# ANNALES Universitatis Scientiarum BUDAPESTINENSIS <br> de Rolando EÖTVÖS NOMINATAE 

## SECTIO MATHEMATICA

TOMUS XLIII.

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## ANNALES

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SECTIO PHILOSOPHICAETSOCIOLOGICA incepit anno MCMLXII

# AN ALTERNATIVE PRINCIPLE FOR CONNECTED ORDERED SPACES 

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## 1. Introduction

In [R1], B. Ricceri considered spaces admitting a continuous bijection onto $[0,1]$ (simply, we will call them $[0,1]$-spaces) and, based on a new alternative principle for multifunctions involving $[0,1]$-spaces, obtained new mini-max theorems in full generality and transparence. Several further consequences of the principle have been investigated in successive works; see [R2, R3, Ci, CB].

In our previous work [P1], we deduced some fixed point theorems for connected [ 0,1$]$-spaces from Ricceri's alternative principle. Even though these theorems were consequences of known theorems for an interval $[a, b]$, in general, they seem to be quite new. More general theorems for connected ordered spaces were recently obtained in [P2] with a different method.

In the present paper, we are mainly concerned with connected topological spaces which admit continuous bijections onto a connected ordered spaces with two end points. In Section 2, we deduce a Ricceri type alternative principle and fixed point theorems. Section 3 deals with the incorrect proof of Maćkowiak [M, Theorem 3.1], which was the main tool in [C]. In Section 4, we show that our alternative principle is a generalization of Szabó's theorem [S]. Section 5 deals with the coincidence theorems of Charatonik [C]. In fact, we give new and correct proofs based on our principle and the affirmative answer to Charatonik's question.

## 2. A Ricceri type alternative principle

A multifunction $T: X \rightarrow 2^{Y}$ is a function from $X$ into the power set of $Y$ with nonempty values, and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

For topological spaces $X$ and $Y$, a multifunction $T: X \rightarrow 2^{Y}$ is said to be closed if its graph $\operatorname{Gr}(T)=\{(x, y): x \in X, y \in T(x)\}$ is closed in $X \times Y$, and compact if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in $Y$.

A multifunction $T: X \rightarrow 2^{Y}$ is said to be upper semicontinuous (u.s.c) if for each closed set $B \subset Y$, the set $T^{-1}(B)=\{x \in X: T(x) \cap B \neq \emptyset\}$ is a closed subset of $X$; lower semicontinuous (1.s.c.) if for each open set $B \subset Y$, the set $T^{-1}(B)$ is open; and continuous if it is u.s.c. and 1.s.c. Note that every u.s.c. multifunction $T$ with closed values is closed.

We need the following [ H , Theorem 3.1]:
Lemma. Let $\Gamma: X \rightarrow 2^{Y}$ be 1.s.c. (or u.s.c.). Suppose that $C \subset X$ is connected and that $\Gamma(x)$ is connected for all $x \in C$. Then the image of $C$ under $\Gamma$ is connected.

A linearly ordered set $(X, \leq)$ is called an ordered space if it has the order topology whose subbase consists of all sets of the form $\{x \in X: x<s\}$ and $\{x \in X: x>s\}$ for $s \in X$. Note that an ordered space $X$ is connected iff it is Dedekind complete (that is, every subset of $X$ having an upper bound has a supremum) and whenever $x<y$ in $X$, then $x<z<y$ for some $z$ in $X$; for details, see Willard [Wi].

We give some examples of connected ordered space $X$ with two end points:
(a) Connected $[0,1]$-spaces; that is, connected spaces admitting a continuous bijection onto the unit interval; see [R1], [P1].
(b) An arc is a homeomorphic image of the unit interval $[0,1]$. A generalized arc is a continuum (not necessarily metrizable) having exactly two non-cut points. It is well-known that a generalized arc is an arc if and only if it is metrizable. It is known [W] that cut points are used to define a natural order on a connected set, and that any generalized arc admits a natural linear order which will be denoted by $\leq$.

A connected ordered space with two end points $a, b$ with $a<b$ will be denoted by $[a, b]$.

Motivated by Ricceri [R1], we obtain the following alternative principle:

THEOREM 1. Let $X$ be a connected topological space, and $Y$ a topological space admitting a continuous bijection onto a connected ordered space $[a, b]$. Let $F, G: X \rightarrow 2^{Y}$ be maps satisfying one of the following two conditions:
(i) $F, G$ are l.s.c. with connected values;
(ii) $F, G$ are u.s.c. with compact connected values.

Under such assumptions, at least one of the following two assertions does hold:
(a) $F(X) \neq Y$ and $G(X) \neq Y$.
(b) There exists some $\tilde{x} \in X$ such that $F(\tilde{x}) \cap G(\tilde{x}) \neq \emptyset$.

Proof. Let $\varphi$ be a continuous bijection from $Y$ onto a connected ordered space $[a, b]$. Suppose that $F(x) \cap G(x)=\emptyset$ for all $x \in X$ and $F(X)=Y$. Let

$$
H(x)=\varphi(F(x)) \times \varphi(G(x)) \quad \text { for } \quad x \in X
$$

We claim that $H(X)$ is connected.
Case (i). The multifunctions $\varphi \circ F$ and $\varphi \circ G$ are l.s.c. with connected values. It is easily checked that $H$ is l.s.c.; see [B].

Case (ii). The multifunctions $\varphi \circ F$ and $\varphi \circ G$ are u.s.c. with compact connected values. Then $H$ is u.s.c. with compact connected values; here, the compact-valuedness of $\varphi \circ F$ and $\varphi \circ G$ are essential in order to assure the u.s.c. of $H$; see [B].

Then, by Lemma, $H(X)$ is connected in any case.
Now we show that $H(X)$ is also disconnected: Let $A, B \subset[a, b] \times[a, b]$ such that

$$
A:=\{(s, t): s<t\} \quad \text { and } \quad B:=\{(s, t): s>t\} .
$$

Then $A$ and $B$ are open and disjoint, and we clearly have

$$
H(X) \subset A \cup B
$$

Choose $x_{a}, x_{b} \in X$ such that $\varphi^{-1}(a) \in F\left(x_{a}\right)$ and $\varphi^{-1}(b) \in F\left(x_{b}\right)$. Pick $y_{a} \in G\left(x_{a}\right)$ and $y_{b} \in G\left(x_{b}\right)$. Then we have $\varphi\left(y_{a}\right)>a$; for, otherwise, we would have $F\left(x_{a}\right) \cap G\left(x_{a}\right) \neq \emptyset$. Likewise, we have $\varphi\left(y_{b}\right)<b$.

Consequently,

$$
\left(a, \varphi\left(y_{a}\right)\right) \in A \cap H(X) \text { and }\left(b, \varphi\left(y_{b}\right)\right) \in B \cap H(X)
$$

Then $H(X)$ becomes the union of two disjoint nonempty open subsets $A \cap$ $\cap H(X)$ and $B \cap H(X)$. This contradicts the connectivity of $H(X)$.

REMARKS. 1. We followed the proof of Ricceri [R1, Theorem 2.1], which is the case of Theorem 1 for a [0,1]-space $Y$.
2. However, our result is already known by Ricceri, since he noted that his result [R1, Theorem 2.1] is still true if [0, 1] is replaced by any topological space $T$ having the following property: there are two open (or closed) subsets $A, B$ of $T \times T$ and two points $s_{0}, t_{0} \in T$ such that $(T \times T) \backslash \Delta \subset A \cup B$, $A \cap B \subset \Delta,\left\{s_{0}\right\} \times\left(T \backslash\left\{s_{0}\right\}\right) \subset A$, and $\left\{t_{0}\right\} \times\left(T \backslash\left\{t_{0}\right\}\right) \subset B$, where $\Delta$ is the diagonal of $T \times T$.

From Theorem 1, we have the following:
THEOREM 2. Let $X$ be a topological space, $Y$ a topological space admitting a continuous bijection onto a connected ordered space $[a, b]$, and $S$ a connected subset of $X \times Y$. Moreover, let $\Phi: X \rightarrow 2^{Y}$ be a multifunction which is either l.s.c. with connected values, or u.s.c. with compact connected values. Then, at least one of the following holds:
$\left(\mathrm{a}_{1}\right) p_{Y}(S) \neq Y$ and $\Phi\left(p_{X}(S)\right) \neq Y$, where $p_{X}$ and $p_{Y}$ are projections from $X \times Y$ to $X$ and $Y$, resp.
$\left.\mathrm{a}_{2}\right)$ There exists some $(\tilde{x}, \tilde{y}) \in S$ such that $\tilde{y} \in \Phi(\tilde{x})$.
Proof. We may assume $S \neq \emptyset$. Define $F, G: S \rightarrow 2^{Y}$ by

$$
F(x, y)=\{y\} \quad \text { and } \quad G(x, y)=\Phi(x) \quad \text { for }(x, y) \in S
$$

Then the conclusion follows from Theorem 1.

REMARK. For a [0, 1]-space $Y$ Theorem 2 reduces to Ricceri [R1, Theorem 2.2].

From Theorems 1 and 2, we deduce the following fixed point theorem on multifunctions:

THEOREM 3. Let $X$ be a connected ordered space with two end points. Then a multifunction $F: X \rightarrow 2^{X}$ has a fixed point if it satisfies one of the following conditions:
(I) F has connected graph.
(II) $F$ is l.s.c. with connected values.
(III) $F$ is u.s.c. with compact connected values.
(IV) $F(x)$ is connected and $F^{-1}(y)$ is open for each $x, y \in X$.
(V) $F$ is a closed compact multifunction with connected values.

Proof. (I)-(V) are all simple consequences of Theorems 1 and 2 as follows:
(I) Theorem 2 with $X=Y, S=\operatorname{Gr}(F)$, and $\Phi=i d_{X}$, the identity map on $X$.
(II) Theorem 1(i) with $X=Y$ and $G=i d_{X}$.
(III) Theorem 1(ii) with $X=Y$ and $G=i d_{X}$.
(IV) Since $F^{-1}(y)$ is open for each $y \in X, F$ is l.s.c. Indeed, for each open set $\Omega \subset X$, we have

$$
F^{-1}(\Omega)=\{x \in X: F(x) \cap \Omega \neq \emptyset\}=\bigcup_{y \in \Omega} F^{-1}(y)
$$

is open. Therefore, (IV) follows from (II).
(V) It is well-known that a closed compact multifunction is u.s.c. with compact values. Therefore, (V) follows from (III).

REMARK. Theorem 3 was given in [P2] with different proof.

## 3. On a coincidence theorem of Maćkowiak

In 1981, Maćkowiak [M] introduduced componentwise continuous (c.c.) multifunctions and used them to obtain some fixed point theorems which generalize most known fixed point theorems for trees, dendroids, and $\lambda$-dendroids. Moreover, he obtained a coincidence theorem [M, Theorem 3.1] for two c.c. multifunctions from a connected Hausdorff space $X$ into a generalized arc $I$.

For Hausdorff compact spaces $X$ and $Y$, a multifunction $F: X \rightarrow 2^{Y}$ is said to be componentwise continuous (c.c.) [M] if $x=\lim \left\{x_{\sigma}\right\}$ implies that (a) $\operatorname{Ls}\left\{C_{\alpha}\right\} \cap F(x) \neq \emptyset$, where $C_{\alpha}$ is a component of $F\left(x_{\sigma}\right)$ for each $\sigma\left[\operatorname{Ls}\left\{C_{\sigma}\right\}\right.$ is the superior limit of the net $\left.\left\{C_{\sigma}\right\}\right]$; and
(b) every component of $F(x)$ intersects $\operatorname{Ls}\left\{F\left(x_{\sigma}\right)\right\}$.

In [M], many examples of c.c. multifunctions were given and, among them are
(1) lower semicontinuous (l.s.c.) multifunctions with connected values, and
(2) upper semicontinuous (u.s.c.) multifunctions with closed connected values.

Recall that a generalized arc is a continuum which has exactly two noncut points.

ThEOREM M. [M, Theorem 3.1] Let c.c. multifunctions $F$ and $G$ map a connected space $X$ into a generalized arc $I$. Assume that one of the following conditions holds:
(i) $F$ is a surjection with connected values.
(ii) $F$ and $G$ are both surjections.

Then there is an $x \in X$ such that $F(x) \cap G(x) \neq \emptyset$.

On the other hand, in an unpublished work of the present author, he tried to apply Theorem M to obtain a common generalization of Theorem M and Ricceri's alternative principle [R1]. However, an excellent referee of that work realized and informed the present author that, unfortunately, the proof of Theorem M is wrong. The referee wrote as follows:

Indeed, using the same notations as in [M], take:

$$
\begin{gathered}
X=I=[0,1], \\
F(x)=\{x\}, \\
G(x)= \begin{cases}\{x\} & \text { if } x \in[0,1) \\
{[0,1]} & \text { if } x=1 .\end{cases}
\end{gathered}
$$

Observe that both the multifunctions $F, G$ are upper semicontinuous, with compact and connected values. So, they are c.c. Moreover, $F$ and $G$ are both surjections. Hence, all the assumptions of Theorem M are satisfied. Now, consider the set $A$ introduced in the proof. Namely,

$$
A=\{x \in[0,1]: G(x) \subset[x, 1]\}
$$

In the proof, it is claimed that $A$ is closed. In the present case, this is not true. Indeed, we clearly have

$$
A=[0,1) .
$$

Knowing that $A$ is closed is absolutely necessary in the approach adopted in [M]. Consequently, Theorem $M$, in the absence of a correct proof, should be considered as a conjecture.

## 4. Generalizations of Szabó's theorem

In 1994, Szabó [S] obtained a coincidence theorem for two continuous multifunctions from a connected space into the set of closed connected subsets of $[0,1]$. Further, he raised a problem how one can generalize his result, in particular for other space than [0, 1]. In 1997, the present author [P1] and Charatonik [C] gave affirmative solutions to the problem, independently. However, in [C], it was noted that [M, Theorem 3.1] is much stronger than Szabó's theorem.

Let $X$ be a connected topological space. According to Szabó [S], let $K[0,1]$ denote the set of closed connected subsets of [0, 1] , and a function $F: X \rightarrow K[0,1]$ is said to be continuous if each $F(x)$ for $x \in X$ is [ $\left.f_{0}(x), f_{1}(x)\right]$ where $f_{0}, f_{1}: X \rightarrow[0,1]$ are continuous.

The following is due to Szabó [S]:

THEOREM S. Let $F, G: X \rightarrow K[0,1]$ be continuous functions and assume that

$$
\bigcup_{x \in X} F(x)=[0,1] .
$$

Then there exists $x_{0} \in X$ such that $F\left(x_{0}\right) \cap G\left(x_{0}\right) \neq \emptyset$.

Moreover, Szabó [S] raised the following:

Problem S. How can we generalize Theorem $S$ for other spaces instead of $[0,1]$ ?

In our previous work [P1], we showed that Theorem $S$ is still true if $[0,1]$ is replaced by any space $T$ in Remark 2 of Theorem 1.

More precisely, Theorem 1 generalizes Theorem $S$ and is an affirmative solution of Problem S.

## 5. On coincidence theorems of Charatonik

Since any generalized arc admits a natural linear order $\leq$, it can be denoted by $[a, b]$.

Given a generalized arc $[a, b]$, Charatonik [C] denoted by $K[a, b]$ the set of closed connected subsets of $[a, b]$. Thus each nondegenerate element of $K[a, b]$ is a generalized arc $[c, d]$ with $a \leq c<d \leq b$. Adopting the definition from Szabó [S] to this more general case, Charatonik [C] defined the following:

DEFINITION C. Let a generalized arc $[a, b]$ be fixed. A multifunction $F: X \rightarrow K[a, b]$ is said to be continuous provided that if $F(x)=\left[f_{0}(x), f_{1}(x)\right]$, then the function $f_{0}: X \rightarrow[a, b]$ and $f_{1}: X \rightarrow[a, b]$ are continuous.

Then Charatonik [C] obtained the following:
Statement C. Let $X$ be a space and $Y=[a, b]$ a generalized arc. Then a multifunction $F: X \rightarrow K[a, b] \subset 2^{Y}$ is continuous (u.s.c. and l.s.c.) if and only if it is continuous in the sense of Definition $C$.

Note that in the above argument, a generalized arc can be replaced by any connected ordered space with two end points.

Analyzing carefully assumptions of Theorem S, Charatonic [C] showed that, in the light of Statment C, the theorem can be reformulated as follows:

Theorem C. Let $X$ and $Y$ be topological spaces and let multifunctions $F, G: X \rightarrow 2^{Y}$ be given. Assume that
(1) $X$ is connected;
(2) $Y$ is a generalized arc (or more generally, a connected ordered space with two end points);
(3) $Y$ is metrizable;
(4) $F$ is l.s.c.;
(5) $F$ is u.s.c.;
(6) F has compact values;
(7) $F$ has connected values;
(8) $F$ is surjective;
(9) $G$ is I.s.c.;
(10) $G$ is u.s.c.;
(11) G has compact values;
(12) G has connected values.

## Then

(13) There exists $x_{0} \in X$ such that $F\left(x_{0}\right) \cap G\left(x_{0}\right) \neq \emptyset$.

Note that we replaced the closedness in (6) and (11) by compactness.
Charatonik [C] formulated the following as a possible generalization of Theorem S (or, equivalently, of Theorem C) for Hausdorff spaces $X$ :

Proposition $\mathrm{C}_{1}$. If $X$ is a Hausdorff space, then in Theorem $C$ assumptions (3), (5), (6), (10), and (11) can be omitted.

In the proof of Proposition $\mathrm{C}_{1}$, the author applied Theorem M. In view of Section 3 of the present paper, the proof can not be complete.

However, without assuming Hausdorffness of $X$, Proposition $\mathrm{C}_{1}$ follows immediately from Theorem 1(i).

Another modification of Theorem $S$ is the following in [C].
Proposition $\mathrm{C}_{2}$. If $X$ is a Hausdorff space, then in Theorem $C$ assumptions (3), (4), and (9) can be omitted.

This also follows from Theorem 1(ii) without assuming Hausdorffness of $X$.

Therefore, we answered affirmatively to the following raised in [C]:
Question C. Can the assumption that the space $X$ is Hausdorff be omitted in Propositions $\mathrm{C}_{1}$ and $\mathrm{C}_{2}$ ?

Note that the Hausdorffness in Propositions $\mathrm{C}_{1}$ and $\mathrm{C}_{2}$ came from [M, Theorem 3.1] and is not necessary because our proofs are based on Theorem 1.

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# GENERALISATIONS OF ROBERTSON-WALKER SPACES 

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Dedicated to the memory of Professor Dr. Halina Pidek-Lopuszañska

## 1. Introduction

A warped product $\bar{M} \times{ }_{F} N$, of a 1-dimensional base manifold ( $\bar{M}, \bar{g}$ ), $\bar{g}_{11}=-1$, with a warping function $F$ and an $(n-1)$-dimensional Riemannian fiber manifold ( $N, \tilde{g}$ ), $n \geq 4$, is said to be a generalized Robertson-Walker spacetime [1] [11]. This is a generalisation to arbitrary dimension of the well-known Robertson-Walker spacetimes $\bar{M} \times{ }_{F} \bar{N}$, which are a particular case of the above in 4 dimensions, when ( $N, \tilde{g}$ ) a 3-dimensional Riemannian space of constant curvature.

Within the framework of curvature theory, curvature properties of pseudosymmetry type of Robertson-Walker spacetimes have been investigated from several points of view. Pseudosymmetric manifolds constitute a generalization of spaces of constant (sectional) curvature, along the line of locally symmetric ( $\nabla R=0$ ) and semisymmetric spaces ( $R \cdot R=0[14]$ ), consecutively. The profound investigation of several properties of semisymmetric manifolds, gave rise to their next generalization: the (properly) pseudosymmetric manifolds (see $(*)_{1}$, or (1)). For more detailed information on the geometric motivation for the introduction of pseudosymmetric manifolds see [7], see also [15].

More generally, one also considers tensors of the form $R \cdot T$ and $Q(A, T)$, with $A$ a symmetric ( 0,2 )-tensor and $T$ a generalized curvature tensor; see

[^0]e.g. $(*)_{2}$ or (2), and $(*)_{3}$ or (3). For precise definitions of the used symbols, we refer to Section 2. Curvature conditions involving these will be called curvature conditions of pseudosymmetry type. For a review of results on different aspects of pseudosymmetric spaces, we refer to the survey article [7]; see also [15]. Let us just mention here the following application. Curvature conditions of pseudosymmetry type often appear in the theory of general relativity. Many well-known spacetime metrics have been shown to satisfy a curvature condition of pseudosymmetry type: e.g. Schwarzschild, RobertsonWalker, Kottler and Reissner-Nordström metrics are all pseudosymmetric in the proper sense; for more on this, see [10] and [5].

In particular, w.r.t. Robertson-Walker spacetimes specifically, we recall the following properties. Lemma 2.3 and Theorem 3.5 of [6] imply that every Robertson-Walker spacetime is a pseudosymmetric manifold with the tensor $R \cdot R-Q(S, R)$ vanishing identically. Moreover, it is well-known that a 4-dimensional Robertson-Walker spacetime is also conformally flat. Therefore, on every Robertson-Walker spacetime $\bar{M} \times{ }_{F} N$, the following curvature conditions are satisfied:

$$
\begin{aligned}
R \cdot R & =L_{R} Q(g, R), \\
R \cdot R-Q(S, R) & =L_{2} Q(g, C) \\
R \cdot C & =L_{3} Q(S, C)
\end{aligned}
$$

We thus observe that in 4 dimensions the Robertson-Walker spacetimes simultaneously realise 3 independent curvature conditions of pseudosymmetry type, namely $(*)_{1},(*)_{2}$, and $(*)_{3}$. Moreover, in [4] it was shown that $(*)_{2}$ is even satisfied at every point of a generalized 4-dimensional RobertsonWalker spacetime $\bar{M} \times{ }_{F} N, \operatorname{dim} N=3$; and recently, 4-dimensional generalized Robertson-Walker spacetimes $\bar{M} \times{ }_{F} N$ realizing at every point the condition $(*)_{3}$ were investigated in [9].

In the present paper, we enlarge this study to warped products with 1-dimensional base and semi-Riemannian fiber, and consider the possibility to single out nontrivial such spaces of any dimension, simultaneously realizing the same set of pseudosymmetric curvature conditions. These curvature conditions might be good candidates to indicate an appropriate subclass if one is to expect for the generalized spaces in any dimension geometrical and physical properties more closely related to the ones the prototypes in 4 dimensions have. Indeed, the generalisation allows to consider a vastly greater class of warped product spaces, which may significantly increase their usefullness w.r.t. applications of various type, in the sense as e.g. mentioned in [11]. However, with applications of physical nature in mind, for example, one may
wish to impose some assumed symmetry of the underlying space. This is very much in the line of the exposition in [5], where curvature conditions of pseudosymmetry type were put forward to do so. Here, in this particular case, the known curvature properties of the 4 -dimensional special case, indicate which one can take.

This leads us to a particular class of warped product spaces, with additional conditions imposed on the warped product and having conformally flat fiber space in particular. Finally, we give an explicit example of a family of nontrivial spaces of this kind which exist in any dimension; the spaces of the example are moreover quasi-Einstein. The paper is organized as follows. In Section 2 we give precise definitions of the symbols used, and give the necessary background material concerning the concepts we use. In Section 3, we derive for later use some formulas for warped products. We also perform some preliminary calculations to be applied in the proof of the theorems; we organize them into a technical lemma, in order not the overload the proof of the main theorems later on. Finally, in Section 4, we prove the main results.

## 2. Curvature conditions of pseudosymmetry type

Let $(M, g), n=\operatorname{dim} M \geq 3$, be a connected semi-Riemannian manifold of class $C^{\infty}$ and let $\nabla$ be its Levi-Civita connection. We define on $M$ the endomorphisms $X \wedge_{A} Y, \mathscr{R}(X, Y)$ and $\mathscr{C}(X, Y)$ by

$$
\begin{gathered}
\left(X \wedge_{A} Y\right) Z=A(Y, Z) X-A(X, Z) Y, \\
\mathcal{R}(X, Y) Z=\left[\nabla_{X}, \nabla_{Y}\right] Z-\nabla_{[X, Y]} Z, \\
\mathscr{C}(X, Y)=\mathscr{R}(X, Y)-\frac{1}{n-2}\left(X \wedge_{g} \varphi Y+\mathscr{Y} X \wedge_{g} Y-\frac{\kappa}{n-1} X \wedge_{g} Y\right),
\end{gathered}
$$

where the Ricci operator $\mathscr{J}$ is defined by $S(X, Y)=g(X, \varphi \mathrm{Y}), S$ is the Ricci tensor, $\kappa$ the scalar curvature, $A$ a symmetric ( 0,2 )-tensor and $X, Y$, $Z \in \Xi(M), \Xi(M)$ being the Lie algebra of vector fields of $M$. Next, we define the tensor $G$, the Riemann-Christoffel curvature tensor $R$ and the Weyl conformal curvature tensor $C$ of $(M, g)$ by

$$
\begin{aligned}
& G\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=g\left(\left(X_{1} \wedge_{g} X_{2}\right) X_{3}, X_{4}\right), \\
& R\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=g\left(\mathcal{R}\left(X_{1}, X_{2}\right) X_{3}, X_{4}\right), \\
& C\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=g\left(\mathscr{C}\left(X_{1}, X_{2}\right) X_{3}, X_{4}\right) .
\end{aligned}
$$

For a $(0, k)$-tensor $T, k \geq 1$, and a symmetric $(0,2)$-tensor $A$, we define the ( $0, k+2$ )-tensors $R \cdot T$ and $Q(A, T)$ by

$$
\begin{aligned}
& (R \cdot T)\left(X_{1}, \ldots, X_{k} ; X, Y\right)=(\mathscr{R}(X, Y) \cdot T)\left(X_{1}, \ldots, X_{k}\right)= \\
& \quad=-T\left(\mathcal{R}(X, Y) X_{1}, X_{2}, \ldots, X_{k}\right)-\ldots-T\left(X_{1}, \ldots, X_{k-1}, \mathscr{R}(X, Y) X_{k}\right) \\
& \quad Q(A, T)\left(X_{1}, \ldots, X_{k} ; X, Y\right)=\left(\left(X \wedge_{A} Y\right) \cdot T\right)\left(X_{1}, \ldots, X_{k}\right)= \\
& \quad=-T\left(\left(X \wedge_{A} Y\right) X_{1}, X_{2}, \ldots, X_{k}\right)-\ldots-T\left(X_{1}, \ldots, X_{k-1},\left(X \wedge_{A} Y\right) X_{k}\right)
\end{aligned}
$$

For (0,2)-tensors $A$ and $B$ we define its Kulkarni-Nomizu product $A \wedge B$ by

$$
\begin{aligned}
(A \wedge B)\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=A( & \left.X_{1}, X_{4}\right) B\left(X_{2}, X_{3}\right)+A\left(X_{2}, X_{3}\right) B\left(X_{1}, X_{4}\right)- \\
& -A\left(X_{1}, X_{3}\right) B\left(X_{2}, X_{4}\right)-A\left(X_{2}, X_{4}\right) B\left(X_{1}, X_{3}\right)
\end{aligned}
$$

Putting in the last formulas $T=R, T=S, T=C$ or $T=G$ and $A=g$ or $A=S$, we obtain the tensors $R \cdot R, R \cdot S, R \cdot C, Q(g, R), Q(g, S), Q(g, C)$, $Q(S, R), Q(S, C)$ and $Q(S, G)$, respectively. The tensor $C \cdot C$ we define in the same way as the tensor $R \cdot R$.

Curvature conditions involving tensors of the form $R \cdot T$ and $Q(A, T)$, are called curvature conditions of pseudosymmetry type; examples are e.g. $(*)_{1},(*)_{2}$, and $(*)_{3}$; one of the advantages w.r.t. their applicability is that, whenever a geometrical property can be captured by a curvature condition of this type, their tensorial character permits a particular elegant transfer of the concept to other dimensions and signatures.

A semi-Riemannian manifold $(M, g)$ is said to be (properly) pseudosymmetric [8] if at every point of $M$ the following condition is satisfied:
$(*)_{1} \quad$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.
The condition $(*)_{1}$ is equivalent to

$$
\begin{equation*}
R \cdot R=L_{R} Q(g, R) \tag{1}
\end{equation*}
$$

on the set $U_{R}=\left\{x \in M \left\lvert\, R-\frac{\kappa}{(n-1) n} G \neq 0\right.\right.$ at $\left.x\right\}$, where $L_{R}$ is a function on $U_{R}$.

The condition
$(*)_{2} \quad$ the tensors $R \cdot R-Q(S, R)$ and $Q(g, C)$ are linearly dependent.
is equivalent to

$$
\begin{equation*}
R \cdot R-Q(S, R)=L_{2} Q(g, C) \tag{2}
\end{equation*}
$$

on the set $U_{C}=\{x \in M \mid C \neq 0$ at $x\}$, where $L_{2}$ is a function on $U_{C}$. General results on warped products realizing $(*)_{2}$ were obtained in [4].

The condition
$(*)_{3} \quad$ the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent is equivalent to

$$
\begin{equation*}
R \cdot C=L_{3} Q(S, C) \tag{3}
\end{equation*}
$$

on the set $\mathcal{U}=\{x \in M \mid Q(S, C) \neq 0$ at $x\}$, where $L_{3}$ is a function on $\mathcal{U}$.
Finally, we will also need the following generalization of the Einstein metric condition $S=\frac{\kappa}{n} g$. Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds determined by a curvature condition imposed on their Ricci tensor [2] (Table, pp. 432-433). For example, Einstein manifolds do also form a natural subclass of the class of quasi-Einstein manifolds. A semi-Riemannian manifold ( $M, g$ ), $n \geq 3$, is called a quasi-Einstein manifold if at every point of $M$ its Ricci tensor $S$ is decomposed on two terms: a metrical term and a term of rank at most one, i.e. if the following relation is satisfied at every point $x \in M$ :

$$
\begin{equation*}
S=\alpha g+\beta w \otimes w, \quad w \in T_{x}^{*}(M), \quad \alpha, \beta \in \mathbb{R} \tag{4}
\end{equation*}
$$

For example, the 4-dimensional Robertson-Walker spacetimes are quasiEinstein manifolds.

## 3. Warped products

Let now $(\bar{M}, \bar{g})$ and $(N, \tilde{g}), \operatorname{dim} \bar{M}=p, \operatorname{dim} N=n-p, 1 \leq p<n$, be semi-Riemannian manifolds covered by systems of charts $\left\{U_{1} ; x^{a}\right\}$ and $\left\{U_{2} ; y^{\alpha}\right\}$, respectively. Let $F$ be a positive smooth function on $\bar{M}$. The warped product $\bar{M} \times{ }_{F} N$ of $(\bar{M}, \bar{g})$ and ( $N, \tilde{g}$ ) [3] is the product manifold $\bar{M} \times N$ with the metric $g=\bar{g} \times{ }_{F} \tilde{g}$ defined by

$$
\bar{g} \times{ }_{F} \tilde{g}=\pi_{1}^{*} \bar{g}+\left(F \circ \pi_{1}\right) \pi_{2}^{*} \tilde{g}
$$

where $\pi_{1}: \bar{M} \times N \rightarrow M$ and $\pi_{2}: \bar{M} \times N \rightarrow N$ are the natural projections on $\bar{M}$ and $N$, respectively. Let $\left\{U_{1} \times U_{2} ; x^{1}, \ldots, x^{p}, x^{p+1}=y^{1}, \ldots, x^{n}=y^{n-p}\right\}$ be a product chart for $\bar{M} \times N$. The local components of the metric $g=\bar{g} \times{ }_{F} \tilde{g}$ with respect to this chart are the following $g_{r s}=\bar{g}_{a b}$ if $r=a$ and $s=b$, $g_{r s}=F \tilde{g}_{\alpha \beta}$ if $r=\alpha$ and $s=\beta$, and $g_{r s}=0$ otherwise, where $a, b, c$, $\ldots \in\{1, \ldots, p\}, \alpha, \beta, \gamma, \ldots \in\{p+1, \ldots, n\}$ and $r, s, t, \ldots \in\{1,2, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\bar{g}$ (resp., $\tilde{g}$ ). The local components $R_{r s t u}=g_{r w} R_{s t u}^{w}=g_{r w}\left(\partial_{u} \Gamma_{s t}^{w}-\partial_{t} \Gamma_{s u}^{w}+\Gamma_{s t}^{v}+\Gamma_{v u}^{w}-\right.$ $\left.-\Gamma_{s u}^{v} \Gamma_{v t}^{w}\right), \partial_{u}=\frac{\partial}{\partial x^{u}}$, of the Riemann-Christoffel curvature tensor $R$ and the
local components $S_{t s}$ of the Ricci tensor $S$ of the warped product $\bar{M} \times{ }_{F} N$ which may not vanish identically are the following [12]:
(5) $R_{a b c d}=\bar{R}_{a b c d}, R_{\alpha a b \beta}=-\frac{1}{2} T Y_{a b} \tilde{g}_{\alpha \beta}, R_{\alpha \beta \gamma \delta}=F \tilde{R}_{\alpha \beta \gamma \delta}-\frac{1}{4} \Delta_{1} F \tilde{G}_{\alpha \beta \gamma \delta}$,
(6) $S_{a b}=\bar{S}_{a b}-\frac{n-p}{2} \frac{1}{F} T_{a b}, S_{\alpha \beta}=\tilde{S}_{\alpha \beta}-\frac{1}{2}\left(\operatorname{tr}(T)+\frac{n-p-1}{2 F} \Delta_{1} F\right) \tilde{g}_{\alpha \beta}$,

$$
\begin{gather*}
T_{a b}=\bar{\nabla}_{b} F_{a}-\frac{1}{2 F} F_{a} F_{b}, \quad \operatorname{tr}(T)=\bar{g}^{a b} T_{a b} \\
\Delta_{1} F=\Delta_{1 \bar{g}} F=\bar{g}^{a b} F_{a} F_{b}, \quad F_{a}=\frac{\partial F}{\partial x^{a}} \tag{7}
\end{gather*}
$$

From now we assume that $\operatorname{dim} \bar{M} \times{ }_{F} N=n \geq 4$ and $\operatorname{dim} \bar{M}=1$. The scalar curvature $\kappa$ of $\bar{M} \times{ }_{F} N$ is then given by the following relation

$$
\begin{equation*}
\kappa=\frac{1}{F} \tilde{\kappa}-\frac{n-1}{F}\left(\operatorname{tr}(T)+\frac{n-2}{4} \frac{\Delta_{1} F}{F}\right) . \tag{8}
\end{equation*}
$$

The local components of the Weyl conformal curvature tensor of the warped product manifold $\bar{M} \times{ }_{F} N$ are given by

$$
\begin{align*}
C_{r s t u}= & R_{r s t u}-\frac{1}{n-2}\left(g_{r u} S_{s t}-g_{r t} S_{s u}+g_{s t} S_{r u}-g_{s u} S_{r t}\right)+ \\
& +\frac{\kappa}{(n-2)(n-1)} G_{r s t u} \tag{9}
\end{align*}
$$

where $G_{r s t u}=g_{r u} g_{s t}-g_{r t} g_{s u}$, and $r, s, t, u \in\{1,2, \ldots, n\}$.
Applying now (5), (6) and (8) into (9) we get

$$
\begin{align*}
C_{\alpha 11 \beta} & =-\frac{1}{n-2} \bar{g}_{11}\left(\tilde{S}_{\alpha \beta}-\frac{\tilde{\kappa}}{n-1} \tilde{g}_{\alpha \beta}\right)  \tag{10}\\
C_{\alpha \beta \gamma \delta} & =F \tilde{R}_{\alpha \beta \gamma \delta}-\frac{F}{n-2} \tilde{U}_{\alpha \beta \gamma \delta}+\frac{\tilde{\kappa} F}{(n-1)(n-2)} \tilde{G}_{\alpha \beta \gamma \delta} \tag{11}
\end{align*}
$$

Using (5), (6), (8), (10) and (11) we can verify that the local components of the tensors $R \cdot C$ and $Q(S, C)$ of $\bar{M} \times{ }_{F} N$, which do not vanish identically are the following:

$$
\begin{equation*}
(R \cdot C)_{\alpha \beta \gamma \delta \lambda \mu}= \tag{12}
\end{equation*}
$$

$$
=F(\tilde{R} \cdot \tilde{R})_{\alpha \beta \gamma \delta \lambda \mu}-\frac{F}{n-2}(\tilde{R} \cdot(\tilde{g} \wedge \tilde{S}))_{\alpha \beta \gamma \delta \lambda \mu}-\frac{1}{4} \frac{\Delta_{1} F}{F} Q(\tilde{g}, C)_{\alpha \beta \gamma \delta \lambda \mu}
$$

$$
Q(S, C)_{\alpha \beta \gamma \delta \lambda \mu}=F Q(\tilde{S}, \tilde{R})_{\alpha \beta \gamma \delta \lambda \mu}-\frac{F}{n-2} Q(\tilde{S}, \tilde{g} \wedge \tilde{S})_{\alpha \beta \gamma \delta \lambda \mu}+
$$

$$
\begin{equation*}
+\frac{1}{2(n-2)}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right) F Q(\tilde{g}, \tilde{g} \wedge \tilde{S})_{\alpha \beta \gamma \delta \lambda \mu} \tag{13}
\end{equation*}
$$

(14) $(R \cdot C)_{\alpha 11 \beta \gamma \delta}=-\frac{1}{n-2} \frac{1}{F} \bar{g}_{11}\left(F(\tilde{R} \cdot \tilde{S})_{\alpha \beta \gamma \delta}-\frac{1}{4} \Delta_{1} F Q(\tilde{g}, \tilde{S})_{\alpha \beta \gamma \delta}\right)$,

$$
Q(S, C)_{\alpha 11 \beta \gamma \delta}=
$$

$(15)=-\frac{1}{n-2} \bar{g}_{11}\left(\frac{1}{n-1} \tilde{\kappa}-\frac{1}{2}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right)\right) Q(\tilde{g}, \tilde{S})_{\alpha \beta \gamma \delta}$, $(R \cdot C)_{1 \alpha \beta \gamma 1 \delta}=-\frac{1}{2} \frac{1}{F} \operatorname{tr}(T) \bar{g}_{11} C_{\delta \alpha \beta \gamma}+\frac{1}{n(n-1)} \tilde{\kappa} \operatorname{tr}(T) \bar{g}_{11} \tilde{G}_{\delta \alpha \beta \gamma}-$

$$
\begin{gather*}
-\frac{1}{2(n-2)} \operatorname{tr}(T) \bar{g}_{11}\left(\tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta}-\tilde{g}_{\beta \delta} \tilde{S}_{\alpha \gamma}\right)  \tag{16}\\
Q(S, C)_{1 \alpha \beta \gamma 1 \delta}=-\frac{n-1}{2} \frac{1}{F} \operatorname{tr}(T) \bar{g}_{11} C_{\delta \alpha \beta \gamma}+ \\
+\frac{1}{n-2} \bar{g}_{11}\left(\tilde{S}_{\delta \gamma} \tilde{S}_{\alpha \beta}-\tilde{S}_{\delta \beta} \tilde{S}_{\alpha \gamma}-\frac{1}{n-1} \tilde{\kappa}\left(\tilde{g}_{\alpha \beta} \tilde{S}_{\delta \gamma}-\tilde{g}_{\alpha \gamma} \tilde{S}_{\delta \beta}\right)\right)+ \\
+\frac{1}{n(n-1)} \tilde{\kappa}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right) \bar{g}_{11} \tilde{G}_{\delta \alpha \beta \gamma}- \\
-\frac{1}{2(n-2)} \bar{g}_{11}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right)\left(\tilde{g}_{\delta \gamma} \tilde{S}_{\alpha \beta}-\tilde{g}_{\delta \beta} \tilde{S}_{\alpha \gamma}\right. \tag{17}
\end{gather*}
$$

Lemma 3.1. Let $(N, \tilde{g}), \operatorname{dim} N=n-1 \geq 3$, be a conformally flat semiRiemannian manifold such that its scalar curvature $\tilde{\kappa}$ vanishes and $\operatorname{rank} \tilde{S}=1$. Then the warped product $\bar{M} \times{ }_{F} N$, of a 1-dimensional manifold $(\bar{M}, \bar{g})$ and the manifold $(N, \tilde{g})$ satisfies the condition $R \cdot C=L Q(S, C)$ if at only if for every point $x \in \mathcal{U}$ there exist a chart $U_{1} \times U_{2} \subset \mathcal{U}$ such that the following relations are satisfied on $U_{1} \times U_{2}$ :

$$
\begin{align*}
& -\frac{1}{4} \frac{\Delta_{1} F}{F} Q(\tilde{g}, C)_{\alpha \beta \gamma \delta \lambda \mu}=L\left(-\frac{1}{2}\left(\operatorname{tr}(T)+\frac{n-2}{n} \frac{\Delta_{1} F}{F}\right) F Q(\tilde{g}, \tilde{R})_{\alpha \beta \gamma \delta \lambda \mu}-\right. \\
& \text { (18) } \left.\quad-\frac{L}{2(n-2)}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right) F Q(\tilde{g}, \tilde{g} \wedge \tilde{S})_{\alpha \beta \gamma \delta \lambda \mu}\right), \tag{18}
\end{align*}
$$

$$
\begin{gather*}
\frac{1}{4(n-2)} \frac{\Delta_{1} F}{F}=L \frac{1}{2(n-2)}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right)  \tag{19}\\
-\frac{1}{2} \frac{1}{F} \operatorname{tr}(T) C_{\delta \alpha \beta \gamma}-\frac{1}{2(n-2)} \operatorname{tr}(T)\left(\tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta}-\tilde{g}_{\beta \delta} \tilde{S}_{\alpha \gamma}\right)= \\
=L\left(-\frac{n-1}{2} \frac{\operatorname{tr}(T)}{F} C_{\delta \alpha \beta \gamma}-\right. \\
\left.\quad-\frac{1}{2(n-2)}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right)\left(\tilde{g}_{\delta \gamma} \tilde{S}_{\alpha \beta}-\tilde{g}_{\delta \beta} \tilde{S}_{\alpha \gamma}\right)\right) . \tag{20}
\end{gather*}
$$

Proof. We note that $\mathcal{U} \subset U_{S}$. In fact, if we had at a point $x \in \mathcal{U}$ $\tilde{S}=\frac{\tilde{\kappa}}{n-1} \tilde{g}$ then $\tilde{C}=0$ implies $\tilde{R}=\frac{\tilde{\kappa}}{(n-2)(n-1)} \tilde{G}$. Applying this into (10) and (11) we obtain $C=0$, a contradiction. Now, from $(R \cdot C)_{\alpha 11 \beta \gamma \delta}=$ $=L Q(S, C)_{\alpha 11 \beta \gamma \delta}$, by an application of the above remark, (14) and (15), we get (19). Further, in view of Lemma 2.1(iv) of [9], it follows that $\tilde{R} \cdot \tilde{R}=0$. Furthermore, using the following identities

$$
\begin{gathered}
\tilde{C}=\tilde{R}-\frac{1}{n-2} \tilde{g} \wedge \tilde{S}+\frac{\tilde{\kappa}}{(n-1)(n-2)} \tilde{G}=\tilde{R}-\frac{1}{n-2} \tilde{g} \wedge \tilde{S} \\
Q(\tilde{S}, \tilde{g} \wedge \tilde{S})=-\frac{1}{2} Q(\tilde{g}, \tilde{S} \wedge \tilde{S})
\end{gathered}
$$

and the assumption that $\operatorname{rank}(\tilde{S})=1$, we obtain

$$
\begin{aligned}
& 0=Q(\tilde{S}, \tilde{C})=Q(\tilde{S}, \tilde{R})-\frac{1}{n-2} Q(\tilde{S}, \tilde{g} \wedge \tilde{S})= \\
& =Q(\tilde{S}, \tilde{R})+\frac{1}{2(n-2)} Q(\tilde{g}, \tilde{S} \wedge \tilde{S})=Q(\tilde{S}, \tilde{R})
\end{aligned}
$$

Finally (12), (13), (16) and (17) lead to (18) and (20), completing the proof.

## 4. Main results

We show that on a particular subclass of the warped product manifolds under consideration the same three curvature conditions of pseudosymmetry type $(*)_{1},(*)_{2},(*)_{3}$ are satisfied as by the 4 -dimensional Robertson-Walker spacetimes. Proposition 4.1 first indicates this subset among the warped products which were subject of Lemma 3.1; Proposition 4.2 then proves that the conditions are indeed realized.

PROPOSITION 4.1. Let $(N, \tilde{g}), \operatorname{dim} N=n-1 \geq 3$, be a conformally flat semi-Riemannian manifold such that its scalar curvature $\tilde{\kappa}$ vanishes and $\operatorname{rank} \tilde{S}=1$. Then the warped product $\bar{M} \times{ }_{F} N$, of a 1-dimensional manifold $(\bar{M}, \bar{g})$ and the manifold $(N, \tilde{g})$ satisfies the condition $R \cdot C=L Q(S, C)$ if at only if for every point $x \in \mathcal{U}$ there exists a chart $U_{1} \times U_{2} \subset \mathcal{U}$ such that the following relations are satisfied on $U_{1} \times U_{2}$ :

$$
\begin{equation*}
F=F\left(x^{1}\right)=a \exp \left(b x^{1}\right), \quad a=\text { const. }>0, \quad b=\text { const } \neq 0 \tag{21}
\end{equation*}
$$

Proof. Let $U_{1} \times U_{2} \subset \mathcal{U}$ be a chart around a point $x \in \mathcal{U}$. We assume that (18), (19) and (20) are satisfied on $U_{1} \times U_{2}$. From (20), by symmetrization with $\alpha$ and $\beta$ we get

$$
\begin{gathered}
-\frac{1}{2(n-2)} \operatorname{tr}(T)\left(\tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta}-\tilde{g}_{\beta \delta} \tilde{S}_{\alpha \gamma}+\tilde{g}_{\alpha \gamma} \tilde{S}_{\beta \delta}-\tilde{g}_{\alpha \beta} \tilde{S}_{\gamma \delta}\right)= \\
=-\frac{L}{2(n-2)}\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right)\left(\tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta}-\tilde{g}_{\beta \delta} \tilde{S}_{\alpha \gamma}+\tilde{g}_{\alpha \gamma} \tilde{S}_{\beta \delta}-\tilde{g}_{\alpha \beta} \tilde{S}_{\gamma \delta}\right)
\end{gathered}
$$

whence immediately follows that

$$
\begin{equation*}
\operatorname{tr}(T)=L\left(\operatorname{tr}(T)+\frac{n-2}{2} \frac{\Delta_{1} F}{F}\right) \tag{22}
\end{equation*}
$$

holds on $U_{1} \times U_{2}$. Applying this into (20) we get $\left.\left(L-\frac{1}{n-1}\right) \operatorname{tr}(T) C\right)_{\delta \alpha \beta \gamma}=$ $=0$, which reduces to $\left(L-\frac{1}{n-1}\right) \operatorname{tr}(T)=0$. If $L \neq \frac{1}{n-1}$ then on some neighbourhood $V \subset U_{1} \times U_{2}$ of the point $x$ we have $\operatorname{tr}(T)=0$, and, by (19) and (22) we obtain $\frac{\Delta_{1} F}{F}=(n-2) L \frac{\Delta_{1} F}{F}$ and $L \frac{\Delta_{1} F}{F}=0$. From the last two relations we have $\frac{\Delta_{1} F}{F}=0$, which yields $F^{\prime}=0$ on $V$ and, in a consequence, $Q(S, C)=0$ holds on $V$, a contradiction. Thus we have: $L=\frac{1}{n-1}$ and

$$
\begin{equation*}
\operatorname{tr}(T)=\frac{1}{2} \frac{\Delta_{1} F}{F} \tag{23}
\end{equation*}
$$

which is equivalent to $L=\frac{1}{n-1}$ and $F F^{\prime \prime}=\left(F^{\prime}\right)^{2}$, where $F^{\prime}=\frac{d F}{d x^{1}}$ and $F^{\prime \prime}=\frac{d F^{\prime}}{d x^{1}}$. From this we get easily (21). Conversly, we can check that (21) implies (18), (19) and (20), with $L=\frac{1}{n-1}$. Our proposition is thus proved.

Proposition 4.2. Let $\bar{M} \times{ }_{F} N$ be the warped product of a 1-dimensional manifold $(\bar{M}, \bar{g})$, with $\bar{g}_{11}=\epsilon= \pm 1$, the warping function $F, F\left(x^{1}\right)=$
$=a \exp \left(b x^{1}\right), a=$ const. $>0, b=$ const. $\neq 0$ and a conformally flat semiRiemannian manifold $(N, \tilde{g}), \operatorname{dim} N=n-1 \geq 3$, such that its scalar curvature $\tilde{\kappa}$ vanishes and $\operatorname{rank} \tilde{S}=1$. Then the following relations are satisfied on $\bar{M} \times{ }_{F} N:$

$$
\begin{align*}
R \cdot R & =\frac{\kappa}{(n-1) n} Q(g, R),  \tag{24}\\
R \cdot R-Q(S, R) & =-\frac{(n-2) \kappa}{(n-1) n} Q(g, C),  \tag{25}\\
R \cdot C & =\frac{1}{n-1} Q(S, C) . \tag{26}
\end{align*}
$$

Proof. First, we remark that in view of Lemma 2.1 (iv) of [8] the manifold $(N, \tilde{g})$ is semisymmetric, i.e. $\tilde{R} \cdot \tilde{R}=0$ holds on $(N, \tilde{g})$. Thus, in view of Corollary 4.2 of [8], the manifold $\bar{M} \times{ }_{F} N$ is pseudosymmetric. Moreover, (3.17) of [8] implies

$$
L_{R}=-\frac{\epsilon}{4 F^{2}}\left(2 F^{\prime \prime}-\left(F^{\prime}\right)^{2}\right)
$$

whence

$$
\begin{equation*}
L_{R}=-\frac{1}{4} \epsilon a^{2} \tag{27}
\end{equation*}
$$

Further, applying in (8) the relation (23) and $\tilde{\kappa}=0$, we get

$$
\begin{equation*}
\kappa=-\frac{n(n-1)}{4} \frac{\Delta_{1} F}{F^{2}}=-\frac{n(n-1)}{4} \epsilon a^{2} . \tag{28}
\end{equation*}
$$

From (27) and (28) follows now immediately the proof of (24).
Further, the proof of Lemma 3.1 shows that $Q(\tilde{S}, \tilde{R})=0$ is satisfied on $(N, \tilde{g})$. Now Lemma 4.1 of [4] implies that (2) is satisfied, with

$$
\begin{equation*}
L_{2}=\frac{n-2}{2} \frac{\operatorname{tr}(T)}{F} \tag{29}
\end{equation*}
$$

Applying to (29) the relation (23), we obtain $L_{2}=\frac{n-2}{4} \frac{\Delta_{1} F}{F^{2}}$, whence $L_{2}=$ $\frac{n-2}{4} \epsilon a^{2}$, which, by making use of (28), turns into $L_{2}=-\frac{(n-2) \kappa}{(n-1) n}$, completing the proof of (25).

Finally, we see that the conditions of Proposition 4.1 are satisfied by our assumptions; whence (3) is realised. Moreover, from the proof of Proposition 4.1 follows that necessarily $L_{3}=\frac{1}{n-1}$, which yields (26).

This finishes the proof of Proposition 4.2.
We now give an explicit example of a family of such spaces satisfying the conditions of Proposition 4.2 and thus realizing $(*)_{1},(*)_{2}$, and $(*)_{3}$ in an nontrivial way. The examples exist in any dimension, and are moreover also quasi-Einstein, making the analogy with the 4-dimensional Robertson-Walker spacetimes still closer.

EXAMPLE 4.1. (i) First, we have to construct an appropriate conformally flat manifold $(\tilde{M}, \tilde{g}), \operatorname{dim} \tilde{M}=n-1 \geq 3$, such that $\tilde{\kappa}=0$ and $\operatorname{rank} \tilde{S}=1$. Let $M_{2}=\left\{\left(x^{2}, x^{3}\right): x^{2}, x^{3} \in \mathbb{R}\right\}$ be a connected, non-empty, open subset of $\mathbb{R}^{2}$, equipped with the metric tensor $g_{2}$ defined by $g_{2,22}=g_{2,33}=0$, $g_{2,23}=g_{2,32}=1$, and let $H=H\left(x^{2}\right)$ be a smooth function on $M_{2}$. Moreover, let $\left(M_{3}, g_{3}\right)$ be a semi-Euclidean space $\mathbb{E}_{s}^{n-3}$ and let $g_{3 y z}=e_{z} \delta_{y z}, e_{z}= \pm 1$, $y, z \in\{4,5, \ldots, n\}$. In [13] (see p. 177) it was shown that the rank of the Ricci tensor $\tilde{S}$ of the warped product $M_{2} \times{ }_{H} M_{3}$ is equal to one and that its scalar curvature $\tilde{\kappa}$ vanishes. Moreover, $\tilde{S}_{22}=-\frac{1}{H}\left(\frac{d H_{2}}{d x^{2}}-\frac{1}{2 H} H_{2} H_{3}\right)$, where $H_{2}=\frac{d H}{d x^{2}}$ and other local components of $\tilde{S}$ are equal to zero [13] (p. 177).
(ii) Next, we consider the warped product $\bar{M} \times{ }_{F} N$, where $\bar{M}$ is an open, non-empty interval of $\mathbb{R}, \bar{g}_{11}$ is the metric tensor of $\bar{M}$, the warping function $F, F\left(x^{1}\right)=a \exp \left(b x^{1}\right), a=$ const. $>0, b=$ const. $\neq 0$, and $(\tilde{M}, \tilde{g})$ be the manifold defined in (i). Using (6), (8) and (23) we can verify that the tensor $S-\frac{\kappa}{n} g$ of the manifold $\bar{M} \times{ }_{F} N$ is of rank one, i.e. this warped product is also a quasi-Einstein manifold.

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# ON UPPER AND LOWER WEAKLY $\beta$-CONTINUOUS MULTIFUNCTIONS 

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## 1. Introduction

Abd El-Monsef et al. [1] defined $\beta$-continuity in topological spaces as a generalization of quasi-continuity [14] and precontinuity [15]. Recently, BORSIK and DOBOS [6] have introduced the notion of almost quasi-continuity and obtained a decomposition of quasi-continuity. The authors [25] of the present paper obtained several characterizations of $\beta$-continuity and showed that almost quasi-continuity is equivalent to $\beta$-continuity. The equivalence of almost quasi-continuity and a $\beta$-continuity was also shown by BORSIK [5] and EwERT [10], independently. In [28, 20], the present authors have introduced the notion of weakly a $\beta$-continuous functions and upper (lower) weakly $\beta$-continuous multifunctions. The purpose of the present paper is to obtain many characterizations and several properties concerning upper (lower) weakly $\beta$-continuous multifunctions.

## 2. Preliminaries

Let $X$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\operatorname{Int}(A)$, respectively. A subset $A$ is said to be regular open (resp. regular closed) if $A=\operatorname{Int}(\mathrm{Cl}(A))$ (resp. $A=\mathrm{Cl}(\operatorname{Int}(A))$ ). A subset $A$ is said to be an $\alpha$-open [16] (resp. semi-open [13], preopen [15], $\beta$-open [1] or semi-preopen [4]) if $A \subset \operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\operatorname{Int}(A)), A \subset \operatorname{Int}(\mathrm{Cl}(A)), A \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(A))))$. The family of all $\alpha$-open (resp, semi-open, $\beta$-open) sets of $X$ containing a point $x \in X$ is denoted by $\alpha(X, x)$ (resp. $\operatorname{SO}(X, x), \beta(X, x)$ ). The family of all $\alpha$-open (resp. semi-open, preopen, $\beta$-open) sets of $X$ is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X), \beta(X)$ ). The complement of an $\alpha$-open (resp. semi-open,
preopen, $\beta$-open) set is said to be an $\alpha$-closed (resp, semi-closed, preclosed, $\beta$-closed or semi-preclosed). The intersection of all $\alpha$-closed (resp, semiclosed, preclosed, $\beta$-closed or semi-preclosed) sets of $X$ containing $A$ is called the a $\alpha$-closure [16] (resp. semi-closure [8], preclosure [9], $\beta$-closure [2] or semi preclosure [4]) of $A$ and is denoted by $\alpha \mathrm{Cl}(A)(r \operatorname{resp}, \operatorname{sCl}(A)$, $\mathrm{pCl}(A),{ }_{\beta} \mathrm{Cl}(A)$ or $\left.\operatorname{spCl}(A)\right)$. The union of all $\beta$-open sets of $X$ contained in $A$ is called the $\beta$-interior of $A$ and is denoted by $\beta \operatorname{Int}(A)$.

The following basic properties of the $\beta$-closure are useful in the sequel:
Lemma 2.1. (Abd El-Monsef et al. [2], Andriuević [4]). Let A be a subset of a space $X$. Then the following properties hold:
(1) ${ }_{\beta} \mathrm{Cl}(X-A)=X-{ }_{\beta} \operatorname{Int}(A)$,
(2) $x \in{ }_{\beta} \mathrm{Cl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $U \in \beta(X, x)$,
(3) $\beta \operatorname{Int}(A)=A \cap \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(A)))$,
(4) ${ }_{\beta} \mathrm{Cl}(A)=A \cup \operatorname{Int}(\mathrm{Cl}(\operatorname{Int}(A)))$.

The $\theta$-closure [33] of $A$, denoted by $\mathrm{Cl}_{\theta}(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood $U$ of $x$. If $A=\mathrm{Cl}_{\theta}(A)$, then $A$ is said to be $\theta$-closed. The complement of a $\theta$-closed set is said to be $\theta$-open. It is shown in [33] that $\mathrm{Cl}_{\theta}(A)$ is closed in $X$ for each subset $A$ of $X$ and $\mathrm{Cl}(U)=\mathrm{Cl}_{\theta}(U)$ for each open set $U$ of $X$.

Throughout this paper, spaces $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$ ) always mean topological spaces and $F: X \rightarrow Y$ (resp. $f: X \rightarrow Y$ ) presents a multivalued (resp. single valued) function. For a multifunction $F: X \rightarrow Y$, we shall denote the upper and lower inverse of a set $G$ of $Y$ by $F^{+}(G)$ and $F^{-}(G)$, respectively, that is
$F^{+}(G)=\{x \in X: F(x) \subset G\} \quad$ and $\quad F^{-}(G)=\{x \in X: F(x) \cap G \neq \emptyset\}$.

## 3. Characterizations

DEFInITION 3.1. A multifunction $F: X \rightarrow Y$ is said to be
(a) upper weakly $\beta$-continuous [20] (resp, almost $\beta$-continuous [20], ( $\beta$ continuous [27]) at a point $x \in X$ if for each open set $V$ containing $F(x)$, there exists $U \in \beta(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$ (resp. $F(U) \subset \operatorname{Int}(\mathrm{Cl}(V))$, $F(U) \subset V)$,
(b) lower weakly $\beta$-continuous [20] (resp. almost $\beta$-continuous [20], $\beta$-continuous [27]) at a point $x \in X$ if for each open set $V$ such that $F(x) \cap$ $\cap V \neq \emptyset$ there exists $U \in \beta(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ (resp. $F(u) \cap$ $\cap \operatorname{Int}(\mathrm{Cl}(V)) \neq \emptyset, F(u) \cap V \neq \emptyset)$ for every $u \in U$,
(c) upper (lower) weakly $\beta$-continuous (almost $\beta$-continuous, $\beta$-continuous) if $F$ has this property at every point of $X$.

Remark 3.1. By u.w. $\beta$.c. (resp. l.w. $\beta$.c.), we denote the term "upper weakly $\beta$-continuous" (resp. "lower weakly $\beta$-continuous").

Theorem 3.1. For a multifunction $F: X \rightarrow Y$, the following are equivalent
(1) $F$ is u.w. $\beta$.c. at a point $x \in X$;
(2) $x \in \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$ containing $F(x)$;
(3) for each open neighborhood $U$ of $x$ and each open set $V$ of $Y$ containing $F(x)$; there exists an open set $G$ of $X$ such that $\emptyset \neq G \subset U$ and $G \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right) ;$
(4) for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in \operatorname{SO}(X, x)$ such that $U \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$;
(5) $x \in{ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$ for every open set $V$ of $Y$ containing $F(x)$.

Proof. (1) $\Rightarrow$ (2): Let $V$ be an open set of $Y$ containing $F(x)$. Then there exists $U \in \beta(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Then $x \in U \subset$ $\subset F^{+}(\mathrm{Cl}(V))$. Since $U$ is $\beta$-open, we have $x \in U \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(U))) \subset$ $\subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)$.
(2) $\Rightarrow$ (3): Let $V$ be an open set of $Y$ containing $F(x)$ and $U$ an open set of $X$ containing $x$. Since $x \in \mathrm{Cl}\left(\operatorname{Int}\left(\operatorname{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)$, we have $U \cap \operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(C l(V))\right)\right) \neq \emptyset$. Put $G=U \cap \operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)$, then $G$ is an open set, $\emptyset \neq G \subset U$ and

$$
G \subset \operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right) \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right) .
$$

(3) $\Rightarrow$ (4): Let $V$ be an open set of $Y$ containing $F(x)$. By $\mathbf{U}(x)$, we denote the family of all open neighborhoods of $x$. For each $U \in U(x)$, there exists an open set $G_{U}$ of $X$ such that $\emptyset \neq G_{U} \subset U$ and $G_{U} \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$. Put $W=\cup\left\{G_{U}: U \in \mathbf{U}(x)\right\}$, then $W$ is an open set of $X, x \in \mathrm{Cl}(W)$ and $W \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$. Moreover, we put $U_{0}=W \cup\{x\}$, then we obtain $U_{0} \in \operatorname{SO}(X, x)$ and $U_{0} \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$.
(4) $\Rightarrow$ (5): Let $V$ be an open set of $Y$ containing $F(x)$. There exists $G \in \mathrm{SO}(X, x)$ such that $G \subset \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$. Then, by Lemma 2.1 we have

$$
\begin{gathered}
x \in F^{+}(V) \cap G \subset F^{+}(\mathrm{Cl}(V)) \cap \mathrm{Cl}(\operatorname{Int}(G)) \subset \\
\subset F^{+}(\mathrm{Cl}(V)) \cap \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)={ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)
\end{gathered}
$$

(5) $\Rightarrow$ (1): Let $V$ be any open set of $Y$ containing $F(x)$. Then, by (5) we have $x \in \beta \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$. Put $U=\beta \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$. Then $U \in \beta(X, x)$ and $F(U) \subset \mathrm{Cl}(V)$. Therefore, $F$ is u.w. $\beta$.c. at $x$.

THEOREM 3.2. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is l.w. $\beta . c$. at a point $x$ of $X$;
(2) $x \in \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset ;$
(3) for any open neighborhood $U$ of $x$ and any open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open set $G$ of $X$ such that

$$
\emptyset \neq G \subset U \quad \text { and } \quad G \subset \mathrm{Cl}\left(F^{-}(\mathrm{Cl}(V))\right)
$$

(4) for any open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in$ $\in \mathrm{SO}(X, x)$ such that $U \subset \mathrm{Cl}\left(F^{-}(\mathrm{Cl}(V))\right)$;
(5) $x \in \beta \operatorname{Int}\left(F^{-}(\mathrm{Cl}(V))\right)$ for every open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$.

PROOF. The proof is similar to that of Theorem 3.1 and is thus omitted.
DEFINITION 3.2. A function $f: X \rightarrow Y$ is said to be weakly $\beta$-continuous at a point $x \in X$ if for each open set $V$ of $Y$ containing $f(x)$ there exists $U \in \beta(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. If $f$ has this property at each point of $X$, it is said to be weakly $\beta$-continuous [28].

Corollary 3.1 (Popa and Noiri [28]). For a function $f: X \rightarrow Y$, the following are equivalent:
(1) $f$ is weakly $\beta$-continuous at a point $x$ of $X$;
(2) $x \in \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(f^{-1}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$ containing $f(x)$;
(3) for any open neighborhood $U$ of $x$ and any open set $V$ of $Y$ containing $f(x)$; there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $G \subset \mathrm{Cl}\left(f^{-1}(\mathrm{Cl}(V))\right) ;$
(4) for any open set $V$ of $Y$ containing $f(x)$, there exists $U \in \operatorname{SO}(X, x)$ such that $U \subset \mathrm{Cl}\left(f^{-1}(\mathrm{Cl}(V))\right)$;
(5) $\left.x \in{ }_{\beta} \operatorname{Int}\left(f^{-1}(\mathrm{Cl}(V))\right)\right)$ ) for every open set $V$ of $Y$ containing $f(x)$.

Theorem 3.3. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is u.w. $\beta$.c.;
(2) $F^{+}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$;
(3) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) \subset F^{-}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$;
(4) $\operatorname{Int}\left(\operatorname{Cl}\left(\operatorname{Int}\left(F^{-}(\operatorname{Int}(K))\right)\right)\right) \subset F^{-}(K)$ for every closed set $K$ of $Y$;
(5) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(K))\right) \subset F^{-}(K)$ for every closed set $K$ of $Y$;
(6) $\beta_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset F^{-}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(7) $F^{+}(\operatorname{Int}(B)) \subset{ }_{\beta} \operatorname{Int}\left(F^{+}(\operatorname{Cl}(\operatorname{Int}(B)))\right)$ for every subset $B$ of $Y$;
(8) $F^{+}(V) \subset{ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$ for every open set $V$ of $Y$;
(9) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right) \subset F^{-}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2): Let $V$ be any open set of $Y$ and $x \in F^{+}(V)$. Then $F(x) \subset V$ and by Theorem $3.1 x \in \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right)$. Therefore, we obtain

$$
F^{+}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right) .
$$

(2) $\Rightarrow$ (3): Let $V$ be any open set of $Y$. Then we have

$$
\begin{gathered}
X-F^{-}(\mathrm{Cl}(V))=F^{+}(Y-\mathrm{Cl}(V)) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(\mathrm{Cl}(Y-\mathrm{Cl}(V)))\right)\right)\right)= \\
=\mathrm{Cl}\left(\operatorname { I n t } \left(\mathrm { Cl } \left(F^{+}(Y-\operatorname{Int}(\mathrm{Cl}(V))) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(Y-V)\right)\right)\right)=\right.\right.\right. \\
=\mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(X-F^{-}(V)\right)\right)\right)=X-\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) .
\end{gathered}
$$

Therefore, we obtain $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) \subset F^{-}(\mathrm{Cl}(V))$.
(3) $\Rightarrow$ (4): Let $K$ be any closed set of $Y$, then $\operatorname{Int}(K)$ is open in $Y$ and hence

$$
\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(\operatorname{Int}(K))\right)\right)\right) \subset F^{-}(\mathrm{Cl}(\operatorname{Int}(K))) \subset F^{-}(K)
$$

(4) $\Rightarrow$ (5): Let $K$ be any closed set of $Y$. Then we have

$$
\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(\operatorname{Int}(K))\right)\right)\right) \subset F^{-}(K) \quad \text { and } \quad F^{-}(\operatorname{Int}(K)) \subset F^{-}(K) .
$$

Therefore, by Lemma 2.1 we obtain a $\mathrm{Cl}\left(F^{-}(\operatorname{Int}(K))\right) \subset F^{-}(K)$.
(5) $\Rightarrow$ (6): Let $B$ be any subset of $Y$, then $\mathrm{Cl}(B)$ is closed in $Y$. Thus we obtain

$$
{ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset F^{-}(\mathrm{Cl}(B)) .
$$

(6) $\Rightarrow$ (7): Let $B$ be any subset of $Y$. Then, we obtain

$$
\begin{gathered}
F^{+}(\operatorname{Int}(B))=X-F^{-}(\mathrm{Cl}(Y-B)) \subset X-{ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(Y-B)))\right)= \\
={ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(\operatorname{Int}(B)))\right) .
\end{gathered}
$$

(7) $\Rightarrow$ (8): The proof is obvious.
(8) $\Rightarrow$ (9): Let $V$ be any open set of $Y$. Then, by (8) we have

$$
\begin{gathered}
{ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right) \subset{ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)={ }_{\beta} \mathrm{Cl}\left(X-F^{+}(Y-\operatorname{Int}(\mathrm{Cl}(V)))\right)= \\
=X-{ }_{\beta} \operatorname{Int}\left(F^{+}(Y-\operatorname{Int}(\mathrm{Cl}(V)))\right)=X-{ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(Y-\mathrm{Cl}(V)))\right) \subset \\
\subset X-F^{+}(Y-(\mathrm{Cl}(V)))=F^{-}(\mathrm{Cl}(V)) .
\end{gathered}
$$

Therefore, we obtain ${ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right) \subset F^{-}(\mathrm{Cl}(V))$.
(9) $\Rightarrow$ (1): Let $x$ be any point of $X$ and $V$ be any open set of $Y$ containing $F(x)$. Now, we set $U={ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$. Then, we have

$$
\begin{aligned}
U & =\beta \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)= \\
& =X-{ }_{\beta} \mathrm{Cl}\left(F^{-}(Y-\mathrm{Cl}(V))\right) \supset X-F^{-}(\mathrm{Cl}(Y-\mathrm{Cl}(V)))= \\
& =F^{+}(\operatorname{Int}(\mathrm{Cl}(V))) \supset F^{+}(V) \ni x .
\end{aligned}
$$

Therefore, we obtain that $U \in \beta(X, x)$ and $F(U) \subset \mathrm{Cl}(V)$. This shows that $F$ is u.w. $\beta$.c.

Theorem 3.4. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) F is l.w. $\beta . c$. ;
(2) $F^{-}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$;
(3) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{+}(V)\right)\right)\right) \subset F^{+}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$;
(4) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{+}(\operatorname{Int}(K))\right)\right)\right) \subset F^{+}(K)$ for every closed set $K$ of $Y$;
(5) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(K))\right) \subset F^{+}(K)$ for every closed set $K$ of $Y$;
(6) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset F^{+}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(7) $F^{-}(\operatorname{Int}(B)) \subset{ }_{\beta} \operatorname{Int}\left(F^{-}(\operatorname{Cl}(\operatorname{Int}(B)))\right)$ for every subset $B$ of $Y$;
(8) $F^{-}(V) \subset{ }_{\beta} \operatorname{Int}\left(F^{-}(\mathrm{Cl}(V))\right)$ for every open set $V$ of $Y$;
(9) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(V)\right) \subset F^{+}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$.

Proof. The proof is similar to that of Theorem 3.3 and is thus omitted.
COROLLARY 3.2 (POPA and NOIRI [28]). For a function $f: X \rightarrow Y$, the following are equivalent:
(1) $f$ is weakly $\beta$-continuous;
(2) $f^{-1}(V) \subset{ }_{\beta} \operatorname{Int}\left(f^{-1}(\mathrm{Cl}(V))\right)$ for every open set $V$ of $Y$;
(3) ${ }_{\beta} \mathrm{Cl}\left(f^{-1}(\operatorname{Int}(K))\right) \subset f^{-1}(K)$ for every closed set $K$ of $Y$;
(4) $\beta_{\beta} \operatorname{Cl}\left(f^{-1}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(5) $f^{-1}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(f^{-1}(\mathrm{Cl}(V))\right)\right)\right)$ for every open set $V$ of $Y$;
(6) ${ }_{\beta} \mathrm{Cl}\left(f^{-1}(V)\right) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$.

THEOREM 3.5. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is u.w. $\beta . c$. ;
(2) ${ }_{\beta} \mathrm{Cl}\left(F^{-}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(B)\right)\right)\right) \subset F^{-}\left(\mathrm{Cl}_{\theta}(B)\right)$ for every subset $B$ of $Y$;
(3) $\beta_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset F^{-}\left(\mathrm{Cl}_{\theta}(B)\right)$ for every subset $B$ of $Y$;
(4) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right) \subset F^{-}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$;
(5) $\beta \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{-}(\mathrm{Cl}(G))$ for every preopen set $G$ of $Y$;
(6) $\beta \mathrm{Cl}\left(F^{-}(\operatorname{Int}(K))\right) \subset F^{-}(K)$ for every regular closed set $K$ of $Y$;
(7) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{-}(\mathrm{Cl}(G))$ for every $G \in \beta(Y)$;
(8) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{-}(\mathrm{Cl}(G))$ for every $G \in \mathrm{SO}(Y)$.

Proof. (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Then $\mathrm{Cl}_{\theta}(B)$ is closed in $Y$. Therefore, by Lemma 2.1 and Theorem 3.3 we obtain

$$
\beta \mathrm{Cl}\left(F^{-}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(B)\right)\right)\right)=
$$

$\left.=F^{-}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(B)\right)\right)\right) \cup \operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(B)\right)\right)\right)\right)\right) \subset F^{-}\left(\mathrm{Cl}_{\theta}(B)\right)$.
(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_{\theta}(B)$ for every subset $B$ of $Y$.
$(3) \Rightarrow$ (4): This is obvious since $\mathrm{Cl}(V)=\mathrm{Cl}_{\theta}(V)$ for every open set $V$ of $Y$.
(4) $\Rightarrow$ (5): Let $G$ be any preopen set of $Y$. Then we have $G \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(G))$ and hence $\mathrm{Cl}(G)=\mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(G)))$. Now, set $V=\operatorname{Int}(\mathrm{Cl}(G))$, then $V$ is open in $Y$ and $\mathrm{Cl}(G)=\mathrm{Cl}(V)$. Therefore, by (4) we have ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{-}(\mathrm{Cl}(G))$.
(5) $\Rightarrow$ (6): Let $K$ be any regular closed set of $Y$. Then we have $\operatorname{Int}(K) \in$ $\in \mathrm{PO}(Y)$ and hence by $(5){ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(K))\right)={ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(K))))\right) \subset$ $\subset F^{-}(\mathrm{Cl}(\operatorname{Int}(K)))=F^{-}(K)$.
(6) $\Rightarrow$ (7): Let $G \in \beta(Y)$. Then $G \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(G)))$. Since $\mathrm{Cl}(G)$ is regular closed in $Y$, by (6) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{-}(\mathrm{Cl}(G))$.
$(7) \Rightarrow$ (8): This is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.
$(8) \Rightarrow(1)$ : Let $V$ be any open set of $Y$. Then, since $V \in \operatorname{SO}(Y)$, by (8) we have ${ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right) \subset{ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right) \subset F^{-}(\mathrm{Cl}(V))$. Hence, by Theorem 3.3, $F$ is u.w. $\beta$.c.

THEOREM 3.6. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is l.w. $\beta$.c.;
(2) ${ }_{\beta} \mathrm{Cl}\left(F^{+}\left(\operatorname{Int}\left(\mathrm{Cl}_{\theta}(B)\right)\right)\right) \subset F^{+}\left(\mathrm{Cl}_{\theta}(B)\right)$ for every subset $B$ of $Y$;
(3) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset F^{+}\left(\mathrm{Cl}_{\theta}(B)\right)$ for every subset $B$ of $Y$;
(4) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(V)))\right) \subset F^{+}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$;
(5) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{+}(\mathrm{Cl}(G))$ for every preopen set $G$ of $Y$;
(6) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(K))\right) \subset F^{+}(K)$ for every regular closed set $K$ of $Y$;
(7) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{+}(\mathrm{Cl}(G))$ for every $G \in \beta(Y)$;
(8) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(\operatorname{Int}(\mathrm{Cl}(G)))\right) \subset F^{+}(\mathrm{Cl}(G))$ for every $G \in \mathrm{SO}(Y)$.

Proof. The proof is similar to that of Theorem 3.5.
Corollary 3.3 (POPA and Noiri [28]). For a function $f: X \rightarrow Y$, the following are equivalent:
(1) $f$ is weakly $\beta$-continuous;
(2) $\beta_{\beta} \mathrm{Cl}\left(f^{-1}(\operatorname{Int}(\mathrm{Cl}(B)))\right) \subset f^{-1}(\mathrm{Cl}(B))$ for every subset $B$ of $Y$;
(3) ${ }_{\beta} \mathrm{Cl}\left(f^{-1}(\operatorname{Int}(F))\right) \subset f^{-1}(F)$ for every regular closed set $F$ of $Y$;
(4) $\beta_{\beta} \mathrm{Cl}\left(f^{-1}(V)\right) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set $V$ of $Y$.

THEOREM 3.7. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is u.w. $\beta . c$. ;
(2) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right) \subset F^{-}(\mathrm{Cl}(V))\right.\right.$ for every $V \in \mathrm{PO}(Y)$;
(3) ${ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right) \subset F^{-}(\mathrm{Cl}(V))$ for every $V \in \mathrm{PO}(Y)$;
(4) $F^{+}(V) \subset{ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$ for every $V \in \mathrm{PO}(Y)$.

Proof. (1) $\Rightarrow$ (2): Let $V$ be any preopen set of $Y$. Since $F$ is u.w. $\beta$.c., by Theorem 3.3 we obtain

$$
\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) \subset \operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(\operatorname{Int}(\mathrm{Cl}(V)))\right)\right)\right) \subset F^{-}(\mathrm{Cl}(V))
$$

(2) $\Rightarrow$ (3): Let $V$ be any preopen set of $Y$. By Lemma 2.1, we have ${ }_{\beta} \mathrm{Cl}\left(F^{-}(V)\right)=F^{-}(V) \cup \operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) \subset F^{-}(\mathrm{Cl}(V))$.
(3) $\Rightarrow$ (4): Let $V$ be any preopen set of $Y$. Then, we have $V \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(V))$ and $Y-V \supset \mathrm{Cl}(\operatorname{Int}(Y-V))$. By (3), we have $X-F^{+}(V)=F^{-}(Y-V) \supset F^{-}(\mathrm{Cl}(\operatorname{Int}(Y-V))) \supset{ }_{\beta} \mathrm{Cl}\left(F^{-}(\operatorname{Int}(Y-V))\right)=$ $={ }_{\beta} \mathrm{Cl}\left(F^{-}(Y-\mathrm{Cl}(V))\right)={ }_{\beta} \mathrm{Cl}\left(X-F^{+}(\mathrm{Cl}(V))\right)=X-{ }_{\beta} \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$. Therefore, we obtain $F^{+}(V) \subset \beta \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$.
$(4) \Rightarrow(1)$ : Since every open set is preopen, this follows from Theorem 3.3.

THEOREM 3.8. For a multifunction $F: X \rightarrow Y$, the following are equivalent:
(1) $F$ is $1 . w . \beta . c . ;$
(2) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{+}(V)\right)\right)\right) \subset F^{+}(\mathrm{Cl}(V))$ for every $V \in \mathrm{PO}(Y)$;
(3) ${ }_{\beta} \mathrm{Cl}\left(F^{+}(V)\right) \subset F^{+}(\mathrm{Cl}(V))$ for every $V \in \mathrm{PO}(Y)$;
(4) $F^{-}(V) \subset{ }_{\beta} \operatorname{Int}\left(F^{-}(\mathrm{Cl}(V))\right)$ for every $V \in \mathrm{PO}(Y)$.

Proof. The proof is similar to that of Theorem 3.7 and is thus omitted.
Corollary 3.4. For a function $f: X \rightarrow Y$, the following are equivalent:
(1) $f$ is weakly $\beta$-continuous;
(2) $\operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(f^{-1}(V)\right)\right)\right) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathrm{PO}(Y)$;
(3) ${ }_{\beta} \mathrm{Cl}\left(f^{-1}(V)\right) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathrm{PO}(Y)$;
(4) $f^{-1}(V) \subset{ }_{\beta} \operatorname{Int}\left(f^{-1}(\mathrm{Cl}(V))\right)$ for every $V \in \mathrm{PO}(Y)$.

For a multifunction $F: X \rightarrow Y$, the graph multifunction $G_{F}: X \rightarrow$ $\rightarrow X \times Y$ is defined as follows: $G_{F}(x)=\{x\} \times F(x)$ for every $x \in X$.

Lemma 3.1 (Noiri and Popa [18]). For a multifunction $F: X \rightarrow Y$, the following hold:
(a)

$$
G_{F^{+}}(A \times B)=A \cap F^{+}(B)
$$

and
(b)

$$
G_{F^{-}}(A \times B)=A \cap F^{-}(B)
$$

for any subsets $A \subset X$ and $B \subset Y$.
Theorem 3.9. Let $F: X \rightarrow Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then $F$ is u.w. $\beta$.c. if and only if $G_{F}: X \rightarrow X \times Y$ is u.w. $\beta$.c.

Proof. Necessity. Suppose that $F: X \rightarrow Y$ is u.w. $\beta$.c. Let $x \in X$ and $W$ be any open set of $X \times Y$ containing $G_{F}(x)$. For each point $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times$ $\times V(y) \subset W$. The family $\{V(y): y \in F(x)\}$ is an open cover of $F(x)$ and $F(x)$ is compact. Therefore, there exist a finite number of points, say, $y_{1}, y_{2}$, $\ldots, y_{n}$ in $F(x)$ such that $F(x) \subset \cup\left\{V\left(y_{i}\right): 1 \leq i \leq n\right\}$. Set

$$
U=\cap\left\{U\left(y_{i}\right): 1 \leq i \leq n\right\} \quad \text { and } \quad V=\cup\left\{V\left(y_{i}\right): 1 \leq i \leq n\right\} .
$$

Then $U$ and $V$ are open sets of $X$ and $Y$, respectively, and $G_{F}(x)=\{x\} \times$ $\times F(x) \subset U \times V \subset W$. Since $F$ is u.w. $\beta$.c., there exists $U_{0} \in \beta(X, x)$ such that $F\left(U_{0}\right) \subset \mathrm{Cl}(V)$. It follows from [4, Theorem 2.7] that $U \cap U_{0} \in \beta(X, x)$. By Lemma 3.1, we have

$$
U \cap U_{0} \subset \mathrm{Cl}(U) \cap F^{+}(\mathrm{Cl}(V))=G_{F^{+}}(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) \subset G_{F^{*}}(\mathrm{Cl}(W)) .
$$

Therefore, we obtain $U \cap U_{0} \in \beta(X, x)$ and $G_{F}\left(U \cap U_{0}\right) \subset \mathrm{Cl}(W)$. This shows that $G_{F}$ is u.w. $\beta$.c.

Sufficiency. Suppose that $G_{F}: X \rightarrow X \times Y$ is u.w. $\beta$.c. Let $x \in X$ and $V$ be any open set of $Y$ containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_{F}(x) \subset X \times V$, there exists $U \in \beta(X, x)$ such that $G_{F}(U) \subset \mathrm{Cl}(X \times V)=$ $=X \times \mathrm{Cl}(V)$. By Lemma 3.1, we have $U \subset G_{F^{+}}(X \times \mathrm{Cl}(V))=F^{+}(\mathrm{Cl}(V))$. This shows that $F$ is u.w. $\beta$.c.

THEOREM 3.10. A multifunction $F: X \rightarrow Y$ is 1.w. $\beta$.c. if and only if $G_{F}: X \rightarrow X \times Y$ is l.w. $\beta$.c.

Proof. Necessity. Suppose that $F$ is l.w. $\beta$.c. Let $x \in X$ and $W$ be any open set of $X \times Y$ such that $x \in G_{F^{-}}(W)$. Since $W \cap(\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since $F$ is l.w. $\beta$.c. and $F(x) \cap V \neq \emptyset$, there exists $U_{0} \in \beta(X, x)$ such that $U_{0} \subset F^{-}(\mathrm{Cl}(V))$. By Lemma 3.1, we have

$$
U \cap U_{0} \subset U \cap F^{-}(\mathrm{Cl}(V))=G_{F^{-}}(U \times \mathrm{Cl}(V)) \subset G_{F^{-}}(\mathrm{Cl}(W))
$$

Moreover, we have $U \cap U_{0} \in \beta(X, x)$ by [4, Theorem 2.7] and hence $G_{F}$ is l.w. $\beta$.c.

Sufficiency. Suppose that $G_{F}$ is l.w. $\beta$.c. Let $x \in X$ and $V$ be an open set of $Y$ such that $x \in F^{-}(V)$. Then $X \times V$ is open in $X \times Y$ and

$$
G_{F}(x) \cap(X \times V)=(\{x\} \times F(x)) \cap(X \times V)=\{x\} \times(F(x) \cap V) \neq \emptyset
$$

Since $G_{F}$ is l.w. $\beta$.c., there exists $U \in \beta(X, x)$ such that $U \subset G_{F^{-}}(\mathrm{Cl}(X \times$ $\times V)$ ). By Lemma 3.1, we obtain $U \subset G_{F^{-}}(\mathrm{Cl}(X \times V))=F^{-}(\mathrm{Cl}(V))$. This shows that $F$ is l.w. $\beta$.c.

COROLLARY 3.5. (Popa and Noiri [28]). Let $f: X \rightarrow Y$ be a function and $g: X \rightarrow X \times Y$ the graph function defined as follows: $g(x)=(x, f(x))$ for each $x \in X$. Then $f$ is weakly $\beta$-continuous if and only if $g$ is weakly $\beta$-continuous.

THEOREM 3.11. Let $\left\{U_{\alpha}: \alpha \in \nabla\right\}$ be an $\alpha$-open cover of a topological space $X$. A multifunction $F: X \rightarrow Y$ is u.w. $\beta$.c. if and only if the restriction $F \mid U_{\alpha}: U_{\alpha} \rightarrow Y$ is u.w. $\beta$.c. for each $\alpha \in \nabla$.

Proof. Necessity. Suppose that $F$ is u.w. $\beta$.c. Let $\alpha \in \nabla$ and $x \in U_{\alpha}$. Let $V$ be an open set of $Y$ such that $\left(F \mid U_{\alpha}\right)(x) \subset V$. Since $F$ is u.w. $\beta$.c. and $F(x)=\left(F \mid U_{\alpha}\right)(x) \subset V$, there exists $U_{0} \in \beta(X, x)$ such that $F\left(U_{0}\right) \subset \mathrm{Cl}(V)$. Set $U=U_{0} \cap U_{\alpha}$, then by [1, Lemma 2.5] we have $U \in \beta\left(U_{\alpha}, x\right)$. Then $\left(F \mid U_{\alpha}\right)(U)=F(U) \subset \mathrm{Cl}(V)$. Therefore, $F \mid U_{\alpha}$ is u.w. $\beta$.c.

Sufficiency. Suppose that $F \mid U_{\alpha}: U_{\alpha} \rightarrow Y$ is u.w. $\beta$.c. for each $\alpha \in \nabla$. Let $x \in X$ and $V$ be any open set of $Y$ such that $F(x) \subset V$. There exists $\alpha \in V$ such that $x \in U_{\alpha}$. Since $F \mid U_{\alpha}$ is u.w. $\beta$.c. and $\left(F \mid U_{\alpha}\right)(x)=F(x) \subset V$, there exists $U \in \beta\left(U_{\alpha}, x\right)$ such that $\left(F \mid U_{\alpha}\right)(U) \subset$ $\subset \mathrm{Cl}(V)$. Since $U_{\alpha}$ is $\alpha$-open, we have $U \in \beta(X, x)$ [1, Lemma 2.7] and $F(U)=\left(F \mid U_{\alpha}\right)(U) \subset \mathrm{Cl}(V)$. This shows that $F$ is u.w. $\beta$.c.

THEOREM 3.12. Let $\left\{U_{\alpha}: \alpha \in \nabla\right\}$ be an $\alpha$-open cover of a topological space $X$. A multifunction $F: X \rightarrow Y$ is l.w. $\beta$.c. if and only if the restriction $F \mid U_{\alpha}: U_{\alpha} \rightarrow Y$ is l.w. $\beta$.c. for each $\alpha \in \nabla$.

Proof. The proof is similar to that of Theorem 3.11.

## 4. Weak $\beta$-continuity and almost $\beta$-continuity

DEFINITION 4.1. A multifunction $F: X \rightarrow Y$ is said to be almost $\beta$-open if $F(U) \subset \operatorname{Int}(\mathrm{Cl}(F(U)))$ for every $U \in \beta O(X, x)$.

Lemma 4.1 (POPA and Noiri [26]). If $X$ is a topological space and $U$ is open in $X$, then $\mathrm{sCl}(U)=\operatorname{Int}(\mathrm{Cl}(U))$.

THEOREM 4.1. If a multifunction $F: X \rightarrow Y$ is u.w. $\beta . c$, and almost $\beta$-open, then $F$ is upper almost $\beta$-continuous.

Proof. Let $V$ be any open set of $Y$ containing $F(x)$. There exists $U \in \beta(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. Since $F$ is almost $\beta$-open, $F(U) \subset$ $\subset \operatorname{Int}(\mathrm{Cl}(F(U))) \subset \operatorname{Int}(\mathrm{Cl}(V))=\mathrm{sCl}(V)$. It follows from [20, Theorem 3] that $F$ is upper almost $\beta$-continuous.

COROLLARY 4.5 (NOIRI and POPA [19]). If a function $f: X \rightarrow Y$ is weakly $\beta$-continuous and almost $\beta$-open, then $f$ is almost $\beta$-continuous.

THEOREM 4.2. Let $F: X \rightarrow Y$ be a multifunction such that $F(x)$ is an open set of $Y$ for each $x \in X$. Then the following are equivalent:
(1) $F$ is lower $\beta$-continuous;
(2) $F$ is lower almost $\beta$-continuous;
(3) $F$ is l.w. $\beta$.c.

PROOF. The implications (1) $\Rightarrow(2)$ and $(2) \Rightarrow(3)$ are obvious and we show that
(3) implies (1). Let $x \in X$ and $V$ be an open set such that $F(x) \cap V \neq \emptyset$. Then there exists $U \in \beta(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence $F$ is lower $\beta$-continuous.

Definition 4.2. A subset $A$ of a topological space $X$ is said to be
(a) $\alpha$-paracompact [34] if every cover of $A$ by open sets of $X$ is refined by a cover of $A$ which consists of open sets of $X$ and is locally finite in $X$,
(b) $\alpha$-regular [11] (resp. $\alpha$-almost-regular [12]) if for each $a \in A$ and each open (resp, regular open) set $U$ of $X$ containing $a$, there exists an open set $G$ of $X$ such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

DEFINITION 4.3. A topological space $X$ is said to be almost regular [31] if for each $x \in X$ and each regular closed set $F$ of $X$ not containing $x$, there exist disjoint open sets $U$ and $V$ of $X$ such that $x \in U$ and $F \subset V$.

LEMMA 4.2 (POPA and NOIRI [26]). If $A$ is an $\alpha$-almost regular $\alpha$ paracompact set of a topological space $X$ and $U$ is a regular open neighborhood of $A$, then there exists an open set $G$ of $X$ such that $A \subset G \subset$ $\subset \mathrm{Cl}(G) \subset U$.

THEOREM 4.3. If a multifunction $F: X \rightarrow Y$ is u.w. $\beta . c$. and $F(x)$ is an $\alpha$-almost regular and $\alpha$-paracompact set for each $x \in X$, then $F$ is upper almost $\beta$-continuous.

Proof. Let $V$ be a regular open set of $Y$ and $x \in F^{+}(V)$. Then $F(x) \subset V$ and by Lemma 4.2. there exists an open set $W$ of $Y$ such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since $F$ is u.w. $\beta$.c., there exists $U \in \beta(X, x)$ such that $F(U) \subset \mathrm{Cl}(W) \subset V$. Therefore, we have $x \in U \subset F^{+}(V)$. This shows that $F^{+}(V) \in \beta(X, x)$. It follows from [20, Theorem 3] that $F$ is upper almost $\beta$-continuous.

Corollary 4.2. Let $F: X \rightarrow Y$ be an u.w. $\beta$.c. multifunction such that $F(x)$ is compact for each $x \in X$ and $Y$ is almost regular. Then, $F$ is upper almost $\beta$-continuous.

COROLLARY 4.3 (NOIRI and POPA [19]). If $f: X \rightarrow Y$ is a weakly $\beta$-continuous function and $Y$ is almost regular, then $f$ is almost $\beta$-continuous.

LEMMA 4.3 (POPA [24]). If $A$ is an $\alpha$-almost regular set of a space $X$, then for every regular open set $U$ which intersects $A$, there exists an open set $G$ such that $A \cap G \neq \emptyset$ and $\mathrm{Cl}(G) \subset U$.

THEOREM 4.4. If $F: X \rightarrow Y$ is a l.w. $\beta$.c. multifunction such that $F(x)$ is an $\alpha$-almost regular set of $Y$ for every $x \in X$, then $F$ is lower almost $\beta$-continuous.

PROOF. Let $V$ be a regular open set of $Y$ and $x \in F^{-}(V)$. Since $F(x)$ is $\alpha$-almost regular, by Lemma 4.3 there exists an open set $W$ of $Y$ such that
$F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. Since $F$ is l.w. $\beta$.c., there exists $U \in \beta(X, x)$ such that $F(u) \cap \mathrm{Cl}(W) \neq \emptyset$; hence $F(u) \cap V \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^{-}(V)$. This shows that $F^{-}(V) \in \beta(X, x)$. It follows from [20, Theorem 4] that $F$ is lower almost $\beta$-continuous.

## 5. $\beta$-continuity and weak $\beta$-continuity

THEOREM 5.1. Let $Y$ be a regular space. Then, for a multifunction $F$ : $X \rightarrow Y$, the following are equivalent:
(1) $F$ is upper $\beta$-continuous;
(2) $F^{-}\left(\mathrm{Cl}_{\theta}(B)\right)$ is $\beta$-closed in $X$ for every subset $B$ of $Y$;
(3) $F^{-}(K)$ is $\beta$-closed in $X$ for every $\theta$-closed set $K$ of $Y$;
(4) $F^{+}(V) \in \beta(X)$ for every $\theta$-open set $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Then $\mathrm{Cl}_{\theta}(B)$ is closed in $Y$ and it follows from [30, Theorem 3.3] that $F^{-}\left(\mathrm{Cl}_{\theta}(B)\right)$ is $\beta$-closed in $X$.
$(2) \Rightarrow(3)$ : This is obvious.
(3) $\Rightarrow$ (4): Let $V$ be any $\theta$-open set of $Y$. By (3), $F(Y-V)$ is $\beta$-closed in $X$ and $F(Y-V)=X-F^{+}(V)$. Therefore, we obtain $F^{+}(V) \in \beta(X)$.
(4) $\Rightarrow$ (1): Let $V$ be any open set of $Y$. Since $Y$ is regular, $V$ is $\theta$-open in $Y$ and by (4) $F^{+}(V) \in \beta(X)$. It follows from [30, Theorem 3.3] that $F$ is upper $\beta$-continuous.

THEOREM 5.2. Let $Y$ be a regular space. Then, for a multifunction $F$ : $X \rightarrow Y$, the following are equivalent:
(1) $F$ is lower $\beta$-continuous;
(2) $F^{+}\left(\mathrm{Cl}_{\theta}(B)\right)$ is $\beta$-closed in $X$ for every subset $B$ of $Y$;
(3) $F^{+}(K)$ is $\beta$-closed in $X$ for every $\theta$-closed set $K$ of $Y$;
(4) $F^{-}(V) \in \beta(X)$ for every $\theta$-open set $V$ of $Y$;
(5) $F$ is l.w. $\beta . c$.

Proof. We prove only the implication $(5) \Rightarrow(1)$, the proof of the other being similar to that of Theorem 5.1. The proof of the implication (4) $\Rightarrow$ (5) is obvious.
(5) $\Rightarrow$ (1): Let $V$ be any open set of $Y$ and $x \in F^{-}(V)$. Then we have $F(x) \cap V \neq \emptyset$. Since $Y$ is regular, there exists an open set $W$ of $Y$ such that $F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. Since $F$ is l.w. $\beta$.c., there exists $U \in \beta(X, x)$
such that $U \subset F^{-}(\mathrm{Cl}(W)) \subset F(V)$. Thus, we have $x \in U \subset \mathrm{Cl}(\operatorname{Int}(\mathrm{Cl}(U)))$ and hence $F^{-}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}(V)\right)\right)\right)$. This shows that $F^{-}(V) \in \beta(X)$ and it follows from [30, Theorem 3.4] that $F$ is lower $\beta$-continuous.

Corollary 5.1 (POPA and Noiri [28]). Let $Y$ be a regular space. Then, for a function $f: X \rightarrow Y$, the following are equivalent:
(1) $f$ is $\beta$-continuous;
(2) $f^{-1}\left(\mathrm{Cl}_{\theta}(B)\right)$ is $\beta$-closed in $X$ for every subset $B$ of $Y$;
(3) $f$ is weakly $\beta$-continuous.

DEFINITION 5.1. A multifunction $F: X \rightarrow Y$ is said to be nearly almost open [18] if there exists an open basis $\Sigma=\left\{V_{\alpha}: \alpha \in \nabla\right\}$ of the topology for $Y$ such that $F^{-}\left(\mathrm{Cl}\left(V_{\alpha}\right)\right) \subset \mathrm{Cl}\left(F^{-}\left(V_{\alpha}\right)\right)$ for every $\alpha \in \nabla$.

THEOREM 5.3. If $F: X \rightarrow Y$ is 1.w. $\beta$.c. and nearly almost open, then $F$ is lower $\beta$-continuous.

Proof. Let $\Sigma=\left\{V_{\alpha}: \alpha \in \nabla\right\}$ be an open basis of the topology for $Y$ such that $F^{-}\left(\mathrm{Cl}\left(V_{\alpha}\right)\right) \subset \mathrm{Cl}\left(F^{-}\left(V_{\alpha}\right)\right)$ for every $\alpha \in \nabla$. For any open set $V$ of $Y$, there exists a subset $\nabla_{0}$ of $\nabla$ such that $V=\cup\left\{V_{\alpha}: \alpha \in \nabla_{0}\right\}$. Therefore, by Theorem 3.4 we obtain

$$
\begin{aligned}
& F^{-}(V)= F^{-}\left(\cup\left\{V_{\alpha}: \alpha \in \nabla_{0}\right\}\right)=\cup\left\{F^{-}\left(V_{\alpha}\right): \alpha \in \nabla_{0}\right\} \subset \\
& \subset \cup\left\{\mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}\left(\mathrm{Cl}\left(V_{\alpha}\right)\right)\right)\right)\right): \alpha \in \nabla_{0}\right\} \subset \\
& \subset \cup\left\{\mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}\left(V_{\alpha}\right)\right)\right)\right): \alpha \in \nabla_{0}\right\} \subset \\
&\left.\subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(\cup\left\{F^{-}\left(V_{\alpha}\right): \alpha \in \nabla_{0}\right\}\right)\right)\right)\right)= \\
&=\mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}\left(\cup\left\{V_{\alpha}: \alpha \in \nabla_{0}\right\}\right)\right)\right)\right)=\mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}(V)\right)\right)\right) .
\end{aligned}
$$

This shows that $F^{-}(V) \in \beta(X)$. It follows from [30, Theorem 3.4] that $F$ is lower $\beta$-continuous.

DEFINITION 5.2. A multifunction $F: X \rightarrow Y$ is said to be almost open [18] if for each open set $U$ of $X \quad F(U) \subset \operatorname{Int}(\mathrm{Cl}(F(U)))$.

Lemma 5.1 (NOIRI and POPA [18]). A multifunction $F: X \rightarrow Y$ is almost open if and only if $F^{-}(\mathrm{Cl}(V)) \subset \mathrm{Cl}\left(F^{-}(V)\right)$ for each open set $V$ of $Y$.

COROLLARY 5.2. If a multifunction $F: X \rightarrow Y$ is 1.w. $\beta$.c. and almostopen, then $F$ is lower $\beta$-continuous.

THEOREM 5.4. If a multifunction $F: X \rightarrow Y$ is u.w. $\beta . c$. and satisfies $F^{+}(\mathrm{Cl}(V)) \subset \mathrm{Cl}\left(F^{+}(V)\right)$ for each open set $V$ of $Y$, then $F$ is upper $\beta$ continuous.

Proof. Let $V$ be any open set of $Y$. Since $F$ is u.w. $\beta$.c., by Theorem 3.3 we have $F^{+}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\operatorname{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)\right)\right.$ and hence $F^{+}(V) \subset$ $\subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{+}(V)\right)\right)\right.$. Thus, $F^{+}(V) \in \beta(X)$ and it follows from [30, Theorem 3.3] that $F$ is upper $\beta$-continuous.

DEFINITION 5.3. A multifunction $F: X \rightarrow Y$ is said to be complementary continuous [22] if for each open set $V$ of $Y, F^{-}(\operatorname{Fr}(V))$ is a closed set of $X$.

THEOREM 5.5. If $F: X \rightarrow Y$ is u.w. $\beta$.c. and complementary continuous, then it is upper $\beta$-continuous.

Proof. Let $x \in X$ and $V$ be any open set of $Y$ such that $F(x) \subset V$. There exists $G \in \beta(X, x)$ such that $F(G) \subset \mathrm{Cl}(V)$. Put $U=G \cap[X-$ $\left.-F^{-}(\operatorname{Fr}(V))\right]$. Since $F^{-}(\operatorname{Fr}(V))$ is closed in $X, U \in \beta(X)$ [4, Theorem 2.7]. Moreover, we have

$$
F(x) \cap \operatorname{Fr}(V) \subset[V \cap \mathrm{Cl}(V)] \cap(Y-V)=\emptyset
$$

and hence

$$
x \in X-F^{-}(\operatorname{Fr}(V))
$$

Thus, we obtain $U \in \beta(X, x)$ and $F(U) \subset V$ since

$$
F(U) \subset F(G) \subset \mathrm{Cl}(V) \quad \text { and } \quad F(u) \cap \operatorname{Fr}(V)=\emptyset \quad \text { for each } u \in U
$$

Therefore, $F$ is upper $\beta$-continuous.
DEFINITION 5.4. A multifunction $F: X \rightarrow Y$ is said to be
(a) upper weakly continuous [22] at a point $x$ of $X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists an open set $U$ containing $x$ such that $F(U) \subset \mathrm{Cl}(V) ;$
(b) lower weakly continuous ([22]) at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open set $U$ containing $x$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$;
(c) upper (or lower) weakly continuous ([22]) if it is upper (or lower) weakly continuous at every point of $X$.

DEFINITION 5.5. A multifunction $F: X \rightarrow Y$ is said to be complementary $\beta$-continuous if for each open set $V$ of $Y, F^{-}(\operatorname{Fr}(V))$ is a $\beta$-closed set of $X$.

THEOREM 5.6. If $F: X \rightarrow Y$ is u.w.c. and complementary $\beta$-continuous, then it is upper $\beta$-continuous.

Proof. It is similar to the proof of Theorem 5.5.
COROLLARY 5.3 (Noiri and Popa [21]). If $: X \rightarrow Y$ is weakly continuous and complementary $\beta$-continuous, then it is $\beta$-continuous.

## 6. Weak $\beta$-continuity and other forms of weak continuity

DEFINITION 6.1. A multifunction $F: X \rightarrow Y$ is said to be
(a) upper almost continuous [23] at $x \in X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists an open set $U$ of $X$ containing $x$ such that $F(U) \subset \operatorname{Int}(\mathrm{Cl}(V)) ;$
(b) lower almost continuous [23] at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open set $U$ of $X$ containing $x$ such that $F(u) \cap \operatorname{Int}(\operatorname{Cl}(V)) \neq \emptyset$ for every $u \in U$;
(c) upper (or lower) almost continuous [23] if it is upper (or lower) almost continuous at every point of $X$.

DEFINITION 6.2. A multifunction $F: X \rightarrow Y$ is said to be
(1) upper weakly quasicontinuous [17] if for each point $x \in X$, each open set $U$ of $X$ containing $x$ and each open set $V$ containing $F(x)$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(G) \subset \mathrm{Cl}(V)$,
(2) lower weakly quasicontinuous [17] if for each point $x \in X$, each open set $U$ of $X$ containing $x$ and each open set $V$ such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(g) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $g \in G$.

DEFINITION 6.3. A multifunction $F: X \rightarrow Y$ is said to be
(a) upper $\alpha$-continuous [27] at a point $x$ of $X$ if for each open set $V$ of $Y$ containing $F(x)$, there exists $U \in \alpha(X, x)$ such that $F(U) \subset V$,
(b) lower $\alpha$-continuous [27] at $x \in X$ if for each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,
(c) upper (or lower) $\alpha$-continuous [27] if it is upper (or lower) $\alpha$ continuous at every point of $X$.

THEOREM 6.1. If a multifunction $F: X \rightarrow Y$ is u.w. $\beta$.c, and lower almost continuous, then $F$ is upper weakly quasicontinuous.

Proof. Let $V$ be any open set of $Y$. Since $F$ is u.w. $\beta$.c., by Theorem 3.3 we have $\left.F^{+}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{( } \mathrm{Cl}(V)\right)\right)\right)\right)$. Since $\mathrm{Cl}(V)$ is regular closed, it follows from [23, Theorem 2.2] that $F^{+}(\mathrm{Cl}(V))$ is closed in $X$. Thus, we have $F^{+}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)\right)$. It follows from [17, Theorem 3.1] that $F$ is upper weakly quasicontinuous.

THEOREM 6.2. If a multifunction $F: X \rightarrow Y$ is l.w. $\beta$.c. and upper almost continuous, then $F$ is lower weakly quasicontinuous.

Proof. The proof is similar to that of Theorem 6.1.
THEOREM 6.3. If a multifunction $F: X \rightarrow Y$ is lower $\alpha$-continuous and u.w. $\beta$.c., then $F$ is lower weakly continuous.

Proof. Let $V$ be any open set of $Y$. Since $F$ is lower $\alpha$-continuous, by [27, Theorem 3.4] $F^{-}(V)$ is $\alpha$-open in $X$. Since $F$ is u.w. $\beta$.c., we have by Theorem 3.3

$$
F^{-}(V) \subset \operatorname{Int}\left(\mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)\right) \subset F^{-}(\mathrm{Cl}(V))
$$

Therefore, we obtain $F^{-}(V) \subset \operatorname{Int}\left(F^{-}(\mathrm{Cl}(V))\right)$. It follows from [22, Theorem 4] that $F$ is lower weakly continuous.

COROLLARY 6.1 (POPA and NOIRI [27]). If a multifunction $F: X \rightarrow Y$ is lower $\alpha$-continuous and upper $\beta$-continuous, then $F$ is lower weakly continuous.

THEOREM 6.4. If a multifunction $F: X \rightarrow Y$ is upper $\alpha$-continuous and 1.w. $\beta$.c., then $F$ is upper weakly continuous.

PROOF. The proof is similar to that of Theorem 6.3.
COROLLARY 6.2 (POPA and NOIRI [27]). If a multifunction $F: X \rightarrow Y$ is upper $\alpha$-continuous and lower $\beta$-continuous, then $F$ is upper weakly continuous.

LEMMA 6.1 (CLAY and JOSEPH [7]). A multifunction $F: X \rightarrow Y$ is upper weakly continuous (resp. lower weakly continuous) if and only if
$\mathrm{Cl}\left(F^{-}(V)\right) \subset F^{-}(\mathrm{Cl}(V))\left(\right.$ resp. $\mathrm{Cl}\left(F^{+}(V)\right) \subset F^{+}(\mathrm{Cl}(V))$ ) for every open set $V$ of $Y$.

THEOREM 6.5. If a multifunction $F: X \rightarrow Y$ is upper weakly continuous, l.w. $\beta$.c. and $F(x)$ is $\alpha$-regular for each $x \in X$, then $F$ is lower quasicontinuous.

Proof. Let $V$ be any open set of $Y$ and $x \in F^{-}(V)$. Then $F(x) \cap$ $\cap V \neq \emptyset$. There exists $y \in F(x) \cap V$ and hence $y \in W \subset \mathrm{Cl}(W) \subset V$ for some open set $W$ of $Y$ since $F(x)$ is $\alpha$-regular. By [20, Theorem 16], $F$ is lower $\beta$-continuous and by [30, Theorem 3.2] $x \in \operatorname{Cl}\left(\operatorname{Int}\left(\mathrm{Cl}\left(F^{-}(W)\right)\right)\right)$. Since $F$ is upper weakly continuous, by Lemma 6.1 we have $\mathrm{Cl}\left(F^{-}(W)\right) \subset$ $\subset F^{-}(\mathrm{Cl}(W)) \subset F^{-}(V)$. Therefore, we have $x \in \mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)$ and $F^{-}(V) \subset \mathrm{Cl}\left(\operatorname{Int}\left(F^{-}(V)\right)\right)$. This shows that $F^{-}(V)$ is semi-open in $X$ and $F$ is lower quasicontinuous.

## 7. Some separation axioms and u.w. $\beta$.c. multifunctions

DEFINITION 7.1. A multifunction $F: X \rightarrow Y$ is said to be
(1) upper weakly $\alpha$-continuous [29] at a point $x \in X$ if for each $U \in$ $\mathrm{SO}(X, x)$ and each open set $V$ of $Y$ containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset \mathrm{sCl}(V)$,
(2) lower weakly $\alpha$-continuous [29] at a point $x \in X$ if for each $U \in$ $\in \mathrm{SO}(X, x)$ and each open set $V$ such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set $G \subset U$ such that $F(g) \cap \operatorname{sCl}(V) \neq \emptyset$ for every $g \in G$,
(3) upper (lower) weakly $\alpha$-continuous if $F$ has this property at every point of $X$.

Lemma 7.1 (Smithson [32]). If $A$ and $B$ are disjoint compact subsets of a Urysohn space $X$, then there exist open sets $U$ and $V$ of $X$ such that $A \subset U, B \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V)=\emptyset$.

THEOREM 7.1. Let $F, G: X \rightarrow Y$ be multifunctions into a Urysohn space $Y$ and $F(x), G(x)$ be compact in $Y$ for each $x \in X$. If $F$ is upper weakly $\alpha$-continuous and $G$ is u.w. $\beta$.c., then $A=\{x \in X: F(x) \cap G(x) \neq \emptyset\}$ is $\beta$-closed in $X$.

Proof. Let $x \in X-A$. Then we have $F(x) \cap G(x)=\emptyset$. By Lemma 7.1, there exist open sets $V$ and $W$ of $X$ such that $F(x) \subset V, G(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W)=\emptyset$. Since $F$ is upper weakly $\alpha$-continuous, there exists ([29,

Theorem 1]) $U_{1} \in \alpha(X, x)$ such that $F\left(U_{1}\right) \subset \operatorname{Cl}(V)$. Since $G$ is u.w. $\beta$.c., there exists $U_{2} \in \beta(X, x)$ such that $G\left(U_{2}\right) \subset \mathrm{Cl}(W)$. Now set $U=U_{1} \cap U_{2}$, then we have $U \in \beta(X, x)$ and $U \cap A=\emptyset$. Therefore, $A$ is $\beta$-closed in $X$.

COROLLARY 7.1 (POPA and NOIRI [28]). Let $f, g: X \rightarrow Y$ be functions into a Urysohn space $Y$. If $f$ is weakly $\alpha$-continuous and $g$ is weakly $\beta$ continuous, then $A=\{x \in X: f(x)=g(x)\}$ is $\beta$-closed in $X$.

THEOREM 7.2. Let $F_{1}: X_{1} \rightarrow Y$ and $F_{2}: X_{2} \rightarrow Y$ be multifunctions into a Urysohn space $Y$ and $F_{1}(x)$ compact in $Y$ for each $x \in X_{i}$ and each $i=1$, 2. If $F_{1}$ and $F_{2}$ are u.w.c. $\beta$., then $A=\left\{\left(x_{1}, x_{2}\right) \in X_{1} \times X_{2}: F_{1}\left(x_{1}\right) \cap F_{2}\left(x_{2}\right) \neq \emptyset\right\}$ is a $\beta$-closed set of the product space $X_{1} \times X_{2}$.

Proof. We shall show that $X_{1} \times X_{2}-A$ is $\beta$-open in $X_{1} \times X_{2}$. Let $\left(x_{1}, x_{2}\right) \in X_{1} \times X_{2}-A$. Then we have $F_{1}\left(x_{1}\right) \cap F_{2}\left(x_{2}\right)=\emptyset$. By Lemma 7.1, there exist open sets $V_{i}$, such that $F_{i}\left(x_{i}\right) \subset V_{i}$ for $i=1,2$ and $\mathrm{Cl}\left(V_{1}\right) \cap$ $\cap \mathrm{Cl}\left(V_{2}\right)=\emptyset$. Since $F_{i}$ is u.w. $\beta$.c., there exists $U_{i} \in \beta\left(X_{i}, x_{i}\right)$ such that $F_{i}\left(U_{i}\right) \subset \mathrm{Cl}\left(V_{i}\right)$ for $i=1$, 2. Now, put $U=U_{1} \times U_{2}$, then we have $U \in \beta\left(X_{1} \times X_{2}\right)$ [2, Lemma 2.2] and $\left(x_{1}, x_{2}\right) \in U \subset X_{1} \times X_{2}-A$. Therefore, $X_{1} \times X_{2}-A$ is $\beta$-open in $X_{1} \times X_{2}$ and $A$ is $\beta$-closed in $X_{1} \times X_{2}$.

DEFINITION 7.2. A multifunction $F: X \rightarrow Y$ is said to be injective if $x \neq y$ implies $F(x) \cap F(y)=\emptyset$.

DEFINITION 7.3. A space $X$ is said to be $\beta-T_{2}$ if for each distinct points $x_{1}, x_{2}$ of $X$ there exist $\beta$-open sets $U_{i}$ such that $x_{i} \in U_{i}$ for $i=1,2$ and $U_{1} \cap U_{2}=\emptyset$.

THEOREM 7.3. If $F: X \rightarrow Y$ is an u.w. $\beta$.c. injective multifunction into a Urysohn space $Y$ and $F(x)$ is compact in $Y$ for each $x \in X$, then $X$ is $\beta-T_{2}$.

PROOF. For any distinct points $x_{1}, x_{2}$ of $X$, we have $F\left(x_{1}\right) \cap F\left(x_{2}\right)=0$ since $F$ is injective. Since $F\left(x_{i}\right)$ is compact in a Urysohn space $Y$, by Lemma 7.1 there exist open sets $V_{i}$ such that $F\left(x_{i}\right) \subset V_{i}$ for $i=1,2$ and $\mathrm{Cl}\left(V_{i}\right) \cap \mathrm{Cl}\left(V_{2}\right)=\emptyset$. Since $F$ is u.w. $\beta$.c., there exists $U_{i} \in \beta\left(X, x_{i}\right)$ such that $F\left(U_{i}\right) \subset \mathrm{Cl}\left(V_{i}\right)$ for $i=1,2$. Therefore, we obtain $U_{1} \cap U_{2}=\emptyset$ and hence $X$ is $\beta-T_{2}$.

COROLLARY 7.2. If $: X \rightarrow Y$ is a weakly $\beta$-continuous injection and $Y$ is Urysohn, then $X$ is $\beta-T_{2}$.

THEOREM 7.4. Let $F_{1}, F_{2}: X \rightarrow Y$ be multifunctions into a Urysohn space $Y$ and $F_{i}(x)$ compact in $Y$ for each $x \in X$ and each $i=1$, 2. If
$F_{i}(x) \cap F_{2}(X) \neq \emptyset$ for each $x \in X, F_{1}$ is u.w. $\beta . c$. and $F_{2}$ is upper weakly $\alpha$-continuous, then a multifunction $F: X \rightarrow Y$, defined as follows $F(x)=$ $=F_{1}(x) \cap F_{2}(x)$ for each $x \in X$, is u.w. $\beta$.c.

Proof. Let $x \in X$ and $V$ be an open set of $Y$ such that $F(x) \subset V$. Then, $A=F_{1}(x)-V$ and $B=F_{2}(x)-V$ are disjoint compact sets. By Lemma 7.1, there exist open sets $V_{1}$ and $V_{2}$ such that $A \subset V_{1}, B \subset V_{2}$ and $\mathrm{Cl}\left(V_{1}\right) \cap \mathrm{Cl}\left(V_{2}\right)=\emptyset$. Since $F_{1}$ is u.w. $\beta$.c., there exists $U_{1} \in \beta(X, x)$ such that $F_{1}\left(U_{1}\right) \subset \mathrm{Cl}\left(V_{i} \cup V\right)$. Since $F_{2}$ is upper weakly $\alpha$-continuous, there exists an $\alpha$-open set $U_{2}$ containing $x$ such that $F_{2}\left(U_{2}\right) \subset \mathrm{Cl}\left(V_{2} \cup V\right)$. Set $U=U_{1} \cap U_{2}$, then $U \in \beta(X, x)$. If $y \in F\left(x_{0}\right)$ for any $x_{\in} U$, then $y \in \operatorname{Cl}\left(V_{1} \cup\right.$ $\cup V) \cap \mathrm{Cl}\left(V_{1} \cup V\right)=\left(\mathrm{Cl}\left(V_{1}\right) \cap \mathrm{Cl}\left(V_{2}\right)\right) \cup \mathrm{Cl}(V)$. Since $\mathrm{Cl}\left(V_{1}\right) \cap \mathrm{Cl}\left(V_{2}\right)=\emptyset$, we have $y \in \mathrm{Cl}(V)$ and hence $F(U) \subset \mathrm{Cl}(V)$. Therefore, $F$ is u.w. $\beta$.c.

DEFINITION 7.4. For a multifunction $F: X \rightarrow Y$, the graph $G(F)=$ $=\{(x, F(x)): x \in X\}$ is said to be strongly $\beta$-closed if for each $(x, y) \in(X \times$ $\times Y)-G(F)$, there exist $U \in \beta(X, x)$ and an open set $V$ containing $y$ such that $\left[U \times{ }_{\beta} \mathrm{Cl}(V)\right] \cap G(F)=\emptyset$.

LEMMA 7.2. A multifunction $F: X \rightarrow Y$ has a strongly $\beta$-closed graph if and only if for each $(x, y) \in(X \times Y)-G(F)$, there exist $U \in \beta(X, x)$ and an open set $V$ containing $y$ such that $F(U) \cap \mathrm{sCl}(V)=\emptyset$.

Proof. This follows from Lemmas 2.1 and 4.1.
THEOREM 7.5. If $F: X \rightarrow Y$ is an u.w. $\beta$.c. multifunction into a Hausdorff space $Y$ such that $F(x)$ is $\alpha$-paracompact for each $x \in X$, then $G(F)$ is strongly $\beta$-closed in $X \times Y$.

Proof. Let $\left(x_{0}, y_{0}\right) \in(X \times Y)-G(F)$, then $y_{0} \in Y-F\left(x_{0}\right)$. Since $Y$ is Hausdorff, for each $y \in F\left(x_{0}\right)$ there exist open sets $V(y)$ and $W(y)$ of $Y$ such that $y \in V(y), y_{0} \in W(y)$ and $V(y) \cap W(y)=\emptyset$, The family $\left\{V(y): y \in F\left(x_{0}\right)\right\}$ is an open cover of $F\left(x_{0}\right)$ which is $\alpha$-paracompact. Thus it has a locally finite open refinement $\mathbf{U}=\left\{U_{i}: i \in I\right\}$ which covers $F\left(x_{0}\right)$. Let $W_{0}$ be an open neighborhood of $y_{0}$ such that $W_{0}$ intersects only finitely many members, say, $U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}$. Choose $y_{1}, y_{2}, \ldots$, $y_{n}$ in $F\left(x_{0}\right)$ such that $U_{i(k)} \subset V\left(y_{k}\right)$ for each $k(1 \leq k \leq n)$ and set $W=W_{0} \cap\left(\cap\left\{W\left(y_{k}\right): 1 \leq k \leq n\right\}\right)$. Then $W$ is an open neighborhood of $y_{0}$ with $W \cap\left(\cup\left\{U_{i}: i \in I\right\}\right)=\emptyset$ which implies $\operatorname{sCl}(W) \cap \mathrm{Cl}\left(\cup\left\{U_{i}: i \in I\right\}\right)=\emptyset$ by Lemma 4.1. Since $F$ is u.w. $\beta$.c., there exists $U \in \beta\left(X, x_{0}\right)$ such that
$F(U) \subset \mathrm{Cl}\left(\cup\left\{U_{i}: i \in I\right\}\right)$. Therefore, we have $F(U) \cap \operatorname{sCl}(W)=\emptyset$ and $G(F)$ is strongly $\beta$-closed in $X \times Y$.

COROLLARY 7.3. If $F: X \rightarrow Y$ is an u.w. $\beta$.c. multifunction such that $F(x)$ is compact for each $x \in X$ and $Y$ is a Hausdorff space, then $G(F)$ is strongly $\beta$-closed.

## 8. Other properties of u.(l.)w. $\beta$.c. multifunctions

Definition 8.1. The $\beta$-frontier [3] of a subset $A$ of a space $X$, denoted by ${ }_{\beta} \operatorname{Fr}(A)$, is defined by ${ }_{\beta} \operatorname{Fr}(A)={ }_{\beta} \operatorname{Cl}(A) \cap_{\beta} \operatorname{Cl}(X-A)={ }_{\beta} \mathrm{Cl}(A)-{ }_{\beta} \operatorname{Int}(A)$

THEOREM 8.1. The set of all points $x$ of $X$ at which a multifunction $F: X \rightarrow Y$ is not u.w. $\beta$.c. (resp. l.w. $\beta . c$. ) is identical with the union of the $\beta$-frontiers of the upper (resp. lower) inverse images of the closures of open sets containing (resp. meeting) $F(x)$.

Proof. Let $x$ be a point of $X$ at which $F$ is not u.w. $\beta$.c. Then, there exists an open set $V$ containing $F(x)$ such that $U \cap\left(X-F^{+}(\mathrm{Cl}(V))\right) \neq \emptyset$ for every $U \in \beta(X, x)$. Therefore, we have $x \in={ }_{\beta} \mathrm{Cl}\left(X-F^{+}(\mathrm{Cl}(V))\right)$. Since $x \in F^{+}(V)$, we have $x \in, \beta \mathrm{Cl}\left(F^{+}(\mathrm{Cl}(V))\right)$ and hence $x \in{ }_{\beta} \operatorname{Fr}\left(F^{+}(\mathrm{Cl}(V))\right)$.

Conversely, if $F$ is u.w. $\beta$.c. at $x$, then for any open set $V$ of $Y$ containing $F(x)$ there exists $U \in \beta(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^{+}(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \beta \operatorname{Int}\left(F^{+}(\mathrm{Cl}(V))\right)$. This contradicts that $x \in \beta \operatorname{Fr}\left(F^{+}(\mathrm{Cl}(V))\right)$. The case of l.w. $\beta$.c, is similarly shown.

Corollary 8.1 (Noiri and Popa [21]). The set of all points $x$ of $X$ at which a function $f: X \rightarrow Y$ is not $\beta$-continuous is identical with the union of the $\beta$-frontiers of the inverse images of open sets containing $f(x)$.

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# CHARACTERIZATION OF THE ISOSCELES AND THE EQUILATERAL TRIANGLE BY ALGEBRAIC RELATIONS 

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1. Let us consider the triangle $A B C$ with sides $a=B C, b=A C$ and $c=A B$. We denote by $h_{a}, h_{b}$ and $h_{c}$ the length of the altitudes which correspond to the sides $a, b$ and $c$, respectively.

THEOREM 1. The triangle $A B C$ is isosceles $(a=b)$ or rectangular $(\hat{C}=$ $=90^{\circ}$ ) if and only if

$$
a^{\alpha}+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha} \quad \text { for } \quad \alpha \in \mathbf{R} \backslash\{0\}
$$

Proof. If we denote by $S$ the area of the triangle $A B C$ then $S=\frac{a \cdot h_{a}}{2}=$ $=\frac{b \cdot h_{b}}{2}$. We mention that $a>b$ implies $h_{a}<h_{b}$. If $a=b$ then $h_{a}=h_{b}$, i.e. a well known fact from the elementary geometry, that in the isosceles triangle the altitudes which correspond to the equal sides are equal. Consequently, if $a=b$ then $a^{\alpha}+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha}$. Now, if $\hat{C}=90^{\circ}$ then the triangle $A B C$ is a rectangular triangle, so the legs are altitudes at the same time: $h_{a}=b$ and $h_{b}=a$. Consequently if $\hat{C}=90^{\circ}$ then $a^{\alpha}+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha}$.

Conversely, we suppose $a^{\alpha}+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha}$ for some $\alpha \in \mathbf{R} \backslash\{0\}$ and we show that the triangle $A B C$ is isosceles or $\hat{C}=90^{\circ}$. We can suppose $\alpha>0$. Indeed, if $\alpha<0$ then $\beta=-\alpha>0$ and

$$
\begin{aligned}
a^{\alpha}+h_{a}^{\alpha} & =b^{\alpha}+h_{b}^{\alpha} \Leftrightarrow a^{-\beta}+h_{a}^{-\beta}=b^{-\beta}+h_{b}^{-\beta} \Leftrightarrow \frac{a^{\beta}+h_{a}^{\beta}}{\left(a \cdot h_{a}\right)^{\beta}}=\frac{b^{\beta}+h_{b}^{\beta}}{\left(b \cdot h_{b}\right)^{\beta}} \Leftrightarrow \\
& \Leftrightarrow a^{\beta}+h_{a}^{\beta}=b^{\beta}+h_{b}^{\beta}
\end{aligned}
$$

So let $\alpha>0$. We have the following equivalent equalities:

$$
\begin{aligned}
a^{\alpha}+h_{a}^{\alpha} & =b^{\alpha}+h_{b}^{a} \Leftrightarrow a^{\alpha}+\left(\frac{2 S}{a}\right)^{\alpha}=b^{\alpha}+\left(\frac{2 S}{b}\right)^{\alpha} \Leftrightarrow \\
& \Leftrightarrow a^{\alpha}-b^{\alpha}=(2 S)^{\alpha}\left(\frac{1}{b^{\alpha}}-\frac{1}{a^{\alpha}}\right) \Leftrightarrow\left(a^{\alpha}-b^{\alpha}\right)\left(1-\frac{(2 S)^{\alpha}}{a^{\alpha} b^{\alpha}}\right)=0
\end{aligned}
$$

Consequently either $a^{\alpha}=b^{\alpha}$, i.e. $a=b$, or $1-\frac{(2 S)^{\alpha}}{a^{\alpha} b^{\alpha}}=0$, i.e. $S=\frac{a b}{2}$. But in every triangle $A B C$ we have $S=\frac{a b \sin C}{2}$, so in this last case $\sin C=1$, i.e. $\hat{C}=90^{\circ}$.

THEOREM 2. The triangle $A B C$ is equilateral if and only if

$$
a^{\alpha}+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha}=c^{\alpha}+h_{c}^{\alpha} \quad \text { for } \quad \alpha \in \mathbf{R} \backslash\{0\} .
$$

Proof. In every triangle $A B C$, at least two angles are acute, so their measure is strictly less then $90^{\circ}$. Indeed, if we suppose the contrary then the sum of the measure of the angles in a triangle would be strictly greater then $180^{\circ}$, which is a contradiction. Without loss of generality, we can suppose for example $\hat{A}, \hat{C} \in\left(0^{\circ}, 90^{\circ}\right)$. Using Theorem 1 we have $a=b \Leftrightarrow a^{\alpha}+$ $+h_{a}^{\alpha}=b^{\alpha}+h_{b}^{\alpha}$ and $b=c \Leftrightarrow b^{\alpha}+h_{b}^{\alpha}=c^{\alpha}+h_{c}^{a}$, which in turn yields the theorem.
2. We will study similar results when we consider instead of altitudes the medians. Let $m_{a}, m_{b}$ and $m_{c}$ be the length of medians which correspond to the sides $a, b$ and $c$, respectively. We will study the following hypotheses: the triangle $A B C$ is isosceles $(a=b)$ if and only if $a^{\alpha}+m_{a}^{\alpha}=b^{\alpha}+m_{b}^{\alpha}$ for $\alpha \in \mathbf{R} \backslash\{0\}$, and the triangle $A B C$ is equilateral if and only if $a^{\alpha}+m_{a}^{\alpha}=$ $=b^{\alpha}+m_{b}^{\alpha}=c^{\alpha}+m_{c}^{\alpha}$ for $\alpha \in \mathbf{R} \backslash\{0\}$. We solve the problem only for the values $\alpha=1, \alpha=2$ and $\alpha=4$, the other cases are open problems.

THEOREM 3. The condition that the triangle $A B C$ is isosceles $(a=b)$ is not equivalent with $a+m_{a}=b+m_{b}$.

PROOF. If we write the formulas $m_{a}=\sqrt{\frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}}$, etc. then we mention that $a>b$ implies $m_{a}=\sqrt{\frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}}<\sqrt{\frac{2\left(a^{2}+c^{2}\right)-b^{2}}{4}}=m_{b}$, and $a=b$ implies $m_{a}=\sqrt{\frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}}=\sqrt{\frac{2\left(a^{2}+c^{2}\right)-b^{2}}{4}}=m_{b}$, i.e. a well known fact from the elementary geometry, that in the isosceles triangle the
medians which correspond to the equal sides are equal. Consequently if $A B C$ is an isosceles triangle $(a=b)$ then $a+m_{a}=b+m_{b}$.

Conversely we prove that the condition $a+m_{a}=b+m_{b}$ does not imply that $A B C$ is isosceles. Indeed, we will show the existence of a triangle $A B C$ such that $a>b>c$ and still $a+m_{a}=b+m_{b}$.

We have the following equivalent equalities:

$$
\begin{aligned}
a & +m_{a}=b+m_{b} \Leftrightarrow 2 a+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}=2 b+\sqrt{2\left(a^{2}+c^{2}\right)-b^{2}} \Leftrightarrow \\
& \Leftrightarrow 2(a-b)=\sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}-\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}} \Leftrightarrow \\
& \Leftrightarrow 2(a-b)=\frac{2\left(a^{2}+c^{2}\right)-b^{2}-2\left(b^{2}+c^{2}\right)+a^{2}}{\sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}} \Leftrightarrow \\
& \Leftrightarrow 2(a-b) \cdot\left[\sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}\right]=3\left(a^{2}-b^{2}\right) \Leftrightarrow \\
& \Leftrightarrow(a-b) \cdot\left[\sqrt{2 \cdot\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}-\frac{3}{2}(a+b)\right]=0 .
\end{aligned}
$$

We suppose that $a>b>c$ and we denote by $y=\frac{a}{c}$ and $z=\frac{b}{c}$. So we have the resctictions $1<z<y<z+1$ (from the conditions $a>b$ and $a>c$ results automatically that $b<a+c$ and $c<a+b$ ). According to the above we have

$$
\begin{aligned}
a+m_{a}=b+m_{b} & \Leftrightarrow \sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}-\frac{3}{2}(a+b)=0 \Leftrightarrow \\
& \Leftrightarrow \sqrt{2\left(y^{2}+1\right)-z^{2}}+\sqrt{2\left(z^{2}+1\right)-y^{2}}-\frac{3}{2}(y+z)=0
\end{aligned}
$$

We must show that for a fixed $z_{0}>1$ there exists $y_{0} \in\left(z_{0}, z_{0}+1\right)$ such that the following equality is true:

$$
\sqrt{2\left(y_{0}^{2}+1\right)-z_{0}^{2}}+\sqrt{2\left(z_{0}^{2}+1\right)-y_{0}^{2}}-\frac{3}{2}\left(y_{0}+z_{0}\right)=0
$$

One possibility is to solve the irrational equation but another way is the following: let us consider the function $g:[z, z+1] \rightarrow \mathbf{R}, g(y)=$ $=\sqrt{2\left(y^{2}+1\right)-z^{2}}+\sqrt{2\left(z^{2}+1\right)-y^{2}}-\frac{3}{2}(y+z)$. Then $g$ is well defined, it is continuous, $g(z)=2 \sqrt{z^{2}+2}-3 z>0$, if and only if $z \in\left(1, \frac{2 \sqrt{10}}{5}\right)$,
(which results from an elementary calculus) and

$$
\begin{aligned}
g(z+1) & =\sqrt{2 \cdot\left[(z+1)^{2}+1\right]-z^{2}}+\sqrt{2 \cdot\left(z^{2}+1\right)-(z+1)^{2}}-\frac{3}{2}(2 z+1)= \\
& =(z+2)+(z-1)-\frac{3}{2}(2 z+1)=-z-\frac{1}{2}<0
\end{aligned}
$$

So if $z_{0} \in\left(1, \frac{2 \sqrt{10}}{5}\right)$ then the Darboux property of the function $g$ assures us the existence of $y_{0} \in\left(z_{0}, z_{0}+1\right)$ such that $g\left(y_{0}\right)=0$. Consequently there exists a triangle $A B C$ so that $a>b>c, b=z_{0} \cdot c<\frac{2 \sqrt{10}}{5} \cdot c$, $a=y_{0} \cdot c<\left(z_{0}+1\right) \cdot c=b+c<\left(1+\frac{2 \sqrt{10}}{5}\right) \cdot c$ and $a+m_{a}=b+m_{b}$.

THEOREM 4. The condition that the triangle $A B C$ is equilateral is not equivalent with $a+m_{a}=b+m_{b}=c+m_{c}$.

PROOF. If $A B C$ is an equilateral triangle then $a+m_{a}=b+m_{b}=c+m_{c}$.
Conversely, we suppose that the relations $a+m_{a}=b+m_{b}=c+m_{c}$ are true in a triangle $A B C$ and we show that the triangle $A B C$ is isosceles, but is not sure that it is equilateral.

Using the proof of Theorem 3 we have the following equivalent equalities:

$$
\begin{aligned}
a & +m_{a}=b+m_{b} \Leftrightarrow \\
& \Leftrightarrow(a-b) \cdot\left[\sqrt{2 \cdot\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}-\frac{3}{2}(a+b)\right]=0 \Leftrightarrow \\
& \Leftrightarrow(a-b) \cdot\left[2 \cdot m_{a}+2 \cdot m_{b}-\frac{3}{2}(a+b)\right]=0
\end{aligned}
$$

If we suppose that $a \neq b, b \neq c$ and $c \neq a$ then from the equalities $a+m_{a}=$ $=b+m_{b}=c+m_{c}$ and from the above we obtain that $2 m_{a}+2 m_{b}-\frac{3}{2}(a+b)=0$ and the other two same equalities $2 m_{b}+2 m_{c}-\frac{3}{2}(b+c)=0$ and $2 m_{c}+2 m_{a}-$ $-\frac{3}{2}(c+a)=0$, respectively. We substract the first two equalities from each other and we obtain $2 m_{a}-2 m_{c}-\frac{3}{2}(a-c)=0$ and $2 m_{a}+2 m_{c}-\frac{3}{2}(c+a)=0$. Taking there sum results $4 m_{a}-3 a=0$. But $4 m_{a}=3 a \Leftrightarrow m_{a}^{2}=\frac{9}{16} a^{2} \Leftrightarrow$ $\Leftrightarrow \frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}=\frac{9}{16} a^{2} \Leftrightarrow b^{2}+c^{2}=\frac{13}{8} a^{2}$. Analogously, by the permutation of the elements $a, b, c$ we obtain $c^{2}+a^{2}=\frac{13}{8} b^{2}$. If we eliminate $c$ from these
two relations we get $c^{2}=\frac{13}{8} a^{2}-b^{2}=\frac{13}{8} b^{2}-a^{2}$. Consequently $a=b$, which is a contradiction. We deduce that two numbers from $a, b, c$ are equal, i.e. it is sure that the triangle $A B C$ is isosceles.

Without loss of generality we can suppose $b=c$. Now we prove the existence of an isosceles triangle $A B C$ such that $a>b=c$ and still $a+m_{a}=$ $=b+m_{b}=c+m_{c}$. The condition $b=c$ implies $b+m_{b}=c+m_{c}$, so it remains to verify the condition $a+m_{a}=b+m_{b}$. According to the above we have $a+m_{a}=b+m_{b} \Leftrightarrow \sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}-\frac{3}{2}(a+b)=0$, because $a \neq b$. We denote by $x=\frac{a}{c}$ and $y=\frac{b}{c}=1$. So $x=\frac{a}{c}>1$ and from the triangle inequality we find $a<b+c=2 c$, i.e. $x=\frac{a}{c}<2$ (from the conditions $a>b$ and $a>c$ results automatically that $b<a+c$ and $c<a+b$ ). But

$$
\begin{aligned}
& \sqrt{2\left(a^{2}+c^{2}\right)-b^{2}}+\sqrt{2\left(b^{2}+c^{2}\right)-a^{2}}-\frac{3}{2}(a+b)=0 \Leftrightarrow \\
& \Leftrightarrow \sqrt{2 a^{2}+c^{2}}+\sqrt{4 c^{2}-a^{2}}-\frac{3}{2}(a+c)=0 \Leftrightarrow \\
& \Leftrightarrow \sqrt{2 x^{2}+1}+\sqrt{4-x^{2}}-\frac{3}{2}(x+1)=0
\end{aligned}
$$

So we must assure the existence of a real value $x_{0} \in(1,2)$ such that $\sqrt{2 x_{0}^{2}+1}+\sqrt{4-x_{0}^{2}}-\frac{3}{2}\left(x_{0}+1\right)=0$. One possibility is to solve the irrational equation, but another way is the following: let us consider the function $f:[1,2] \rightarrow \mathbf{R}, f(x)=\sqrt{2 x^{2}+1}+\sqrt{4-x^{2}}-\frac{3}{2}(x+1)$. Then $f$ is well defined and it is continuous, $f(1)=2 \sqrt{3}-3>0$ and $f(2)=-\frac{3}{2}<0$. The Darboux property for $f$ assures us the existence of a value $x_{0} \in(1,2)$ such that $f\left(x_{0}\right)=0$. So there exists a triangle $A B C$ which is not equilateral such that $a=b \cdot x_{0}, b=c$ and $a+m_{a}=b+m_{b}=c+m_{c}$.

THEOREM 5. The triangle $A B C$ is isosceles $(a=b)$ if and only if $a^{2}+$ $+m_{a}^{2}=b^{2}+m_{b}^{2}$.

PROOF. We have the following equivalent equalities: $a^{2}+m_{a}^{2}=b^{2}+m_{b}^{2} \Leftrightarrow$ $\Leftrightarrow a^{2}+\frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}=b^{2}+\frac{2\left(a^{2}+c^{2}\right)-b^{2}}{4} \Leftrightarrow a^{2}=b^{2} \Leftrightarrow a=b$.

THEOREM 6. The triangle $A B C$ is equilateral if and only if $a^{2}+m_{a}^{2}=$ $=b^{2}+m_{b}^{2}=c^{2}+m_{c}^{2}$.

Proof. The statement of this theorem we obtain immediatly from the Theorem 5.

Theorem 7. The condition that the triangle $A B C$ is isosceles $(a=b)$ is not equivalent with $a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4}$.

Proof. If $a=b$ it is immediatly that $a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4}$.
Conversely we show that the condition $a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4}$ does not imply that $a=b$. We have the following sequence of equivalent equalities:

$$
\begin{aligned}
& a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4} \Leftrightarrow \\
& \Leftrightarrow a^{4}+\left[\frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}\right]^{2}=b^{4}+\left[\frac{2\left(a^{2}+c^{2}\right)-b^{2}}{4}\right]^{2} \Leftrightarrow \\
& \Leftrightarrow 16 a^{4}+\left[4\left(b^{2}+c^{2}\right)^{2}+a^{4}-4\left(b^{2}+c^{2}\right) a^{2}\right]> \\
& >16 b^{4}+\left[4\left(a^{2}+c^{2}\right)^{2}+b^{4}-4\left(a^{2}+c^{2}\right) b^{2}\right] \Leftrightarrow \\
& \Leftrightarrow 13\left(a^{4}-b^{4}\right)-12 c^{2}\left(a^{2}-b^{2}\right)=0 \Leftrightarrow\left(a^{2}-b^{2}\right)\left[13\left(a^{2}+b^{2}\right)-12 c^{2}\right]=0 .
\end{aligned}
$$

If we choose $a=3, b=4$ and $c=\sqrt{\frac{13}{12}} \cdot 5$ then there exists such a triangle $A B C$, because $a<b<c<a+b$. We can observe that for these values $13\left(a^{2}+b^{2}\right)-12 c^{2}=0$, so $a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4}$, but the triangle $A B C$ is not isosceles.

THEOREM 8. The triangle $A B C$ is equilateral if and only if $a^{4}+m_{a}^{4}=$ $=b^{4}+m_{b}^{4}=c^{4}+m_{c}^{4}$.

Proof. If $a=b=c$ it is immediatly that $a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4}=c^{4}+m_{c}^{4}$. We show conversely. In the proof of Theorem 7 we show that:

$$
a^{4}+m_{a}^{4}=b^{4}+m_{b}^{4} \Leftrightarrow\left(a^{2}-b^{2}\right)\left[13\left(a^{2}+b^{2}\right)-12 c^{2}\right]=0 .
$$

Analogously: $b^{4}+m_{b}^{4}=c^{4}+m_{c}^{4} \Leftrightarrow\left(b^{2}-c^{2}\right) \cdot\left[13\left(b^{2}+c^{2}\right)-12 a^{2}\right]=0$. We have the following discussion:

1. if $13\left(a^{2}+b^{2}\right)-12 c^{2}=0$ and $13\left(b^{2}+c^{2}\right)-12 a^{2}=0$ then the sum of these two relations is $a^{2}+26 b^{2}+c^{2}=0$, which is a contradiction;
2. if $13\left(a^{2}+b^{2}\right)-12 c^{2} \neq 0$ and $13\left(b^{2}+c^{2}\right)-12 a^{2}=0$ then $a=b$, so $13\left(b^{2}+c^{2}\right)-12 a^{2}=b^{2}+13 c^{2}=0$, which is a contradiction;
3. if $13\left(a^{2}+b^{2}\right)-12 c^{2}=0$ and $13\left(b^{2}+c^{2}\right)-12 a^{2} \neq 0$ then $b=c$, so $13\left(a^{2}+b^{2}\right)-12 c^{2}=13 a^{2}+b^{2}=0$, which is a contradiction;
4. if $13\left(a^{2}+b^{2}\right)-12 c^{2} \neq 0$ and $13\left(b^{2}+c^{2}\right)-12 a^{2} \neq 0$ then $a=b=c$.
5. We denote by $l_{a}, l_{b}$ and $l_{c}$ the length of interior bisectrices which correspond to the sides $a, b$ and $c$, respectively. We will study the following hypotheses: the triangle $A B C$ is isosceles $(a=b)$ if and only if $a^{\alpha}+l_{a}^{\alpha}=$ $=b^{\alpha}+l_{b}^{\alpha}$ for $\alpha \in \mathbf{R} \backslash\{0\}$, and the triangle $A B C$ is equilateral if and only if $a^{\alpha}+l_{a}^{\alpha}=b^{\alpha}+l_{b}^{\alpha}=c^{\alpha}+l_{c}^{\alpha}$ for $\alpha \in \mathbf{R} \backslash\{0\}$.

We consider here only the case $\alpha=1$, the other cases are open problems.
THEOREM 9. The condition that the triangle $A B C$ is isosceles $(a=b)$ is not equivalent with the relation $a+l_{a}=b+l_{b}$.

Proof. If we write the formulas $l_{a}=\frac{2 b c}{b+c} \cdot \cos \frac{A}{2}$, etc. then for example $a=b$ implies $\hat{A}=\hat{B}$ and $l_{a}=\frac{2 b c}{b+c} \cdot \cos \frac{A}{2}=\frac{2 a c}{a+c} \cdot \cos \frac{B}{2}=l_{b}$, i.e. a well-known fact from the elementary geometry, that in the isosceles triangle the interior bisectrices which correspond to the equal sides are equal. Consequently from $a=b$ results $a+l_{a}=b+l_{b}$.

We mention that $a>b$ implies $\hat{A}>\hat{B}$, so $\frac{2 b c}{b+c}<\frac{2 a c}{a+c}$ and $\cos \frac{A}{2}<\cos \frac{B}{2}$. Consequently $l_{a}<l_{b}$.

Conversely, we suppose that the condition $a+l_{a}=b+l_{b}$ does not imply that $A B C$ is isosceles.

We consider the set of the triangles $A B C$ for which $a>b>c$. Without loss of generality, we can suppose that $c=1$.

First we consider the triangle $A_{1} B_{1} C_{1}$ such that $\hat{A}=75^{\circ}, \hat{B}=60^{\circ}$, $\hat{C}=45^{\circ}, c_{1}=A_{1} B_{1}=1<b_{1}=A_{1} C_{1}=\frac{\sqrt{6}}{2}<a_{1}=B_{1} C_{1}=\frac{\sqrt{3}+1}{2}$ (see figure 1).

In [1] the author proved by elementary calculus that in the triangle $A_{1} B_{1} C_{1} a_{1}+l_{a_{1}}>b_{1}+l_{b_{1}}>c_{1}+l_{c_{1}}$.

If we try by computer, we obtain a simple program written in MATHCAD:

$$
\begin{array}{lll}
A:=75 \cdot \frac{\pi}{180} & B:=60 \cdot \frac{\pi}{180} & C:=\pi-(A+B) \\
a:=\frac{\sin (A)}{\sin (C)} & b:=\frac{\sin (B)}{\sin (C)} & c:=1
\end{array}
$$



Fig. 1

$$
\begin{aligned}
\alpha & :=\sqrt{\frac{(b+c-a)(b+c+a)}{4 \cdot b \cdot c}} \\
\gamma & :=\sqrt{\frac{(b+a-c)(b+a+c)}{4 \cdot a \cdot b}} \\
y & :=\frac{2 \cdot a \cdot c}{a+c} \cdot \beta
\end{aligned}
$$

$$
a=1.3660254038 \quad b=1.2247448714 \quad c=1
$$

$$
a+x=2.2395237066 \quad b+y=2.2247448714 \quad c+z=2.1932208719
$$

In the second part we consider the triangle $A_{2} B_{2} C_{2}$ so that $c_{2}=A_{2} B_{2}=1<$ $<b_{2}=A_{2} C_{2}<a_{2}=B_{2} C_{2}$ (see figure 2). The vertex $C_{2}$ is in the drawing region obtained by the intersection of the outside of the circle wich center $A_{2}$ and radius 1 , the half plane obtained by the mid perpendicular of the segment $A_{2} B_{2}$ which containes $A_{2}$ and the half-plain determined by the perpendicular in $A_{2}$ to the segment $A_{2} B_{2}$ which containes $B_{2}$. In [1] the author showed that if the vertex $C_{2}$ is "near" the mid perpendicular of the segment $A_{2} B_{2}$ and at sufficiently "long" distance from the segment $A_{2} B_{2}$ then in the triangle $A_{2} B_{2} C_{2}$ we obtain $b_{2}+l_{b_{2}}>a_{2}+l_{a_{2}}>c_{2}+l_{c_{2}}$. Indeed, if we choose for example $c_{2}=1<b_{2}=10^{n}<a_{2}=10^{n}+\varepsilon$, where $n \geq 1$ is a natural number, then for sufficiently small $\varepsilon>0$ we can realise the following sequence of inequalities: $b_{2}+l_{b_{2}}>a_{2}+l_{a_{2}}>c_{2}+l_{c_{2}}$.

The following program written in MATHCAD underlines this fact from the numerical point of view:

$$
\begin{aligned}
& A:=89.999 \cdot \frac{\pi}{180} \quad B:=89.00 \cdot \frac{\pi}{180} \quad C:=\pi-(A+B) \\
& a:=\frac{\sin (A)}{\sin (C)} \quad b:=\frac{\sin (B)}{\sin (C)} \quad c:=1 \\
& \alpha:=\sqrt{\frac{(b+c-a)(b+c+a)}{4 \cdot b \cdot c}} \quad \beta:=\sqrt{\frac{(a+c-b)(a+c+b)}{4 \cdot a \cdot c}} \\
& \gamma:=\sqrt{\frac{(b+a-c)(b+a+c)}{4 \cdot b \cdot a}} \quad x:=\frac{2 \cdot b \cdot c}{b+c} \cdot \alpha \\
& y:=\frac{2 \cdot a \cdot c}{a+c} \cdot \beta \\
& a=5208.707259674 \\
& a+x=5210.1212141178 \\
& c=1 \quad c+z=5209.707196058
\end{aligned}
$$

Now we consider the triangles $A_{1} B_{1} C_{1}$ and $A_{2} B_{2} C_{2}$ such that $A_{1} \equiv$ $\equiv A_{2}, B_{1} \equiv B_{2}$ and the vertex $C_{2}$ is at greater distance from $A_{1} B_{1}=A_{2} B_{2}$ then the vertex $C_{1}$. We consider at the same time a moving triangle $A^{\prime} B^{\prime} C^{\prime}$ such that $A^{\prime} \equiv A_{1} \equiv A_{2}, B^{\prime} \equiv B_{1} \equiv B_{2}$ and the vertex $C^{\prime}$ goes continuously on the segment $C_{1} C_{2}$ from $C_{1}$ to $C_{2}$. Every triangle $A^{\prime} B^{\prime} C^{\prime}$ has the property that $a^{\prime}=B^{\prime} C^{\prime}>b^{\prime}=A^{\prime} C^{\prime}>c^{\prime}=A^{\prime} B^{\prime} \equiv A_{1} B_{1} \equiv A_{2} B_{2}=1$. When the vertex $C^{\prime}$ describes the segment $C_{1} C_{2}$ continuously then the expresion $\left(a^{\prime}+l_{a^{\prime}}\right)-\left(b^{\prime}+l_{b^{\prime}}\right)$ changes continuously. But for $C=C_{1} a_{1}+l_{a_{1}}>b_{1}+l_{b_{1}}$ and for $C=C_{2} a_{2}+l_{a_{2}}<b_{2}+l_{b_{2}}$. So there exists a position $C_{0}$ of the vertex $C^{\prime}$ on the segment $C_{1} C_{2}$ such that for the triangle $A_{0} B_{0} C_{0}\left(A_{0} \equiv A_{1} \equiv\right.$ $\equiv A_{2}, B_{0} \equiv B_{1} \equiv B_{2}$ ) we have $a_{0}+l_{a_{0}}=b_{0}+l_{b_{0}}$

THEOREM 10. The condition that the triangle $A B C$ is equilateral is not equivalent with the relation $a+l_{a}=b+l_{b}=c+l_{c}$.

PROOF. If $A B C$ is an equilateral triangle then $a+l_{a}=b+l_{b}=c+l_{c}$.
Conversely, we suppose that the relations $a+l_{a}=b+l_{b}=c+l_{c}$ are true is a triangle $A B C$. The author does not know whether the triangle $A B C$ is isosceles or not, but we can prove it is sure that the triangle $A B C$ is not equilateral.

First we consider the rectangular isosceles triangle $A_{3} B_{3} C_{3}\left(\hat{C}_{3}=90^{\circ}\right)$ such that $c_{3}=A_{3} B_{3}=1<b_{3}=A_{3} C_{3}=a_{3}=B_{3} C_{3}=\frac{\sqrt{2}}{2}$. In this triangle by elementary calculus results that

$$
c_{3}+l_{c_{3}}=1+\frac{1}{2}>\frac{\sqrt{2}}{2}+\sqrt{2-\sqrt{2}}=b_{3}+l_{b_{3}}=a_{3}+l_{a_{3}} .
$$

Secondly we consider the isosceles triangle $A_{4} B_{4} C_{4}$ such that $c_{4}=$ $=A_{4} B_{4}=1<b_{4}=A_{4} C_{4}=a_{4}=B_{4} C_{4}=10$. In this triangle $a_{4}>l_{c_{4}}$ and by elementary calculus we get $l_{a_{4}}=\frac{\sqrt{210}}{11}>1=c_{4}$. Consequently in the triangle $A_{4} B_{4} C_{4} \quad a_{4}+l_{a_{4}}=b_{4}+l_{b_{4}}>c_{4}+l_{c_{4}}$. We consider the moving triangle $A^{\prime} B^{\prime} C^{\prime}$ such that $A^{\prime} \equiv A_{3}=A_{4}, B^{\prime}=B_{3} \equiv B_{4}$ and the vertex $C^{\prime}$ goes continuously on the segment $C_{3} C_{4}$ from $C_{3}$ to $C_{4}$. Every triangle $A^{\prime} B^{\prime} C^{\prime}$ is isosceles. When the vertex $C^{\prime}$ describes the segment $C_{3} C_{4}$ continuously then the expression $\left(a+l_{a}\right)-\left(c+l_{c}\right)$ changes continuously. But for $C=C_{3}$ $c_{3}+l_{c_{3}}>a_{3}+l_{a_{3}}$ and for $C=C_{4} c_{4}+l_{c_{4}}<a_{4}+l_{a_{4}}$. So there exists a position $C_{5}$ of the vertex $C^{\prime}$ on the segment $C_{3} C_{4}$ such for the isosceles triangle $A_{5} B_{5} C_{5}\left(a_{5}=B_{5} C_{5}=b_{5}=A_{5} C_{5}\right)\left(A_{5} \equiv A_{3} \equiv A_{4} ; B_{5} \equiv B_{3} \equiv B_{4}\right)$ we have $c_{5}+l_{c_{5}}=a_{5}+l_{a_{5}}=b_{5}+l_{b_{5}}$.
4. In the last part we show some similar results but the algebraic conditions we take in the multiplicativ form.

In every triangle $A B C$ we have $a \cdot h_{a}=b \cdot h_{b}=c \cdot h_{c}$.
In case of the medians we obtain:
Theorem 11. The condition that the triangle $A B C$ is isosceles $(a=b)$ is not equivalent with the relation $a \cdot m_{a}=b \cdot m_{b}$.

Proof. We have the following equivalent equalities: $a \cdot m_{a}=b \cdot m_{b} \Leftrightarrow$ $\Leftrightarrow a^{2} \cdot m_{a}^{2}=b^{2} \cdot m_{b}^{2} \Leftrightarrow a^{2} \cdot \frac{2\left(b^{2}+c^{2}\right)-a^{2}}{4}=b^{2} \cdot \frac{2\left(a^{2}+c^{2}\right)-b^{2}}{4} \Leftrightarrow 2 a^{2} c^{2}-a^{4}=$ $=2 b^{2} c^{2}-b^{4} \Leftrightarrow\left(a^{2}-b^{2}\right)\left(a^{2}+b^{2}-2 c^{2}\right)=0$. If we put $a=3, b=4$ and $c=\frac{5 \sqrt{2}}{2}$ then for this triangle $(a<b<c<a+b)$ we have $a^{2}+b^{2}-2 c^{2}=0$.

Theorem 12. The triangle $A B C$ is equilateral if and only if $a \cdot m_{a}=$ $=b \cdot m_{b}=c \cdot m_{c}$.

Proof. Using the proof of Theorem 11 we have $a \cdot m_{a}=b \cdot m_{b} \Leftrightarrow$ $\Leftrightarrow\left(a^{2}-b^{2}\right)\left(a^{2}+b^{2}-2 c^{2}\right)=0$. Analogously $b \cdot m_{b}=c \cdot m_{c} \Leftrightarrow$ $\Leftrightarrow\left(b^{2}-c^{2}\right)\left(b^{2}+c^{2}-2 a^{2}\right)=0$. We have the following possibilities:

1. if $a^{2}+b^{2}-2 c^{2} \neq 0$ and $b^{2}+c^{2}-2 a^{2} \neq 0$ then $a=b=c$;
2. if $a^{2}+b^{2}-2 c^{2} \neq 0$ and $b^{2}+c^{2}-2 a^{2}=0$, then $a=b$ and $b^{2}+c^{2}-2 a^{2}=0$, so $a=b=c$;
3. if $a^{2}+b^{2}-2 c^{2}=0$ and $b^{2}+c^{2}-2 a^{2} \neq 0$, then $a^{2}+b^{2}-2 c^{2}=0$ and $b=c$, so $a=b=c$;
4. if $a^{2}+b^{2}-2 c^{2}=0$ and $b^{2}+c^{2}-2 a^{2}=0$ by substruction we obtain $3 a^{2}-3 c^{2}=0$, i.e. $a=c$ and $a^{2}+b^{2}-2 c^{2}=0$, so $a=b=c$.

In the case of bisectrices we obtain:
THEOREM 13. The condition that the triangle $A B C$ is isosceles $(a=b)$ is not equivalent with the relation $a \cdot l_{a}=b \cdot l_{b}$.

Proof. We have the following equivalent equalities:
$a \cdot l_{a}=b \cdot l_{b} \Leftrightarrow a \cdot \frac{2 b c}{b+c} \cdot \cos \frac{A}{2}=b \cdot \frac{2 a c}{a+c} \cdot \cos \frac{B}{2} \Leftrightarrow$
$\Leftrightarrow(a+c) \cdot \cos \frac{A}{2}=(b+c) \cdot \cos \frac{B}{2} \Leftrightarrow$
(using the sinus theorem)
$\Leftrightarrow(2 R \cdot \sin A+2 R \cdot \sin C) \cdot \cos \frac{A}{2}=(2 R \cdot \sin B+2 R \cdot \sin C) \cdot \cos \frac{B}{2} \Leftrightarrow$ $\Leftrightarrow 2 \sin \frac{A+C}{2} \cdot \cos \frac{A-C}{2} \cdot \cos \frac{A}{2}=2 \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \cos \frac{B}{2} \Leftrightarrow$ $\Leftrightarrow \sin \frac{\pi-B}{2} \cdot \cos \frac{A-C}{2} \cdot \cos \frac{A}{2}=\sin \frac{\pi-A}{2} \cdot \cos \frac{B-C}{2} \cdot \cos \frac{B}{2} \Leftrightarrow$
$\Leftrightarrow \cos \frac{A-C}{2}=\cos \frac{B-C}{2} \Leftrightarrow \frac{A-C}{2}=\frac{B-C}{2}$
or $\frac{A-C}{2}=-\frac{B-C}{2} \Leftrightarrow A=B$ or $A+B=2 C \Leftrightarrow A=B$ or $C=60^{\circ}$. If we choose the triangle $A B C A=45^{\circ}, B=75^{\circ}$ and $C=60^{\circ}$ then for this triangle $a<c<b$ and $a \cdot l_{a}=b \cdot l_{b}$.

THEOREM 14. The triangle $A B C$ is equilateral if and only if $a \cdot l_{a}=$ $=b \cdot l_{b}=c \cdot l_{c}$.

PROOF. Using the proof of Theorem 13 we have $a \cdot l_{a}=b \cdot l_{b} \Leftrightarrow A=B$ or $C=60^{\circ}$. Analogously $b \cdot l_{b}=c \cdot l_{c} \Leftrightarrow B=C$ or $A=60^{\circ}$. We have the following possibilities:

1. if $A=B$ and $B=C$ then $a=b=c$;
2. if $A=B$ and $A=60^{\circ}$ then $A=B=60^{\circ}$, so $C=60^{\circ}$, too, i.e. $a=b=c$;
3. if $C=60^{\circ}$ and $B=C$ then $B=C=60^{\circ}$, so $A=60^{\circ}$, too, i.e. $a=b=c$;
4. if $C=60^{\circ}$ and $A=60^{\circ}$ then $B=60^{\circ}$, too, i.e. $a=b=c$.

## Reference

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# A GENERAL APPROACH TO STRONG LAWS OF LARGE NUMBERS FOR FIELDS OF RANDOM VARIABLES 

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## 1. Introduction and Notation

There are several methods to obtain almost sure (a.s.) convergence results for random fields (see e.g. [14], [9], [8], [5] and the literature cited there). The aim of our paper is to present a general approach to obtain strong laws of large numbers (SLLN) for random fields. Our method is an extension of the one given in [6]. In [6] only random sequences (i.e. not fields) were considered.

The paper is organized as follows. Section 2 contains the main result (Theorem 3). Once a maximal inequality is known, Theorem 3 easily implies an SLLN and it helps to obtain appropriate normalizing constants in the SLLN. The remaining sections contain applications. In Section 3 an SLLN is presented for logarithmically weighted sums. We remark that such kind of SLLN's are useful to prove almost sure central limit theorems (see e.g. [3]). In Section 4 the case of fields with superadditive moment structure is studied. In Section 5 a Brunk-Prokhorov type SLLN is presented. Section 6 is devoted to mixingales.

In the following $\mathbb{N}_{0}$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively. Let $d$ be a fixed positive integer. Throughout the paper $I, J, K, L, M$ and $N$ denote elements of $\mathbb{N}_{0}^{d}$ (in particular, elements of $\mathbb{N}^{d}$ ). If an element of $\mathbb{N}_{0}^{d}\left(\right.$ or $\left.\mathbb{N}^{d}\right)$ is denoted by a capital letter, then its coordinates are denoted by the lower case of the same letter, i.e. N always means the vector

[^1]$\left(n_{1}, \ldots, n_{d}\right) \in \mathbb{N}_{0}^{d}$. We also use $\mathbf{1}=(1, \ldots, 1) \in \mathbb{N}^{d}$ and $\mathbf{0}=(0, \ldots, 0) \in \mathbb{N}_{0}^{d}$. In $\mathbb{N}_{0}^{d}$ we consider the coordinate-wise partial ordering: $M \leq N$ means $m_{i} \leq n_{i}, i=1, \ldots, d(M<N$ means $M \leq N$ and $N \neq M) . \quad N \rightarrow \infty$ is interpreted as $n_{i} \rightarrow \infty, i=1, \ldots, d, \lim _{N} a_{N}$ is meant in this sense. In $\mathbb{N}_{0}^{d}$ the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $N=\left(n_{1}, \ldots, n_{d}\right) \in \mathbb{N}_{0}^{d}$ then $\langle N\rangle=\prod_{i=1}^{d} n_{i}$.

A numerical sequence $a_{N}, N \in \mathbb{N}_{0}^{d}$, is called $d$-sequence. If $a_{N}$ is a $d$-sequence then its difference sequence, i.e. the $d$-sequence $b_{N}$ for which $\sum_{M \leq N} b_{M}=a_{N}, N \in \mathbb{N}^{d}$, will be denoted by $\Delta a_{N}$.

We shall say that a $d$-sequence $a_{N}$ is of product type if $a_{N}=\prod_{i=1}^{d} a_{n_{i}}^{(i)}$, where $a_{n_{i}}^{(i)}\left(n_{i}=0,1,2, \ldots\right)$ is a (single) sequence for each $i=1, \ldots, d$. Our consideration will be confined to normalizing constants of product type: $b_{N}$ will always denote $b_{N}=\prod_{i=1}^{d} b_{n_{i}}^{(i)}$, where $b_{n_{i}}^{(i)}, n_{i}=0,1,2, \ldots$, is a nondecreasing sequence of positive numbers for each $i=1, \ldots, d$. In this case we shall say that $b_{N}$ is a positive nondecreasing $d$-sequence of product type. Moreover, if for each $i=1, \ldots, d$ the sequence $b_{n_{i}}^{(i)}$ is unbounded, then $b_{N}$ is called positive, nondecreasing, unbounded $d$-sequence of product type.

The random field will be denoted by $X_{N}, N \in \mathbb{N}_{0}^{d} . S_{N}$ is the partial sum: $S_{N}=\sum_{M \leq N} X_{M}$ for $N \in \mathbb{N}_{0}^{d}$. As $X_{N}$ is a field with lattice indices we shall say that $X_{N}, N \in \mathbb{N}_{0}^{d}$, is a $d$-sequence of random variables (r.v.'s). Remark that a sum or a maximum over the empty set will be interpreted as zero (i.e. $\sum_{N \in \mathscr{H}} X_{N}=\max _{N \in \mathscr{H}} X_{N}=0$ if $\mathscr{H}=\emptyset$ ). As usual, $\log ^{+}(x)=$ $=\max \{1, \log (x)\}, x>0$.

## 2. The Basic SLLN

The proposition and lemma below are useful for proving Theorem 3. Proposition 1 and its proof are straightforward generalizations of Theorem 1.1 and its proof in [6]. Note that there are several other ways to obtain maximal inequalities of this type: see for example [8].

Proposition 1. (Hájek-Rényi type maximal inequality.) Let $N \in \mathbb{N}^{d}$ be fixed. Let $r$ be a positive real number, $a_{N}$ be a nonnegative $d$-sequence.

Suppose that $b_{M}$ is a positive, nondecreasing $d$-sequence of product type. Then

$$
\mathbb{E}\left\{\max _{L \leq M}\left|S_{L}\right|^{r}\right\} \leq \sum_{L \leq M} a_{L} \quad \forall M \leq N
$$

implies

$$
\mathbb{E}\left\{\max _{M \leq N}\left|\frac{S_{M}}{b_{M}}\right|^{r}\right\} \leq 4^{d} \sum_{M \leq N} \frac{a_{M}}{b_{M}^{r}} .
$$

Proof. Without loss of generality we can assume that $b_{\mathbf{1}}=1$. Fix an $N \in \mathbb{N}^{d}$ and for a moment a real number $c>1$. For $I=\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{N}_{0}^{d}$ let us define the set

$$
\mathscr{A}_{I}=\left\{J \in \mathbb{N}^{d}: J \leq N \text { and } c^{i_{k}} \leq b_{j_{k}}^{(k)}<c^{i_{k}+1}, \quad k=1, \ldots, d\right\} .
$$

Now we can form

$$
D_{I}=\sum_{J \in \mathscr{A}_{I}} a_{J} \text { and } K=\max \left\{I: \mathscr{A}_{I} \neq \emptyset\right\},
$$

where $D_{I}$ as we mentioned above is considered to be zero if $\mathscr{A}_{I}=\emptyset$. Note that $K$ is well defined because of product form of $b_{N}$. It is easy to see that each nonempty $\mathscr{A}_{I}$ has a maximal element. Therefore if $\mathscr{A}_{I} \neq \emptyset$ let

$$
M_{I}=\max \left\{J: J \in \mathscr{A}_{I}\right\}
$$

otherwise set $M_{I}=\mathbf{0}$. Since $\bigcup_{I \leq K} \mathscr{A}_{I}$ covers the rectangle $\left\{M \in \mathbb{N}^{d}: M \leq\right.$ $\leq N\}$ so

$$
\mathbb{E}\left\{\max _{M \leq N}\left|\frac{S_{M}}{b_{M}}\right|^{r}\right\} \leq \sum_{J \leq K} \mathbb{E}\left\{\max _{I \in A_{J}}\left|\frac{S_{I}}{b_{I}}\right|^{r}\right\}
$$

By the definition of $\mathscr{A}_{I}, M_{I}$ and $D_{I}$ we get

$$
\begin{aligned}
& \sum_{J \leq K} \mathbb{E}\left\{\max _{I \in A_{J}}\left|\frac{S_{I}}{b_{I}}\right|^{r}\right\} \leq \sum_{J \leq K}\left\{\prod_{m=1}^{d} c^{-r j_{m}}\right\} \mathbb{E}\left\{\max _{I \in \mathcal{A}_{J}}\left|S_{I}\right|^{r}\right\} \leq \\
\leq & \sum_{J \leq K}\left\{\prod_{m=1}^{d} c^{-r j_{m}}\right\} \mathbb{E}\left\{\max _{I \leq M_{J}}\left|S_{I}\right|^{r}\right\} \leq \sum_{J \leq K}\left\{\prod_{m=1}^{d} c^{-r j_{m}}\right\} \sum_{I \leq M_{J}} a_{I} \leq \\
\leq & \sum_{J \leq K}\left\{\prod_{m=1}^{d} c^{-r j_{m}}\right\} \sum_{I \leq J} D_{I} \leq \sum_{I \leq K} D_{I} \sum_{I \leq J \leq K}\left\{\prod_{m=1}^{d} c^{-r j_{m}}\right\} \leq
\end{aligned}
$$

$$
\begin{aligned}
& \leq \sum_{I \leq K} D_{I} \prod_{m=1}^{d}\left\{\sum_{j=i_{m}}^{k_{m}} c^{-r j}\right\} \leq \sum_{I \leq K} D_{I} \prod_{m=1}^{d} \frac{c^{-r i_{m}}}{1-c^{-r}} \leq \\
& \leq\left\{\frac{c^{r}}{1-c^{-r}}\right\}^{d} \sum_{I \leq K} D_{I} \prod_{m=1}^{d} c^{-r\left(i_{m}+1\right)} \leq \\
& \leq\left\{\frac{c^{r}}{1-c^{-r}}\right\}^{d} \sum_{I \leq K}\left\{\sum_{J \in \mathscr{A}_{I}} a_{J}\right\} \prod_{m=1}^{d} c^{-r\left(i_{m}+1\right)} \leq \\
& \leq\left\{\frac{c^{r}}{1-c^{-r}}\right\}^{d} \sum_{I \leq K} \sum_{J \in \mathscr{A}_{I}} \frac{a_{J}}{b_{J}^{r}} \leq\left\{\frac{c^{r}}{1-c^{-r}}\right\}^{d} \sum_{J \leq N} \frac{a_{J}}{b_{J}^{r}}
\end{aligned}
$$

This proves the proposition because $\inf _{c>1} \frac{c^{r}}{1-c^{-r}}=4$.
LEMMA 2. Let $a_{N}$ be a nonnegative $d$-sequence and let $b_{N}$ be a positive, nondecreasing, unbounded $d$-sequence of product type. Suppose that $\sum_{N} \frac{a_{N}}{b_{N}^{r}}<+\infty$ with a fixed real $r>0$. Then there exists a positive, nondecreasing, unbounded $d$-sequence $\beta_{N}$ of product type for which

$$
\lim _{N} \frac{\beta_{N}}{b_{N}}=0 \quad \text { and } \quad \sum_{N} \frac{a_{N}}{\beta_{N}^{r}}<+\infty
$$

Proof. Clearly it is enough to prove for $r=1$. In case of $d=1$ one can find our proposition in [6, Lemma 2.2]. Let $d \geq 2$. Then

$$
+\infty>\sum_{N} \frac{a_{N}}{b_{N}}=\sum_{n_{1}} \frac{1}{b_{n_{1}}^{(1)}} \sum_{n_{2}, \ldots, n_{d}} \frac{a_{N}}{\prod_{m=2}^{d} b_{n_{m}}^{(m)}}=\sum_{n_{1}} \frac{1}{b_{n_{1}}^{(1)}} T_{n_{1}}
$$

with $T_{n_{1}}=\sum_{n_{2}, \ldots, n_{d}} \frac{a_{N}}{\prod_{m=2}^{d} b_{n_{m}}^{(m)}}$. Applying the above mentioned lemma of [6], we get that there exists an unbounded, positive, nondecreasing sequence $\beta_{n}^{(1)}$ so that

$$
\lim _{n} \frac{\beta_{n}^{(1)}}{b_{n}^{(1)}}=0 \quad \text { and } \quad \sum_{n_{1}} \frac{1}{\beta_{n_{1}}^{(1)}} T_{n_{1}}<+\infty
$$

If we have already obtained $\beta_{n}^{(m)}$ for $m=1, \ldots, k, k<d$ then replacing in the above procedure $b_{N}$ by $\prod_{m=1}^{k} \beta_{n_{m}}^{(m)} \prod_{m=k+1}^{d} b_{n_{m}}^{(m)}$ and coordinate 1 by
coordinate $k+1$ we get an appropriate $\beta_{n}^{(k+1)}$. Finally, by setting $\beta_{N}=$ $=\prod_{m=1}^{d} \beta_{n_{m}}^{(m)}$, it obviously satisfies the requirements.

The following theorem is an extension of Theorem 2.1 of [6].
THEOREM 3. Let $a_{N}, b_{N}$ be nonnegative $d$-sequences and let $r>0$. Suppose that $b_{N}$ is a positive, nondecreasing, unbounded $d$-sequence of product type. Then

$$
\sum_{N} \frac{a_{N}}{b_{N}^{r}}<+\infty \quad \text { and } \quad \mathbb{E}\left\{\max _{M \leq N}\left|S_{M}\right|^{r}\right\} \leq \sum_{M \leq N} a_{M} \quad \forall N \in \mathbb{N}^{d}
$$

imply

$$
\lim _{N} \frac{S_{N}}{b_{N}}=0 \quad \text { a.s. }
$$

Proof. Let $\beta_{N}$ be the sequence obtained in the previous lemma. According to Proposition 1:

$$
\mathbb{E}\left\{\max _{M \leq N}\left|\frac{S_{M}}{\beta_{M}}\right|^{r}\right\} \leq 4^{d} \sum_{M \leq N} \frac{a_{M}}{\beta_{M}^{r}} \quad \forall N \in \mathbb{N}^{d}
$$

Hence

$$
\mathbb{E}\left\{\sup _{n_{d}} \ldots \sup _{n_{1}}\left|\frac{S_{N}}{\beta_{N}}\right|^{r}\right\} \leq 4^{d} \sum_{N} \frac{a_{N}}{\beta_{N}^{r}}
$$

Since

$$
\sup _{n_{d}} \ldots \sup _{n_{1}}\left|\frac{S_{N}}{\beta_{N}}\right|^{r}=\sup _{N}\left|\frac{S_{N}}{\beta_{N}}\right|^{r}
$$

it follows from the foregoing that

$$
\sup _{N}\left|\frac{S_{N}}{\beta_{N}}\right|^{r}<+\infty \quad \text { a.s. }
$$

We have

$$
\left|\frac{S_{N}}{b_{N}}\right|=\frac{\beta_{N}}{b_{N}}\left|\frac{S_{N}}{\beta_{N}}\right| \leq \frac{\beta_{N}}{b_{N}} \sup _{K}\left|\frac{S_{K}}{\beta_{K}}\right| .
$$

This proves the theorem because $\lim _{N} \frac{\beta_{N}}{b_{N}}=0$.
DEFINITION 4. A function $g$ on $\mathbb{N}^{d} \times \mathbb{N}^{d}$ is said to be superadditive if $\left.g\left(I,\left(j_{1}, \ldots, j_{m-1}, k, j_{m+1}, \ldots, j_{d}\right)\right)+g\left(i_{1}, \ldots, i_{m-1}, k+1, i_{m+1}, \ldots, i_{d}\right), J\right)$
can be majorized by $g(I, J)$ for any $m=1, \ldots, d$ and for any $i_{m} \leq k<j_{m}$. A $d$-sequence of random variables is said to have $r$-th moment function of superadditive structure (MFSS) if

$$
\mathbb{E}\left\{\left|\sum_{I \leq K \leq J} X_{K}\right|^{r}\right\} \leq g(I, J)^{\alpha} \quad \forall I, J \in \mathbb{N}^{d}
$$

where $g$ is superadditive on $\mathbb{N}^{d} \times \mathbb{N}^{d}, r>0$ and $\alpha>1$. Remark that the notion of $r$-th MFSS was used by Móricz in [11].

REMARK 5. Maximal inequalities play important role in proving SLLN's. We shall frequently use the following result of Móricz (see [9, Corollary 1] or [12, Theorem 7]):

Suppose that $r \geq 1$ and $X_{N}$ has $r$-th MFSS. Then there is a constant $A_{r, \alpha, d}$ for which

$$
\mathbb{E}\left\{\max _{K \leq N}\left|S_{K}\right|^{r}\right\} \leq A_{r, \alpha, d} g(1, N)^{\alpha} \quad \forall N \in \mathbb{N}^{d}
$$

The reader can verify that the above proposition is true in the case of $0<r<1$, too.

## 3. Logarithmically Weighted Sums

Móri in [15, Theorem 1] proved that the sequence

$$
\frac{1}{\log ^{+} n} \sum_{k=1}^{n} \frac{X_{k}}{k}
$$

converges a.s. to zero under general assumptions. With their general method in [6] Fazekas and Klesov proved a special case of Móri's theorem. Now we extend this case to fields of random variables. Our method is a generalization that of [6]. Our Lemma 6 and Theorem 7 are extensions of Lemma 9.1 and Theorem 9.1 of [6], respectively. In the lemma below $[x], x \geq 0$ denotes the integer part of $x$, i.e. $[x]$ is the largest integer for which $[x] \leq x$.

LEMMA 6. (a) Let $n \in \mathbb{N}$ and $0<\beta<1$. Then there is a constant $C_{d, \beta}$ depending only on $d$ and $\beta$ such that:

$$
\sum_{m_{1}=1}^{n} \sum_{m_{2}=1}^{\left[\frac{n}{m_{1}}\right]} \cdots \sum_{m_{d}=1}^{\left[\frac{n}{m_{1} m_{2} \cdots m_{d-1}}\right]} \frac{1}{\langle\boldsymbol{M}\rangle^{1-\beta}} \leq C_{d, \beta} n^{\beta}\left(\log ^{+} n\right)^{d-1}
$$

(b) Let $0<\beta<1,1<\gamma<2, I, M, J \in \mathbb{N}^{d}, \quad I \leq M \leq J$. Then there is a constant $C_{d, \beta}$ depending only on $d$ and $\beta$ such that:

$$
\sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\\langle\boldsymbol{K}\rangle \leq\langle\boldsymbol{M}\rangle}} \frac{1}{\langle\boldsymbol{M}\rangle^{1+\beta}} \frac{1}{\left(\log ^{+}\langle\boldsymbol{M}\rangle\right)^{d-1}} \frac{1}{\langle\boldsymbol{K}\rangle^{1-\beta}} \leq C_{d, \beta}\left\{\sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle}\right\}^{\gamma}
$$

Proof. (a) The case $d=1$ is well known from elementary analysis. We prove by induction on $d$. Suppose that the statement is true for $d=f$. Let $n \in \mathbb{N}$ and $0<\beta<1$. Then

$$
\begin{aligned}
& \sum_{m_{1}=1}^{n} \sum_{m_{2}=1}^{\left[\frac{n}{m_{1}}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_{1} m_{2} \cdots m_{f}}\right]} \frac{1}{\langle M\rangle^{1-\beta}}= \\
= & \sum_{m_{1}=1}^{n} \frac{1}{m_{1}^{1-\beta}} \sum_{m_{2}=1}^{\left[\frac{n}{m_{1}}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_{1} m_{2} \cdots m_{f}}\right]} \frac{1}{\left(m_{2} \cdots m_{f}\right)^{1-\beta}}
\end{aligned}
$$

Now applying the hypothesis for $\left[\frac{n}{m_{1}}\right]$ we get that the above expression is majorized by:

$$
\begin{aligned}
& C_{f, \beta} \sum_{m_{1}=1}^{n} \frac{1}{m_{1}^{1-\beta}}\left[\frac{n}{m_{1}}\right]^{\beta}\left\{\log ^{+}\left[\frac{n}{m_{1}}\right]\right\}^{f-1} \leq C_{f, \beta} n^{\beta}\left(\log ^{+} n\right)^{f-1} \sum_{m_{1}=1}^{n} \frac{1}{m_{1}} \leq \\
& \leq C_{f, \beta} n^{\beta}\left(\log ^{+} n\right)^{f-1} C \log ^{+} n
\end{aligned}
$$

with certain $C>0$ (here we used the fact $\left[\frac{1}{c}\left[\frac{a}{b}\right]\right]=\left[\frac{a}{b c}\right]$ for $a, b, c \in \mathbb{N}$ ).
(b) In case $\sum_{I \leq M \leq J} \frac{1}{\langle M\rangle} \leq 1$ we get that

$$
\begin{gathered}
\sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\
\langle\boldsymbol{K}\rangle \leq\langle\boldsymbol{M}\rangle}} \frac{1}{\langle\boldsymbol{M}\rangle^{1+\beta}\left(\log ^{+}\langle\boldsymbol{M}\rangle\right)^{d-1}\langle\boldsymbol{K}\rangle^{1-\beta}} \leq \\
\leq \sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\
\langle\boldsymbol{K}\rangle \leq \boldsymbol{M}\rangle}} \frac{1}{\langle\boldsymbol{M}\rangle^{1+\beta}\langle\boldsymbol{K}\rangle^{1-\beta}} \leq \\
\leq \sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\
\langle\boldsymbol{K}\rangle \leq\langle\boldsymbol{M}\rangle}} \frac{1}{\langle\boldsymbol{M}\rangle\langle\boldsymbol{K}\rangle} \leq\left\{\sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle}\right\}^{2} \leq\left\{\sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle}\right\}^{\gamma} .
\end{gathered}
$$

In case $\sum_{I \leq M \leq J} \frac{1}{\langle M\rangle}>1$ using part (a) and the simple fact, that

$$
\sum_{m_{1}=1}^{n} \sum_{m_{2}=1}^{\left[\frac{n}{m_{1}}\right]} \cdots \sum_{m_{d}=1}^{\left[\frac{n}{m_{1} m_{2} \cdots m_{d-1}}\right]} \frac{1}{\langle\boldsymbol{M}\rangle^{1-\beta}}=\sum_{\substack{M \in \mathbb{N}^{d} \\\langle M\rangle \leq n}} \frac{1}{\langle\boldsymbol{M}\rangle^{1-\beta}}
$$

holds for all $n, d \in \mathbb{N}$, we get that

$$
\begin{gathered}
\sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle^{1+\beta}\left(\log ^{+}\langle\boldsymbol{M}\rangle\right)^{d-1}} \sum_{\substack{I \leq K \leq J \\
\langle K\rangle \leq\langle\boldsymbol{M}\rangle}} \frac{1}{\langle\boldsymbol{K}\rangle^{1-\beta}} \leq \\
\leq C_{d, \beta} \sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle^{1+\beta}\left(\log ^{+}\langle\boldsymbol{M}\rangle\right)^{d-1}}\langle\boldsymbol{M}\rangle^{\beta}\left(\log ^{+}\langle\boldsymbol{M}\rangle\right)^{d-1}= \\
=C_{d, \beta} \sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle} \leq C_{d, \beta}\left\{\sum_{I \leq M \leq J} \frac{1}{\langle\boldsymbol{M}\rangle}\right\}^{\gamma} .
\end{gathered}
$$

THEOREM 7. Let $X_{N}, N \in \mathbb{N}^{d}$, be a d-sequence of random variables and suppose that for some $C>0, \beta>0$

$$
\left|\mathbb{E}\left(X_{K} X_{L}\right)\right| \leq C\left\{\frac{\langle K\rangle}{\langle L\rangle}\right\}^{\beta} \frac{1}{\left(\log ^{+}\langle L\rangle\right)^{d-1}} \quad \text { if }\langle K\rangle \leq\langle L\rangle
$$

Then

$$
\lim _{N} \frac{1}{\prod_{i=1}^{d} \log ^{+} n_{i}} \sum_{K \leq N} \frac{X_{K}}{\langle K\rangle}=0 \text { a.s. }
$$

Proof. Clearly it is enough to prove for $0<\beta<1$. Let $I, J \in \mathbb{N}^{d}, I \leq J$. Using the assumptions we get:

$$
\begin{aligned}
& \mathbb{E}\left\{\left|\sum_{I \leq K \leq J} \frac{X_{K}}{\langle K\rangle}\right|^{2}\right\} \leq 2 \sum_{I \leq L \leq J} \sum_{\substack{I \leq K \leq J \\
\langle K\rangle \leq\langle L\rangle}} \frac{1}{\langle K\rangle\langle L\rangle}\left|\mathbb{E}\left(X_{K} X_{L}\right)\right| \leq \\
& \leq 2 C \sum_{I \leq L \leq J} \sum_{\substack{I \leq K \leq J \\
\langle K\rangle \leq\langle L\rangle}} \frac{1}{\langle K\rangle^{1-\beta}\langle L\rangle^{1+\beta}\left(\log ^{+}\langle L\rangle\right)^{d-1}} .
\end{aligned}
$$

Let $1<\gamma<2$. It follows from Lemma $6(b)$ that

$$
\mathbb{E}\left\{\left|\sum_{I \leq K \leq J} \frac{X_{K}}{\langle K\rangle}\right|^{2}\right\} \leq D_{d, \beta}\left\{\sum_{I \leq L \leq J} \frac{1}{\langle L\rangle}\right\}^{\gamma}
$$

where $D_{d, \beta}>0$ depends only on $d$ and $\beta$. Now, from Remark 5 we get that

$$
\mathbb{E}\left\{\max _{I \leq J}\left|\sum_{K \leq I} \frac{X_{K}}{\langle K\rangle}\right|^{2}\right\} \leq C_{d, \beta, \gamma}\left\{\sum_{K \leq J} \frac{1}{\langle K\rangle}\right\}^{\gamma} \quad \forall J
$$

where $C_{d, \beta, \gamma}>0$ depends only on $d, \beta$ and $\gamma$. From the Hölder inequality we have:

$$
\mathbb{E}\left\{\max _{I \leq J}\left|\sum_{K \leq I} \frac{X_{K}}{\langle K\rangle}\right|^{\frac{2}{\gamma}}\right\} \leq\left(C_{d, \beta, \gamma}\right)^{\frac{1}{\gamma}} \sum_{K \leq J} \frac{1}{\langle K\rangle} \quad \forall J
$$

Now we can apply Theorem 3 because

$$
\sum_{N} \frac{1}{\left(\prod_{m=1}^{d} \log n_{m}\right)^{\frac{2}{\gamma}}} \frac{1}{\langle N\rangle}<+\infty .
$$

Now we state some analogues of Theorem 7.
REMARK 8. Let $X_{N}$ be an orthogonal $d$-sequence of random variables, $r>0$ and $s>\frac{1+r}{2}$. Suppose that for some $C>0$

$$
\mathbb{E}\left(X_{K}^{2}\right) \leq C\langle K\rangle^{r}
$$

Then for any $\rho>1$

$$
\lim _{N} \frac{1}{\left\{\prod_{i=1}^{d} \log ^{+} n_{i}\right\}^{\rho}} \sum_{K \leq N} \frac{X_{K}}{\langle K\rangle^{s}}=0 \quad \text { a.s. }
$$

For the proof one can use the $d$-multiple version of the RademacherMenšov inequality [9, Corollary 3a].

REMARK 9. Let $0<r<1$ and $0<s \leq \frac{2}{3-r}$. Suppose that for some $C>0$

$$
\left|\mathbb{E}\left(X_{K} X_{L}\right)\right| \leq \frac{C\langle K\rangle^{s r}}{\langle L\rangle^{s}\left(\log ^{+}\langle L\rangle\right)^{d-1}} \quad \text { if }\langle K\rangle \leq\langle L\rangle
$$

Then

$$
\lim _{N} \frac{1}{\langle N\rangle^{1-s}} \sum_{K \leq N} \frac{X_{K}}{\langle K\rangle^{s}}=0 \text { a.s. }
$$

The proof is similar to that of Theorem 7.

## 4. Sequences with superadditive moment structure

In this section we prove a Marcinkiewicz-Zygmund type SLLN for $d$-sequences with superadditive moment structure. Our Proposition 11 is a generalization of Theorem 8.1 of [6]. For the sake of completness we start with a simple technical lemma on partial summation.

LEMMA 10. Let $a_{N}, b_{N}$ be nonnegative $d$-sequences such that $b_{N}=\frac{1}{\langle N\rangle^{\alpha}}$ for some $\alpha>0$. Then

$$
\sum_{N}(-1)^{d} \Lambda_{N} \Delta b_{N+1}<+\infty
$$

implies

$$
\sum_{N} a_{N} b_{N}<+\infty
$$

where $\Lambda_{N}=\sum_{M \leq N} a_{M}$.

Proposition 11. Let $r>0, \alpha>1$ and suppose that $X_{N}$ has $r$-th MFSS and $\Delta g(\mathbf{1}, N)^{\alpha}$ is nonnegative for any $N \in \mathbb{N}^{d}$. Then for arbitrary $q>0$

$$
\begin{equation*}
\sum_{N} \frac{g(\mathbf{1}, N)^{\alpha}}{\langle N\rangle^{1+\frac{r}{q}}}<+\infty \tag{I}
\end{equation*}
$$

implies

$$
\lim \frac{S_{N}}{\langle N\rangle^{\frac{1}{q}}}=0 \quad \text { a.s. }
$$

Proof. Using Remark 5 we get for all $N \in \mathbb{N}^{d}$ that:

$$
\mathbb{E}\left\{\max _{M \leq N}\left|S_{M}\right|^{r}\right\} \leq A_{r, \alpha, d} g(\mathbf{1}, N)^{\alpha}
$$

Let us introduce the notation $b_{N}=\frac{1}{\langle N\rangle^{\frac{T}{q}}}$. Since $\prod_{m=1}^{d}\left\{\frac{1}{\frac{\frac{\Gamma}{G}}{n_{m}}}-\frac{1}{\left(n_{m}+1\right)^{\frac{T}{q}}}\right\} \leq$ $\leq C \frac{1}{\langle N\rangle^{1+\frac{r}{q}}}$ for some $C>0$, so (I) implies

$$
\sum_{N}(-1)^{d} g(\mathbf{1}, N)^{\alpha} \Delta b_{N+1}<+\infty
$$

Finally, we apply Lemma 10 and Theorem 3 to obtain the result.

## 5. A Brunk-Prokhorov Type Theorem

Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let $X_{N}$ and $\mathscr{A}_{N}$ be a $d$-sequence of random variables and be a $d$-sequence of $\sigma$-subalgebras of $\mathscr{A}$, respectively. We shall say that the pair $\left(X_{N}, \mathcal{A}_{N}\right)$ has property $(e x)$ if (ex) $\left.\quad \mathbb{E}\left(\mathbb{E}\left(X_{L} \mid \mathcal{A}_{M}\right) \mid \mathscr{A}_{N}\right)\right)=\mathbb{E}\left(X_{L} \mid \mathscr{A}_{\min (M, N)}\right) \quad L, M, N \in \mathbb{N}^{d}$.
This property is widely used in the theory of multiindex martingales (see e.g. [5]). Let $X_{N}$ be a $d$-sequence of random variables and $\mathscr{A}_{N}$ a nondecreasing $d$-sequence of sub $\sigma$-algebras of $\mathscr{A}$. We say that $X_{N}$ is a martingale difference if

$$
\begin{aligned}
& X_{N} \text { is measurable with respect to } \mathscr{A}_{N}, \quad N \in \mathbb{N}^{d}, \\
& \mathbb{E}\left(X_{\mathbf{1}}\right)=0 \text { and } \mathbb{E}\left(X_{N} \mid \mathscr{A}_{M}\right)=0 \text { if } M<N .
\end{aligned}
$$

In this section we shall use the Doob and the Burkholder inequalities for $d$-sequences of random variables. For the sake of completeness we state and prove these inequalities in the lemma below.

LEMMA 12. (a) (Doob's $L^{p}$-inequality.) Let $p>1$. Then for any martingale $\left(X_{N}, \mathscr{A}_{N}\right)$ having property (ex) for arbitrary $N \in \mathbb{N}^{d}$

$$
\mathbb{E}\left\{\max _{M \leq N}\left|X_{M}\right|^{p}\right\} \leq\left\{\frac{p}{p-1}\right\}^{p d} \mathbb{E}\left(\left|X_{N}\right|^{p}\right)
$$

(b) (Burkholder's inequality) Let $p>1$. Then there is a constant $D_{p, d}$ such that for any martingale difference $X_{N}$ having property (ex)

$$
\mathbb{E}\left(\left|S_{N}\right|^{2 p}\right) \leq D_{p, d} \mathbb{E}\left(\left\{\sum_{M \leq N} X_{M}^{2}\right\}^{p}\right) \quad \forall N \in \mathbb{N}^{d}
$$

PROPOSITION 13. Let $X_{N}$ be a martingale difference having property (ex) and $p \geq 1$. Suppose that $\sum_{M \leq N} E\left(\left|X_{M}\right|^{2 p}\right) \leq C\langle N\rangle^{r}$ for some $C>0$ and $r<p+1$. Then $\lim _{N} \frac{S_{N}}{\langle N\rangle}=0$ a.s.

Proof. From Burkholder's inequality (Lemma 12(b)) and Hölder's inequality

$$
\begin{aligned}
& \mathbb{E}\left(\left|S_{N}\right|^{2 p}\right) \leq D_{2 p, 2} \mathbb{E}\left\{\left\{\sum_{M \leq N} X_{M}^{2}\right\}^{p}\right\} \leq \\
\leq & D_{2 p, 2}\langle N\rangle^{p-1} \sum_{M \leq N} \mathbb{E}\left(\left|X_{M}\right|^{2 p}\right) \leq D_{2 p, 2}\langle N\rangle^{p+r-1}
\end{aligned}
$$

Thus, by Doob's inequality (Lemma 12(a)),

$$
\mathbb{E}\left\{\max _{M \leq N}\left|S_{M}\right|^{2 p}\right\} \leq F_{2 p, 2} \sum_{M \leq N} \Delta\langle M\rangle^{p+r-1}
$$

for some constant $F_{2 p, 2}>0$. Now $\Delta\langle\boldsymbol{M}\rangle^{p+r-1} \leq C\langle\boldsymbol{M}\rangle^{p+r-2}$ and Theorem 3 implies the result.

PROPOSITION 14. Let $X_{N}$ be a martingale difference having property (ex) and let $p \geq 1$. Suppose that $\mathbb{E}\left(\left|X_{N}\right|^{2 p}\right)$ is $d$-sequence of product type. Then

$$
\sum_{N} \frac{\mathbb{E}\left(\left|X_{N}\right|^{2 p}\right)}{b_{N}^{2 p}}\langle N\rangle^{p-1}<+\infty
$$

implies $\lim _{N} \frac{S_{N}}{b_{N}}=0$ a.s., provided that $b_{N}$ is a nondecreasing, positive, unbounded $d$-sequence of product type and either $p=1$ or $\frac{\langle N\rangle^{\delta}}{b_{N}}$ is nonincreasing for some $\delta>\frac{p-1}{2 p}$.

Proof. Applying Lemma 12(b), Hölder's inequality and Lemma 12(a) we get

$$
\mathbb{E}\left\{\max _{M \leq N}\left|S_{M}\right|^{2 p}\right\} \leq C_{p, d}\langle N\rangle^{p-1} \sum_{M \leq N} \mathbb{E}\left(\left|X_{M}\right|^{2 p}\right)
$$

for some $C_{p, d}>0$. In case $p=1$ our main theorem and the above inequality imply the result. Let $p>1$. Introduce the notation $c_{N}=$ $=\langle N\rangle^{p-1} \sum_{M \leq N} \mathbb{E}\left(\left|X_{M}\right|^{2 p}\right)$. It is easy to see that

$$
\begin{aligned}
\Delta c_{N}= & \prod_{l=1}^{d}\left\{n_{l}^{p-1} \sum_{k=1}^{n_{l}} a_{k}^{(l)}-\left(n_{l}-1\right)^{p-1} \sum_{k=1}^{n_{l}-1} a_{k}^{(l)}\right\}= \\
= & \prod_{l=1}^{d}\left\{n_{l}^{p-1} a_{n_{l}}^{(l)}+\left\{n_{l}^{p-1}-\left(n_{l}-1\right)^{p-1}\right\} \sum_{k=1}^{n_{l}-1} a_{k}^{(l)}\right\} \leq \\
& \leq \prod_{l=1}^{d}\left\{n_{l}^{p-1} a_{n_{l}}^{(l)}+C n_{l}^{p-2} \sum_{k=1}^{n_{l}-1} a_{k}^{(l)}\right\}
\end{aligned}
$$

for some $C>0$, where $\prod_{l=1}^{d} a_{n_{l}}^{(l)}=\mathbb{E}\left(\left|X_{N}\right|^{2 p}\right)$. Using the assumptions we get

$$
\begin{aligned}
& \sum_{m=1}^{n} \frac{m^{p-2}}{b_{m}^{(l) 2 p}} \sum_{k=1}^{m-1} a_{k}^{(l)}=\sum_{k=1}^{n-1} a_{k}^{(l)} \sum_{m=k+1}^{n} \frac{m^{p-2}}{b_{m}^{(l) 2 p}} \leq \\
\leq & \sum_{k=1}^{n} a_{k}^{(l)} \sum_{m=k}^{\infty} \frac{m^{p-2}}{b_{m}^{(l) 2 p}}==\sum_{k=1}^{n} a_{k}^{(l)} \sum_{m=k}^{\infty} \frac{1}{m^{r}} \frac{m^{p+r-2}}{b_{m}^{(l) 2 p}} \leq \\
\leq & \sum_{k=1}^{n} a_{k}^{(l)} \frac{k^{p+r-2}}{b_{k}^{(l) 2 p}} \sum_{m=k}^{\infty} \frac{1}{m^{r}} \leq \sum_{k=1}^{n} a_{k}^{(l)} \frac{k^{p+r-2}}{b_{k}^{(l) 2 p}} C k^{1-r}
\end{aligned}
$$

for some $r>1, C_{r}>0$ and for each $1 \leq l \leq d$. This means that $\sum_{N} \Delta \frac{c_{N}}{b_{N}^{2 p}}<$ $<+\infty$, hence one can apply Theorem 3.

We remark that a similar proposition can be proved in a similar manner for $d$-sequences having maximal coefficient of correlation strictly smaller than 1. For this, one can use [14, Lemma 4] instead of Burkholder's inequality.

## 6. Mixingales

In this chapter we define multiindex $L^{r}$ mixingales and prove an SLLN for a special class of such random variables. Remark that the notion of $L^{r}$ mixingales was introduced by McLeish [10] and Andrews [1]. Let $\mathbb{Z}$ denote the set of integers and let
$\mathscr{E}_{N}=\left\{M \in \mathbb{Z}^{d}: 0 \leq n_{k}-m_{k} \leq 1, k=1, \ldots, d\right.$ and $\sum_{k=1}^{d}\left(n_{k}-m_{k}\right)$ is even $\}$, $\mathcal{O}_{N}=\left\{M \in \mathbb{Z}^{d}: 0 \leq n_{k}-m_{k} \leq 1, k=1, \ldots, d\right.$ and $\sum_{k=1}^{d}\left(n_{k}-m_{k}\right)$ is odd $\}$, if $N \in \mathbb{Z}^{d}$.

DEFINITION 15. Let $r \geq 1,(\Omega, \mathcal{A}, P)$ be a probability space, $X_{N}$ be a $d$-sequence of random variables with finite $r$-th moment, $\mathscr{A}_{N}\left(N \in \mathbb{Z}^{d}\right)$ be a nondecreasing $d$-sequence of $\sigma$-subalgebras of $\mathscr{A}$. The pair $\left(X_{N}, \mathscr{A}_{M}\right)$ $\left(N \in \mathbb{N}^{d}, M \in \mathbb{Z}^{d}\right)$ is called $L^{r}$-mixingale if
(a) $\left\|\mathbb{E}\left(X_{N} \mid \mathscr{A}_{N-M}\right)\right\|_{r} \leq c_{N} \Psi_{-M}$ if $m_{i} \geq 0$ for some $i=1, \ldots, d$,

$$
\begin{equation*}
\left\|X_{N}-\mathbb{E}\left(X_{N} \mid \mathscr{A}_{N+M}\right)\right\|_{r} \leq c_{N} \Psi_{M} \quad \text { if } \quad M \geq \mathbf{0} \tag{b}
\end{equation*}
$$

where $c_{N}\left(N \in \mathbb{N}^{d}\right), \Psi_{N}\left(N \in \mathbb{Z}^{d}\right)$ are $d$-sequences with $\Psi_{N} \rightarrow 0$ as $n_{i} \rightarrow-\infty$ for some $i=1, \ldots, d, \Psi_{N} \rightarrow 0$ as $n_{i} \rightarrow \infty$ for each $i=1, \ldots, d$, and there is a constant $C>0$ for which

$$
\Psi_{M} \leq C \Psi_{N}
$$

for any $N \in \mathbb{Z}^{d}$ and $M \in \mathscr{E}_{N} \cup \mathcal{O}_{N}$.
The following lemma is a straightforward generalization of Lemma 1 and Lemma 2 of [7].

LEMMA 16. (a) Let $r \geq 2$ and $\left(X_{N}, \mathcal{A}_{M}\right) \quad\left(N \in \mathbb{N}^{d}, M \in \mathbb{Z}^{d}\right)$ be an $L^{r}$ mixingale, having property (ex). Then there exists an $F_{r, d}>0$ such that

$$
\left\|\max _{M \leq N}\left|S_{M}\right|\right\|_{r} \leq F_{r, d} \sum_{K \in \mathbb{Z}^{d}}\left\{\sum_{M \leq N}\left\|X_{M}^{(K)}\right\|_{r}^{2}\right\}^{\frac{1}{2}}
$$

where $X_{M}^{(K)}=\Delta \mathbb{E}\left(X_{M} \mid \mathscr{A}_{M-K}\right)$ and here the difference is taken according to the subscript of $\mathscr{A}$ while the subscript of $X$ remains fixed.
(b) Let $r \geq 2$ and $\left(X_{N}, \mathcal{A}_{M}\right) \quad\left(N \in \mathbb{N}^{d}, M \in \mathbb{Z}^{d}\right)$ be an $L^{r}$ mixingale, having property (ex) such that $\sum_{K \in \mathbb{Z}^{d}} \Psi_{K}<+\infty$. Then

$$
\left\|\max _{M \leq N}\left|S_{M}\right|\right\|_{r} \leq C_{r, d}\left\{\sum_{M \leq N} c_{M}^{2}\right\}^{\frac{1}{2}}
$$

for some $C_{r, d}$.
Proof. (a) Let $N, K \in \mathbb{N}^{d}$. Then

$$
\begin{gathered}
\sum_{-K \leq M \leq K} X_{N}^{(M)}=\sum_{-K \leq M \leq K} \Delta \mathbb{E}\left(X_{N} \mid \mathscr{A}_{N-M}\right)= \\
=\mathbb{E}\left(X_{N} \mid \mathscr{A}_{N+K}\right)+\sum_{L \in \mathscr{L}_{K}^{+}} \mathbb{E}\left(X_{N} \mid \mathscr{A}_{N+L}\right)+(-1) \sum_{L \in \mathscr{L}_{K}^{-}} \mathbb{E}\left(X_{N} \mid \mathscr{A}_{N+L}\right),
\end{gathered}
$$

where

$$
\begin{gathered}
\mathscr{L}_{K}^{+}=\left\{L \in \mathbb{Z}^{d}: l_{i}=k_{i} \text { if } i \notin I \text { and } l_{i}=-\left(k_{i}+1\right) \text { if } i \in I,\right. \\
\text { for some } I \subset\{1, \ldots, d\}, \text { with } I \neq \emptyset \text { and } \operatorname{card}(I) \text { is even }\}, \\
\mathscr{L}_{K}^{-}=\left\{L \in \mathbb{Z}^{d}: l_{i}=k_{i} \text { if } i \notin I \text { and } l_{i}=-\left(k_{i}+1\right) \text { if } i \in I,\right. \\
\text { for some } I \subset\{1, \ldots, d\}, \text { with } \operatorname{card}(I) \text { is odd }\} .
\end{gathered}
$$

By the definition of the $L^{r}$-mixingale, one can see that

$$
\lim _{K}\left\{\sum_{-K \leq M \leq K} X_{N}^{(M)}-\mathbb{E}\left(X_{N} \mid \mathscr{A}_{N+K}\right)\right\}=\mathbf{0} \quad \text { in } L^{r}
$$

and so

$$
\lim _{K}\left\{\sum_{-K \leq M \leq K} X_{N}^{(M)}-X_{N}\right\}=\mathbf{0} \quad \text { in } L^{r}
$$

Hence, using the triangle inequality in $L^{r}$, we get

$$
\left\|\max _{M \leq N}\left|S_{M}\right|\right\|_{r}=\left\|\max _{M \leq N}\left|\sum_{L \leq M} \sum_{K \in Z^{d}} X_{L}^{(K)}\right|\right\| \|_{r} \leq
$$

$$
\leq\left\|\max _{M \leq N} \sum_{K \in Z^{d}}\left|\sum_{L \leq M} X_{L}^{(K)}\right|\right\|_{r} \leq\left.\sum_{K \in Z^{d}}\left\|\max _{M \leq N}\left|\sum_{L \leq M} X_{L}^{(K)}\right|\right\|\right|_{r}=(I)
$$

Let $K \in \mathbb{N}^{d}$ be fixed. With the help of property (ex) it is easy to check that the pair $\left(Z_{M}, \mathscr{F}_{M}\right)$ is martingale difference, where

$$
Z_{M}=X_{M}^{(K)} \quad \text { and } \quad \mathscr{F}_{M}=\mathscr{A}_{M-K}
$$

Hence by Lemma 12 (a), (b) and by the triangle inequality in the space $L^{\frac{r}{2}}$, we have

$$
\begin{aligned}
& (I) \leq D_{r, d} \sum_{K \in Z^{d}}\left\|\sum_{L \leq N} X_{L}^{(K)}\right\|_{r} \leq F_{r, d} \sum_{K \in Z^{d}}\left\|\left\{\sum_{L \leq N}\left|X_{L}^{(K)}\right|^{2}\right\}^{\frac{1}{2}}\right\|_{r}= \\
= & F_{r, d} \sum_{K \in Z^{d}}\left\|\left\{\sum_{L \leq N}\left|X_{L}^{(K)}\right|^{2}\right\}\right\|_{r}^{\frac{1}{2}} \leq F_{r, d} \sum_{K \in Z^{d}}\left\{\sum_{L \leq N}\left\|\left|X_{L}^{(K)}\right|^{2}\right\|_{\frac{r}{2}}\right\}^{\frac{1}{2}} .
\end{aligned}
$$

(b) Let us consider $X_{L}^{(K)}$. If $k_{m} \geq 0$ for some $m=1, \ldots, d$ then

$$
\left\|\Delta \mathbb{E}\left(X_{L} \mid \mathcal{A}_{L-K}\right)\right\|_{r} \leq c_{L} 2^{d} C \Psi_{-K}
$$

Otherwise, if $k_{m} \leq-1$ for each $m=1, \ldots, d$, then by Definition 15 ,

$$
\left\|\Delta \mathbb{E}\left(X_{L} \mid \mathscr{A}_{L-K}\right)\right\|_{r} \leq \sum_{M \in \mathscr{E}_{L-K} \cup \mathscr{O}_{L-K}}\left\|X_{L}-\mathbb{E}\left(X_{L} \mid \mathcal{A}_{M}\right)\right\|_{r} \leq c_{L} 2^{d} C \Psi_{-K}
$$

Hence, by part (a),

$$
\begin{gathered}
\left\|\max _{M \leq N} \mid S_{M}\right\|_{\|_{r}} \leq F_{r, d} \sum_{K \in Z^{d}}\left\{\sum_{L \leq N} c_{L}^{2} 2^{2 d} C^{2} \Psi_{-K}^{2}\right\}^{\frac{1}{2}}= \\
=F_{r, d} 2^{d} C\left\{\sum_{K \in Z^{d}} \Psi_{K}\right\}\left\{\sum_{L \leq N} c_{L}^{2}\right\}^{\frac{1}{2}}
\end{gathered}
$$

PROPOSITION 17. Let $r \geq 2$ and $\left(X_{N}, \mathcal{A}_{M}\right)\left(N \in \mathbb{N}^{d}, M \in \mathbb{Z}^{d}\right)$ be an $L^{r}$ mixingale of property (ex). Then

$$
\sum_{N \in \mathbb{Z}^{d}} \Psi_{N}<\infty \quad \text { and } \quad \sum_{N \in \mathbb{N}^{d}} \frac{1}{\langle N\rangle^{1+\frac{r}{q}}}\left\{\sum_{M \leq N} c_{M}^{2}\right\}^{\frac{r}{2}}<\infty
$$

imply

$$
\lim _{N} \frac{S_{N}}{\langle N\rangle^{\frac{1}{q}}}=0 \quad \text { a.s. }
$$

provided that the $d$-sequence $c_{N}$ is of product type.
Proof. Easy consequence of Proposition 11 and Lemma 16(b).

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# ÜBER DIE DICKE VON $\langle p, q\rangle$-PUNKTSYSTEMEN IN DER EUKLIDISCHEN EBENE 

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## 1. Einleitung

Es seien $p, q \geq 1$ ganze Zahlen. Die Punktmenge $\Sigma$ bildet ein $\langle p, q\rangle$ Punktsystem in der euklidischen Ebene $\mathbf{E}^{2}$, wenn solche positive reelle Zahlen $r$ und $R$ existieren, für die die folgenden zwei Bedingungen erfüllt sind.
1.1. Jeder offene Kreis vom Radius $r$ in $\mathbf{E}^{2}$ enthält höchstens $p$ Punkte von $\Sigma$.
1.2. Mindestens $q$ Punkte von $\Sigma$ gehören zu einem beliebigen abgeschlossenen Kreis vom Radius $R$.

Bildet $\Sigma$ ein $\langle p, q\rangle$-Punktsystem, dann verwenden wir die Bezeichnung $\Sigma(p, q)$.

Wir betrachten alle mögliche Zahlen $r$ und $R$, für die das angegebene Punktsystem $\Sigma$ ein $\langle p, q\rangle$-Punktsystem ist. Es seien $r_{p}=\sup r$ und $R_{q}=\inf R$ solche reelle Zahlen, für die $\Sigma$ noch ein $\langle p, q\rangle$-Punktsystem ist. Der Quotient $\frac{r_{p}}{R_{q}}$ wird als die $\langle p, q\rangle$-Dicke von $\Sigma$ genannt.

Die Aufgabe ist, das Supremum der $\langle p, q\rangle$-Dicken zu bestimmen und die extremalen $\langle p, q\rangle$-Punktsysteme anzugeben. Es sei

$$
\kappa(p, q)=\sup _{\Sigma \in \Sigma(p, q)} \frac{r_{p}}{R_{q}}
$$

Es bezeichne $\Gamma$ ein $\langle p, q\rangle$-Punktgitter und $\kappa_{\Gamma}(p, q)$ die extremale Dicke für $\langle p, q\rangle$-Punktgitter.

HorvÁth [3], [4] hat die Definition der $\langle p, q\rangle$-Punktsysteme und der $\langle p, q\rangle$-Dicke in Räumen konstanter Krümmung und in der Minkowskischen Ebene angegeben.

Im Fall $p=q=1$ erhalten wir die von Delone [1] definierten $(r, R)$ Punktsysteme. RYSKOV [8], [9] beschäftigte sich mit der Bestimmung der Dichten von $(r, R)$-Punktsystemen. In der euklidischen Ebene findet man Ergebnisse für $p, q \geq 1$, vielleicht in einer anderen Formulierung, in [2], [3, 4], [9], [5], [7], [12].

In dieser Arbeit untersuchen wir $\langle p, q\rangle$-Punktgitter. Wir geben eine Methode zur Bestimmung der extremalen $\langle p, q\rangle$-Punktgitter (H. TEMESVÁRI) und lösen das obene Problem für $p=5$ und $q=1,2,3,4$ (H. TEMESVÁRI); $p=1,2,3,4$ und $q=5$ (VÉGH). In den Fällen $p=1, q \in \mathbf{Z}^{+}$und $p=3$, $q \in \mathbf{Z}^{+}$(VÉGH) wird der Kreis der möglichen extremalen Gitter bedeutend reduziert.

## 2. Bezeichnungen

2.1. Es sei $O$ ein belibiger Punkt der Ebene $\mathbf{E}^{2}$. Mit $A$ bezeichnen wir den Ortsvektor $\overrightarrow{\mathrm{OA}}$ und seinen Endpunkt. Es seien $A$ und $B$ linear unabhängige Vektoren die Basisvektoren des Gitters $\Gamma$ (Abb. 1a), d.h. $\Gamma=$ $=\{X \mid X=m A+n B, m, n \in \mathbf{Z}\}$. Im folgenden werden wir immer nach Minkowski reduzierte Basen für die Gitter angeben. Dann gelten

$$
\begin{equation*}
|A| \leq|B| \leq|B-A| \quad \text { und } \quad \angle(A O B) \leq \frac{\pi}{2} \tag{1}
\end{equation*}
$$

Mit den Bezeichnungen $x=\frac{|A|}{|B|}, \alpha=\angle(A O B), y=\cos \alpha$ haben wir die folgenden äquivalenten Ungleichungen:

$$
\begin{equation*}
0<x \leq 1 \quad \text { und } \quad 0 \leq y \leq \frac{x}{2} \tag{2}
\end{equation*}
$$

Zu jedem nach MinKowski reduzierten Gitter können wir ein geordnetes Zahlenpaar $(x, y)$ mit den Bedingungen (2) zuordnen. Und umgekehrt, zu jedem Zahlenpaar $(x, y) \neq(0,0)$ gehört ein nach Minkowski reduziertes Gitter bis auf Ähnlichkeit. Wir betrachten das kartesische Koordinatensystem $x, y$ mit dem Anfangspunkt $\bar{O}$. Es sei $P(1,0), Q\left(1, \frac{1}{2}\right)$ (Abb. 1b). Auf Grund der Vorhergehenden existiert eine eineindeutige Zuordnung zwischen den Punkten des Dreiecks $\bar{O} P Q$ und den nach MiNKOWSKI reduzierten Gittern


Abb. 1/a


Abb. 1/b
bis auf Ähnlichkeit. Es ist offenbar, dass die $\langle p, q\rangle$-Dicke für ähnliche Gitter gleich ist.

Mit $\Delta_{i}$ bezeichnen wir das durch die nicht kollinearen Gitterpunkte $X_{i}, Y_{i}, Z_{i}$ bestimmte Gitterdreieck. Es sei $k\left(\Delta_{i}\right)$ der Umkreis von $\Delta_{i}$ und $\partial k\left(\Delta_{i}\right)$ die entsprechende Kreislinie (Abb. 2a). Der Kreis mit Durchmesser $S_{i} T_{i}$ wird mit $k\left(S_{i} T_{i}\right)$ und die entsprechende Kreislinie mit $\partial k\left(S_{i} T_{i}\right)$ (Abb. 2b) bezeichnet, wobei $S_{i}, T_{i} \in \Gamma$.


Abb. 2/a


Abb. $2 / b$

Die Gitterdreiecke $\Delta_{i}$ und $\Delta_{j}$ des Gitters $\Gamma$ sind äquivalent, wenn eine Verschiebung oder eine Spiegelung an einem Gitterpunkt oder die Aufeinanderfolge dieser Abbildungen das eine Gitterdreieck in das andere überführt. Auf ähnliche Weise kann man die Äquivalenz der Gitterstrecken $S_{i} T_{i}$ und $S_{k} T_{k}$ definieren.

Der Gitterkreis $k\left(\Delta_{i}\right)$ ist von der p-Eigenschaft, wenn
2.1.1. das Gitterdreieck $\Delta_{i}$ nicht stumpfwinklig ist;
2.1.2. $k\left(\Delta_{i}\right)$ mindestens $p+1$, der offene Kreis höchstens $p-2$ Gitterpunkte enthält.

Der Gitterkreis $k\left(S_{i} T_{i}\right)$ ist von der p-Eigenschaft, wenn
2.1.3. $k\left(S_{i} T_{i}\right)$ mindestens $p+1$, der offene Kreis höchstens $p-1$ Gitterpunkte enthält.

Es bezeichne $k_{i}^{p}\left(\Delta_{i}\right)$ bzw. $k_{i}^{p}\left(S_{i} T_{i}\right)$ den Kreis $k\left(\Delta_{i}\right)$ bzw. $k\left(S_{i} T_{i}\right)$ von der $p$-Eigenschaft und $r_{i}^{p}$ den Radius dieser Kreise.

Der Gitterkreis $\tilde{k}\left(\Delta_{j}\right)$ ist von der $q$-Eigenschaft, wenn
2.1.3. das Gitterdreieck $\Delta_{j}$ nicht stumpfwinklig ist;
2.1.4. $\tilde{k}\left(\Delta_{j}\right)$ mindestens $q+2$, der offene Kreis höchstens $q-1$ Gitterpunkte enthält.

Es bezeichne $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ den Kreis $\tilde{k}\left(\Delta_{i}\right)$ von der $q$-Eigenschaft und $\tilde{r}_{j}^{q}$ den Radius dieses Kreises.

Mit $\mathbf{L}(\Gamma, \rho)$ bezeichnen wir die Anordnung von offenen Kreisen mit Radius $\rho$, wobei $\Gamma$ das Gitter der Kreismittelpunkte sind. Es sei $\overline{\mathbf{L}}(\Gamma, \rho)$ die entsprechende Anordnung von den abgeschlossenen Kreisen.

Die Kreisanordnung $\mathbf{L}(\Gamma, \rho)$ ist eine $p$-fache Packung, wenn ein beliebiger Punkt der Ebene durch die Kreise von $\mathbf{L}(\Gamma, \rho)$ höchstens $p$-fach überdeckt ist. Die Kreisanordnung $\overline{\mathbf{L}}(\Gamma, \rho)$ ist eine $q$-fache Überdeckung, wenn ein beliebiger Punkt der Ebene mit den Kreisen von $\overline{\mathbf{L}}(\Gamma, \rho)$ höchstens $q$-fach überdeckt ist.

## 3. Die Methode zur Bestimmung von $\kappa_{\Gamma}(p, q)$ und der extremalen $\langle p, q\rangle$-Punktgitter in $\mathbf{E}^{2}$

3.1. Es sei $\Gamma$ ein beliebiges $\langle p, q\rangle$-Punktgitter und seine Basis $A, B$ nach MINKOWSKI reduziert. Wir schlagen offene Kreise vom Radius $r$ um die Gitterpunkte.

Aus 1.1 folgt, dass die Kreisanordnung $\mathbf{L}(\Gamma, r)$ eine $p$-fache Kreispackung ist. Wir betrachten einen beliebigen Gitterkreis von der $p$-Eigenschaft, d.h. den Kreis $k_{i}^{p}\left(\Delta_{i}\right)$ bzw. $k_{i}^{p}\left(S_{i} T_{i}\right)$. Für den Kreisradius $r_{i}^{p}$ gilt $r \leq r_{p} \leq r_{i}^{p}$ (vgl. [10]). Dann ist $r_{p}=\inf r_{i}^{p}$. Die Kreisanordnung $\mathbf{L}\left(\Gamma, r_{p}\right)$ ist eine
dichteste $p$-fache Kreipackung für das fixe Gitter $\Gamma$. Die Existenz von $\mathbf{L}\left(\Gamma, r_{p}\right)$ ist in [10] bewiesen.

Wir betrachten die Kreisanordnung $\overline{\mathbf{L}}(\Gamma, R)$. Aus 1.2. ergibt sich, dass $\overline{\mathbf{L}}(\Gamma, R)$ eine $q$-fache Überdeckung ist. Es sei $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ ein beliebiger Gitterkreis von $\Gamma$ mit der $q$-Eigenschaft. Es gilt $R \geq R_{q} \geq \tilde{r}_{j}^{q}$ (vgl. [6]). Dann ist $R_{q}=$ $=\sup \tilde{r}_{j}^{q}$. Die Kreisanordnung $\overline{\mathbf{L}}\left(\Gamma, R_{q}\right)$ ist eine dünnste $q$-fache Überdeckung für das fixes Gitter $\Gamma$. Die Existenz von $\overline{\mathbf{L}}\left(\Gamma, R_{q}\right)$ ist in [6] bewiesen.

Es sei $G_{i j}^{p q}=\frac{r_{i}^{p}}{\tilde{r}_{j}^{q}}$. Für die Dicke von $\Gamma$ gilt $\frac{r_{p}}{R_{q}}=G_{i j}^{p q}$. Es ist zu beweisen, dass $G_{i j}^{p q}$ nur von $x$ und $y$ hängt. Nach [10] existiert eine Zerlegung von $\bar{O} P Q$ in endlich vielen Bereiche $H_{i}^{p}(1 \leq i \leq t)$, dass derselbe offene Gitterkreis $k_{i}^{p}\left(\Delta_{i}\right)$ bzw. $k_{i}^{p}\left(S_{i} T_{i}\right)$ für jedes $(x, y) \in H_{i}^{p}$ die $p$-Eigenschaft hat. Es gibt auch eine Zerlegung (vgl. [6]) von $\bar{O} P Q$ in endlich viele Bereiche $\tilde{H}_{j}^{q}(1 \leq j \leq s)$, dass derselbe abgeschlossene Gitterkreis $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ für jedes $(x, y) \in \tilde{H}_{j}^{q}$ die $q$-Eigenschaft hat. Nach [10] und [6] existieren endlich viele Gittertypen nach den Gitterkreisen $k_{i}^{p}\left(\Delta_{i}\right)$ bzw. $k_{i}^{p}\left(S_{i} T_{i}\right)$ mit Radius $r_{p}$ und $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ mit Radius $R_{q}$.

Auf Grund der Vorhergehenden ist die Zerlegung von $\bar{O} P Q$ in Bereiche $H_{i j}^{p q}=H_{i}^{p} \bigcap \tilde{H}_{j}^{q},(1 \leq i \leq t, 1 \leq j \leq s)$ vom Gesichtspunkt der Dicke interressant. Die Anzahl der Funktionen $(x, y) \mapsto G_{i j}^{p q},(x, y) \in H_{i j}^{p q}(1 \leq i \leq t$, $1 \leq j \leq s)$ sind endlich, deshalb müssen wir die Maxima von endlich vielen Funktionen mit zweien Veränderlichen bestimmen und dann das Extremum auswählen.

In den konkreten Fällen konnte man mit geometrischer Methode (vgl. [7]), mit Anwendung von Gittertransformationen, das Problem zur Bestimmung der Extrema von endlich vielen Funktionen mit einer Veränderlichen zurückführen.

Das entsprechende extremale (oder die extremalen) Punktgitter $\Gamma$ ist(sind) einfach rekonstruierbar (bis auf Ähnlichkeit). Es ist klar, wenn $(x, y) \in H_{i j}^{p q}$ für das extremale Punktgitter $\Gamma$ gilt, dann ist $\Gamma$ mit den Kreisradien $r_{i}^{p}=r_{p}$ und $\tilde{r}_{j}^{q}=R_{q}$ wirklich ein $\langle p, q\rangle$-Punktsystem.
3.2. Aus den obigen folgt, dass die Bestimmung der Gitterkreise vom minimalen bzw. maximalen Radius, die von der $p$-bzw. $q$-Eigenschaft haben, wichtig ist.

Für die Auswahl der möglichen Gitterkreise brauchen wir die folgenden Hilfssätze.

Hilfssatz 1. Es seien $A$ und $B$ die Basisvektoren eines nach Minkowski reduzierten Gitters $\Gamma$. Wir nehmen an, dass das Gitterdreieck $\triangle=O X Y$ nicht stumpfwinklig, $O T$ eine Gitterstrecke und $k \in \mathbf{Z}^{+}$ist. Nehmen wir an, dass die offenen Kreise $k(\triangle)$ und $k(O T)$ höchstens $k-1$ Gitterpunkte enthalten. Dann gehören die Gitterpunkte X, Y, T zum zentralsymmetrischen Sechseck mit Ecken $k A, k B, k(B-A),-k A,-k B,-k(B-A)$.

Beweis. Die Behauptung folgt aus dem Hilfssatz 3 von [6].
Im folgenden untersuchen wir, wie man entscheidet, ob ein Gitterpunkt zu einem Gitterkreis gehört oder nicht.

Es sei das Gitterdreieck $\triangle=O X Y$ angegeben, wobei $X=n A+m B$ und $Y=u A+v B$ Gitterpunkte sind. Mit $K$ bezeichnen wir den Mittelpunkt des Gitterkreises $k(\triangle)$. Dann gelten die Gleichungen

$$
\begin{equation*}
|K-(n A+m B)|^{2}=|K|^{2}, \quad|K-(u A+v B)|^{2}=|K|^{2} \tag{3}
\end{equation*}
$$

Aus (3) kann man die skalaren Produkte ausdrücken. Es gelten

$$
\begin{align*}
& K A=\frac{\left(n^{2} v-m u^{2}\right) A^{2}+2 v m(n-u) A B+\left(v m^{2}-v^{2} m\right) B^{2}}{2(n v-m u)}  \tag{4}\\
& K B=\frac{\left(n^{2} u-n u^{2}\right) A^{2}+2 n u(m-v) A B+\left(u m^{2}-v^{2} n\right) B^{2}}{2(m u-n v)}
\end{align*}
$$

wobei $n v-m u \neq 0$ ( $\triangle$ ist ein Gitterdreieck).
Ein beliebiger Gitterpunkt $Z=s A+t B$ gehört zum abgeschlossen Kreis $k(\triangle)$ nur im Fall, wenn

$$
\begin{equation*}
|K-(s A+t B)| \leq|K| \tag{6}
\end{equation*}
$$

Aus (6) folgt

$$
\begin{equation*}
-2 s K A-2 t K B+s^{2} A^{2}+2 s t A B+t^{2} B^{2} \leq 0 \tag{7}
\end{equation*}
$$

Auf Grund von (4), (5) und (7) erhält man

$$
\begin{align*}
& y \leq \frac{\left[s\left(n^{2} v-m u^{2}\right)-t\left(n^{2} u-n u^{2}\right)-s^{2}(n v-m u)\right] x^{2}}{[-2 s v m(n-u)+2 \operatorname{tn} u(m-v)+2 \operatorname{st}(n v-m u)] x}+ \\
& \quad+\frac{\left[s\left(m^{2} v-m v^{2}\right)-t\left(m^{2} u-n v^{2}\right)-t^{2}(n v-m u)\right]}{[-2 s v m(n-u)+2 \operatorname{tn} u(m-v)+2 s t(n v-m u)] x} \tag{8}
\end{align*}
$$

für $\frac{-2 \operatorname{svm}(n-u)+2 t n u(m-v)+2 s t(n v-m u)}{n v-m u}>0$;

$$
\begin{align*}
& y \leq \frac{\left[s\left(n^{2} v-m u^{2}\right)-t\left(n^{2} u-n u^{2}\right)-s^{2}(n v-m u)\right] x^{2}}{[-2 \operatorname{svm} v(n-u)+2 \operatorname{tn} u(m-v)+2 \operatorname{st}(n v-m u)] x}+ \\
& \quad+\frac{\left[s\left(m^{2} v-m v^{2}\right)-t\left(m^{2} u-n v^{2}\right)-t^{2}(n v-m u)\right]}{[-2 s v m(n-u)+2 \operatorname{tn} u(m-v)+2 \operatorname{st}(n v-m u)] x} \tag{9}
\end{align*}
$$

$$
\text { für } \frac{-2 \operatorname{svm}(n-u)+2 t n u(m-v)+2 s t(n v-m u)}{n v-m u}<0 \text {; }
$$

$$
\begin{align*}
0 \leq & {\left[s\left(n^{2} v-m u^{2}\right)-t\left(n^{2} u-n u^{2}\right)-s^{2}(n v-m u)\right] x^{2}+} \\
& +\left[s\left(m^{2} v-m v^{2}\right)-t\left(m^{2} u-n v^{2}\right)-t^{2}(n v-m u)\right] \tag{10}
\end{align*}
$$

für $-2 \operatorname{svm}(n-u)+2 \operatorname{tn} u(m-v)+2 \operatorname{st}(n v-m u)=0$.
Aus den Vorhergehenden folgt
HilfsSatz 2. Es seien $A$ und $B$ die Basisvektoren eines nach MinkowSKI reduzierten Gitters $\Gamma$ und $O$ der Anfangspunkt der Basisvektoren. Der abgeschlossene Umkreis des Gitterdreiecks $O X Y$ mit $X=n A+m B$ und $Y=u A+v B$ enthält den Gitterpunkt $Z=s A+t B$, wenn die Koordinaten des dem Gitter $\Gamma$ entsprechende Punkt im Dreieck $\bar{O} P Q$ die Ungleichungen (8), (9) oder (10) befriedigen.

Man kann ein Programm für Computer schreiben, womit in den konkreten Fällen wirklich entscheiden kann, ob ein Gitterpunkt zum angegebenen Gitterkreis gehört, oder nicht.

Auf Grund von Hilfssatz 1 und 2 wählen wir die Gitterkreise von der $p$ bzw. $q$-Eigenschaft für $p, q=1,2, \ldots, 5$ aus. Dann kann man die Gitterkreise vom minimalen bzw. maximalen Radius unter den möglichen Gitterkreisen von der $p$ - bzw. $q$-Eigenschaft bestimmen. In der Tabelle I findet man die möglichen extremalen Gitterkreise für $p=1,2, \ldots, 5$ und $q=1,2, \ldots, 4$. Der Fall $q=5$ ist schon kompliziert. Hier wurden Gitterkreise von der $q$-Eigenschaft angegeben, die nicht unbedingt vom maximalen Radius sind. Mit den Radien dieser nicht unbedingt extremalen Gitterkreisen kann man aber eine entsprechende obere Abschätzung für die Dicke geben, die immer für das im Satz 3 angegebene Gitter genau ist. In diesen Fällen kann man die Methode in [7] anwenden.

Mit der Anwendung von Hilfssatz 2 kann man auch die Bedingungen angeben, für welche Teilmenge des Dreiecks $\bar{O} P Q$ ein Gitterkreis extremal ist.

Tabelle I

| $p$ |  | $q$ |  |
| :---: | :---: | :---: | :---: |
| 1 | $k_{1}^{1}=k(O, A)$ | 1 | $\tilde{k}_{1}^{1}=k(O A B)$ |
| 2 | $\begin{aligned} k_{1}^{2} & =k(O A B) \\ k_{2}^{2} & =k(O(2 A)) \end{aligned}$ | 2 | $\tilde{k}_{1}^{2}=k(O(2 A)(A+B))$ |
| 3 | $\begin{aligned} & k_{1}^{3}=k(O(A+B)) \\ & k_{2}^{3}=k(O(3 A)) \end{aligned}$ | 3 | $\begin{aligned} & \tilde{k}_{1}^{3}=k(O(3 A)(A+B)) \\ & \tilde{k}_{2}^{3}=k(O A(2 B)) \\ & \tilde{k}_{3}^{3}=k(O(A+B)(2 A-B)) \end{aligned}$ |
| 4 | $\begin{aligned} & k_{1}^{4}=k(O(4 A)) \\ & k_{2}^{4}=k(O(2 A) B) \\ & k_{3}^{4}=k(O(2 B)) \end{aligned}$ | 4 | $\begin{aligned} & \tilde{k}_{1}^{4}=k(O(4 A)(2 A+B)) \\ & \tilde{k}_{2}^{4}=k(O(3 A)(2 A+B)) \\ & \tilde{k}_{3}^{4}=k(O(2 A)(2 B)) \\ & \tilde{k}_{4}^{4}=k(O A(2 B)) \\ & \tilde{k}_{5}^{4}=k(O(2 A-B)(2 A+B)) \\ & \tilde{k}_{6}^{4}=k(O(A-B)(2 A+B)) \end{aligned}$ |
| 5 | $\begin{aligned} & k_{1}^{5}=k(O(5 A)) \\ & k_{2}^{5}=k(O(2 A+B)) \\ & k_{3}^{5}=k(O A(2 B)) \\ & k_{4}^{5}=k(O(A+B)(2 A-B)) \end{aligned}$ | 5 | $\begin{aligned} & \tilde{k}_{1}^{5}=k(O(3 A) B) \\ & \tilde{k}_{2}^{5}=k(O(4 A)(A+B)) \\ & \tilde{k}_{3}^{5}=k(O(5 A)(2 A+B)) \\ & \tilde{k}_{4}^{5}=k(O A(2 B)) \\ & \tilde{k}_{5}^{5}=k(O(2 A+B)(A-B)) \\ & \tilde{k}_{6}^{5}=k(O(2 A+B)(3 A-B)) \\ & \tilde{k}_{7}^{5}=k(O(2 A)(2 B)) \\ & \tilde{k}_{8}^{5}=k(O(2 A)(A+2 B)) \\ & \tilde{k}_{9}^{5}=k(O(3 A)(2 A+B)) \\ & \tilde{k}_{10}^{5}=k(O(A+B)(3 A-B)) \\ & \hline \end{aligned}$ |

Z.B. im Fall $p=5$ ist die Zerlegung von $\bar{O} P Q$ nach den extremalen Gitterkreisen von der $p$-Eigenschaft das folgende (Abb. 3.).

Es seien
(h): $4 x y_{h}^{3}+\left(4 x^{2}+1\right) y_{h}^{2}-8 x y_{h}+3-3 x^{2}=0$
und


Abb. 3
(g): $68 x^{3} y_{g}^{3}-3 x^{2}\left(5-4 x^{2}\right) y_{g}^{2}-6 x\left(2 x^{4}+7 x^{2}+2\right) y_{g}+4 x^{6}-15 x^{4}+12 x^{2}+4=0$

Kurven in impliziter Form, wobei (2) für $x$ und $y_{h}$, weiterhin für $x$ und $y_{g}$ gilt.

Es ist zu zeigen, dass die $\operatorname{Kurven}(h),(g)$ und $y=\frac{2 x^{2}-1}{2 x}, x \in\left[\sqrt{\frac{1}{2}} ; 1\right]$ im Dreieck $\bar{O} P Q$ einen gemeinsamen Punkt haben. Es sei $x_{5}$ die $x$ Koordinate dieses Punktes. Jeder Punkt von $\bar{O} P Q$ (ausser $\bar{O}$ ) gehört zu mindenstens einem dieser Bereiche:

$$
\begin{array}{rlr}
H_{1}^{5}=\left\{(x, y) \left\lvert\, x \in\left[0, \sqrt{\frac{1}{21}}\right]\right.,\right. & y \in\left[0, \frac{x}{2}\right] \text { oder } \\
x \in\left[\sqrt{\frac{1}{21}}, \sqrt{\frac{1}{19}}\right], & \left.y \in\left[\frac{21 x^{2}-1}{4 x}, \frac{x}{2}\right]\right\} \\
H_{2}^{5}=\{(x, y) \mid & x \in\left[\sqrt{\frac{1}{21}}, \sqrt{\frac{1}{19}}\right], & y \in\left[0, \frac{21 x^{2}-1}{4 x}\right] \text { oder } \\
x \in\left[\sqrt{\frac{1}{19}}, \sqrt{\frac{1}{2}}\right], & y \in\left[0, \frac{x}{2}\right] \text { oder } \\
& x \in\left[\sqrt{\frac{1}{2}}, x_{5}\right], & y \in\left[0, y_{h}\right] \text { oder } \\
& x \in\left[x_{5}, 1\right], & \left.y \in\left[0, y_{g}\right]\right\} \\
H_{3}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{2}}, x_{5}\right]\right.,\right. & y \in\left[y_{h}, \frac{x}{2}\right] \text { oder }
\end{array}
$$

$$
\begin{array}{cc}
x \in\left[x_{5}, 1\right], & \left.y \in\left[\frac{2 x^{2}-1}{2 x}, \frac{x}{2}\right]\right\} \\
H_{4}^{5}=\left\{(x, y) \mid x \in\left[x_{5}, 1\right],\right. & \left.y \in\left[y_{g}, \frac{2 x^{2}-1}{2 x}\right]\right\} .
\end{array}
$$

Est ist offenbar, dass die obigen Bereiche paarweise keinen gemeinsamen inneren Punkt haben.

## 4. Sätze

In diesem Abschnitt beweisen wir drei Sätze.
SATZ 1. Es seien $A$ und $B$ die Basisvektoren eines nach MinKowski reduzierte $\langle p, q\rangle$-Punktgitters $\Gamma$, wobei $p=1$ und $q \in \mathbf{Z}^{+}$. Die Gleichung $|A|=|B|$ ist die notwendige Bedingung dafür, dass $\Gamma$ die extremale $\langle p, q\rangle$ Dicke $\kappa_{\Gamma}(p, q)$ hat.

Beweis. Das $\langle p, q\rangle$-Punktgitter $\Gamma$ ist nach MinKowski reduziert, deshalb enthält der offene Gitterkreis $k_{1}^{1}=k(O A)$ keinen Gitterpunkt und ist vom minimalen Radius unter den offenen Gitterkreisen, die von der $p$-Eigenschaft $p=1$ sind. Es sei $(x, y)$ der dem Gitter $\Gamma$ entsprechende Punkt im Dreieck $\bar{O} P Q$. Die Zerlegung von $\bar{O} P Q$ besteht also aus dem Bereich $H_{1}^{1}$. Wir betrachten die Zerlegung von $\bar{O} P Q$ in die Bereiche $\tilde{H}_{j}^{q}(1 \leq j \leq s)$ nach den extremalen Gitterkreisen (vgl. 3.) von der $q$-Eigenschaft. Auf Grund der Vorhergehenden ergibt sich die Zerlegung von $\bar{O} P Q$ in Bereiche $H_{1 j}^{1 q}=$ $=H_{1}^{1} \cap \tilde{H}_{j}^{q}(1 \leq j \leq s)$.

Die entsprechenden Funktionen für die Dicken sind $(x, y) \mapsto G_{1 j}^{1 q}$, $(x, y) \in H_{1 j}^{1 q}(1 \leq j \leq s)$. Zur Vergrösserung der Dicke definieren wir eine Gittertransformation $t_{1}$. Bei der Anwendung von $t_{1}$ bleibt die Gittergerade $O A$ fix und der Endpunkt des Basisvektors $B$ bewegt sich auf einer zu $O A$ ortogonalen Geraden neben der Abnahme von $|B|$, d.h., $x$ und $y$ nehmen zu. Der Kreisradius ist offenbar ändert sich nicht. Wir müssen beweisen, dass $\tilde{r}_{j}^{q}(1 \leq j \leq s)$ während der Anwendung von $t_{1}$ abnimmt. Wir wenden die Transformation $t_{1}$ nur im Fall an, wenn die Bedingungen (1) für die entstehenden Gitter gelten.

Das Gitterdreieck $\Delta_{j}$ (mit Umkreis $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ ist nicht stumpfwinklig. Die Ecken von $\Delta_{j}$ kann man immer derart wählen, dass eine der Ecken $O$ ist, d.h. $\Delta_{j}=O X_{j} Y_{j}$. Ist z. B. $X_{j}=k A k \in \mathbf{Z}^{+}$(Abb. 4a), dann kommt der Gitterpunkt $Y_{j}$ ins Innere des ursprünglichen Gitterdreiecks $\Delta_{j}$ bei der Anwendung von $t_{1}$. Der Umkreisradius $\tilde{r}_{j}^{q}$ nimmt offenbar ab. Sonst kann man immer ein unter den äquivalenten Gitterdreiecken wählen, bei dem die Ecken $X_{j}, Y_{j}$ von $O A$ abgetrennt (Abb. 4b) sind. Es ist einfach nachzuweisen, dass $X_{j}$ und $Y_{j}$ ins Innere des ursprünglichen Gitterkreises $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ kommen, wenn wir die Gittertransformation $t_{1}$ anwenden, d.h., $\tilde{r}_{j}^{q}$ abnimmt.


Abb. 4/a


Abb. 4/b

Endlich erreichen wir einen Randpunkt von $H_{1 j}^{1 q}$ oder einen Randpunkt von $\bar{O} P Q$. Im ersten Fall kann man die Transformation weiter anwenden und die Dicke weiter vergrössern. (Nach [6] sind die Dicken für die Randpunkte von zwei Bereichen in der Zerlegung gleich.) Gilt $|B|=|B-A|$, dann können wir im obigen Sinne $t_{1}$ anwenden. Endlich erreichen wir einen Punkt von $P Q$. In diesem Fall gilt aber $|A|=|B|$.

Satz 2. Es sei $\Gamma$ ein $\langle p, q\rangle$-Punktgitter, wobei $p=3$ und $q \in \mathbf{Z}^{+}$. Der dem Gitter $\Gamma$ entsprechende Punkt in $\bar{O} P Q$ ist $(x, y)$. Das Gitter $\Gamma$ hat die extremale $\langle p, q\rangle$-Dicke $\kappa_{\Gamma}(p, q)$ nur im Fall, wenn $y=\frac{x}{2}, x \in\left[\sqrt{\frac{1}{7}}, 1\right]$ oder $y=\frac{8 x^{2}-1}{2 x}, x \in\left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right]$ gelten.


Abb. 5
BewEIS. Die Abbildung 5 zeigt die Zerlegung von $\bar{O} P Q$ nach den extremalen Gitterkreisen von der $p$-Eigenschaft. (Der Beweis findet man in [11].) Der Kreis $k_{1}^{3}=k(O(A+B))$ bzw. $k_{2}^{3}=k(O(3 A))$ besitzt den minimalen Radius unter den Gitterkreisen von der $p$-Eigenschaft im Bereich $H_{1}^{3}$ bzw. $H_{2}^{3}$. Der Durchschnitt von $H_{1}^{3}$ und $H_{2}^{3}$ ist $y=\frac{8 x^{2}-1}{2 x}, x \in\left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right]$ im Dreieck $\bar{O} P Q$.

Nun betrachten wir die Zerlegung von $\bar{O} P Q$ in die Bereiche $\tilde{H}_{j}^{q}(1 \leq$ $j \leq s$ ) nach den extremalen Gitterkreisen (vgl. 3.) von der $q$-Eigenschaft. Auf Grund der Vorhergehenden ergibt sich die Zerlegung von $\bar{O} P Q$ in Bereiche $H_{1 j}^{3 q}=H_{1}^{3} \cap \tilde{H}_{j}^{q}$ und $H_{2 j}^{3 q}=H_{2}^{3} \bigcap \tilde{H}_{j}^{q}(1 \leq j \leq s)$.

Die entsprechenden Funktionen für die Dicken $\operatorname{sind}(x, y) \mapsto G_{1 j}^{3 q}$, $(x, y) \in H_{1 j}^{3 q}$ und $(x, y) \mapsto G_{2 j}^{3 q},(x, y) \in H_{2 j}^{3 q}(1 \leq j \leq s)$. Zur Vergrösserung der Dicke in $H_{2 j}^{3 q}$ verwenden wir die Gittertransformation $t_{1}$. Bei der Anwendung von $t_{1}$ bleibt die Gittergerade $O A$ fix, d.h., der Radius von $k_{2}^{3}=$ $=k(O(3 A))$ ist konstant. Wie im SATZ 1 kann man einsehen, dass die Radien der Kreise $\tilde{k}_{j}^{q}\left(\Delta_{j}\right)$ für $(x, y) \in H_{2 j}^{3 q}(1 \leq j \leq s)$ abnehmen, folglich nimmt die Dicke zu. Endlich erreichen wir den Punkt von $y=\frac{8 x^{2}-1}{2 x}, x \in\left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right]$.

Für die Funktionen $(x, y) \mapsto G_{1 j}^{3 q},(x, y) \in H_{1 j}^{3 q}$ definieren wir eine weitere Gittertransformation $t_{2}$. Bei der Anwendung von $t_{2}$ bleibt die Gittergerade $O(A+B)$ fix und die Endpunkt des Basisvektors $B$ bewegt sich auf einer zu $O(A+B)$ orthogonalen Geraden neben der Abnahme von $|B|$. Dann nimmt $y$ offenbar zu. Durch einfache Rechnungen ergibt sich die Abnahme von $x$ auf Grund von (1).

Wie im Fall von $t_{1}$ kann man beweisen, dass die entsprechenden Kreisradien $\tilde{r}_{j}^{q}(1 \leq j \leq s)$ für $(x, y) \in H_{1 j}^{3 q}$ abnehmen, d.h., die Dicke zunimmt. Das bedeutet, dass wir am Ende einen Punkt der im Satz erwähnten Kurven erreichen.

Es sei $\bar{y}$ die Lösung der Gleichung $68 y^{3}-3 y^{2}-66 y+5=0$ für $y \in$ $\in\left[0, \frac{1}{2}\right]$. Im folgenden Satz geben wir die extremalen Dicken für $p=5$ und $q=1,2,3,4$, weiterhin in den Fällen $p=1,2,3,4$ und $q=5$. In Klammern findet man die extremalen Gitter (bis auf Ähnlichkeit).

Satz 3. Es gelten

$$
\begin{aligned}
& \kappa_{\Gamma}(1,5)=\sqrt{\frac{1}{7}} \quad\left((x, y)=\left(1, \frac{1}{2}\right)\right), \\
& \kappa_{\Gamma}(2,5)=\sqrt{4 \frac{15-2 \sqrt{46}}{(18-2 \sqrt{46})^{2}}} \quad((x, y)=(\sqrt{14-2 \sqrt{46}}, 0)), \\
& \kappa_{\Gamma}(3,5)=\sqrt{\frac{7}{16}} \quad\left((x, y)=\left(\sqrt{\frac{1}{2}}, \frac{1}{2} \sqrt{\frac{1}{2}}\right)\right), \\
& \kappa_{\Gamma}(4,5)=\sqrt{\frac{4}{7}} \quad\left((x, y)=\left(\sqrt{\frac{1}{12}}, 0\right) \quad \text { oder } \quad(x, y)=\left(1, \frac{1}{2}\right)\right), \\
& \kappa_{\Gamma}(5,1)=\sqrt{\frac{7}{2}} \quad\left((x, y)=\left(\sqrt{\frac{1}{2}}, \frac{1}{2} \sqrt{\frac{1}{2}}\right)\right), \\
& \kappa_{\Gamma}(5,2)=\frac{\sqrt{7}}{2} \quad\left((x, y)=\left(\sqrt{\frac{1}{2}}, \frac{1}{2} \sqrt{\frac{1}{2}}\right)\right), \\
& \kappa_{\Gamma}(5,3)=\frac{3 \sqrt{2}}{4} \quad\left((x, y)=\left(\sqrt{\frac{1}{2}}, \frac{1}{2} \sqrt{\frac{1}{2}}\right) \quad \text { oder } \quad(x, y)=\left(1, \frac{1}{8}\right)\right), \\
& \kappa_{\Gamma}(5,4)=\sqrt{\frac{9(1+\bar{y})}{2(5+4 \bar{y})}} \quad((x, y)=(1, \bar{y})) .
\end{aligned}
$$

Beweis. Wir verwenden die Methode in 3. Es ist sehr lang, die Rechnungen durchzuführen, deshalb behandeln wir nur einige Fälle. Die Zerlegungen von $\bar{O} P Q$ für $p, q=1,2,3,4$ in [7] sind extremal wie es in [11] bewiesen
wurde. Im Fall $p=5$ haben wir die extremale Zerlegung nach den Gitterkreisen von der $p$-Eigenschaft angegeben. Für $q=5$ werden Gitterkreise von der $q$-Eigenschaft angegeben, die nicht unbedingt vom maximalen Radius sind. Mit den Radien dieser nicht unbedingt extremalen Gitterkreisen kann man aber eine entsprechende obere Abschätzung für die Dicke geben, die immer für das im Satz 3 angegebene Gitter genau ist. In diesen Fällen kann man die Methode in [7] anwenden.


Abb. 6
4.1. Es seien (Abb. 6)

$$
\begin{gathered}
\tilde{H}_{1}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{14-2 \sqrt{46}}, \sqrt{\frac{6}{11}}\right]\right., y \in\left[0, \frac{3-\sqrt{21-28 x^{2}+x^{4}}}{4 x}\right]\right. \text { oder } \\
x \in\left[\sqrt{\frac{1}{2}}, \sqrt{\frac{6}{11}}\right], y \in\left[\frac{3-5 x^{2}}{2 x}, \frac{1-x^{2}}{2 x}\right] \text { oder } \\
\left.x \in\left[\sqrt{\frac{6}{11}}, 1\right], y \in\left[0, \frac{1-x^{2}}{2 x}\right]\right\} \\
\tilde{H}_{2}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\frac{1}{2}, \frac{\sqrt{-18+6 \sqrt{21}}}{6}\right]\right., y \in\left[\frac{x^{2}-\sqrt{3 x^{2}-1+4 x^{4}}}{2 x}, \frac{2-7 x^{2}}{2 x}\right]\right. \text { oder } \\
x
\end{gathered} \begin{aligned}
x & \left.\in\left[\frac{\sqrt{-18+6 \sqrt{21}}}{6}, \sqrt{\frac{2}{7}}\right], y \in\left[0, \frac{2-7 x^{2}}{2 x}\right]\right\} \\
\tilde{H}_{3}^{5}=\{(x, y) \mid x & \left.\in] 0, \sqrt{\frac{2}{13}}\right], y \in\left[0, \frac{x}{2}\right] \text { oder } \\
x & \left.\in\left[\sqrt{\frac{2}{13}}, \sqrt{\frac{1}{6}}\right], y \in\left[0, \frac{1-6 x^{2}}{x}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{H}_{4}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{6}}, \frac{1}{2}\right]\right., y \in\left[0, \frac{6 x^{2}-1}{4 x}\right]\right. \text { oder } \\
& \left.x \in\left[\frac{1}{2}, \frac{\sqrt{-18+6 \sqrt{21}}}{6}\right], y \in\left[0, \frac{x^{2}-\sqrt{3 x^{2}-1+4 x^{4}}}{2 x}\right]\right\} \\
& \tilde{H}_{5}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\frac{\sqrt{161}+\sqrt{14}}{21}, \sqrt{\frac{2}{3}}\right]\right., y \in\left[\frac{7 x^{2}-29+6 \sqrt{79 x^{4}+15 x^{2}+15}}{86 x}, \frac{2 x^{2}-1}{x}\right]\right. \\
& \text { oder } \left.x \in\left[\sqrt{\frac{2}{3}}, \frac{2}{7} \sqrt{-14+7 \sqrt{11}}\right], y \in\left[\frac{7 x^{2}-29+6 \sqrt{79 x^{4}+15 x^{2}+15}}{86 x}, \frac{2-x^{2}}{4 x}\right]\right\} \\
& \tilde{H}_{6}^{\prime 5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{6}}\right]\right., y \in\left[\frac{1-6 x^{2}}{x}, \frac{x}{2}\right]\right. \text { oder } \\
& \left.x \in\left[\sqrt{\frac{1}{6}}, \frac{1}{2}\right], y \in\left[\frac{6 x^{2}-1}{4 x}, \frac{x}{2}\right]\right\} \\
& \tilde{H}_{6}^{\prime \prime} 5=\left\{(x, y) \left\lvert\, x \in\left[\frac{1}{2}, \sqrt{\frac{2}{7}}\right]\right., y \in\left[\frac{2-7 x^{2}}{2 x}, \frac{1-x^{2}}{6 x}\right]\right. \text { oder } \\
& \left.x \in\left[\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{5}}\right], y \in\left[\frac{7 x^{2}-2}{8 x}, \frac{1-x^{2}}{6 x}\right]\right\} \\
& \tilde{H}_{7}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right]\right., y \in\left[\frac{x}{4}, \frac{3 x^{2}-1}{2 x}\right]\right. \text { oder } \\
& \left.x \in\left[\sqrt{\frac{1}{2}}, \sqrt{\frac{6}{11}}\right], y \in\left[\frac{x}{4}, \frac{3-5 x^{2}}{2 x}\right]\right\} \\
& \tilde{H}_{8}^{\prime 5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{5}}\right]\right., y \in\left[0, \frac{7 x^{2}-2}{8 x}\right]\right. \text { oder } \\
& x \in\left[\sqrt{\frac{2}{5}}, \sqrt{14-2 \sqrt{46}}\right], y \in\left[0, \frac{x}{4}\right] \text { oder } \\
& \left.x \in\left[\sqrt{14-2 \sqrt{46}}, \sqrt{\frac{6}{11}}\right], y \in\left[\frac{3-\sqrt{21-28 x^{2}+x^{4}}}{4 x}, \frac{x}{4}\right]\right\} \\
& \tilde{H}_{8}^{\prime \prime} 5=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{3}}, \frac{2}{7} \sqrt{-14+7 \sqrt{11}}\right]\right., y \in\left[\frac{2-x^{2}}{4 x}, \frac{x}{2}\right]\right. \text { oder } \\
& \left.x \in\left[\frac{2}{7} \sqrt{-14+7 \sqrt{11}}, 1\right], y \in\left[f, \frac{x}{2}\right]\right\} \\
& \tilde{H}_{9}^{5}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{2}}, \frac{\sqrt{161}+\sqrt{14}}{21}\right]\right., y \in\left[\frac{-x^{2}-1+2 \sqrt{-5 x^{4}+5 x^{2}+1}}{6 x}, \frac{x}{2}\right]\right. \text { oder } \\
& \left.x \in\left[\frac{\sqrt{161}+\sqrt{14}}{21}, \sqrt{\frac{2}{3}}\right], y \in\left[\frac{2 x^{2}-1}{x}, \frac{x}{2}\right]\right\}
\end{aligned}
$$

$$
\begin{gathered}
\tilde{H}_{10}^{\prime 5}=\left\{(x, y) \left\lvert\, x \in\left[\frac{1}{2}, \sqrt{\frac{2}{5}}\right]\right., y \in\left[\frac{1-x^{2}}{6 x}, \frac{x}{2}\right]\right. \text { oder } \\
\left.x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right], y \in\left[\frac{3 x^{2}-1}{2 x}, \frac{x}{2}\right]\right\} \\
\tilde{H}_{10}^{\prime \prime} 5=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{2}}, \frac{\sqrt{161}+\sqrt{14}}{21}\right]\right., y \in\left[\frac{1-x^{2}}{2 x}, \frac{-x^{2}-1+2 \sqrt{-5 x^{4}+5 x^{2}+1}}{6 x}\right]\right.
\end{gathered}
$$

oder

$$
\begin{aligned}
& x \in\left[\frac{\sqrt{161}+\sqrt{14}}{21}, \frac{2}{7} \sqrt{-14+7 \sqrt{11}}\right], \\
& y \in\left[\frac{1-x^{2}}{2 x}, \frac{7 x^{2}-29+6 \sqrt{79 x^{4}+15 x^{2}+15}}{86 x}\right]
\end{aligned}
$$

oder

$$
\left.x \in\left[\frac{2}{7} \sqrt{-14+7 \sqrt{11}}, 1\right], y \in\left[\frac{1-x^{2}}{2 x}, f\right]\right\}
$$

wobei $y_{f}$ die Gleichung

$$
\begin{aligned}
& (f)-32 x^{6}-44 x^{4}+20 x^{2}+80 x^{4} y_{f}^{2}+24 x^{5} y_{f}+ \\
& +48 x^{3} y_{f}-4+24 x y_{f}+16 x^{2} y_{f}^{2}-96 x^{3} y_{f}^{3}=0
\end{aligned}
$$

erfüllt.
In Tabelle I findet man die Gitterkreise, nach denen die obige Zerlegung von $\bar{O} P Q$ angefertigt wurde.
4.2. Wir betrachten ein beliebiges $\langle 1,5\rangle$-Punktgitter. Nach SATZ 1 kann die Dicke nur für die Gitter mit $x=1$ extremal. Dann müssen wir das Maximum der Funktion $x \mapsto G_{1,10}^{15}=\frac{r_{1}^{1}}{\tilde{r}_{10}^{5}}, x=1, y \in\left[0, \frac{1}{2}\right]$ bestimmen. Der Umkreisradius eines Dreiecks wird mit den Seiten und mit dem Inhalt ausgedrückt. Auf Grund dieser Bemerkung ist $G_{1,10}^{15}=$ $=\sqrt{\frac{16 x^{4}\left(1-y^{2}\right)}{\left(x^{2}+1+2 x y\right)\left(9 x^{2}+1-6 x y\right)\left(4 x^{4}+4-8 x y\right)}}$. An der Stelle $x=1$ erhält man $G_{1,10}^{15}=\sqrt{\frac{1}{2(5-3 y)}}$, deren Maximum für $y=\frac{1}{2}$ eintritt. Der Maximumwert, d.h. die maximale Dicke ist $\sqrt{\frac{1}{7}}$. Als extremales Punktgitter $\Gamma^{15}$ ergibt sich das reguläre Dreiecksgitter $\left((x, y)=\left(1, \frac{1}{2}\right)\right)$, wobei die Länge des kürzesten Basisvektors $|A|=2 r_{1}^{1}$ ist.

Wir müssen noch einsehen, dass $\Gamma^{15}$ wirklich ein $\langle 1,5\rangle$-Punktsystem ist. (Es wurde nämlich nicht bewiesen, dass die Zerlegung des Dreiecks $\bar{O} P Q$ für $q=5$ nach den extremalen Gitterkreisen ist.) Es ist klar, dass jeder Kreis mit Radius $r_{1}^{1}$ höchstens einen Gitterpunkt enthält. Es gilt $\tilde{r}_{10}^{5}=\sqrt{7} r_{1}^{1}$ für $\Gamma^{15}$. Es genügt zu zeigen, dass die um die Gitterpunkte geschlagenen Kreisen vom Radius $\tilde{r}_{10}^{5}$ das Dreieck $\Delta^{15}=A(A-B)(2 A-B) 5$-fach überdecken. Die um die Gitterpunkte $A, A-B$ und $2 A-B$ geschlagenen Kreise vom Radius $\tilde{r}_{10}^{5}$ bedecken $\Delta^{15}$ je einfach. Es gelten $O(3 A-B)=(-B)(2 A)=$ $=(A-2 B)(2 A)=2 \tilde{r}_{10}^{5}$. Dann überdecken die Kreise mit Mittelpunkt $O$ und $2 A-2 B$ gemeinsam mindestens einfach das Dreieck $\Delta^{15}$. Obwohl die Kreise mit Mittelpunkt $-B, A-2 B, 2 A$ ein Teil des Dreiecks $\Delta^{15}$ nicht bedecken, werden die nicht überdeckte Punkte des Dreiecks von den Kreisen mit Mittelpunkt $O$ und $2 A-2 B$ zweifach bedeckt.
4.3. Es sei $p=3, q=5$. Nach SATZ 2 und 4.1 die Dicke eines $\langle 3,5\rangle$-Punktgitters kann nur für $y=\frac{x}{2}, x \in\left[\sqrt{\frac{1}{7}}, 1\right]$ oder für $y=\frac{8 x^{2}-1}{2 x}$, $x \in\left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right]$ maximal. Wir müssen die Maxima der folgenden Funktionen unter den angegebenen Bedingungen bestimmen.

$$
\begin{aligned}
& G_{13}^{35}=\sqrt{\frac{\left(x^{2}+1+2 x y\right)\left(1-y^{2}\right)}{\left(4 x^{2}+1+4 x y\right)\left(9 x^{2}+1-6 x y\right)}} \\
& y=\frac{8 x^{2}-1}{2 x}, x \in\left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right] \quad \text { und } y=\frac{x}{2}, x \in\left[\sqrt{\frac{1}{7}}, \sqrt{\frac{2}{13}}\right] \\
& G_{16}^{35}=\sqrt{\frac{25 x^{2}\left(x^{2}+1+2 x y\right)\left(1-y^{2}\right)}{\left(4 x^{2}+1+4 x y\right)\left(9 x^{2}+1-6 x y\right)\left(x^{2}+4-4 x y\right)}} \quad y=\frac{x}{2}, x \in\left[\sqrt{\frac{2}{13}}, \sqrt{\frac{1}{2}}\right] \\
& G_{18}^{35}=\sqrt{\frac{4\left(x^{2}+1+2 x y\right)\left(1-y^{2}\right)}{\left(x^{2}+4+4 x y\right)\left(x^{2}+4-4 x y\right)}} \quad y=\frac{x}{2}, x \in\left[\sqrt{\frac{2}{3}}, 1\right] \\
& G_{19}^{35}=\sqrt{\frac{\left(x^{2}+1+2 x y\right)\left(1-y^{2}\right)}{\left(4 x^{2}+1+4 x y\right)\left(x^{2}+1-2 x y\right)}} \quad y=\frac{x}{2}, x \in\left[\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}\right] \\
& G_{1,10}^{35}=\sqrt{\frac{4 x^{2}\left(1-y^{2}\right)}{\left(x^{2}+1-2 x y\right)\left(9 x^{2}+1-6 x y\right)}} \quad y=\frac{x}{2}, x \in\left[\frac{1}{2}, \sqrt{\frac{1}{2}}\right]
\end{aligned}
$$

Unter den Maxima der obigen Funktionen von einer Veränderlichen tritt das absolute Maximum für $(x, y)=\left(\frac{1}{\sqrt{2}}, \frac{1}{2 \sqrt{2}}\right)$ ein. Es ist wie in 4.2 zu
beweisen, dass das dem Punkt $\left(\frac{1}{\sqrt{2}}, \frac{1}{2 \sqrt{2}}\right)$ entsprechende Gitter $\Delta^{35}$ mit den Kreisradien $r=r_{1}^{3}$ und $R=\tilde{r}_{9}^{5}=\tilde{r}_{7}^{5}=\tilde{r}_{1}^{5}=\tilde{r}_{10}^{5}$ ein $\langle 3,5\rangle$-Punktgitter ist. Die maximale Dicke ist $\sqrt{\frac{7}{16}}$.
4.4. Als letztes Beispiel skizzieren wir die Lösung des Problems für $p=5, q=3$. Die extremalen Gitterkreise von der $p$ - bzw. $q$-Eigenschaft findet man in Tabelle I. Die Zerlegung von $\bar{O} P Q$ ist für $q=3$ nach den extremalen Gitterkreisen (vgl. [11]) das folgende.

$$
\begin{aligned}
&\left.\tilde{H}_{1}^{3}=\{(x, y) \mid x \in] 0, \sqrt{\frac{2}{5}}\right], y \in\left[0, \frac{x}{2}\right] \text { oder } \\
& x\left.x\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right], y \in\left[0, \frac{1-2 x^{2}}{x}\right]\right\} \\
& \tilde{H}_{2}^{3}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{2}}, 1\right]\right., y \in\left[0, \frac{2 x^{2}-1}{2 x}\right]\right\} \\
& \tilde{H}_{3}^{3}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right]\right., y \in\left[\frac{1-2 x^{2}}{x}, \frac{x}{2}\right]\right. \text { oder } \\
&\left.x \in\left[\sqrt{\frac{1}{2}}, 1\right], y \in\left[\frac{2 x^{2}-1}{2 x}, \frac{x}{2}\right]\right\}
\end{aligned}
$$



Abb. 7

Nach Hilfssatz 2 findet man die entsprechende Zerlegung von $\bar{O} P Q$ nach den extremalen Gitterkreisen von der $p$-Eigenschaft für $p=5$ (Abb. 3). Wir haben also für die Bestimmung der Dicken die folgenden Mengen (Abb. 7).

$$
\left.H_{11}^{53}=\{(x, y) \mid x \in] 0, \sqrt{\frac{1}{21}}\right], y \in\left[0, \frac{x}{2}\right] \text { oder }
$$

$$
\begin{gathered}
\left.x \in\left[\sqrt{\frac{1}{21}}, \sqrt{\frac{1}{19}}\right], y \in\left[\frac{21 x^{2}-1}{4 x}, \frac{x}{2}\right]\right\} \\
H_{21}^{53}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{21}}, \sqrt{\frac{1}{19}}\right]\right., y \in\left[0, \frac{21 x^{2}-1}{4 x}\right]\right. \text { oder } \\
x \in\left[\sqrt{\frac{1}{19}}, \sqrt{\frac{2}{5}}\right], y \in\left[0, \frac{x}{2}\right] \text { oder } \\
\left.x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right], y \in\left[0, \frac{1-2 x^{2}}{x}\right]\right\} \\
H_{22}^{53}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{1}{2}}, x_{5}\right]\right., y \in\left[0, \frac{2 x^{2}-1}{2 x}\right]\right. \text { oder } \\
H_{23}^{53}=\left\{(x, y) \left\lvert\, x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right]\right., y \in\left[\frac{1-2 x^{2}}{x}, \frac{x}{2}\right]\right. \text { oder } \\
x
\end{gathered} \begin{aligned}
x & \left.\in\left[\sqrt{\frac{1}{2}}, x_{5}\right], y \in\left[\frac{2 x^{2}-1}{2 x}, y_{h}\right]\right\} \\
H_{33}^{53}=\{(x, y) \mid x & \in\left[\sqrt{\frac{1}{2}}, x_{5}\right], y \in\left[y_{h}, \frac{x}{2}\right] \text { oder } \\
x & \left.\in\left[x_{5}, 1\right], y \in\left[\frac{2 x^{2}-1}{2 x}, \frac{x}{2}\right]\right\} \\
H_{42}^{53}=\{(x, y) \mid x & \left.\in\left[x_{5}, 1\right], y \in\left[y_{g}, \frac{2 x^{2}-1}{2 x}\right]\right\}
\end{aligned}
$$

Die Dicken sind für einen Randpunkt von zwei Mengen gleich.
Für die Vergrösserung der Dicken verwenden wir drei Gittertransformationen.

Wir geben das $\langle 5,3\rangle$-Punktgitter $\Gamma$ mit den Basisvektoren $A$ und $B$ nach (1) an. Bei der Anwendung der Gittertransformation $t_{k}$ bleibt die Gittergerade $g_{k}$ fest und der Gitterpunkt $G_{k}$ bewegt sich auf der zu $g_{k}$ parallelen Geraden neben der Zunahme von $\left|V_{k}\right|$. Es ist leicht zu zeigen, dass die in der Tabelle II angegebene Monotonie von $x$ und $y$ während der Anwendung von $t_{k}$ richtig ist. Wir wenden $t_{k}$ nur im Fall an, wenn die Bedingungen (1) für die entstehenden Gitter noch gelten. Endlich geben wir die Mengen $H_{i j}^{53}$ ( $i=1,2,3,4, j=1,2,3$ ) an, wo wir die Gittertransformation $g_{k}$ anwenden. In der Menge $H_{42}^{53}$ verwenden wir die inverse Transformation von $t_{2}$.

Es ist zu beweisen, dass die Kreisradien $r_{i}^{5}, i=1,2,3,4$ bei der Anwendung der entsprechenden Transformation $t_{k}$ konstant sind oder zunehmen und die Kreisradien $\tilde{r}_{j}^{3}, i=1,2,3$ abnehmen. Wegen der Monotonie von $x$ und $y$ (Tabelle II) erreichen wir endlich einen Punkt der folgenden Kurven. (Die entsprechenden Kurven im Dreieck $\bar{O} P Q$ wurden dick bezeichnet.)

Tabelle II

| $k$ | $g_{k}$ | $G_{k}$ | $\left\|V_{k}\right\|$ | $x$ | $y$ | $H_{i j}^{53}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | $O A$ | $B$ | $\|B\|$ | fallend | wachsend | $H_{11}^{53}, H_{21}^{53}$ |
| 4 | $A(2 B)$ | $O$ | $\|B\|$ | fallend | wachsend | $H_{23}^{53}, H_{33}^{53}$ |
| 5 | $O B$ | $A$ | $\|A\|$ | wachsend | wachsend | $H_{22}^{53}$ |

4.3.1. $x \in] 0,1], y=\frac{x}{2}$
4.3.2. $x=1, y \in\left[0, \frac{1}{2}\right]$
4.3.3. $x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right], y=\frac{1-2 x^{2}}{x}$
4.3.4. $x \in\left[x_{5}, 1\right]$, die Kurve ( $g$ )

In Tabelle III sieht man wir die Dickenfunktionen, die wir unter den angegebenen Bedingungen untersuchen müssen. Es bezeichne $y_{g}(1)$ die $y$ Koordinate des Punktes in dem die Kurve ( $g$ ) die Strecke $P Q$ schneidet.

In jedem der obigen Fälle erhalten wir eine Funktion von einer Veränderlichen. Es ist auszurechnen, dass das Maximum der Dicken $\frac{3 \sqrt{2}}{4}$ ist. Zwei extremale $\langle 5,3\rangle$-Punktgitter existieren, nämlich die den Punkten $\left(\sqrt{\frac{1}{2}}, \frac{1}{2} \sqrt{\frac{1}{2}}\right)(S)$ und $\left(1, y_{g}(1)\right)(N)$ entsprechenden Gitter.

Die Rechnungen wurden mit Programm Maple V Release 4 kontrolliert.

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Tabelle III

$$
\begin{array}{llll}
G_{11}^{53}=\sqrt{\frac{25 x^{2}\left(1-y^{2}\right)}{\left(x^{2}+1+2 x y\right)\left(4 x^{2}+1-4 x y\right)}} & \left.x \in] 0, \sqrt{\frac{1}{19}}\right] & y=\frac{x}{2} & \bar{O} J \\
G_{21}^{53}=\sqrt{\frac{\left(4 x^{2}+1+4 x y\right)\left(1-y^{2}\right)}{\left(x^{2}+1+2 x y\right)\left(4 x^{2}+1-4 x y\right)}} & x \in\left[\sqrt{\frac{1}{19}}, \sqrt{\frac{2}{5}}\right] & y=\frac{x}{2} & J U \\
& x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right] & y=\frac{1-2 x^{2}}{x} & U V \\
G_{23}^{53}=\sqrt{\frac{9 x^{2}\left(1-y^{2}\right)\left(4 x^{2}+1+4 x y\right)}{\left(x^{2}+1+2 x y\right)\left(4 x^{2}+1-4 x y\right)\left(x^{2}+4-4 x y\right)}} & x \in\left[\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}\right] & y=\frac{x}{2} & U S \\
& x \in\left[\sqrt{\frac{1}{2}}, 1\right] & y=\frac{x}{2} & S Q \\
G_{33}^{53}=\sqrt{\frac{9 x^{2}}{\left(x^{2}+1+2 x y\right)\left(4 x^{2}+1-4 x y\right)}} & x \in\left[x_{5}, 1\right] & (g) & M N \\
G_{22}^{53}=\sqrt{\frac{\left(4 x^{2}+1+4 x y\right)\left(1-y^{2}\right)}{\left(x^{2}+4-4 x y\right)}} & x \in N
\end{array}
$$

$$
\begin{array}{r}
G_{42}^{53}=\sqrt{\frac{\left(x^{2}+1+2 x y\right)\left(4 x^{4}+1-4 x y\right)}{9 x^{2}}} \quad x=1 \\
\quad \text { Literaturverzeichnis }
\end{array}
$$

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# A NOTE ON THE SUNOUCHI OPERATOR WITH RESPECT TO VILENKIN SYSTEM 

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## 1. Introduction

The so-called Sunouchi operator $S$ was introduced and firstly investigated by Sunouchi [12], [13] in Walsh-Fourier analysis. He proved among others that his operator characterizes the $L^{p}$ spaces for $p>1$ but this characterization fails to hold for $p=1$. In SIMON [8] we showed that $S$ is a bounded map from the dyadic Hardy space $H^{1}$ into $L^{1}$. Furthermore, we formulated a conjecture, namely that $H^{1}$ can be characterized by $S$. This conjecture was proved by DALY and PHILLIPS [1], that is, the $H^{1}$-norm of a function $f$ with mean value zero is equivalent to the $L^{1}$-norm of $S f$. For the $H^{p}(0<p \leq 1)$ version of Simon's and Daly and Phillips's results see Simon [9], [10], [11] and Daly and Phillips [2].

The Vilenkin analogue of the Sunouchi operator was given by Gát [3]. He investigated the boundedness of $S$ as map from (Vilenkin) $H^{1}$ into $L^{1}$ and proved that if the Vilenkin group has an unbounded structure and $H^{1}$ is defined by means of the usual maximal function then $S$ is not bounded. Furthermore, if we consider a modified $H^{1}$ space (introduced by SimOn [7]), then a necessary and sufficient condition can be given for the Vilenkin group $S: H^{1} \rightarrow L^{1}$ to be bounded. All Vilenkin groups with bounded structure and also certain groups without this boundedness property staisfy Gát's condition.

Thus in the so-called bounded case the $\left(H^{1}, L^{1}\right)$-boundedness of $S$ remains true also for Vilenkin systems. In this note we extend this result, showing the $\left(H^{p}, L^{p}\right)$-boundedness of $S$ for all $0<p \leq 1$. Moreover, the equivalence $\|f\|_{H^{p}} \sim\|S f\|_{p}(0<p \leq 1)$ will be proved for $f$ with mean value zero. We investigate also the role of the bounded structure.

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## 2. Preliminaries and notations

In this section we introduce the most important definitions and notations and formulate some known results with respect to the Vilenkin system, which play a basic role in the further investigations. For details see Vilenkin [14] and the book Schipp-Wade-Simon and Pál [6].

Let $m=\left(m_{0}, m_{1}, \ldots, m_{k}, \ldots\right)$ be given as a sequence of natural numbers such that $m_{k} \geq 2(k \in \mathbf{N}:=\{0,1, \ldots\})$. For all $k \in \mathbf{N}$ we denote by $Z_{m_{k}}$ the $m_{k}$ th discrete cyclic group, where $Z_{m_{k}}$ is represented by $\left\{0,1, \ldots, m_{k}-1\right\}$. The so-called Vilenkin group $G_{m}$ is the complete direct product of $Z_{m_{k}}$ 's. The group $G_{m}$ is a compact Abelian group with Haar measure 1, its elements are of the form ( $x_{0}, x_{1}, \ldots, x_{k}, \ldots$ ), where $x_{k} \in Z_{m_{k}}(k \in \mathbf{N})$. The group operation $\dot{+}$ in $G_{m}$ is the $\bmod m_{k}(k \in \mathbf{N})$ addition. The inverse of $\dot{+}$ will be denoted by - . The topology of the group $G_{m}$ is completely determined by the sets

$$
I_{n}:=I_{n}(0):=\left\{\left(x_{0}, x_{1}, \ldots, x_{k}, \ldots\right) \in G_{m}: x_{j}=0 \quad(j=0, \ldots, n-1)\right\}
$$

$\left(0 \neq n \in \mathbf{N}, I_{0}:=G_{m}\right)$. Let $I_{n}(x):=x \dot{+} I_{n}(n \in \mathbf{N})$ be the coset of $I_{n}$ by a given $x \in G_{m}$. The Haar measure of $I_{n}(x)$ is $M_{n}$, where the generalized powers $M_{n}$ $(n \in \mathbf{N})$ are defined in the following way: $M_{0}:=1, M_{n}:=\prod_{j=0}^{n-1} m_{j}(0<$ $<n \in \mathbf{N})$. The symbol $L^{p}(0<p \leq \infty)$ will denote the usual Lebesgue space of complex-valued functions $f$ defined on $G_{m}$ with the norm (or quasinorm) $\|f\|_{p}:=\left(\int|f|^{p}\right)^{1 / p}(p<\infty),\|f\|_{\infty}:=$ ess sup $|f|$.

It is well-known that the characters of $G_{m}$ (the so-called Vilenkin system) form a complete orthonormal system $\widehat{G}_{m}$ in $L^{1}$. To the description of $\widehat{G}_{m}$ let

$$
r_{n}(x):=\exp \frac{2 \pi i x_{n}}{m_{n}}
$$

$\left(n \in \mathbf{N}, x=\left(x_{0}, x_{1}, \ldots\right) \in G_{m}, i:=\sqrt{-1}\right)$ and

$$
\Psi_{n}:=\prod_{k=0}^{\infty} r_{k}^{n_{k}}
$$

where $n=\sum_{k=0}^{\infty} n_{k} M_{k}\left(n_{k} \in Z_{m_{k}}(k \in \mathbf{N})\right)$. Then $\widehat{G}_{m}$ is none other than $\left\{\Psi_{n}: n \in \mathbf{N}\right\}$. In the special case $m_{n}=2(n \in \mathbf{N})$ we get the Walsh-Paley system.

The kernels of Dirichlet type will be denoted by

$$
D_{n}:=\sum_{k=0}^{n-1} \Psi_{k} \quad(n \in \mathbf{N})
$$

The most important property of $D_{n}$ 's is the equality

$$
D_{M_{n}}(x)=\left\{\begin{array}{ll}
M_{n} & \left(x \in I_{n}\right)  \tag{1}\\
0 & \left(x \in G_{m} \backslash I_{n}\right)
\end{array} \quad(n \in \mathbf{N})\right.
$$

The kernels of Fejér type are defined as in the classical case, namely let

$$
K_{n}:=\frac{1}{n} \sum_{k=1}^{n} D_{k} \quad(0<n \in \mathbf{N})
$$

If $m$ is bounded then we get the following estimation for $K_{n}$ 's (see Pál and Simon [5]):

$$
\begin{equation*}
\left|K_{n}(x)\right| \leq C \frac{1}{M_{s}} \sum_{l=0}^{s-1} M_{l} \sum_{i=l}^{s-1} \sum_{j=0}^{m_{l}-1} D_{M_{i}}\left(x \dot{+j} e_{l}\right) \quad\left(x \in G_{m}\right) \tag{2}
\end{equation*}
$$

where $0<n \in \mathbf{N}, M_{s-1} \leq n<M_{s}$ for some $0<s \in \mathbf{N}, j e_{l} \in G_{m}$ is determined by $\left(j e_{l}\right)_{k}=0(\mathbf{N} \ni k \neq l)$ and $\left(j e_{l}\right)_{l}=j$, the constant $C$ is independent on $n$ and $x$.

If $f \in L^{1}$ then $\hat{f}(k):=\int_{G_{m}} f \overline{\Psi_{k}}(k \in \mathbf{N})$ is the usual Fourier coefficient of $f$ with respect to $\widehat{G}_{m}$. Let $S_{n} f(n \in \mathbf{N})$ be the $n$-th partial sum of $f$, i.e.

$$
S_{n} f:=\sum_{k=0}^{n-1} \hat{f}(k) \Psi_{k}
$$

Furthermore, let

$$
\sigma_{n} f:=n^{-1} \sum_{k=1}^{n} S_{k} f \quad(0<n \in \mathbf{N})
$$

be the $n$-th Fejér's mean of $f$. It is clear that $S_{n} f(x)=\int f(t) D_{n}(x \dot{-} t) d t$ and $\sigma_{n} f(x)=\int f(t) K_{n}(x-t) d t\left(n \in \mathbf{N}, x \in G_{m}\right)$. The next equality will be used also in our investigations (PÁL and SIMON [5], see also GÁt [3]):

$$
\begin{equation*}
S_{M_{n}} f(x)-\sigma_{M_{n}} f(x)= \tag{3}
\end{equation*}
$$

$$
=\frac{\boldsymbol{M}_{n}-1}{2} \int_{I_{n}(x)} f-\sum_{k=0}^{n-1} \sum_{y=1}^{m_{k}-1} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{n}\left(x+y e_{k}\right)} f \quad\left(x \in G_{m}, n \in \mathbf{N}\right)
$$

If $f \in L^{1}$ then the so-called (martingale) maximal function of $f$ is given by

$$
f^{*}(x)=\sup _{n}\left|S_{M_{n}} f(x)\right|=\sup _{n} M_{n}\left|\int_{I_{n}(x)} f\right| \quad\left(x \in G_{m}\right)
$$

It is known (see e.g. Weisz [17]) that the maximal operator $L^{p} \ni f \rightarrow f^{*}$ $(1<p \leq \infty)$ is $L^{p}$-bounded, that is,

$$
\begin{equation*}
\left\|f^{*}\right\|_{p} \leq C_{p}\|f\|_{p} \quad\left(f \in L^{p}\right) \tag{4}
\end{equation*}
$$

(From now on $c_{p}, C_{p}, C$ will denote positive constants depending at most on $p$, not always the same at different occurences.)

Define the (martingale) Hardy space $H^{p}$ for $0<p \leq 1$ as the space of $f$ 's for which

$$
\|f\|_{H^{p}}:=\left\|f^{*}\right\|_{p}<\infty
$$

(For details on Hardy spaces as well for the historical background see e.g. the book WEISZ [17].) Then $\|f\|_{H^{p}}$ is equivalent to $\|Q f\|_{p}$, i.e.

$$
c_{p}\|Q f\|_{p} \leq\|f\|_{H^{p}} \leq C_{p}\|Q f\|_{p}
$$

where $Q f$ is the quadratic variation of $f$ :

$$
Q f:=\left(|\hat{f}(0)|^{2}+\sum_{n=0}^{\infty}\left|S_{M_{n+1}} f-S_{M_{n}} f\right|^{2}\right)^{1 / 2}
$$

It is well-known that for bounded $m$ the atomic description of $H^{p}$ plays an important part in the investigations with respect to Hardy spaces. To give this description we recall first the concept of the atoms as follows: the function $a \in L^{\infty}$ is called a $p$-atom if either $a$ is identically equal to 1 or there exists $J:=I_{n}(x)\left(x \in G_{m}, n \in \mathbf{N}\right)$ (called the support of $a$ ) for which

$$
\begin{cases}\text { i) } & \operatorname{supp} a \subset J  \tag{5}\\ \text { ii) } & \|a\|_{\infty} \leq|J|^{-1 / p}:=M_{n}^{1 / p} \\ \text { iii) } & \int a=0\end{cases}
$$

Then $f$ belongs to $H^{p}(0<p \leq 1)$ iff $f$ is given by $f=\sum_{k=0}^{\infty} \lambda_{k} a_{k}$, where each $a_{k}$ is a $p$-atom and $\sum_{k=0}^{\infty}\left|\lambda_{k}\right|^{p}<\infty$. Furthermore, there exist positive constants $c_{p}, C_{p}$ depending only on $p$ such that

$$
c_{p}\|f\|_{H^{p}} \leq \inf \left(\sum_{k=0}^{\infty}\left|\lambda_{k}\right|\right)^{1 / p} \leq C_{p}\|f\|_{H^{p}}
$$

where the infimum is taken over all such decompositions of $f$.
As we remarked above the atomic structure of $H^{p}(0<p \leq 1)$ is very useful in the investigations of Hardy spaces. For example let $\mathscr{J}$ be an operator given at least on $L^{2}$ and assume that $\mathscr{J}$ is $L^{2}$-bounded: $\|\mathscr{T} f\|_{2} \leq C\|f\|_{2}(f \in$ $\in L^{2}$ ). Furthermore, let $\mathscr{J}^{*} f:=\sup _{n}\left|S_{M_{n}}(\mathcal{T} f)\right|$. Then to the $H^{p}$-boundedness
of $\mathscr{T}$, i.e. to $\|\mathscr{T} f\|_{H^{p}} \leq C_{p}\|f\|_{H^{p}}\left(f \in H^{p}\right)$ it is enough to prove that $\mathscr{T}^{*}$ is $p$-quasi-local (see WeISz [16]). This last property means that

$$
\begin{equation*}
\int_{G_{m} \backslash J}\left(\mathcal{T}^{*} a\right)^{p} \leq C_{p}, \tag{6}
\end{equation*}
$$

where $a$ is an arbitrary $p$-atom with support $J$ (see the definition of atoms). For the sake of the completeness we show here this statement. It is clear that we need only to prove $\sup \int\left(\mathcal{I}^{*} a\right)^{p} \leq C_{p}$, where the supremum is taken over all $p$-atoms $a$. Taking into account (6) the last integral can be computed only on the support $J$ of $a$. Then applying Hölder's inequality, (4), the $L^{2}$-boundedness of $\mathcal{I}$ and (5) it follows that

$$
\begin{aligned}
& \int_{J}\left(\mathscr{J}^{*} a\right)^{p} \leq\left(\int_{J}\left(\mathscr{J}^{*} a\right)^{2}\right)^{p / 2}|J|^{1-p / 2} \leq\left\|\mathscr{T}^{*} a\right\|_{2}^{p}|J|^{1-p / 2} \leq \\
& \leq C_{p}\|\mathscr{T} a\|_{2}^{p}|J|^{1-p / 2} \leq C_{p}\|a\|_{2}^{p}|J|^{1-p / 2} \leq\|a\|_{\infty}^{p}|J| \leq C_{p} .
\end{aligned}
$$

## 3. Results

The purpose of this note is to investigate the operator $S$ given by

$$
S f:=\left(\sum_{n=0}^{\infty}\left|S_{M_{n}} f-\sigma_{M_{n}} f\right|^{2}\right)^{1 / 2} \quad\left(f \in L^{1}\right)
$$

introduced and firstly investigated in the Walsh case (i.e. when $m_{n}=2$ for all $n \in \mathbf{N}$ ) by Sunouchi [12], [13]. (For a short history of Sunouchi operator see e.g. Simon [9], Daly and Phillips [1].) In Simon [9] it was proved that in the Walsh case $S: H^{p} \rightarrow L^{p}(0<p \leq 1)$ is bounded. This is the extension of the case $p=1$ (Simon [8]). Moreover, if $1 / 2<p \leq 1$ then for the functions $f \in H^{p}$ with $\hat{f}(0)=0$ the norm $\|f\|_{H^{p}}$ is equivalent to $\|S f\|_{p}$. For $p=1$ this is due to Daly and Phillips [1]. (We remark that by another argument Daly and PHillips [2] showed also the equivalence for all $0<p \leq 1$.) The ( $H^{1}, L^{1}$ )-boundedness of $S$ for Vilenkin system was investigated by GÁt [3].

The first statement of this work is
Theorem 1. Let $m$ be bounded and $0<p \leq 1$. Then $S: H^{p} \rightarrow L^{p}$ is bounded. Moreover, there exist positive constants $c_{p}, C_{p}$ depending only on $p$ such that for all $f \in H^{p}$ with $\hat{f}(0)=0$ we have

$$
\begin{equation*}
c_{p}\|f\|_{H^{p}} \leq\|S f\|_{p} \leq C_{p}\|f\|_{H^{p}} \tag{7}
\end{equation*}
$$

Let $f \in L^{1}$ and write $S f$ in the following form:

$$
\begin{aligned}
S f & =\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{M_{n+1}-1} \frac{k}{M_{n+1}} \hat{f}(k) \Psi_{k}\right|^{2}\right)^{1 / 2}= \\
& =\left(\sum_{n=0}^{\infty}\left|\sum_{j=0}^{n} \sum_{k=M_{j}}^{M_{j+1}-1} \frac{k}{M_{n+1}} \hat{f}(k) \Psi_{k}\right|^{2}\right)^{1 / 2}= \\
& =\left(\sum_{n=0}^{\infty}\left|\sum_{j=0}^{n} \frac{M_{j}}{M_{n+1}} \sum_{k=M_{j}}^{M_{j+1}-1} \frac{k}{M_{j}} \hat{f}(k) \Psi_{k}\right|\right)^{1 / 2} .
\end{aligned}
$$

If

$$
T g:=\sum_{l=0}^{\infty} \sum_{i=M_{l}}^{M_{l+1}-1} \frac{i}{M_{l}} \hat{f}(i) \Psi_{i}
$$

and $\Delta_{j} g:=\sum_{i=M_{j}}^{M_{j+1}-1} \hat{g}(i) \Psi_{i}\left(j \in \mathbf{N}, g \in L^{1}\right)$, then

$$
S f=\left(\sum_{n=0}^{\infty}\left|\sum_{j=0}^{n} \frac{M_{j}}{M_{n+1}} \Delta_{j}(T f)\right|^{2}\right)^{1 / 2}
$$

Define the mapping $\tau$ on the set of the complex-valued sequences as follows: if $a=\left(a_{n}, n \in \mathbf{N}\right)$ is such a sequence then let the sequence $\tau(a):=$ $=b=\left(b_{n}, n \in \mathbf{N}\right)$ be given by

$$
b_{n}:=\frac{1}{M_{n+1}} \sum_{j=0}^{n} M_{j} a_{j}
$$

It is not hard to see that $\tau$ is $\ell_{2}$-bounded:

$$
\begin{aligned}
\|\tau(a)\|_{\ell_{2}} \leq & \left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{M_{j}}{M_{n+1}}\left|a_{j}\right|\right)^{2}\right)^{1 / 2} \leq\left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{1}{2^{n-j}}\left|a_{j}\right|\right)^{2}\right)^{1 / 2}= \\
& =\left\|\left(2^{-n}\right) *\left(\left|a_{n}\right|\right)\right\|_{\ell_{2}} \leq\left\|\left(2^{-n}\right)\right\|_{\ell_{1}}\|a\|_{\ell_{2}}=2\|a\|_{\ell_{2}}
\end{aligned}
$$

where $*$ stands for the usual convolution in $\ell_{2}$. A simple calculation shows that $\tau$ is invertible and if $m$ is bounded then $\tau^{-1}$ is also $\ell_{2}$-bounded.

Therefore

$$
S f=\left\|\tau\left(\Delta_{n}(T f)\right)\right\|_{\ell_{2}} \leq 2\left\|\left(\Delta_{n}(T f)\right)\right\|_{\ell_{2}}=2 Q(T f)
$$

which implies $\|S f\|_{p} \leq C_{p}\|T f\|_{H^{p}}$. Thus if $T: H^{p} \rightarrow H^{p}$ is bounded, then $\|S f\|_{p} \leq C_{p}\|f\|_{H^{p}}$. Now, define $R f$ in the following manner:

$$
R f:=\sum_{l=0}^{\infty} \sum_{i=M_{l}}^{M_{l+1}-1} \frac{M_{l}}{i} \hat{f}(i) \Psi_{i}
$$

Assume that $T, R: H^{p} \rightarrow H^{p}$ are bounded. Then $\|f\|_{H^{p}}=\|R(T f)\|_{H^{p}} \leq$ $C_{p}\|T f\|_{H^{p}}$, i.e. $\|f\|_{H^{p}}$ is equivalent to $\|T f\|_{H^{p}}$. Moreover, in the case $\sup _{n} m_{n}<\infty$ this leads to the next estimation:

$$
\begin{gathered}
\|f\|_{H^{p}} \leq C_{p}\|T f\|_{H^{p}} \leq C_{p}\|Q(T f)\|_{p}=C_{p}\| \|\left(\Delta_{n}(T f)\right)\left\|_{\ell_{2}}\right\|_{p}= \\
=C_{p}\| \| \tau^{-1}\left(\tau\left(\Delta_{n}(T f)\right)\right)\left\|_{\ell_{2}}\right\|_{p} \leq \\
\left.\leq C_{p}\| \| \tau\left(\Delta_{n}(T f)\right)\right)\left\|_{\ell_{2}}\right\|_{p}=C_{p}\|S f\|_{p} \quad\left(f \in H^{p}, \hat{f}(0)=0\right)
\end{gathered}
$$

In other words $\|f\|_{H^{p}}$ is equivalent to $\|S f\|_{p}$. Thus Theorem 1 follows from
THEOREM 2. Let $m$ be bounded, $0<p \leq 1, \mathcal{T} \in\{T, R\}$. Then $\mathcal{J}: H^{p} \rightarrow$ $\rightarrow H^{p}$ is bounded.

The next theorem shows that in general the boundedness of $m$ in the previous theorems is essential.

THEOREM 3. Let $0<p \leq 1$ and assume that $\sup _{n} m_{n}=\infty$. Then $S$ doesn't $\left(H^{p}, L^{p}\right)$-bounded.

We remark that for all unbounded $m$ 's GÁt [3] has constructed a function $f \in H^{1}$ such that $\|S f\|_{1}=\infty$. Therefore in case $\mathrm{p}=1$ Theorem 3 follows from Gát's result. (It is not hard to see that also the converse of this remark is true.) Furthermore, in GÁt [4] the left hand side inequality of (7) is proved for some unbounded $m$ 's in the case $p=1$, namely for $m$ such that $\sum_{n=0}^{\infty} m_{n}^{-2}<\infty$. This means that the boundedness of $m$ doesn't necessary in the corresponding part of Theorem 1. Moreover, it seems that in this connection also the sequences $m$ "unbounded enough" can be considered.

As we have seen the $H^{p}$-boundedness of $T$ is enough for $S: H^{p} \rightarrow L^{p}$ to be bounded. Thus Theorem 3 implies

Corollary 1. Let $0<p \leq 1$ and assume that $\sup _{n} m_{n}=\infty$. Then $T$ doesn't $H^{p}$-bounded.

## 4. Proofs

To the proof of Theorem 2 it is enough to show that the maximal operator

$$
\mathcal{T}^{*} f:=\sup _{n}\left|S_{M_{n}}\left(\mathscr{T}_{f}\right)\right| \quad\left(f \in L^{2}\right)
$$

is $p$-quasi local. First let $\mathcal{G}=T$ and $a$ be a $p$-atom. It can be assumed that the support of $a$ is $I_{N}$ for some $N \in \mathbf{N}$, i.e. $\|a\|_{\infty} \leq M_{N}^{1 / p}$ and $\int_{I_{N}} a=0$. From this it follows evidently that $\hat{a}(j)=0$ for all $j=0, \ldots, M_{N}-1$, thus

$$
T a=\sum_{l=N}^{\infty} M_{l}^{-1} \sum_{j=M_{l}}^{M_{l+1}-1} j \hat{a}(j) \Psi_{j} .
$$

Moreover, $S_{M_{n}}(T a)=0$ if $\mathbf{N} \ni n \leq N$. For $\mathbf{N} \ni n>N$ we have

$$
S_{M_{n}}(T a)=\sum_{l=N}^{n-1} M_{l}^{-1} \sum_{j=M_{l}}^{M_{l+1}-1} j \hat{a}(j) \Psi_{j}
$$

and by (5)

$$
\begin{gathered}
S_{M_{n}}(T a)(x)=\int_{I_{N}} a(t) \sum_{l=N}^{n-1} M_{l}^{-1} \sum_{j=M_{l}}^{M_{l+1}-1} j \Psi_{j}(x \dot{-} t) d t= \\
=\int_{I_{N}} a(t) \sum_{l=N}^{n-1} M_{l}^{-1}\left(M_{l+1}\left(D_{M_{l+1}}(x \dot{-} t)-K_{M_{l+1}}(x \dot{-} t)\right)-\right. \\
\left.\quad-M_{l}\left(D_{M_{l}}(x \dot{-} t)-K_{M_{l}}(x \dot{-} t)\right)\right) d t= \\
=\sum_{l=N}^{n-1}\left(m_{l}\left(S_{M_{l+1}} a(x)-\sigma_{M_{l+1}} a(x)\right)-\left(S_{M_{l}} a(x)-\sigma_{M_{l}} a(x)\right)\right) .
\end{gathered}
$$

Taking into account (3) we get

$$
\begin{aligned}
& S_{M_{n}}(T a)(x)= \\
& =\sum_{l=N}^{n-1}\left(m_{l}\left(\frac{M_{l+1}-1}{2} \int_{I_{l+1}(x)} a-\sum_{k=0}^{l} \sum_{y=1}^{m_{k}-1} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l+1}\left(x+y e_{k}\right)} a\right)-\right. \\
& \left.\quad-\left(\frac{M_{l}-1}{2} \int_{I_{l}(x)} a-\sum_{k=0}^{l-1} \sum_{y=1}^{m_{k}-1} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l}\left(x+y e_{k}\right)} a\right)\right) .
\end{aligned}
$$

If $\mathbf{N} \ni v \geq N$ and $x \notin I_{N}$ then $I_{v}(x) \cap I_{N}=\emptyset$. Similarly, when $\mathbf{N} \ni v, k \geq N$, $y=1, \ldots, m_{k}-1$ then $I_{v}\left(x \dot{+} y e_{k}\right) \cap I_{N}=\emptyset$. This means by supp $a \subset I_{N}$ that $\int_{I_{l+1}\left(x+y e_{k}\right)} a=\int_{I_{l}\left(x+y e_{k}\right)} a=0$ for all $l=N, \ldots, n-1 ; k=N, \ldots, l$; $y=0, \ldots, m_{k}-1$, that is

$$
\begin{aligned}
S_{M_{n}}(T a)(x)= & \sum_{l=N}^{n-1}\left(\sum_{k=0}^{N-1} \sum_{y=1}^{m_{k}-1} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l}\left(x+y e_{k}\right)} a-\right. \\
& \left.-m_{l} \sum_{k=0}^{N-1} \sum_{y=1}^{m_{k}-1} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l+1}\left(x+y e_{k}\right)} a\right) .
\end{aligned}
$$

Denote by $J_{k y}$ the set $I_{N}\left(\left(m_{k}-y\right) e_{k}\right)\left(k=0, \ldots, N-1 ; y=1, \ldots, m_{k}-1\right)$. If $x \notin \cup_{k=0}^{N-1} \cup \sum_{y=1}^{m_{k}-1} J_{k y}$ then $I_{l}\left(x \dot{+} y e_{k}\right) \cap I_{N}=\emptyset$ for $l=N, N+1, \ldots$. Furthermore, if $x \in J_{k y}$ for some $k=0, \ldots, N-1 ; y=1, \ldots, m_{k}-1$ then $\left|S_{M_{n}}(T a)(x)\right|=$
$=\left|\sum_{l=N}^{n-1}\left(\frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l}\left(x+y e_{k}\right)} a-m_{l} \frac{M_{k}}{1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)} \int_{I_{l+1}\left(x+y e_{k}\right)} a\right)\right| \leq$
$\leq 2\|a\|_{\infty} \frac{M_{k}}{\left|1-r_{k}\left(\left(m_{k}-y\right) e_{k}\right)\right|} \sum_{l=N}^{\infty} \frac{1}{M_{l}}=$
$=\|a\|_{\infty} \frac{M_{k}}{\sin \left(\pi y / m_{k}\right)} \sum_{l=N}^{\infty} \frac{1}{M_{l}} \leq C M_{k} M_{N}^{1 / p-1}$.
In other words for $x \in J_{k y}\left(k=0, \ldots, N-1 ; y=1, \ldots, m_{k}-1\right)$ we have $T^{*} a(x) \leq C M_{k} M_{N}^{1 / p-1}$. This implies the next estimation:

$$
\int_{G_{m} \backslash I_{N}}\left(T^{*} a\right)^{p}=\sum_{k=0}^{N-1} \sum_{y=1}^{m_{k}-1} \int_{J_{k y}}\left(T^{*} a\right)^{p} \leq C_{p} M_{N}^{1-p} \sum_{k=0}^{N-1} M_{k}^{p} M_{N}^{-1} \leq C_{p}
$$

which proves our theorem for $\mathscr{J}=T$.
Now, let $\mathcal{J}=R$ and $a$ be a $p$-atom as above. Therefore $S_{M_{n}}(R a)=0$ when $\mathbf{N} \ni n<N$ while

$$
S_{M_{n}}(R a)=\sum_{l=N}^{n-1} M_{l} \sum_{j=M_{l}}^{M_{l+1}-1} \hat{a}(j) \Psi_{j} / j
$$

for $\mathbf{N} \ni n \geq N$. Applying Abel transformation it follows that

$$
\begin{aligned}
& S_{M_{n}}(R a)(x)=\sum_{l=N}^{n-1} M_{l} \int_{I_{N}} a(t)\left(\sum_{j=M_{l}}^{M_{l+1}-3}\left(\frac{1}{j}-\frac{1}{j+2}\right) K_{j+1}(x \dot{-} t)-\right. \\
& -\frac{1}{M_{l}+1} K_{M_{l}}(x \dot{-} t)+\frac{1}{M_{l+1}-2} K_{M_{l+1}-1}(x \dot{-} t)-\frac{1}{M_{l}} D_{M_{l}}(x \dot{-} t)+ \\
& \left.+\frac{1}{M_{l+1}-1} D_{M_{l+1}}(x \dot{-} t)\right) d t \quad\left(x \in G_{m}\right)
\end{aligned}
$$

As in the case $\mathcal{J}=T$ we get $\int_{I_{N}} a(t) D_{M_{v}}(x \dot{-} t) d t=0$ for $x \notin I_{N}$ and for $\mathbf{N} \ni v \geq N$. Thus $S_{M_{n}}(R a)(x)$ has the next decomposition:

$$
\begin{gathered}
S_{M_{n}}(R a)(x)=\sum_{l=N}^{n-1} M_{l} \int_{I_{N}} a(t) \sum_{j=M_{l}}^{M_{l+1}-3}\left(\frac{1}{j}-\frac{1}{j+2}\right) K_{j+1}(x \dot{-} t) d t- \\
-\sum_{l=N}^{n-1} \frac{M_{l}}{M_{l}+1} \int_{I_{N}} a(t) K_{M_{l}}(x \dot{-} t) d t+\frac{M_{l}}{M_{l+1}-2} \int_{I_{N}} a(t) K_{M_{l+1}-1}(x \dot{-} t) d t=: \\
=: U_{n}(x)+V_{n}(x)+Z_{n}(x)
\end{gathered}
$$

Since $\int_{I_{N}} a(t) D_{M_{l}}(x \dot{-} t) d t=0$ it can be written

$$
V_{n}(x)=\sum_{l=N}^{n-1} \frac{M_{l}}{M_{l}+1} \int_{I_{N}} a(t)\left(D_{M_{l}}(x \dot{-} t)-K_{M_{l}}(x \dot{-} t)\right) d t
$$

From this we can deduce the estimation $\underset{G_{m} \backslash I_{N}}{\int}\left(\sup _{n}\left|V_{n}\right|\right)^{p} \leq C_{p}$ in analogous way as in the firs part of the proof.

To show the previous inequality with $Z_{n}$ instead of $V_{n}$ we need the next estimation for $K_{M_{l+1}-1}$ (see Pál-Simon [5]):

$$
\begin{aligned}
& \left|K_{M_{l+1}-1}(u)\right| \leq \\
& \quad \leq C\left(\frac{1}{M_{l+1}} \sum_{k=0}^{l} M_{k} D_{M_{k}}(u)++\frac{1}{M_{l+1}} \sum_{k=0}^{l} M_{k} \sum_{i=k+1}^{l} \sum_{j=1}^{m_{k}-1} D_{M_{i}}\left(u \dot{j} e_{k}\right)\right) \\
& \quad\left(u \in G_{m}\right) .
\end{aligned}
$$

Hence

$$
\begin{gathered}
\left|Z_{n}(x)\right| \leq C \sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{k=0}^{l} M_{k} \int_{I_{N}}|a(t)| D_{M_{k}}(x \dot{-} t) d t+ \\
+\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{k=0}^{l} M_{k} \sum_{i=k+1}^{l} \sum_{j=1}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+} j e_{k} \dot{-} t\right) d t=: \\
=: Z_{n}^{(0)}(x)+Z_{n}^{(1)}(x) \quad\left(x \in G_{m}\right) .
\end{gathered}
$$

Let $x \notin I_{N}$ then (see the considerations of the first part of the proof)

$$
\begin{aligned}
& Z_{n}^{(0)}(x)=\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{k=0}^{N-1} M_{k} \int_{I_{N}}|a(t)| D_{M_{k}}(x \dot{-} t) d t= \\
= & \sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{k=0}^{N-1} M_{k} D_{M_{k}}(x) \int_{I_{N}}|a| \leq C \frac{1}{M_{N}} \sum_{k=0}^{N-1} M_{k} D_{M_{k}}(x) .
\end{aligned}
$$

There exists a unique $v=0, \ldots, N-1$ such that $x \in I_{v} \backslash I_{v+1}$ which implies by (1) $Z_{n}^{(0)}(x) \leq C M_{N}^{-1} \sum_{k=0}^{v} M_{k}^{2}$. Therefore the same estimation holds also for $\sup _{n} Z_{n}^{(0)}(x)$, i.e.

$$
\begin{gathered}
\int_{G_{m} \backslash I_{N}}\left(\sup _{n} Z_{n}^{(0)}\right)^{p}=\sum_{v=0}^{N-1} \int_{I_{v} \backslash I_{v+1}}\left(\sup _{n} Z_{n}^{(0)}\right)^{p} \leq \\
\leq C_{p} \frac{1}{M_{N}^{p}} \sum_{v=0}^{N-1} \frac{1}{M_{v}} \sum_{k=0}^{v} M_{k}^{2 p} \leq C_{p} \frac{1}{M_{N}^{p}} \sum_{v=0}^{N-1} M_{v}^{2 p-1} \leq C_{p} .
\end{gathered}
$$

Decompose $Z_{n}^{(1)}(x)$ as follows:

$$
\begin{gathered}
Z_{n}^{(1)}(x)=\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=1}^{l} \sum_{k=0}^{i-1} M_{k} \sum_{j=1}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+j} e_{k} \dot{-} t\right) d t= \\
=\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=1}^{N-1} \sum_{k=0}^{i-1} M_{k} \sum_{j=1}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+j} e_{k} \dot{-} t\right) d t+ \\
+\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=N}^{l} \sum_{k=0}^{i-1} M_{k} \sum_{j=1}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+} j e_{k} \dot{-} t\right) d t=: Z_{n}^{(10)}(x)+Z_{n}^{(11)}(x) .
\end{gathered}
$$

Taking into consideration the basic property (1) of the kernels it follows for $Z_{n}^{(10)}(x)$ that

$$
Z_{n}^{(10)}(x)=\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=1}^{N-1} \sum_{k=0}^{i-1} M_{k} \sum_{j=1}^{m_{k}-1} D_{M_{i}}\left(x \dot{+j} e_{k}\right) \int_{I_{N}}|a(t)| d t
$$

This yields the next estimation for $\sup _{n} Z_{n}^{(10)}$ :

$$
\sup _{n} Z_{n}^{(10)}(x) \leq C \frac{1}{M_{N}} \sum_{i=1}^{N-1} \sum_{k=0}^{i-1} M_{k} \sum_{j=1}^{m_{k}-1} D_{M_{i}}\left(x \dot{+} j e_{k}\right)
$$

Let $x \in I_{v} \backslash I_{v+1}(v=0, \ldots, N-1)$ then $D_{M_{i}}\left(x \dot{+} j e_{k}\right)=0(i=1, \ldots, v$; $\left.k=0, \ldots, i-1 ; j=1, \ldots, m_{k}-1\right)$. Furthermore, the same equality holds if $i=v+1, \ldots, N-1$ and $k \neq v$, which implies

$$
\sup _{n} Z_{n}^{(10)}(x) \leq C \frac{1}{M_{N}} \sum_{i=v+1}^{N-1} M_{v} M_{i}
$$

and

$$
\begin{aligned}
\int_{G_{m} \backslash I_{N}}\left(\sup _{n} Z_{n}^{(10)}\right)^{p} & =\sum_{v=0}^{N-1} \int_{I v \backslash I_{v+1}}\left(\sup _{n} Z_{n}^{(10)}\right)^{p} \leq \\
& \leq C \frac{1}{M_{N}^{p}} \sum_{v=0}^{N-1} \sum_{i=v+1}^{N-1} \boldsymbol{M}_{v}^{p-1} \boldsymbol{M}_{i}^{p} \leq C_{p}
\end{aligned}
$$

To the estimation of $Z_{n}^{(11)}(x)$ let $x \in J_{k v}(k=0, \ldots, N-1 ; v=$ $\left.=1, \ldots, m_{k}-1\right)$ then

$$
\begin{aligned}
Z_{n}^{(11)}(x) & =\sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=N}^{l} M_{k} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+} v e_{k} \dot{-} t\right) d t \leq \\
& \leq \sum_{l=N}^{n-1} \frac{1}{M_{l+1}} \sum_{i=N}^{l} M_{k}\|a\|_{\infty} \leq \\
& \leq M_{k}\|a\|_{\infty} \sum_{l=N}^{\infty} \frac{l-N+1}{M_{l+1}} \leq C \frac{M_{k}}{M_{N}}\|a\|_{\infty} \leq C M_{k} M_{N}^{1 / p-1}
\end{aligned}
$$

It follows from the last inequality that $\sup _{n} Z_{n}^{(11)}(x) \leq C M_{k} M_{N}^{1 / p-1}$, that is,

$$
\begin{aligned}
\int_{G_{m} \backslash I_{N}}\left(\sup _{n} Z_{n}^{(11)}\right)^{p} & =\sum_{k=0}^{N-1} \sum_{v=1}^{m_{k}-1} \int_{J_{k v}}\left(\sup _{n} Z_{n}^{(11)}\right)^{p} \leq \\
& \leq C_{p} \sum_{k=0}^{N-1} \boldsymbol{M}_{k}^{p} \boldsymbol{M}_{N}^{1-p} \boldsymbol{M}_{N}^{-1} \leq C_{p}
\end{aligned}
$$

Finally, the estimation of $U_{n}(x)\left(x \notin I_{N}\right)$ is needed. To this end we recall the estimation (2) with respect to Fejér's kernels which leads to

$$
\begin{aligned}
& \left|U_{n}(x)\right| \leq \\
& \leq C \sum_{l=N}^{n-1} M_{l} \sum_{j=M_{l}}^{M_{l+1}-3} \frac{2}{j(j+1)} \int_{I_{N}}|a(t)| \sum_{k=0}^{l} \frac{M_{k}}{M_{l+1}} \sum_{i=k}^{l} \sum_{s=0}^{m_{k}-1} D_{M_{i}}\left(x \dot{+} s e_{k} \dot{-} t\right) d t \leq \\
& \leq C \sum_{l=N}^{n-1} \frac{1}{M_{l}} \sum_{i=0}^{l} \sum_{k=0}^{i} M_{k} \sum_{s=0}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+} s e_{k} \dot{-} t\right) d t \leq \\
& \leq C \sum_{l=N}^{n-1} \frac{1}{M_{l}} \sum_{i=0}^{N-1} \sum_{k=0}^{i} M_{k} \sum_{s=0}^{m_{k}-1} D_{M_{i}}\left(x \dot{+} s e_{k}\right)+ \\
& +C \sum_{l=N}^{n-1} \frac{1}{M_{l}} \sum_{i=N}^{l} \sum_{k=0}^{N-1} M_{k} \sum_{s=0}^{m_{k}-1} \int_{I_{N}}|a(t)| D_{M_{i}}\left(x \dot{+} s e_{k} \dot{-} t\right) d t .
\end{aligned}
$$

From this on the estimation can be finished in the same way as with respect to $Z_{n}^{(1 i)}(i=0,1)$ and we get $\int_{G_{m} \backslash I_{N}}\left(\sup _{n}\left|U_{n}\right|\right)^{p} \leq C_{p}$.

This completes the proof of Theorem 2.
Proof of Theorem 3. Let $n \in \mathbf{N}$ and define $f_{n}$ in the following way:

$$
f_{n}:=\sum_{j=M_{n}}^{M_{n+1}-1} \Psi_{j}=D_{M_{n+1}}-D_{M_{n}}
$$

Since $S_{M_{k}}\left(f_{n}\right)=0(\mathbf{N} \ni k \leq n)$ and $S_{M_{k}}\left(f_{n}\right)=f_{n}(\mathbf{N} \ni k>n)$ it follows that $f_{n}^{*}=\left|f_{n}\right|$. Therefore by (1)

$$
\left\|f_{n}\right\|_{H^{p}}=\left\|f_{n}\right\|_{p}=\left(\frac{\left(M_{n+1}-M_{n}\right)^{p}}{M_{n+1}}+M_{n}^{p}\left(\frac{1}{M_{n}}-\frac{1}{M_{n+1}}\right)\right)^{1 / p} \leq
$$

$$
\leq \frac{\boldsymbol{M}_{n}^{1-1 / p}}{m_{n}^{1 / p}}\left(m_{n}^{p}+m_{n}\right)^{1 / p}
$$

On the other hand $S f_{n} \geq M_{n+1}^{-1}\left|\sum_{j=M_{n}}^{M_{n+1}-1} j \Psi_{j}\right|$, therefore

$$
\begin{aligned}
S f_{n} & \geq \frac{1}{M_{n+1}}\left|\sum_{l=1}^{m_{n}-1} \sum_{j=0}^{M_{n}-1}\left(l M_{n}+j\right) \Psi_{l M_{n}+j}\right|= \\
& =\frac{1}{M_{n+1}}\left|\sum_{l=1}^{m_{n}-1} r_{n}^{l} \sum_{j=0}^{M_{n}-1}\left(l M_{n}+j\right) \Psi_{j}\right|= \\
& =\left|\frac{1}{m_{n}}\left(\sum_{l=1}^{m_{n}-1} l r_{n}^{l}\right) D_{M_{n}}+\frac{1}{M_{n+1}}\left(\sum_{l=1}^{m_{n}-1} r_{n}^{l}\right) \sum_{j=0}^{M_{n}-1} j \Psi_{j}\right|
\end{aligned}
$$

Then the $L^{p}$-norm of $S f_{n}$ can be estimated from below as follows:

$$
\begin{aligned}
\left\|S f_{n}\right\|_{p}^{p} \geq & \int_{I_{n}}\left|S f_{n}\right|^{p}=\sum_{k=0}^{m_{n}-1} \int_{I_{n}\left(k e_{n}\right)}\left|S f_{n}\right|^{p} \geq \\
\geq & \sum_{k=0}^{m_{n}-1} \frac{1}{M_{n+1}} \left\lvert\,\left(\sum_{l=1}^{m_{n}-1} l \exp \frac{2 \pi i k l}{m_{n}}\right) \frac{M_{n}}{m_{n}}+\right. \\
& +\left.\frac{1}{M_{n+1}}\left(\sum_{l=1}^{m_{n}-1} \exp \frac{2 \pi i k l}{m_{n}}\right) \frac{M_{n}\left(M_{n}-1\right)}{2}\right|^{p} \geq \\
\geq & \frac{1}{M_{n+1}} \sum_{k=1}^{m_{n}-1}\left|\frac{M_{n}}{\exp \frac{2 \pi i k}{m_{n}}-1}-\frac{M_{n}-1}{2 m_{n}}\right|^{p} \geq \\
\geq & \frac{M_{n}^{p-1}}{m_{n}} \sum_{1 \leq k \leq m_{n} / \pi}\left|\frac{1}{\left|\exp \frac{2 \pi i k}{m_{n}}-1\right|}-\frac{M_{n}-1}{2 M_{n+1}}\right|^{p} \geq \\
\geq & C_{p} \frac{M_{n}^{p-1}}{m_{n}} \sum_{1 \leq k \leq m_{n} / \pi}\left(\frac{m_{n}}{\pi k}-\frac{1}{2}\right)^{p} \geq \\
\geq & C_{p} M_{n+1}^{p-1} \sum_{1 \leq k \leq m_{n} / \pi}\left(\frac{1}{k}-\frac{\pi}{2 m_{n}}\right)^{p} \geq
\end{aligned}
$$

$$
\geq C_{p} M_{n+1}^{p-1} \sum_{1 \leq k \leq m_{n} / \pi} \frac{1}{k^{p}} \geq \begin{cases}C_{1} \log m_{n} & (p=1) \\ C_{p} M_{n}^{p-1} & (0<p<1)\end{cases}
$$

Therefore $\left\|S f_{n}\right\|_{p} \geq C_{p} M_{n}^{1-1 / p}(0<p<1)$ and $\left\|S f_{n}\right\|_{1} \geq C_{1} \log m_{n}$. Now, assume that $S: H^{p} \rightarrow L^{p}$ is bounded. Then the next inequalities would be true:
$\boldsymbol{M}_{n}^{1-1 / p} \leq C_{p} \frac{\boldsymbol{M}_{n}^{1-1 / p}}{m_{n}^{1 / p}}\left(m_{n}^{p}+m_{n}\right) \quad(0<p<1)$ and $\log m_{n} \leq C_{1} \quad(p=1)$,
that is, $m_{n}^{1 / p} \leq C_{p}\left(m_{n}^{p}+m_{n}\right)(0<p<1, n \in \mathbf{N})$ and $\log m_{n} \leq C_{1}$ ( $p=1, n \in \mathbf{N}$ ), resp. It is clear that these fail to hold if $\sup _{n} m_{n}=\infty$. This contradiction shows Theorem 3.

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# NOTE ON NORMAL NUMBERS AND UNIFORM DISTRIBUTION OF SEQUENCES 

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## 1. Introduction

A theorem due to Wall states [1], [2] that a real number $s$ expressed in base $b$ is normal in base $b$ if and only if the sequence $\left(u_{n}\right)=\left(b^{n} s\right)$ is uniformly distributed modulo one. We will give sequences based on partial sums that define $s$, which are $U D \bmod 1$ if and only if $s$ is normal. Suppose that $s=\lim _{n \rightarrow \infty} s_{n}, s_{n}=\sum_{0}^{n} h_{i}$. If $s_{n}$ converges rapidly enough to $s$ then the sequence ( $b^{n} s_{n}$ ) might be expected to be a good enough approximation to $\left(b^{n} s\right)$ to itself be $U D \bmod 1$. We give a condition that this is the case, $s$ normal. If $h_{i} \in 2$ (2 denotes the rational numbers) then the sequence may be usefully computable. Long strings of digits of $s_{n}$ may change on carries on the addition of the next general term to the partial sum and we give a sequence that corrects these digits. This gives examples of sequences that have suffered some modification, yet remain $U D \bmod 1$. The quantity $b^{n} s$ has an integer part that grows without bound. If these lost digits are collected in a certain sequence of terminating rationals, their uniform distribution is shown to be equivalent to the normality of $s$.

These new sequences are of interest in dynamical systems theory where the shift operation $(x \rightarrow b x \bmod 1)$ has long been a fundamental tool. Thus Wall's sequence can be found in a direct interpretation of abstract symbols as digits in the symbolic dynamics of the shift on sequences [3] and in the baker's automorphism [4].

## 2. The case of partial sums

In base $b$ arithmetic, suppose that $s=\lim _{n \rightarrow \infty} s_{n}, s_{n}=\sum_{0}^{n} h_{i}$. Then if the partial sum converges rapidly enough to a normal number $s$ the sequence ( $b^{n} s_{n}$ ) is indeed $U D \bmod 1$ as follows. Recall that Weyl's criterion [5] for uniform distribution states that a real sequence $\left(\omega_{n}\right)$ is $U D \bmod 1$ if and only if $\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N} \exp \left(2 \pi i k \omega_{n}\right)=0$, for all $k \in \mathcal{N}$ (natural numbers without zero).

THEOREM 1. In base $b$ arithmetic, suppose $s=\sum_{i=0}^{\infty} h_{i}, h_{i} \geq 0, s=s_{n}+r_{n}$ where $s_{n}$ is the $n$ 'th partial sum and $r_{n}$ is the remainder. Suppose in addition that $b^{n} r_{n} \rightarrow 0$ as $n \rightarrow \infty$. Then $\left(b^{n} s_{n}\right)$ is $U D \bmod 1$ if and only if $s$ is normal in base $b$.

Proof. From Wall's sequence we have $\left(b^{n} s\right)=\left(b^{n} s_{n}+b^{n} r_{n}\right)$ and from Weyl's criterion we get

$$
\begin{gathered}
\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N} \exp \left(2 \pi i k b^{n} s\right)= \\
=\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N} \exp \left(2 \pi i k b^{n} s_{n}\right)\left(\exp \left(2 \pi i k b^{n} r_{n}\right)-1\right)+ \\
+\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N} \exp \left(2 \pi i k b^{n} s_{n}\right), \quad \text { for all } k \in \mathcal{N} .
\end{gathered}
$$

Suppose that $s$ is normal in base $b$. Then Wall's sequence is $U D \bmod 1$ and by Weyl's criterion, the left hand side is zero. But if $b^{n} r_{n} \rightarrow 0$ then the first term on the right is zero since $\exp \left(2 \pi i k b^{n} r_{n}\right)-1 \rightarrow 0$ ([5], Part 1, II, lemma 1), and so by Weyl's criterion $\left(b^{n} s_{n}\right)$ is $U D \bmod 1$.

Suppose that $\left(b^{n} s_{n}\right)$ is $U D \bmod 1$ and that $b^{n} r_{n} \rightarrow 0$ as $n \rightarrow \infty$. Then the right hand side is zero by the lemma of [5] and Weyl's criterion. But then $\left(b^{n} s\right)$ is $U D \bmod 1$ by Weyl's criterion and so $s$ is normal in base $b$ by Wall's theorem.

This result is stronger than Wall's theorem as a given completed real can be regarded as an infinitely rapidly converging sequence, that is convergence
is achieved in a finite number of terms. If $h_{n} \in \mathscr{Q}$ and is an explicitly given function of $n$, then each term in the new sequence can certainly be exactly computed.

Now however fast a series might converge, the addition of the next general term may alter digits a long way back up the digit string of the partial sum. Indeed, all normal numbers have arbitrarily long strings of zeros infinitely often and these may occasionally be born in the event of a carry up a long string of $b-1$ 's (nines in base 10) in the next partial sum. It happens then that $\left(b^{n} s_{n}\right)$ sometimes has $b-1$ 's where $\left(b^{n} s\right)$ has zeros. Indeed, these $b-1$ 's can appear at the decimal point of $b^{n} s_{n} \bmod 1$ when the string grows from time $n$, with at least $n \quad b-1$ 's and we expect a bias towards counts of visits to intervals of the form $\left[0 . b^{\prime} \ldots, 0 . b^{\prime} \ldots\right.$ ) where $b^{\prime}=b-1$. We note that there is no compensating tendency for long strings of zeros to become $b-1$ 's in the case that $h_{i} \geq 0$. This difference between Wall's sequence ( $b^{n} s$ ) and the sequence for partial sums ( $b^{n} s_{n}$ ) does not affect the uniform distribution of the latter, by the previous result.

Now it is possible to see ahead down the list of partial sums to test for this event and adjust the sequence $\left(b^{n} s_{n}\right)$ accordingly. Such a sequence essentially accepts the effects of sets of general terms at each time step rather than just one term, the associated partial sums thus converge faster than the standard sum and so we again expect a uniformly distributed sequence.

Place an upper bound on the general term by

$$
h_{n}<b^{-f_{n}}
$$

where $f_{n}: \mathcal{N} \rightarrow \mathscr{R}$ and is monotone increasing. It is straightforward to show that if

$$
\begin{gather*}
f_{n+1}-f_{n}>n \\
r_{n} \leq \frac{1}{1-b^{-1}} b^{-f_{n+1}} \leq b^{1-f_{n+1}} \tag{2}
\end{gather*}
$$

Then $b^{n} r_{z} \rightarrow 0$ as $n \rightarrow \infty$ if $f_{n}>n$, that is $\left(b^{n} s_{n}\right)$ is $U D \bmod 1$ under the constraints

$$
\begin{equation*}
f_{n}>n \quad \text { and } \quad f_{n+1}-f_{n}>n . \tag{3}
\end{equation*}
$$

Now note from (2) that the number of zeros in the remainder is $f_{n+1}-2$ or more. But then if (here index $f_{n}$ is integer part of $f_{n}$ )

$$
s_{n}=0 \cdot a_{1} a_{2} \ldots a_{f_{n+1}-2} a_{f_{n+1}-1} a_{f_{n+1}} \ldots
$$

and

$$
r_{n}=0.00 \ldots 00 b_{f_{n+1}-1} b_{f_{n+1}} \ldots
$$

then the only way that $a_{f_{n+1}-2}$ can be affected by all subsequent terms is if $a_{f_{n+1}-1}=b-1$ because a carry is then possible. Now if $a_{f_{n+1}-1} \neq b-1$ then all digits up to and including $a_{f_{n+1}-2}$ can be validated as digits of $s$ and we accept $b^{n} s_{n}$ as a term in the sequence. If $a_{f_{n+1}-1}=b-1$ then go to $s_{n+1}$ and check $a_{f_{n+2}-1}$; if it is not $b-1$ skip $s_{n}$ and accept $b^{n} s_{n+1}$ and $b^{n+1} s_{n+1}$ as the next two terms in the sequence; if it is $b-1$ then go to $s_{n+2}$ and so on. We are thus searching for the least $v_{n} \geq 0$ such that the digit $a_{f_{n+1+v_{n}}-1} \neq b-1$. We borrow the unbounded minimization operator $\mu$ from recursive function theory [6] and write

$$
v_{n}=\mu v_{n}\left(a_{f_{n+1+v_{n}}} \neq b-1\right)
$$

read ' $v_{n}$ is the least $v_{n}$ such that in the representation of $s_{n+v_{n}}$ the digit $a_{f_{n+1+v_{n}}} \neq b-1$ '. The operator is unbounded because a real number may end in repeated $b-1$ 's (that is it is a terminating rational) and the value of $v_{n}$ may diverge. This of course cannot happen here where $s$ is a normal number, that is $s \in \mathscr{R} \backslash \mathscr{Q}$.

Now where Wall's sequence has terms

$$
u_{n}=b^{n} s
$$

where the partial sums give the sequence with terms

$$
v_{n}=b^{n} s_{n}
$$

we have the new sequence with terms

$$
w_{j}=b^{j} s_{n+v_{n}}, \quad n \leq j \leq n+v_{n}
$$

This advances regularly with $j=2,3, \ldots$ (so that the first digit of the partial sum can be checked) but selects groups of general terms in periods of $1+v_{n}$ as it meets the condition on $b-1$ 's; $v_{n}$ may be irregular. It ensures that the first $f_{n+1+v_{n}}-2$ digits of $s$ are correct and that long strings of $b-1$ 's have been switched to zeros.

THEOREM 2. In base $b$ arithmetic, suppose $s=\sum_{i=0}^{\infty} h_{i}=s_{n}+r_{n}$ where $s_{n}$, is the $n$ 'th partial sum and $r_{n}$ is the remainder. Suppose in addition that $b^{n} r_{n} \rightarrow 0$ as $n \rightarrow \infty$. Then $\left(w_{n}\right)$ is $U D \bmod 1$ if and only if $s$ is normal in base $b$.

Proof. Within the period $n \leq j \leq n+v_{n}$, we have

$$
w_{j}=b^{j} s_{n+v_{n}}=b^{j} s-b^{j} r_{n+v_{n}}
$$

The proof follows as in Theorem 1 with the only change that in (1) we substitute $r_{n}$ with $r_{n+v_{n}}$ and take the sum in groups of terms of length $1+v_{n}$; since $r_{n+v_{n}} \leq r_{n}$, the lemma of [5] again applies.

Note that if $f_{n} \equiv n$, on the margin of the constraint (3), we have $r_{n} \leq b^{-n}$ and for $n=2,3, \ldots$ we see that in general we can only validate the $n-2$ 'th digit and that after $n$ shifts we cannot validate any digits of $s$ in $w_{j}$. It is in this case that the uniform distribution of the sequence is not ensured. The constraint $f_{n}>n$ ensures that digits of $s$ survive in $w_{j}$. The constraint $f_{n+1}-f_{n}>n$ ensures that the number of valid digits of $s$ in $w_{j}$ increases with time. In these cases $\left(v_{n}\right)$ and $\left(w_{n}\right)$ are $U D \bmod 1$.

## 3. A theorem for lost digits

That Wall's sequence is $U D \bmod 1$ is equivalent to saying that the fractional part $\left\{b^{n} s\right\}$ is $U D \bmod 1$. Let Wall's sequence $\left(b^{n} s\right)$ be given in the equivalent form $\left(u_{n}\right)=\left(\left\{b^{n} s\right\}\right)$. Construct the sequence $\left(t_{n}\right)$ by the recurrence relation

$$
t_{n}=\frac{1}{b}\left(t_{n-1}+\left\lfloor b u_{n-1}\right\rfloor\right), \quad u_{0}=0
$$

where $\lfloor r\rfloor$ indicates integer part of $r$. Then if $s=0 . a_{1} a_{2} a_{3} \ldots$ where the $a_{i}$ are digits, $b^{n}$ shifts the decimal point of $s, n$ places to the right thus

$$
b^{n} s=a_{1} a_{2} a_{3} \ldots a_{n} \cdot a_{n+1} \ldots
$$

We see that

$$
u_{n}=0 \cdot a_{n+1} a_{n+2} \ldots\left(=b^{n} s\right)
$$

(where $\left(u_{n}\right)$ is $\left.U D \bmod 1\right)$ and

$$
t_{n}=0 . a_{n} a_{n-1} \ldots a_{2} a_{1}
$$

Clearly the sequence $\left(t_{n}\right)$ is composed exactly of those digits of the integer part of $u_{n}$ which are discarded modulo one in Wall's sequence. This sequence is familiar in the baker's automorphism in dynamical systems theory [4].

Recall [2] that $s$ is normal in base $b$ in the sense of Borel if

$$
\lim _{n \rightarrow \infty} \frac{1}{n} N\left(B_{k}, A_{n}\right)=b^{-k}, \quad \text { any natural number } k
$$

Here, if $s=0 . a_{1} a_{2} a_{3} \ldots$, then $A_{n}$ is a block $a_{1} a_{2} a_{3} \ldots a_{n}$ of the first $n$ digits of $a, B_{k}$ is a subblock of $k \leq n$ arbitrary digits and $N\left(B_{k}, A_{n}\right)$ is the number of occurrences of $B_{k}$ in $A_{n}$.

Recall [5] that a real sequence $\left(\omega_{n}\right)$ is $U D \bmod 1$ if

$$
\lim _{n \rightarrow \infty} \frac{1}{n} N(p)=l(p) .
$$

Here $p$ is any subinterval of the unit interval $I=[0,1)$ and $N(p)$ is the number of terms of a sequence $\omega_{1}, \omega_{2}, \omega_{3}, \ldots, \omega_{n}$ of length $n, 0 \leq \omega_{i}<1$, that lie in $p ; l(p)$ is the length of $p$.

Lemma 3. Wall's sequence $\left(u_{n}\right)$ is $U D \bmod 1$ if and only if the sequence of lost digits $\left(t_{n}\right)$ is $U D \bmod 1$.

Proof. For $k \geq 1, b_{i} \in\{0,1,2, \ldots, b-1\}$ consider any semi-open interval of the form

$$
p=\left[0 . b_{1} b_{2} \ldots b_{k}, 0 . b_{1} b_{2} \ldots \overline{b_{k}+1}\right) .
$$

Then for each $p$ there is a unique interval $\bar{p}$ given by

$$
\bar{p}=\left[0 . b_{k} b_{k-1} \ldots b_{2} b_{1}, 0 . b_{k} b_{k-1} \ldots b_{2} \overline{b_{1}+1}\right)
$$

and vise versa. Both $p$ and $\bar{p}$ are of length $b^{-k}$. Now suppose that $\left(u_{n}\right)$ is $U D \bmod 1$. Then $\left(u_{n}\right)$ appears in $p$ with limiting frequency $\frac{1}{n} N(p) \rightarrow b^{-k}$. But for every count of ( $u_{n}$ ) in $p$ (when the first $k$ digits of $u_{n}$ are $b_{1} b_{2} \ldots b_{k}$ ) a count is made of $t_{n+k}$ in $\bar{p}$ (when the first $k$ digits of $v_{n}$ are $b_{k} b_{k-1} \ldots b_{2} b_{1}$ ) so $\frac{1}{n} N(\bar{p}) \rightarrow b^{-k}$ for $\left(t_{n}\right)$. This holds for all $p$ and hence $\bar{p}$ and so $\left(t_{n}\right)$ is $U D \bmod 1$ [5]. If $\left(t_{n}\right)$ is $U D \bmod 1$ then $\left(u_{n}\right)$ is similarly $U D \bmod 1$.

Theorem 4. A real number is normal in base $b$ if and only if the sequence of lost digits ( $t_{n}$ ) is $U D \bmod 1$.

Proof. By Wall's theorem and Lemma 3.
Only one digit is added to $t_{n}$ at a time and so we generate a sequence of terminating rational numbers of growing length. Again these may in principle be exactly computed. If $s$ has finite integer part then $u_{0}$ is suitably adjusted to take these digits. For partial sums, $b-1$ 's can be carried through to a sequence $\left(t_{n}^{\prime}\right)$ of lost digits before the carry takes place in which case $\left(t_{n}^{\prime}\right)$ has $b-1$ 's where $\left(t_{n}\right)$ has zeros. Although $\left(v_{n}\right)$ is $U D \bmod 1$ we do not have a proof that $\left(t_{n}^{\prime}\right)$ is $U D \bmod 1$. However the lost digits of $\left(w_{n}\right)$ are the same as those of $\left(u_{n}\right)$ and they are $U D \bmod 1$.

The quantitative measure of uniformity of distribution of the first $n$ terms of a sequence $\omega$ is the discrepancy of the sequence [5]. The bias towards $b-$ -1 's will presumably be reflected in the discrepancy of $\left(w_{n}\right)$ while the relative
discrepancies of $\left(t_{n}\right),\left(u_{n}\right),\left(v_{n}\right),\left(w_{n}\right)$ are of obvious interest, especially in relation to computation.

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# ON INHOMOGENEOUS VELOCITY BOUNDARY CONDITIONS IN THE FÖRSTE MODEL OF A RADIATING, VISCOUS, HEAT CONDUCTING 

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## 1. Introduction

In FORSTE [2], a model has been proposed for the stationary flow of a radiating, viscous and heat conducting fluid. Apparently, this is the only paper in which, simultaneously, such important characteristics of real industrial processes have been taken into account, as: three-dimensionality, influence of temperature and radiation on fluid flow.

The paper of FÖRSTE shows a way to prove existence and uniqueness of a weak solution under homogeneous velocity boundary conditions, and also contains the assertion that the approach ensures uniqueness for heat conduction and viscosity coefficients sufficiently large and for absorption coefficients and solution domain sufficiently small; moreover, it announces that it should be possible to handle inhomogeneous Dirichlet boundary conditions for the velocity.

When going through the arguments of J. FORSTE in our paper GERGO, StOYAN [3], we found it necessary to inspect all constants in the estimates in order to prove the uniqueness. The result was that uniqueness can be shown under the single condition of a sufficiently small solution domain; uniqueness for appropriate coefficients remained unclear, and the question of inhomogeneous velocity boundary conditions was not tackled.

In the present paper we generalize the existence theorem of FORSTE [2] to the case of inhomogeneous Dirichlet data for the velocity. Moreover, for this more general case we prove also uniqueness. Our result shows that the

[^2]original conjecture of Förste (though concerning homogeneous boundary conditions: uniqueness for sufficiently large physical parameters and sufficiently small diameter of the domain) consists of two independent parts (uniqueness for either sufficiently large parameters or for a sufficiently small domain).

## 2. The Förste model and its weak solution

Let $\Omega \subset R^{3}$ be a bounded domain with Lipschitz boundary $\Gamma$. For $x=\left(x_{1}, x_{2}, x_{3}\right) \in \Omega$ we consider the following system of equations, see FORSTE [2], which represent the physical conservation laws of impulse, mass, inner and radiated energy:

$$
\begin{equation*}
\rho(\vec{v} \operatorname{grad}) \vec{v}+\operatorname{grad} p=\mu \Delta \vec{v}+\vec{f}_{0}\left(T-T_{0}\right) \tag{1}
\end{equation*}
$$

$$
\begin{align*}
(\vec{v} \mathrm{grad}) T & =\lambda \Delta T-4 \alpha_{P}\left(\sigma T^{4}-\pi I_{m}\right)  \tag{3}\\
0 & =\Delta I_{m}+\frac{3 \alpha_{R} \alpha_{P}}{\pi}\left(\sigma T^{4}-\pi I_{m}\right)
\end{align*}
$$

Along with these differential equations, the following boundary conditions are considered:

$$
\vec{v}=\left.\vec{q}\right|_{\Gamma}, \quad T=\vartheta, \quad I_{m}=I_{m, 0}, \quad x \in \Gamma .
$$

Above, we have used the following notations for the unknowns to be determined:

- $\vec{v}=\left(v_{1}, v_{2}, v_{3}\right)^{T}$ is the velocity vector and $\left.\vec{q}\right|_{\Gamma}$ its boundary value,
- $p$ is pressure,
- $T$ is temperature,
- $I_{m}$ denotes the radiation intensity (mean value over all frequencies and directions).
Moreover, the following constants are occurring:
- $\rho$ is the density of the fluid, $\mu$ its viscosity, $\vec{f}_{0}$ the vector of earth acceleration multiplied by the extension coefficient (as resulting from the Boussinesq approximation),
- $\lambda$ is the coefficient of heat conductivity, $\alpha_{P}$ and $\alpha_{R}$ are the Planck and the Rosseland absorption coefficients, respectively,
- $\sigma$ is the Stefan-Boltzmann constant.

With the usual notations for Sobolev spaces, see ADAMS [1], we assume that $I_{m, 0}, \vartheta \in H^{1 / 2}(\Gamma)$ and hence can be continued into all of $\Omega$ to define functions of $H^{1}(\Omega)$; we further suppose $T_{0} \in L_{2}(\Omega)$.

In order to be able to take into account inflow and outflow across $\Gamma$, as a generalization of the boundary condition $\left.\vec{v}\right|_{\Gamma}=0$, in this paper we consider the inhomogeneous boundary condition $\left.\vec{v}\right|_{\Gamma}=\left.\vec{q}\right|_{\Gamma}$. In GUNZBURGER, PETERSON [4] a similar investigation has been performed for the NavierStokes equations.

The velocity space is then

$$
\vec{V}:=\left\{\vec{v} \in\left(H^{1}(\Omega)\right)^{3},(\operatorname{div} \vec{v}, p)_{0}=0 \quad \text { for all } p \in L_{2}(\Omega)\right\}
$$

instead of

$$
\vec{V}_{0}:=\left\{\vec{u} \in\left(H_{0}^{1}(\Omega)\right)^{3} ;(\operatorname{div} \vec{u}, p)_{0}=0 \quad \text { for all } P \in L_{2}(\Omega)\right\}
$$

which latter space serves here as the space of the velocity test functions $\vec{w}$.
Concerning the boundary data $\left.\vec{q}\right|_{\Gamma}$ for the velocity, we assume that $\left.\vec{q}\right|_{\Gamma} \in$ $\in\left(H^{1 / 2}(\Gamma)\right)^{3}$ and satisfies the solvability condition $\left.\int_{\Gamma} \vec{n} \cdot \vec{q}\right|_{\Gamma} \mathrm{ds}=0$. Then $\left.\vec{q}\right|_{\Gamma}$ can be continued into $\Omega$ defining there a function $\vec{q} \in \vec{V}$ with trace $\left.\vec{q}\right|_{\Gamma}$ and with the property

$$
\begin{equation*}
\|\vec{q}\|_{\vec{V}} \leq c_{1 / 2}\left\|\left.\vec{q}\right|_{\Gamma}\right\|_{1 / 2, \Gamma} \tag{5}
\end{equation*}
$$

We now look for weak solutions $\vec{v}, T, I_{m}$ with $\vec{v}=\vec{q}+\vec{z} \in \vec{V}$ and $T$, $I_{m} \in H^{1}(\Omega)$. Then $\vec{z}:=\vec{v}-\vec{q} \in \vec{V}_{0}, \tau:=T-\vartheta, i:=I_{m}-I_{m, 0} \in H_{0}^{1}(\Omega)$.

We shall denote both the $L_{2}(\Omega)$ and the $\left(L_{2}(\Omega)\right)^{3}$ scalar products by $(\cdot, \cdot)_{0}$, and both the $H_{0}^{1}(\Omega)$ and $\left(H_{0}^{1}(\Omega)\right)^{3}$ scalar products by $(\cdot, \cdot)_{1}$, e.g.

$$
\begin{align*}
(\tau, t)_{1} & :=\int \sum_{k=1}^{3} \operatorname{grad} \tau \operatorname{grad} t \mathrm{~d} \Omega, \quad \tau, t \in H_{0}^{1}  \tag{6}\\
(\vec{v}, \vec{w})_{1} & :=\int \sum_{k=1}^{3} \operatorname{grad} v_{k} \operatorname{grad} w_{k} \mathrm{~d} \Omega, \quad \vec{v}, \vec{w} \in \vec{V}_{0}
\end{align*}
$$

and for the corresponding norms, we use the notation $\|\cdot\|_{L_{2}}$ and $|\cdot|_{1}$. Further, when $|\cdot|$ is applied to a constant vector resp. to $\Omega$, then it denotes the Euclidean norm resp. the volume.

Finally, for $\vec{u}, \vec{v}, \vec{w} \in \vec{V}$ we introduce the trilinear form

$$
\begin{equation*}
a_{1}(\vec{u}, \vec{v}, \vec{w}):=\int \sum_{k=1}^{3}\left(\vec{u} \cdot \operatorname{grad} v_{k}\right) w_{k} \mathrm{~d} \Omega . \tag{8}
\end{equation*}
$$

Then, the weak solution $(\vec{v}, \tau, i) \in \vec{V} \times H_{0}^{1} \times H_{0}^{1}$ is defined by the following variational problem in which $\vec{v}=\vec{q}+\vec{z}$ with $\vec{z} \in H_{0}^{1}$, and $\vec{w}, t, j$ are test functions from $\vec{V}_{0} \times H_{0}^{1} \times H_{0}^{1}$ :

$$
\begin{align*}
\mu(\vec{v}, \vec{w})_{1}= & -\rho a_{1}(\vec{v}, \vec{v}, \vec{w})+\left(\vec{f}_{0}\left(\tau+\vartheta-T_{0}\right), \vec{w}\right)_{0},  \tag{9}\\
\lambda(\tau, t)_{1}= & ((\tau+\vartheta) \vec{v}, \operatorname{grad} t)_{0}-\lambda(\vartheta, t)_{1}- \\
& -\alpha\left(\sigma|\tau+\vartheta|^{3}(\tau+\vartheta)-\pi\left(i+I_{m, 0}\right), t\right)_{0},  \tag{10}\\
(i, j)_{1}= & \alpha \beta\left(\sigma|\tau+\vartheta|^{3}(\tau+\vartheta)-\pi\left(i+I_{m, 0}\right), j\right)_{0} . \tag{11}
\end{align*}
$$

In the variational problem (9)-(11), we have introduced the constants $\alpha:=$ $=4 \alpha_{P}$ and $\beta:=\frac{3}{4 \pi} \alpha_{R}$; the equation of mass conservation has been absorbed into the definition of $\vec{V}_{0}$. We remark that in Gergó, Stoyan [3] instead of $\alpha \beta$ the notation $\beta$ was used in the $i$-equation, here labeled (11).

In our investigation below an important role play continuous embeddings. For an introduction into this concept see Adams [1]. We collect here some information about the constants in the embedding inequalities.

1. The embedding $H_{0}^{1_{c}} \rightarrow L_{q}$ is continuous for $1 \leq q \leq 6$. For the constant $c_{q}$ of this embedding, i.e. the constant $c_{q}$ in

$$
\begin{equation*}
\|u\|_{L_{q}} \leq c_{q}|u|_{1}, \tag{12}
\end{equation*}
$$

there holds

$$
\begin{equation*}
c_{q} \leq O\left(d^{(6-q) /(2 q)}\right), \quad 1 \leq q \leq 6 \tag{13}
\end{equation*}
$$

Here $d$ is the diameter of $\Omega$. For a proof see Gergo, Stoyan [3] (taking into account that there $d^{2} / 6$ has been used erroneously as an upper estimate of $c_{2}$, the constant in (12) for $q=2$. In fact, $d^{2} / 6$ is an upper estimate of $c_{2}^{2}$.
2. For the constants $\bar{c}_{q}$ of the continuous embedding $H^{1} \hookrightarrow L_{q}$ :

$$
\begin{equation*}
\|u\|_{L_{q}} \leq \bar{c}_{q}\|u\|_{H^{1}}=\bar{c}_{q}\left\{\|u\|_{L_{2}}^{2}+|u|_{1}^{2}\right\}^{1 / 2} \tag{14}
\end{equation*}
$$

we have $\bar{c}_{q} \geq|\Omega|^{\frac{2-q}{2 q}}$ for $q \geq 2$ as follows from (14) by inserting $u \equiv 1$. This fact must be taken into account when we are going to prove a uniqueness
theorem like in GERGÓ, StOYAN [3] for a sufficiently small diameter of $\Omega$ : we must avoid using (14) and, instead, split functions from $H^{1}$ like $T=\vartheta+\tau$ into a boundary part $\vartheta$ and a $H_{0}^{1}$-part $\tau$ and use embedding only for this latter part, see (12) and (13).
3. Let $q>p$. For the embedding constant $c_{p, q}$ of $L_{q} \hookrightarrow L_{p}$ :

$$
\begin{equation*}
\|u\|_{L_{p}(\Omega)} \leq c_{p, q}\|u\|_{L_{q}(\Omega)} \quad \text { for all } u \in L_{q}(\Omega) \tag{15}
\end{equation*}
$$

we have

$$
\begin{equation*}
c_{p, q}=|\Omega|^{\frac{q-p}{q p}} . \tag{16}
\end{equation*}
$$

In fact, the upper estimate $c_{p, q} \leq|\Omega|^{\frac{q-p}{q p}}$ follows from an application of a Hölder inequality to $\|u\|_{L_{p}}$, whereas the corresponding lower estimate is obtained by inserting $u \equiv 1$ into (15).

## 3. Existence in case of inhomogeneous boundary values of the velocity

THEOREM 1. Let the velocity boundary data $\left.\vec{q}\right|_{\Gamma}$ satisfy $\left.\vec{q}\right|_{\Gamma} \in\left(H^{1 / 2}(\Gamma)\right)^{3}$, $\left.\int_{\Gamma} \vec{n} \cdot \vec{q}\right|_{\Gamma} \mathrm{ds}=0$ and have sufficiently small norm, see condition (18) below. Then the equations of the Förste-model in variational form, i.e. (9)-(11), possess a solution $\vec{v}=\vec{q}+\vec{z} \in \vec{V}, T=\theta+\tau, I_{m}=I_{m, 0}+i \in H^{1}(\Omega)$.

Proof. Our proof parallels that of GERGÓ, StOYAN [3] for homogeneous boundary values of the velocity.

On the basis of (9)-(11), operator equations in $\vec{V} \times H_{0}^{1} \times H_{0}^{1}$ can be introduced, see Gergó, Stoyan [3]. Since the presence of inhomogeneous boundary conditions for the velocity does not influence the complete continuity of those operators, we may concentrate on the boundedness of possible weak solutions. Their existence then follows from the Leray-Schauder fixed point theorem, see FÖrste [2], Gergó, Stoyan [3].

To derive an estimate for possible solutions of (9)-(11), we remember that, in (9), $\vec{v}=\vec{z}+\vec{q}$ with $\vec{z} \in \vec{V}_{0}$, and inserting $\vec{w}=\vec{z}$ we obtain

$$
\begin{align*}
\mu(\vec{z}, \vec{z})_{1}=- & \rho\left\{a_{1}(\vec{z}, \vec{z}, \vec{z})+a_{1}(\vec{q}, \vec{z}, \vec{z})+a_{1}(\vec{z}, \vec{q}, \vec{z})+a_{1}(\vec{q}, \vec{q}, \vec{z})\right\}+ \\
& +\left(\vec{f}_{0}\left(T-T_{0}\right), \vec{z}\right)_{0}-\mu(\vec{q}, \vec{z})_{1} . \tag{17}
\end{align*}
$$

Since it is well known that for $\vec{u}, \vec{q} \in V$ and $\vec{z} \in \vec{V}_{0}$ there holds $a_{1}(\vec{u}, \vec{z}, \vec{q})=$ $=-a_{1}(\vec{u}, \vec{q}, \vec{z})$, we have $a_{1}(\vec{u}, \vec{z}, \vec{z})=0$, and it then follows from (17) that

$$
\left(\mu-\rho c_{4}^{2}\|\vec{q}\|_{\vec{V}}\right)|\vec{z}|_{1}^{2} \leq \rho\|\vec{q}\|_{\left(L_{4}\right)^{3}}^{2}|\vec{z}|_{1}+\left|\vec{f}_{0}\right|\left\|T-T_{0}\right\|_{L_{2}} c_{2}|\vec{z}|_{1}+\mu\|\vec{q}\|_{\vec{V}}|\vec{z}|_{1}
$$

As earlier, here $c_{q}$ denotes the embedding constant of $H_{0}^{1} \rightarrow L_{q}$. We assume now

$$
\begin{equation*}
\left\|\left.\vec{q}\right|_{\Gamma}\right\|_{1 / 2, \Gamma}<\frac{\mu}{\rho c_{4}^{2} c_{1 / 2}} \tag{18}
\end{equation*}
$$

where $c_{1 / 2}$ is the embedding constant (5), have then

$$
\mu-\rho c_{4}^{2}\|\vec{q}\|_{\vec{V}} \geq \mu-\rho c_{4}^{2} c_{1 / 2}\left\|\left.\vec{q}\right|_{\Gamma}\right\|_{1 / 2, \Gamma}>0
$$

and therefore find

$$
\begin{gather*}
|\vec{z}|_{1} \leq c_{2} \overline{\gamma_{1}}\|T\|_{L_{2}}+\overline{\gamma_{2}}, \quad \overline{\gamma_{1}}:=\frac{\left|\vec{f}_{0}\right|}{\mu-\rho c_{4}\|\vec{q}\|_{\left(L_{4}\right)^{3}}},  \tag{19}\\
\overline{\gamma_{2}}:=c_{2} \overline{\gamma_{1}}\left\|T_{0}\right\|_{L_{2}}+\frac{\rho\|\vec{q}\|_{\left(L_{4}\right)^{3}}^{2}+\mu\|\vec{q}\|_{\vec{V}}}{\mu-\rho c_{4}\|\vec{q}\|_{\left(L_{4}\right)^{3}}}
\end{gather*}
$$

If $\lambda_{1}=\lambda_{1}(-\Delta)$ is the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions, then $c_{2}=\lambda_{1}^{-1 / 2}$, see (12) for $q=2$, and in that case (12) is equivalent to the Friedrichs inequality. From there we have further

$$
\|\vec{z}\|_{\vec{V}} \leq \sqrt{1+c_{2}^{2}}|\vec{z}|_{1}
$$

and since $\|\vec{z}\|_{\vec{V}} \leq\|\vec{q}\|_{\vec{V}}+\|\vec{z}\|_{\vec{V}}$, we get from (19)

$$
\begin{equation*}
\|\vec{v}\|_{\vec{V}} \leq \gamma_{1}\|T\|_{L_{2}}+\gamma_{2} \tag{20}
\end{equation*}
$$

$$
\gamma_{1}:=c_{2} c_{v, 1} \overline{\gamma_{1}}, \quad \gamma_{2}:=c_{v, 1} \overline{\gamma_{2}}+\|\vec{q}\|_{\vec{V}}, \quad c_{v, 1}:=\sqrt{1+c_{2}^{2}}
$$

Adding next (10) multiplied by $\beta$ to (11) and substituting $t=j=\beta \lambda \tau+i$, we get the inequality

$$
|\beta \lambda \tau+i|_{1}^{2} \leq\left(\beta\|\vec{v}\|_{\left(L_{4}\right)^{3}}\|T\|_{L_{4}}+\beta \lambda|\vartheta|_{1}\right)|\beta \lambda \tau+i|_{1}
$$

and hence

$$
\begin{aligned}
|\beta \lambda \tau+i|_{i} & \leq \beta\|\vec{v}\|_{\left(L_{4}\right)^{3}}\|T\|_{L_{4}}+\gamma_{3} \leq \\
& \leq \beta\left(\|\vec{q}\|_{\left(L_{4}\right)^{3}}+c_{4}\left(c_{2} \overline{\gamma_{1}}\|T\|_{L_{2}}+\overline{\gamma_{2}}\right)\right)\|T\|_{L_{4}}+\gamma_{3} \leq \\
& \leq \gamma_{4}\|T\|_{L_{2}}\|T\|_{L_{4}}+\gamma_{5}\|T\|_{L_{4}}+\gamma_{3}, \\
\gamma_{3} & :=\beta \lambda|\vartheta|_{1}, \quad \gamma_{4}:=\beta c_{4} c_{2} \overline{\gamma_{1}}, \quad \gamma_{5}:=\beta\left(\|\vec{q}\|_{\left(L_{4}\right)^{3}}+c_{4} \overline{\gamma_{2}}\right) .
\end{aligned}
$$

Then, using the triangle inequality, there follows

$$
\begin{equation*}
|i|_{1} \leq \gamma_{4}\|T\|_{L_{2}}\|T\|_{L_{4}}+\gamma_{5}\|T\|_{L_{4}}+\gamma_{6}|\tau|_{1}+\gamma_{3} \tag{21}
\end{equation*}
$$

where $\gamma_{6}:=\beta \lambda$.
Concerning $\gamma_{3}$ we remark that $\vartheta$ in general is not zero on $\Gamma$ (we obtained $\vartheta$ just by continuation into $\Omega$ from the boundary values for the temperature $T$ ). Hence, $|\vartheta|_{1}$ is only a semi-norm and zero for constant $\vartheta$.

From (21) we get like in GERGO, Stoyan [3] the estimate

$$
\begin{align*}
|i|_{2}^{1} & \leq \gamma_{7}\left(\lambda|\tau|_{1}^{2}+\alpha \sigma\|T\|_{L_{5}}^{5}+\gamma_{8}\right) .  \tag{22}\\
\gamma_{7} & :=\frac{6 c_{4,5}}{5 \alpha \sigma}\left(\gamma_{4}^{2} c_{2,5}^{2}+\gamma_{5}^{2}\right)+\frac{\gamma_{6}^{2}}{\lambda}+1, \quad \gamma_{8}:=\gamma_{3}^{2}+\frac{2}{3} \alpha \sigma .
\end{align*}
$$

We now find an estimate of $|\tau|_{1}$ by putting $t=\tau$ in (10):

$$
\lambda|\tau|_{1}^{2}=\int\left\{T \vec{v} \operatorname{grad} \tau-\lambda \operatorname{grad} \vartheta \operatorname{grad} \tau-\alpha\left(\sigma|T|^{3} T-\pi\left(i+I_{m, 0}\right)\right) \tau\right\} \mathrm{d} \Omega
$$

Here, the first term on the right-hand side contains $\vec{v} \tau \operatorname{grad} \tau=\vec{v} \operatorname{grad}\left(\frac{1}{2} \tau^{2}\right)$ the integral of which is zero since even in the presence of inhomogeneous boundary conditions of $\vec{v}$ we have

$$
\begin{align*}
\int \vec{v} \tau \operatorname{grad} \tau \mathrm{~d} \Omega & =\int \vec{v} \operatorname{grad}\left(\frac{1}{2} \tau^{2}\right) \mathrm{d} \Omega= \\
& =\frac{1}{2} \int_{\Gamma} \tau^{2} \vec{v} \cdot \vec{n} \mathrm{ds}-\frac{1}{2} \int \tau^{2} \operatorname{div} \vec{v} \mathrm{~d} \Omega=0 \tag{23}
\end{align*}
$$

because of $\tau \in H_{0}^{1}$ and $\vec{v} \in \vec{V}$. Hence

$$
\begin{align*}
& \lambda|\tau|_{1}^{2} \leq\|\vartheta\|_{L_{4}}\|\vec{v}\|_{\left(L_{4}\right)^{3}}|\tau|_{1}+\lambda|\vartheta|_{1}|\tau|_{1}+ \\
& \quad+\alpha \pi\left|\int\left(i+I_{m, 0}\right) \tau \mathrm{d} \Omega\right|-\alpha \sigma \int|T|^{3} T \tau \mathrm{~d} \Omega \tag{24}
\end{align*}
$$

$$
\begin{align*}
& \leq\|\vartheta\|_{L_{4}}\|\vec{v}\|_{\left(L_{4}\right)^{3}}|\tau|_{1}+\lambda|\vartheta|_{1}|\tau|_{1}+\alpha \pi\left\|i+I_{m, 0}\right\|_{L_{5 / 4}}\|T\|_{L_{5}}+ \\
& \quad+\alpha \pi\left\|i+I_{m, 0}\right\|_{L_{5 / 4}}\|\vartheta\|_{L_{5}}-\alpha \sigma\|T\|_{L_{5}}^{5}+\alpha \sigma\|T\|_{L_{5}}^{4}\|\vartheta\|_{L_{5}} \tag{25}
\end{align*}
$$

where the two last terms in (24) have been estimated using Hölder inequalities and $\tau=T-\vartheta$. Next, using (19) and embedding theorems, we find for the first term in (25)

$$
\|\vartheta\|_{L_{4}}\|\vec{v}\|_{\left(L_{4}\right)^{3}}|\tau|_{1} \leq\|\vartheta\|_{L_{4}}\left(\|\vec{q}\|_{\left(L_{4}\right)^{3}}+c_{4} c_{2} \overline{\gamma_{1}} c_{2,5}\|T\|_{L_{5}}+c_{4} \overline{\gamma_{2}}\right)|\tau|_{1}
$$

Together with (25), this gives

$$
\begin{align*}
& \lambda|\tau|_{1}^{2}+\alpha \sigma\|T\|_{L_{5}}^{5} \leq \gamma_{9}\|T\|_{L_{5}}|\tau|_{1}+\gamma_{10}|\tau|_{1}+\alpha \sigma\|\vartheta\|_{L_{5}}\|T\|_{L_{5}}^{4}+  \tag{26}\\
& +\alpha \pi\left(\|i\|_{L_{5 / 4}}+\left\|I_{m, 0}\right\|_{L_{5 / 4}}\right)\|T\|_{L_{5}}+\alpha \pi\|\vartheta\|_{L_{5}}\|i\|_{L_{5 / 4}}+\gamma_{11}
\end{align*}
$$

where

$$
\begin{aligned}
\gamma_{9} & :=\|\vartheta\|_{L_{4}} c_{4} c_{2} \overline{\gamma_{1}} c_{2,5} \\
\gamma_{10} & :=\|\vartheta\|_{L_{4}}\left(\|\vec{q}\|_{\left(L_{4}\right)^{3}}+c_{4} \overline{\gamma_{2}}\right)+\lambda|\vartheta|_{1}, \\
\gamma_{11} & :=\alpha \pi\left\|I_{m, 0}\right\|_{L_{5 / 4}}\|\vartheta\|_{L_{5}} .
\end{aligned}
$$

We apply now the inequality $a b c \leq \frac{1}{r} a^{r}+\frac{1}{s} b^{2}+\frac{1}{t} c^{t}$ (valid for positive $a, b$, $c, r, s, t$ satisfying $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$; we shall take $r=10 / 3, s=5, t=2$ ), to the first term on the right-hand side of (26), writing it as

$$
a b c=\left(\frac{\gamma_{9}}{\sqrt{\epsilon \lambda / 2}}\right)\left(\sqrt{\epsilon}\|T\|_{L_{5}}\right)\left(\sqrt{\lambda / 2}|\tau|_{1}\right)
$$

where $\epsilon$ will later be chosen appropriately. Similarly, to the further terms (not counting $\gamma_{11}$ ) on the right-hand side of (26) we apply the inequality $a b<\frac{1}{p} a^{p}+\frac{1}{q} b^{q}$ (valid for positive $a, b, p, q$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ ), taking $p=2$ for the estimation of $\gamma_{10}|\tau|_{1}=\left(\gamma_{10} \sqrt{2 / \lambda}\right)\left(\sqrt{\lambda / 2}|\tau|_{1}\right)$ and $p=5 / 4$ for the remaining terms (including $\epsilon$ at appropriate places). In this way, we arrive at

$$
\begin{equation*}
\lambda|\tau|_{1}^{2}+\alpha \sigma\|T\|_{L_{5}}^{5} \leq 2 \gamma_{12}(\epsilon)\|i\|_{L_{5 / 4}}^{5 / 4}+\gamma_{13}(\epsilon) \leq \gamma_{14}|i|_{1}^{5 / 4}+\gamma_{13}(\epsilon) \tag{27}
\end{equation*}
$$

where

$$
\begin{aligned}
& \gamma_{14}:=2 \gamma_{12} c_{5 / 4}^{5 / 4} \\
& \gamma_{12}:=\frac{4 \alpha \pi}{5}\left(\frac{1}{\epsilon^{5 / 4}}+\epsilon^{5 / 4}\right) \\
& \gamma_{13}:=2\left(\frac{2^{2 / 3} 3}{5}\left(\frac{\gamma_{9}^{2}}{\lambda \epsilon}\right)^{5 / 3}+\frac{\gamma_{10}^{2}}{\lambda}+\frac{\alpha(\sigma+\pi)}{5 \epsilon^{5}}\|\vartheta\|_{L_{5}}^{5}+\frac{4 \alpha \pi}{5 \epsilon^{5 / 4}}\left\|I_{m, 0}\right\|_{L_{5 / 4}}^{5 / 4}+\gamma_{11}\right)
\end{aligned}
$$

and where $\epsilon$ is the unique positive solution of

$$
\left.k(\epsilon):=\frac{1}{5}\left(\epsilon^{5 / 2}+4 \alpha \sigma \epsilon^{5 / 4}+2 \alpha \pi \epsilon^{5}\right)\right)=\frac{\alpha \sigma}{2} .
$$

From (27) and (22) we get

$$
\begin{gather*}
|i|_{1}^{2} \leq \gamma_{15}|i|_{1}^{5 / 4}+\gamma_{16}  \tag{28}\\
\gamma_{15}:=\gamma_{7} \gamma_{14}, \quad \gamma_{16}:=\gamma_{7}\left(\gamma_{13}+\gamma_{8}\right)
\end{gather*}
$$

and further, like in GERGO, STOYAN [3], there follows a bound on $|i|_{1}$ :

$$
\begin{equation*}
|i|_{1} \leq \max \left(\gamma_{16}^{1 / 2}\left(\frac{8}{3}\right)^{4 / 5},\left(\gamma_{15}^{5 / 3}+\frac{5}{3} \gamma_{16}^{5 / 8}\right)^{4 / 5}\right)=: K_{i} \tag{29}
\end{equation*}
$$

Now we get a bound $K_{\tau}$ for $|\tau|_{1}$ from (27), whereas, for $1<q \leq 6$,

$$
\begin{align*}
\|T\|_{L_{q}} & \leq\|\vartheta\|_{L_{q}}+\|\tau\|_{L_{q}} \leq\|\vartheta\|_{L_{q}}+c_{q}|\tau|_{1} \leq \\
& \leq\|\vartheta\|_{L_{1}}+c_{q} K_{\tau}=: K_{T, q} . \tag{30}
\end{align*}
$$

Finally, from (19) and (20), we have the estimates

$$
\begin{align*}
|\vec{z}|_{1} & \leq c_{2} \overline{\gamma_{1}} K_{T, 2}+\overline{\gamma_{2}}=: K_{z}, \\
\|\vec{v}\|_{\vec{V}} & \leq \gamma_{1} K_{T, 2}+\gamma_{2}=: K_{V}, \\
\|\vec{v}\|_{\left(L_{q}\right)^{3}} & \leq\|\vec{q}\|_{\left(L_{q}\right)^{3}}+\|\vec{z}\|_{\left(L_{q}\right)^{3}} \leq\|\vec{q}\|_{\left(L_{q}\right)^{3}}+c_{q} K_{z}=: K_{v, q} . \tag{31}
\end{align*}
$$

The estimates (29)-(31) prove our theorem.
REMARK. Concerning the condition of sufficiently small boundary data (18) we remark that the expression $\mu-\rho c_{4}^{2} c_{1 / 2}\left\|\left.\vec{q}\right|_{\Gamma}\right\|_{1 / 2, \Gamma}$ comes from estimating $\rho a_{1}(\vec{z}, \vec{q}, \vec{z})$ on the right-hand side of (17) by $\rho c_{4}^{2}\left\|\left.\vec{q}\right|_{\Gamma}\right\|_{1 / 2, \Gamma}$ and then using (5).

Instead, we may also use Lemma 1.8 in TEMAM [6] or the corresponding result in LADYZHENSKAYA [5] stating that the function $\vec{q} \in \vec{V}$ which continues the boundary values of $\left.\vec{q}\right|_{\Gamma}$ into $\Omega$ can be chosen in such a way as to satisfy $\left|a_{1}(\vec{z}, \vec{q}, \vec{z})\right| \leq \epsilon|\vec{z}|_{1}^{2}$ for any positive $\epsilon$. Hence (18) can be weakened, but it has the advantage to stress that there is a condition on the possible boundary values.

## 4. Uniqueness

To solve the question of uniqueness of a weak solution under more general conditions than in GERGO, STOYAN [3], we modify and generalize the approach taken there.

Assume that the conditions of Theorem 1 hold, consider two solutions $(\vec{v}, \tau, i)$ and $\left(\vec{v}^{\prime}, \tau^{\prime}, i^{\prime}\right)$ of (9)-(11) in $\vec{V} \times H_{0}^{1} \times H_{0}^{1}$, subtract the corresponding variational equations, define

$$
\vec{U}:=\vec{v}-\vec{v}^{\prime}, \quad \Theta:=\tau-\tau^{\prime}, \quad J:=i-i^{\prime}
$$

and put $\vec{w}=\vec{U}, t=\Theta, j=J$ in (9)-(11). Then, there results

$$
\begin{align*}
\mu|\vec{U}|_{1}^{2} & =\int\left\{\rho \sum_{k=1}^{3}\left(U_{k} \vec{v}+v_{k}^{\prime} \vec{U}\right) \operatorname{grad} U_{k}+\vec{U} \vec{f}_{0} \Theta\right\} \mathrm{d} \Omega  \tag{32}\\
\lambda|\Theta|_{1}^{2} & \left.=\int\left\{T \vec{U}+\Theta \vec{v}^{\prime}\right) \operatorname{grad} \Theta-\alpha\left[\sigma\left(|T|^{3} T-\left|T^{\prime}\right|^{3} T^{\prime}\right)-\pi J\right] \Theta\right\} \mathrm{d} \Omega  \tag{33}\\
|J|_{1}^{2} & =\alpha \beta \int\left[\sigma\left(|T|^{3} T-\left|T^{\prime}\right|^{3} T^{\prime}\right)-\pi J\right] J \mathrm{~d} \Omega \tag{34}
\end{align*}
$$

Here we have used

$$
\begin{gathered}
\Theta=T-T^{\prime}, \quad J=I_{m}-I_{m}^{\prime} \\
v_{k} \vec{v}-v_{k}^{\prime} \vec{v}^{\prime}=U_{k} \vec{v}+v_{k}^{\prime} \vec{U}, \quad T \vec{v}-T^{\prime} \vec{v}^{\prime}=T \vec{U}+\Theta \vec{v}^{\prime}
\end{gathered}
$$

In (32) resp. in (33), the integrals over $\sum_{k=1}^{3} U_{k} \vec{v} \operatorname{grad} U_{k}$ resp. over $\Theta \vec{v}^{\prime} \operatorname{grad} \Theta$ are zero since $\vec{v} \in \vec{V}$ and $U_{k}, \Theta \in H_{0}^{1}$, compare with (23). Taking this into account, we first estimate the right-hand side of (32):

$$
\begin{equation*}
\mu|\vec{U}|_{1}^{2} \leq \rho\|\vec{U}\|_{\left(L_{4}\right)^{3}}\left\|\vec{v}^{\prime}\right\|_{\left(L_{4}\right)^{3}}|\vec{U}|_{1}+\left|\overrightarrow{f_{0}}\right|\|\vec{U}\|_{\left(L_{2}\right)^{3}}\|\Theta\|_{L_{2}} \tag{35}
\end{equation*}
$$

Remember that $\left|\vec{f}_{0}\right|$ denotes the Euclidean norm of $\vec{f}_{0}$. To derive an estimate from (33), we remark that

$$
\begin{equation*}
\left||T|^{3} T-\left|T^{\prime}\right|^{3} T^{\prime}\right| \leq\left|T-T^{\prime}\right| P_{3}\left(|T|,\left|T^{\prime}\right|\right)=|\Theta| P_{3}\left(|T|,\left|T^{\prime}\right|\right), \tag{36}
\end{equation*}
$$

where $P_{3}(x, y):=x^{3}+x^{2} y+x y^{2}+y^{3}$. Then we obtain

$$
\begin{align*}
\lambda|\Theta|_{1}^{2} \leq & \|T\|_{L_{4}}\|\vec{U}\|_{\left(L_{4}\right)^{3}}|\Theta|_{1^{+}} \\
& +\alpha \sigma\left\|P_{3}\left(|T|,\left|T^{\prime}\right|\right)\right\|_{L_{2}}\|\Theta\|_{L_{4}}^{2}+\alpha \pi\|J\|_{L_{2}}\|\Theta\|_{L_{2}} . \tag{37}
\end{align*}
$$

Using Lemma 1 from Gergó, Stoyan [3], we have

$$
\begin{equation*}
\left\|P_{3}\left(|T|,\left|T^{\prime}\right|\right)\right\|_{L_{2}} \leq P_{3}\left(\|T\|_{L_{6}},\left\|T^{\prime}\right\|_{L_{6}}\right) \tag{38}
\end{equation*}
$$

Applying to (37) also the theorem on continuous embedding (12), it follows that

$$
\begin{equation*}
\lambda|\Theta|_{1} \leq c_{4}\|T\|_{L_{4}}|\vec{U}|_{1}+\alpha \sigma P_{3}\left(\|T\|_{L_{6}},\left\|T^{\prime}\right\|_{L_{6}}\right) c_{4}^{2}|\Theta|_{1}+\alpha \pi c_{2}^{2}|J|_{1} . \tag{39}
\end{equation*}
$$

Next, applying (12) also to (35), we find

$$
\begin{equation*}
\mu|\vec{U}|_{1} \leq \rho c_{4}\left\|\vec{v}^{\prime}\right\|_{\left(L_{4}\right)^{3}}|\vec{U}|_{1}+c_{2}^{2}\left|\overrightarrow{f_{0}}\right||\Theta|_{1} . \tag{40}
\end{equation*}
$$

Finally, we see from (34), (36) and from (38) that

$$
\begin{align*}
|J|_{1}^{2} & \leq \alpha \beta \sigma\|\Theta\|_{L_{4}}\left\|P_{3}\left(|T|,\left|T^{\prime}\right|\right)\right\|_{L_{2}}\|J\|_{L_{4}} \leq \\
& \leq \alpha \beta \sigma c_{4}^{2}|\Theta|_{1} P_{3}\left(\|T\|_{L_{6}},\left\|T^{\prime}\right\|_{L_{6}}\right)|J|_{1} . \tag{41}
\end{align*}
$$

We now introduce the vector

$$
y:=\left(|\vec{U}|_{1},|\Theta|_{1},|J|_{1}\right)^{T}
$$

with the aim to show that under suitable conditions there holds $y=0$. For this, we first take into account in (39)-(41) the boundedness estimates (30)-(31) which we here summarize as

$$
\begin{array}{r}
\|T\|_{L_{4}},\left\|T^{\prime}\right\|_{L_{4}},\|T\|_{L_{6}},\left\|T^{\prime}\right\|_{L_{6}} \leq K_{T}, \\
\|\vec{v}\|_{\left(L_{4}\right)^{3}},\left\|\overrightarrow{v^{\prime}}\right\|_{\left(L_{4}\right)^{3}} \leq K_{v} .
\end{array}
$$

E.g., (38) can now be continued by $P_{3}\left(\|T\|_{L_{6}},\left\|T^{\prime}\right\|_{L_{6}}\right) \leq 4 K_{T}^{3}$ due to the definition of $P_{3}$.

Then we rewrite the estimates (39)-(41) in vector form as follows

$$
0 \leq y \leq \mathscr{A} y, \quad \text { where } \mathscr{A}:=\left(\begin{array}{ccc}
\frac{\rho}{\mu} c_{4} K_{v} & \frac{1}{\mu} c_{2}^{2}\left|\vec{f}_{0}\right| & 0  \tag{42}\\
\frac{1}{\lambda} c_{4} K_{T} & 4 \frac{\alpha \sigma}{\lambda} c_{4}^{2} K_{T}^{3} & \frac{\alpha}{\lambda} \pi c_{2}^{2} \\
0 & 4 \alpha \beta \sigma c_{4}^{2} K_{T}^{3} & 0
\end{array}\right) .
$$

These inequalities are to be understood componentwise. From (42) we have

$$
0 \leq y \leq \mathscr{A} y \leq \mathscr{A}^{2} y \leq \ldots \leq \mathscr{A}^{k} y
$$

for $k \geq 1$, but, as is well known, $\mathcal{A}^{k} \rightarrow 0$ elementwise for $k \rightarrow \infty$ iff the spectral radius of $\mathscr{A}$ is less than 1 . But then this chain of inequalities is possible only for $y=0$ which means uniqueness. A weaker condition is that an induced norm of $\mathscr{A}$ is less than 1 . We use this idea to prove the final theorem where it turns out that the conditions listed by Förste as sufficient uniqueness conditions can be separated.

THEOREM 2. Assume that the conditions of Theorem 1 are satisfied. Then the Förste model (9)-(11) has at most one solution $\vec{v} \in \vec{V}, T=\tau+\vartheta \in H^{1}$, $I_{m}=i+I_{m, 0} \in H^{1}$ when either of the following two conditions holds, additionally:

1) the diameter $d$ of $\Omega$ is sufficiently small;
2) $\alpha$ and $\beta$ are sufficiently small, and $\lambda$ and $\mu$ are sufficiently large, moreover, for some positive constants $\kappa_{1}, \kappa_{2}$ there holds

$$
\begin{equation*}
\kappa_{1} \leq \alpha \lambda, \quad \text { and } \quad \beta^{2} \lambda \leq \kappa_{2} \lambda^{\kappa} \tag{43}
\end{equation*}
$$

where $\kappa:=3 / 10$.
Proof. 1) As (42) shows, every nonzero element of $\mathcal{A}$ contains an embedding constant which goes to zero when d goes to zero, see (13). We must therefore clarify the possible growth of $K_{T}, K_{v}$ for decreasing $d$.

Hence, taking into account (13) and (16) and considering values like $\|\vec{q}\|_{\left(L_{4}\right)^{3}},|\vartheta|_{1}$ as $O(1)$, we check all constants $\gamma_{i}$ for their dependence on $d$ and find that $\gamma_{1}, \gamma_{4}, \gamma_{9}, \gamma_{14}, \gamma_{15}$ with $d$ go to zero whereas the remainder and $K_{i}, K_{\tau}, K_{T}, K_{v}$ are $O(1)$.

Thus, the spectral radius of $\mathscr{A}$ becomes less 1 for sufficiently small diameter of $\Omega$, and then there follows $y=0$ from (42).

Similarly, for fixed $d$, since every nonzero element of $\mathscr{A}$ contains either the absorption coefficients $\alpha$ or $\beta$ or $1 / \lambda$ or $1 / \mu$, we also have uniqueness in the second case, provided the bounds $K_{T}$ and $K_{v}$ don't grow with $\lambda, \mu$, $1 / \alpha, 1 / \beta$. For this, we trace the constants $\gamma_{j}$ of the estimates in Section 3 under condition 2 above (thus taking $\mu^{-1}, \alpha \leq O(1)$ for granted) and find the following relations when assuming $\kappa \geq 0$ in (43):

$$
\gamma_{1}, \gamma_{9} \leq O\left(\mu^{-1}\right), \quad \gamma_{2} \leq O\left(1+\mu^{-1}\right)=O(1), \quad \gamma_{3}, \gamma_{6} \leq O(\beta \lambda)
$$

$$
\begin{aligned}
\gamma_{4} & \leq O\left(\beta \mu^{-1}\right), \quad \gamma_{5} \leq O\left(\beta+\beta \mu^{-1}\right) \leq O(\beta) \\
\gamma_{7} & \leq O\left(1+\beta^{2} \lambda+\beta^{2} \alpha^{-1}+\beta^{2} \mu^{-2} \alpha^{-1}\right) \leq O\left(1+\beta^{2} \lambda\right) \\
\gamma_{8} & \leq O\left(\beta^{2} \lambda^{2}+\alpha\right) \leq O\left(\lambda^{1+\kappa}\right), \quad \gamma_{10} \leq O\left(1+\mu^{-1}+\lambda\right) \leq O(\lambda) \\
\gamma_{11} & \leq O(\alpha), \quad \gamma_{12}, \gamma_{14} \leq O\left(\alpha^{1 / 2}\right), \quad \epsilon=O\left(\alpha^{2 / 5}\right) \\
\gamma_{13} & \leq O\left(\lambda+\alpha^{-1}\right) \leq O(\lambda) \\
\gamma_{15} & \left.\leq O\left(\alpha^{1 / 2}\left(1+\beta^{2} \lambda\right)\right) \leq O\left(1+\beta^{2} \lambda\right)\right) \leq O\left(\lambda^{\kappa}\right) \\
\gamma_{16} & \leq O\left(\left(1+\beta^{2} \lambda\right)\left(\beta^{2} \lambda^{2}+\lambda+\alpha^{-1}\right)\right) \leq O\left(\lambda^{1+2 \kappa}\right)
\end{aligned}
$$

Then, in (29), $\gamma_{15}^{5 / 3} \leq O\left(\lambda^{\frac{5}{3} \kappa}\right) \leq O\left(\lambda^{\frac{5}{8}+\frac{5}{4} \kappa}\right)$ for $\kappa \leq \frac{3}{2}$, and $\gamma_{16}^{5 / 8} \leq$ $\leq O\left(\lambda^{\frac{5}{8}+\frac{5}{4} \kappa}\right)$, and from (29)-(31) there result the estimates

$$
\begin{align*}
|i|_{1} & \leq O\left(\lambda^{\kappa+1 / 2}\right)=: K_{i} \\
\lambda|\tau|_{1}^{2} & \leq O\left(\alpha^{1 / 2} \lambda^{\left(\kappa+\frac{1}{2}\right) \frac{5}{4}}+\lambda\right) \leq O\left(\lambda^{\left(\kappa+\frac{1}{2}\right) \frac{5}{4}}+\lambda\right)=O(\lambda)  \tag{44}\\
|\tau|_{1} & \leq O(1), \quad\|T\|_{L_{q}} \leq O(1)=: K_{T} \\
\|\vec{v}\|_{\left(L_{q}\right)^{3}} & \leq O(1)=: K_{v}
\end{align*}
$$

The specific value of $\kappa$ arises from (44) when requiring $\left(\kappa+\frac{1}{2}\right) \frac{5}{4}=1$.
Observe finally that (due to the linearity of the Förste model in $I_{m}$ ), $K_{i}$ does not appear in $\mathcal{A}$.

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# THE GRADIENT-FOURIER METHOD FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN SOBOLEV SPACE 

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## 1. Introduction

Our paper deals with the iterative solution of a certain class of nonlinear elliptic partial differential equations. It incorporates two methods: the gradient method and the Fourier method.

There are various ways to solve nonlinear elliptic problems numerically. The finite element method and the method of finite differences transform the boundary value problem into an algebraic system of equations in $\mathbb{R}^{N}$ (see $[2,16])$. A commonly used approach for solving these algebraic systems is the (classical) gradient method in finite dimensional spaces, which-together with its applications-is described in [11] and [13]. Another way is Newton's method (see for example in [6]), which provides faster convergence but requires more computations.

The gradient method in Hilbert spaces (both the linear and the nonlinear case) is described in [6, Chapter XV]. Iterations using Hilbert space theory are found further for example in [3], [5], [6], [15], and especially in Sobolev space in [1], [7], [10] and [14]. Sobolev space theory related to the finite element method is used to establish error estimates (see [2]).

In order to apply the gradient method, the given partial differential equation is written as $F(u)=0$ with the weak differential operator $F$, then a variational principle is used.

In Section 2 the gradient method is applied to the given boundary value problem. The iterative sequence obtained by the method consists of elements of the corresponding Sobolev space. While this section summarizes previously known facts, Section 3 brings out new result.

The construction of the iterative sequence requires the stepwise solution of linear boundary value problems. In contrast with Newton's method, these auxiliary problems are of fixed (Poisson) type. In Section 3 these linear problems are solved analytically using the Fourier method, which is essentially based on Fourier series expansion. (A different approach can be found for example in [4].) In the end, we give an error estimate of the combined Gradient-Fourier method. The main advantage of the combined method is the easy algorithmic and numerical realization when the eigenfunctions and eigenvalues of the Laplacian operator on the given domain are known.

## 2. The gradient method in Sobolev space

In this section we first formulate the boundary value problem, then quote the convergence result on the gradient method in Sobolev space that we will rely on (cf. [4] and [7]).

Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain which is $C^{2}$-diffeomorphic to a convex one and let $H:=\left(L^{2}(\Omega),\langle\cdot, \cdot\rangle_{L^{2}(\Omega)}\right)$, where $\langle f, g\rangle_{L^{2}(\Omega)}:=\int_{\Omega} f g$ $\left(f, g \in L^{2}(\Omega)\right)$.
(Throughout the paper $L^{2}(\Omega)$ is considered as a real Hilbert space.) We define a differential operator $T$ with domain

$$
\operatorname{dom} T:=D:=H^{2}(\Omega) \cap H_{0}^{1}(\Omega)
$$

( $H^{2}(\Omega)$ and $H_{0}^{1}(\Omega)$ denote the usual Sobolev spaces.)
Let

$$
T(u):=-\operatorname{div} f(\cdot, \nabla u) \quad(u \in D)
$$

that is

$$
(T(u))(x):=-\sum_{i=1}^{N} \partial_{x_{i}}\left[f_{i}\left(x, \partial_{x_{1}} u(x), \ldots \partial_{x_{N}} u(x)\right)\right]
$$

with $x=\left(x_{1}, x_{2}, \ldots, x_{N}\right)$ and $f=\left(f_{1}, f_{2}, \ldots, f_{N}\right)$, where $f_{i}$ 's are all real-valued functions, satisfying $f_{i} \in C^{1}\left(\bar{\Omega} \times \mathbb{R}^{N}\right)$.

Moreover, suppose that

$$
\partial_{p_{j}} f_{i}(x, p)=\partial_{p_{i}} f_{j}(x, p)
$$

for all $x \in \bar{\Omega}$ and $p \in \mathbb{R}^{N}(i, j=1,2, \ldots, N)$.

Assume that there exist constants $m$ and $M(0<m<M)$ such that

$$
\begin{equation*}
m|\xi|^{2} \leq \sum_{i, j=1}^{N} \partial_{p_{j}} f_{i}(x, p) \zeta_{i} \xi_{j} \leq M|\xi|^{2} \tag{1}
\end{equation*}
$$

for any $(x, p) \in \bar{\Omega} \times \mathbb{R}^{N}$ and any $\zeta=\left(\xi_{1}, \ldots, \zeta_{N}\right) \in \mathbb{R}^{N}$.
(We remark that (1) is equivalent to demanding that all eigenvalues of $\partial_{p} f(x, p)$ lie in the interval $[m, M]$, where $\partial_{p} f(x, p)$ is the $N \times N$ (symmetric) matrix with $\partial_{p_{j}} f_{i}(x, p)$ standing at the $(i, j)$ th position.)

Finally, let

$$
B:=-\Delta, \quad \operatorname{dom} B:=D
$$

(Throughout the paper $(-\Delta)$ is considered as an $L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ operator with domain $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$.)

The following boundary value problem will be considered:

$$
\left.\begin{array}{c}
T(u)=g  \tag{2}\\
\left.u\right|_{\partial \Omega}=0
\end{array}\right\}
$$

where $g \in L^{2}(\Omega)$ is given and $u \in D$ is the unknown function.
EXAMPLE. We briefly mention an example, which arises in plasticity theory (for details, see [4]).

Let $\Omega$ be the unit square in $\mathbb{R}^{2}$, that is $\Omega:=[0,1] \times[0,1]$. Define the operator $T$ as

$$
\begin{gathered}
T(u):=-\operatorname{div}(\bar{g}(|\nabla u|) \nabla u), \\
\left.u\right|_{\partial \Omega}=0,
\end{gathered}
$$

where $\bar{g}$ is a given scalar-valued function (the strain-stress function) satisfying $0<m \leq \bar{g}(r) \leq \bar{g}(r)+r \cdot \bar{g}^{\prime}(r) \leq M(r \geq 0)$ with suitable constants $m$ and $M$. For example, $\bar{g}$ can have the form

$$
\bar{g}(t):=\frac{\text { constant }}{1+\sqrt{1-\frac{t^{2}}{3}}},
$$

if $0 \leq t \leq t_{0}$, with a suitable $t_{0}<\sqrt{3}$ and $\bar{g}(t):=\bar{g}\left(t_{0}\right)$, if $t>t_{0}$.
We now quote a theorem on the convergence of the Sobolev space gradient method for (2).

THEOREM 2.1. (cf. [4] and [7].) Consider the boundary value problem

$$
\left.\begin{array}{c}
T(u)=g  \tag{3}\\
\left.u\right|_{\partial \Omega}=0
\end{array}\right\}
$$

where $T$ is defined as above. Let $g \in L^{2}(\Omega)$ be arbitrary. Then problem (3) admits a unique weak solution $u^{*} \in H_{0}^{1}(\Omega)$, that is for all $v \in H_{0}^{1}(\Omega)$

$$
\int_{\Omega}\left\langle f\left(x, \nabla u^{*}(x)\right), \nabla v(x)\right\rangle_{\mathbb{R}^{N}} \mathrm{~d} x=\int_{\Omega} g(x) v(x) \mathrm{d} x
$$

holds. (If $u^{*} \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, then $T\left(u^{*}\right)=g$.)
Moreover, let $u_{0} \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ be arbitrary, then the following sequence

$$
u_{n+1}:=u_{n}-\frac{2}{M+m}(-\Delta)^{-1}\left(T\left(u_{n}\right)-g\right) \quad(n \in \mathbb{N})
$$

converges to the solution $u^{*}$ and

$$
\left\|u_{n}-u^{*}\right\|_{H_{0}^{1}(\Omega)} \leq \frac{1}{m \sqrt{\lambda_{1}}}\left\|T\left(u_{0}\right)-g\right\|_{L^{2}(\Omega)} \cdot\left(\frac{M-m}{M+m}\right)^{n}
$$

where $\lambda_{1}$ is the smallest eigenvalue of $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$.
We note that the theorem also holds if we add a term $q(\cdot, u)$ in $T(u)$, which may have some polynomial growth (see [8] and [9]).

The following lemma will also be needed. (For its proof, see for example [12].)

LEMMA 2.2. For any $w \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ the following inequality holds for $\lambda_{1}$, i.e. the smallest eigenvalue of $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ :

$$
\langle-\Delta w, w\rangle_{L^{2}(\Omega)} \geq \lambda_{1}\|w\|_{L^{2}(\Omega)}^{2}
$$

The main advantage of the method so far is that the solution of the nonlinear equation (i.e. determining $T^{-1}$ ) is reduced to the solution of a sequence of linear equations of fixed type: the inverse of $(-\Delta)$ is to be calculated in each iteration step, which is-in general-a much easier task.

## 3. The Gradient-Fourier method

In the previous section a sequence $u_{n}$ has been constructed-using the gradient method-which converges to the unique solution $u^{*}$. Obtaining the next term $u_{n+1}$ of the sequence requires a linear Poisson equation to be solved at each step. (These equations will be referred to as auxiliary equations.) There are various ways to deal with Poisson equations ([4] for example describes the gradient-finite element method).

In this paper we introduce the Gradient-Fourier method: solutions to the auxiliary equations will be determined by using the Fourier method. Our approach succeeds in cases when both the eigenvalues and eigenfunctions of the operator $(-\Delta)$ are known. In exchange, this enables easy algorithmic and numerical realization.

Formally, solving a Poisson equation is equivalent to applying the inverse of the Laplacian operator to the right hand side of the equation. In the vast majority of cases, however, symbolic computation is not possible, so in general the exact sequence $u_{n}$ is only theoretically known. A numerical method (now the Fourier method) yields an approximation $\bar{u}_{n}$ of the sequence. Then our goal is to guarantee that the limit of the approximating sequence $\bar{u}_{n}$ still lies near the exact solution $u^{*}$.

Let us first introduce some notations. For $n=0,1, \ldots$, let

$$
\begin{aligned}
\bar{u}_{0} & =u_{0}, \\
\bar{u}_{n+1} & :=\bar{u}_{n}-\frac{2}{M+m} \bar{z}_{n},
\end{aligned}
$$

where $\bar{z}_{n}$ is the solution to the auxiliary equation obtained by the Fourier method. (As in the previous section, homogeneous Dirichlet boundary conditions are imposed on the auxiliary equations.) We also define two other sequences

$$
\begin{aligned}
& z_{n}:=(-\Delta)^{-1}\left(T\left(u_{n}\right)-g\right), \\
& z_{n}^{*}:=(-\Delta)^{-1}\left(T\left(\bar{u}_{n}\right)-g\right),
\end{aligned}
$$

i.e. $z_{n}$ is the exact solution to the auxiliary equation with exact (but only theoretically known) right hand side, and $z_{n}^{*}$ is the exact solution to the auxiliary equation with the approximate (i.e. numerically computed) right hand side of the corresponding Poisson equation. Because of the presence of the inverse of Laplacian, these functions are both unknown, in practice only $\bar{z}_{n}$ is available. Finally, let

$$
E_{n}:=\left\|\bar{u}_{n}-u_{n}\right\| .
$$

(The norms without indices are all understood to be $H_{0}^{1}(\Omega)$-norms throughout the section.)

### 3.1. Convergence estimate of the approximating sequence

In this section the distance of $\bar{u}_{n}$ and $u^{*}$ will be estimated provided that $\bar{z}_{n}$ and $z_{n}^{*}$ remain sufficiently close through the iteration process. These estimates are independent of the way of solving the auxiliary equations. The next section will discuss how this can be achieved in the actual construction of $\bar{z}_{n}$ using the Fourier method.

For the proof of the first lemma, see [4].
LEMMA 3.1. If $\left\|\bar{z}_{n}-z_{n}^{*}\right\| \leq \delta_{n}$ for some $\delta_{n}(n \in \mathbb{N})$, then

$$
E_{n+1} \leq \frac{M-m}{M+m} E_{n}+\frac{2}{M+m} \delta_{n}
$$

where $m$ and $M$ are given in (1).
Lemma 3.2. Let $\varepsilon>0$ be arbitrary and $\delta_{n}:=m \varepsilon(n \in \mathbb{N})$, then

$$
E_{n} \leq \varepsilon
$$

PROOF. If $n=0$ then $0=\left\|\bar{u}_{0}-u_{0}\right\|=E_{0} \leq \varepsilon$. Suppose the statement is true for some $n \in \mathbb{N}$. Then by the previous lemma we have

$$
E_{n+1} \leq \frac{M-m}{M+m} E_{n}+\frac{2}{M+m} \delta_{n} \leq \frac{M-m}{M+m} \varepsilon+\frac{2}{M+m} m \varepsilon=\frac{M+m}{M+m} \varepsilon=\varepsilon
$$

COROLLARY 3.3. Let $\varepsilon>0$. If $\left\|\bar{z}_{n}-\bar{z}_{n}^{*}\right\| \leq m \varepsilon(n \in \mathbb{N})$, then

$$
\left\|\bar{u}_{n}-u^{*}\right\| \leq \varepsilon+\frac{1}{m \sqrt{\lambda_{1}}}\left\|T\left(u_{0}\right)-g\right\|_{L^{2}(\Omega)} \cdot\left(\frac{M-m}{M+m}\right)^{n}
$$

where $\lambda_{1}$ is the smallest eigenvalue of $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$.
PROOF.

$$
\begin{gathered}
\left\|\bar{u}_{n}-u^{*}\right\| \leq\left\|\bar{u}_{n}-u_{n}\right\|+\left\|u_{n}-u^{*}\right\| \leq \\
\left.\leq E_{n}+\frac{1}{m \sqrt{\lambda_{1}}} \| T\left(u_{0}\right)-g\right) \|_{L^{2}(\Omega)} \cdot\left(\frac{M-m}{M+m}\right)^{n} \leq \\
\leq \varepsilon+\frac{1}{m \sqrt{\lambda_{1}}}\left\|T\left(u_{0}\right)-g\right\|_{L^{2}(\Omega)} \cdot\left(\frac{M-m}{M+m}\right)^{n}
\end{gathered}
$$

using Theorem 2.1 and the previous lemma.

### 3.2. The Fourier method for the auxiliary equations

Now let us focus on a single iteration step (i.e. $n \in \mathbb{N}$ is fixed in the section). The Fourier method implemented here can be found for example in [12].

According to the definition of $z_{n}^{*}$, the auxiliary equation takes the form

$$
\left.\begin{array}{l}
-\Delta z_{n}^{*}=r_{n}^{*} \\
\left.z_{n}^{*}\right|_{\partial \Omega}=0
\end{array}\right\}
$$

with $r_{n}^{*}:=T\left(\bar{u}_{n}\right)-g$. Instead of $r_{n}^{*}$, the right hand side will be replaced by its Fourier series approximation $\bar{r}_{n}$. The modified equation is

$$
\left.\begin{array}{l}
-\Delta \bar{z}_{n}=\bar{r}_{n}  \tag{4}\\
\left.\bar{z}_{n}\right|_{\partial \Omega}=0
\end{array}\right\}
$$

which now can be solved by a formula.
Let $\lambda_{i}$ and $e_{i}(i=1,2, \ldots)$ denote the eigenvalues and eigenfunctions of $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, respectively. Let $c_{i}(i=1,2, \ldots)$ be the coefficients of $r_{n}^{*}$ in its Fourier series expansion, that is

$$
c_{i}:=\int_{\Omega} r_{n}^{*} e_{i} .
$$

(We simply use $c_{i}$ instead of $c_{i, n}$ since $n$ is fixed.) Then

$$
r_{n}^{*}=\sum_{i=1}^{\infty} c_{i} e_{i} .
$$

Now define $\bar{r}_{n}$ as a partial sum of the above infinite series. Let $l$ (also depending on $n$ ) be a positive integer and

$$
\bar{r}_{n}:=\sum_{i=1}^{l} c_{i} e_{i} .
$$

Define $\bar{z}_{n}$, as

$$
\bar{z}_{n}:=\sum_{i=1}^{l} \frac{c_{i}}{\lambda_{i}} e_{i} .
$$

A simple calculation shows that these satisfy (4):

$$
-\Delta \bar{z}_{n}=\sum_{i=1}^{l} \frac{c_{i}}{\lambda_{i}}\left(-\Delta e_{i}\right)=\sum_{i=1}^{l} \frac{c_{i}}{\lambda_{i}} \lambda_{i} e_{i}=\sum_{i=1}^{l} c_{i} e_{i}=\bar{r}_{n}
$$

We will now determine how the requirement of Corollary 3.3 can be fulfilled by the appropriate choice of $l$. The basis for this is the fact that the weak solution to the equation $(-\Delta)\left(\bar{z}_{n}-z_{n}^{*}\right)=\bar{r}_{n}-r_{n}^{*}$ depends continuously on the right hand side, i.e. there exists a constant $c_{*}$ such that

$$
\left\|\bar{z}_{n}-z_{n}^{*}\right\| \leq c_{*}\left\|\bar{r}_{n}-r_{n}^{*}\right\|_{L^{2}(\Omega)}
$$

A suitable choice for $c_{*}$ is $\frac{1}{\sqrt{\lambda_{1}}}$ due to the following
LEMMA 3.4. If

$$
\left.\begin{array}{l}
-\Delta w=f \\
\left.w\right|_{\partial \Omega}=0
\end{array}\right\}
$$

then

$$
\|w\| \leq \frac{1}{\sqrt{\lambda_{1}}}\|f\|_{L^{2}}
$$

PROOF. If $w \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, then according to the Green formula

$$
\int_{\Omega}(-\Delta w) w=\int_{\Omega}|\nabla w|^{2}
$$

that is $\|w\|^{2}=\langle-\Delta w, w\rangle_{L^{2}}$. Using Lemma 2.2 we get

$$
\|w\|^{2}=\langle-\Delta w, w\rangle_{L^{2}}=\langle f, w\rangle_{L^{2}} \leq\|f\|_{L^{2}}\|w\|_{L^{2}} \leq\|f\|_{L^{2}} \frac{1}{\sqrt{\lambda_{1}}}\|w\|
$$

thus

$$
\|w\| \leq \frac{1}{\sqrt{\lambda_{1}}}\|f\|_{L^{2}}
$$

which was to be proved.
Hence for Corollary 3.3 it is sufficient to guarantee that

$$
\begin{equation*}
\left\|\bar{r}_{n}-r_{n}^{*}\right\|_{L^{2}} \leq m \varepsilon \sqrt{\lambda_{1}} \tag{5}
\end{equation*}
$$

Taking into consideration that

$$
\left\|r_{n}^{*}-\bar{r}_{n}\right\|_{L^{2}}^{2}=\sum_{i=l+1}^{\infty}\left|c_{i}\right|^{2}=\sum_{i=1}^{\infty}\left|c_{i}\right|^{2}-\sum_{i=1}^{l}\left|c_{i}\right|^{2}=\left\|r_{n}^{*}\right\|_{L^{2}}^{2}-\left\|\bar{r}_{n}\right\|_{L^{2}}^{2}
$$

we see that for (5) it is sufficient that

$$
\left\|r_{n}^{*}\right\|_{L^{2}}^{2}-\sum_{i=1}^{l}\left|c_{i}\right|^{2} \leq \lambda_{1} m^{2} \varepsilon^{2}
$$

Recalling the definition of $r_{n}^{*}$ we get the following
COROLLARY 3.5. If $l$ is a positive integer such that

$$
\sum_{i=1}^{l}\left|c_{i}\right|^{2} \geq\left\|T\left(\bar{u}_{n}\right)-g\right\|_{L^{2}}^{2}-\lambda_{1} m^{2} \varepsilon^{2}
$$

then

$$
\left\|\bar{z}_{n}-z_{n}^{*}\right\| \leq m \varepsilon .
$$

### 3.3 Final convergence estimate of the algorithm

Now the results of our paper will be summarized. The solution to the given boundary value problem is approximated by an iterative sequence $\bar{u}_{n}$, which is obtained by the gradient method. Computing each term of the sequence is effectively reduced to a series of numerical integrations via Fourier series expansion. In order to realize the algorithm, one needs to know the eigenvalues and eigenfunctions of the operator $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$. With this information, exact error control can be given in $H_{0}^{1}(\Omega)$-norm.

THEOREM 3.6. Let $\Omega \subset \mathbb{R}^{N}$ be the bounded domain and $T$ be the nonlinear elliptic differential operator defined in Section 2. The lower and upper bounds of the coefficients of $T$ are denoted by $m$ and $M$, respectively, as in (1). Let $g \in L^{2}(\Omega)$ be arbitrary. Then the unique weak solution $u^{*} \in H_{0}^{1}(\Omega)$ to the boundary value problem

$$
\left.\begin{array}{r}
T(u)=g \\
\left.u\right|_{\partial \Omega}=0
\end{array}\right\}
$$

can be approximated by the following iterative sequence $\bar{u}_{n} \in H^{2}(\Omega) \cap$ $\cap H_{0}^{1}(\Omega)$. Let $\bar{u}_{0} \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ be arbitrary, and for $n \in \mathbb{N}$

$$
\bar{u}_{n+1}:=\bar{u}_{n}-\frac{2}{M+m} \bar{z}_{n},
$$

where

$$
\bar{z}_{n}:=\sum_{i=1}^{l(n)} \frac{c_{i, n}}{\lambda_{i}} e_{i},
$$

and $l(n)$ is a suitable positive integer, further $\lambda_{i}$ and $e_{i}(i=1,2, \ldots)$ denote the eigenvalues and eigenfunctions of $(-\Delta)$ on $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$, respectively, and

$$
c_{i, n}:=\int_{\Omega}\left(T\left(\bar{u}_{n}\right)-g\right) e_{i}
$$

Let $\varepsilon>0$ be arbitrary. If the numbers $l(n)$ are chosen such that

$$
\sum_{i=1}^{l(n)}\left|c_{i, n}\right|^{2} \geq\left\|T\left(\bar{u}_{n}\right)-g\right\|_{L^{2}(\Omega)}^{2}-\lambda_{1} m^{2} \varepsilon^{2} \quad(n \in \mathbb{N})
$$

then

$$
\left\|\bar{u}_{n}-u^{*}\right\|_{H_{0}^{1}(\Omega)} \leq \varepsilon+\frac{1}{m \sqrt{\lambda_{1}}}\left\|T\left(\bar{u}_{0}\right)-g\right\|_{L^{2}(\Omega)} \cdot\left(\frac{M-m}{M+m}\right)^{n} .
$$

Proof. The proof simply follows by applying Corollary 3.5 and Corollary 3.3.

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# SPHERE PACKINGS IN THE REGULAR CROSSPOLYTOPE 

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## 1. Introduction

The problem of packing equal spheres inside a given container has been considered for a long time. SCHAER, MEIR and WENGERODT determined the densest circle packing for a small number of circles within a square, whereas Pirl, Kravitz, Goldberg and Fodor concluded the same within a circle (see Fodor [2] for references, and MELISSEN [4] for a detailed survey on circle packings inside a given domain). In the three space, sphere packings inside a cube were investigated by SCHAER who determined the optimal arrangement for up to ten equal spheres (see SCHAER [5]). In addition, BEZDEK [1] and GOLSER [3] found the maximal radius of $n$ equal non-overlapping spheres within a tetrahedron and an octahedron for small $n$.

Let $v_{1}, \ldots, v_{d}$ be an orthonormal base of the $d$-space, and we write $O^{d}$ to denote the $d$-dimensional regular crosspolytope whose vertices are $\pm v_{1}, \ldots, \pm v_{d}$, and hence the edge length of $O^{d}$ is $\sqrt{2}$. In this paper, we determine the maximal radius $r(n, d)$ of $n$ equal non-overlapping spheres within $O^{d}$ for $n \leq 2 d+1$.

We note that it is equivalent to consider the maximum $\varphi(n, d)$ of the minimal distance of $n$ points in $O^{d}$; more precisely,

$$
r(n, d)=\frac{\varphi(n, d)}{\varphi(n, d) \sqrt{d}+2} .
$$

[^3]We present the results and arguments in terms of point sets in $O^{d}$ because the formulation and the proofs are more transparent this way.

In the planar case, it is an elementary exercise to show that $\varphi(3,2)=$ $=2(\sqrt{3}-1)$, one of the points in the optimal configuration is a vertex of $O^{2}$, and the three points are vertices of some regular triangle. The optimal configuration of four points are the vertices of $O^{2}$, and hence $\varphi(4,2)=\sqrt{2}$. Finally, the centre and the vertices of $O^{2}$ form the optimal configuration of five points, thus $\varphi(5,2)=1$.

It is probably surprising but these estimates generalize to any dimension $d$; namely, we verify that $\varphi(n, d)$ is $2(\sqrt{3}-1)$ if $n=3$, equals $\sqrt{2}$ if $4 \leq n \leq 2 d$, and equals one if $n=2 d+1$ :

THEOREM 1. For $d \geq 3$, let the minimal distance among three points in $O^{d}$ be maximal. Then one of the points is a vertex, say $v_{i}$, and the three points form a regular triangle that is contained in the square with vertices $\pm v_{i}, \pm v_{j}$ for some $v_{j} \neq v_{i}$.

Theorem 1 was proved in GolSER [3] and BEZDEK [1] if $d=3$. Now if the number $n$ of the points in $O^{d}$ is between 4 and $2 d$ then one can not do better than placing them at the vertices:

THEOREM 2. For $d \geq 3$, let the minimal distance among $n$ points in $O^{d}$ be maximal where $4 \leq n \leq 2 d$. Then either each point is a vertex of $O^{d}$, or $n=4$, three of the points are the vertices of a two-face of $O^{d}$, and the fourth point is the centroid of the opposite two-face.

Theorem 2 was proved in GOLSER [3] and BEZDEK [1] if $d=3$. In case of ball packings, Theorem 2 has the following interesting corollary:

If a d-dimensional regular crosspolytope contains four equal solid balls then it can host even $2 d$ solid balls of the same radius.

Finally there exists only one optimal configuration if $n=2 d+1$ :
THEOREM 3. For $d \geq 3$, let the minimal distance among $2 d+1$ points in $O^{d}$ be maximal. Then one of the points is the centre of $O^{d}$, and the other points are the vertices of $O^{d}$.

Theorem 3 was proved in GOLSER [3] and BEZDEK [1] if $d=3$.

Some of ideas in the proofs below are taken from BEZDEK [1] and GOLSER [3]. In order to to simplify notation, we define

$$
v_{i+d}=-v_{i}, \quad i=1, \ldots, d
$$

The scalar product is denoted by $\langle\cdot, \cdot\rangle$, and the origin is denoted by $o$.

## 2. The case of three points

Let $x_{1}, x_{2}, x_{3} \in O^{d}$ be in optimal position, and hence the minimal distance between any two of them is at least $2(\sqrt{3}-1)$. Only at most one out of $x_{1}, x_{2}, x_{3}$ is a vertex of $O^{d}$ because two neighbouring vertices are too close to each other, and the spheres of radius $\sqrt{2}$ around two opposing vertices cover the whole crosspolytope.

First we suppose that the triangle $x_{1}, x_{2}, x_{3}$ is not regular, and seek a contradiction. We may assume that $d\left(x_{1}, x_{2}\right)>d\left(x_{1}, x_{3}\right)$ and $x_{1}$ is not a vertex. Then $x_{1}$ can be moved into a position $x_{1}^{\prime} \in O^{d}$ such that both distances $d\left(x_{1}^{\prime}, x_{2}\right)$ and $d\left(x_{1}^{\prime}, x_{3}\right)$ are larger than $d\left(x_{1}, x_{3}\right)$. Now either $x_{2}$ or $x_{3}$ is not a vertex, and this point can be moved into a new position such that the distances between the pairs of the resulting system of three points in $O^{d}$ are larger than $\varphi(3, d)$. This contradiction yields that any optimal triple determines a regular triangle.

Next we suppose that $x_{1}$ lies in the relative interior of a $k$-face $F$ of $O^{d}$ with $k \geq 2$, and seek a contradiction. Let $H$ be the hyperplane that is the perpendicular bisector of the segment $x_{2} x_{3}$, and let $G$ be the $(d-2)$-plane that is orthogonal to the two-plane $x_{1} x_{2} x_{3}$, and passes through $x_{1}$. Now $G$ is contained in $H$, and we write $G^{+}$to denote the half-hyperplane of $H$ that is bounded by $G$ and does not intersect the segment $x_{2} x_{3}$. Then aff $F \cap G^{+}$ contains a half line $h$ emanating from $x_{1}$, and hence translating $x_{1}$ along $h$ we obtain an optimal triangle that is not regular. This is absurd; therefore each $x_{i}$ is contained in an edge $e_{i}, i=1,2,3$.

Since the convex hull of $e_{i}$ and $e_{j}$ is of diameter greater than $\sqrt{2}$, we deduce that

$$
\begin{equation*}
e_{i} \cap-e_{j} \neq \emptyset \tag{*}
\end{equation*}
$$

First we consider the case when $e_{1}, e_{2}, e_{3}$ are pairwise disjoint. Then we may assume that $e_{1}, e_{2}$ and $e_{3}$ are the segments $v_{1} v_{d+2}, v_{2} v_{d+3}$ and $v_{3} v_{d+1}$,
respectively, and $x_{1}$ is not further from $v_{1}$ than from $v_{d+2}$. We write $m_{1}$ and $m_{2}$ to denote the midpoints of the segments $v_{1} v_{d+2}$ and $v_{2} v_{d+3}$, respectively, and define $m_{3}=\frac{3}{4} \cdot v_{3}+\frac{1}{4} \cdot v_{d+1}$. Since $x_{1}$ is contained in the segment $v_{1} m_{1}$, and the diameter of the tetrahedron $v_{1} m_{1} v_{3} m_{3}$ is

$$
d\left(v_{1}, m_{3}\right)=\frac{\sqrt{34}}{4}<2 \cdot(\sqrt{3}-1)
$$

we deduce that $x_{3}$ is contained in the segment $v_{d+1} m_{3}$. Similarly, the diameter of the tetrahedron $v_{1} m_{1} v_{d+3} m_{2}$ is $\sqrt{2}$, and hence $x_{2}$ is contained in the segment $v_{2} m_{2}$. On the other hand, the diameter of the tetrahedron $v_{2} m_{2} v_{d+1} m_{3}$ is $\sqrt{2}$, therefore $d\left(x_{2}, x_{3}\right)<2 \cdot(\sqrt{3}-1)$. This contradiction yields that say $e_{1}$ and $e_{2}$ have a common vertex $v_{1}$.

According to $(*)$, we may assume that the other endpoints of $e_{1}$ and $e_{2}$ are $v_{2}$ and $v_{d+2}$, respectively. Now one endpoint of $e_{3}$ is $-v_{1}$. If $x_{3}$ is not contained in the square $S$ with vertices $\pm v_{1}, \pm v_{2}$ then we may assume that the other endpoint of $e_{3}$ is $v_{3}$. Now for $i=1,2$, the longest side of the triangle $x_{i} x_{3} v_{3}$ is $x_{i} x_{3}$, and hence the angle $\angle x_{i} x_{3} v_{d+1}$ is obtuse. It follows that $d\left(x_{i}, x_{3}\right)<d\left(x_{i}, v_{d+1}\right)$, or in other words, the triangle $x_{1} x_{2} v_{d+1}$ is optimal but not regular. Therefore $x_{3}$ does lie in $S$. If $S$ has a side $e$ that does not contain any of $x_{1}, x_{2}, x_{3}$ then the triangle $x_{1} x_{2} x_{3}$ can be translated orthogonally to $e$ in a way that one of the $x_{i}$ 's arrives into the interior of $S$. We conclude that one of the $x_{i}$ 's is a vertex of $S$, which in turn yields Theorem 1.

## 3. The case when $4 \leq n \leq 2 d$

In order to verify Theorem 2, it is sufficient to consider the case $n=4$. Therefore let $x_{1}, x_{2}, x_{3}, x_{4} \in O^{d}$ be an optimal set of four points, and hence $d\left(x_{i}, x_{j}\right) \geq \sqrt{2}$ holds for any $i \neq j$.

Let $G_{i}$ denote the convex hull of $v_{i}$ and the midpoints of the edges of $v_{i} v_{j}, j \neq i+d$, and we define

$$
P_{0}=O^{d} \backslash \cup_{i=1}^{2 d} G_{i}
$$

Then the closure of $P_{0}$ is a polytope whose vertices are the midpoints of the edges of $O^{d}$ but these midpoints do not belong to $P_{0}$. Since the maximal distance between the midpoints of the edges of $O^{d}$ is $\sqrt{2}$, we deduce that at most one $x_{i}$ is contained in $P_{0}$.

Now the diameter of any $G_{j}$ is one, and hence any $G_{j}$ contains at most one $x_{i}$. Thus we may assume that $x_{1} \in G_{1}$ and $x_{2} \in G_{2}$. For $0 \leq t \leq \frac{1}{2}$ and $i=1, \ldots, 2 d$, we write $G_{i}^{t}$ to denote the image of $G_{i}$ by the homothety whose centre is $v_{i}$, and the quotient is $2 \cdot t$. Therefore the vertices of $G_{i}^{t}$ are

$$
v_{i} \quad \text { and } \quad(1-t) \cdot v_{i}+t \cdot v_{j} \text { for } v_{j} \neq \pm v_{i}
$$

Now we distinguish two cases:
a) Either $x_{3}$ or $x_{4}$ is contained in some $G_{j}$ with $j \neq d+1, d+2$.

We may assume that $x_{3} \in G_{3}$. Let $1-t$ be the minimum of $\left\langle x_{i}, v_{i}\right\rangle$ for $i=1,2,3$, where $0 \leq t \leq \frac{1}{2}$. We assume that $\left\langle x_{1}, v_{1}\right\rangle=1-t$ and $\left\langle x_{1}, v_{2}\right\rangle \geq\left\langle x_{1}, v_{3}\right\rangle$. Then $x_{1}$ is contained in the part $F$ of the facet of $G_{1}^{t}$ opposite to $v_{1}$ such that the second coordinate of its points is not less than the third coordinate. Thus $F$ is the $(d-1)$-polytope with vertices

$$
\left\{\begin{array}{l}
(1-t) \cdot v_{1} \pm t \cdot v_{i} \\
(1-t) \cdot v_{1}+t \cdot v_{2} \\
(1-t) \cdot v_{1}-t \cdot v_{3} \\
(1-t) \cdot v_{1} \pm \frac{t}{2} \cdot\left(v_{2}+v_{3}\right) .
\end{array} \quad \text { for } i=4, \ldots, d\right.
$$

On the other hand, $x_{2}$ is contained in $G_{2}^{t}$. If $0<t \leq \frac{1}{2}$ then some elementary calculations show for any vertex of $F$ and any vertex of $G_{2}^{t}$ that the distance between the two points is less than $\sqrt{2}$. Therefore $t=0$; namely, $x_{i}=v_{i}$ for $i=1,2,3$. It follows that $x_{4}$ is either another vertex of $O^{d}$ or the centre of the two face $v_{d+1} v_{d+2} v_{d+3}$.
b) Neither $x_{3}$ nor $x_{4}$ is contained in some $G_{j}$ with $j \neq d+1, d+2$.

Then we may assume that $x_{3} \in G_{d+1}$.
We suppose that $x_{4} \in P_{0}$, and seek a contradiction. We may assume that $\left\langle x_{4}, v_{1}\right\rangle \geq\left\langle x_{4}, v_{d+1}\right\rangle$, and hence $x_{4}$ lies in the convex hull of the midpoints of edges of $O^{d}$ that do not contain $v_{d+1}$. Since the distance of any of these midpoints from the vertices of $G_{1}$ is at most $\sqrt{2}$, and $x_{4}$ does not coincide with any of these midpoints, we deduce that $d\left(x_{1}, x_{4}\right)<\sqrt{2}$. This is absurd, and hence $x_{4} \in G_{d+2}$.

The rest of the argument is similar to the one in a). Let $1-t$ be the minimum of $\left\langle x_{1}, v_{1}\right\rangle,\left\langle x_{2}, v_{2}\right\rangle,\left\langle x_{3}, v_{d+1}\right\rangle$ and $\left\langle x_{4}, v_{d+2}\right\rangle$ where $0 \leq t \leq \frac{1}{2}$. We may assume that $\left\langle x_{1}, v_{1}\right\rangle=1-t$ and $\left\langle x_{1}, v_{2}\right\rangle \geq\left\langle x_{1}, v_{d+2}\right\rangle$. Then $x_{1}$ is contained in the part $F^{\prime}$ of the facet of $G_{1}^{t}$ opposite to $v_{1}$ such that the second
coordinate of its points is non-negative, and hence $F^{\prime}$ is the $(d-1)$-polytope with vertices

$$
(1-t) \cdot v_{1}+t \cdot v_{2} \quad \text { and } \quad(1-t) \cdot v_{1} \pm t \cdot v_{i} \quad \text { for } i=3, \ldots, d
$$

On the other hand, $x_{2}$ is contained in $G_{2}^{t}$. If $0<t \leq \frac{1}{2}$ then some simple calculations show for any vertex of $F^{\prime}$ and any vertex of $G_{2}^{t}$ that the distance between the two points is less than $\sqrt{2}$. Therefore $t=0$; namely, $x_{1}=v_{1}$, $x_{2}=v_{2}, x_{3}=v_{d+1}$ and $x_{4}=v_{d+2}$.

## 4. The case of $\mathbf{2 d} \mathbf{+ 1}$ points

We define the Dirichlet-Voronoi cell $D_{i}$ of $v_{i}$ as

$$
D_{i}:=\left\{x \in O^{d}:\left\langle x, v_{i}\right\rangle \leq\left\langle x, v_{j}\right\rangle \text { for all } j \neq i\right\}
$$

Then the vertices of $D_{i}$ are the centroids of the faces of $O^{d}$ that contain $v_{i}$ (including $O^{d}$ as a $d$-face). We note that if $F$ is an $(m-1)$-face with vertices $v_{i_{1}}, \ldots, v_{i_{m}}$ for some $2 \leq m \leq d$ then its centroid is $\frac{v_{i_{1}}+\ldots+v_{i_{m}}}{m}$.

Lemma. Let $x, y \in D_{i}$. Then $d(x, y) \leq 1$, and $d(x, y)=1$ occurs only if either $\{x, y\}=\left\{o, v_{i}\right\}$ or $\{x, y\}=\left\{\frac{v_{i}+v_{j}}{2}, \frac{v_{i}-v_{j}}{2}\right\}$ where $v_{j} \neq \pm v_{i}$.

PROOF. The points $o, v_{i}, \frac{v_{i}+v_{j}}{2}$ and $\frac{v_{i}-v_{j}}{2}$ are all vertices of $O^{d}$, thus we may assume that $x$ and $y$ are vertices of $O^{d}$, as well. We may also assume that $i=1$.

If either $x$ or $y$ is the origin or $v_{1}$ then the Lemma readily holds, and hence let $x$ and $y$ be contained in an $(m-1)$-face $F$ and $(k-1)$-face $G$ of $O^{d}$ for $2 \leq m, k \leq d$, respectively. We assume that $k \leq m$. We write $p$ to denote for the number of common vertices of $F$ and $G$, and $q$ to denote for the number of vertices of $F$ whose opposite is a vertex of $G$. Then

$$
d(x, y)^{2}=p \cdot\left(\frac{1}{k}-\frac{1}{m}\right)^{2}+q \cdot\left(\frac{1}{k}+\frac{1}{m}\right)^{2}+\frac{m-q-p}{m^{2}}+\frac{k-q-p}{k^{2}}
$$

Now $p \geq 1$, and hence for given $k$ and $m$, the possible maximum of $d(x, y)^{2}$ is attained if $p=1$ and $q=k-1$; namely, when

$$
d(x, y)^{2}=\frac{1}{m} \cdot\left(3-\frac{4}{k}\right)+\frac{1}{k} \leq \frac{4}{k} \cdot\left(1-\frac{1}{k}\right)
$$

Therefore $d(x, y)^{2}$ is maximal if $m=k=2$, and $F$ is an edge $v_{i} v_{j}$, while $G$ is the edge $v_{i}\left(-v_{j}\right)$.

Let us prove Theorem 3. For $x \in O^{d}$ and $i=1, \ldots, 2 d$, we define the weight $w_{i}(x)$ of $x$ as

$$
w_{i}(x)=\left\{\begin{array}{ll}
\frac{1}{\#\left\{j: x \in D_{j}\right\}} & \text { if } x \in D_{i} \\
0 & \text { if } x \notin D_{i}
\end{array} .\right.
$$

We note that $\sum_{i=1}^{2 d} w_{i}(x)=1$ holds for any $x \in O^{d}$.
Now the vertices and the centre of $O^{d}$ form $2 d+1$ points whose mutual distances are at least one. Thus let $x_{1}, \ldots, x_{n}$ be $n$ points in $O^{d}$ such that the mutual distances are at least one. The Lemma yields for any $i=1, \ldots, 2 d$ that

$$
\sum_{j=1}^{n} w_{i}\left(x_{j}\right) \leq 1+\frac{1}{2 d}
$$

and equality holds if and only if the points of $\left\{x_{1}, \ldots, x_{n}\right\}$ in $D_{i}$ are the origin and $v_{i}$. Therefore

$$
n=\sum_{i=1}^{2 d} \sum_{j=1}^{n} w_{i}\left(x_{j}\right) \leq 2 d+1
$$

and equality holds if and only if $x_{1}, \ldots, x_{n}$ consists of the vertices and the centre of $O^{d}$.

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# A PROOF OF ESCHER'S (ONLY?) THEOREM 

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The famous Dutch artist, M. C. ESCHER created amazing patterns and tiles in the plane. His drawings (as we can learn from [1]) were preceded by mathematical, or more precisely geometrical studies. While doing so, he stated a theorem (or rather a conjecture):

ESCHER'S THEOREM. When dividing a side of a triangle into three seqments of equal length another side into four segments of equal length and the third into five segments of equal length we get a set of dividing points (including the vertices). If we join these points to form segments, in certain cases there will be three of them intersecting in one point, and that inside the triangle (Fig. 1).


Fig. 1

There are 17 such cases. In these cases the common point of the segments intersects them in a "nice" ratio (see Fig. 2).

Escher himself didn't give a proper proof of the fact. He made drawings, excluding the cases where there were no intersection points inside the triangle. In the rest of the cases he made more accurate drawings to see if the seemingly intersection point was really one.

Out of these 17 cases 15 are proved, as we can read about it in [1].
In this article we give a proof different from the type one would most probably choose: instead of elementary geometric means we use linear algebraic methods.


Case 1
$A S: S D=2: 1$
$B S: S E=1: 5$
$C S: S F=1: 1$


Case 11
$A S: S D=1:$
$B S: S E=3:$
$C S: S F=3: 2$


Case 15
$A S: S D=3: 1$
$B S: S E=1: 1$

$C S: S F=3: 7$


Case 2
$A S: S D=8: 3$;
$B S: S E=6: 5$;
$C S: S F=9: 2$;


Case 4
$A S: S D=1: 1$
$B S$
$C S$
$C S: S F=4: 1$


Case 8
$A S: S D=2: 1$
$B S: S E=1: 1$
$C S: S F=3: 2$


Case 12
$A S: S D=3: 4$
$B S: S E=2: 5$
$C S: S F=9: 5$


Case 16
$A S: S D=2: 3$
$B S: S E=1: 4$
$C S: S F=3: 7$


Case 5
$A S: S D=1: 1$
$B S: S E=3: 5$
CS:SF$=4: 1$


Case 9
$A S: S D=4: 1$
$B S: S E=2: 3$
$C S: S F=9: 16$


Case 13
$S: S D=1: 1$
$A S: S D=1: 1$
$B S: S E=1: 5$
$C S: S F=3: 5$


Case 17
$A S: S D=1: 2$
$B S: S E=1: 3$
$C S: S F=1: 1$

Fig. 2
The 17 cases of Escher's theorem

The theorem actually consists of three statements:
i) the line-segments intersect each other in one point in the mentioned 17 cases;
ii) in the rest of the cases no common intersection point can be obtained;
iii) the intersection point divides the segments into rational parts of the segment.

We begin with statement iii)
Proof of iii). Let us denote the set of dividing points and the vertices by $S$.

Let the triangle be positioned in the plane as shown on Fig. 3. Let $\mathbf{i}$ denote the $1 / 60$ th of the vector pointing to vertex $B$ and $\mathbf{j}$ the $1 / 60$ th of the vector pointing to vertex $C$. We then consider the vector field generated by $\mathbf{i}$ and $\mathbf{j}$ over the field of rational numbers: $V$.

It is clear that all points in $S$ can be represented as a rational linear combination of $\mathbf{i}$ and


Fig. 3 $\mathbf{j}$, that is, $S \subset\langle\mathbf{i}, \mathbf{j}\rangle$.

If $a_{1}, a_{2}, b_{1}, b_{2}, c_{1}, c_{2} \in S$ are the endpoints of line-segments satisfying the "Escher-condition", then their intersection point is also in the vector field $V$, since we can calculate the coordinates of it by solving a linear system of equations in $V$. This means that taking the two endpoints of a line-segment as a new base of $V$, the intersection point of the segments (a point of all three segments) is a rational linear combination of these new basis vectors, thus proving iii).

We now continue with proving statement i):
Proof of i). Let $a_{1}, a_{2}, b_{1}, b_{2}, c_{1}, c_{2} \in S$ satisfy the "Escher-condition". $a_{1}=\alpha_{11} \mathbf{i}+\alpha_{12} \mathbf{j}, a_{2}=\alpha_{21} \mathbf{i}+\alpha_{22} \mathbf{j}, b_{1}=\beta_{11} \mathbf{i}+\beta_{12} \mathbf{j}, b_{2}=\beta_{21} \mathbf{i}+\beta_{22} \mathbf{j}$, $c_{1}=\gamma_{11} \mathbf{i}+\gamma_{12} \mathbf{j}, c_{2}=\gamma_{21} \mathbf{i}+\gamma_{22} \mathbf{j}$.

The common intersection point divides $a_{1} a_{2}$ in the ratio $a, b_{1} b_{2}$ in the ratio $b$ and $c_{1} c_{2}$ in the ratio $c$. We have shown that $a, b, c$ are rational
numbers. So $a \cdot\left(\vec{a}_{2}-\vec{a}_{1}\right), b \cdot\left(\vec{b}_{2}-\vec{b}_{1}\right)$ and $c \cdot\left(\vec{c}_{2}-c_{1}\right)$ are the same points. Then the intersection point can be written as

$$
\begin{gathered}
a \cdot\left(\alpha_{21} \mathbf{i}+\alpha_{22} \mathbf{j}-\alpha_{11} \mathbf{i}-\alpha_{12} \mathbf{j}\right)= \\
b \cdot\left(\beta_{21} \mathbf{i}+\beta_{22} \mathbf{j}-\beta_{11} \mathbf{i}-\beta_{12} \mathbf{j}\right)= \\
c \cdot\left(\gamma_{21} \mathbf{i}+\gamma_{22} \mathbf{j}-\gamma_{11} \mathbf{i}-\gamma_{12} \mathbf{j}\right) .
\end{gathered}
$$

The above chain of equations can be rewritten two by two forming a linear system of three equations. The solutions (if they exist) are also rational numbers.

In each of the 17 cases we write the chosen points as the rational linear combination of $\mathbf{i}$ and $\mathbf{j}$. Then we can write a rational linear system of two unknowns to define the intersection points of two line-segments.

To check the 17 cases we have to write these 17 systems of equations. We are going to check only three cases ( 3,4 and 5 , since there are minor mistakes about these results in [1]).

We express the intersection points with vectors:
(3) $20 \mathbf{j}+a \cdot(30 \mathbf{i}+10 \mathbf{j})=36 \mathbf{i}+b \cdot(60 \mathbf{j}-36 \mathbf{i})=60 \mathbf{i}+c \cdot(40 \mathbf{j}-60 \mathbf{i})$
(4) $a \cdot(15 \mathbf{i}+45 \mathbf{j})=12 \mathbf{i}+b \cdot(60 \mathbf{j}-12 \mathbf{i})=30 \mathbf{i}+30 \mathbf{j}+c \cdot(-30 \mathbf{i}-10 \mathbf{j})$
(5) $a \cdot(30 \mathbf{i}+30 \mathbf{j})=24 \mathbf{i}+b \cdot(40 \mathbf{j}-24 \mathbf{i})=60 \mathbf{i}+c \cdot(20 \mathbf{j}-60 \mathbf{i})$

We then write the corresponding systems of equations and the given ratios.
(3) $a \cdot 30=36-b \cdot 36=60-c \cdot 60$

| $20+a \cdot 10=b \cdot 60=c \cdot 40$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ |
| :--- | :--- | :--- | :--- |
| $a \cdot 45=b \cdot 60=30-c \cdot 10$ | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{4}{5}$ |
| $a \cdot 30=b \cdot 40=c \cdot 20$ | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{5}{5}$ |

Verifying by substitution we find that the results are not correct. Solving the equations we get:
in case (3) $a=\frac{2}{3}, b=\frac{4}{9}(!), c=\frac{2}{3}(!) ; \quad$ in case (4) $a=\frac{1}{2}, b=\frac{3}{8}, c=\frac{3}{4}(!)$; in case (5) $a=\frac{1}{2}, b=\frac{3}{8}, c=\frac{3}{4}(!)$.

The ratios we got can be obtained by means of elementary geometric tools, but it seems that we have to give a different proof for each case. Our proof is using linear algebraic methods and give a universal result.

As the matter of statement ii) we only show how we can verify that the rest of the cases do not satisfy Escher's condition.

We check it the other way round: Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}$ be 5 out of the 6 chosen points. Taking two and another two of them as the end points of two
line segments, we can check whether the line connecting their intersection point to the fifth point will intersect the perimeter of the triangle in one of the points of set $S$ or not.

In case of taking the points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}$ we get the vector equation

$$
\mathbf{p}+a \cdot(\mathbf{q}-\mathbf{p})=\mathbf{r}+b \cdot(\mathbf{s}-\mathbf{r})
$$

from which we express $\mathbf{i}$ and $\mathbf{j}$. The solution is the two coordinates of the intersection point of segments $\mathbf{p}-\mathbf{q}$ and $\mathbf{r}-\mathbf{s}$ : $\mathbf{u}$. The line through $\mathbf{u}$ and $\mathbf{t}$ intersects the line of the sides of the triangle, we get three equations, one equation to each side:

$$
\begin{equation*}
\mathbf{t}+c_{1}(\mathbf{u}-\mathbf{t})=d_{1}(60 \mathbf{i}+0 \mathbf{j}) \tag{1}
\end{equation*}
$$

Good values for $d_{1}$ are: $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$. Good values for $d_{2}$ are: $0, \frac{1}{3}, \frac{2}{3}, 1$. Good values for $d_{3}$ are: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

If we do not get the appropriate values, then the corresponding intersection point is not in set $S$.

What can be said in general? We can take any triangle in the plane, we can take any dividing points, the same problem can always be solved the same way. Only, we cannot be sure that there will be segments intersecting in one point.

We cannot take any $n$-gon, since we cannot be sure that the dividing points can be expressed in the vector field over the set of rational numbers as the linear combination of two sides we choose. However, if we make sure that the dividing points of the $n$-gon are linear combinations of some sides we choose, the same question can be answered.

In space we can take a tetrahedron, we can divide its vertices and ask the same question. In this case we use a 3-dimensional vector field over the rational numbers. Instead of the dividing points of the vertices we might want to take points on the faces of the tetrahedron. We can only choose points that can be expressed as a rational linear combination of the three basis vectors. We still cannot be certain that there will be three line segments intersecting in one point at all.

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# ON THE RIGIDITY OF RAMANUJAN GRAPHS 

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## 1. Introduction

A graph $G$ is generically [4] rigid in dimension one if and only if it contains a spanning tree, that is, a spanning subgraph assembled by inductively joining 1 -simplices along 0 -simplices. The analogous property is sufficient but not necessary for the generic rigidity of graphs in higher dimensions, that is, a generically rigid graph in $\mathbb{R}^{n}$ need not contain a spanning subgraph consisting of $n$-simplices joined along $(n-1)$-simplices, see Figure 1a. Indeed, a graph which is generically rigid in the plane need not contain any triangles at all. For example the graph in Figure 1b, $K_{3,3}$ is generically isostatic in $\mathbb{R}^{2}$, and its shortest cycle is of length four. Observe further that the graph in Figure 1a behaves tree-like with respect to rigidity in the sense that the removal of any single edge cuts the graph into two rigid components. In sharp contrast, the removal of any edge of $K_{3,3}$ produces a graph of degree of freedom one in which any edge can move nontrivially relative to any other edge, which is to say that the set of maximal rigid subgraphs equals the edge set.

b


Fig. 1

The two graphs in Figure 1 also behave quite differently with respect to the addition of a single edge. For the graph in Figure 1a the addition of an edge yields minimally dependent sets of various sizes depending on where it is placed. On the other hand, the addition of any edge to $K_{3,3}$ produces a single minimally dependent edge set comprising all 10 edges, i.e. the graph can be globally reinforced by the addition of a single edge.

It is our aim to construct rigid graphs in the plane of large girth and show what they possess the feature properties of $K_{3,3}$.

## 2. The Ramanujan Graph $X^{p, q}$

The length of the shortest cycle in a graph is called the girth of the graph. If we fix the number of vertices and try to construct an edge maximal graph of large girth, we expect the connectivity to be low which tends to procedure non-rigidity. A graph theoretic concept that might be more intimately related to rigidity than connectivity is toughness. A graph is $t$-tough if the removal of at least $t x$ vertices is necessary to disconnect the graph into $x$ connected components (where $x>1$ ). Note that $t$-toughness implies $2 t$-connectivity but the reverse implication is not true.

We now describe the construction of a class of Cayley graphs given in [7]: Let $p$ and $q$ be primes, $p \equiv q-1(\bmod 4) . X^{p, q}$ will be a $(p+1)$-regular graph, namely the Cayley graph of $\operatorname{PSL}(2, q)$ if $\left(\frac{p}{q}\right)=1$ (where $\left(\frac{p}{q}\right)$ is the Legendre symbol) and $\operatorname{PGL}(2, q)$ if $\left(\frac{p}{q}\right)=-1$. The generators correspond to the $p+1$ ways of presenting $p$ as a sum of four squares under the following normalizing conditions: $p=a_{0}^{2}+a_{1}^{2}+a_{2}^{2}+a_{3}^{2}$ (with $a_{0}>0, a_{0}$ odd and $a_{j}$ even for $j \in\{1,2,3\}$.)

The number of representations of integrs by certain quarternary quadratic forms is needed in the construction and in the proofs that the constructions work. Progress on one of Ramanujan's conjectures was a necessary ingredient in the work of Lubotzky, Phillips and Sarnak, [7] hence the name Ramanujan graph was chosen by them. Ramanujan graphs possess, among other nice extremal properties large chromatic number, incidence number and girth, $g>2 \log _{p}(q)$, and good expansion properties. The second largest eigenvalue of their adjacency matrix equals $2 \sqrt{p}$.

In [1] an explicit proof is given that the toughness $t$ of $X^{p, q}$ satisfies

$$
t>\frac{1}{3}\left(\frac{(p+1)^{2}}{\sqrt{2}(p+1)+p}-1\right) .
$$

Therefore we can choose $p$ large enough so that $t>3$. Choose $q$ large enough so that $2 \log _{p}(q) \geq g$. Then $X^{p, q}$ will be 3-tough, therefore 6 -connected, hence generically rigid in $\mathbb{R}^{2}$ by [6], and of girth at least $g$ by the bounds in [7]. We have proved the following.

Theorem 2.1. Given a natural number $g$ there exists a graph which is generically rigid in the plane and has girth at least $g$.

While upper and lower bounds of $X^{p, q}$, see [2], [7] are quite close, the bound on the touhgness is not tight. Looking for a triangle free rigid graph in the plane using these bounds we would need to construct $X^{401, q}$ where $q$ is a prime number larger than $401^{3}=64,481,201$, so the number of vertices is on the order of $10^{23}$. Note that $K_{3,3}$ does the job with only 6 vertices. Thus it would be of great interest to study the rigidity properties of the Ramanujan graphs directly.

## 3. An Example: $X^{5,13}$

We now construct the Ramanujan graph $X^{5,13}$. There are $8(p+1)=48$ solutions to $a_{0}^{2}+a_{1}^{2}+a_{2}^{2}+a_{3}^{2}=5$, with 6 of them having the property that $a_{0}>0$ and $a_{1}, a_{2}, a_{3}$ even and $a_{0}$ odd. To each of these solutions $\alpha$ we associate a matrix $\tilde{\alpha}$ in $\operatorname{PGL}(2, q)$ as follows:

$$
\tilde{\alpha}=\left[\begin{array}{cc}
a_{0}+i a_{1} & a_{2}+i a_{3} \\
-a_{2}+i a_{3} & a_{0}-i a_{1}
\end{array}\right]
$$

where $i^{2} \equiv-1(\bmod 3)$.

$$
\begin{array}{ll}
\alpha_{1}=(1,0,0,-2) & \tilde{\alpha}_{1}=\left[\begin{array}{ll}
1 & 3 \\
3 & 1
\end{array}\right] \\
\alpha_{2}=(1,0,-2,0) & \tilde{\alpha}_{2}=\left[\begin{array}{cc}
1 & 11 \\
2 & 1
\end{array}\right] \\
\alpha_{3}=(1,-2,0,0) & \tilde{\alpha}_{3}=\left[\begin{array}{cc}
4 & 0 \\
0 & 11
\end{array}\right] \\
\alpha_{4}=(1,0,0,2) & \tilde{\alpha}_{4}=\left[\begin{array}{cc}
1 & 10 \\
10 & 1
\end{array}\right] \\
\alpha_{5}=(1,0,2,0) & \tilde{\alpha}_{5}=\left[\begin{array}{cc}
1 & 2 \\
11 & 1
\end{array}\right] \\
\alpha_{6}=(1,2,0,0) & \tilde{\alpha}_{6}=\left[\begin{array}{cc}
11 & 0 \\
0 & 4
\end{array}\right]
\end{array}
$$

The six matrices $\tilde{a}_{i}$ are the generators for $X^{5,13}$, the Cayley graph of the group PGL $(2,13)$. The Cayley graph is bipartite and has $n=q\left(q^{2}-1\right)=2,184$ vertices. It is 6-regular and hence has 6,552 edges. The rigidity matrix in $\mathbb{R}^{3}$ is a square matrix of size 6,552 . It follows that $X^{5,13}$ is generically dependent in $\mathbb{R}^{3}$.

In [9], $X^{5,13}$ was randomly embedded in $\mathbb{R}^{3}$ and the rank of the corresponding rigidity matrix was computed to be 6,546 , which shows that $X^{5,13}$ is generically rigid in dimension three. The girth was computed to be 8 . (The theory of cages [8] yields that the girth is at most 10 , the bounds from [7] and [2] imply that 6 and 8 are the only possible values.) $X^{5,13}$ is not only rigid, it remains rigid even after the removal of any two vertices, or after the removal of six "random" edges.

## 4. Open Problems

Is $X^{5, q}$ rigid (vertex bririgid) for all $q$ ? If one could show that they are Hamiltonian, in fact, possibly even the union of three disjoint Hamiltonian cycles, one might be able to use the 6T3 decompositions obtained by deleting 6 of the edges of the graphs (avoiding the removal of more than 3 incident with one vertex) to show rigidity.

Is there a realization of $X^{5, q}$ in $\mathbb{R}^{3}$ such that the ratio of the longest to shortest edge is small, and the ratio of the diameter to the length of the longest edge is large?

An example of an embedding of a regular vertex birigid graph in 2-space is the following: $G=(V, E), V=\{1,2, \ldots, n\}, E=\{(1,(i+3) \bmod n)\} \cup$ $\cup\{(i,(i+1) \bmod n)\}$. If the vertices are embedded on a regular polygon, the graph is realized with two edges lengths and, as $n$ approaches infinity, the ratio of the diameter to either of these edge lengths approaches infinity also, while the retio of the longest to shortest length approaches 3 . These graphs are the edge disjoint union of two Hamiltonian cycles, as indicated by the thick and thin edges of Figure 2. One can use this partition to quickly get a 3 T 2 decomposition of the graph (after the deletion of three non-mutually-incident edges).


Fig. 2

Recently, in [5], an algorithm was published which generates random $k$-regular graphs on $n$ vertices quickly. Is a random 6-regular graph rigid with probability converging to 1 as $n$ goes to infinity? Given a random embedding of a 6-regular graph, can anything be said about the proportion of long edges to short edges as described in the previous problem?

Given $t$ and $g$, one can construct a graph which is $2 t$-connected (in fact $t$-tough) and has girth at least $g$. For $t=3$ this provides a class of rigid graphs in $\mathbb{R}^{2}$ which has arbitrarily large girth. Is $t=3$ best possible for $\mathbb{R}^{2}$ ? What $t$ works for the same result in $\mathbb{R}^{3}$ ?

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