Rightarrow$$

$$\Rightarrow \left| \frac{f_i(z, x_n(z))}{\|x_n\|_{1,p}^{p-1}} \right| \to 0 \text{ a.e. on } Z_0 \text{ as } n \to \infty,\ i = 0, 1.$$

Therefore we conclude that (4) and (5) hold.

Next, let $\hat{G} : L^P(Z) \to \mathbf{R}$ be defined by

$$\hat{G}(x) = \int_Z \int_0^{x(z)} f(z, r) dr dz = \int_Z F(z, x(z)) dz.$$

Evidently $G = \hat{G}|_{W_0^{1,p}(Z)}$ and from CHANG [7], we know that $G(\cdot)$ is locally Lipschitz too. So Theorem 2.2 of CHANG [7] implies that $\partial G(x) \subseteq \subseteq \partial \hat{G}(x) \subseteq L^q(Z)$ for all $x \in W_0^{1,p}(Z)$. By definition (see section 2) we have

$$\partial \hat{G}(x) = \left\{ v \in L^q(Z) : \int_Z v(z) u(z) dz \le \hat{G}^0(x; u) \text{ for every } u \in L^p(Z) \right\}$$

where

$$\hat{G}^0(x; u) = \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda} \left[\hat{G}(x + h + \lambda y) - \hat{G}(x + h) \right] =$$

$$= \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda} \int_Z \int_{(x+h)(z)}^{(x+h+\lambda u)(z)} f(z, r) dr dz.$$

by using the substitution $r = x(z) + h(z) + \eta\lambda y(z)$ we have

$$\hat{G}^0(x; y) = \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda} \int_Z \int_0^1 f(z, x(z) + h(z) + \eta\lambda y(z)) \lambda y(z) d\eta dz \le$$

$$\le \int_Z \overline{\lim_{\substack{h \to 0 \\ \lambda \downarrow 0}}} \int_0^1 f(z, x(z) + h(z) + \eta\lambda y(z)) y(z) d\eta dz \text{ (by Fatou's lemma)} \le$$

$$\le \int_{\{y>0\}} f_1(z, x(z)) y(z) dz + \int_{\{y<0\}} f_0(z, x(z)) y(z) dz < \infty.$$

Therefore if $v \in \partial \hat{G}(x)$, we have

$$\int_Z v(z)y(z)dz \leq$$

$$\leq \int_{\{y>0\}} f_1(z,x(z))y(z)dz + \int_{\{y<0\}} f_0(z,x(z))y(z)dz \text{ for all } y \in L^p(Z) \Rightarrow$$

$$\Rightarrow \; v(z) \in [f_0(z,x(z)), f_1(z,x(z))] \text{ a.e. on } Z.$$

Hence $g_n(z) \in [f_0(z,x(z)), f_1(z,x(z))]$ a.e. on Z. Combining this with (4) and (5) we obtain that

$$\frac{g_n(z)}{\|x_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\longrightarrow} \beta(z)|y^+(z)|^{p-2}y^+(z) - \alpha(z)|y^-(z)|^{p-2}y^-(z) \text{ a.e. on } Z.$$

Moreover, from the growth property of $f(z,r)$ (which of course is also valid for f_0 and f_1), we have

$$\left|\frac{g_n(z)}{\|x_n\|_{1,p}^{p-1}}\right| \leq \frac{\alpha_1(z) + c_1|x_n(z)|^{p-1}}{\|x_n\|_{1,p}^{p-1}} \text{ a.e. on } Z.$$

Since the sequence of functions $y_n = \frac{x_n}{\|x_n\|_{1,p}}$ is convergent in $L^p(Z)$, we can apply Vitali's convergence theorem to obtain that

$$\frac{g_n}{\|x_n\|_{1,p}^{p-1}} \overset{n\to\infty}{\longrightarrow} \beta|y^+|^{p-2}y^+ - \alpha|y^-|^{p-2}y^- \text{ in } L^q(Z).$$

Recall from the choice of $\{x_n\}_{n\geq 1}$ that we have $A(x_n) - g_n \overset{n\to\infty}{\longrightarrow} 0$ in $W^{-1,q}(Z)$. So for all $v \in W_0^{1,p}(Z)$ we have

$$\left|\int_Z \frac{\|Dx_n(z)\|^{p-2}(Dx_n(z), Dv(z))_{\mathbf{R}^N}}{\|x_n\|_{1,p}^{p-1}}dz - \int \frac{g_n(z)v(z)}{\|x_n\|_{1,p}^{p-1}}dz\right| \leq$$

(6) $$\leq \frac{\varepsilon_n\|v\|_{1,p}}{\|x_n\|_{1,p}^{p-1}} \quad \text{with } \lim \varepsilon_n = 0. \Rightarrow$$

$$\Rightarrow \left|\int_Z \|Dy_n(z)\|^{p-2}(Dy_n(z), Dv(z))_{\mathbf{R}^N}dz - \int_Z \frac{g_n(z)v(z)}{\|x_n\|_{1,p}^{p-1}}dz\right| \leq \varepsilon_n \frac{\|v\|_{1,p}}{\|x_n\|_{1,p}^{p-1}} \Rightarrow$$

$$\Rightarrow \left| \langle A(y_n), v\rangle - \left(\frac{g_n}{\|x_n\|_{1,p}^{p-1}}, v \right)_{pq} \right| \leq \frac{\varepsilon_n \|v\|_{1,p}}{\|x_n\|_{1,p}^{p-1}}$$

Here by $\langle \ , \ \rangle$ we denote the duality brackets for the pair

$$\left(W^{-1,q}(Z), W_0^{1,p}(Z) \right)$$

and by $(\cdot, \cdot)_{pq}$ the duality brackets for the pair $\left(L^q(Z), L^p(Z)\right)$. Let $v = y_n -$ $- y$, $n \geq 1$. We have

$$\langle A(y_n), y_n - n\rangle \leq \left(\frac{g_n}{\|x_n\|_{1,p}^{p-1}}, y_n - y \right)_{pq} + \varepsilon_n \frac{\|y_n - y\|}{\|x_n\|_{1,p}^{p-1}} \to 0.$$

As we mentioned earlier $A(\cdot)$ is monotone, demicontinuous, hence it has the property (M) (see ZEIDLER [141, pp. 583–584). So we have $\langle A(y_n), y_n\rangle =$ $\|Dy_n\|_p^p \to \langle A(y), y\rangle = \|Dy\|_p^p$. Since $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$ and the latter being uniformly convex, it has the Kadec–Klee property, we find that $y_n \overset{n\to\infty}{\longrightarrow} y$ in $W_0^{1,p}(Z)$. Therefore for all $v \in W_0^{1,p}(Z)$, we have by (6)

$$\int_Z \|Dy_n(z)\|^{p-2}(Dy_n(z), Dv(z))_{\mathbf{R}^N} dz - \int_Z \frac{g_n(z)}{\|x_n\|_{1,p}^{p-1}} v(z) dz \overset{n\to\infty}{\longrightarrow}$$

$$\overset{n\to\infty}{\longrightarrow} \int_z \|Dy(z)\|^{p-2}(Dy(z), Dv(z))_{\mathbf{R}^N} dz -$$

$$- \int_Z (\beta(z)|y^+(z)|^{p-2}y^+(z) - \alpha(z)|y^-(z)|^{p-2}y^-(z))v(z)dz = 0.$$

So $y \in W_0^{1,p}(Z)$ is a weak solution of
(7)

$$\left\{ \begin{array}{l} -\operatorname{div}\left(\|Dy(z)\|^{p-2}Dy(z)\right) = \beta(z)|y^+(z)|^{p-2}y^+(z) - \alpha(z)|y^-(z)|^{p-2}y^-(z) \\ y|_\Gamma = 0. \end{array} \right\}$$

We will show that problem (7) has only the trivial solution. For this purpose, let $y \in W_0^{1,p}(Z)$ be a solution of (7). Apply to (7) the function $v = y^- \in W_0^{1,p}(Z)$ (see GILBARG–TRUDINGER [10], p. 145) Recall that

$Dy^-(z) = 0$ if $y(z) \geq 0$ and $Dy^-(z) = -Dy(z)$ if $y(z) < 0$ (see GILBARG–TRUDINGER [10], Lemma 7.6, p.145). So we obtain

$$\int_Z \|Dy^-(z)\|^p dz = -\int_Z \alpha(z)|y^-(z)|^p dz.$$

Since $\alpha(z) > \lambda_1 > 0$ a.e. on Z, from the last equality it follows that $y^-(z) = 0$ a.e. on Z. On the other hand, since $\beta(z) < \lambda_1$ a.e. on Z, from theorem 3.3 of BOCCARDO–DRABEK–GIACHETTI–KUČERA [4], we know that the quasilinear eigenvalue problem

$$\left\{ \begin{array}{l} -\operatorname{div}\left(\|Dy^+(z)\|^{p-2}Dy^+(z)\right) = \beta(z)|y^+(z)|^{p-2}y^+(z) \text{ a.e. on } Z \\ y^+|_\Gamma = 0 \end{array} \right\}$$

has only the trivial solution. So $y^+(z) = 0$ a.e. on Z and we conclude that $y = 0$. Thus $y_n \overset{n\to\infty}{\longrightarrow} y = 0$ in $W_0^{1,p}(Z)$ which is impossible since $\|y_n\|_{1,p} = 1$. This proves the $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Thus by passing to a subsequence if necessary, we may assume that $x_n \overset{w}{\rightarrow} x$ in $W_0^{1,p}(Z)$ and since the latter space is compactly embedded in $L^p(Z)$ we also have that $x_n \overset{n\to\infty}{\longrightarrow} x$ in $L^p(Z)$. Also we have $|\langle A(x_n) - g_n, x_n - x\rangle| \leq \|A(x_n) - g_n\|_* \|x_n - - x\|_{1,p} \overset{n\to\infty}{\longrightarrow} 0$. Therefore

$$\overline{\lim_{n\to\infty}} \langle A(x_n), x_n - x\rangle \leq \lim_{n\to\infty} \langle g_n, x_n - x\rangle = \lim_{n\to\infty} (g_n, x_n - x)_{pq} = 0.$$

As before using the property (M) of the operator $A(\cdot)$ and the Kadec–Klee property of the space $W_0^{1,p}(Z)$ we conclude that $x_n \overset{n\to\infty}{\longrightarrow} x$ in $W_0^{1,p}(Z)$, hence $R(\cdot)$ satisfies the (PS)-condition. ∎

This proposition allows us to prove the following existence theorem for problem (2):

THEOREM 3. *If hypotheses* H(f)$_1$ *hold, then problem* (2) *admits a nontrivial solution.*

PROOF. By virtue of hypothesis H(f)$_1$(ii), uniformly for $z \in Z \setminus N_1$, $|N_1| = 0$, we have

$$\lim_{r\to\infty} \frac{f(z,r)}{|r|^{p-2}r} = \alpha(z) \geq ess \inf \alpha(z) > \lambda_1.$$

Hence if $\lambda_1 < \mu < ess\inf\alpha(z)$, we can find $M > 0$ such that for all $z \in Z\setminus N_1$ and all $r < -M$, we have

$$\frac{f(z,r)}{|r|^{p-2}r} \geq \mu \Rightarrow f(z,r) \leq \mu|r|^{p-2}r = (-\mu)|r|^{p-1} \quad \text{since } r < -M < 0.$$

Also for $z \in Z \setminus N_2$, $|N_2| = 0$ and $-M \leq r \leq 0$ we have

$$|f(z,r) \leq \alpha_1(z) + c_1 M^{p-2} = \gamma_1(z).$$

Therefore if $N_3 = N_1 \cup N_2$ ($|N_3| = 0$), for all $z \in Z \setminus N_3$ and all $r \leq 0$, we have

$$F(z,r) = \int_0^r f(z,v)dv = -\int_r^0 f(z,v)dv = -\int_r^{-M} f(z,v)dv - \int_{-M}^0 f(z,v)dv \geq$$

$$\geq -\int_r^{-M} (-\mu)|v|^{p-1}dv - \int_{-M}^0 \gamma_1(z)dv = -\int_r^{-M} \mu(-v)^{p-1}d(-v) + \gamma_1(z)M =$$

$$= \frac{\mu}{p}|r|^p + \gamma_2(z)$$

with $\gamma_2(z) = \gamma_1(z)M - \frac{\mu}{p}M^p$, $\gamma_2 \in L^p(Z)$. Let u_1 be the eigenfunction corresponding to the principal eigenvalue $\lambda_1 > 0$. Recall (see section 2) that $u_1(z) > 0$ a.e. on Z. So for $\theta > 0$ we can write

$$R(\theta(-u_1)) = \frac{\theta^p}{p}\|Du_1\|_p^p - \int_Z F(z,\theta(-u_1)(z))dz \leq$$

$$\leq \frac{\theta^p}{p}\|Du_1\|_p^p - \frac{\mu\theta^p}{p}\|u_1\|_p^p - \|\gamma_2\|_1 = \frac{\theta^p}{p}\left(1 - \frac{\mu}{\lambda_1}\right)\|Du_1\|_p^p - \|\gamma_2\|_1.$$

Since $\lambda_1 < \mu$, we see that $R(\theta(-u_1)) \overset{\theta\to\infty}{\longrightarrow} -\infty$, i.e. $R(\cdot)$ is unbounded from below.

Next let $\eta(z) = \alpha(z) + 1$. Then using hypothesis $H(f)_1$(ii) we can find $M_1 \geq 1$ such that for all $z \in Z \setminus N_4$ and all $r \leq -M_1$ we have

$$\frac{f(z,r)}{|r|^{p-2}r} \leq \eta(z) \Rightarrow f(z,r) \geq \eta(z)|r|^{p-2}r \geq \eta(z)|r|^{p_1-2}r$$

since $M_1 \geq 1$. Also because of hypothesis $H(f)_1$(iii), we can find $c_1 > 0$ large enough and $\delta > 0$ such that for all $z \in Z \setminus N_5$, $|N_5| = 0$, and all $|r| \leq \delta$

$$|f(z,r)| \leq c_1|r|^{p_1-1}.$$

Finally for $\delta \leq |r| \leq M_1$, we have

$$|f(z,r)| \leq \gamma_{M_1}(z) \leq \|\gamma_{M_1}\|_\infty \quad \text{a.e. on } Z.$$

We can always choose $c_1 > 0$ large enough so that for $\delta \leq |r| \leq M_1$ we have

$$\|\gamma_{M_1}\|_\infty \leq c_1 |r|^{p_1-1}.$$

Therefore for $z \in Z \setminus N_6$, $|N_6| = 0$, and for $0 \leq |r| \leq M_1$ we have

$$|f(z,r)| \leq c_1 |r|^{p_1-1}.$$

Thus we can say that for all $z \in Z \setminus N_7$, $|N_7| = |N_4 \cup N_5 \cup N_6| = 0$ and all $r \leq 0$ we have

$$f(z,r) \geq -\eta(z)|r|^{p_1-2}r - c_1|r|^{p_1-1} \;\Rightarrow$$

$$\Rightarrow\; F(z,r) = \int_0^r f(z,v)dv = -\int_r^0 f(z,v)dv \leq \Rightarrow$$

$$\Rightarrow \leq c_1 \int_r^0 |v|^{p_1-1}dv - \int_r^0 \eta(z)|v|^{p_1-2}v\,dv = \Rightarrow$$

$$\Rightarrow = -c_1 \int_r^0 (-v)^{p_1-1}d(-v) + \eta(z)\int_r^0 |v|^{p_1-1}d(-v) = \Rightarrow$$

$$\Rightarrow = -c_1 \int_r^0 (-v)^{p_1-1}d(-v) - \eta(z)\int_r^0 (-v)^{p_1-1}d(-v) = \frac{c_1+\eta(z)}{p_1}|r|^{p_1}. \tag{9}$$

On the other hand for all $z \in Z \setminus N_8$, $|N_8| = 0$ and all $r \geq 0$, we have for $\mu_0 > 0$ such that $\mathit{ess}\sup\beta < \mu_0 < \lambda_1$,

$$f(z,r) \leq \mu_0|r|^{p-1} + c_1|r|^{p_1-1} \Rightarrow F(z,r) \leq \frac{\mu_0}{p}r^p + \frac{c_1}{p}r^{p_1}. \tag{10}$$

From (9) and (10), it follows that for all $z \in Z \setminus N_9$, $|N_9| = 0$, and all $r \in \mathbf{R}$, we have

$$F(z,r) < \frac{\mu_0}{p}|r|^p + c_2|r|^{p_1}, \qquad c_2 > 0.$$

Hence we have

$$R(x) \geq \frac{1}{p}\|Dx\|_p^p - \frac{\mu_0}{p}\|x\|_p^p - c_2\|x\|_{p_1}^{p_1}.$$

Since by hypothesis H(f)$_1$(iii) $p < p_1 \leq \frac{Np}{N-p}$, from the Sobolev embedding theorem we have that $W_0^{1,p}(Z)$ is continuously embedded in $L^{p_1}(Z)$. Recalling that $\|Dx\|_p$ is an equivalent norm on $W_0^{1,p}(Z)$, we can find $c_3 > 0$ such that $\|x\|_{p_1} \leq c_3\|Dx\|_p$. So we have

$$R(x) \geq \frac{1}{p}\left(1 - \frac{\mu_1}{\lambda_1}\right)\|Dx\|_p^p - c_4\|Dx\|_p^{p_1}, \qquad c_4 > 0.$$

Since $\mu_0 < \lambda_1$ and $2 \leq p < p_1$, we can find $\rho > 0$ such that if $\|Dx\|_p = \rho$, then $R(x) \geq \xi > 0$ and $R(x) > 0$ if $0 < \|Dx\|_p < \rho$. Further, $R(\theta(-u_1)) \overset{\theta\to\infty}{\longrightarrow} -\infty$. Thus we can apply Theorem 1 and find $x \in W_0^{1,p}(Z)$ such that

$$0 \in \partial R(x) \subseteq A(x) - \partial G(x) \;\Rightarrow\; A(x) = g, \quad g \in \partial G(x) \subseteq L^q(Z).$$

So for all $\phi \in C_0^\infty(Z)$ we have

$$\langle A(x), \phi\rangle = \langle g, \phi\rangle = (g, \phi)_{pq} \;\Rightarrow$$

$$\Rightarrow \int_Z \|Dx(z)\|^{p-2}(Dx(Z), D\phi(z))_{\mathbf{R}^N}dz = \int_Z g(z)\phi(z)dz.$$

Recall that $g(z) \in \hat{f}(z, x(z))$ a.e. on Z. Therefore $x \in W_0^{1,p}(Z)$ is a solution of (2). Since $R(x) \geq \xi > 0 = R(0)$ (see theorem 1), we conclude that x is a nontrivial solution. ∎

We have another existence theorem with a nonresonance condition below the first eigenvalue $\lambda_1 > 0$. The exact hypotheses on $f(z,x)$ are the following:

H(f)$_2$: $f : Z \times \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that

(i) $f_0(z,r), f_1(z,r)$ are finite and N-measurable;

(ii) for almost all $z \in Z$ and all $r \in \mathbf{R}$ we have $|f(z,r)| \leq \alpha(z) + c|r|^{p_1-1}$ with $\alpha \in L^{\frac{p^*}{p^*-p_1}}$, $p < p_1 \leq p^* = \frac{Np}{N-p}$, $c > 0$;

(iii) there exist $\theta > p$ and $r_0 > 0$ such that for almost all $z \in Z$ and all $|r| \geq r_0$ we have $0 < \theta F(z,r) \leq rf(z,r)$ and $F(z,r_0) \geq \hat{c} > 0$ a.e. on Z;

(iv) $\lim\limits_{r\to 0} \frac{|f(z,r)|}{|r|^{p-1}} < \lambda_1$ uniformly for almost all $z \in Z$.

REMARK. Hypothesis H(f)$_2$(iii) together with the nonresonance condition H(f)$_2$(iv), were first used by AMBROSETTI–RABINOWITZ [2l in the context of semilinear problems with smooth data.

PROPOSITION 4. *If hypotheses* $\mathrm{H(f)}_2$ *hold, then* $R(\cdot)$ *satisfies the (PS)-condition.*

PROOF. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be such that $\{R(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) \stackrel{n\to\infty}{\longrightarrow} 0$. We have to produce a strongly convergent subsequence of $\{x_n\}_{n\geq 1}$. Let $g_n \in \partial G(x_n)$, $n \geq 1$, such that $m(x_n) = \|A(x_n) - g_n\|_* \stackrel{n\to\infty}{\longrightarrow} 0$, where as before $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the monotone, demicontinuous operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbf{R}^N}\, dz$$

for all $y \in W_0^{1,p}(Z)$ and $\|\cdot\|_*$ denotes the norm of $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$.

Let $\eta > 0$ be such that for all $n \geq 1$

$$|R(x_n)| = \left|\frac{1}{p}\|Dx_n\|_p^p - \int_Z F(z, x_n(z))dz\right| \leq \eta \ \Rightarrow$$

$$\Rightarrow \left|\frac{\theta}{p}\|Dx_n\|_p^p - \int_Z \theta F(z, x_n(z))dz\right| \leq \theta\eta. \tag{11}$$

From the choice of g_n, we can find $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that

$$|\langle A(x_n) - g_n, v\rangle| = \tag{12}$$

$$= \left|\int_Z \|Dx_n(z)\|^{p-2}(Dx_n(z), Dv(z))_{\mathbf{R}^N} dz - \int_Z g_n(z)v(z)dz\right| \leq \varepsilon_n\|v\|_{1,p}$$

for all $v \in W_0^{1,p}(Z)$. So if in (12) we set $v = x_n$ and then combine it with (11), we obtain

$$\left(\frac{\theta}{p} - 1\right)\|Dx_n\|_p^p \leq \int_Z (\theta F(z, x_n(z)) - g_n(z)x_n(z))dz + \varepsilon_n\|x_n\|_{1,p} + \theta\eta. \tag{13}$$

But by virtue of hypothesis $\mathrm{H(f)}_2$(iii) for almost all $z \in Z$ and all $|r| \geq r_0$ we have

$$0 < \theta F(z, r) \leq rf_0(z, r) \quad \text{and} \quad 0 < \theta F(z, r) \leq rf_1(z, r). \tag{14}$$

Using (14) and (15), we obtain

$$\left(\frac{\theta}{p}-1\right)\|Dx\|_p^p \le \int\limits_{\{|x_n|<r_0\}} (\theta F(z,x_n(z))-g_n(z)x_n(z))dz+\varepsilon_n\|x_n\|_{1,p}+\theta\eta \le$$

$$\le M_1+\varepsilon_n\|x_n\|_{1,p}+\theta\eta, \quad M_1>0 \quad \text{(see hypothesis H(f)}_2\text{(ii)).}$$

Recalling that $\|Dx\|_p$ is equivalent to the norm $\|x\|_{1,p}$ in $W_0^{1,p}(Z)$, from the above inequality it follows that $\{x_n\}_{n\ge1}$ is bounded in $W_0^{1,p}(Z)$. Then as in the proof of Proposition 2, using the fact that $A(\cdot)$ has the property (M) (being monotone, demicontinuous), we can have a strongly convergent subsequence of $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$. Therefore $R(\cdot)$ satisfies the (PS)-condition. ∎

Now note that hypothesis H(f)$_2$(iv) implies that $f(z,0)=0$ a.e. on Z. So $x=0$ is a solution of problem (2) (trivial solution). Our goal is to establish the existence of a nontrivial solution. By integrating the inequality of hypothesis H(f)$_2$(iii) and then taking exponentials, we see that for almost all $z\in Z$ and all $r\ge r_0$, we have

$$F(z,r_0)\frac{|r|^\theta}{r_0^\theta} \le F(z,r). \tag{15}$$

We can now state and prove our second existence theorem for problem (2)

THEOREM 5. *If hypotheses* H(f)$_2$ *hold, then problem* (2) *admits a nontrivial solution.*

PROOF. Let $u_1\in W_0^{1,p}(Z)\cap L^\infty(Z)$ be the eigenfunction corresponding to the eigenvalue $\lambda_1>0$ of the p-Laplacian. Recall that $u_1>0$ a.e. on Z. From (15) we have that for almost all $z\in Z$ and all $r\ge r_0$ we have

$$F(z,r_0)\frac{r^\theta}{r_0^\theta} \le rf(z,r).$$

Hence it follows that $\lim\limits_{r\to+\infty}\frac{f(z,r)}{r^{p-1}}=+\infty$ uniformly for almost all $z\in Z$ (recall that by hypothesis H(f)$_2$(iii) $\theta>p$ and $F(z,r_0)\ge\hat{c}>0$ a.e. on Z). So, given $\mu>\lambda_1$ we can find $M_1>0$ such that for almost all $z\in Z$ and all $r\ge M_1$ we have $f(z,r)\ge\mu|r|^{p-1}$. Using this and hypothesis H(f)$_2$ (ii) we can deduce that for almost all $z \in Z$ and all $r \geq 0$ we have $F(z,r) \geq$ $\geq \frac{\mu}{p}|r|^p - \gamma_1(z)$ for some $\gamma_1 \in L^{\frac{p^*}{p^*-p_1}}(Z)$. Therefore for every $\lambda > 0$ we have

$$R(\lambda u_1) = \frac{\lambda^p}{p}\|Du_1\|_p^p - \int_Z F(z,\lambda u_1(z))dz \leq$$

$$\leq \frac{\lambda^p}{p}\|Du_1\|_p^p - \frac{\mu\lambda^p}{\lambda_1 p}\|Du_1\|_p^p + \|\gamma_1\|_1 =$$

$$= \frac{\lambda^p}{p}\left(1 - \frac{\mu}{\lambda_1}\right)\|Du_1\|_p^p + \|\gamma_1\|_1 \overset{\lambda\to\infty}{\longrightarrow} -\infty.$$

because $\mu > \lambda_1$.

Next, from hypothesis H(f)$_2$(iv), given $0 < \mu_1 < \lambda_1$, we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|r| < \delta$ we have

$$|f(z,r)| \leq \mu_1|r|^{p-1} \;\Rightarrow\; |F(z,r)| \leq \frac{\mu_1}{p}|r|^p.$$

Also from hypothesis H(f)$_2$(ii) we have that for almost all $z \in Z$ and all $|r| > \delta$

$$|f(z,r)| \leq \alpha(z) + c|x|^{p_1-1} \;\Rightarrow\; |F(z,r)| \leq \alpha(z)|r| + \frac{c}{p}|r|^{p_1}.$$

Let $\alpha_3(z) = \frac{1}{\delta^{p_1-1}}\left(\alpha(z) + \frac{c}{p}\delta^{p_1-1}\right)$, $\alpha_3 \in L^{\frac{p^*}{p^*-p_1}}(Z)$. Then it is easy to see that $\alpha_3(z)|r|^{p_1} \geq \alpha(z)|r| + \frac{c}{p}|r|^{p_1}$ a.e. on Z and so for almost all $z \in Z$ and all $|r| \geq \delta$ we have

$$|F(z,r)| \leq \alpha_3(z)|r|^{p_1} \;\Rightarrow$$

$$\Rightarrow\; F(z,r) \leq \frac{\mu_1}{p}|r|^p + \alpha_3(z)|r|^{p_1} \quad \text{a.e. on } Z \text{ for all } r \in \mathbf{R}.$$

Therefore for every $x \in W_0^{1,p}(Z)$ we have

$$R(x) \geq \frac{1}{p}\|Dx\|_p^p - \frac{\mu_1}{p}\|x\|_p^p - \int_Z \alpha_3(z)|x(z)|^{p_1}dz.$$

Note that $|x(\,.\,)|^{p_1} \in L^p(Z)$. Hence by Hölders inequality we have

$$\int_Z \alpha_3(z)|x(z)|^{p_1}dz \leq \|\alpha_3\|_\eta \|x\|_{p^*}^{p_1} \qquad \left(\eta = \frac{p^*}{p^* - p_1}\right).$$

So we have

$$R(x) \geq \frac{1}{p}\left(1 - \frac{\mu_1}{\lambda_1}\right)\|Dx\|_p^p - \|\alpha_3\|_\eta \|x\|_{p^*}^{p_1}.$$

Since $W_0^{1,p}(Z)$ is embedded compactly in $L^{p^*}(Z)$ (Sobolev embedding theorem) we have

$$R(x) \geq \frac{1}{p}\left(1 - \frac{\mu_1}{\lambda_1}\right)\|Dx\|_p^p - c_1\|Dx\|_{p^*}^{p_1}, \qquad c_1 > 0.$$

Thus we can find $\rho > 0$ small enough such that $R(x) \geq \xi > 0$ for all $x \in W_0^{1,p}(Z)$ with $\|Dx\|_p = \rho$. Since $R(0) = 0$, $R(x) > 0$ if $0 < \|Dx\|_p < \rho$ and $R(\lambda u_1) \overset{\lambda\to\infty}{\longrightarrow} -\infty$ we can apply Theorem 1 and obtain $x \in W_0^{1,p}(Z)$ such that $0 \in \partial R(x) \subseteq A(x) - \partial G(x) \;\Rightarrow\; A(x) \subseteq \partial G(x)$. As in the proof of Theorem 3, we can check that x is the desired nontrivial solution of (2). ∎

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PATTERN FORMATION IN BOUNDED SPATIAL DOMAINS

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1. Introduction

This paper is primarily concerned with the existence and stability properties of spatially homogeneous respectively nonhomogeneous stationary solutions to a system of reaction-diffusion equations. These equations can be very important in the modelling and study of population growth and ecological processes and last but not least in nuclear and chemical reactions, still, due to the form of the kinetic system this one is rather a predator-prey system.

We are concerned with the system of reaction-diffusion equations

$$\frac{\partial S_j}{\partial t} = d_j \cdot \Delta S_j + f_j(\mathbf{S}) \tag{1.1}$$

($j \in \{1,2\}$) in $\Omega_i \times \mathbf{R}_0^+$ ($i \in \{H, R, T\}$) where

$$\Omega_H := \left\{ (x,y) \in \mathbf{R}^2 \;\middle|\; |x| < \frac{H\sqrt{3}}{2},\ \left| y + \frac{x}{\sqrt{3}} \right| < H \right\} \quad (H > 0),$$

$$\Omega_R := \left\{ (x,y) \in \mathbf{R}^2 \mid 0 < x < a,\ 0 < y < b \right\} \quad (a, b > 0),$$

$$\Omega_T := \left\{ (x,y) \in \mathbf{R}^2 \mid 0 < y < x < b \right\} \quad (a > 0)$$

(trivially with piecewise smooth boundary in $\mathbf{R}^2$) and the diffusion coefficients are positive and unequal, $f_j \in C^1(\mathbf{R}^2, \mathbf{R})$ ($j \in \{1,2\}$). We are looking for solutions $S_j : \Omega_i \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$ that satisfy the no-flux boundary conditions (homogeneous Neumann conditions) on $\partial\Omega_i \times \mathbf{R}_0^+$:

$$(\mathbf{n} \cdot \nabla) S_j = 0 \tag{1.2}$$

where $\mathbf{n}$ is the outer unit normal on $\partial\Omega_i$ $((i \in \{H, R, T\}),\ j \in \{1,2\})$. If $\mathbf{S}(\mathbf{r}, 0) \geq \mathbf{0}$ $(\mathbf{r} \in \Omega_i)$, then our initial-boundary-value problem has a unique, globally defined solution $\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ which satisfies $\mathbf{S}(\mathbf{r}, t) \geq \mathbf{0}$ for $(\mathbf{r}, t) \in$ $\in \Omega_i \times \mathbf{R}_0^+$ (see CHUEH, CONLEY and SMOLLER [4]).

2. The kinetic system

Let

$$f_1(S_1, S_2) := S_1 \cdot M_1(S_1, S_2),$$
$$f_2(S_1, S_2) := S_2 \cdot M_2(S_1, S_2),$$

where

$$M_1(S_1, S_2) := \alpha_{11} \cdot \left(1 - \frac{S_1}{\alpha_{13}}\right) - \alpha_{12} \cdot \frac{S_2}{\alpha_{12} + S_1},$$
$$M_2(S_1, S_2) := \alpha_{12} \cdot \frac{S_1}{\alpha_{12} + S_1} - \frac{\alpha_{21} + \alpha_{22} S_2}{1 + S_2},$$
$$\alpha_{jk} > 0 \qquad (j \in \{1,2\},\ k \in \{1,2,3,\})$$

and we assume that

$$\alpha_{21} < \alpha_{12} \leq \alpha_{22}, \tag{2.1}$$
$$\alpha_{12} < \alpha_{13}, \tag{2.2}$$
$$\alpha_{21} < \frac{\alpha_{12} \cdot \alpha_{13}}{\alpha_{12} + \alpha_{13}}. \tag{2.3}$$

The interaction terms f_1, f_2 are written in Kolmogorov form (see MAY [7]). This form is useful, since one can check the following properties of the kinetic system

$$\dot{\mathbf{S}} = \begin{pmatrix} S_1 \cdot M_1(\mathbf{S}) \\ S_2 \cdot M_2(\mathbf{S}) \end{pmatrix} : \tag{2.4}$$

1) M_1 and M_2 are smooth functions, therefore the positive quadrant of the phase space $[S_1, S_2]$ is an invariant region (see SMOLLER [11] pp. 198–203 and 230–231).

2) $\frac{\partial M_1}{\partial S_2}(S_1, S_2) = -\frac{\alpha_{12}}{\alpha_{12}+S_1} < 0$ and $\frac{\partial M_2}{\partial S_1}(S_1, S_2) = \alpha_{12} \cdot \frac{\alpha_{12}+S_1-S_1}{(\alpha_{12}+S_1)^2} =$ $= \frac{\alpha_{12}^2}{(\alpha_{12}+S_1)^2} > 0$, therefore (2.4) is a predator-prey system, where S_1 and S_2 represent the population densities of the prey and the predator, respectively,

and the functions M_1 and M_2 are considered as per capita growth rates for S_1 and S_2, respectively, depending on S_1 and S_2, further the prey growth rate M_1 decreases as the predator population increases, and that an increase in prey is favorable for the growth rate M_2 of the predator. The constants $\alpha_{jk} > 0$ ($j \in \{1,2\}$, $k \in \{1,2,3,\}$) can be interpreted as the specific growth rate of prey, the conversion rate and the carrying capacity with respect to the prey (α_{1k}, $k \in \{1,2,3\}$), respectively, further as the minimal mortality and the limiting mortality of the predator (α_{2k}, $k \in \{1,2\}$), respectively.

The following statements are discussed and proved in CAVANI–FARKAS [2]:

1) (2.1) ensures that predator mortality is increasing with density, and the zero set of S_2: $S_2 = \frac{(\alpha_{12}-\alpha_{21})\cdot S_1-\alpha_{12}\cdot\alpha_{21}}{(\alpha_{22}-\alpha_{12})\cdot S_1+\alpha_{12}\cdot\alpha_{22}}$ has a reasonable concave down shape;

2) (2.3) ensures that for the prey an Allée-effect zone exists (the interval $\left(0, \frac{\alpha_{13}-\alpha_{12}}{2}\right)$) where the increase of prey density is favourable to its growth rate;

3) there are at least three equilibria in the nonnegative quadrant: $(0,0)$ and $(\alpha_{13}, 0)$ on the boundary, $(\overline{S}_1, \overline{S}_2)^T$ in the interior of this invariant region, latter as intersection of the zero sets of S_1 and S_2, where of S_1 is given by $S_2 = \frac{(\alpha_{13}-S_1)\cdot(\alpha_{12}+S_1)\cdot\alpha_{11}}{\alpha_{12}\cdot\alpha_{13}}$. The equilibria on the boundary are unstable and the equilibrium in the interior is asymptotically stable if it is on the descending branch of the zero set of S_1, if it is in the ascending branch in the Allée-effect zone, then it may or may not be stable.

3. Diffusion driven instability

Denoting $D := \operatorname{diag}(d_1, d_2)$ and $\mathbf{F}(\mathbf{S}) := \begin{pmatrix} f_1(\mathbf{S}) \\ f_1(\mathbf{S}) \end{pmatrix}$, the initial-boundary-value problem has the following form:

$$\frac{\partial \mathbf{S}}{\partial t} = D \cdot \Delta \mathbf{S} + \mathbf{F}(\mathbf{S}) \quad \text{in} \quad \Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}), \tag{3.1}$$

$$(\mathbf{n} \cdot \nabla)\mathbf{S} = \mathbf{0} \quad \text{on} \quad \partial\Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}), \tag{3.2}$$

and

$$\mathbf{S}(\mathbf{r}, 0) = \mathbf{S}_0(\mathbf{r}) \quad \text{on} \quad \Omega_i \times \{0\} \quad (i \in \{H, R, T\}), \tag{3.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_i$.

Similarly to the theory of ordinary differential equations one can introduce some concepts of stability (see CASTEN and HOLLAND [1], SMOLLER [11]):

DEFINITION 3.1. Let $\overline{\mathbf{S}}(\mathbf{r})$ be an equilibrium solution of the system (3.1)–(3.2), i.e. $D \cdot \Delta\overline{\mathbf{S}}(\mathbf{r}) + \mathbf{F}(\overline{\mathbf{S}}(\mathbf{r})) = \mathbf{0}$ and $(\mathbf{n} \cdot \nabla)\overline{\mathbf{S}}(\mathbf{r}) = \mathbf{0}$. Then

1) $\overline{\mathbf{S}}(\mathbf{r})$ is called *stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathbf{S}$ is a solution of (3.1) with $\|\mathbf{S}(\mathbf{r}, 0) - \overline{\mathbf{S}}(\mathbf{r})\| < \delta$, then $\mathbf{S}$ exists for all $t > 0$ and $\|\mathbf{S}(\mathbf{r}, t) - \overline{\mathbf{S}}(\mathbf{r})\| < \varepsilon$ $(t \geq 0)$ where $\| \cdot \| := \| \cdot \|_\infty$.

2) $\overline{\mathbf{S}}(\mathbf{r})$ is called *asymptotically* stable if it is stable and there is $\eta > 0$ such that $\|\mathbf{S}(\mathbf{r}, 0) - \overline{\mathbf{S}}(\mathbf{r})\| < \eta$ implies $\lim_{t\to\infty}(\|\mathbf{S}(\mathbf{r}, t) - \overline{\mathbf{S}}(\mathbf{r})\|) = 0$.

3) If η can be chosen „arbitrary large”, then the solution is *globally asymptotically stable*.

4) The solution is *unstable* if it is not stable.

5) $\overline{\mathbf{S}}(\mathbf{r})$ is called *linearly stable* if it is an asymptotically stable solution of the linearized system $\frac{\partial \mathbf{Z}}{\partial t} = D \cdot \Delta\mathbf{Z} + A\mathbf{Z}$ with the same initial and boundary conditions as given with (3.1) where $A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the community matrix with coefficients $a_{11} = \frac{\partial f_1}{\partial S_1}(\overline{\mathbf{S}})$, $a_{12} = \frac{\partial f_1}{\partial S_2}(\overline{\mathbf{S}})$, $a_{21} = \frac{\partial f_2}{\partial S_1}(\overline{\mathbf{S}})$, $a_{22} = \frac{\partial f_2}{\partial S_2}(\overline{\mathbf{S}})$, and $\mathbf{Z}(\mathbf{r}, t)$ can be thought of as an approximation to $\mathbf{S}(\mathbf{r}, t) - \overline{\mathbf{S}}(\mathbf{r})$.

REMARK. A spatially constant solution $\mathbf{S}(t) = (S_1(t), S_2(t))^T$ of the system (3.1) satisfies the boundary conditions (3.2) and the kinetic system $\dot{\mathbf{S}} = (\mathbf{F}(\mathbf{S})$. The equilibrium of the kinetic system $\overline{\mathbf{S}} := (\overline{S}_1, \overline{S}_2)^T$ is a constant solution of (3.1)–(3.2) at the same time.

6) The equlibrium $\mathbf{S}$ of (3.1) is Turing (diffusionally) unstable if it is an asymptotically stable equilibrium of the kinetic system but is not asymptotically stable with respect to the solutions (3.1)–(3.2) (see OKUBO [9], SVIREZHEV–LOGOFET [12], CAVANI–FARKAS [3]).

THEOREM 3.1. *If* $\overline{\mathbf{S}}(\mathbf{r})$ *is a linearly stable solution of* (3.1)*, then* $\overline{\mathbf{S}}(\mathbf{r})$ *is asymptotically stable.*

PROOF. See SMOLLER [11] pp. 121–122.

Hence, we study the stability of an equilibrium solution of (3.1)–(3.2) via linearized stability. We have our system linearized at the point $\overline{\mathbf{S}} := (\overline{S}_1, \overline{S}_2)^T$ with the new coordinates $\mathbf{Z} = (Z_1, Z_2) = (S_1 - \overline{S}_1, S_2 - \overline{S}_2)$

$$\frac{\partial \mathbf{Z}}{\partial t} = D \cdot \Delta \mathbf{Z} + A\mathbf{Z} \quad \text{in} \quad \Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}), \tag{3.4}$$

with boundary conditions

$$(\mathbf{n} \cdot \nabla)\mathbf{Z} = 0 \quad \text{on} \quad \partial\Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}) \tag{3.5}$$

and initial conditions

$$\mathbf{Z}(\mathbf{r}, 0) = \mathbf{Z}_0(\mathbf{r}) \quad \text{on} \quad \Omega_i \times \{0\} \quad (i \in \{H, R, T\}) \tag{3.6}$$

where the community coefficiens are (see CAVANI–FARKAS [3]) $a_{11} = bc_1c_2$, $a_{12} = -c_1$, $a_{21} = \alpha_{12}^2 bc_3$, $a_{22} = -bc_3c_4$, where $b := \frac{\alpha_{11}}{\alpha_{13}\cdot\alpha_{12}}$, $c_1 := \frac{\alpha_{12}\cdot\overline{S}_1}{\alpha_{12}+\overline{S}_1}$, $c_2 := \alpha_{13} - \alpha_{12} - 2\overline{S}_1$, $c_3 := \frac{\alpha_{13}-\overline{S}_1}{\alpha_{12}+\overline{S}_1}$, $c_4 := \frac{((\alpha_{12}-\alpha_{22})\overline{S}_1+\alpha_{22}\cdot\alpha_{12})^2}{\alpha_{12}-\alpha_{21}}$.

We solve this linearized system with boundary and initial conditions (3.5) and (3.6) by the method of eigenfunction expansions for the domains Ω_R, Ω_C and Ω_T. We introduce the functions $\varphi : \mathbf{R}_0^+ \to \mathbf{R}^2$ and $\psi : \Omega_i \to \mathbf{R}$ $(i \in \{H, R, T\})$ satisfying

$$\dot{\varphi} = (A - \lambda D)\varphi) \tag{3.7}$$

and

$$\Delta\psi = -\lambda\psi, \qquad \frac{\partial\psi}{\partial\mathbf{n}} \Big| \, \partial\Omega_i = 0. \tag{3.8}$$

Clearly, the solutions of (3.4)–(3.5) can be written in the form $\mathbf{Z}(\mathbf{r}, t) = \psi(\mathbf{r}) \cdot \cdot \varphi(t)$.

Obviously, the eigenvalues of the boundary value problem (3.6) are non-negative, namely if we apply the antisymmetric Green-law for the form of the equation $\Delta\psi + \lambda\psi = 0$ multiplied by ψ, then we obtain that

$$\iint_{\Omega_i} \left(\left(\frac{\partial\psi}{\partial x} \right)^2 + \left(\frac{\partial\psi}{\partial y} \right)^2 - \lambda\psi^2 \right) dx\,dy =$$

$$= \oint_{\partial\Omega_i} (\psi \cdot \nabla\psi) d\mathbf{s} = \oint_{\partial\Omega_i} (\psi \cdot (\mathbf{n} \cdot \nabla\psi)) ds = \oint_{\partial\Omega_i} \left(\psi \cdot \frac{\partial\psi}{\partial\mathbf{n}} \right) ds = 0,$$

i.e. $\iint\limits_{\Omega_i} \left(\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right) dx\,dy = \lambda \iint\limits_{\Omega_i} \psi^2 dx\,dy$, therefore no eigenvalue can be negative. From the latter identity one can see that for the eigenvalue zero the eigenfunction is the constant function. For these domains denote by $\lambda_{m,n}$ the eigenvalues and by $\psi_{m,n}$ the corresponding normalised eigenfuctions. Define the vector coefficient $\mathbf{Z}_{0_{m,n}}$ of the eigenfunction $\psi_{m,n}$ in the expansion of the initial condition $\mathbf{Z}_0(\mathbf{r})$ by

$$\mathbf{Z}_{0_{m,n}} := \int\limits_{\Omega_i} \mathbf{Z}_0(\mathbf{r}) \psi_{m,n}(\mathbf{r}) d\mathbf{r} \tag{3.9}$$

with coordinates $z^1_{0_{m,n}}$, $z^2_{0_{m,n}}$, and let $\exp(A_{m,n}t)$ be the matrix solution of the differential equation (3.7) with two linearly independent columns $\varphi_{1;m,n}(t)$, $\varphi_{2;m,n}(t)$, where $A_{m,n} := A(\lambda_{m,n})$ with $A(\lambda) := A - \lambda D$ and with initial conditions $\exp(A_{m,n}0) = I$. Then the solution of the linearized problem (3.4)–(3.6) may be written in the form

$$\mathbf{Z}(\mathbf{r}, t) = \sum_{m,n=0}^{\infty} \psi_{m,n}(\mathbf{r}) \exp(A_{m,n}t) \mathbf{Z}_{0_{m,n}}. \tag{3.10}$$

The individual eigenvalues and the corresponding eigenfunctions are:

1) In the case of Ω_H the eigenvalues are $\lambda_{m,n} = \frac{16\pi^2}{9H^2} m^2$ with correspon-ding eigenfunctions

$$\psi_{m,n}(x, y) = \frac{1}{3} \left(\cos\left(\frac{2m\pi}{3H} (\sqrt{3}x + y) \right) + \cos\left(\frac{2m\pi}{3H} (\sqrt{3}x - y) \right) + \right.$$
$$\left. + \cos\left(\frac{4m\pi}{3H} y \right) \right) \quad (m \in \mathbf{N}_0).$$

2) For Ω_R: $\lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$ with $\psi_{m,n}(x, y) = \cos\left(\frac{m\pi}{a} x \right) \cos\left(\frac{n\pi}{b} y \right)$ $(m,\ n \in \mathbf{N}_0)$.

3) In the case of Ω_T the eigenvalues are $\lambda_{m,n} = \frac{\pi^2}{a^2}(m^2 + n^2)$ with corresponding eigenfunctions

$$\psi_{m,n}(x, y) = \cos\left(\frac{m\pi}{a} x \right) \cos\left(\frac{n\pi}{a} y \right) + \cos\left(\frac{n\pi}{a} x \right) \cos\left(\frac{m\pi}{a} y \right)$$
$$(m, n \in \mathbf{N}_0,\ m \geq n).$$

REMARK 1. These eigenfunctions constitute a complete orthogonal system in the weighted spaces $L^2(\Omega_H)$, $L^2(\Omega_R)$ and $L^2(\Omega_T)$ (see SZŐKEFALVI-NAGY, B. [13], MAKAI [6] and SIMON–BADERKO [10]).

REMARK 2. The solution in the case of Ω_H has some interesting properties. Firstly, since Ω_H has complete symmetry about the diagonals, the eigenvalues and hence the eigenfunctions form a one parameter sequence. Secondly, the values of $\psi_{m,n}(x,y)$ on the lines

$$\left(y \pm \frac{3}{4}H\right)\left(x + \frac{y}{\sqrt{3}} \pm \frac{\sqrt{3}}{2}H\right)\left(x - \frac{y}{\sqrt{3}} \pm \frac{\sqrt{3}}{2}H\right) = 0$$

when m is odd are everywhere constant.

REMARK 3. Let $f(x,y)$ be a symmetric function of its variables $(x,y) \in$ $\in \{(\xi,\eta) \in \mathbf{R}^2 \mid 0 < \xi < a,\ 0 < \eta < a\}$ $(f(x,y) = f(y,x))$, then the normal derivative of f with respect to the line $x = y$, if it exists, vanishes on that line, and $f(x,y) := \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) + \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)$ is a symmetric function.

REMARK 4. The smallest positive eigenvalues are $\lambda_{1,n} = \frac{16\pi^2}{9H^2}$ (for the hexagon) $\lambda_{0,1} = \frac{\pi^2}{b^2}$ (for the rectangle, assumed that $b > a$), $\lambda_{0,1} = \frac{\pi^2}{a^2}$ (for the rectangular triangle).

The following three lemmata (see CASTEN and HOLLAND [1], CONWAY [5]) are useful in what follows:

LEMMA 3.1. *If for each nonnegative integer m, n both eigenvalues of $A_{m,n}$ have negative real part, then the equilibrium $\overline{\mathbf{S}}$ of* (3.1)–(3.2) *is asymptotically stable, if for some m, n there exists an eigenvalue of $A_{m,n}$ with positive real part, then $\overline{\mathbf{S}}$ is unstable.*

LEMMA 3.2. *If $\overline{\mathbf{S}}$ is Turing unstable, then it lies in the Allée-effect zone, i.e.* $0 < \overline{S}_2 < \frac{\alpha_{13}-\alpha_{12}}{2}$.

LEMMA 3.3. *Suppose that $\overline{\mathbf{S}}$ lies in the Allée-effect zone, and that both eigenvalues of the matrix A have negative real parts. If $\overline{\mathbf{S}}$ is Turing unstable, then $d_2 > d_1$ must hold.*

PROOFS.

STEP 1. The proof of Lemma 3.1 is not necessary for our considerations, see CASTEN and HOLLAND [1]. We remark only that from the conditions formulated above the stability of the equilibrium of the linear system (3.4)–(3.5) follows, and this property can be extended, with the help of the eigenfunction expansion of $\mathbf{Z}$, to the equilibrium of the corresponding non-linear system.

STEP 2. The equilibrium $\overline{\mathbf{S}}$ lies in the Allée-effect zone $0 < \overline{S}_1 < \frac{\alpha_{13}-\alpha_{12}}{2}$, i.e. $c_2 = \alpha_{13} - \alpha_{12} - 2\overline{S}_1 > 0$, therefore if $\overline{\mathbf{S}}$ lies outside the Allée-effect zone, then $c_2 \leq 0$. Since

$$\operatorname{Tr}(A) = b(c_1c_2 - c_3c_4), \tag{3.11}$$

$$\det(A) = bc_1c_3(\alpha_{12}^2 - bc_2c_4), \tag{3.12}$$

$$\operatorname{Tr}(A_{m,n}) = \operatorname{Tr}(A) - (d_1 + d_2)\lambda_{m,n}, \tag{3.13}$$

$$\det(A_{m,n}) = d_1d_2\lambda_{m,n}^2 + b(d_1c_3c_4 - c_1c_2d_2)\lambda_{m,n} + \det(A) \tag{3.14}$$

and obviously c_1, c_3, c_4 are positive, therefore if $c_2 \leq 0$, then $\operatorname{Tr}(A) < 0$, $\det(A) > 0$, $\operatorname{Tr}(A_{m,n}) < 0$, $\det(A_{m,n}) > 0$, i.e. all eigenvalues of the matrices A, $A_{m,n}$ $(m, n \in \mathbf{N}_0)$ have negative real parts, so in view of Lemma 3.1 $\overline{\mathbf{S}}$ is not Turing unstable.

STEP 3. Since the matrix A has both eigenvalues with negative real parts, therefore $\operatorname{Tr}(A) < 0$, $\det(A) > 0$ hold, what implies that $\operatorname{Tr}(A_{m,n}) < 0$. To have Turing instability the quadratic polynomial $\det(A_{m,n})$ must be nonpositive for some $m, n \in \mathbf{N}$. Suppose that $d_2 \leq d_1$. Then we will show that $\det(A_{m,n}) > 0$ for all $\lambda_{m,n}$. Since the quadratic polynomial $\det(A(\lambda))$ has a positive leading coefficient $d_1 \cdot d_2$, and $\det(A(0)) = \det(A) > 0$, therefore it must be shown that $\det(A(\lambda)) > 0$ for all $\lambda > 0$. It is clear that

$$\frac{d}{d\lambda}(\det(A(0))) = b(d_1c_3c_4 - d_2c_1c_2). \tag{3.15}$$

Since $\operatorname{Tr}(A) < 0$, i.e. $c_1c_2 < c_3c_4$ and $\overline{\mathbf{S}}$ lies in the Allée-effect zone, i.e. $c_2 > 0$, we have $0 < c_1c_2 < c_3c_4$. So with $d_2 \leq d_1$

$$\frac{d}{d\lambda}(\det(A(0))) > bc_1c_2(d_1 - d_2) \geq 0$$

holds, what is sufficient for $\det(A(\lambda)) > 0$ $(\lambda > 0)$. ∎

REMARK. In view of the latter proof a necessary condition for the Turing instability is $\frac{d}{d\lambda}(\det(A(0))) = b(d_1c_3c_4 - d_2c_1c_2) < 0$.

EXAMPLE 1. Set $a_{12} := 0.5000$, $\alpha_{21} := 0.1000$, $\alpha_{22} := 0.5000$, $\alpha_{11} :=$ $= 0.1000$, $\alpha_{13} := 5.000$. The unique positive equilibrium is $(\overline{S}_1, \overline{S}_2)^T =$ $(0.2100, 0.1360)^T$. This point is in the Allée-effect zone $(0.2100 < 4.5000/2)$ and it is an asymptotically stable equilibrium of the kinetic system (2.4).

EXAMPLE 2. Set $\alpha_{12} := 0.1065$, $\alpha_{21} := 0.0085$, $\alpha_{22} := 0.1065$, $\alpha_{11} :=$ $= 1.600$, $\alpha_{13} := 35.3500$. The unique positive equilibrium is $(\overline{S}_1, \overline{S}_2)^T =$ $= (15.1736, 131.0240)^T$. This point is in the Allée-effect zone $(15.1736 <$ $< 35.2435/2)$ and it is an unstable equilibrium of the kinetic system (2.4).

As we have seen, for Turing instability the part of the Allée-effect zone is of interest, where the positive equilibrium of the kinetic system is asymptotically stable (outside this zone the equilibrium is asymptotically stable). This means that the following three inequalities must hold:

$$0 < \overline{S}_1 < \frac{\alpha_{13} - \alpha_{12}}{2}, \tag{3.16}$$

$$\mathrm{Tr}(A) = \frac{\alpha_{11}\overline{S}_1(\alpha_{13} - \alpha_{12} - 2\overline{S}_1)}{\alpha_{13}(\alpha_{12} + \overline{S}_1)} - \frac{(\alpha_{22} - \alpha_{21})\overline{S}_2}{(1 + \overline{S}_2)^2} < 0 \tag{3.17}$$

and

$$\det(A) = \frac{\alpha_{12}\overline{S}_1\overline{S}_2}{\alpha_{12} + \overline{S}_1}\left(\frac{\alpha_{12}^2}{(\alpha_{12} + \overline{S}_1)^2} - \frac{\alpha_{11}(\alpha_{22} - \alpha_{21})(\alpha_{13} - \alpha_{12} - 2\overline{S}_1)}{\alpha_{13}\alpha_{12}(1 + \overline{S}_2)^2}\right) < 0 \tag{3.18}$$

Therefore in according to CAVANI–FARKAS [3] we need the following

DEFINITION 3.2. The parameters $\alpha_{jk} > 0$ ($j \in \{1, 2\}$, $k \in \{1, 2, 3\}$) are said to be *Turing-admissible* if

1) the inequalities (2.1)–(2.3) hold;

2) the kinetic system (2.4) has an equilibrium point with positive coordinates in the Allée-effect zone and it is linearly asymptotically stable, i.e. (3.16)–(3.18).

LEMMA 3.4. *If the parameters $\alpha_{jk} > 0$ ($j \in \{1, 2\}$, $k \in \{1, 2, 3\}$) are Turing-admissible, then for a fixed $d_2 > 0$ a pair $m, n \in \mathbf{N}$ can be found such that*

$$\frac{\lambda_{m,n} b c_1 c_2 d_2 - \det(A)}{\lambda_{m,n}(b c_3 c_4 + \lambda_{m,n} d_2)} \tag{3.19}$$

is positive and if

$$0 < d_1 < \frac{\lambda_{m,n} bc_1c_2d_2 - \det(A)}{\lambda_{m,n}(bc_3c_4 + \lambda_{m,n}d_2)}, \tag{3.20}$$

then $\overline{\mathbf{S}} := (\overline{S}_1, \overline{S}_2)^T$ *is Turing unstable.*

PROOF. For the Turing instability is necessary that A be a stable matrix and because of Lemma 3.1 $\det(A_{m,n}) < 0$. From (3.14) it is easy to see that

$$\det(A(\lambda)) = \det(A) - \lambda bc_1c_2d_2 + \lambda d_1(bc_3c_4 + \lambda d_2).$$

Since $\lim_{m,n\to\infty} \lambda_{m,n} = \infty$, for fixed $d_2 > 0$ there exists $m, n \in \mathbf{N}$ such that $\lambda_{m,n} bc_1c_2d_2 - \det(A) > 0$. Hence $\frac{\lambda_{m,n}bc_1c_2d_2 - \det(A)}{\lambda_{m,n}(bc_3c_4+\lambda_{m,n}d_2)} > 0$ ($\lambda_{m,n}(bc_3c_4 + {}+ \lambda_{m,n}d_2) > 0$), and if for d_1: $0 < d_1 < \frac{\lambda_{m,n}bc_1c_2d_2 - \det(A)}{\lambda_{m,n}(bc_3c_4+\lambda_{m,n}d_2)}$ holds, therefore $\det(A_{m,n}) < 0$. ∎

REMARK 1. In this case we can see that $d_1 \in (0, d_2)$ can be chosen such that $\overline{\mathbf{S}} := (\overline{S}_1, \overline{S}_2)^T$ is Turing unstable.

REMARK 2. If we fix $m = n = 1$, for example, i.e. $\lambda_{1,1} = \frac{16\pi^2}{9H^2}$ (for the hexagon), $\lambda_{1,1} = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$ (for the rectangle) and $\lambda_{1,1} = \frac{2\pi^2}{a^2}$ (for the rectangular triangle), then for sufficiently large d_2 the numerator of (3.19) will be positive. As d_2 tends to infinity the rational function of d_2 in (3.19) is monotone increasing and bounded by $\frac{bc_1c_2}{\lambda_{1,1}}$ as the least upper bound. Since

$$\frac{bc_1c_2}{\lambda_{1,1}} = \frac{c_1c_2}{\lambda_{1,1}} \cdot \frac{\alpha_{11}}{\alpha_{13}\cdot\alpha_{12}} = \frac{\alpha_{11}}{\lambda_{1,1}} \cdot \frac{c_1c_2}{\alpha_{13}\cdot\alpha_{12}} = \frac{\alpha_{11}}{\lambda_{1,1}} \cdot \frac{\overline{S}_1(\alpha_{13} - \alpha_{12} - 2\overline{S}_1)}{\alpha_{13}(\alpha_{12} + \overline{S}_1)} =$$

$$= \frac{\alpha_{11}}{\lambda_{1,1}} \cdot \frac{\overline{S}_1\alpha_{13} - \overline{S}_1\alpha_{12} - 2\overline{S}_1^2}{\alpha_{13}\alpha_{12} + \alpha_{13}\overline{S}_1} < \frac{\alpha_{11}}{\lambda_{1,1}} \cdot \frac{\overline{S}_1\alpha_{13}}{\alpha_{13}\alpha_{12} + \alpha_{13}\overline{S}_1} < \frac{\alpha_{11}}{\lambda_{1,1}},$$

therefore this least upper bound for the fraction in (3.19) is less than $\alpha_{11} \cdot \frac{9H^2}{16\pi^2}$ (for the hexagon), $\alpha_{11} \cdot \frac{a^2b^2}{\pi^2(a^2+b^2)}$ (for the rectangle) and $\alpha_{11} \cdot \frac{a^2}{2\pi^2}$ (for the rectangular triangle). This means that irrespective of how large the predator diffusion rate d_2 is, in order to have Turing instability the prey diffusion rates d_1 must satisfy

$$d_1 < \alpha_{11} \cdot \frac{9H^2}{16\pi^2}, \tag{3.21a}$$

(3.21b)
$$d_1 < \alpha_{11} \cdot \frac{a^2 b^2}{\pi^2 (a^2 + b^2)},$$

(3.21c)
$$d_1 < \alpha_{11} \cdot \frac{a^2}{2\pi^2}$$

for the respective domains. Since, clearly, for all $m, n \in \mathbf{N}$

$$\frac{bc_1c_2}{\lambda_{m,n}} \le \frac{bc_1c_2}{\lambda_{1,1}} \le \alpha_{11} \cdot \min\left\{\frac{9H^2}{16\pi^2}, \frac{a^2b^2}{\pi^2(a^2+b^2)}, \frac{a^2}{2\pi^2}\right\}$$

the inequalities (3.21a–c) are respective necessary conditions of Turing instability.

4. Pattern formation

As we have seen, our system of reaction-diffusion equations contains seven parameters, including the diffusion coefficients. Varying the value of these parameters we get different stability relations. If we vary one of $\alpha_{jk} > 0$ ($j \in \{1,2\}$, $k \in \{1,2,3\}$) then it can easily happen that the positive equilibrium of the kinetic system is not asymptotically stable and in this case Turing instability cannot occur. It seems suitable to vary one of the diffusional coefficients, namely d_2 because of Lemma 3.3. We shall choose this coefficient (the predator diffusion coefficient) as a bifurcation parameter. First we establish conditions of a Turing bifurcation of the zero solution of the linearized system (3.4)–(3.5), then we shall extend this result to the non-linear problem (3.1)–(3.2).

DEFINITION 4.1. Let Ω_i be the same as in Section 1, further $S_j : \Omega_i \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying the no-flux boundary conditions on $\partial\Omega_i \times \mathbf{R}_0^+$:

(4.1)
$$(\mathbf{n} \cdot \nabla) S_j = 0,$$

$\mathbf{n}$ is the outer unit normal on $\partial\Omega_i$ ($i \in \{H, R, T\}$, $j \in \{1,2\}$); $\mathbf{F} \in C^1(\mathbf{R}^2 \times [0,\infty), \mathbf{R}^2)$, $\overline{\mathbf{S}} \in \mathbf{R}^2$, with $\mathbf{F}(\overline{\mathbf{S}}, \mu) = \mathbf{0}$ ($\mu \in [0,\infty)$); $D \in C^1([0,\infty), (\mathbf{R}_0^+)^{2\times 2})$ a diagonal matrix; and let us consider the system of reaction-diffusion equations

(4.2)
$$\frac{\partial \mathbf{S}}{\partial t} = D(\mu) \cdot \Delta \mathbf{S} + \mathbf{F}(\mathbf{S}, \mu).$$

We say that the constant stationary solution $\overline{\mathbf{S}}$ of (4.1)–(4.2) undergoes a *Turing bifurcation* at $\mu_T \in (0,\infty)$ if

1) for $\mu \in (0, \mu_T)$ the solution $\overline{\mathbf{S}}$ is asymptotically stable and for $\mu \in (\mu_T, \infty)$ is unstable (*supercritical bifurcation*), or for $\mu \in (0, \mu_T)$ the solution $\overline{\mathbf{S}}$ is unstable and for $\mu \in (\mu_T, \infty)$ is asymptotically stable (*subcritical bifurcation*);

2) in every neighbourhood of μ_T there are values of μ such that the problem (4.1)–(4.2) has non-constant stationary solutions.

In this section for the sake of simplicity let us denote λ_{s+1} the smallest positive eigenvalue of (3.6), i.e. $\lambda_{s+1} = \lambda_{0,1}$ for the rectangle and the rectangular triangle and $\lambda_{s+1} = \lambda_{1,1}$ for the hexagon, respectively; and λ_{s+l} the l-th positive eigenvalue in the sequence ($0 =: \lambda_s$, λ_{s+l} form a monotone nondecreasing sequence for $l \in \mathbf{N}$).

THEOREM 4.1. *If the parameters $\alpha_{jk} > 0$ ($j \in \{1,2\}$, $k \in \{1,2,3\}$) are Turing-admissible, then the following two statements hold:*

1) *If*

$$d_1 \geq \frac{bc_1c_2}{\lambda_{s+1}} \tag{4.3}$$

then the zero solution of the linear problem (3.4)–(3.5) *is asymptotically stable for all $d_2 > 0$.*

2) *If*

$$\frac{bc_1c_2}{\lambda_{s+p}} > d_1 \geq \frac{bc_1c_2}{\lambda_{s+p+1}} \qquad (p \in \mathbf{N}) \tag{4.4}$$

then at

$$d_2 = d_T := \frac{\lambda_{s+p} d_1 bc_1c_2 + \det(A)}{\lambda_{s+p} bc_1c_2 - \lambda_{s+p}^2 d_1} \qquad (p \in \mathbf{N}) \tag{4.5}$$

the zero solution of the linear problem (3.4)–(3.5) *undergoes a supercritical Turing bifurcation.*

PROOF.

STEP 1. In view of the proof of Lemma 3.1. we need $\mathrm{Tr}(A_{s+l}) < 0$ and $\det(A_{s+l}) > 0$ ($l \in \mathbf{N}_0$). Since the parameters $\alpha_{jk} > 0$ ($j \in \{1,2\}$, $k \in \{1,2,3\}$) are Turing-admissible, $\mathrm{Tr}(A) < 0$ must hold, therefore, because of (3.11), $\mathrm{Tr}(A_{s+l}) < 0$ for all $l \in \mathbf{N}_0$. From (3.12) and (3.14) we have

$$\det(A_{s+l}) = (\lambda_{s+l} d_1 - bc_1c_2)(\lambda_{s+l} d_2 + bc_3c_4) + b\alpha_{12}^2 c_1c_3.$$

Since λ_{s+1} is the smallest positive eigenvalue of (3.6), therefore (4.3) implies $\lambda_{s+1}d_1 - bc_1c_2 \geq 0$, i.e. $\det(A_{s+l}) > 0$ for all $l \in \mathbf{N}$. For the eigenvalue 0 we have $\det(A_s) = \det(A) > 0$. So for all $l \in \mathbf{N}_0$ $\det(A_{s+l}) > 0$, hence the zero solution of (3.4)–(3.5) is asymptotically stable.

STEP 2. If d_1 satisfies (4.4) and $d_2 = d_T$, then an easy calculation shows that $\det(A_{s+p}) = 0$; for $d_2 \in (0, d_T)$ we have $\det(A_{s+p}) > 0$, for $d_2 \in (d_T, \infty)$: $\det(A_{s+p}) < 0$; and in all these cases $\det(A_{s+l}) > 0$ $(l \neq p)$. Thus in view of Lemma 3.1. for $d_2 \in (0, d_T)$ the zero solution is asymptotically stable, and for $d_2 \in (d_T, \infty)$ it is unstable. If $d_2 = d_T$, then the matrix A_{s+p} is negative semidefinite, i.e. one of the two eigenvalues of A_{s+p} is zero and the other is negative. Denoting the eigenvector corresponding to the zero eigenvalue by $\varphi_{1,s+p} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, i.e.

$$A_{s+p}\varphi_{1,s+p} = (A - \lambda_{s+p})\varphi_{1,s+p} = 0,$$

we have a spatially non-constant solution of the linearized problem (3.4)–(3.5):

$$\mathbf{z}_{s+p}(\mathbf{r}) = \varphi_{1,s+p} \cdot \psi_{s+p}(\mathbf{r}) = \tag{4.6a}$$

$$= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \frac{\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x + y)\right) + \cos\left(\frac{2m\pi}{3H}(\sqrt{3}x - y)\right) + \cos\left(\frac{4m\pi}{3H}y\right)}{3}$$

for the hexagon,

$$\mathbf{z}_{s+p}(\mathbf{r}) = \varphi_{1,s+p} \cdot \psi_{s+p}(\mathbf{r}) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{4.6b}$$

for the rectangle,

$$\mathbf{z}_{s+p}(\mathbf{r}) = \varphi_{1,s+p} \cdot \psi_{s+p}(\mathbf{r}) = \tag{4.6c}$$

$$= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \left(\cos\left(\frac{m\pi}{a}x\right) \cdot \cos\left(\frac{n\pi}{a}y\right) + \cos\left(\frac{n\pi}{a}x\right) \cdot \cos\left(\frac{m\pi}{a}y\right)\right)$$

for the rectangular triangle, where m and n are integers for which the p-th eigenvalue of (3.8) is $\lambda_{m,n} = \lambda_{s+p}$. ∎

In the remaining part of this section — to extend the latter result about the Turing bifurcation of the zero solution of the linearized system to the non-linear problem (3.1)–(3.2) — we need the following

THEOREM 4.2. *Let X, Y be Banach spaces, $U := R \times V$ an open subset of $\mathbf{R} \times X$, and $f \in C^2(U, Y)$ such that $f(\mu, 0) = 0$ for all $\mu \in R \subset \mathbf{R}$. Denote the linear operators obtained by differentiating f with respect to its* *second, resp. first and second variables at* $\mu_0 \in S$, $0 \in V$ *by* $L_{02} := \partial f(\mu_0, 0)$ *and* $L_{12} := \partial_{12} f(\mu_0, 0)$, *respectively; and assume that the following conditions hold:*

(i) *the null space of* L_{02}, *the subspace* $N(L_{02})$ *of* X *is one dimensional, spanned by* $\mathbf{z}_{s+p} \in X$;

(ii) *the range of* L_{02}, *the subspace* $R(L_{02})$ *of* Y *has codimension* 1, *i.e.* $\dim(Y/R(L_{02})) = 1$;

(iii) $L_{12}\mathbf{z}_{s+p} \notin R(L_{02})$.

Let further W *be an arbitrary closed subspace of* X *such that* $X = \operatorname{Span}(\mathbf{z}_{s+p}) \oplus W$ *(i.e. any* $\mathbf{x} \in X$ *can be uniquely written as* $\mathbf{x} = \alpha \cdot \mathbf{z}_{s+p} + \mathbf{w}$, $\alpha \in \mathbf{R}$, $\mathbf{w} \in W$*). Then there is a* $\delta > 0$ *and a* C^1*-curve* $(\mu, \varrho) : (-\delta, \delta) \to R \times W$ *such that:*

1) $\mu(0) = \mu_0$;

2) $\varrho(0) = \mathbf{0}$;

3) $f(\mu(s), s \cdot \mathbf{z}_{s+p} + s \cdot \varrho(s)) = 0$ *for* $s \in (-\delta, \delta)$;

furthermore, there is a neighbourhood of $(\mu_0, 0)$ *such that any zero of* f *either lies on this curve or is of the form* $(\mu, 0)$.

PROOF. The idea of the proof is to introduce a new parameter s which enables us to apply immediately the implicit function theorem for the function $F \in C^1(U \times W, Y)$, $F(s, \mu, \mathbf{w}) := \begin{cases} f(\mu, s \cdot \mathbf{z}_{s+p} + s \cdot \mathbf{w})/s, & \text{if } s \neq 0, \\ L_{02}(\mathbf{z}_{s+p} + \mathbf{w}), & \text{if } s = 0 \end{cases}$ (see SMOLLER [11] pp. 172–173). ∎

REMARK. In what follows the role of the space X will be played by

$$X := \left\{ \mathbf{Z} \in C^2(\Omega_i, \mathbf{R}^2) \mid (\mathbf{n} \cdot \nabla)\mathbf{Z} = \mathbf{0},\ i \in \{H, R, T\} \right\} \tag{4.7}$$

with the norm $\|f\|_X := \sum_{0 \leq |\alpha| \leq 2} \sup_{\Omega_i}(|\partial^\alpha f|)$, where $|\cdot|$ denotes the usual vector resp. matrix-norm, while $Y := C^0(\Omega_i, \mathbf{R}^2)$ with the norm $\|f\|_Y := \sup_{\Omega_i}(|f|)$ ($i \in \{H, R, T\}$). However, in choosing the subspace W we shall use the orthogonality induced by the inner product

$$\langle \mathbf{Z}, \mathbf{Z}' \rangle := \int_{\Omega_i} (Z_1(\mathbf{r})Z_1'(\mathbf{r}) + Z_2(\mathbf{r})Z_2'(\mathbf{r}))d\mathbf{r}. \tag{4.8}$$

THEOREM 4.3. *If the parameters* $\alpha_{jk} > 0$ *(*$j \in \{1,2\}$, $k \in \{1,2,3\}$*) are Turing-admissible, then the following two statements hold:*

(1) *If* (4.3) *holds then the constant solution* $\overline{\mathbf{S}} := (\overline{S}_1, \overline{S}_2)^T$ *of the nonlinear problem* (3.1)–(3.2) *is asymptotically stable.*

(2) *If* $\begin{pmatrix} 0 \\ \xi_2 \end{pmatrix}$ *is not parallel to the second eigenvector* $\varphi_{2,s+p}$ *of the matrix* A_{s+p} *(corresponding to the negative eigenvalue) and* d_1 *satisfies* (4.4) *then at* $d_2 = d_T$*, as given by* (4.5)*, the constant solution* $\overline{\mathbf{S}}$ *undergoes a supercritical Turing bifurcation.*

PROOF.

STEP 1. Since because of Theorem 4.1 the zero solution of the linear problem (3.4)–(3.5) is asymptotically stable, therefore in view of Lemma 3.1. the above statement is clear.

STEP 2. From Theorem 4.1. just like in 1) we have that for $d_2 \in (0, d_T)$ $\overline{\mathbf{S}}$ is asymptotically stable, while for $d_2 \in (d_T, \infty)$ it is unstable. So, we have to show the existence of a stationary non-constant solution in some neighbourhood of the critical value d_T of the bifurcation parameter d_2. Such a stationary solution satisfies the following two dimensional system of second order partial differential equations

$$D \cdot \Delta \mathbf{S} = \mathbf{F}(\mathbf{S}) = \mathbf{0} \quad \text{in} \quad \Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}), \tag{4.9}$$

with zero-flux boundary conditions

$$(\mathbf{n} \cdot \nabla)\mathbf{S} = \mathbf{0} \quad \text{on} \quad \partial\Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}). \tag{4.10}$$

Introducing the new vector of the variation $\mathbf{Z} := \mathbf{S} - \overline{\mathbf{S}}$ (4.9)–(4.10) assumes the equivalent form

$$D \cdot \Delta \mathbf{Z} + A\mathbf{Z} + \mathbf{G}(\mathbf{Z}) = \mathbf{0} \quad \text{in} \quad \Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}), \tag{4.11}$$

$$(\mathbf{n} \cdot \nabla)\mathbf{Z} = \mathbf{0} \quad \text{on} \quad \partial\Omega_i \times \mathbf{R}_0^+ \quad (i \in \{H, R, T\}) \tag{4.12}$$

where A is the the Jacobian of $\mathbf{F}$ evaluated at $\overline{\mathbf{S}}$, and

$$\mathbf{G}(\mathbf{Z}) := \mathbf{F}(\overline{\mathbf{S}} + \mathbf{Z}) - A\mathbf{Z} \quad \text{with} \quad \mathbf{G}(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \mathbf{G}_Z(\mathbf{0}) = \mathbf{0}. \tag{4.13}$$

Let $f \in C^2(\mathbf{R} \times X, Y)$, $f(d_2, \mathbf{Z}) := D \cdot \Delta \mathbf{Z} + A\mathbf{Z} + \mathbf{G}(\mathbf{Z})$ and $L_{02} \in \mathrm{Lin}(X, Y)$, $L_{02} := \partial_2 f(d_T, 0)$, then the following relation of spectra holds: $\sigma(L_{02}) \supset$ $\supset \sigma(A_{s+l})$ ($l \in \mathbf{N}_0$). Denote the eigenvalues of the matrices A_{s+l} by $\sigma_{q,l}$ ($q \in \{1,2\}; l \in \mathbf{N}_0$), then the corresponding eigenfunctions are $\psi_{s+l}(\mathbf{r}) \cdot \varphi_{q,s+l}$ ($q \in \{1,2\}$; $l \in \mathbf{N}_0$), where ψ_{s+l} is given as in (4.6a–c) and $\varphi_{q,s+l}$ are the eigenvectors of the matrix A_{s+l} corresponding to the eigenvalue $\sigma_{q,l}$. Now, all matrices $A_{s+l} = A - \lambda_{s+l} D$ are to be taken at $d_2 = d_T$. As it can be seen from the proof of Theorem 4.1 and from (3.13)–(3.14) for $q \in \{1, 2\}$; $l \in \mathbf{N}_0$ all $\sigma_{q,l}$ have negative real parts. For $l = p$ one eigenvalue, $\sigma_{1,p}$ say, is zero the other one is negative. The eigenfunction corresponding to $\sigma_{1,p}$ is $\mathbf{z}_{s+p}(\mathbf{r})$ (given by (4.6a–c)). Thus, the null space of the operator $L_{02} := \partial_2 f(d_T, 0)$ is one dimensional, spanned by $\mathbf{z}_{s+p}$, and the range of this operator is, because of the orthogonality and completeness of the eigenfunction system of the minus Laplacian, given by

$$R(L_{02}) = \{\mathbf{Z} \in C^0(\Omega_i, \mathbf{R}^2) \mid \text{the eigenfunction expansion of } \mathbf{Z} \\ \text{does not contain } \Theta \text{ term}\} \cup \{\varphi_{2,s+p} \cdot \Theta\} \quad (i \in \{H, R, T\})$$

where

$$\Theta := \frac{1}{3}\left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x + y)\right) + \cos\left(\frac{2m\pi}{3H}\sqrt{3}x - y)\right) + \cos\left(\frac{4m\pi}{3H}y\right)\right),$$

in case of the hexagon, $\Theta := \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$, in case of the rectangle, $\Theta := \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) + \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)$, in case of the rectangular triangle where m and n are integers for which the p-th eigenvalue of (3.8) is $\lambda_{m,n} = \lambda_{s+p}$. So the condition $\operatorname{co\,dim}(R(L_{02})) = 1$ is satisfied.

Let $L_{12} \in \operatorname{Lin}(X, Y)$ and $L_{12} := \partial f(d_T, 0)$, then $L_{12} = \frac{\partial D}{\partial d_2} \cdot \Delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \Delta$. Because of (4.6a–c)

$$L_{12}\mathbf{z}_{s+p} = -\frac{\partial D}{\partial d_2} \cdot \varphi_{1,s+p} \cdot \frac{16m^2\pi^2}{27H^2}\left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x + y)\right) + \right. \\ \left. + \cos\left(\frac{2m\pi}{3H}(\sqrt{3}x - y)\right) + \cos\left(\frac{4m\pi}{3H}y\right)\right) = -\frac{16m^2\pi^2}{27H^2} \cdot \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix} \cdot \\ \cdot \left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x + y)\right) + \cos\left(\frac{2m\pi}{3H}(\sqrt{3}x - y)\right) + \cos\left(\frac{4m\pi}{3H}y\right)\right),$$

for the hexagon,

$$L_{12}\mathbf{z}_{s+p} = -\frac{\partial D}{\partial d_2} \cdot \varphi_{1,s+p} \cdot \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{n}\right)^2\right] \cdot \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) = \\ = -\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \cdot \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix} \cdot \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right),$$

for the rectangle,

$$L_{12}\mathbf{z}_{s+p} = -2\frac{\partial D}{\partial d_2}\cdot\varphi_{1,s+p}\cdot\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cdot\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) =$$
$$= -2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cdot\begin{pmatrix}0\\ \xi_2\end{pmatrix}\cdot\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right),$$

for the rectangular triangle. So $L_{12}\mathbf{z}_{s+p}$ is not parallel to $\varphi_{2,s+p}\cdot\Theta$ and

$$\left\langle \mathbf{z}_{s+p}, L_{12}\mathbf{z}_{s+p}\right\rangle = -\frac{16m2\pi^2}{27H^2}\cdot\int\limits_{\Omega_H}(\xi_2)^2\left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x+y)\right)+\right.$$
$$\left.+\cos\left(\frac{2m\pi}{3H}\sqrt{3}x-y)\right)+\cos\left(\frac{4m\pi}{3H}y\right)\right)d\mathbf{r}\neq 0,$$

$$\left\langle \mathbf{z}_{s+p}, L_{12}\mathbf{z}_{s+p}\right\rangle =$$
$$= -\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cdot\int\limits_{\Omega_R}(\xi_2)^2\cos^2\left(\frac{m\pi}{a}x\right)\cos^2\left(\frac{n\pi}{b}y\right)d\mathbf{r}\neq 0$$

and

$$\left\langle \mathbf{z}_{s+p}, L_{12}\mathbf{z}_{s+p}\right\rangle =$$
$$= -2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cdot\int\limits_{\Omega_T}(\xi_2)^2\cos^2\left(\frac{m\pi}{a}x\right)\cos^2\left(\frac{n\pi}{b}y\right)d\mathbf{r}\neq 0$$

because $\xi_2\neq 0$. Thus, the condition $L_{12}\mathbf{z}_{s+p}\notin R(L_{02})$ is satisfied.

If we define the closed subspace W of X by $W := R(L_{02})$, then all the hypotheses of Theorem 4.2 hold, and $(d_T, 0)$ is a bifurcation point, further there exits a $\delta > 0$, a function $d_2 : (-\delta,\delta)\to\mathbf{R}$ and for $s\in(-\delta,\delta)$ a solution of (4.11) with $d_2 = d_2(s)$ ($|s| < \delta$) substituted

$$\mathbf{Z}(s,\mathbf{r}) = s\cdot\varphi_{1,s+p}\cdot\frac{1}{3}\left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x+y)\right)+\right.$$
$$\left.+\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x-y)\right)+\cos\left(\frac{4m\pi}{3H}y\right)\right)+s\,\varrho(s,\mathbf{r})$$

(for the hexagon)

$$\mathbf{Z}(s,\mathbf{r}) = s\cdot\varphi_{1,s+p}\cdot\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)+s\,\varrho(s,\mathbf{r})$$

(for the rectangle)

$\mathbf{Z}(s, \mathbf{r}) =$
$$= s \cdot \varphi_{1,s+p} \cdot \left(\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) + \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)\right) + s\,\varrho(s, \mathbf{r})$$

(for the rectangular triangle) such that $d_2(0) = d_T$, $\varrho(0, \mathbf{r}) = 0$, $d_2 \in C^1$, $\varrho(\,\cdot\,, \mathbf{r}) \in C^1$, and $\varrho(s,\,\cdot\,) \in W$. ∎

REMARK 1. The corresponding solution of (4.9), i.e. the nonconstant stationary solution of the nonlinear problem (3.1)–(3.2) is

$$\text{(4.14a)} \qquad \mathbf{S}(s, \mathbf{r}) = \overline{\mathbf{S}} + s \cdot \varphi_{1,s+p} \cdot \frac{1}{3}\left(\cos\left(\frac{2m\pi}{3H}(\sqrt{3}x + y)\right) + \right.$$
$$\left. + \cos\left(\frac{2m\pi}{3H}(\sqrt{3}x - y)\right) + \cos\left(\frac{4m\pi}{3H}y\right)\right) + O(s^2),$$

for the hexagon,

$$\text{(4.14b)} \qquad \mathbf{S}(s, \mathbf{r}) = \overline{\mathbf{S}} + s \cdot \varphi_{1,s+p} \cdot \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) + O(s^2),$$

for the rectangle,

$\text{(4.14c)} \qquad \mathbf{S}(s, \mathbf{r}) =$
$$= \overline{\mathbf{S}} + s \cdot \varphi_{1,s+p} \cdot \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) + \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right) + O(s^2),$$

for the rectangular triangle (corresponding to the choice $d_2 = d_2(s)$, $|s| < \delta$). Since s is considered to be small here, this solution is called as a *small amplitude pattern*.

REMARK 2. Because of Theorem 4.2 (3.1)–(3.2) has no other stationary solution apart from $(\overline{S}_1, \overline{S}_2)^T$ and (4.14a–c) in a neighbourhood of $(d_T, \overline{\mathbf{S}}) \in \in \mathbf{R} \times X$.

REMARK 3. In the linear case (by Theorem 4.1) for the function d_2 holds: $d_2(s) \equiv d_T$, and a corresponding one parameter family of solutions is $\overline{\mathbf{S}} + s \cdot \cdot \mathbf{z}_{s+p}(\mathbf{r})$ $(s \in \mathbf{R})$.

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